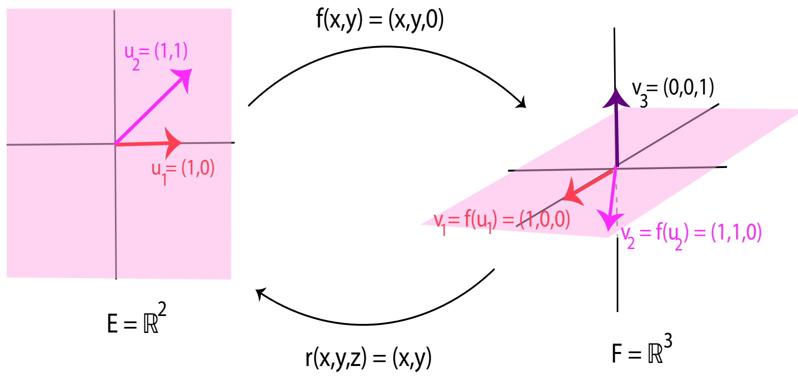

Figure 6.2: Let $f\colon E \to F$ be the injective linear map from $\mathbb{R}^2$ to $\mathbb{R}^3$ given by $f(x, y) = (x, y, 0)$. Then a surjective retraction is given by $r\colon \mathbb{R}^3 \to \mathbb{R}^2$ is given by $r(x, y, z) = (x, y)$. Observe that $r(v_1) = u_1$, $r(v_2) = u_2$, and $r(v_3) = 0$.

6.2 Matrices of Linear Maps and Multiplication by Blocks

Direct sums yield a fairly easy justification of matrix block multiplication. The key idea is that the representation of a linear map $f\colon E \to F$ over a basis $(u_1, \ldots, u_n)$ of E and a basis $(v_1, \ldots, v_m)$ of F by a matrix $A = (a_{ij})$ of scalars (in K) can be generalized to the representation of f over a direct sum decomposition $E = E_1 \oplus \cdots \oplus E_n$ of E and a direct sum decomposition $F = F_1 \oplus \cdots \oplus F_m$ of F in terms of a matrix (f_{ij}) of linear maps $f_{ij}\colon E_j \to F_i$. Futhermore, matrix multiplication of scalar matrices extends naturally to matrix multiplication of matrices of linear maps. We simply have to replace multiplication of scalars in K by the composition of linear maps.

Let E and F be two vector spaces and assume that they are expressed as direct sums

$$E = \bigoplus_{j=1}^{n} E_j, \quad F = \bigoplus_{i=1}^{m} F_i.$$

Definition 6.6. Given any linear map $f\colon E \to F$, we define the linear maps $f_{ij}\colon E_j \to F_i$ as follows. Let $pr_i^F\colon F \to F_i$ be the projection of $F = F_1 \oplus \cdots \oplus F_m$ onto F_i. If $f_j\colon E_j \to F$ is the restriction of f to E_j, which means that for every vector $x_j \in E_j$,

$$f_j(x_j) = f(x_j),$$

then we define the map $f_{ij}\colon E_j \to F_i$ by

$$f_{ij} = pr_i^F \circ f_j,$$