than m) the idea is to generate the $n \times n$ Hessenberg submatrix H_n of the full Hessenberg matrix H (such that $A = UHU^*$) consisting of its first n rows and n columns; the matrix U_n consisting of the first n columns of U is also produced. The Rayleigh–Ritz method consists in computing the eigenvalues of H_n using the QR- method with shifts. These eigenvalues, called *Ritz values*, are approximations of the eigenvalues of A. Typically, extreme eigenvalues are found first.

Arnoldi iteration can also be viewed as a way of computing an orthonormal basis of a *Krylov subspace*, namely the subspace $\mathcal{K}_n(A, b)$ spanned by $(b, Ab, \ldots, A^n b)$. We can also use Arnoldi iteration to find an approximate solution of a linear equation $Ax = b$ by minimizing $\|b - Ax_n\|_2$ for all x_n is the Krylov space $\mathcal{K}_n(A, b)$. This method named GMRES is discussed in Section 18.5.

The special case where H is a symmetric (or Hermitian) tridiagonal matrix is discussed in Section 18.6. In this case, Arnoldi's algorithm becomes *Lanczos' algorithm*. It is much more efficient than Arnoldi iteration.

We close this chapter by discussing two classical methods for computing a single eigenvector and a single eigenvalue: power iteration and inverse (power) iteration; see Section 18.7.

18.1 The Basic QR Algorithm

Let A be an $n \times n$ matrix which is assumed to be diagonalizable and invertible. The basic QR algorithm makes use of two very simple steps. Starting with $A_1 = A$, we construct sequences of matrices (A_k), (Q_k) (R_k) and (P_k) as follows:

$$\begin{array}{ll} \text{Factor} & A_1 = Q_1 R_1 \\ \text{Set} & A_2 = R_1 Q_1 \\ \text{Factor} & A_2 = Q_2 R_2 \\ \text{Set} & A_3 = R_2 Q_2 \\ & \vdots \\ \text{Factor} & A_k = Q_k R_k \\ \text{Set} & A_{k+1} = R_k Q_k \\ & \vdots \end{array}$$

Thus, A_{k+1} is obtained from a QR-factorization $A_k = Q_k R_k$ of A_k by swapping Q_k and R_k. Define P_k by

$$P_k = Q_1 Q_2 \cdots Q_k.$$

Since $A_k = Q_k R_k$, we have $R_k = Q_k^* A_k$, and since $A_{k+1} = R_k Q_k$, we obtain

$$A_{k+1} = Q_k^* A_k Q_k. \tag{$*_1$}$$