the notion of basis so that a "Hilbert basis" is an orthogonal family that is also dense in E, i.e., every $v \in E$ is the limit of a sequence of finite combinations of vectors from the Hilbert basis, then we can recover most of the "nice" properties of finite-dimensional Hermitian spaces. For instance, if $(u_k)_{k \in K}$ is a Hilbert basis, for every $v \in E$, we can define the Fourier coefficients $c_k = \langle v, u_k \rangle / \|u_k\|$, and then, v is the "sum" of its Fourier series $\sum_{k \in K} c_k u_k$. However, the cardinality of the index set K can be very large, and it is necessary to define what it means for a family of vectors indexed by K to be summable. We will do this in Section A.1. It turns out that every Hilbert space is isomorphic to a space of the form $\ell^2(K)$, where $\ell^2(K)$ is a generalization of the space of Example 48.1 (see Theorem A.8, usually called the Riesz-Fischer theorem).

Our first goal is to prove that a closed subspace of a Hilbert space has an orthogonal complement. We also show that duality holds if we redefine the dual E' of E to be the space of *continuous* linear maps on E. Our presentation closely follows Bourbaki [27]. We also were inspired by Rudin [140], Lang [111, 112], Schwartz [150, 149], and Dixmier [51]. In fact, we highly recommend Dixmier [51] as a clear and simple text on the basics of topology and analysis. To achieve this goal, we must first prove the so-called projection lemma.

Recall that in a metric space E, a subset X of E is *closed* iff for every convergent sequence (x_n) of points $x_n \in X$, the limit $x = \lim_{n \to \infty} x_n$ also belongs to X. The *closure* $\overline{X}$ *of* X is the set of all limits of convergent sequences (x_n) of points $x_n \in X$. Obviously, $X \subseteq \overline{X}$. We say that the subset X of E is *dense in* E iff $E = \overline{X}$, the closure of X, which means that every $a \in E$ is the limit of some sequence (x_n) of points $x_n \in X$. Convex sets will again play a crucial role. In a complex vector space E, a subset $C \subseteq E$ is convex if $(1 - \lambda)x + \lambda y \in C$ for all $x, y \in C$ and all *real* $\lambda \in [0, 1]$. Observe that a subspace is convex.

First we state the following easy "parallelogram law," whose proof is left as an exercise.

Proposition 48.2. *If E is a Hermitian space, for any two vectors $u, v \in E$, we have*

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2).$$

From the above, we get the following proposition:

Proposition 48.3. *If E is a Hermitian space, given any $d, \delta \in \mathbb{R}$ such that $0 \leq \delta < d$, let*

$$B = \{u \in E \mid \|u\| < d\} \quad \textit{and} \quad C = \{u \in E \mid \|u\| \leq d + \delta\}.$$

For any convex set such A that $A \subseteq C - B$, we have

$$\|v - u\| \leq \sqrt{12 d\delta},$$

for all $u, v \in A$ (see Figure 48.1).