obtained by taking the quotient of the (upper) half-sphere S^n_+, by the equivalence relation identifying antipodal points a_+ and a_- on the boundary of the half-sphere. Another interesting fact is that the complex projective line $\mathbb{CP}^1 = \mathbf{P}(\mathbb{C}^2)$ is homeomorphic to the (real) 2-sphere S^2, and that the real projective space $\mathbb{RP}^3$ is homeomorphic to the group of rotations $\mathbf{SO}(3)$ of $\mathbb{R}^3$.

(2) If H is a hyperplane in E, recall from Proposition 11.7 that there is some nonnull linear form $f \in E^*$ such that $H = \mathrm{Ker}\, f$. Also, given any nonnull linear form $f \in E^*$, its kernel $H = \mathrm{Ker}\, f = f^{-1}(0)$ is a hyperplane, and if $\mathrm{Ker}\, f = \mathrm{Ker}\, g = H$, then $g = \lambda f$ for some $\lambda \neq 0$. These facts can be concisely stated by saying that the map

$$[f]_\sim \mapsto \mathrm{Ker}\, f$$

mapping the equivalence class $[f]_\sim = \{\lambda f \mid \lambda \neq 0\}$ of a nonnull linear form $f \in E^*$ to the hyperplane $H = \mathrm{Ker}\, f$ in E is a bijection between the projective space $\mathbf{P}(E^*)$ and the set of hyperplanes in E. When E is of finite dimension, this bijection yields a useful duality, which will be investigated in Section 26.12.

We now define projective subspaces.

26.3 Projective Subspaces

Projective subspaces of a projective space $\mathbf{P}(E)$ are induced by subspaces of the vector space E.

Definition 26.2. Given a nontrivial vector space E, a *projective subspace (or linear projective variety) of* $\mathbf{P}(E)$ is any subset W of $\mathbf{P}(E)$ such that there is some subspace $V \neq \{0\}$ of E with $W = p(V - \{0\})$. The dimension $\dim(W)$ of W is defined as follows: If V is of infinite dimension, then $\dim(W) = \dim(V)$, and if $\dim(V) = p \geq 1$, then $\dim(W) = p - 1$. We say that a family $(a_i)_{i\in I}$ of points of $\mathbf{P}(E)$ is *projectively independent* if there is a linearly independent family $(u_i)_{i\in I}$ in E such that $a_i = p(u_i)$ for every $i \in I$.

Remark: If we allow the empty subset to be a projective subspace, then if assign the empty subset to the trivial subspace $\{0\}$, we obtain a bijection between the subspaces of E and the projective subspaces of $\mathbf{P}(E)$. If $\mathbf{P}(V)$ is the projective space induced by the vector space V, we also denote $p(V - \{0\})$ by $\mathbf{P}(V)$, or even by $p(V)$, even though $p(0)$ is undefined.

A projective subspace of dimension 0 is a called a *(projective) point*. A projective subspace of dimension 1 is called a *(projective) line*, and a projective subspace of dimension 2 is called a *(projective) plane*. If H is a hyperplane in E, then $\mathbf{P}(H)$ is called a *projective hyperplane*. It is easily verified that any arbitrary intersection of projective subspaces is a projective subspace.