Now if A is invertible, then $\sigma_1 \geq \cdots \geq \sigma_n > 0$, and it is easy to show that the eigenvalues of $(A^*A)^{-1}$ are $\sigma_n^{-2} \geq \cdots \geq \sigma_1^{-2}$, which shows that

$$\left\|A^{-1}\right\|_2 = \sigma_n^{-1},$$

and thus

$$\mathrm{cond}_2(A) = \frac{\sigma_1}{\sigma_n}.$$

(3) This follows from the fact that $\|A\|_2 = \rho(A)$ for a normal matrix.

(4) If A is a unitary matrix, then $A^*A = AA^* = I$, so $\rho(A^*A) = 1$, and $\|A\|_2 = \sqrt{\rho(A^*A)} = 1$. We also have $\|A^{-1}\|_2 = \|A^*\|_2 = \sqrt{\rho(AA^*)} = 1$, and thus $\mathrm{cond}(A) = 1$.

(5) This follows immediately from the unitary invariance of the spectral norm. $\square$

Proposition 9.17 (4) shows that unitary and orthogonal transformations are very well-conditioned, and Part (5) shows that unitary transformations preserve the condition number.

In order to compute $\mathrm{cond}_2(A)$, we need to compute the top and bottom singular values of A, which may be hard. The inequality

$$\|A\|_2 \leq \|A\|_F \leq \sqrt{n}\, \|A\|_2\,,$$

may be useful in getting an approximation of $\mathrm{cond}_2(A) = \|A\|_2 \|A^{-1}\|_2$, if A^{-1} can be determined.

Remark: There is an interesting geometric characterization of $\mathrm{cond}_2(A)$. If $\theta(A)$ denotes the least angle between the vectors Au and Av as u and v range over all pairs of orthonormal vectors, then it can be shown that

$$\mathrm{cond}_2(A) = \cot(\theta(A)/2)).$$

Thus if A is nearly singular, then there will be some orthonormal pair u, v such that Au and Av are nearly parallel; the angle $\theta(A)$ will the be small and $\cot(\theta(A)/2))$ will be large. For more details, see Horn and Johnson [95] (Section 5.8 and Section 7.4).

It should be noted that in general (if A is not a normal matrix) a matrix could have a very large condition number even if all its eigenvalues are identical! For example, if we consider the $n \times n$ matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 2 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 & 2 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & 2 \\ 0 & 0 & \dots & 0 & 0 & 0 & 1 \end{pmatrix},$$