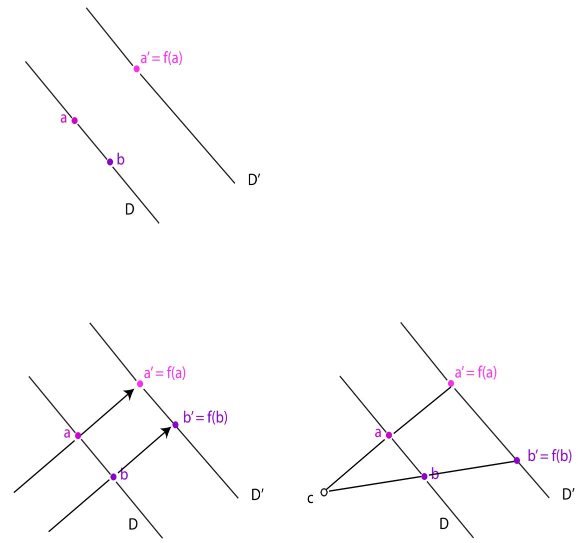


Figure 24.22: An illustration of Proposition 24.11. The bottom left diagram illustrates a translation, while the bottom right illustrates a central dilation through $c$.

*Proof.* Pappus's theorem is illustrated in Figure 24.23. If $D$ and $D'$ are not parallel, let $d$ be their intersection. Let $f$ be the dilatation of center $d$ such that $f(a) = b$, and let $g$ be the dilatation of center $d$ such that $g(b) = c$. Since the lines $\langle a, b'\rangle$ and $\langle a', b\rangle$ are parallel, and the lines $\langle b, c'\rangle$ and $\langle b', c\rangle$ are parallel, by Proposition 24.11 we have $a' = f(b')$ and $b' = g(c')$. However, we observed that dilatations with the same center commute, and thus $f \circ g = g \circ f$, and thus, letting $h = g \circ f$, we get $c = h(a)$ and $a' = h(c')$. Again, by Proposition 24.11, the lines $\langle a, c'\rangle$ and $\langle a', c\rangle$ are parallel. If $D$ and $D'$ are parallel, we use translations instead of dilatations. $\square$

There is a converse to Pappus's theorem, which yields a fancier version of Pappus's theorem, but it is easier to prove it using projective geometry. It should be noted that in axiomatic presentations of projective geometry, Pappus's theorem is equivalent to the commutativity of the ground field $K$ (in the present case, $K = \mathbb{R}$). We now prove an affine version of Desargues's theorem.

**Proposition 24.13.** *Given any affine space $E$, and given any two triangles $(a, b, c)$ and $(a', b', c')$, where $a, b, c, a', b', c'$ are all distinct, if $\langle a, b\rangle$ and $\langle a', b'\rangle$ are parallel and $\langle b, c\rangle$ and $\langle b', c'\rangle$ are parallel, then $\langle a, c\rangle$ and $\langle a', c'\rangle$ are parallel iff the lines $\langle a, a'\rangle$, $\langle b, b'\rangle$, and $\langle c, c'\rangle$ are either parallel or concurrent (i.e., intersect in a common point).*

*Proof.* We prove half of the proposition, the direction in which it is assumed that $\langle a, c\rangle$ and $\langle a', c'\rangle$ are parallel, leaving the converse as an exercise. Since the lines $\langle a, b\rangle$ and $\langle a', b'\rangle$ are