and assume that the functions φ_i are differentiable at u for all $i \in I(u)$ and continuous at u for all $i \notin I(u)$. If J is differentiable at u, has a local minimum at u with respect to U, and if the constraints are qualified at u, then there exist some scalars $\lambda_i(u) \in \mathbb{R}$ for all $i \in I(u)$, such that

$$J'_u + \sum_{i \in I(u)} \lambda_i(u)(\varphi'_i)_u = 0, \quad \textit{and} \quad \lambda_i(u) \geq 0 \ \textit{for all } i \in I(u).$$

The above conditions are called the Karush–Kuhn–Tucker optimality conditions. Equivalently, in terms of gradients, the above conditions are expressed as

$$\nabla J_u + \sum_{i \in I(u)} \lambda_i(u) \nabla(\varphi_i)_u = 0, \quad \textit{and} \quad \lambda_i(u) \geq 0 \ \textit{for all } i \in I(u).$$

Proof. By Proposition 50.1(2), we have

$$J'_u(w) \geq 0 \quad \text{for all } w \in C(u), \tag{$*_1$}$$

and by Proposition 50.2(2), we have $C(u) = C^*(u)$, where

$$C^*(u) = \{v \in V \mid (\varphi'_i)_u(v) \leq 0, \ i \in I(u)\}, \tag{$*_2$}$$

so $(*_1)$ can be expressed as: for all $w \in V$,

$$\text{if } w \in C^*(u) \text{ then } J'_u(w) \geq 0,$$

or

$$\text{if } -(\varphi'_i)_u(w) \geq 0 \text{ for all } i \in I(u), \text{ then } J'_u(w) \geq 0. \tag{$*_3$}$$

Under the isomorphism $\sharp$, the vector $(J'_u)^\sharp$ is the gradient ∇J_u, so that

$$J'_u(w) = \langle w, \nabla J_u \rangle, \tag{$*_4$}$$

and the vector $((\varphi'_i)_u)^\sharp$ is the gradient $\nabla(\varphi_i)_u$, so that

$$(\varphi'_i)_u(w) = \langle w, \nabla(\varphi_i)_u \rangle. \tag{$*_5$}$$

Using Equations $(*_4)$ and $(*_5)$, Equation $(*_3)$ can be written as: for all $w \in V$,

$$\text{if } \langle w, -\nabla(\varphi_i)_u \rangle \geq 0 \text{ for all } i \in I(u), \text{ then } \langle w, \nabla J_u \rangle \geq 0. \tag{$*_6$}$$

By the Farkas–Minkowski proposition (Proposition 50.4), there exist some sacalars $\lambda_i(u)$ for all $i \in I(u)$, such that $\lambda_i(u) \geq 0$ and

$$\nabla J_u = \sum_{i \in I(u)} \lambda_i(u)(-\nabla(\varphi_i)_u),$$

that is

$$\nabla J_u + \sum_{i \in I(u)} \lambda_i(u) \nabla(\varphi_i)_u = 0,$$