we have

$$b_{11} = \det(A_{11}) = \begin{vmatrix} -2 & -2 \\ 3 & -3 \end{vmatrix} = 12 \qquad b_{12} = -\det(A_{21}) = -\begin{vmatrix} 1 & 1 \\ 3 & -3 \end{vmatrix} = 6$$
$$b_{13} = \det(A_{31}) = \begin{vmatrix} 1 & 1 \\ -2 & -2 \end{vmatrix} = 0 \qquad b_{21} = -\det(A_{12}) = -\begin{vmatrix} 2 & -2 \\ 3 & -3 \end{vmatrix} = 0$$
$$b_{22} = \det(A_{22}) = \begin{vmatrix} 1 & 1 \\ 3 & -3 \end{vmatrix} = -6 \qquad b_{23} = -\det(A_{32}) = -\begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix} = 4$$
$$b_{31} = \det(A_{13}) = \begin{vmatrix} 2 & -2 \\ 3 & 3 \end{vmatrix} = 12 \qquad b_{32} = -\det(A_{23}) = -\begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix} = 0$$
$$b_{33} = \det(A_{33}) = \begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix} = -4,$$

we find that

$$\widetilde{A} = \begin{pmatrix} 12 & 6 & 0 \\ 0 & -6 & 4 \\ 12 & 0 & -4 \end{pmatrix}.$$

Note the reversal of the indices in

$$b_{i\,j} = (-1)^{i+j} \det(A_{j\,i}).$$

Thus, $\widetilde{A}$ is the *transpose* of the matrix of cofactors of elements of A.

We have the following proposition.

Proposition 7.10. *Let K be a commutative ring. For every matrix $A \in \mathrm{M}_n(K)$, we have*

$$A\widetilde{A} = \widetilde{A}A = \det(A)I_n.$$

As a consequence, A is invertible iff $\det(A)$ is invertible, and if so, $A^{-1} = (\det(A))^{-1}\widetilde{A}$.

Proof. If $\widetilde{A} = (b_{i\,j})$ and $A\widetilde{A} = (c_{i\,j})$, we know that the entry $c_{i\,j}$ in row i and column j of $A\widetilde{A}$ is

$$c_{i\,j} = a_{i\,1}b_{1\,j} + \cdots + a_{i\,k}b_{k\,j} + \cdots + a_{i\,n}b_{n\,j},$$

which is equal to

$$a_{i\,1}(-1)^{j+1}\det(A_{j\,1}) + \cdots + a_{i\,n}(-1)^{j+n}\det(A_{j\,n}).$$

If $j = i$, then we recognize the expression of the expansion of $\det(A)$ according to the i-th row:

$$c_{i\,i} = \det(A) = a_{i\,1}(-1)^{i+1}\det(A_{i\,1}) + \cdots + a_{i\,n}(-1)^{i+n}\det(A_{i\,n}).$$

If $j \neq i$, we can form the matrix A' by replacing the j-th row of A by the i-th row of A. Now the matrix $A_{j\,k}$ obtained by deleting row j and column k from A is equal to the matrix