Problem 10.5. Prove that the converse of Proposition 10.5 holds. That is, if A is an invertible Hermitian matrix with the splitting $A = M - N$ where M is invertible, if the Hermitian matrix $M^* + N$ is positive definite and if $\rho(M^{-1}N) < 1$, then A is positive definite.

Problem 10.6. Consider the following tridiagonal $n \times n$ matrix:

$$A = \frac{1}{(n+1)^2} \begin{pmatrix} 2 & -1 & 0 & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & 0 & -1 & 2 \end{pmatrix}.$$

(1) Prove that the eigenvalues of the Jacobi matrix J are given by

$$\lambda_k = \cos\left(\frac{k\pi}{n+1}\right), \quad k = 1, \ldots, n.$$

Hint. First show that the Jacobi matrix is

$$J = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & & \\ 1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & 1 \\ & & 0 & 1 & 0 \end{pmatrix}.$$

Then the eigenvalues and the eigenvectors of J are solutions of the system of equations

$$\begin{aligned} y_0 &= 0 \\ y_{k+1} + y_{k-1} &= 2\lambda y_k, \quad k = 1, \ldots, n \\ y_{n+1} &= 0, \end{aligned}$$

where the variables y_0 and y_{n+1} are introduced so that the same equation applies for $k = 1, \ldots, n$. It is well known that the general solution to the above recurrence is given by

$$y_k = \alpha z_1^k + \beta z_2^k, \quad k = 0, \ldots, n+1,$$

(with $\alpha, \beta \neq 0$) where z_1 and z_2 are the zeros of the equation

$$z^2 - 2\lambda z + 1 = 0.$$

It follows that $z_2 = z_1^{-1}$ and $z_1 + z_2 = 2\lambda$. The boundary condition $y_0 = 0$ yields $\alpha + \beta = 0$, so $y_k = \alpha(z_1^k - z_1^{-k})$, and the boundary condition $y_{n+1} = 0$ yields

$$z_1^{2(n+1)} = 1.$$