iff

$$\epsilon \geq \langle y, u\rangle - f(x+y) + f(x) = \langle y, u\rangle - h_x(y).$$

Since by definition

$$h_x^*(u) = \sup_{y\in\mathbb{R}^n}(\langle y, u\rangle - h_x(y)),$$

we conclude that

$$\partial_\epsilon f(x) = \{u \in \mathbb{R}^n \mid h_x^*(u) \leq \epsilon\},$$

as claimed. $\square$

Remark: By Fenchel's inequality $h_x^*(y) \geq 0$, and by Proposition 51.28(d), the set of vectors where h_x^* vanishes is $\partial f(x)$.

The equation $\partial_\epsilon f(x) = \{u \in \mathbb{R}^n \mid h_x^*(u) \leq \epsilon\}$ shows that $\partial_\epsilon f(x)$ is a closed convex set. As ϵ gets smaller, the set $\partial_\epsilon f(x)$ decreases, and we have

$$\partial f(x) = \bigcap_{\epsilon>0} \partial_\epsilon f(x).$$

However $\delta^*(y|\partial_\epsilon f(x)) = I^*_{\partial_\epsilon f(x)}(y)$ does not necessarily decrease to $\delta^*(y|\partial f(x)) = I^*_{\partial f(x)}(y)$ as ϵ decreases to zero. The discrepancy corresponds to the discrepancy between $f'(x; y)$ and $\delta^*(y|\partial f(x)) = I^*_{\partial f(x)}(y)$ and is due to the fact that f is not necessarily closed (see Proposition 51.16) as shown by the following result proven in Rockafellar [138] (Theorem 23.6).

Proposition 51.33. *Let f be a closed and proper convex function, and let $x \in \mathbb{R}^n$ such that $f(x)$ is finite. Then*

$$f'(x; y) = \lim_{\epsilon\downarrow 0} \delta^*(y|\partial_\epsilon f(x)) = \lim_{\epsilon\downarrow 0} I^*_{\partial_\epsilon f(x)}(y) \quad \text{for all } y \in \mathbb{R}^n.$$

The theory of convex functions is rich and we have only given a sample of some of the most significant results that are relevant to optimization theory. There are a few more results regarding the minimum of convex functions that are particularly important due to their applications to optimization theory.

51.5 The Minimum of a Proper Convex Function

Let h be a proper convex function on $\mathbb{R}^n$. The general problem is to study the minimum of h over a nonempty convex set C in $\mathbb{R}^n$, possibly defined by a set of inequality and equality constraints. We already observed that minimizing h over C is equivalent to minimizing the proper convex function f given by

$$f(x) = h(x) + I_C(x) = \begin{cases} h(x) & \text{if } x \in C \\ +\infty & \text{if } x \notin C. \end{cases}$$

Therefore it makes sense to begin by considering the problem of minimizing a proper convex function f over $\mathbb{R}^n$. Of course, minimizing over $\mathbb{R}^n$ is equivalent to minimizing over $\mathrm{dom}(f)$.