ADVANCED ANALYSIS OF ALGORITHMS
CPS 5440

UNIT 1: THE ROLE OF ALGORITHMS IN COMPUTING AND FOUNDATIONS REVIEW

ANALYSIS OF ALGORITHMS

The theoretical study of computer-program performance and resource usage.

What's more important than performance?

- modularity
- correctness
- maintainability
- functionality
- robustness
- user-friendliness
- programmer time
- simplicity
- extensibility
- reliability

WHY STUDY ALGORITHMS AND PERFORMANCE?

- Algorithms help us to understand ***scalability***.
- Performance often draws the line between what is feasible and what is impossible.
- Algorithmic mathematics provides a ***language*** for talking about program behavior.
- Performance is the ***currency*** of computing.
- The lessons of program performance generalize to other computing resources.
- Speed is fun!

THE PROBLEM OF SORTING

Input: sequence $\langle a_1, a_2, \ldots, a_n \rangle$ of numbers.

Output: permutation $\langle a'_1, a'_2, \ldots, a'_n \rangle$ such that $a'_1 \leq a'_2 \leq \cdots \leq a'_n$.

Example:

Input: 8 2 4 9 3 6

Output: 2 3 4 6 8 9

INSERTION SORT

"pseudocode"

INSERTION-SORT (A, n) $\triangleleft$ $A[1 \,..\, n]$

for $j \leftarrow 2$ **to** n
  do $key \leftarrow A[j]$
    $i \leftarrow j - 1$
    while $i > 0$ and $A[i] > key$
      do $A[i+1] \leftarrow A[i]$
        $i \leftarrow i - 1$
    $A[i+1] = key$

