Next we apply the Invariant Factors Decomposition Theorem, Theorem 35.31, to E_f. This theorem says that E_f is isomorphic to a direct sum

$$E_f \approx K[X]/(p_1) \oplus \cdots \oplus K[X]/(p_m)$$

of $m \leq n$ cyclic modules, where the p_j are uniquely determined monic polynomials of degree at least 1, such that

$$p_m \mid p_{m-1} \mid \cdots \mid p_1.$$

Each cyclic module $K[X]/(p_i)$ is isomorphic to a cyclic subspace for f, say V_i, whose minimal polynomial is p_i.

It is customary to renumber the polynomials p_i as follows. The n polynomials $q_1, \ldots, q_n$ are defined by:

$$q_j(X) = \begin{cases} 1 & \text{if } 1 \leq j \leq n-m \\ p_{n-j+1}(X) & \text{if } n-m+1 \leq j \leq n. \end{cases}$$

Then we see that

$$q_1 \mid q_2 \mid \cdots \mid q_n,$$

where the first $n-m$ polynomials are equal to 1, and we have the direct sum

$$E = E_1 \oplus \cdots \oplus E_n,$$

where E_i is a cyclic subspace for f whose minimal polynomial is q_i. In particular, $E_i = (0)$ for $i = 1, \ldots, n-m$. Theorem 35.31 also says that the minimal polynomial of f is $q_n = p_1$. We sum all this up in the following theorem.

Theorem 36.5. *(Cyclic Decomposition Theorem, First Version) Let $f \colon E \to E$ be an endomorphism on a K-vector space of dimension n. There exist n monic polynomials $q_1, \ldots, q_n \in K[X]$ such that*

$$q_1 \mid q_2 \mid \cdots \mid q_n,$$

and E is the direct sum

$$E = E_1 \oplus \cdots \oplus E_n$$

of cyclic subspaces $E_i = Z(u_i; f)$ for f, such that the minimal polynomial of the restriction of f to E_i is q_i. The polynomials q_i satisfying the above conditions are unique, and q_n is the minimal polynomial of f.

In view of translation point (4) at the beginning of Section 36.1, we know that over the basis

$$(u_i, f(u_i), \ldots, f^{n_i-1}(u_i))$$