and since $\widetilde{B}$ is matrix whose entries are polynomials in $K[X]$, it makes sense to multiply on the left by $\widetilde{B}$ and we get

$$\widetilde{B} \cdot B \cdot \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix} = (\widetilde{B}B) \cdot \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix} = P_A I \cdot \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix} = \widetilde{B} \cdot \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix};$$

that is,

$$P_A \cdot e_j = 0, \quad j = 1, \ldots, n,$$

which proves that $P_A(f) = 0$, as claimed. $\square$

If K is a field, then the characteristic polynomial of a linear map $f \colon E \to E$ is independent of the basis $(e_1, \ldots, e_n)$ chosen in E. To prove this, observe that the matrix of f over another basis will be of the form $P^{-1}AP$, for some inverible matrix P, and then

$$\begin{aligned}
\det(XI - P^{-1}AP) &= \det(XP^{-1}IP - P^{-1}AP) \\
&= \det(P^{-1}(XI - A)P) \\
&= \det(P^{-1}) \det(XI - A) \det(P) \\
&= \det(XI - A).
\end{aligned}$$

Therefore, the characteristic polynomial of a linear map is intrinsic to f, and it is denoted by P_f.

The zeros (roots) of the characteristic polynomial of a linear map f are called the *eigenvalues* of f. They play an important role in theory and applications. We will come back to this topic later on.

7.8 Permanents

Recall that the explicit formula for the determinant of an $n \times n$ matrix is

$$\det(A) = \sum_{\pi \in \mathfrak{S}_n} \epsilon(\pi) a_{\pi(1)\,1} \cdots a_{\pi(n)\,n}.$$

If we drop the sign $\epsilon(\pi)$ of every permutation from the above formula, we obtain a quantity known as the *permanent*:

$$\mathrm{per}(A) = \sum_{\pi \in \mathfrak{S}_n} a_{\pi(1)\,1} \cdots a_{\pi(n)\,n}.$$

Permanents and determinants were investigated as early as 1812 by Cauchy. It is clear from the above definition that the permanent is a multilinear symmetric form. We also have

$$\mathrm{per}(A) = \mathrm{per}(A^\top),$$