and assume that I is infinite. For every $j \in J$, let $L_j \subseteq I$ be the finite set

$$L_j = \{i \in I \mid v_j = \sum_{i \in I} \lambda_i u_i,\ \lambda_i \neq 0\}.$$

Let $L = \bigcup_{j \in J} L_j$. By definition $L \subseteq I$, and since $(u_i)_{i \in I}$ is a basis of E, we must have $I = L$, since otherwise $(u_i)_{i \in L}$ would be another basis of E, and this would contradict the fact that $(u_i)_{i \in I}$ is linearly independent. Furthermore, J must be infinite, since otherwise, because the L_j are finite, I would be finite. But then, since $I = \bigcup_{j \in J} L_j$ with J infinite and the L_j finite, by a standard result of set theory, $|I| \leq |J|$. If $(v_j)_{j \in J}$ is also a basis, by a symmetric argument, we obtain $|J| \leq |I|$, and thus, $|I| = |J|$ for any two bases $(u_i)_{i \in I}$ and $(v_j)_{j \in J}$ of E.

Definition 3.8. When a vector space E is not finitely generated, we say that E is of infinite dimension. The *dimension* of a finitely generated vector space E is the common dimension n of all of its bases and is denoted by $\dim(E)$.

Clearly, if the field K itself is viewed as a vector space, then every family (a) where $a \in K$ and $a \neq 0$ is a basis. Thus $\dim(K) = 1$. Note that $\dim(\{0\}) = 0$.

Definition 3.9. If E is a vector space of dimension $n \geq 1$, for any subspace U of E, if $\dim(U) = 1$, then U is called a *line*; if $\dim(U) = 2$, then U is called a *plane*; if $\dim(U) = n-1$, then U is called a *hyperplane*. If $\dim(U) = k$, then U is sometimes called a *k-plane*.

Let $(u_i)_{i \in I}$ be a basis of a vector space E. For any vector $v \in E$, since the family $(u_i)_{i \in I}$ generates E, there is a family $(\lambda_i)_{i \in I}$ of scalars in K, such that

$$v = \sum_{i \in I} \lambda_i u_i.$$

A very important fact is that the family $(\lambda_i)_{i \in I}$ is **unique**.

Proposition 3.12. *Given a vector space E, let $(u_i)_{i \in I}$ be a family of vectors in E. Let $v \in E$, and assume that $v = \sum_{i \in I} \lambda_i u_i$. Then the family $(\lambda_i)_{i \in I}$ of scalars such that $v = \sum_{i \in I} \lambda_i u_i$ is unique iff $(u_i)_{i \in I}$ is linearly independent.*

Proof. First, assume that $(u_i)_{i \in I}$ is linearly independent. If $(\mu_i)_{i \in I}$ is another family of scalars in K such that $v = \sum_{i \in I} \mu_i u_i$, then we have

$$\sum_{i \in I} (\lambda_i - \mu_i) u_i = 0,$$

and since $(u_i)_{i \in I}$ is linearly independent, we must have $\lambda_i - \mu_i = 0$ for all $i \in I$, that is, $\lambda_i = \mu_i$ for all $i \in I$. The converse is shown by contradiction. If $(u_i)_{i \in I}$ was linearly dependent, there would be a family $(\mu_i)_{i \in I}$ of scalars not all null such that

$$\sum_{i \in I} \mu_i u_i = 0$$