dependent and A_k would not be invertible, a contradiction. This situation is illustrated by the following matrix for $n = 5$ and $k = 3$:

$$\begin{pmatrix} a_{1\,1}^{(3)} & a_{1\,2}^{(3)} & a_{1\,3}^{(3)} & a_{1\,3}^{(3)} & a_{1\,5}^{(3)} \\ 0 & a_{2\,2}^{(3)} & a_{2\,3}^{(3)} & a_{2\,4}^{(3)} & a_{2\,5}^{(3)} \\ 0 & 0 & 0 & a_{3\,4}^{(3)} & a_{3\,5}^{(3)} \\ 0 & 0 & 0 & a_{4\,4}^{(3)} & a_{4\,n}^{(3)} \\ 0 & 0 & 0 & a_{5\,4}^{(3)} & a_{5\,5}^{(3)} \end{pmatrix}.$$

The first three columns of the above matrix are linearly dependent.

So one of the entries $a_{i\,k}^{(k)}$ with $k \leq i \leq n$ can be chosen as pivot, and we permute the kth row with the ith row, obtaining the matrix $\alpha^{(k)} = (\alpha_{j\,l}^{(k)})$. The new pivot is $\pi_k = \alpha_{k\,k}^{(k)}$, and we zero the entries $i = k+1, \ldots, n$ in column k by adding $-\alpha_{i\,k}^{(k)}/\pi_k$ times row k to row i. At the end of this step, we have A_{k+1}. Observe that the first $k-1$ rows of A_k are identical to the first $k-1$ rows of A_{k+1}.

The process of Gaussian elimination is illustrated in schematic form below:

$$\begin{pmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix} \Longrightarrow \begin{pmatrix} \times & \times & \times & \times \\ \mathbf{0} & \times & \times & \times \\ \mathbf{0} & \times & \times & \times \\ \mathbf{0} & \times & \times & \times \end{pmatrix} \Longrightarrow \begin{pmatrix} \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & \mathbf{0} & \times & \times \\ 0 & \mathbf{0} & \times & \times \end{pmatrix} \Longrightarrow \begin{pmatrix} \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \\ 0 & 0 & \mathbf{0} & \times \end{pmatrix}.$$

8.3 Elementary Matrices and Row Operations

It is easy to figure out what kind of matrices perform the elementary row operations used during Gaussian elimination. The key point is that if $A = PB$, where A, B are $m \times n$ matrices and P is a square matrix of dimension m, if (as usual) we denote the rows of A and B by $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$, then the formula

$$a_{ij} = \sum_{k=1}^{m} p_{ik} b_{kj}$$

giving the (i, j)th entry in A shows that the ith row of A is a *linear combination* of the rows of B:

$$A_i = p_{i1} B_1 + \cdots + p_{im} B_m.$$

Therefore, *multiplication of a matrix on the left by a square matrix performs row operations*. Similarly, multiplication of a matrix on the right by a square matrix performs column operations

The permutation of the kth row with the ith row is achieved by multiplying A on the left by the *transposition matrix* $P(i, k)$, which is the matrix obtained from the identity matrix