which is left as an exercise (use Proposition 15.7 which shows that if $(\lambda_1, \dots, \lambda_n)$ are the (not necessarily distinct) eigenvalues of A, then $(\lambda_1^k, \dots, \lambda_n^k)$ are the eigenvalues of A^k, for $k \geq 1$).

Pick any complex matrix norm $\| \, \|_c$ on $\mathbb{C}^n$ (for example, the Frobenius norm, or any subordinate matrix norm induced by a norm on $\mathbb{C}^n$). The restriction of $\| \, \|_c$ to real matrices is a real norm that we also denote by $\| \, \|_c$. Now by Theorem 9.5, since $\mathrm{M}_n(\mathbb{R})$ has finite dimension n^2, there is some constant $C > 0$ so that

$$\|B\|_c \leq C \|B\|, \quad \text{for all} \quad B \in \mathrm{M}_n(\mathbb{R}).$$

Furthermore, for every $k \geq 1$ and for every real $n \times n$ matrix A, by Proposition 9.6, $\rho(A^k) \leq \left\|A^k\right\|_c$, and because $\| \, \|$ is a matrix norm, $\left\|A^k\right\| \leq \|A\|^k$, so we have

$$(\rho(A))^k = \rho(A^k) \leq \left\|A^k\right\|_c \leq C \left\|A^k\right\| \leq C \|A\|^k,$$

for all $k \geq 1$. It follows that

$$\rho(A) \leq C^{1/k} \|A\|, \quad \text{for all} \quad k \geq 1.$$

However because $C > 0$, we have $\lim_{k \mapsto \infty} C^{1/k} = 1$ (we have $\lim_{k \mapsto \infty} \frac{1}{k} \log(C) = 0$). Therefore, we conclude that

$$\rho(A) \leq \|A\|,$$

as desired. $\square$

We now determine explicitly what are the subordinate matrix norms associated with the vector norms $\| \, \|_1$, $\| \, \|_2$, and $\| \, \|_\infty$.

Proposition 9.10. *For every square matrix $A = (a_{ij}) \in \mathrm{M}_n(\mathbb{C})$, we have*

$$\begin{aligned}
\|A\|_1 &= \sup_{\substack{x \in \mathbb{C}^n \\ \|x\|_1 = 1}} \|Ax\|_1 = \max_j \sum_{i=1}^n |a_{ij}| \\
\|A\|_\infty &= \sup_{\substack{x \in \mathbb{C}^n \\ \|x\|_\infty = 1}} \|Ax\|_\infty = \max_i \sum_{j=1}^n |a_{ij}| \\
\|A\|_2 &= \sup_{\substack{x \in \mathbb{C}^n \\ \|x\|_2 = 1}} \|Ax\|_2 = \sqrt{\rho(A^*A)} = \sqrt{\rho(AA^*)}.
\end{aligned}$$

Note that $\|A\|_1$ is the maximum of the ℓ^1-norms of the columns of A and $\|A\|_\infty$ is the maximum of the ℓ^1-norms of the rows of A. Furthermore, $\|A^*\|_2 = \|A\|_2$, the norm $\| \, \|_2$ is unitarily invariant, which means that

$$\|A\|_2 = \|UAV\|_2$$

for all unitary matrices U, V, and if A is a normal matrix, then $\|A\|_2 = \rho(A)$.