INSERTION SORT

"pseudocode"

```
INSERTION-SORT (A, n)    ◁ A[1 . . n]

    for j ← 2 to n
        do key ← A[ j]
           i ← j – 1
           while i > 0 and A[i] > key
               do A[i+1] ← A[i]
                  i ← i – 1
           A[i+1] = key
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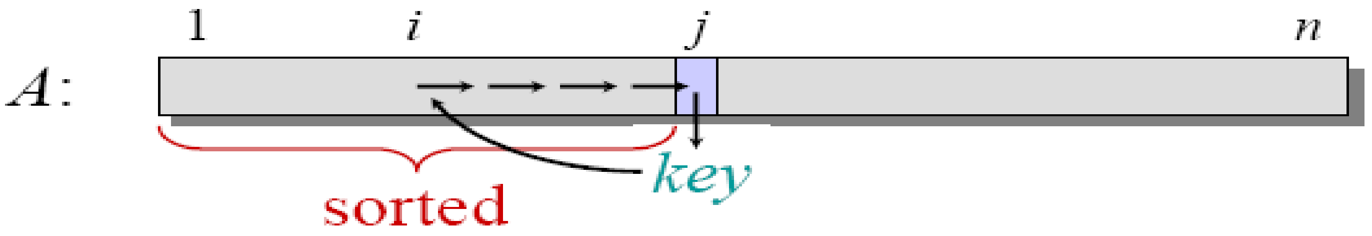

EXAMPLE OF INSERTION SORT

8 2 4 9 3 6

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2 8 4 9 3 6

EXAMPLE OF INSERTION SORT

8	2	4	9	3	6
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2	4	8	9	3	6

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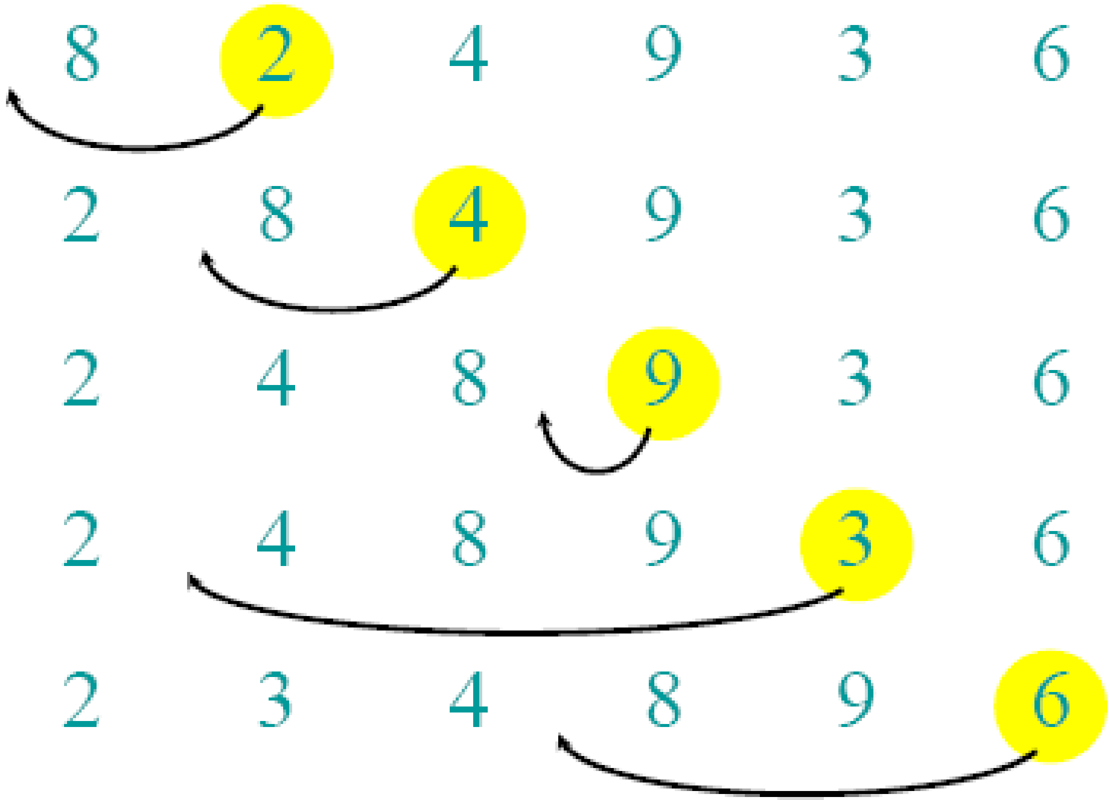
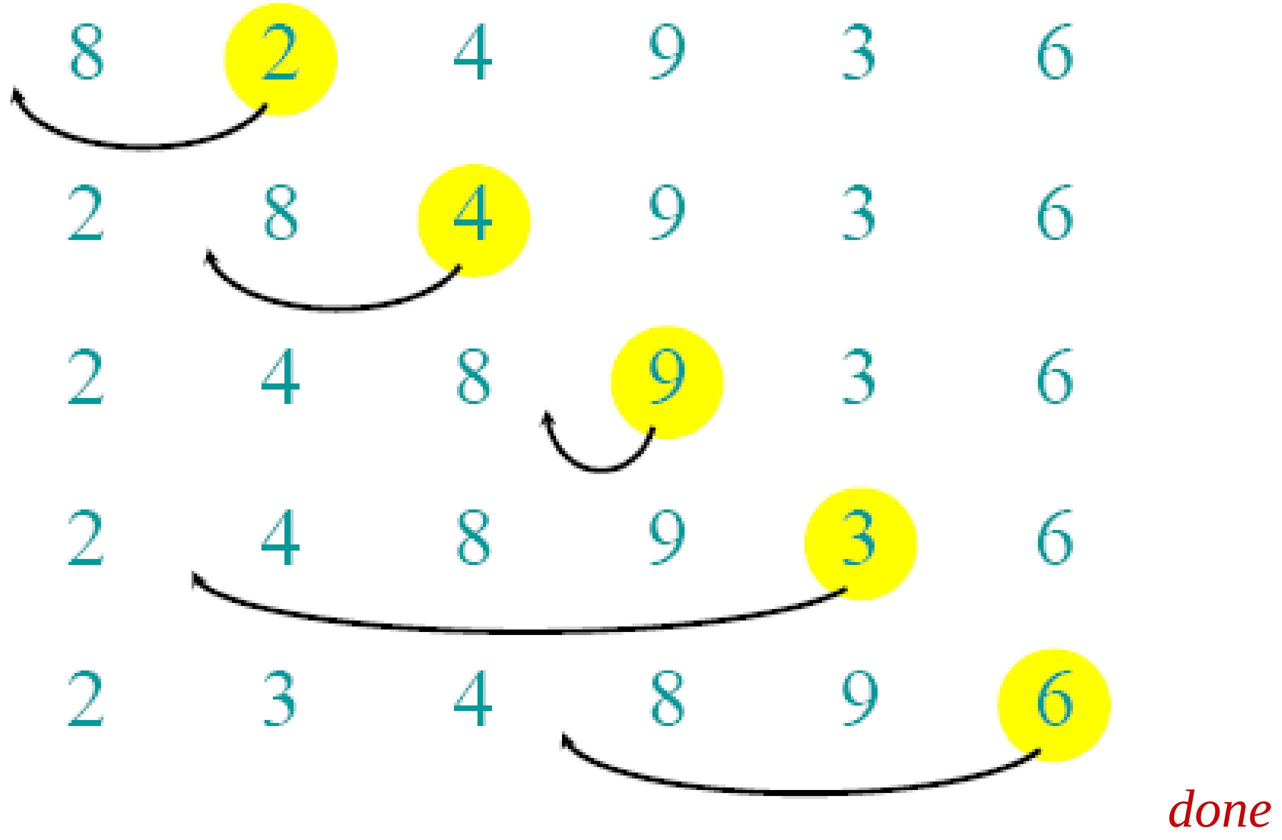
EXAMPLE OF INSERTION SORT



EXAMPLE OF INSERTION SORT



RUNNING TIME

- The running time depends on the input: an already sorted sequence is easier to sort.
- Parameterize the running time by the size of the input, since short sequences are easier to sort than long ones.
- Generally, we seek upper bounds on the running time, because everybody likes a guarantee.

KINDS OF ANALYSIS

