We demonstrate how to calculate $\mathrm{tr}(f)$ where $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ with $f((x_1, y_1), (x_2, y_2)) = x_1x_2 + 2x_2y_1 + 3x_1y_2 - y_1y_2$. Under the standard basis for $\mathbb{R}^2$, the bilinear form $f$ is represented as

$$\begin{pmatrix} x_1 & y_1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}.$$

This matrix representation shows that

$$f^\natural = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}^\top = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix},$$

and hence

$$\mathrm{tr}(f) = \mathrm{tr}(f^\natural) = \mathrm{tr} \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} = 0.$$

We will also need the following proposition to show that various families are linearly independent.

**Proposition 33.4.** *Let $E$ and $F$ be two nontrivial vector spaces and let $(u_i)_{i\in I}$ be any family of vectors $u_i \in E$. The family $(u_i)_{i\in I}$ is linearly independent iff for every family $(v_i)_{i\in I}$ of vectors $v_i \in F$, there is some linear map $f\colon E \to F$ so that $f(u_i) = v_i$ for all $i \in I$.*

*Proof.* Left as an exercise. $\square$

## 33.2 Tensors Products

First we define tensor products, and then we prove their existence and uniqueness up to isomorphism.

**Definition 33.3.** Let $K$ be a given field, and let $E_1, \ldots, E_n$ be $n \geq 2$ given vector spaces. For any vector space $F$, a map $f\colon E_1 \times \cdots \times E_n \to F$ is *multilinear* iff it is linear in each of its argument; that is,

$$\begin{aligned} f(u_1, \ldots u_{i_1}, v + w, u_{i+1}, \ldots, u_n) &= f(u_1, \ldots u_{i_1}, v, u_{i+1}, \ldots, u_n) \\ &\quad + f(u_1, \ldots u_{i_1}, w, u_{i+1}, \ldots, u_n) \\ f(u_1, \ldots u_{i_1}, \lambda v, u_{i+1}, \ldots, u_n) &= \lambda f(u_1, \ldots u_{i_1}, v, u_{i+1}, \ldots, u_n), \end{aligned}$$

for all $u_j \in E_j$ $(j \neq i)$, all $v, w \in E_i$ and all $\lambda \in K$, for $i = 1 \ldots, n$.

The set of multilinear maps as above forms a vector space denoted $\mathrm{L}(E_1, \ldots, E_n; F)$ or $\mathrm{Hom}(E_1, \ldots, E_n; F)$. When $n = 1$, we have the vector space of linear maps $\mathrm{L}(E, F)$ (also denoted $\mathrm{Hom}(E, F)$). (To be very precise, we write $\mathrm{Hom}_K(E_1, \ldots, E_n; F)$ and $\mathrm{Hom}_K(E, F)$.)