We now consider the continuity of multilinear maps. We treat explicitly bilinear maps, the general case being a straightforward extension.

Proposition 37.59. *Given normed vector spaces E, F and G, for any bilinear map $f\colon E \times F \to G$, the following conditions are equivalent:*

(1) The function f is continuous at $\langle 0, 0\rangle$.

2) There is a constant $k \geq 0$ such that,

$$\|f(u,v)\| \leq k, \;\; \textit{for all } u \in E, v \in F \textit{ such that } \|u\|, \|v\| \leq 1.$$

3) There is a constant $k \geq 0$ such that,

$$\|f(u,v)\| \leq k\|u\|\|v\|, \;\; \textit{for all } u \in E, v \in F.$$

4) The function f is continuous at every point of $E \times F$.

Proof. It is similar to that of Proposition 37.56, with a small subtlety in proving that (3) implies (4), namely that two different η's that are not independent are needed. ☐

In contrast to continuous linear maps, which must be uniformly continuous, nonzero continuous bilinear maps are **not** uniformly continuous. Let $f\colon E \times F \to G$ be a continuous bilinear map such that $f(a,b) \neq 0$ for some $a \in E$ and some $b \in F$. Consider the sequences (u_n) and (v_n) (with $n \geq 1$) given by

$$\begin{aligned} u_n &= (x_n, y_n) = (na, nb) \\ v_n &= (x_n', y_n') = \left(\left(n + \frac{1}{n}\right) a, \left(n + \frac{1}{n}\right) b\right). \end{aligned}$$

Obviously

$$\|v_n - u_n\| \leq \frac{1}{n}(\|a\| + \|b\|),$$

so $\lim_{n\mapsto\infty} \|v_n - u_n\| = 0$. On the other hand

$$f(x_n', y_n') - f(x_n, y_n) = \left(2 + \frac{1}{n^2}\right) f(a,b),$$

and thus $\lim_{n\mapsto\infty} \|f(x_n', y_n') - f(x_n, y_n)\| = 2\,\|f(a,b)\| \neq 0$, which shows that f is not uniformly continuous, because if this was the case, this limit would be zero.

If E, F, and G, are normed vector spaces, we denote the set of all continuous bilinear maps $f\colon E \times F \to G$ by $\mathcal{L}_2(E, F; G)$. Using Proposition 37.59, we can define a norm on $\mathcal{L}_2(E, F; G)$ which makes it into a normed vector space.