Worst-case: (usually)

- $T(n)$ = maximum time of algorithm on any input of size n.

Average-case: (sometimes)

- $T(n)$ = expected time of algorithm over all inputs of size n.
- Need assumption of statistical distribution of inputs.

Best-case: (bogus)

- Cheat with a slow algorithm that works fast on *some* input.

MACHINE INDEPENDENT TIME

What is insertion sort's worst-case time?

- It depends on the speed of our computer:
  - relative speed (on the same machine),
  - absolute speed (on different machines).

BIG IDEA:

- Ignore machine-dependent constants.
- Look at ***growth*** of $T(n)$ as $n \to \infty$.

"Asymptotic Analysis"

Θ-NOTATION

Math:

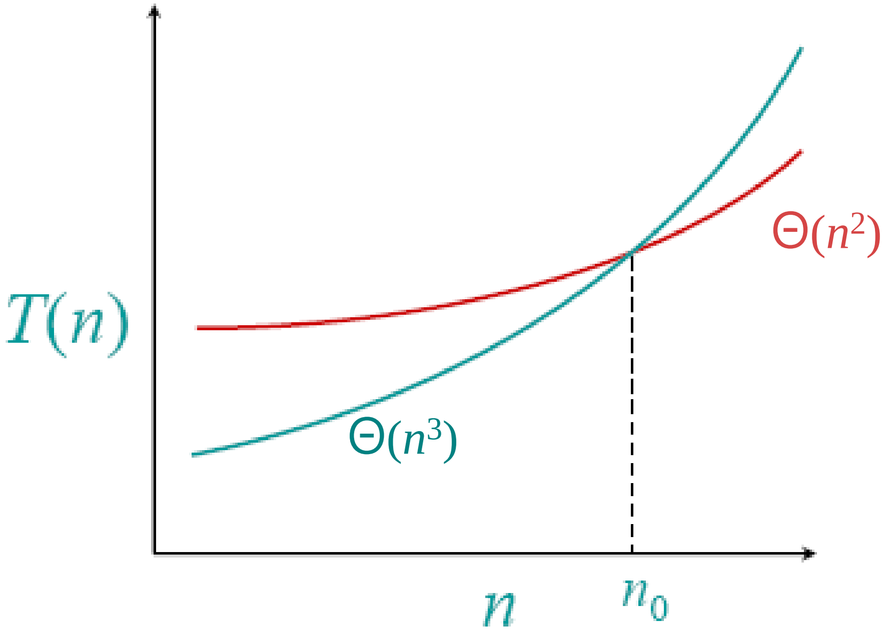
$\Theta(g(n)) = \{ f(n)$: there exist positive constants c_1, c_2, and n_0 such that $0 \le c_1\, g(n) \le f(n) \le c_2\, g(n)$ for all $n \ge n_0$ $\}$

Engineering:

- Drop low-order terms; ignore leading constants.
- Example: $3n^3 + 90n^2 - 5n + 6046 = \Theta(n^3)$

ASYMPTOTIC PERFORMANCE

When n gets large enough, a $\Theta(n^2)$ algorithm *always* beats a $\Theta(n^3)$ algorithm.



