Proof. Since A is positive definite, it is invertible since its eigenvalues are all strictly positive. Let $x_0 = A^{-1}b$, and compute $Q(y) - Q(x_0)$ for any $y \in \mathbb{R}^n$. Since $Ax_0 = b$, we get

$$\begin{aligned} Q(y) - Q(x_0) &= \frac{1}{2}y^\top Ay - y^\top b - \frac{1}{2}x_0^\top Ax_0 + x_0^\top b \\ &= \frac{1}{2}y^\top Ay - y^\top Ax_0 + \frac{1}{2}x_0^\top Ax_0 \\ &= \frac{1}{2}(y - x_0)^\top A(y - x_0). \end{aligned}$$

Since A is positive definite, the last expression is nonnegative, and thus

$$Q(y) \geq Q(x_0)$$

for all $y \in \mathbb{R}^n$, which proves that $x_0 = A^{-1}b$ is a global minimum of $Q(x)$. A simple computation yields

$$Q(A^{-1}b) = -\frac{1}{2}b^\top A^{-1}b.$$

□

Remarks:

(1) The quadratic function $Q(x)$ is also given by

$$Q(x) = \frac{1}{2}x^\top Ax - b^\top x,$$

but the definition using $x^\top b$ is more convenient for the proof of Proposition 42.2.

(2) If $Q(x)$ contains a constant term $c \in \mathbb{R}$, so that

$$Q(x) = \frac{1}{2}x^\top Ax - x^\top b + c,$$

the proof of Proposition 42.2 still shows that $Q(x)$ has a unique global minimum for $x = A^{-1}b$, but the minimal value is

$$Q(A^{-1}b) = -\frac{1}{2}b^\top A^{-1}b + c.$$

Thus when the energy function $Q(x)$ of a system is given by a quadratic function

$$Q(x) = \frac{1}{2}x^\top Ax - x^\top b,$$

where A is symmetric positive definite, finding the global minimum of $Q(x)$ is equivalent to solving the linear system $Ax = b$. Sometimes, it is useful to recast a linear problem $Ax = b$ as a variational problem (finding the minimum of some energy function). However, very often, a minimization problem comes with extra constraints that must be satisfied for all admissible solutions.