Proposition 35.21. *If M is a torsion module over a PID, a submodule N of M is a direct factor of M iff N_p is a direct factor of M_p for every irreducible element $p \in A$.*

Proof. This is because if N and N' are two submodules of M, we have $M = N \oplus N'$ iff, by Proposition 35.20, $M_p = N_p \oplus N'_p$ for every irreducible elements $p \in A$. $\square$

Definition 35.11. An A-module M is said to be *semi-simple* iff for every submodule N of M, there is some submodule N' of M such that $M = N \oplus N'$.

Proposition 35.22. *Let A be a PID which is not a field, and let M be any A-module. Then M is semi-simple iff it is a torsion module and if $M_p = M(p)$ for every irreducible element $p \in A$ (in other words, if $x \in M$ is annihilated by a power of p, then it is already annihilated by p).*

Proof. Assume that M is semi-simple. Let $x \in M$ and pick any irreducible element $p \in A$. Then, the submodule pAx has a supplement N such that

$$M = pAx \oplus N,$$

so we can write $x = pax + y$, for some $y \in N$ and some $a \in A$. But then,

$$y = (1 - pa)x,$$

and since p is irreducible, p is not a unit, so $1 - pa \neq 0$. Observe that

$$p(1 - ap)x = py \in pAx \cap N = (0).$$

Since $p(1 - ap) \neq 0$, x is a torsion element, and thus M is a torsion module. The above argument shows that

$$p(1 - ap)x = 0,$$

which implies that $px = ap^2x$, and by induction,

$$px = a^n p^{n+1} x, \quad \text{for all } n \geq 1.$$

If we pick x in M_p, then there is some $m \geq 1$ such that $p^m x = 0$, and we conclude that

$$px = 0.$$

Therefore, $M_p = M(p)$, as claimed.

Conversely, assume that M is a torsion-module and that $M_p = M(p)$ for every irreducible element $p \in A$. By Proposition 35.21, it is sufficient to prove that a module annihilated by a an irreducible element is semi-simple. This is because such a module is a vector space over the field $A/(p)$ (recall that in a PID, an ideal (p) is maximal iff p is irreducible), and in a vector space, every subspace has a supplement. $\square$

Theorem 35.19 shows that a finitely generated torsion module is a direct sum of p-primary modules M_p. We can do better. In the next section we show that each primary module M_p is the direct sum of cyclic modules of the form $A/(p^n)$.