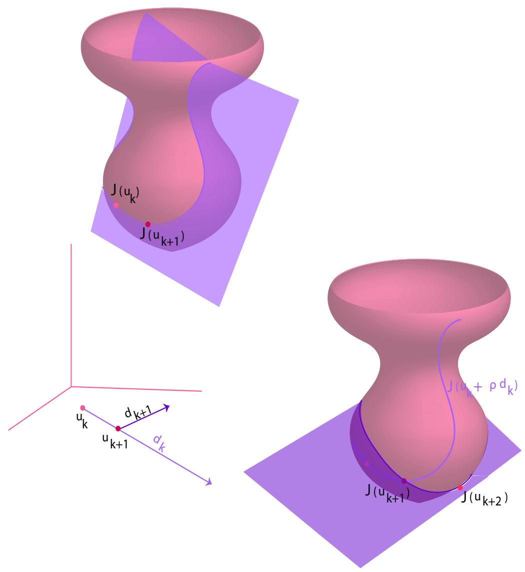

Figure 49.1: Let $J\colon \mathbb{R}^2 \to \mathbb{R}$ be the function whose graph is represented by the pink surface. Given a point u_k in the xy-plane, and a direction d_k, we calculate first u_{k+1} and then u_{k+2}.

Proposition 49.11. *If J is a quadratic elliptic functional of the form*

$$J(v) = \frac{1}{2}a(v,v) - h(v),$$

then given d_k, there is a unique ρ_k solving the line search in Step (2).

Proof. This is because, by Proposition 49.3, we have

$$J(u_k + \rho d_k) = \frac{\rho^2}{2} a(d_k, d_k) + \rho\langle \nabla J_{u_k}, d_k\rangle + J(u_k),$$

and since $a(d_k, d_k) > 0$ (because J is elliptic), the above function of ρ has a unique minimum when its derivative is zero, namely

$$\rho\, a(d_k, d_k) + \langle \nabla J_{u_k}, d_k\rangle = 0. \qquad \square$$

Since Step (2) is often too costly, an alternative is

(3) *Backtracking line search*: Pick two constants α and β such that $0 < \alpha < 1/2$ and $0 < \beta < 1$, and set $t = 1$. Given a descent direction d_k at $u_k \in \mathrm{dom}(J)$,

while $J(u_k + td_k) > J(u_k) + \alpha t\langle \nabla J_{u_k}, d_k\rangle$ **do** $t := \beta t$;
$\rho_k = t$; $u_{k+1} = u_k + \rho_k d_k$.