there exist an integer $k(n)$ such that

$$\left\|u^n_{k(n)} - u\right\| \leq \epsilon_n, \quad \left\|\delta^n_{k(n)}\right\| \leq \epsilon_n.$$

Consider the sequence $(u^n_{k(n)})_{n\geq 0}$. We have

$$u^n_{k(n)} \in U, \;\; u^n_{k(n)} \neq 0, \;\; \text{for all } n \geq 0, \;\; \lim_{n \mapsto \infty} u^n_{k(n)} = u,$$

and we can write

$$u^n_{k(n)} = u + \left\|u^n_{k(n)} - u\right\| \frac{w}{\|w\|} + \left\|u^n_{k(n)} - u\right\| \left(\delta^n_{k(n)} + \left(\frac{w_n}{\|w_n\|} - \frac{w}{\|w\|}\right)\right).$$

Since $\lim_{k \mapsto \infty}(w_n/\left\|w_n\right\|) = w/\left\|w\right\|$, we conclude that $w \in C(u)$. See Figure 50.5.

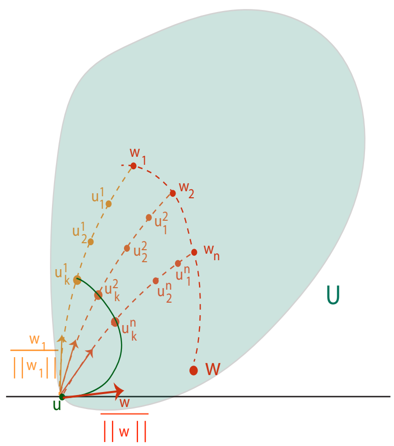


Figure 50.5: Let $U$ be the mint green region in $\mathbb{R}^2$ with $u = (0, 0)$. Let $(w_n)_{n\geq 0}$ be a sequence of vectors (points) along the upper dashed curve which converge to $w$. By following the dashed orange longitudinal curves, and selecting an appropriate vector(point), we construct the dark green curve in $U$, which passes through $u$, and at $u$ has tangent vector proportional to $w$.

(2) Let $w = v - u$ be any nonzero vector in the cone $C(u)$, and let $(u_k)_{k\;\geq 0}$ be a sequence of vectors in $U - \{u\}$ such that

(1) $\lim_{k \mapsto \infty} u_k = u$.

(2) There is a sequence $(\delta_k)_{k\geq 0}$ of vectors $\delta_k \in V$ such that

$$u_k - u = \|u_k - u\| \frac{w}{\|w\|} + \|u_k - u\| \, \delta_k, \quad \lim_{k \mapsto \infty} \delta_k = 0, \;\; w \neq 0,$$

(3) $J(u) \leq J(u_k)$ for all $k \geq 0$.