- We shouldn't ignore asymptotically slower algorithms, however.
- Real-world design situations often call for a careful balancing of engineering objectives.
- Asymptotic analysis is a useful tool to help to structure our thinking.

INSERTION SORT ANALYSIS

Worst case: Input reverse sorted

$$T(n) = \sum_{j=2}^{n} \Theta(j) = \Theta(n^2)$$

[arithmetic series]

Average case: All permutations equally likely.

$$T(n) = \sum_{j=2}^{n} \Theta(j/2) = \Theta(n^2)$$

Is insertion sort a fast sorting algorithm?

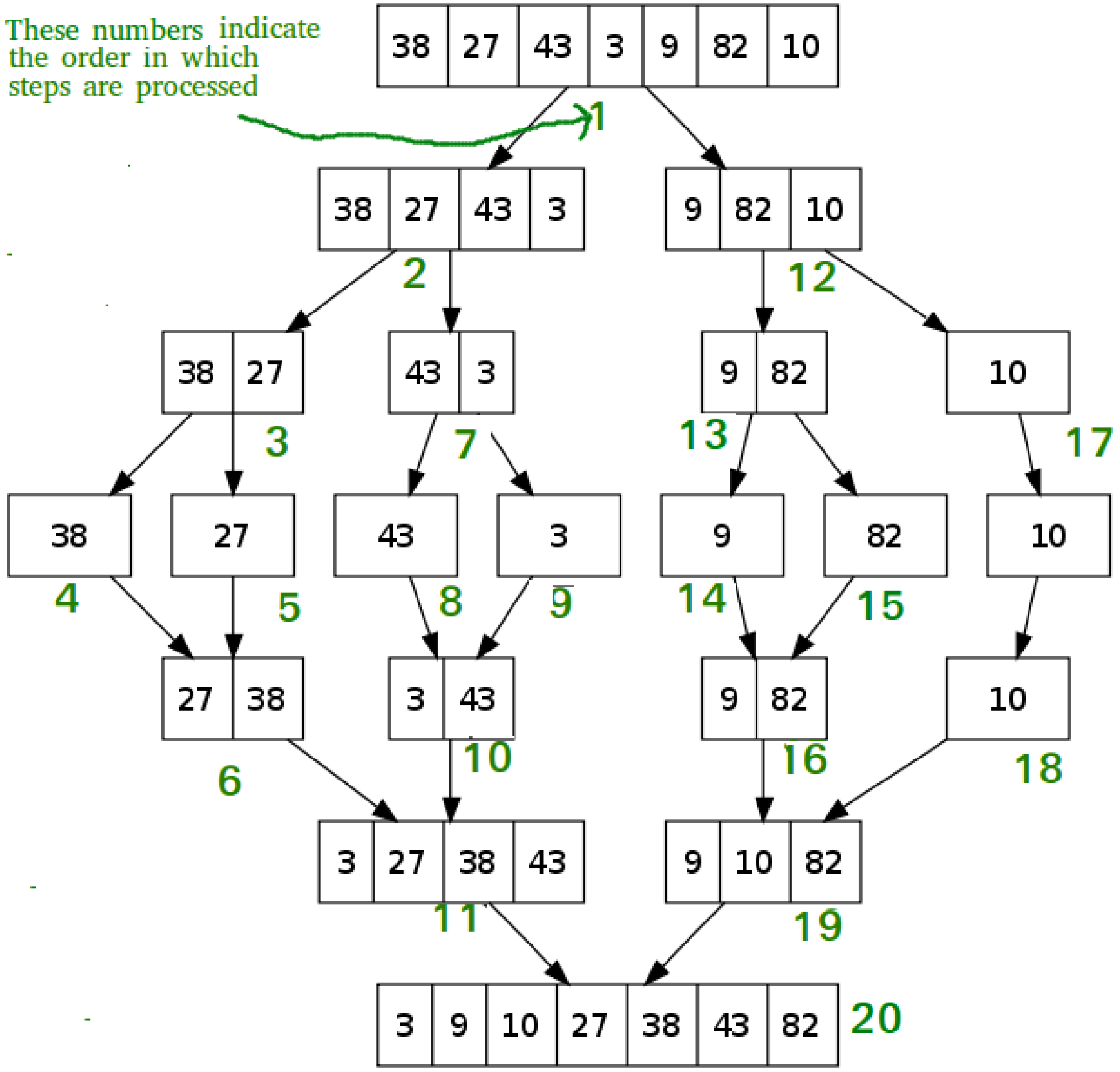
- Moderately so, for small n.
- Not at all, for large n.

MERGE-SORT $A[1 \,.\,.\, n]$

1. If $n = 1$, done.
2. Recursively sort $A[\, 1 \ldots \lceil n/2 \rceil \,]$ and $A[\, \lceil n/2 \rceil + 1 \ldots n \,]$.
3. "***Merge***" the 2 sorted lists.

Key subroutine: **MERGE**

MERGE SORT EXAMPLE



MERGE-SORT $A[1 \,..\, n]$

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Key subroutine: **MERGE**

MERGING TWO SORTED ARRAYS

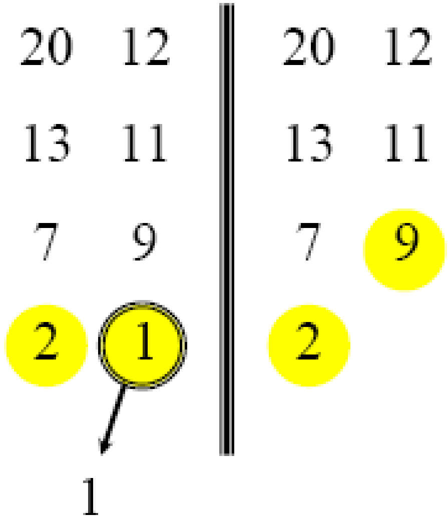
20	12
13	11
7	9
2	1

