Theorem 23.2. *The least squares solution of smallest norm of the linear system $Ax = b$, where A is an $m \times n$ matrix, is given by*

$$x^+ = A^+ b = U D^+ V^\top b.$$

Proof. First assume that A is a (rectangular) diagonal matrix D, as above. Then since x minimizes $\|Dx - b\|_2^2$ iff Dx is the projection of b onto the image subspace F of D, it is fairly obvious that $x^+ = D^+ b$. Otherwise, we can write

$$A = V D U^\top,$$

where U and V are orthogonal. However, since V is an isometry,

$$\|Ax - b\|_2 = \|V D U^\top x - b\|_2 = \|D U^\top x - V^\top b\|_2.$$

Letting $y = U^\top x$, we have $\|x\|_2 = \|y\|_2$, since U is an isometry, and since U is surjective, $\|Ax - b\|_2$ is minimized iff $\|Dy - V^\top b\|_2$ is minimized, and we have shown that the least solution is

$$y^+ = D^+ V^\top b.$$

Since $y = U^\top x$, with $\|x\|_2 = \|y\|_2$, we get

$$x^+ = U D^+ V^\top b = A^+ b.$$

Thus, the pseudo-inverse provides the optimal solution to the least squares problem. $\square$

By Theorem 23.2 and Theorem 23.1, $A^+ b$ is uniquely defined by every b, and thus A^+ depends only on A.

The `Matlab` command for computing the pseudo-inverse B of the matrix A is
`B = pinv(A)`.

Example 23.2. If A is the rank 2 matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix}$$

whose eigenvalues are $-1.1652, 0, 0, 17.1652$, using `Matlab` we obtain the SVD $A = V D U^\top$ with

$$U = \begin{pmatrix} -0.3147 & 0.7752 & 0.2630 & -0.4805 \\ -0.4275 & 0.3424 & 0.0075 & 0.8366 \\ -0.5402 & -0.0903 & -0.8039 & -0.2319 \\ -0.6530 & -0.5231 & 0.5334 & -0.1243 \end{pmatrix},$$

$$V = \begin{pmatrix} -0.3147 & -0.7752 & 0.5452 & 0.0520 \\ -0.4275 & -0.3424 & -0.7658 & 0.3371 \\ -0.5402 & 0.0903 & -0.1042 & -0.8301 \\ -0.6530 & 0.5231 & 0.3247 & 0.4411 \end{pmatrix}, \quad D = \begin{pmatrix} 17.1652 & 0 & 0 & 0 \\ 0 & 1.1652 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$