Since $\lim_{k\mapsto\infty} \widetilde{Q}_k = I$, we deduce that

$$\lim_{k\mapsto\infty} (\widetilde{Q}_k)^* R\Lambda R^{-1}\widetilde{Q}_k = R\Lambda R^{-1} = \begin{pmatrix} \lambda_1 & * & \cdots & * \\ 0 & \lambda_2 & \cdots & * \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

an upper triangular matrix with the eigenvalues of A on the diagonal. Since R is upper triangular, the order of the eigenvalues is preserved. If we let

$$\mathcal{D}_k = (\widetilde{Q}_k)^* R\Lambda R^{-1}\widetilde{Q}_k, \tag{$*_{10}$}$$

then by ($*_9$) we have $A_{k+1} = D_k^*\mathcal{D}_k D_k$, and since the matrices D_k are diagonal matrices, we have

$$(A_{k+1})_{jj} = (D_k^*\mathcal{D}_k D_k)_{ij} = \overline{(D_k)_{ii}}(D_k)_{jj}(\mathcal{D}_k)_{ij},$$

which implies that

$$(A_{k+1})_{ii} = (\mathcal{D}_k)_{ii}, \quad i = 1, \ldots, n, \tag{$*_{11}$}$$

since $|(D_k)_{ii}| = 1$ for $i = 1, \ldots, n$. Since $\lim_{k\mapsto\infty} \mathcal{D}_k = R\Lambda R^{-1}$, we conclude that the strictly lower-triangular part of A_{k+1} converges to zero, and the diagonal of A_{k+1} converges to Λ. $\square$

Observe that if the matrix A is real, then the hypothesis that the eigenvalues have distinct moduli implies that the eigenvalues are all real and simple.

The following `Matlab` program implements the basic QR-method using the function `qrv4` from Section 12.8.

```
function T = qreigen(A,m)
T = A;
for k = 1:m
  [Q R] = qrv4(T);
  T = R*Q;
end
end
```

Example 18.1. If we run the function `qreigen` with 100 iterations on the 8×8 symmetric matrix

$$A = \begin{pmatrix} 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix},$$