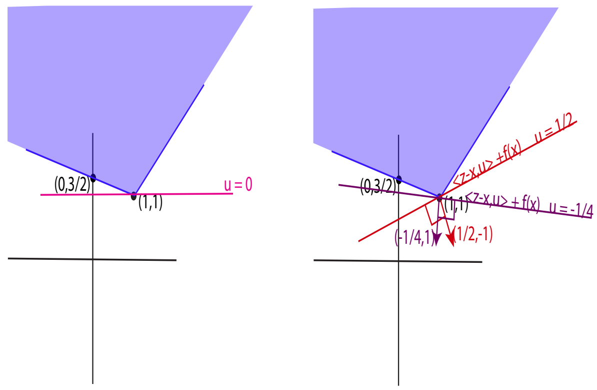

Figure 51.13: Let $f\colon \mathbb{R} \to \mathbb{R} \cup \{-\infty, +\infty\}$ be the piecewise function defined by $f(x) = x+1$ for $x \geq 1$ and $f(x) = -\frac{1}{2}x + \frac{3}{2}$ for $x < 1$. Its epigraph is the shaded blue region in $\mathbb{R}^2$. Since f has minimum at $x = 1$, $0 \in \partial f(1)$, and the graph of $f(x)$ has a horizontal supporting hyperplane at $(1, 1)$. Since $\{\frac{1}{2}, -\frac{1}{4}\} \subset \partial f(1)$, the maroon line $\frac{1}{2}(x-1)+1$ (with normal $(\frac{1}{2}, -1)$) and the violet line $-\frac{1}{4}(x-1)+1$ (with normal $(-\frac{1}{4}, -1)$) are supporting hyperplanes to the graph of $f(x)$ at $(1, 1)$.

See Figure 51.16. We leave it as an exercise to show that f is subdifferentiable (in fact differentiable) at x when $|x| < 1$, but $\partial f(x) = \emptyset$ when $|x| \geq 1$, even though $x \in \mathrm{dom}(f)$ for $|x| = 1$.

Example 51.8. The subdifferential of an indicator function is interesting. Let C be a nonempty convex set. By definition, $u \in \partial I_C(x)$ iff

$$I_C(z) \geq I_C(x) + \langle z - x, u\rangle \quad \text{for all } z \in \mathbb{R}^n.$$

Since C is nonempty, there is some $z \in C$ such that $I_C(z) = 0$, so the above condition implies that $x \in C$ (otherwise $I_C(x) = +\infty$ but $0 \geq +\infty + \langle z - u, u\rangle$ is impossible), so $0 \geq \langle z - x, u\rangle$ for all $z \in C$, which means that z is normal to C at x. Therefore, $\partial I_C(x)$ *is the normal cone* $N_C(x)$ *to* C *at* x.

Example 51.9. The subdifferentials of the indicator function f of the nonnegative orthant of $\mathbb{R}^n$ reveal a connection to complementary slackness conditions. Recall that this indicator function is given by

$$f(x_1, \ldots, x_n) = \begin{cases} 0 & \text{if } x_i \geq 0,\ 1 \leq i \leq n, \\ +\infty & \text{otherwise.} \end{cases}$$

By Example 51.8, the subgradients y of f at $x \geq 0$ form the normal cone to the nonnegative orthant at x. This means that $y \in N_C(x)$ iff

$$\langle z - x, y\rangle \leq 0 \quad \text{for all } z \geq 0$$