Let $U_1, \ldots, U_p$ be any $p \geq 2$ subspaces of some vector space E. Prove that $U_1 + \cdots + U_p$ is a direct sum iff

$$U_i \cap \left(\sum_{j=1}^{i-1} U_j \right) = (0), \quad i = 2, \ldots, p.$$

Problem 6.5. Given any vector space E, a linear map $f\colon E \to E$ is an *involution* if $f \circ f = \mathrm{id}$.

(1) Prove that an involution f is invertible. What is its inverse?

(2) Let E_1 and E_{-1} be the subspaces of E defined as follows:

$$\begin{aligned} E_1 &= \{u \in E \mid f(u) = u\} \\ E_{-1} &= \{u \in E \mid f(u) = -u\}. \end{aligned}$$

Prove that we have a direct sum

$$E = E_1 \oplus E_{-1}.$$

Hint. For every $u \in E$, write

$$u = \frac{u + f(u)}{2} + \frac{u - f(u)}{2}.$$

(3) If E is finite-dimensional and f is an involution, prove that there is some basis of E with respect to which the matrix of f is of the form

$$I_{k,n-k} = \begin{pmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{pmatrix},$$

where I_k is the $k \times k$ identity matrix (similarly for I_{n-k}) and $k = \dim(E_1)$. Can you give a geometric interpretation of the action of f (especially when $k = n - 1$)?

Problem 6.6. An $n \times n$ matrix H is *upper Hessenberg* if $h_{jk} = 0$ for all (j, k) such that $j - k \geq 0$. An upper Hessenberg matrix is *unreduced* if $h_{i+1i} \neq 0$ for $i = 1, \ldots, n - 1$.

Prove that if H is a singular unreduced upper Hessenberg matrix, then $\dim(\mathrm{Ker}\,(H)) = 1$.

Problem 6.7. Let A be any $n \times k$ matrix.

(1) Prove that the $k \times k$ matrix $A^\top A$ and the matrix A have the same nullspace. Use this to prove that $\mathrm{rank}(A^\top A) = \mathrm{rank}(A)$. Similarly, prove that the $n \times n$ matrix $AA^\top$ and the matrix $A^\top$ have the same nullspace, and conclude that $\mathrm{rank}(AA^\top) = \mathrm{rank}(A^\top)$.

We will prove later that $\mathrm{rank}(A^\top) = \mathrm{rank}(A)$.

(2) Let $a_1, \ldots, a_k$ be k linearly independent vectors in $\mathbb{R}^n$ $(1 \leq k \leq n)$, and let A be the $n \times k$ matrix whose ith column is a_i. Prove that $A^\top A$ has rank k, and that it is invertible. Let $P = A(A^\top A)^{-1}A^\top$ (an $n \times n$ matrix). Prove that

$$\begin{aligned} P^2 &= P \\ P^\top &= P. \end{aligned}$$