The above construction also works if $O \in \Delta$; see Figures 26.32 and 26.33.

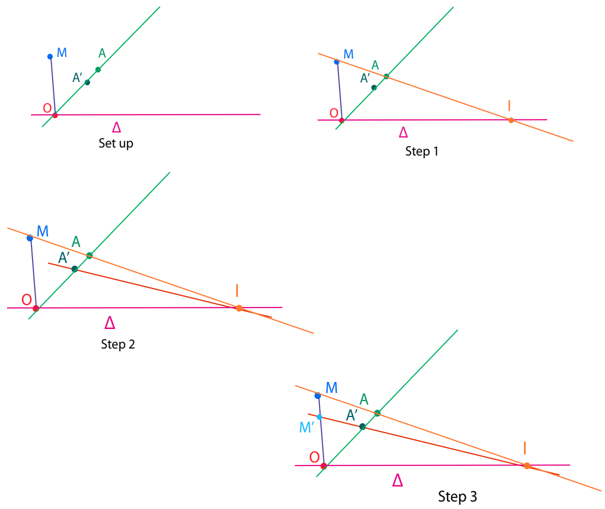

Figure 26.32: The three step process for determining the elation point $h(M) = M'$ when M is not on the line $\langle A, A' \rangle$. Step 1 finds the intersection between the extension of $\langle A, M \rangle$ and Δ. Step 2 forms the line $\langle A', I \rangle$. Step 3 extends $\langle A'I \rangle$ and determines its intersection with $\langle O, M \rangle$. The intersection point is M'.

Another useful property of homologies (here, $O \notin \Delta$) is that for any line d passing through the center O, if I is the intersection point of the line d and Δ, then for any $M \in d$ distinct from O and not on Δ and its image M', the cross-ratio $[O, I, M, M']$ is independent of d. If $[O, I, M, M'] = -1$ for all $M \neq O$, we say that h is a *harmonic homology*. It can be shown that a homography h is a harmonic homology iff h is an involution ($h^2 = \mathrm{id}$); see Silder [161] (Chapter 4, Section 4.4). It can also be shown that any homography of $\mathbb{RP}^2$ can be expressed as the composition of two homologies; see Silder [161] (Chapter 4, Section 4.5).

We now consider the generalization of the notion of homology (and projective transvection) to any projective space $\mathbb{P}(E)$, where E is a vector space of any finite dimension over a field K. We need to review a few concepts from Section 8.15.

Let E be a vector space and let H be a hyperplane in E. Recall from Definition 8.8 that for any nonzero vector $u \in E$ such that $u \notin H$, and any scalar $\alpha \neq 0, 1$, a linear map $f \colon E \to E$ such that $f(x) = x$ for all $x \in H$ and $f(x) = \alpha x$ for every $x \in D = Ku$ is called a *dilatation of hyperplane H, direction D, and scale factor α*. See Figure 26.34.

From Definition 8.9, for any nonzero nonlinear form $\varphi \in E^*$ defining H (which means that $H = \mathrm{Ker}\,(\varphi)$) and any nonzero vector $u \in H$, the linear map $\tau_{\varphi,u}$ given by

$$\tau_{\varphi,u}(x) = x + \varphi(x)u, \quad \varphi(u) = 0,$$

for all $x \in E$ is called a *transvection of hyperplane H and direction u*. See Figure 26.35.