MERGING TWO SORTED ARRAYS

20	12
13	11
7	9
2	1

1

MERGING TWO SORTED ARRAYS



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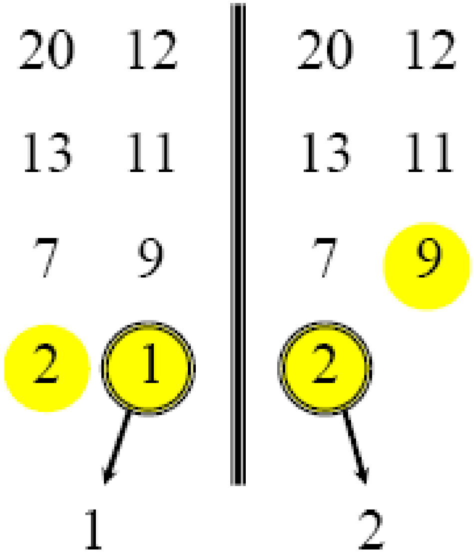

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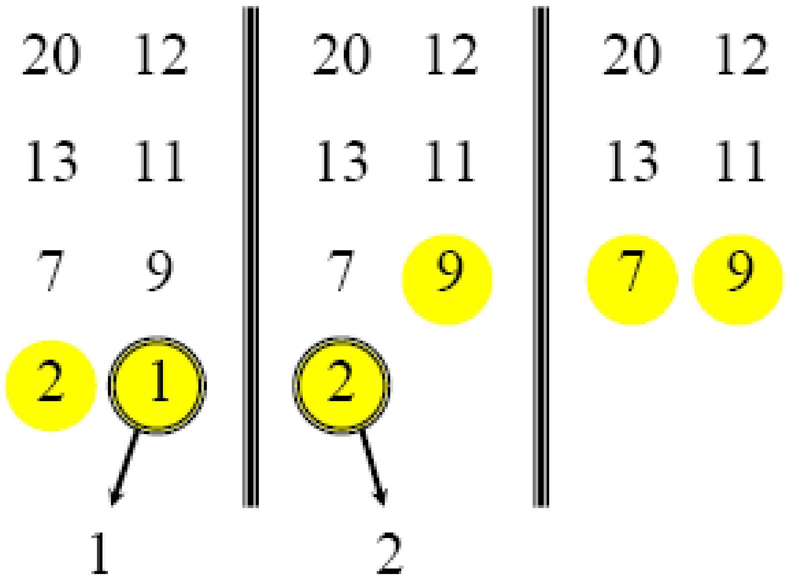

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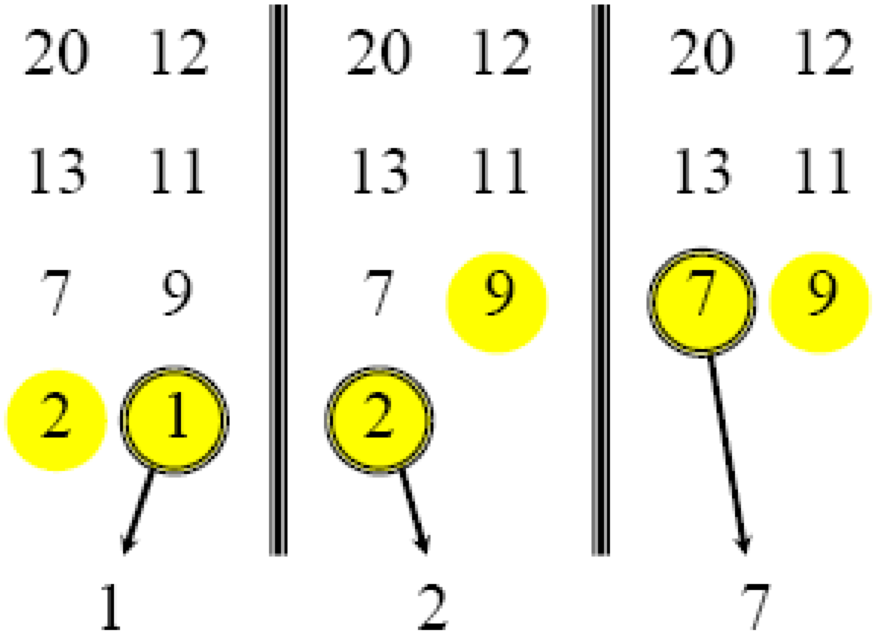

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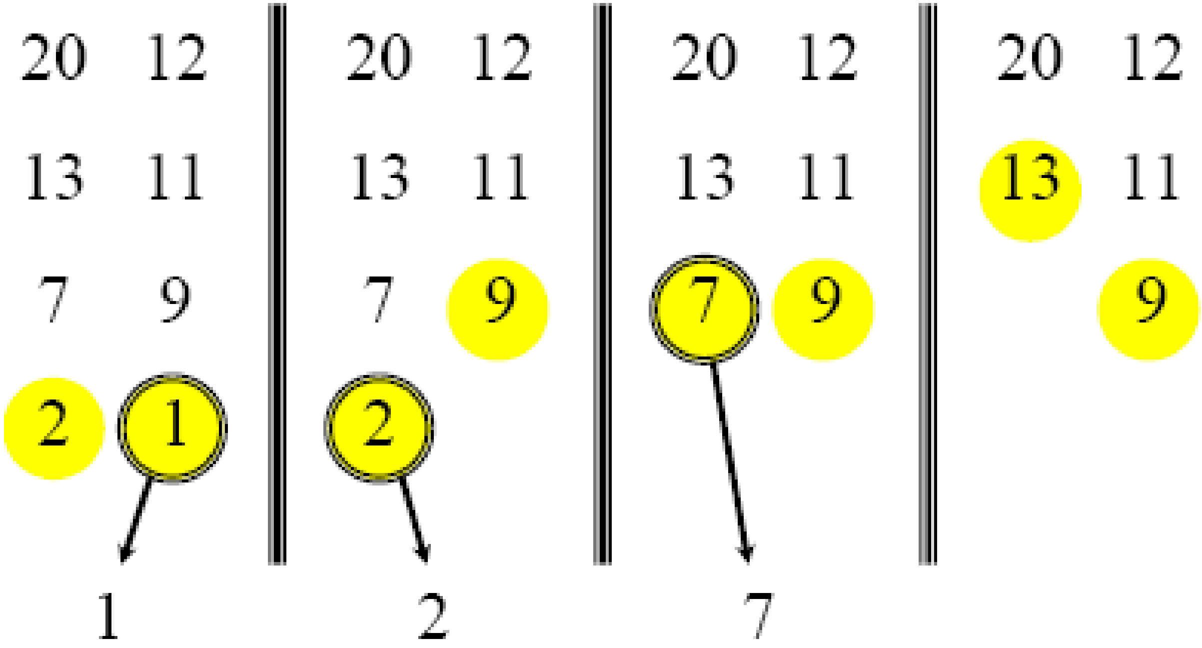

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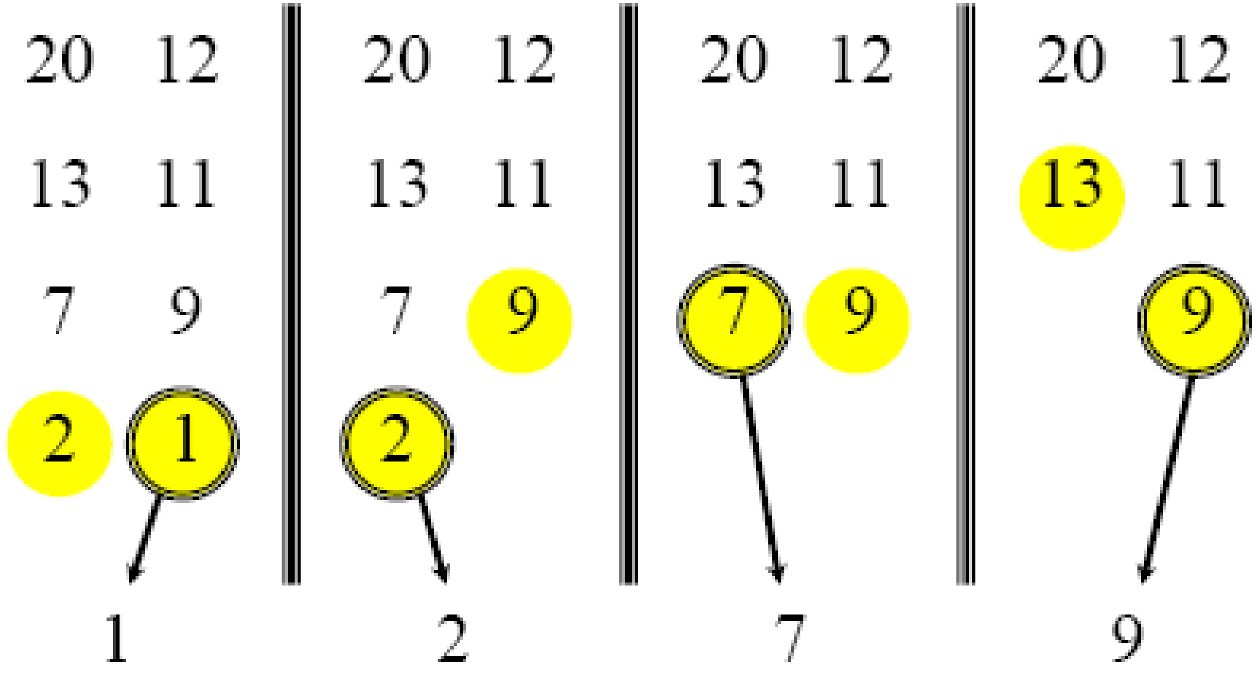

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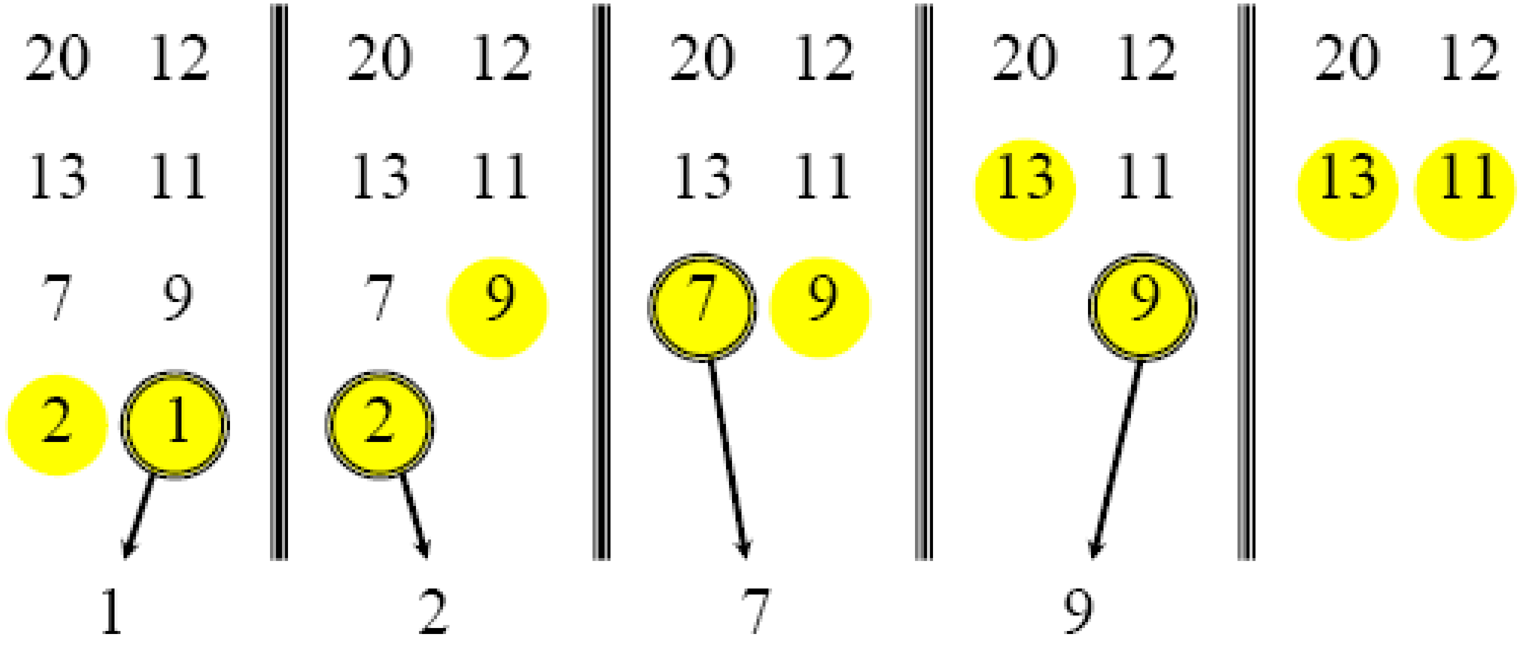

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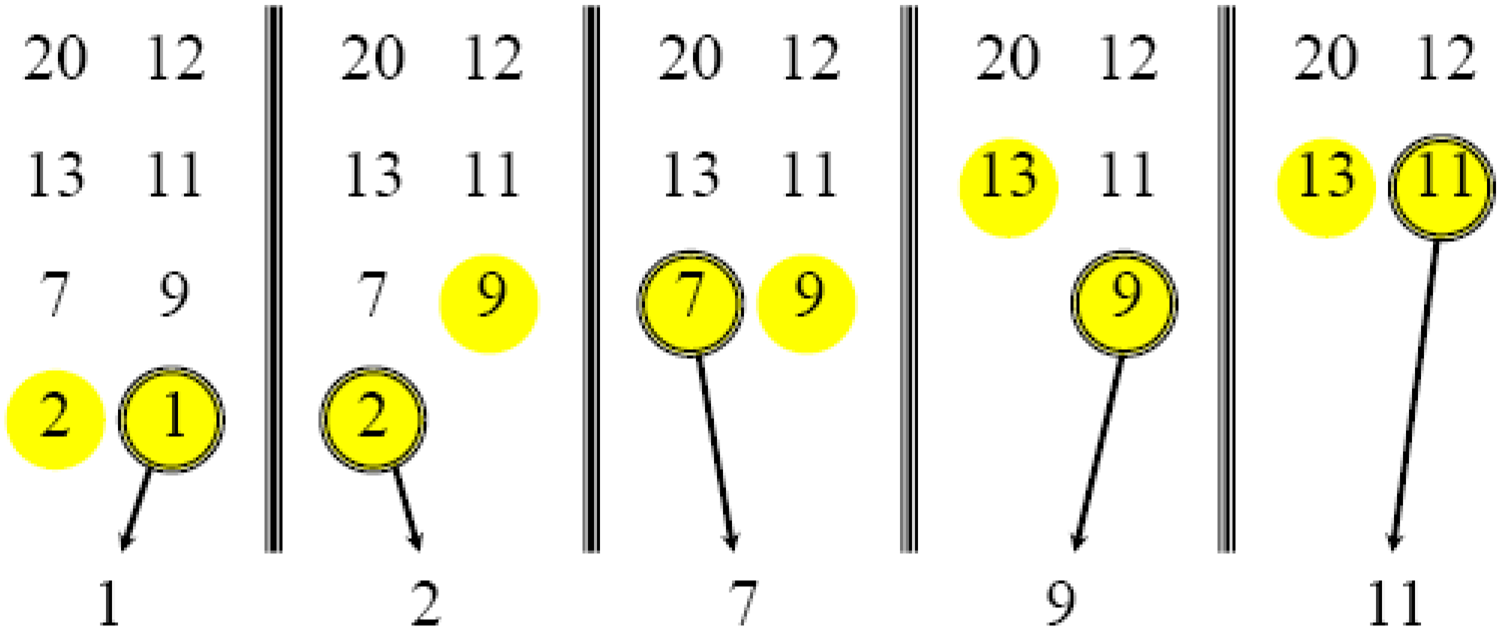

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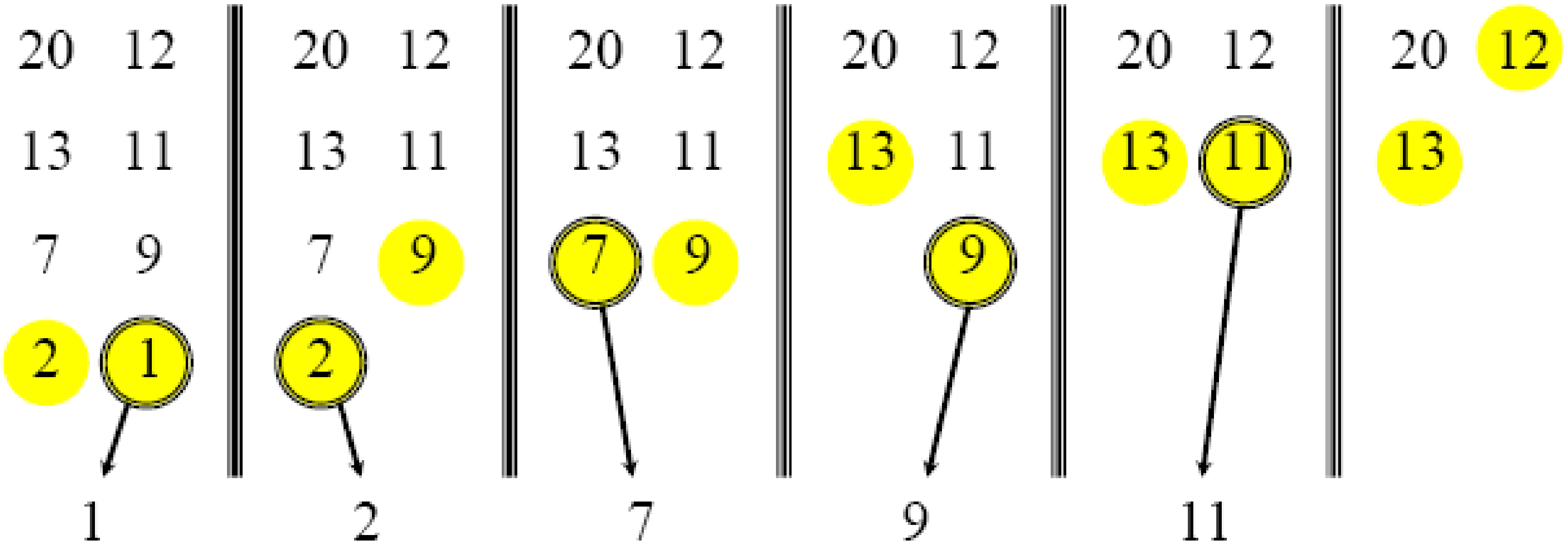

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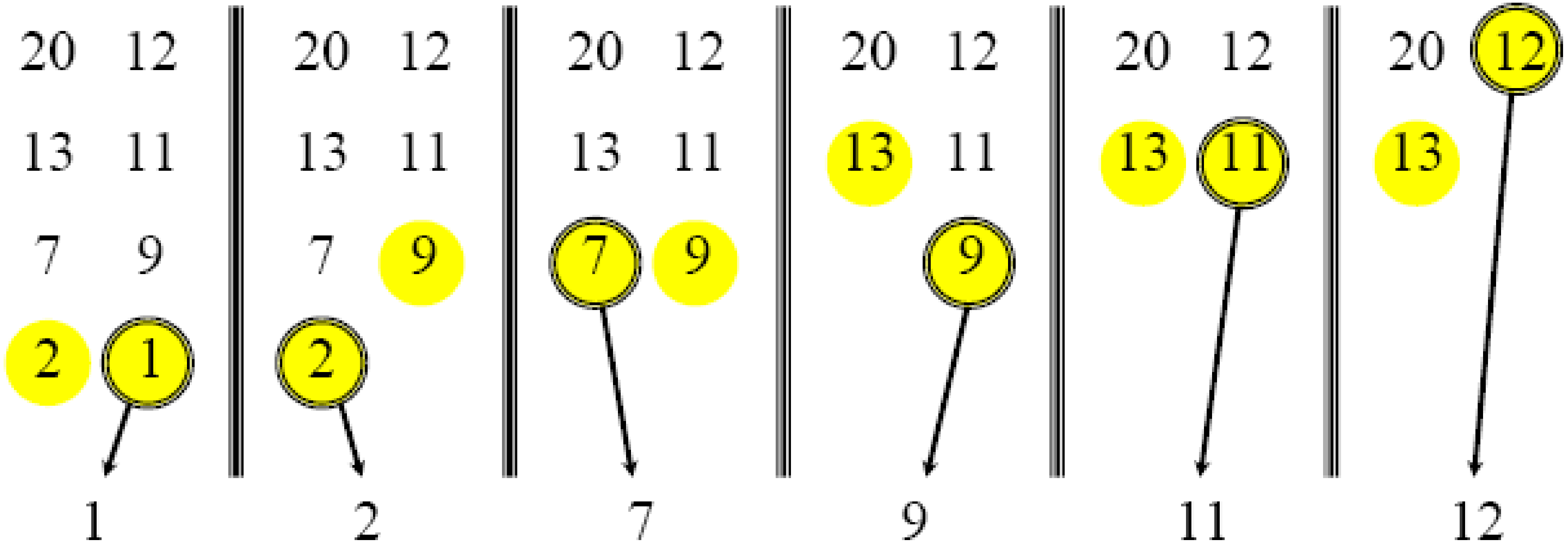

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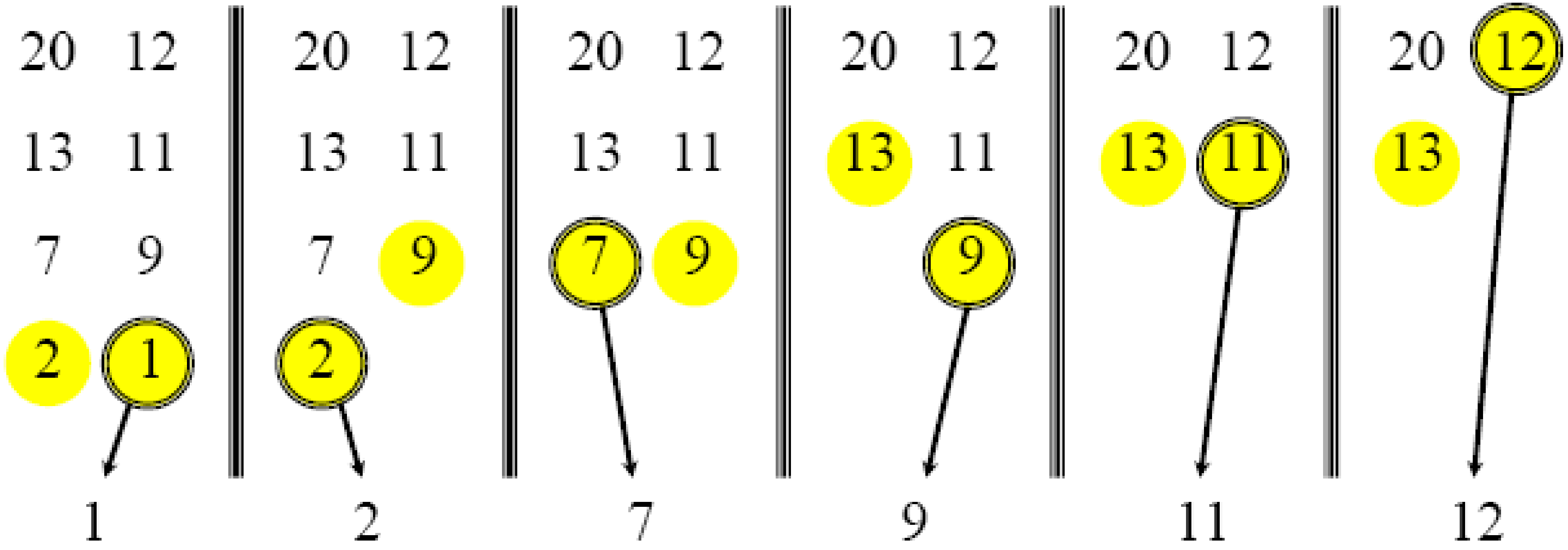


Time = $\Theta(n)$ to merge a total

