it turns out that $\mathrm{cond}_2(A) \geq 2^{n-1}$.

A classical example of matrix with a very large condition number is the *Hilbert matrix* $H^{(n)}$, the $n \times n$ matrix with

$$H_{ij}^{(n)} = \left(\frac{1}{i+j-1}\right).$$

For example, when $n = 5$,

$$H^{(5)} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} \end{pmatrix}.$$

It can be shown that

$$\mathrm{cond}_2(H^{(5)}) \approx 4.77 \times 10^5.$$

Hilbert introduced these matrices in 1894 while studying a problem in approximation theory. The Hilbert matrix $H^{(n)}$ is symmetric positive definite. A closed-form formula can be given for its determinant (it is a special form of the so-called *Cauchy determinant*); see Problem 9.15. The inverse of $H^{(n)}$ can also be computed explicitly; see Problem 9.15. It can be shown that

$$\mathrm{cond}_2(H^{(n)}) = O((1+\sqrt{2})^{4n}/\sqrt{n}).$$

Going back to our matrix

$$A = \begin{pmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{pmatrix},$$

which is a symmetric positive definite matrix, it can be shown that its eigenvalues, which in this case are also its singular values because A is SPD, are

$$\lambda_1 \approx 30.2887 > \lambda_2 \approx 3.858 > \lambda_3 \approx 0.8431 > \lambda_4 \approx 0.01015,$$

so that

$$\mathrm{cond}_2(A) = \frac{\lambda_1}{\lambda_4} \approx 2984.$$

The reader should check that for the perturbation of the right-hand side b used earlier, the relative errors $\|\Delta x\| / \|x\|$ and $\|\Delta x\| / \|x\|$ satisfy the inequality

$$\frac{\|\Delta x\|}{\|x\|} \leq \mathrm{cond}(A) \frac{\|\Delta b\|}{\|b\|}$$

and comes close to equality.