(2) If E has even dimension $n = 2m$, then every improper orthogonal transformation f admits 1 *as an eigenvalue and the eigenspace F of all eigenvectors left invariant under f has an odd dimension $2p+1$. Furthermore, there is an orthonormal basis of E, in which f is represented by a matrix of the form*

$$\begin{pmatrix} S_{2(m-p)-1} & 0 \\ 0 & I_{2p+1} \end{pmatrix},$$

where $S_{2(m-p)-1}$ is an improper orthogonal matrix that does not have 1 *as an eigenvalue.*

Proof. We prove only (1), the proof of (2) being similar. Since f is a rotation and $n = 2m+1$ is odd, by Theorem 27.1, f is the composition of an even number less than or equal to $2m$ of reflections. From Lemma 24.15, recall the Grassmann relation

$$\dim(M) + \dim(N) = \dim(M+N) + \dim(M \cap N),$$

where M and N are subspaces of E. Now, if M and N are hyperplanes, their dimension is $n-1$, and thus $\dim(M \cap N) \geq n-2$. Thus, if we intersect $k \leq n$ hyperplanes, we see that the dimension of their intersection is at least $n-k$. Since each of the reflections is the identity on the hyperplane defining it, and since there are at most $2m = n-1$ reflections, their composition is the identity on a subspace of dimension at least 1. This proves that 1 is an eigenvalue of f. Let F be the eigenspace associated with 1, and assume that its dimension is q. Let $G = F^{\perp}$ be the orthogonal of F. By Lemma 27.2, G is stable under f, and $E = F \oplus G$. Using Lemma 12.10, we can find an orthonormal basis of E consisting of an orthonormal basis for G and orthonormal basis for F. In this basis, the matrix of f is of the form

$$\begin{pmatrix} R_{2m+1-q} & 0 \\ 0 & I_q \end{pmatrix}.$$

Thus, $\det(f) = \det(R)$, and R must be a rotation, since f is a rotation and $\det(f) = 1$. Now, if f left some vector $u \neq 0$ in G invariant, this vector would be an eigenvector for 1, and we would have $u \in F$, the eigenspace associated with 1, which contradicts $E = F \oplus G$. Thus, by the first part of the proof, the dimension of G must be even, since otherwise, the restriction of f to G would admit 1 as an eigenvalue. Consequently, q must be odd, and R does not admit 1 as an eigenvalue. Letting $q = 2p+1$, the lemma is established. □

An example showing that Lemma 27.3 fails for n even is the following rotation matrix (when $n = 2$):

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The above matrix does not have real eigenvalues for $\theta \neq k\pi$.

It is easily shown that for $n = 2$, with respect to any chosen orthonormal basis (e_1, e_2), every rotation is represented by a matrix of form

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$