where D is a diagonal matrix, iff the following equations hold:

$$\begin{aligned}(b+c)\cos(\theta+\varphi) &= (a-d)\sin(\theta+\varphi),\\ (c-b)\cos(\theta-\varphi) &= (a+d)\sin(\theta-\varphi).\end{aligned}$$

(2) Discuss the solvability of the system. Consider the following cases:

Case 1: $a-d=a+d=0$.

Case 2a: $a-d=b+c=0$, $a+d\neq 0$.

Case 2b: $a-d=0$, $b+c\neq 0$, $a+d\neq 0$.

Case 3a: $a+d=c-b=0$, $a-d\neq 0$.

Case 3b: $a+d=0$, $c-b\neq 0$, $a-d\neq 0$.

Case 4: $a+d\neq 0$, $a-d\neq 0$. Show that the solution in this case is

$$\begin{aligned}\theta &= \frac{1}{2}\left[\arctan\left(\frac{b+c}{a-d}\right)+\arctan\left(\frac{c-b}{a+d}\right)\right],\\ \varphi &= \frac{1}{2}\left[\arctan\left(\frac{b+c}{a-d}\right)-\arctan\left(\frac{c-b}{a+d}\right)\right].\end{aligned}$$

If $b=0$, show that the discussion is simpler: basically, consider $c=0$ or $c\neq 0$.

(3) Expressing everything in terms of $u=\cot\theta$ and $v=\cot\varphi$, show that the equations in (2) become

$$\begin{aligned}(b+c)(uv-1) &= (u+v)(a-d),\\ (c-b)(uv+1) &= (-u+v)(a+d).\end{aligned}$$

Problem 13.4. Let A be an $n\times n$ real invertible matrix.

(1) Prove that $A^\top A$ is symmetric positive definite.

(2) Use the Cholesky factorization $A^\top A = R^\top R$ with R upper triangular with positive diagonal entries to prove that $Q=AR^{-1}$ is orthogonal, so that $A=QR$ is the QR-factorization of A.

Problem 13.5. Modify the function `houseqr` so that it applies to an $m\times n$ matrix with $m\geq n$, to produce an $m\times n$ upper-triangular matrix whose last $m-n$ rows are zeros.

Problem 13.6. The purpose of this problem is to prove that given any self-adjoint linear map $f\colon E\to E$ (i.e., such that $f^*=f$), where E is a Euclidean space of dimension $n\geq 3$, given an orthonormal basis $(e_1,\ldots,e_n)$, there are $n-2$ isometries h_i, hyperplane reflections or the identity, such that the matrix of

$$h_{n-2}\circ\cdots\circ h_1\circ f\circ h_1\circ\cdots\circ h_{n-2}$$