29.8 Symplectic Groups

In this section, we are dealing with a nondegenerate alternating form φ on a vector space E of dimension n. As we saw earlier, n must be even, say $n = 2m$. By Theorem 29.24, there is a direct sum decomposition of E into pairwise orthogonal subspaces

$$E = W_1 \overset{\perp}{\oplus} \cdots \overset{\perp}{\oplus} W_m,$$

where each W_i is a hyperbolic plane. Each W_i has a basis (u_i, v_i), with $\varphi(u_i, u_i) = \varphi(v_i, v_i) = 0$ and $\varphi(u_i, v_i) = 1$, for $i = 1, \ldots, m$. In the basis

$$(u_1, \ldots, u_m, v_1, \ldots, v_m),$$

φ is represented by the matrix

$$J_{m,m} = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}.$$

The symplectic group $\mathbf{Sp}(2m, K)$ is the group of isometries of φ. The maps in $\mathbf{Sp}(2m, K)$ are called *symplectic* maps. With respect to the above basis, $\mathbf{Sp}(2m, K)$ is the group of $2m \times 2m$ matrices A such that

$$A^\top J_{m,m} A = J_{m,m}.$$

Matrices satisfying the above identity are called *symplectic* matrices. In this section, we show that $\mathbf{Sp}(2m, K)$ is a subgroup of $\mathbf{SL}(2m, K)$ (that is, $\det(A) = +1$ for all $A \in \mathbf{Sp}(2m, K)$), and we show that $\mathbf{Sp}(2m, K)$ is generated by special linear maps called *symplectic transvections*.

First, we leave it as an easy exercise to show that $\mathbf{Sp}(2, K) = \mathbf{SL}(2, K)$. The reader should also prove that $\mathbf{Sp}(2m, K)$ has a subgroup isomorphic to $\mathbf{GL}(m, K)$.

Next we characterize the symplectic maps f that leave fixed every vector in some given hyperplane H, that is,

$$f(v) = v \quad \text{for all } v \in H.$$

Since φ is nondegenerate, by Proposition 29.22, the orthogonal $H^\perp$ of H is a line (that is, $\dim(H^\perp) = 1$). For every $u \in E$ and every $v \in H$, since f is an isometry and $f(v) = v$ for all $v \in H$, we have

$$\begin{aligned} \varphi(f(u) - u, v) &= \varphi(f(u), v) - \varphi(u, v) \\ &= \varphi(f(u), v) - \varphi(f(u), f(v)) \\ &= \varphi(f(u), v - f(v))) \\ &= \varphi(f(u), 0) = 0, \end{aligned}$$

which shows that $f(u) - u \in H^\perp$ for all $u \in E$. Therefore, $f - \mathrm{id}$ is a linear map from E into the line $H^\perp$ whose kernel contains H, which means that there is some nonzero vector $w \in H^\perp$ and some linear form ψ such that

$$f(u) = u + \psi(u)w, \quad u \in E.$$