Proposition 33.20. *Given a linear map $h\colon V_1 \to V_2$ between two vectors spaces V_1, V_2 over a field K, there is a unique K-algebra homomorphism $\otimes h\colon T(V_1) \to T(V_2)$ making the following diagram commute.*

$$\begin{tikzcd} V_1 \arrow[r, "i_1"] \arrow[d, "h"'] & T(V_1) \arrow[d, "\otimes h"] \\ V_2 \arrow[r, "i_2"] & T(V_2). \end{tikzcd}$$

Most algebras of interest arise as well-chosen quotients of the tensor algebra $T(V)$. This is true for the *exterior algebra* $\bigwedge(V)$ (also called *Grassmann algebra*), where we take the quotient of $T(V)$ modulo the ideal generated by all elements of the form $v \otimes v$, where $v \in V$,and for the *symmetric algebra* $\mathrm{Sym}(V)$, where we take the quotient of $T(V)$ modulo the ideal generated by all elements of the form $v \otimes w - w \otimes v$, where $v, w \in V$.

Algebras such as $T(V)$ are graded in the sense that there is a sequence of subspaces $V^{\otimes n} \subseteq T(V)$ such that

$$T(V) = \bigoplus_{k \geq 0} V^{\otimes n},$$

and the multiplication $\otimes$ behaves well w.r.t. the grading, *i.e.*, $\otimes\colon V^{\otimes m} \times V^{\otimes n} \to V^{\otimes(m+n)}$.

Definition 33.12. A K-algebra E is said to be a *graded algebra* iff there is a sequence of subspaces $E^n \subseteq E$ such that

$$E = \bigoplus_{k \geq 0} E^n,$$

(with $E^0 = K$) and the multiplication $\cdot$ respects the grading; that is, $\cdot\colon E^m \times E^n \to E^{m+n}$. Elements in E^n are called *homogeneous elements of rank (or degree) n.*

In differential geometry and in physics it is necessary to consider slightly more general tensors.

Definition 33.13. Given a vector space V, for any pair of nonnegative integers (r, s), the *tensor space $T^{r,s}(V)$ of type (r, s)* is the tensor product

$$T^{r,s}(V) = V^{\otimes r} \otimes (V^*)^{\otimes s} = \underbrace{V \otimes \cdots \otimes V}_{r} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{s},$$

with $T^{0,0}(V) = K$. We also define the *tensor algebra $T^{\bullet,\bullet}(V)$* as the direct sum (coproduct)

$$T^{\bullet,\bullet}(V) = \bigoplus_{r,s \geq 0} T^{r,s}(V).$$

Tensors in $T^{r,s}(V)$ are called *homogeneous of degree (r, s)*.