since complex multiplication only makes sense over $\mathbb{C}$.

We claim that $\rho^*(M)$ is isomorphic to the $\mathbb{C}$-module $M \times M$ with addition defined by

$$(u_1, v_1) + (u_2, v_2) = (u_1 + u_2, v_1 + v_2)$$

and scalar multiplication by $\lambda + i\mu \in \mathbb{C}$ as

$$(\lambda + i\mu) \cdot (u, v) = (\lambda u - \mu v, \lambda v + \mu u).$$

Define the map $\alpha_0 \colon \rho_*(\mathbb{C}) \times M \to M \times M$ by

$$\alpha_0(\lambda + i\mu, u) = (\lambda u, \mu u).$$

It is easy to check that α_0 is $\mathbb{R}$-linear, so we obtain an $\mathbb{R}$-linear map $\alpha \colon \rho_*(\mathbb{C}) \otimes_{\mathbb{R}} M \to M \times M$ such that

$$\alpha((\lambda + i\mu) \otimes u) = (\lambda u, \mu u).$$

We also define the map $\beta \colon M \times M \to \rho_*(\mathbb{C}) \otimes_{\mathbb{R}} M$ by

$$\beta(u, v) = 1 \otimes u + i \otimes v.$$

Again, it is clear that this map is $\mathbb{R}$-linear. We can now check that α and β are mutual inverses. We have

$$\alpha(\beta(u, v)) = \alpha(1 \otimes u + i \otimes v) = \alpha(1 \otimes u) + \alpha(i \otimes v) = (u, 0) + (0, v) = (u, v),$$

and on generators,

$$\beta(\alpha((\lambda + i\mu) \otimes u)) = \beta(\lambda u, \mu u) = 1 \otimes \lambda u + i \otimes \mu u = \lambda \otimes u + i\mu \otimes u = (\lambda + i\mu) \otimes u.$$

Therefore, $\rho_*(\mathbb{C}) \otimes_{\mathbb{R}} M$ and $M \times M$ are isomorphic as $\mathbb{R}$-module. However, the isomorphism α is also an isomorphism of $\mathbb{C}$-modules. This is because in $\rho_*(\mathbb{C}) \otimes_{\mathbb{R}} M$, on generators we have

$$(\lambda + i\mu) \cdot ((x + iy) \otimes u) = (\lambda + i\mu)(x + iy) \otimes u = (\lambda x - \mu y + i(\lambda y + \mu x)) \otimes u,$$

so

$$\begin{aligned}\alpha((\lambda + i\mu) \cdot ((x + iy) \otimes u) &= \alpha((\lambda x - \mu y + i(\lambda y + \mu x)) \otimes u) \\ &= ((\lambda x - \mu y)u, (\lambda y + \mu x)u),\end{aligned}$$

and by definition of the scalar multiplication by elements of $\mathbb{C}$ on $M \times M$

$$(\lambda + i\mu) \cdot \alpha((x + iy) \otimes u) = (\lambda + i\mu) \cdot (xu, yu) = ((\lambda x - \mu y)u, (\lambda y + \mu x)u).$$

Therefore, α is an isomorphism between the $\mathbb{C}$-modules $\rho^*(M) = \rho_*(\mathbb{C}) \otimes_{\mathbb{R}} M$ and $M \times M$.

The above process of ring extension can also be applied to linear maps. We have the following proposition whose proof is given in Bourbaki [25] (Chapter II, Section 5, Proposition 1).