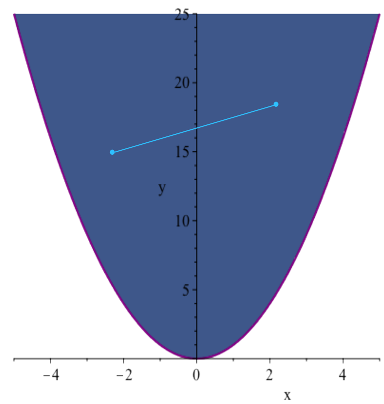

Figure 51.2: Let $f\colon \mathbb{R} \to \mathbb{R} \cup \{-\infty, +\infty\}$ be given by $f(x) = x^2$ for $x \in \mathbb{R}$. Its graph in $\mathbb{R}^2$ is the magenta curve, and its epigraph is the union of the magenta curve and blue region "above" this curve. Observe that $\mathbf{epi}(f)$ is a convex set of $\mathbb{R}^2$ since the aqua line segment connecting any two points is contained within the epigraph.

which means that $(1-\lambda)x_1 + \lambda x_2 \in S$ and

$$f((1-\lambda)x_1 + \lambda x_2) \leq (1-\lambda)y_1 + \lambda y_2. \tag{$*$}$$

Thus S *must be convex* and $f((1-\lambda)x_1 + \lambda x_2) < +\infty$. Condition $(*)$ is a little awkward, since it does not refer explicitly to $f(x_1)$ and $f(x_2)$, as these values may be $-\infty$, in which case it is not clear what the expression $(1-\lambda)f(x_1) + \lambda f(x_2)$ means.

In order to perform arithmetic operations involving $-\infty$ and $+\infty$, we adopt the following conventions:

$$\begin{aligned}
\alpha + (+\infty) = +\infty + \alpha &= +\infty & \quad -\infty < \alpha \leq +\infty \\
\alpha + -\infty = -\infty + \alpha &= -\infty & \quad -\infty \leq \alpha < +\infty \\
\alpha(+\infty) = (+\infty)\alpha &= +\infty & \quad 0 < \alpha \leq +\infty \\
\alpha(-\infty) = (-\infty)\alpha &= -\infty & \quad 0 < \alpha \leq +\infty \\
\alpha(+\infty) = (+\infty)\alpha &= -\infty & \quad -\infty \leq \alpha < 0 \\
\alpha(-\infty) = (-\infty)\alpha &= +\infty & \quad -\infty \leq \alpha < 0 \\
0(+\infty) = (+\infty)0 &= 0 & \quad 0(-\infty) = (-\infty)0 = 0 \\
-(-\infty) &= +\infty & \\
\inf \emptyset &= +\infty & \quad \sup \emptyset = -\infty.
\end{aligned}$$

The expressions $+\infty + (-\infty)$ and $-\infty + (+\infty)$ are *meaningless.*

The following characterizations of convex functions are easy to show.

Proposition 51.1. *Let C be a nonempty convex subset of $\mathbb{R}^n$.*