Proposition 51.28. *For any proper convex function f on $\mathbb{R}^n$ and for any vector $x \in \mathbb{R}^n$, the following conditions on a vector $y \in \mathbb{R}^n$ are equivalent.*

(a) $y \in \partial f(x)$.

(b) The function $\langle z, y\rangle - f(z)$ *achieves its supremum in z at $z = x$.*

(c) $f(x) + f^*(y) \leq \langle x, y\rangle$.

(d) $f(x) + f^*(y) = \langle x, y\rangle$.

If $(\mathrm{cl}(f))(x) = f(x)$, *then there are three more conditions all equivalent to the above conditions.*

(a)* $x \in \partial f^*(y)$.

(b) The function* $\langle x, z\rangle - f^*(z)$ *achieves its supremum in z at $z = y$.*

*(a**)* $y \in \partial(\mathrm{cl}(f))(x)$.

The following results are corollaries of Proposition 51.28; see Rockafellar [138] (Corollaries 23.5.1, 23.5.2, 23.5.3).

Corollary 51.29. *For any proper convex function f on $\mathbb{R}^n$, if f is closed, then $y \in \partial f(x)$ iff $x \in \partial f^*(y)$, for all $x, y \in \mathbb{R}^n$.*

Corollary 51.29 states a sort of adjunction property.

Corollary 51.30. *For any proper convex function f on $\mathbb{R}^n$, if f is subdifferentiable at $x \in \mathbb{R}^n$, then $(\mathrm{cl}(f))(x) = f(x)$ and $\partial(\mathrm{cl}(f))(x) = \partial f(x)$.*

Corollary 51.30 shows that the closure of a proper convex function f agrees with f whereever f is subdifferentiable.

Corollary 51.31. *For any proper convex function f on $\mathbb{R}^n$, for any nonempty closed convex subset C of $\mathbb{R}^n$, for any $y \in \mathbb{R}^n$, the set $\partial\delta^*(y|C) = \partial I_C^*(y)$ consists of the vectors $x \in \mathbb{R}^n$ (if any) where the linear form $z \mapsto \langle z, y\rangle$ achieves its maximum over C.*

There is a notion of approximate subgradient which turns out to be useful in optimization theory; see Bertsekas [19, 17].

Definition 51.17. Let $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be any proper convex function. For any $\epsilon > 0$, for any $x \in \mathbb{R}^n$, if $f(x)$ is finite, then an *ϵ-subgradient of f at x* is any vector $u \in \mathbb{R}^n$ such that

$$f(z) \geq f(x) - \epsilon + \langle z - x, u\rangle, \quad \text{for all } z \in \mathbb{R}^n.$$

See Figure 51.23. The set of all ϵ-subgradients of f at x is denoted $\partial_\epsilon f(x)$ and is called the *ϵ-subdifferential* of f at x.