over the basis $(u_1, \ldots, u_n)$. Since φ is represented by the column vector $\lambda^\top$ over the dual basis $(v_1^*, \ldots, v_m^*)$, we see that $f^\top(\varphi)$ is represented by the column vector

$$(\lambda A)^\top = A^\top \lambda^\top$$

over the dual basis $(u_1^*, \ldots, u_n^*)$. The matrix defining $f^\top$ over the dual bases $(v_1^*, \ldots, v_m^*)$ and $(u_1^*, \ldots, u_n^*)$ is indeed $A^\top$.

Conceptually, we will show later (see Section 30.1) that the linear map $f^\top \colon F^* \to E^*$ is defined by

$$f^\top(\varphi) = \varphi \circ f,$$

for all $\varphi \in F^*$ (remember that $\varphi \colon F \to K$, so composing $f \colon E \to F$ and $\varphi \colon F \to K$ yields a linear form $\varphi \circ f \colon E \to K$).

4.3 Change of Basis Matrix

It is important to observe that the isomorphism $M \colon \mathrm{Hom}(E, F) \to \mathrm{M}_{n,p}$ given by Proposition 4.2 depends on the choice of the bases $(u_1, \ldots, u_p)$ and $(v_1, \ldots, v_n)$, and similarly for the isomorphism $M \colon \mathrm{Hom}(E, E) \to \mathrm{M}_n$, which depends on the choice of the basis $(u_1, \ldots, u_n)$. Thus, it would be useful to know how a change of basis affects the representation of a linear map $f \colon E \to F$ as a matrix. The following simple proposition is needed.

Proposition 4.3. *Let E be a vector space, and let $(u_1, \ldots, u_n)$ be a basis of E. For every family $(v_1, \ldots, v_n)$, let $P = (a_{i\,j})$ be the matrix defined such that $v_j = \sum_{i=1}^n a_{i\,j} u_i$. The matrix P is invertible iff $(v_1, \ldots, v_n)$ is a basis of E.*

Proof. Note that we have $P = M(f)$, the matrix (with respect to the basis $(u_1, \ldots, u_n)$) associated with the unique linear map $f \colon E \to E$ such that $f(u_i) = v_i$. By Proposition 3.18, f is bijective iff $(v_1, \ldots, v_n)$ is a basis of E. Furthermore, it is obvious that the identity matrix I_n is the matrix associated with the identity $\mathrm{id} \colon E \to E$ w.r.t. any basis. If f is an isomorphism, then $f \circ f^{-1} = f^{-1} \circ f = \mathrm{id}$, and by Proposition 4.2, we get $M(f)M(f^{-1}) = M(f^{-1})M(f) = I_n$, showing that P is invertible and that $M(f^{-1}) = P^{-1}$. $\square$

An important corollary of Proposition 4.3 yields the following criterion for a square matrix to be invertible. This criterion was already proven in Proposition 3.14 but Proposition 4.3 yields a shorter proof.

Proposition 4.4. *A square matrix $A \in \mathrm{M}_n(K)$ is invertible iff its columns $(A^1, \ldots, A^n)$ are linearly independent.*

Proof. First assume that A is invertible. If $\lambda_1 A^1 + \cdots + \lambda_n A^n = 0$ for some $\lambda_1, \ldots, \lambda_n \in K$, then

$$A\lambda = \lambda_1 A^1 + \cdots + \lambda_n A^n = 0,$$