Definition 6.7. Let E and F be two vector spaces and assume that they are expressed as direct sums

$$E = \bigoplus_{j=1}^{n} E_j, \quad F = \bigoplus_{i=1}^{m} F_i.$$

Given any linear map $f\colon E \to F$, if $(f_{ij})_{1\leq i\leq m, 1\leq j\leq n}$ is the familiy of linear maps $f_{ij}\colon E_j \to F_i$ defined in Definition 6.6, the $m \times n$ matrix of linear maps

$$M(f) = \begin{pmatrix} f_{1\,1} & \dots & f_{1\,n} \\ \vdots & \ddots & \vdots \\ f_{m\,1} & \dots & f_{m\,n} \end{pmatrix}$$

is called the *matrix of f with respect to the decompositions $\bigoplus_{j=1}^{n} E_j$, and $\bigoplus_{i=1}^{m} F_i$ of E and F as direct sums.*

For any $x = x_1 + \cdots + x_n \in E$ with $x_j \in E_j$ and any $y = y_1 + \cdots + y_m \in F$ with $y_i \in F_i$, we have $y = f(x)$ iff

$$\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} f_{1\,1} & \dots & f_{1\,n} \\ \vdots & \ddots & \vdots \\ f_{m\,1} & \dots & f_{m\,n} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

where the matrix equation above means that the system of m equations

$$y_i = \sum_{j=1}^{n} f_{ij}(x_j), \quad i = 1\dots, m, \tag{†}$$

holds.

But now we can also promote matrix multiplication. Suppose we have a third space G written as a direct sum. It is more convenient to write

$$E = \bigoplus_{k=1}^{p} E_k, \quad F = \bigoplus_{j=1}^{n} F_j, \quad G = \bigoplus_{i=1}^{m} G_i.$$

Assume we also have two linear maps $f\colon E \to F$ and $g\colon F \to G$. Now we have the $n \times p$ matrix of linear maps $B = (f_{jk})$ and the $m \times n$ matrix of linear maps $A = (g_{ij})$. We would like to find the $m \times p$ matrix associated with $g \circ f$.

By definition of $f_k\colon E_k \to F$ and $f_{jk}\colon E_k \to F_j$, if $x_k \in E_k$, then

$$f_k(x_k) = f(x_k) = \sum_{j=1}^{n} f_{jk}(x_k), \quad \text{with } f_{jk}(x_k) \in F_j, \tag{$*_1$}$$

and similarly, by definition of $g_j\colon F_j \to G$ and $g_{ij}\colon F_j \to G_i$, if $y_j \in F_j$, then

$$g_j(y_j) = g(y_j) = \sum_{i=1}^{m} g_{ij}(y_j), \quad \text{with } g_{ij}(y_j) \in G_i. \tag{$*_2$}$$