- Convergence of sequences of vectors in a normed vector space.
- Cauchy sequences, complex normed vector spaces, Banach spaces.
- Convergence of series. Absolute convergence.
- The matrix exponential.
- Skew symmetric matrices and orthogonal matrices.

9.10 Problems

Problem 9.1. Let A be the following matrix:

$$A = \begin{pmatrix} 1 & 1/\sqrt{2} \\ 1/\sqrt{2} & 3/2 \end{pmatrix}.$$

Compute the operator 2-norm $\|A\|_2$ of A.

Problem 9.2. Prove Proposition 9.3, namely that the following inequalities hold for all $x \in \mathbb{R}^n$ (or $x \in \mathbb{C}^n$):

$$\begin{aligned} \|x\|_\infty &\leq \|x\|_1 \leq n\|x\|_\infty, \\ \|x\|_\infty &\leq \|x\|_2 \leq \sqrt{n}\|x\|_\infty, \\ \|x\|_2 &\leq \|x\|_1 \leq \sqrt{n}\|x\|_2. \end{aligned}$$

Problem 9.3. For any $p \geq 1$, prove that for all $x \in \mathbb{R}^n$,

$$\lim_{p \mapsto \infty} \|x\|_p = \|x\|_\infty .$$

Problem 9.4. Let A be an $n \times n$ matrix which is strictly row diagonally dominant, which means that

$$|a_{i\,i}| > \sum_{j \neq i} |a_{i\,j}|,$$

for $i = 1, \ldots, n$, and let

$$\delta = \min_i \left\{ |a_{i\,i}| - \sum_{j \neq i} |a_{i\,j}| \right\}.$$

The fact that A is strictly row diagonally dominant is equivalent to the condition $\delta > 0$.

(1) For any nonzero vector v, prove that

$$\|Av\|_\infty \geq \|v\|_\infty \, \delta.$$

Use the above to prove that A is invertible.