where $\nu_i = 1$ or $\nu_i = 0$; see Theorem 31.16. As a corollary we obtain the *Jordan form*; which involves matrices of the form

$$J_r(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix},$$

called *Jordan blocks*; see Theorem 31.17.

31.1 Annihilating Polynomials and the Minimal Polynomial

Given a linear map $f\colon E \to E$, it is easy to check that the set $\mathrm{Ann}(f)$ of polynomials that annihilate f is an ideal. Furthermore, when E is finite-dimensional, the Cayley–Hamilton Theorem implies that $\mathrm{Ann}(f)$ is not the zero ideal. Therefore, by Proposition 30.10, there is a unique monic polynomial m_f that generates $\mathrm{Ann}(f)$. Results from Chapter 30, especially about gcd's of polynomials, will come handy.

Definition 31.1. If $f\colon E \to E$ is a linear map on a finite-dimensional vector space E, the unique monic polynomial $m_f(X)$ that generates the ideal $\mathrm{Ann}(f)$ of polynomials which annihilate f (the *annihilator* of f) is called the *minimal polynomial* of f.

The minimal polynomial m_f of f is the monic polynomial of smallest degree that annihilates f. Thus, the minimal polynomial divides the characteristic polynomial χ_f, and $\deg(m_f) \geq 1$. For simplicity of notation, we often write m instead of m_f.

If A is any $n \times n$ matrix, the set $\mathrm{Ann}(A)$ of polynomials that annihilate A is the set of polynomials

$$p(X) = a_0 X^d + a_1 X^{d-1} + \cdots + a_{d-1} X + a_d$$

such that

$$a_0 A^d + a_1 A^{d-1} + \cdots + a_{d-1} A + a_d I = 0.$$

It is clear that $\mathrm{Ann}(A)$ is a nonzero ideal and its unique monic generator is called the *minimal polynomial* of A. We check immediately that if Q is an invertible matrix, then A and $Q^{-1}AQ$ have the same minimal polynomial. Also, if A is the matrix of f with respect to some basis, then f and A have the same minimal polynomial.

The zeros (in K) of the minimal polynomial of f and the eigenvalues of f (in K) are intimately related.

Proposition 31.1. *Let $f\colon E \to E$ be a linear map on some finite-dimensional vector space E. Then $\lambda \in K$ is a zero of the minimal polynomial $m_f(X)$ of f iff λ is an eigenvalue of f*