By hypothesis
$$w^\top u_i - (b - \theta) = \eta + \theta \quad \text{for some } i \notin E_\lambda,$$
and by the choice of θ,
$$-w^\top v_j + b - \theta = \eta + \theta \quad \text{for some } j \notin E_\mu.$$

The new value of the objective function is
$$\begin{aligned}\omega(\theta) &= \frac{1}{2} w^\top w - \nu(\eta + \theta) + \frac{1}{p+q}\left(\sum_{i \in E_\lambda} \epsilon_i + \sum_{j \in E_\mu} (\xi_j + 2\theta)\right) \\ &= \frac{1}{2} w^\top w - \nu\eta + \frac{1}{p+q}\left(\sum_{i \in E_\lambda} \epsilon_i + \sum_{j \in E_\mu} \xi_j\right) - \left(\nu - \frac{2q_{sf}}{p+q}\right)\theta.\end{aligned}$$

The rest of the proof is similar to Case 1a with p_{sf} replaced by q_{sf}.

Case 2b. We have $w^\top u_i - b > \eta$ for all $i \notin E_\lambda$. Since by hypothesis $-w^\top v_j + b > \eta$ for all $j \notin E_\mu$, Case 2b is identical to Case 1b, and we are done. ☐

A subtle point here is that Proposition 54.2 shows that if there is an optimal solution, then there is one with a blue and a red support vector, but it does not guarantee that these are support vectors of type 1. Since the dual program does not determine b and η unless these support vectors are of type 1, from a practical point of view this proposition is not helpful.

The proof of Proposition 54.2 reveals that there are three critical values for ν:
$$\frac{2p_{sf}}{p+q}, \ \frac{2q_{sf}}{p+q}, \ \frac{p_{sf} + q_{sf}}{p+q}.$$
These values can be avoided by requiring the strict inequality
$$\max\left\{\frac{2p_{sf}}{p+q}, \frac{2q_{sf}}{p+q}\right\} < \nu.$$
Then the following corollary holds.

Theorem 54.3. *For every optimal solution* $(w, b, \eta, \epsilon, \xi)$ *of Problem* (SVM$_{s2'}$) *with* $w \neq 0$ *and* $\eta > 0$*, if*
$$\max\{2p_f/(p+q), 2q_f/(p+q)\} < \nu < \min\{2p/(p+q), 2q/(p+q)\},$$
then some u_{i_0} *and some* v_{j_0} *is a support vector.*

Proof. We proceed by contradiction. Suppose that for every optimal solution with $w \neq 0$ and $\eta > 0$ no u_i is a blue support vector or no v_j is a red support vector. Since $\nu < \min\{2p/(p+q), 2q/(p+q)\}$, Proposition 54.2 holds, so there is another optimal solution. But since the critical values of ν are avoided, the proof of Proposition 54.2 shows that the value of the objective function for this new optimal solution is strictly smaller than the original optimal value, a contradiction. ☐