of n elements (linear time).

ANALYZING MERGE SORT

$T(n)$	**MERGE-SORT** $A[1\,..\,n]$
$\Theta(1)$	1. If $n = 1$, done.
$2T(n/2)$	2. Recursively sort $A[\,1\,..\,\lceil n/2 \rceil\,]$ and $A[\lceil n/2 \rceil + 1\,..\,n\,]$.
$\Theta(n)$	***3. "Merge"*** the 2 sorted lists

RECURRENCE FOR MERGE SORT

$$T(n) = \begin{cases} \Theta(1) \text{ if } n = 1; \\ 2T(n/2) + \Theta(n) \text{ if } n > 1. \end{cases}$$

- We shall usually omit stating the base case when $T(n) = \Theta(1)$ for sufficiently small n, but only when it has no effect on the asymptotic solution to the recurrence.

MERGE SORT COMPLEXITY ANALYSIS

Iteration Method:
Generate a Guess

$$T(n) = \begin{cases} \Theta(1) \text{ if } n = 1; \\ 2T(n/2) + \Theta(n) \text{ if } n > 1. \end{cases}$$

$T(n) = 2T(n/2) + n$ ➔ Equation 1

Substitute $n/2$ is place of n in Eq 1:

$T(n/2) = 2T(n/4) + n/2$ ➔ Equation 2

Substitute Eq 2 in Eq 1

$T(n) = 2(2T(n/4) + n/2) + n$

$= 2^2 T(n/2^2) + 2n$ ➔ Equation 3

Substitute $n/4$ is place of n in Eq 1

$T(n/4) = 2T(n/8) + n/4$ ➔ Equation 4

$T(n) = 2^2 T(2T(n/8) + n/4) + 2n$

