The derivative $\mathrm{D}_Y X(a)$ is thus the derivative of the vector field X along the curve γ, and it is called the *covariant derivative of X along γ*.

Given an affine frame $(O, (u_1, \ldots, u_n))$ for $(E, \overrightarrow{E})$, it is easily seen that the covariant derivative $\mathrm{D}_Y X(a)$ is expressed as follows:

$$\mathrm{D}_Y X(a) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(Y_j \frac{\partial X_i}{\partial x_j} \right)(a) e_i.$$

Generally, $\mathrm{D}_Y X(a) \neq \mathrm{D}_X Y(a)$. The quantity

$$[X, Y] = \mathrm{D}_X Y - \mathrm{D}_Y X$$

is called the *Lie bracket* of the vector fields X and Y. The Lie bracket plays an important role in differential geometry. In terms of coordinates,

$$[X, Y] = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(X_j \frac{\partial Y_i}{\partial x_j} - Y_j \frac{\partial X_i}{\partial x_j} \right) e_i.$$

39.9 Futher Readings

A thorough treatment of differential calculus can be found in Munkres [130], Lang [112], Schwartz [151], Cartan [34], and Avez [9]. The techniques of differential calculus have many applications, especially to the geometry of curves and surfaces and to differential geometry in general. For this, we recommend do Carmo [52, 53] (two beautiful classics on the subject), Kreyszig [106], Stoker [166], Gray [81], Berger and Gostiaux [13], Milnor [126], Lang [110], Warner [186] and Choquet-Bruhat [38].

39.10 Summary

The main concepts and results of this chapter are listed below:

- *Directional derivative* ($\mathrm{D}_u f(a)$).
- *Total derivative*, *Fréchet derivative*, *derivative*, *total differential*, *differential* ($df(a), df_a$).
- *Partial derivatives*.
- *Affine* functions.
- The *chain rule*.
- *Jacobian matrices* ($J(f)(a)$), *Jacobians*.