As explained in Godsil and Royle [77], we can imagine building a physical model of G by connecting adjacent vertices (in $\mathbb{R}^n$) by identical springs. Then it is natural to consider a representation to be better if it requires the springs to be less extended. We can formalize this by defining the *energy* of a drawing R by

$$\mathcal{E}(R) = \sum_{\{v_i,v_j\}\in E} \|\rho(v_i) - \rho(v_j)\|^2,$$

where $\rho(v_i)$ is the ith row of R and $\|\rho(v_i) - \rho(v_j)\|^2$ is the square of the Euclidean length of the line segment joining $\rho(v_i)$ and $\rho(v_j)$.

Then, "good drawings" are drawings that minimize the energy function $\mathcal{E}$. Of course, the trivial representation corresponding to the zero matrix is optimum, so we need to impose extra constraints to rule out the trivial solution.

We can consider the more general situation where the springs are not necessarily identical. This can be modeled by a symmetric weight (or stiffness) matrix $W = (w_{ij})$, with $w_{ij} \geq 0$. Then our energy function becomes

$$\mathcal{E}(R) = \sum_{\{v_i,v_j\}\in E} w_{ij} \|\rho(v_i) - \rho(v_j)\|^2.$$

It turns out that this function can be expressed in terms of the Laplacian $L = D - W$. The following proposition is shown in Godsil and Royle [77]. We give a slightly more direct proof.

Proposition 21.1. *Let $G = (V, W)$ be a weighted graph, with $|V| = m$ and W an $m \times m$ symmetric matrix, and let R be the matrix of a graph drawing ρ of G in $\mathbb{R}^n$ (a $m \times n$ matrix). If $L = D - W$ is the unnormalized Laplacian matrix associated with W, then*

$$\mathcal{E}(R) = \mathrm{tr}(R^\top L R).$$

Proof. Since $\rho(v_i)$ is the ith row of R (and $\rho(v_j)$ is the jth row of R), if we denote the kth column of R by R^k, using Proposition 20.4, we have

$$\begin{aligned}
\mathcal{E}(R) &= \sum_{\{v_i,v_j\}\in E} w_{ij} \|\rho(v_i) - \rho(v_j)\|^2 \\
&= \sum_{k=1}^{n} \sum_{\{v_i,v_j\}\in E} w_{ij}(R_{ik} - R_{jk})^2 \\
&= \sum_{k=1}^{n} \frac{1}{2} \sum_{i,j=1}^{m} w_{ij}(R_{ik} - R_{jk})^2 \\
&= \sum_{k=1}^{n} (R^k)^\top L R^k = \mathrm{tr}(R^\top L R),
\end{aligned}$$

as claimed. ∎