Proof. The affine space $u_\ell + \mathcal{G}_\ell$ is closed and convex, and since J is a quadratic elliptic functional it is coercive and strictly convex, so by Theorem 49.8(2) it has a unique minimum in $u_\ell + \mathcal{G}_\ell$. This minimum $u_{\ell+1}$ is also the minimum of the problem, find $u_{\ell+1}$ such that

$$u_{\ell+1} \in u_\ell + \mathcal{G}_\ell \quad \text{and} \quad J(u_{\ell+1}) = \inf_{v \in \mathcal{G}_\ell} J(u_\ell + v),$$

and since $\mathcal{G}_\ell$ is a subspace, by Corollary 40.10 we must have

$$dJ_{u_{\ell+1}}(w) = 0 \quad \text{for all } w \in \mathcal{G}_\ell,$$

that is

$$\langle \nabla J_{u_{\ell+1}}, w \rangle = 0 \quad \text{for all } w \in \mathcal{G}_\ell.$$

Since $\mathcal{G}_\ell$ is spanned by $(\nabla J_{u_0}, \nabla J_{u_1}, \ldots, \nabla J_{u_\ell})$, we obtain

$$\langle \nabla J_{u_{\ell+1}}, \nabla J_{u_j} \rangle = 0, \quad 0 \leq j \leq \ell,$$

and since this holds for $\ell = 0, \ldots, k$, we get

$$\langle \nabla J_{u_i}, \nabla J_{u_j} \rangle = 0, \quad 0 \leq i \neq j \leq k+1,$$

which shows the second part of the proposition. $\square$

As a corollary of Proposition 49.15, if $\nabla J_{u_i} \neq 0$ for $i = 0, \ldots, k$, then the vectors ∇J_{u_i} are linearly independent and $\mathcal{G}_k$ has dimension $k+1$. Therefore, the conjugate gradient method terminates in at most n steps. Here is an example of a problem for which the gradient descent with optimal stepsize parameter does not converge in a finite number of steps.

Example 49.2. Let $J\colon \mathbb{R}^2 \to \mathbb{R}$ be the function given by

$$J(v_1, v_2) = \frac{1}{2}(\alpha_1 v_1^2 + \alpha_2 v_2^2),$$

where $0 < \alpha_1 < \alpha_2$. The minimum of J is attained at $(0, 0)$. Unless the initial vector $u_0 = (u_1^0, u_2^0)$ has the property that either $u_1^0 = 0$ or $u_2^0 = 0$, we claim that the gradient descent with optimal stepsize parameter does not converge in a finite number of steps. Observe that

$$\nabla J_{(v_1, v_2)} = \begin{pmatrix} \alpha_1 v_1 \\ \alpha_2 v_2 \end{pmatrix}.$$

As a consequence, given u_k, the line search for finding ρ_k and u_{k+1} yields $u_{k+1} = (0, 0)$ iff there is some $\rho \in \mathbb{R}$ such that

$$u_1^k = \rho \alpha_1 u_1^k \quad \text{and} \quad u_2^k = \rho \alpha_2 u_2^k.$$

Since $\alpha_1 \neq \alpha_2$, this is only possible if either $u_1^k = 0$ or $u_2^k = 0$. The formulae given just before Proposition 49.14 yield

$$u_1^{k+1} = \frac{\alpha_2^2(\alpha_2 - \alpha_1) u_1^k (u_2^k)^2}{\alpha_1^3 (u_1^k)^2 + \alpha_2^3 (u_2^k)^2}, \quad u_2^{k+1} = \frac{\alpha_1^2(\alpha_1 - \alpha_2) u_2^k (u_1^k)^2}{\alpha_1^3 (u_1^k)^2 + \alpha_2^3 (u_2^k)^2},$$

which implies that if $u_1^k \neq 0$ and $u_2^k \neq 0$, then $u_1^{k+1} \neq 0$ and $u_2^{k+1} \neq 0$, so the method runs forever from any initial vector $u_0 = (u_1^0, u_2^0)$ such that $u_1^0 \neq 0$ and, $u_2^0 \neq 0$.