Otherwise, the columns of A_K are linearly dependent, so there is some nonzero vector $v = (v_1, \ldots, v_s)$ such that $A_K v = 0$. Let $w \in \mathbb{R}^n$ be the vector obtained by extending v by setting $w_j = 0$ for all $j \notin K$. By construction,

$$Aw = A_K v = 0.$$

We will derive a contradiction by exhibiting a feasible solution $x(t_0)$ such that $cx(t_0) \geq cx_0$ with more zero coordinates than $\widetilde{x}$.

For this we claim that we may assume that w satisfies the following two conditions:

(1) $cw \geq 0$.

(2) There is some $j \in K$ such that $w_j < 0$.

If $cw = 0$ and if Condition (2) fails, since $w \neq 0$, we have $w_j > 0$ for some $j \in K$, in which case we can use $-w$, for which $w_j < 0$.

If $cw < 0$, then $c(-w) > 0$, so we may assume that $cw > 0$. If $w_j > 0$ for all $j \in K$, since $\widetilde{x}$ is feasible, $\widetilde{x} \geq 0$, and so $x(t) = \widetilde{x} + tw \geq 0$ for all $t \geq 0$. Furthermore, since $Aw = 0$ and $\widetilde{x}$ is feasible, we have

$$Ax(t) = A\widetilde{x} + tAw = b,$$

and thus $x(t)$ is feasible for all $t \geq 0$. We also have

$$cx(t) = c\widetilde{x} + tcw.$$

Since $cw > 0$, as $t > 0$ goes to infinity the objective function $cx(t)$ also tends to infinity, contradicting the fact that is is bounded above. Therefore, some w satisfying Conditions (1) and (2) above must exist.

We show that there is some $t_0 > 0$ such that $cx(t_0) \geq cx_0$ and $x(t_0) = \widetilde{x} + t_0 w$ is feasible, yet $x(t_0)$ has more zero coordinates than $\widetilde{x}$, a contradiction.

Since $x(t) = \widetilde{x} + tw$, we have

$$x(t)_i = \widetilde{x}_i + tw_i,$$

so if we let $I = \{i \in \{1, \ldots, n\} \mid w_i < 0\} \subseteq K$, which is nonempty since w satisfies Condition (2) above, if we pick

$$t_0 = \min_{i \in I} \left\{ \frac{-\widetilde{x}_i}{w_i} \right\},$$

then $t_0 > 0$, because $w_i < 0$ for all $i \in I$, and by definition of K we have $\widetilde{x}_i > 0$ for all $i \in K$. By the definition of $t_0 > 0$ and since $\widetilde{x} \geq 0$, we have

$$x(t_0)_j = \widetilde{x}_j + t_0 w_j \geq 0 \quad \text{for all } j \in K,$$

so $x(t_0) \geq 0$, and $x(t_0)_i = 0$ for some $i \in I$. Since $Ax(t_0) = b$ (for any t), $x(t_0)$ is a feasible solution,

$$cx(t_0) = c\widetilde{x} + t_0 cw \geq cx_0 + t_0 cw \geq cx_0,$$

and $x(t_0)_i = 0$ for some $i \in I$, we see that $x(t_0)$ has more zero coordinates than $\widetilde{x}$, a contradiction. $\square$