MERGE SORT COMPLEXITY ANALYSIS

$$T(n) = 2^3 T(\frac{n}{2^3}) + 3n$$

$$T(n) = 2^4 T(\frac{n}{2^4}) + 4n$$

$$\vdots$$

$$\vdots$$

$$T(n) = 2^i T\left(\frac{n}{2^i}\right) + i * n$$

$$T(1) = 1$$

$$Let \frac{n}{2^i} = 1 \rightarrow n = 2^i$$

$$\log_2 n = \log_2 2^i \rightarrow i = \log_2 n$$

$$T(n) = n\,T(1) + n \log_2 n$$

$$\boldsymbol{T(n) = n + n \log_2 n}$$

$$T(n) = \begin{cases} \Theta(1) \text{ if } n = 1; \\ 2T(n/2) + \Theta(n) \text{ if } n > 1. \end{cases}$$

Iteration Method:

Generate a Guess

But we need to prove it formally:

Substitution method!

RECURSION TREE

Recursion Tree Method: Another method to Generate a Good Guess for solving a recurrence.

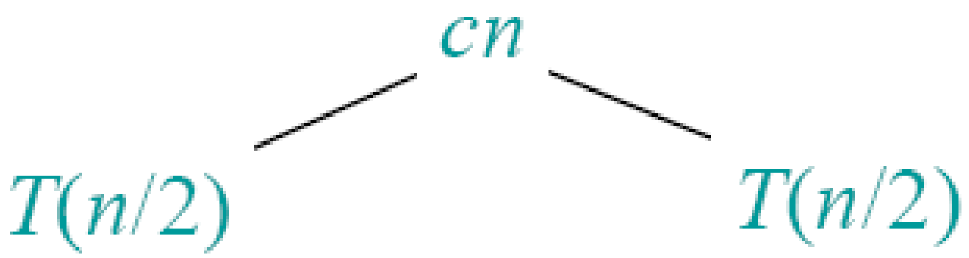
Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

RECURSION TREE

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

$$T(n)$$

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.



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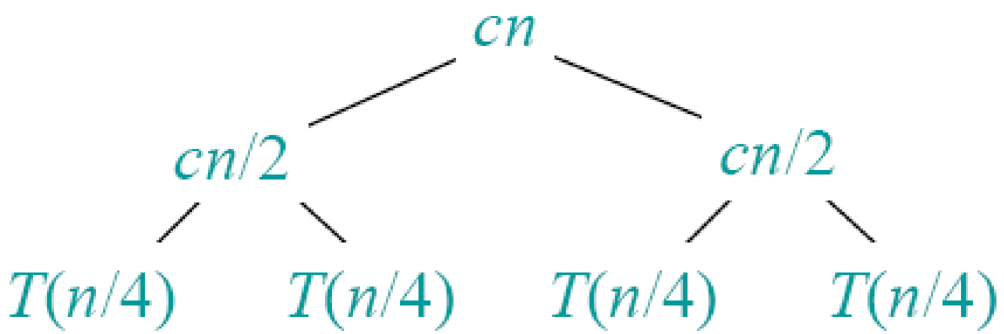

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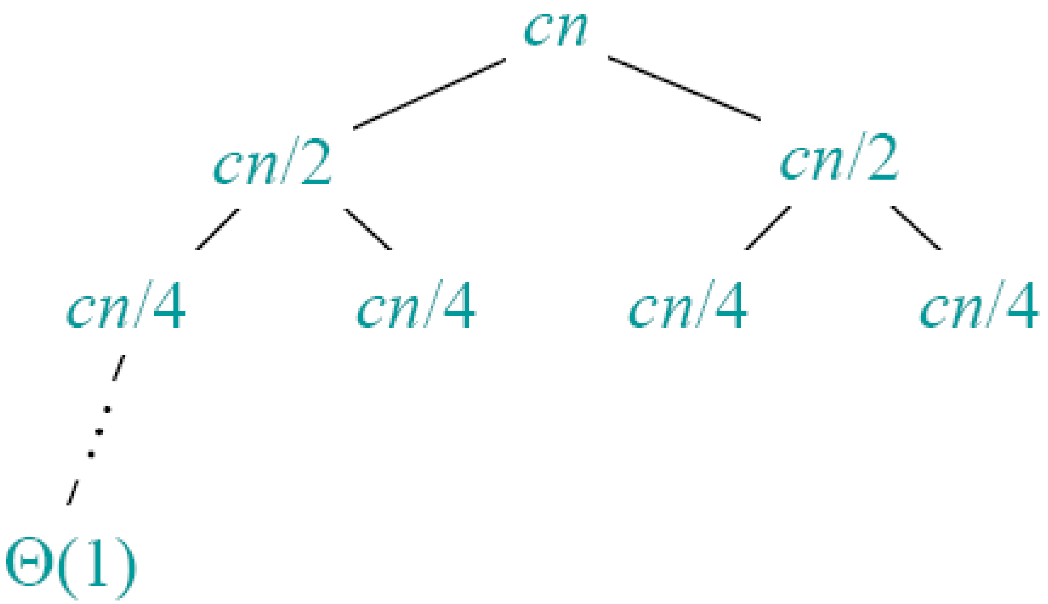

