and so the above inequality is equivalent to

$$\varphi(u,\, v) \leq \sqrt{\Phi(u)\Phi(v)},$$

which is trivial when $\varphi(u,\, v) \leq 0$, and follows from the Cauchy–Schwarz inequality when $\varphi(u,\, v) \geq 0$. Thus, the Minkowski inequality holds. Finally assume that $u \neq 0$ and $v \neq 0$, and that

$$\sqrt{\Phi(u+v)} = \sqrt{\Phi(u)} + \sqrt{\Phi(v)}.$$

When this is the case, we have

$$\varphi(u,\, v) = \sqrt{\Phi(u)\Phi(v)},$$

and we know from the discussion of the Cauchy–Schwarz inequality that the equality holds iff $u$ and $v$ are linearly dependent. The Minkowski inequality is an equality when $u$ or $v$ is null. Otherwise, if $u \neq 0$ and $v \neq 0$, then $u = \lambda v$ for some $\lambda \neq 0$, and since

$$\varphi(u,\, v) = \lambda\varphi(v,\, v) = \sqrt{\Phi(u)\Phi(v)},$$

by positivity, we must have $\lambda > 0$. $\square$

Note that the Cauchy–Schwarz inequality can also be written as

$$|\varphi(u, v)| \leq \sqrt{\Phi(u)}\sqrt{\Phi(v)}.$$

**Remark:** It is easy to prove that the Cauchy–Schwarz and the Minkowski inequalities still hold for a symmetric bilinear form that is positive, but not necessarily definite (i.e., $\varphi(u, v) \geq 0$ for all $u, v \in E$). However, $u$ and $v$ need not be linearly dependent when the equality holds.

The Minkowski inequality

$$\sqrt{\Phi(u+v)} \leq \sqrt{\Phi(u)} + \sqrt{\Phi(v)}$$

shows that the map $u \mapsto \sqrt{\Phi(u)}$ satisfies the convexity inequality (also known as triangle inequality), condition (N3) of Definition 9.1, and since $\varphi$ is bilinear and positive definite, it also satisfies conditions (N1) and (N2) of Definition 9.1, and thus it is a *norm* on $E$. The norm induced by $\varphi$ is called the *Euclidean norm induced by* $\varphi$.

The Cauchy–Schwarz inequality can be written as

$$|u \cdot v| \leq \|u\|\|v\|,$$

and the Minkowski inequality as

$$\|u + v\| \leq \|u\| + \|v\|.$$