and then by induction on $n \geq 1$ than

$$V^{\otimes m} \otimes V^{\otimes n} \cong V^{\otimes(m+n)}.$$

In summary the multiplication $V^{\otimes m} \times V^{\otimes n} \longrightarrow V^{\otimes(m+n)}$ is defined so that

$$(v_1 \otimes \cdots \otimes v_m) \cdot (w_1 \otimes \cdots \otimes w_n) = v_1 \otimes \cdots \otimes v_m \otimes w_1 \otimes \cdots \otimes w_n.$$

(This has to be made rigorous by using isomorphisms involving the associativity of tensor products, for details, see Jacobson [99], Section 3.9, or Bertin [15], Chapter 4, Section 2.)

Definition 33.11. Given a K-vector space V (not necessarily finite dimensional), the vector space

$$T(V) = \bigoplus_{m \geq 0} V^{\otimes m}$$

denoted $T^{\bullet}(V)$ or $\bigotimes V$ equipped with the multiplication operations $V^{\otimes m} \times V^{\otimes n} \longrightarrow V^{\otimes(m+n)}$ defined above is called the *tensor algebra of V*.

Remark: It is important to note that multiplication in $T(V)$ is **not** commutative. Also, in all rigor, the unit $\mathbf{1}$ of $T(V)$ is **not equal** to 1, the unit of the field K. However, in view of the injection $\iota_0 \colon K \to T(V)$, for the sake of notational simplicity, we will denote $\mathbf{1}$ by 1. More generally, in view of the injections $\iota_n \colon V^{\otimes n} \to T(V)$, we identify elements of $V^{\otimes n}$ with their images in $T(V)$.

The algebra $T(V)$ satisfies a universal mapping property which shows that it is unique up to isomorphism. For simplicity of notation, let $i \colon V \to T(V)$ be the natural injection of V into $T(V)$.

Proposition 33.19. *Given any K-algebra A, for any linear map $f \colon V \to A$, there is a unique K-algebra homomorphism $\overline{f} \colon T(V) \to A$ so that*

$$f = \overline{f} \circ i,$$

as in the diagram below.

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[dr, "f"'] & T(V) \arrow[d, "\overline{f}"] \\ & A \end{tikzcd}$$

Proof. Left an an exercise (use Theorem 33.6). A proof can be found in Knapp [104] (Appendix A, Proposition A.14) or Bertin [15] (Chapter 4, Theorem 2.4). $\square$

Proposition 33.19 implies that there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{alg}}(T(V), A) \cong \mathrm{Hom}(V, A),$$

where the algebra A on the right-hand side is viewed as a vector space. Proposition 33.19 also has the following corollary.