$A'_{j\,k}$ obtained by deleting row j and column k from A', since A and A' only differ by the j-th row. Thus,

$$\det(A_{j\,k}) = \det(A'_{j\,k}),$$

and we have

$$c_{i\,j} = a_{i\,1}(-1)^{j+1}\det(A'_{j\,1}) + \cdots + a_{i\,n}(-1)^{j+n}\det(A'_{j\,n}).$$

However, this is the expansion of $\det(A')$ according to the j-th row, since the j-th row of A' is equal to the i-th row of A. Furthermore, since A' has two identical rows i and j, because det is an alternating map of the rows (see an earlier remark), we have $\det(A') = 0$. Thus, we have shown that $c_{i\,i} = \det(A)$, and $c_{i\,j} = 0$, when $j \neq i$, and so

$$A\widetilde{A} = \det(A)I_n.$$

It is also obvious from the definition of $\widetilde{A}$, that

$$\widetilde{A}^\top = \widetilde{A^\top}.$$

Then applying the first part of the argument to $A^\top$, we have

$$A^\top \widetilde{A^\top} = \det(A^\top)I_n,$$

and since $\det(A^\top) = \det(A)$, $\widetilde{A}^\top = \widetilde{A^\top}$, and $(\widetilde{A}A)^\top = A^\top\widetilde{A}^\top$, we get

$$\det(A)I_n = A^\top\widetilde{A^\top} = A^\top\widetilde{A}^\top = (\widetilde{A}A)^\top,$$

that is,

$$(\widetilde{A}A)^\top = \det(A)I_n,$$

which yields

$$\widetilde{A}A = \det(A)I_n,$$

since $I_n^\top = I_n$. This proves that

$$A\widetilde{A} = \widetilde{A}A = \det(A)I_n.$$

As a consequence, if $\det(A)$ is invertible, we have $A^{-1} = (\det(A))^{-1}\widetilde{A}$. Conversely, if A is invertible, from $AA^{-1} = I_n$, by Proposition 7.9, we have $\det(A)\det(A^{-1}) = 1$, and $\det(A)$ is invertible. $\square$

For example, we saw earlier that

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -2 & -2 \\ 3 & 3 & -3 \end{pmatrix} \quad \text{and} \quad \widetilde{A} = \begin{pmatrix} 12 & 6 & 0 \\ 0 & -6 & 4 \\ 12 & 0 & -4 \end{pmatrix},$$