submodule of A^n. Since A^n is finitely generated, the submodule N of A^n is finitely generated, and then $N = \varphi(L)$ is finitely generated.

It remains to prove the theorem for $M = A^n$. We proceed by induction on n. For $n = 1$, a submodule N of A is an ideal, and since A is Noetherian, N is finitely generated. For the induction step where $n > 1$, consider the projection $\pi\colon A^n \to A^{n-1}$ given by

$$\pi(a_1, \ldots, a_n) = (a_1, \ldots, a_{n-1}).$$

The kernel of π is the module

$$\mathrm{Ker}\,(\pi) = \{(0, \ldots, 0, a_n) \in A^n \mid a_n \in A\} \approx A.$$

For any submodule N of A^n, let $\varphi\colon N \to A^{n-1}$ be the restriction of π to N. Since $\varphi(N)$ is a submodule of A^{n-1}, by the induction hypothesis, $\mathrm{Im}(\varphi) = \varphi(N)$ is finitely generated. Also, $\mathrm{Ker}\,(\varphi) = N \cap \mathrm{Ker}\,(\pi)$ is a submodule of $\mathrm{Ker}\,(\pi) \approx A$, and thus $\mathrm{Ker}\,(\varphi)$ is isomorphic to an ideal of A, and thus is finitely generated (since A is Noetherian). Since both $\mathrm{Im}(\varphi)$ and $\mathrm{Ker}\,(\varphi)$ are finitely generated, by Proposition 35.9, the submodule N is also finitely generated. $\square$

As a consequence of Theorem 35.10, every finitely generated A-module over a Noetherian ring A is finitely presented, because if $\varphi\colon A^n \to M$ is a surjection onto the finitely generated module M, then $\mathrm{Ker}\,(\varphi)$ is finitely generated. In particular, if A is a PID, then every finitely generated module is finitely presented.

If the ring A is not Noetherian, then there exist finitely generated A-modules that are not finitely presented. This is not so easy to prove.

We will prove in Proposition 35.35 that if A is a PID then a matrix R can "diagonalized" as

$$R = QDP^{-1}$$

where D is a diagonal matrix (more computational versions of this proposition are given in Theorem 36.18 and Theorem 36.21). It follows from Proposition 35.8 that every finitely generated module M over a PID has a presentation with m generators and r relations of the form

$$\alpha_i e_i = 0,$$

where $\alpha_i \neq 0$ and $\alpha_1 \mid \alpha_2 \mid \cdots \mid \alpha_r$, which shows that M is isomorphic to the direct sum

$$M \approx A^{m-r} \oplus A/(\alpha_1 A) \oplus \cdots \oplus A/(\alpha_r A).$$

This is a version of Theorem 35.25 that will be proved in Section 35.5.