(ii) $\text{rk}(f) + \dim(\text{Ker } f) = \dim(E)$.

(iii) $\text{rk}(f) \leq \min(\dim(E), \dim(F))$.

Proof. Since by Proposition 6.16, $\dim(E) = \dim(\text{Ker } f) + \dim(\text{Im } f)$, and by definition, $\text{rk}(f) = \dim(\text{Im } f)$, we have $\text{rk}(f) = \text{codim}(\text{Ker } f)$. Since $\text{rk}(f) = \dim(\text{Im } f)$, (ii) follows from $\dim(E) = \dim(\text{Ker } f) + \dim(\text{Im } f)$. As for (iii), since $\text{Im } f$ is a subspace of F, we have $\text{rk}(f) \leq \dim(F)$, and since $\text{rk}(f) + \dim(\text{Ker } f) = \dim(E)$, we have $\text{rk}(f) \leq \dim(E)$. ☐

The rank of a matrix is defined as follows.

Definition 6.11. Given a $m \times n$-matrix $A = (a_{i\,j})$ over the field K, the *rank* $\text{rk}(A)$ of the matrix A is the maximum number of linearly independent columns of A (viewed as vectors in K^m).

In view of Proposition 3.8, the rank of a matrix A is the dimension of the subspace of K^m generated by the columns of A. Let E and F be two vector spaces, and let $(u_1, \ldots, u_n)$ be a basis of E, and $(v_1, \ldots, v_m)$ a basis of F. Let $f : E \to F$ be a linear map, and let $M(f)$ be its matrix w.r.t. the bases $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_m)$. Since the rank $\text{rk}(f)$ of f is the dimension of $\text{Im } f$, which is generated by $(f(u_1), \ldots, f(u_n))$, the rank of f is the maximum number of linearly independent vectors in $(f(u_1), \ldots, f(u_n))$, which is equal to the number of linearly independent columns of $M(f)$, since F and K^m are isomorphic. Thus, we have $\text{rk}(f) = \text{rk}(M(f))$, for every matrix representing f.

We will see later, using duality, that the rank of a matrix A is also equal to the maximal number of linearly independent rows of A.

If U is a hyperplane, then $E = U \oplus V$ for some subspace V of dimension 1. However, a subspace V of dimension 1 is generated by any nonzero vector $v \in V$, and thus we denote V by Kv, and we write $E = U \oplus Kv$. Clearly, $v \notin U$. Conversely, let $x \in E$ be a vector such that $x \notin U$ (and thus, $x \neq 0$). We claim that $E = U \oplus Kx$. Indeed, since U is a hyperplane, we have $E = U \oplus Kv$ for some $v \notin U$ (with $v \neq 0$). Then, $x \in E$ can be written in a unique way as $x = u + \lambda v$, where $u \in U$, and since $x \notin U$, we must have $\lambda \neq 0$, and thus, $v = -\lambda^{-1}u + \lambda^{-1}x$. Since $E = U \oplus Kv$, this shows that $E = U + Kx$. Since $x \notin U$, we have $U \cap Kx = 0$, and thus $E = U \oplus Kx$. This argument shows that a hyperplane is a maximal proper subspace H of E.

Theorem 6.16 also yields a characterization of hyperplanes in terms of linear forms. Recall that given a vector space E, a *hyperplane* H in E is subspace of codimension 1, which means that there is a one-dimensional subspace L such that

$$E = H \oplus L.$$

Proposition 6.22. *Given a nontrivial vector space E over a field K, a subspace H of E is a hyperplane iff there is a nonzero linear form $\varphi : E \to K$ such that*

$$H = \text{Ker } \varphi.$$