Remarks:

(1) The conditions $A\,A^* = I_n$, $A^*A = I_n$, and $A^{-1} = A^*$ are equivalent. Given any two orthonormal bases $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$, if P is the change of basis matrix from $(u_1, \ldots, u_n)$ to $(v_1, \ldots, v_n)$, it is easy to show that the matrix P is unitary. The proof of Proposition 14.14 (3) also shows that if f is an isometry, then the image of an orthonormal basis $(u_1, \ldots, u_n)$ is an orthonormal basis.

(2) Using the explicit formula for the determinant, we see immediately that

$$\det(\overline{A}) = \overline{\det(A)}.$$

If f is a unitary transformation and A is its matrix with respect to any orthonormal basis, from $AA^* = I$, we get

$$\det(AA^*) = \det(A)\det(A^*) = \det(A)\overline{\det(A^\top)} = \det(A)\overline{\det(A)} = |\det(A)|^2,$$

and so $|\det(A)| = 1$. It is clear that the isometries of a Hermitian space of dimension n form a group, and that the isometries of determinant $+1$ form a subgroup.

This leads to the following definition.

Definition 14.10. Given a Hermitian space E of dimension n, the set of isometries $f\colon E \to E$ forms a subgroup of $\mathbf{GL}(E, \mathbb{C})$ denoted by $\mathbf{U}(E)$, or $\mathbf{U}(n)$ when $E = \mathbb{C}^n$, called the *unitary group (of E)*. For every isometry f we have $|\det(f)| = 1$, where $\det(f)$ denotes the determinant of f. The isometries such that $\det(f) = 1$ are called *rotations, or proper isometries, or proper unitary transformations*, and they form a subgroup of the special linear group $\mathbf{SL}(E, \mathbb{C})$ (and of $\mathbf{U}(E)$), denoted by $\mathbf{SU}(E)$, or $\mathbf{SU}(n)$ when $E = \mathbb{C}^n$, called the *special unitary group (of E)*. The isometries such that $\det(f) \neq 1$ are called *improper isometries, or improper unitary transformations, or flip transformations*.

A very important example of unitary matrices is provided by Fourier matrices (up to a factor of $\sqrt{n}$), matrices that arise in the various versions of the discrete Fourier transform. For more on this topic, see the problems, and Strang [169, 172].

The group $\mathbf{SU}(2)$ turns out to be the group of *unit quaternions*, invented by Hamilton. This group plays an important role in the representation of rotations in $\mathbf{SO}(3)$ used in computer graphics and robotics; see Chapter 16.

Now that we have the definition of a unitary matrix, we can explain how the Gram–Schmidt orthonormalization procedure immediately yields the QR-decomposition for matrices.

Definition 14.11. Given any complex $n \times n$ matrix A, a *QR-decomposition* of A is any pair of $n \times n$ matrices (U, R), where U is a unitary matrix and R is an upper triangular matrix such that $A = UR$.