If $v \in \pi_i(E)$, then $v = \pi_i(u)$ for some $u \in E$, so

$$\begin{aligned} p_i^{r_i}(f)(v) &= p_i^{r_i}(f)(\pi_i(u)) \\ &= p_i^{r_i}(f)g_i(f)h_i(f)(u) \\ &= h_i(f)p_i^{r_i}(f)g_i(f)(u) \\ &= h_i(f)m(f)(u) = 0, \end{aligned}$$

because m is the minimal polynomial of f. Therefore, $v \in W_i$.

Conversely, assume that $v \in W_i = \mathrm{Ker}\,(p_i^{r_i}(f))$. If $j \neq i$, then $g_j h_j$ is divisible by $p_i^{r_i}$, so

$$g_j(f)h_j(f)(v) = \pi_j(v) = 0, \quad j \neq i.$$

Then since $\pi_1 + \cdots + \pi_k = \mathrm{id}$, we have $v = \pi_i v$, which shows that v is in the range of π_i. Therefore, $W_i = \mathrm{Im}(\pi_i)$, and this finishes the proof of (a).

If $p_i^{r_i}(f)(u) = 0$, then $p_i^{r_i}(f)(f(u)) = f(p_i^{r_i}(f)(u)) = 0$, so (b) holds.

If we write $f_i = f \mid W_i$, then $p_i^{r_i}(f_i) = 0$, because $p_i^{r_i}(f) = 0$ on W_i (its kernel). Therefore, the minimal polynomial of f_i divides $p_i^{r_i}$. Conversely, let q be any polynomial such that $q(f_i) = 0$ (on W_i). Since $m = p_i^{r_i} g_i$, the fact that $m(f)(u) = 0$ for all $u \in E$ shows that

$$p_i^{r_i}(f)(g_i(f)(u)) = 0, \quad u \in E,$$

and thus $\mathrm{Im}(g_i(f)) \subseteq \mathrm{Ker}\,(p_i^{r_i}(f)) = W_i$. Consequently, since $q(f)$ is zero on W_i,

$$q(f)g_i(f) = 0 \quad \text{for all } u \in E.$$

But then qg_i is divisible by the minimal polynomial $m = p_i^{r_i} g_i$ of f, and since $p_i^{r_i}$ and g_i are relatively prime, by Euclid's proposition, $p_i^{r_i}$ must divide q. This finishes the proof that the minimal polynomial of f_i is $p_i^{r_i}$, which is (c). $\square$

To best understand the projection constructions of Theorem 31.10, we provide the following two explicit examples of the primary decomposition theorem.

Example 31.2. First let $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ be defined as $f(x, y, z) = (y, -x, z)$. In terms of the standard basis f is represented by the 3×3 matrix $X_f := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then a simple calculation shows that $m_f(x) = \chi_f(x) = (x^2+1)(x-1)$. Using the notation of the preceding proof set

$$m = p_1 p_2, \qquad p_1 = x^2 + 1, \qquad p_2 = x - 1.$$

Then

$$g_1 = \frac{m}{p_1} = x - 1, \qquad g_2 = \frac{m}{p_2} = x^2 + 1.$$