9.3 Subordinate Norms

We now give another method for obtaining matrix norms using subordinate norms. First we need a proposition that shows that in a finite-dimensional space, the linear map induced by a matrix is bounded, and thus continuous.

Proposition 9.8. *For every norm* $\| \|$ *on* $\mathbb{C}^n$ *(or* $\mathbb{R}^n$*), for every matrix* $A \in \mathrm{M}_n(\mathbb{C})$ *(or* $A \in \mathrm{M}_n(\mathbb{R})$*), there is a real constant* $C_A \geq 0$*, such that*

$$\|Au\| \leq C_A \|u\|,$$

for every vector $u \in \mathbb{C}^n$ *(or* $u \in \mathbb{R}^n$ *if* A *is real).*

Proof. For every basis $(e_1, \ldots, e_n)$ of $\mathbb{C}^n$ (or $\mathbb{R}^n$), for every vector $u = u_1e_1 + \cdots + u_ne_n$, we have

$$\begin{aligned} \|Au\| &= \|u_1A(e_1) + \cdots + u_nA(e_n)\| \\ &\leq |u_1| \|A(e_1)\| + \cdots + |u_n| \|A(e_n)\| \\ &\leq C_1(|u_1| + \cdots + |u_n|) = C_1 \|u\|_1, \end{aligned}$$

where $C_1 = \max_{1 \leq i \leq n} \|A(e_i)\|$. By Theorem 9.5, the norms $\| \|$ and $\| \|_1$ are equivalent, so there is some constant $C_2 > 0$ so that $\|u\|_1 \leq C_2 \|u\|$ for all u, which implies that

$$\|Au\| \leq C_A \|u\|,$$

where $C_A = C_1C_2$. ☐

Proposition 9.8 says that every linear map on a finite-dimensional space is *bounded*. This implies that every linear map on a finite-dimensional space is continuous. Actually, it is not hard to show that a linear map on a normed vector space E is bounded iff it is continuous, regardless of the dimension of E.

Proposition 9.8 implies that for every matrix $A \in \mathrm{M}_n(\mathbb{C})$ (or $A \in \mathrm{M}_n(\mathbb{R})$),

$$\sup_{\substack{x \in \mathbb{C}^n \\ x \neq 0}} \frac{\|Ax\|}{\|x\|} \leq C_A.$$

Since $\|\lambda u\| = |\lambda| \|u\|$, for every nonzero vector x, we have

$$\frac{\|Ax\|}{\|x\|} = \frac{\|x\| \|A(x/\|x\|)\|}{\|x\|} = \|A(x/\|x\|)\|,$$

which implies that

$$\sup_{\substack{x \in \mathbb{C}^n \\ x \neq 0}} \frac{\|Ax\|}{\|x\|} = \sup_{\substack{x \in \mathbb{C}^n \\ \|x\| = 1}} \|Ax\|.$$