where A is a symmetric $n \times n$ matrix and x, b, are vectors in $\mathbb{R}^n$, viewed as column vectors. Actually, for reasons that will be clear shortly, it is preferable to put a factor $\frac{1}{2}$ in front of the quadratic term, so that

$$Q(x) = \frac{1}{2}x^\top Ax - x^\top b.$$

The question is, under what conditions (on A) does $Q(x)$ have a global minimum, preferably unique?

We give a complete answer to the above question in two stages:

1. In this section we show that if A is symmetric positive definite, then $Q(x)$ has a unique global minimum precisely when
$$Ax = b.$$

2. In Section 42.2 we give necessary and sufficient conditions in the general case, in terms of the pseudo-inverse of A.

We begin with the matrix version of Definition 22.2.

Definition 42.1. A symmetric *positive definite matrix* is a matrix whose eigenvalues are strictly positive, and a symmetric *positive semidefinite matrix* is a matrix whose eigenvalues are nonnegative.

Equivalent criteria are given in the following proposition.

Proposition 42.1. *Given any Euclidean space E of dimension n, the following properties hold:*

(1) Every self-adjoint linear map $f\colon E \to E$ is positive definite iff

$$\langle f(x), x\rangle > 0$$

for all $x \in E$ with $x \neq 0$.

(2) Every self-adjoint linear map $f\colon E \to E$ is positive semidefinite iff

$$\langle f(x), x\rangle \geq 0$$

for all $x \in E$.

Proof. (1) First assume that f is positive definite. Recall that every self-adjoint linear map has an orthonormal basis $(e_1, \ldots, e_n)$ of eigenvectors, and let $\lambda_1, \ldots, \lambda_n$ be the corresponding eigenvalues. With respect to this basis, for every $x = x_1e_1 + \cdots + x_ne_n \neq 0$, we have

$$\langle f(x), x\rangle = \Big\langle f\Big(\sum_{i=1}^n x_ie_i\Big), \sum_{i=1}^n x_ie_i\Big\rangle = \Big\langle \sum_{i=1}^n \lambda_ix_ie_i, \sum_{i=1}^n x_ie_i\Big\rangle = \sum_{i=1}^n \lambda_ix_i^2,$$