Chapter 20

Graphs and Graph Laplacians; Basic Facts

In this chapter and the next we present some applications of linear algebra to graph theory. Graphs (undirected and directed) can be defined in terms of various matrices (incidence and adjacency matrices), and various connectivity properties of graphs are captured by properties of these matrices. Another very important matrix is associated with a (undirected) graph: the *graph Laplacian*. The graph Laplacian is symmetric positive definite, and its eigenvalues capture some of the properties of the underlying graph. This is a key fact that is exploited in graph clustering methods, the most powerful being the method of normalized cuts due to Shi and Malik [160]. This chapter and the next constitute an introduction to algebraic and spectral graph theory. We do not discuss normalized cuts, but we discuss graph drawings. Thorough presentations of algebraic graph theory can be found in Godsil and Royle [77] and Chung [39].

We begin with a review of basic notions of graph theory. Even though the graph Laplacian is fundamentally associated with an undirected graph, we review the definition of both directed and undirected graphs. For both directed and undirected graphs, we define the degree matrix D, the incidence matrix B, and the adjacency matrix A. Then we define a *weighted graph*. This is a pair (V, W), where V is a finite set of nodes and W is a $m \times m$ symmetric matrix with nonnegative entries and zero diagonal entries (where $m = |V|$).

For every node $v_i \in V$, the *degree* $d(v_i)$ (or d_i) of v_i is the sum of the weights of the edges adjacent to v_i:

$$d_i = d(v_i) = \sum_{j=1}^{m} w_{i\,j}.$$

The *degree matrix* is the diagonal matrix

$$D = \mathrm{diag}(d_1, \ldots, d_m).$$

The notion of degree is illustrated in Figure 20.1. Then we introduce the (unnormalized)