24.11 Intersection of Affine Spaces

In this section we take a closer look at the intersection of affine subspaces. This subsection can be omitted at first reading.

First, we need a result of linear algebra. Given a vector space E and any two subspaces M and N, there are several interesting linear maps. We have the canonical injections $i\colon M \to M+N$ and $j\colon N \to M+N$, the canonical injections $in_1\colon M \to M\oplus N$ and $in_2\colon N \to M\oplus N$, and thus, injections $f\colon M\cap N \to M\oplus N$ and $g\colon M\cap N \to M\oplus N$, where f is the composition of the inclusion map from $M \cap N$ to M with in_1, and g is the composition of the inclusion map from $M \cap N$ to N with in_2. Then, we have the maps $f + g\colon M \cap N \to M \oplus N$, and $i - j\colon M \oplus N \to M + N$.

Proposition 24.15. *Given a vector space E and any two subspaces M and N, with the definitions above,*

$$0 \longrightarrow M \cap N \xrightarrow{f+g} M \oplus N \xrightarrow{i-j} M + N \longrightarrow 0$$

is a short exact sequence, which means that $f + g$ is injective, $i - j$ is surjective, and that $\mathrm{Im}\,(f + g) = \mathrm{Ker}\,(i - j)$. *As a consequence, we have the Grassmann relation*

$$\dim(M) + \dim(N) = \dim(M + N) + \dim\,(M \cap N).$$

Proof. It is obvious that $i - j$ is surjective and that $f + g$ is injective. Assume that $(i - j)(u + v) = 0$, where $u \in M$, and $v \in N$. Then, $i(u) = j(v)$, and thus, by definition of i and j, there is some $w \in M \cap N$, such that $i(u) = j(v) = w \in M \cap N$. By definition of f and g, $u = f(w)$ and $v = g(w)$, and thus $\mathrm{Im}\,(f + g) = \mathrm{Ker}\,(i - j)$, as desired. The second part of the proposition follows from standard results of linear algebra (see Artin [7], Strang [170], or Lang [109]). ☐

We now prove a simple proposition about the intersection of affine subspaces.

Proposition 24.16. *Given any affine space E, for any two nonempty affine subspaces M and N, the following facts hold:*

(1) $M \cap N \neq \emptyset$ iff $\overrightarrow{ab} \in \overrightarrow{M} + \overrightarrow{N}$ for some $a \in M$ and some $b \in N$.

(2) $M \cap N$ consists of a single point iff $\overrightarrow{ab} \in \overrightarrow{M} + \overrightarrow{N}$ for some $a \in M$ and some $b \in N$, and $\overrightarrow{M} \cap \overrightarrow{N} = \{0\}$.

(3) If S is the least affine subspace containing M and N, then $\overrightarrow{S} = \overrightarrow{M} + \overrightarrow{N} + K\overrightarrow{ab}$ (the vector space $\overrightarrow{E}$ is defined over the field K).