and it is unique, since $\overrightarrow{f} - \mathrm{id}$ is injective. Conversely, if f has a unique fixed point, say a, from

$$\big(\overrightarrow{f} - \mathrm{id}\big)(\overrightarrow{\Omega a}) = -\overrightarrow{\Omega f(\Omega)},$$

we have $\big(\overrightarrow{f} - \mathrm{id}\big)(\overrightarrow{\Omega a}) = 0$ iff $f(\Omega) = \Omega$, and since a is the unique fixed point of f, we must have $a = \Omega$, which shows that $\overrightarrow{f} - \mathrm{id}$ is injective. $\square$

Remark: The fact that E has finite dimension is used only to prove (2), and (1) holds in general.

If an affine isometry f leaves some point fixed, we can take such a point Ω as the origin, and then $f(\Omega) = \Omega$ and we can view f as a rotation or an improper orthogonal transformation, depending on the nature of $\overrightarrow{f}$. Note that it is quite possible that $\mathrm{Fix}(f) = \emptyset$. For example, nontrivial translations have no fixed points. A more interesting example is provided by the composition of a plane reflection about a line composed with a a nontrivial translation parallel to this line.

Otherwise, we will see in Theorem 27.10 that every affine isometry is the (commutative) composition of a translation with an affine isometry that always has a fixed point.

27.4 Affine Isometries and Fixed Points

Let E be an affine space. Given any two affine subspaces F, G, if F and G are orthogonal complements in E, which means that $\overrightarrow{F}$ and $\overrightarrow{G}$ are orthogonal subspaces of $\overrightarrow{E}$ such that $\overrightarrow{E} = \overrightarrow{F} \oplus \overrightarrow{G}$, for any point $\Omega \in F$, we define $q\colon E \to \overrightarrow{G}$ such that

$$q(a) = p_{\overrightarrow{G}}(\overrightarrow{\Omega a}).$$

Note that $q(a)$ is independent of the choice of $\Omega \in F$, since we have

$$\overrightarrow{\Omega a} = p_{\overrightarrow{F}}(\overrightarrow{\Omega a}) + p_{\overrightarrow{G}}(\overrightarrow{\Omega a}),$$

and for any $\Omega_1 \in F$, we have

$$\overrightarrow{\Omega_1 a} = \overrightarrow{\Omega_1 \Omega} + p_{\overrightarrow{F}}(\overrightarrow{\Omega a}) + p_{\overrightarrow{G}}(\overrightarrow{\Omega a}),$$

and since $\overrightarrow{\Omega_1 \Omega} \in \overrightarrow{F}$, this shows that

$$p_{\overrightarrow{G}}(\overrightarrow{\Omega_1 a}) = p_{\overrightarrow{G}}(\overrightarrow{\Omega a}).$$

Then the map $g\colon E \to E$ such that $g(a) = a - 2q(a)$, or equivalently

$$\overrightarrow{ag(a)} = -2q(a) = -2p_{\overrightarrow{G}}(\overrightarrow{\Omega a}),$$