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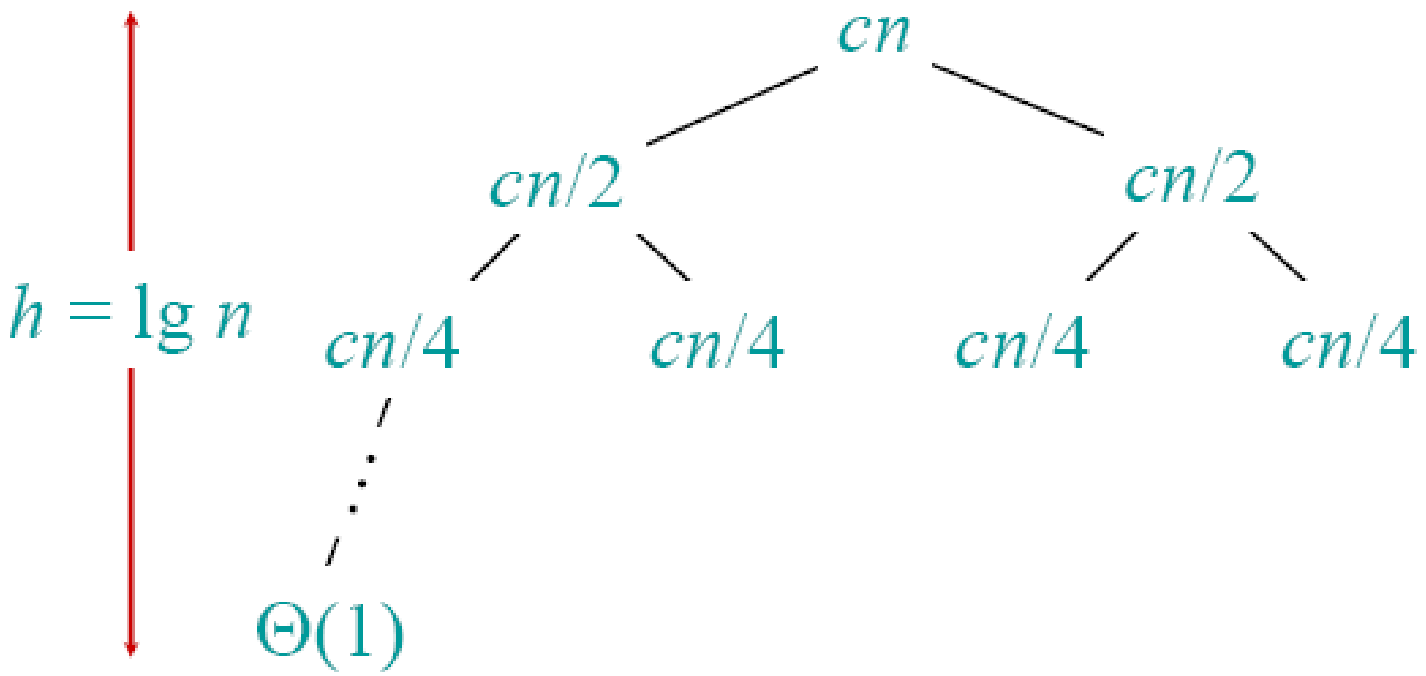

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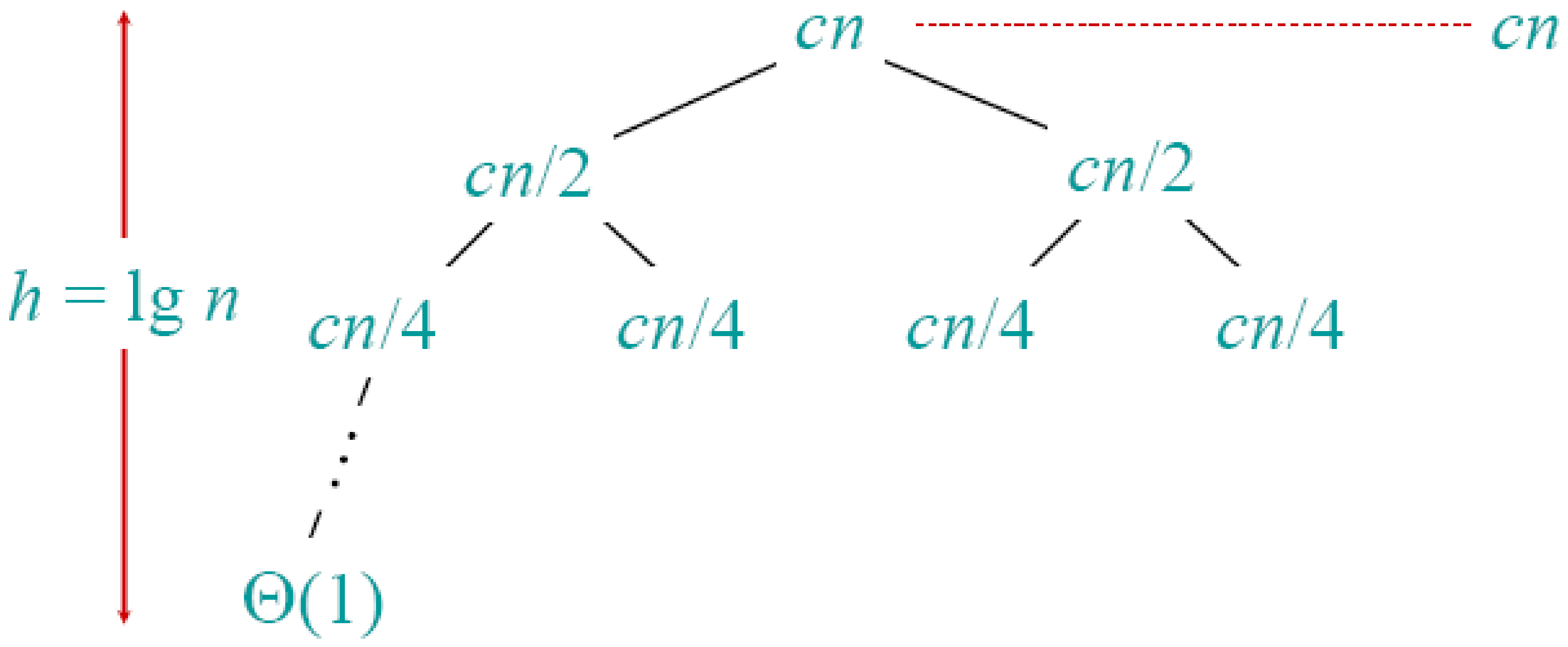

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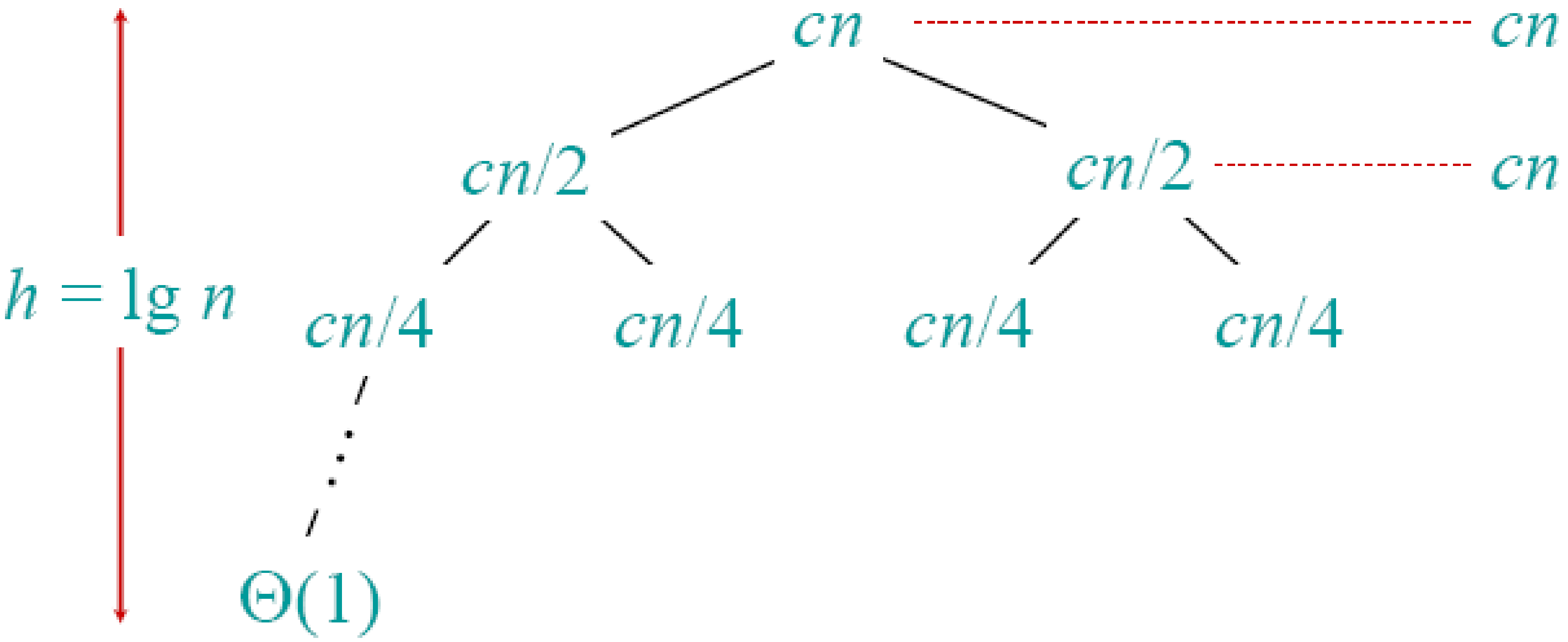

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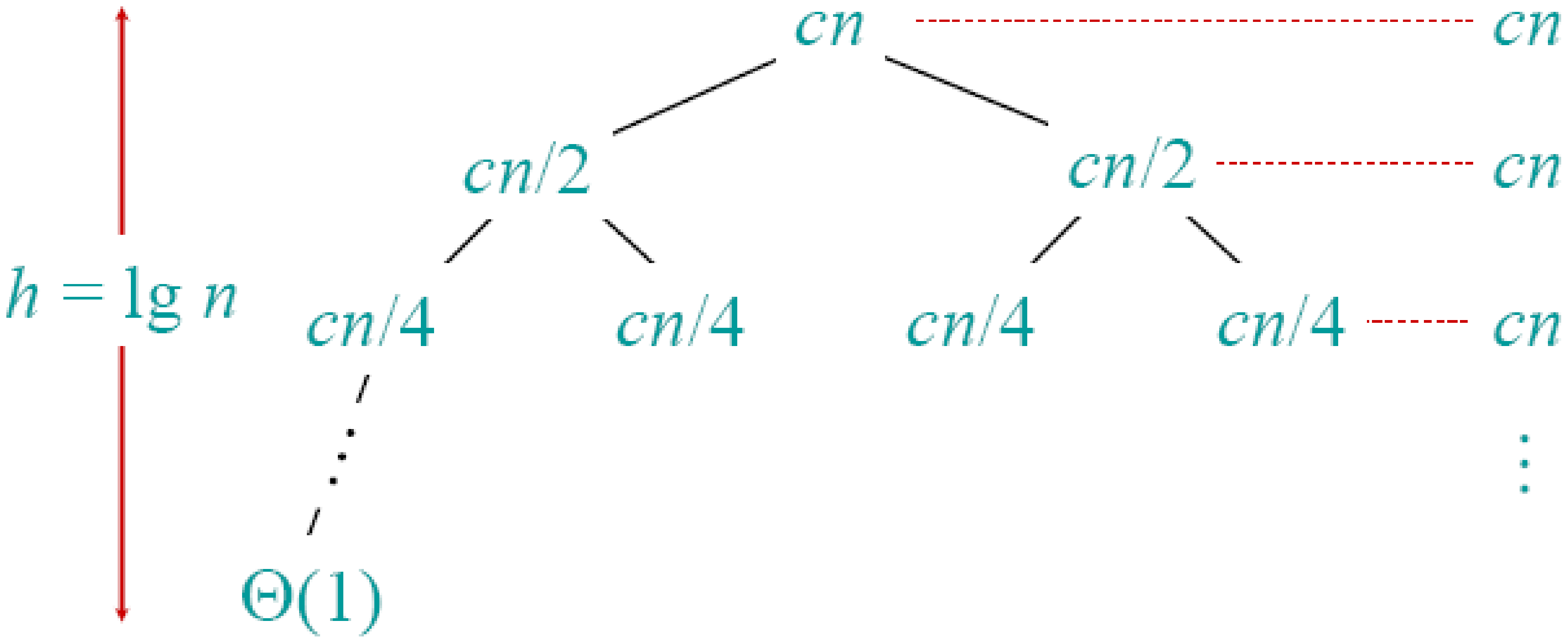

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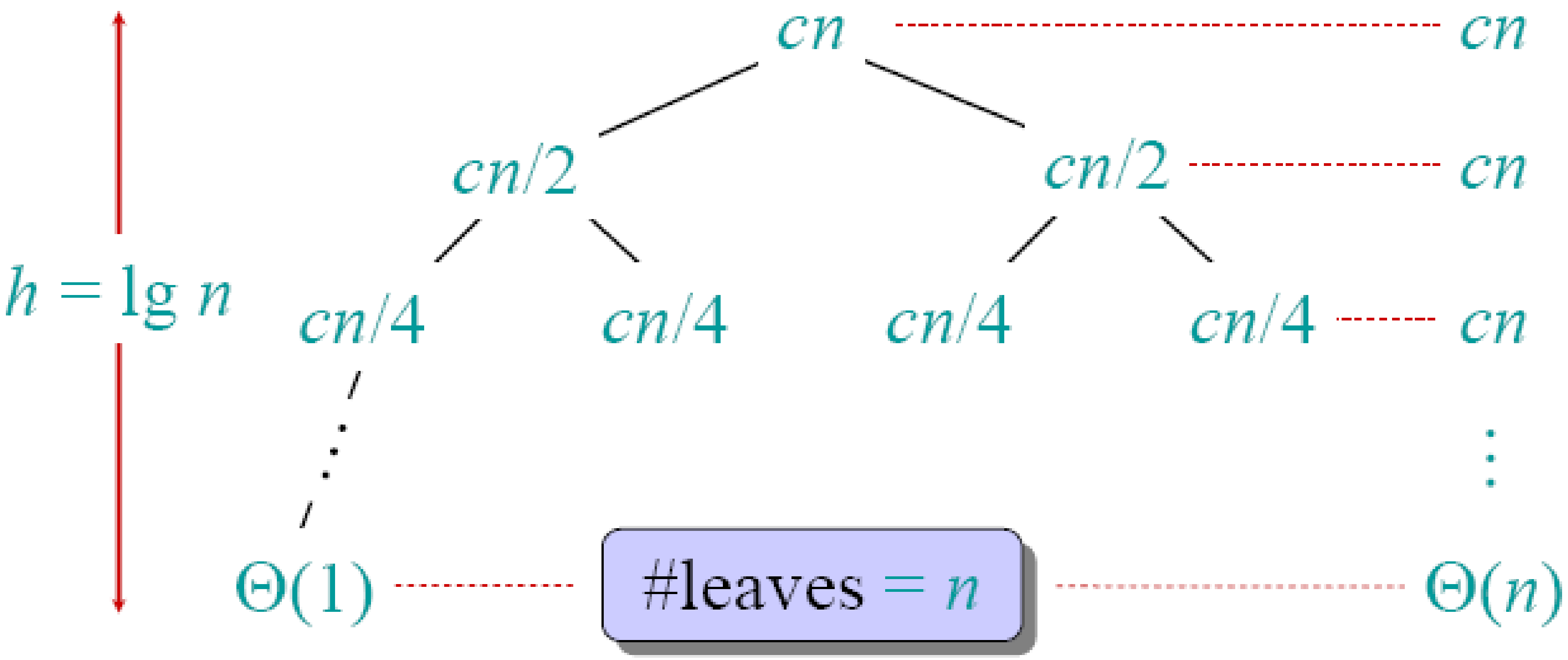

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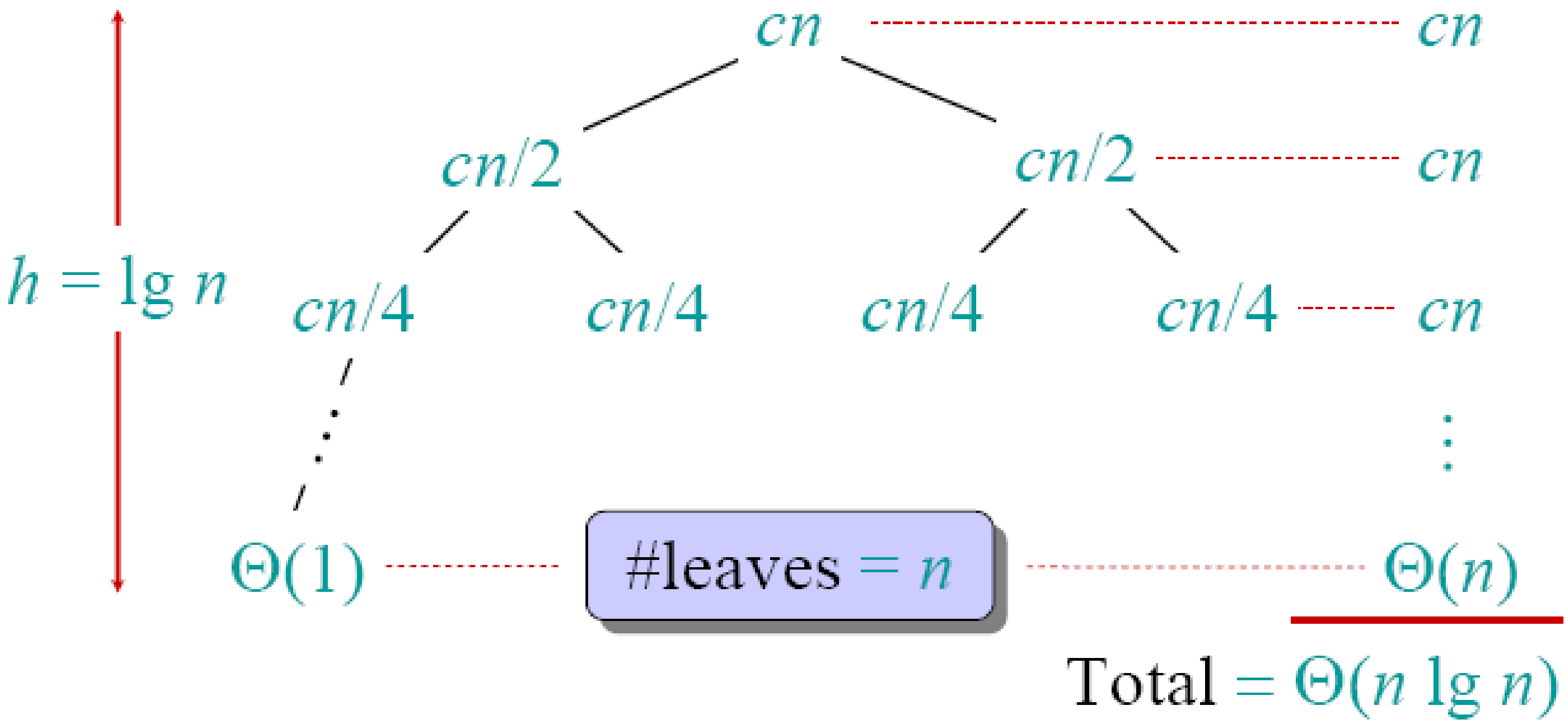

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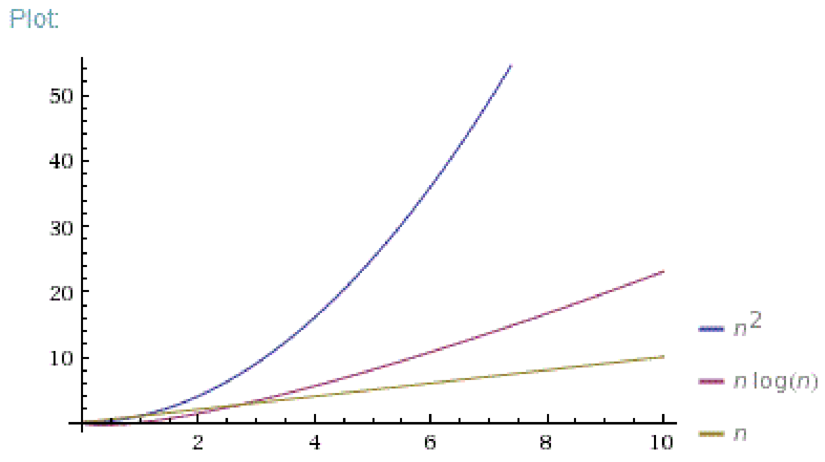
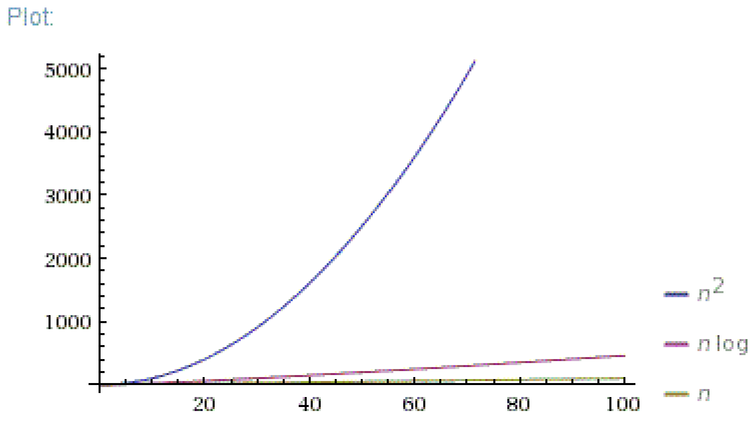
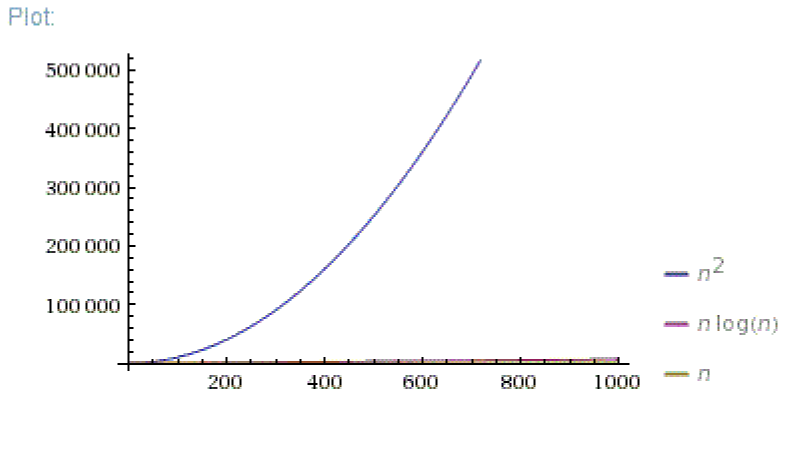


n = [0; 10]



n = [0; 100]



n = [0; 1000]

CONCLUSIONS

- $\Theta(n \lg n)$ grows more slowly than $\Theta(n^2)$.
- Therefore, merge sort asymptotically beats insertion sort in the worst case.
- In practice, merge sort beats insertion sort for $n > 30$ or so.
- Go test it out for yourself!


Q
A