**Proposition 18.7.** *Let $A$ be an $m \times m$ diagonalizable (complex or real) matrix with eigenvalues $\lambda_1, \ldots, \lambda_m$, and let $\lambda = \lambda_\ell$ be an arbitrary eigenvalue of $A$ (not necessary simple). For any $\mu$ such that*

$$\mu \neq \lambda \quad \text{and} \quad |\mu - \lambda| < |\mu - \lambda_j| \quad \text{for all } j \neq \ell,$$

*if $x^0$ does not belong to the subspace spanned by the eigenvectors associated with the eigenvalues $\lambda_j$ with $j \neq \ell$, then*

$$\lim_{k \mapsto \infty} \left( \frac{(\lambda - \mu)^k}{|\lambda - \mu|^k} \right) x^k = v,$$

*where $v$ is an eigenvector associated with $\lambda$. Furthermore, if both $\lambda$ and $\mu$ are real, we have*

$$\begin{aligned} \lim_{k \mapsto \infty} x^k &= v & \text{if } \mu < \lambda, \\ \lim_{k \mapsto \infty} (-1)^k x^k &= v & \text{if } \mu > \lambda. \end{aligned}$$

*Also, if we define the sequence $(\lambda^{(k)})$ by*

$$\lambda^{(k+1)} = (x^{k+1})^* A x^{k+1},$$

*then*

$$\lim_{k \mapsto \infty} \lambda^{(k+1)} = \lambda.$$

The condition of $x^0$ may seem quite stringent, but in practice, a vector $x^0$ chosen at random usually satisfies it.

If $A$ is a Hermitian matrix, then we can say more. In particular, the inverse iteration algorithm can be modified to make use of the newly computed $\lambda^{(k+1)}$ instead of $\mu$, and an even faster convergence is achieved. Such a method is called the *Rayleigh quotient iteration*. When it converges (which is for almost all $x^0$), this method eventually achieves cubic convergence, which is remarkable. Essentially, this means that the number of correct digits is tripled at every iteration. For more details, see Trefethen and Bau [176] (Lecture 27) and Demmel [48] (Section 5.3.2).

## 18.8 Summary

The main concepts and results of this chapter are listed below:

- $QR$ iteration, $QR$ algorithm.
- Upper Hessenberg matrices.
- Householder matrix.