Proposition 14.26. *For any function $p\colon E \to \mathbb{R}$, if we define p^D by*

$$p^D(x) = \sup_{p(z)=1} |\langle z, x\rangle|,$$

then we have

$$p^D(x+y) \leq p^D(x) + p^D(y).$$

Proof. We have

$$\begin{aligned} p^D(x+y) &= \sup_{p(z)=1} |\langle z, x+y\rangle| \\ &= \sup_{p(z)=1} (|\langle z, x\rangle + \langle z, y\rangle|) \\ &\leq \sup_{p(z)=1} (|\langle z, x\rangle| + |\langle z, y\rangle|) \\ &\leq \sup_{p(z)=1} |\langle z, x\rangle| + \sup_{p(z)=1} |\langle z, y\rangle| \\ &= p^D(x) + p^D(y). \end{aligned}$$

□

Definition 14.14. If $p\colon E \to \mathbb{R}$ is a function such that

(1) $p(x) \geq 0$ for all $x \in E$, and $p(x) = 0$ iff $x = 0$;

(2) $p(\lambda x) = |\lambda| p(x)$, for all $x \in E$ and all $\lambda \in \mathbb{C}$;

(3) p is continuous, in the sense that for some basis $(e_1, \ldots, e_n)$ of E, the function

$$(x_1, \ldots, x_n) \mapsto p(x_1 e_1 + \cdots + x_n e_n)$$

from $\mathbb{C}^n$ to $\mathbb{R}$ is continuous,

then we say that p is a *pre-norm*.

Obviously, every norm is a pre-norm, but a pre-norm may not satisfy the triangle inequality.

Corollary 14.27. *The dual norm of any pre-norm is actually a norm.*

Proposition 14.28. *For all $y \in E$, we have*

$$\|y\|^D = \sup_{\substack{x\in E \\ \|x\|=1}} |\langle x, y\rangle| = \sup_{\substack{x\in E \\ \|x\|=1}} \Re\langle x, y\rangle.$$

Proof. Since E is finite dimensional, the unit sphere $S^{n-1} = \{x \in E \mid \|x\| = 1\}$ is compact, so there is some $x_0 \in S^{n-1}$ such that

$$\|y\|^D = |\langle x_0, y\rangle|.$$