Let $J\colon \mathbb{R}^n \to \mathbb{R}$ be the function given by

$$J(v) = \sum_{i,j=1}^{n} a_{iij} v_i v_j + \sum_{i=1}^{n} b_i v_i,$$

where $v = (v_1, \ldots, v_n)$ and

$$a_{ij} = \int_0^1 \varphi_i'(t)\varphi_j'(t)dt, \quad b_i = \int_0^1 \varphi_i(t)dt.$$

(1) Let $U_1$ be the subset of $\mathbb{R}^n$ given by

$$U_1 = \left\{ v \in \mathbb{R}^n \mid \sum_{i=1}^{n} b_i v_i = 0 \right\}.$$

Consider the problem of finding a minimum of $J$ over $U_1$. Prove that the Lagrange multiplier $\lambda$ for which the Lagrangian has a critical point is $\lambda = -1$.

(2) Prove that the map defined on $U_1$ by

$$\|v\| = \left( \int_0^1 \left( \sum_{i=1}^{n} v_i \varphi_i'(x) \right)^2 dx \right)^{1/2}$$

is a norm. Prove that $J$ is elliptic on $U_1$ with this norm. Prove that $J$ has a unique minimum on $U_1$.

(3) Consider the the subset of $\mathbb{R}^n$ given by

$$U_2 = \left\{ v \in \mathbb{R}^n \mid \sum_{i=1}^{n} (\varphi_i(1) + \varphi_i(0)) v_i = 0 \right\}.$$

Consider the problem of finding a minimum of $J$ over $U_2$. Prove that the Lagrange multiplier $\lambda$ for which the Lagrangian has a critical point is $\lambda = -1/2$. Prove that $J$ is elliptic on $U_2$ with the same norm as in (2). Prove that $J$ has a unique minimum on $U_2$.

(4) Consider the the subset of $\mathbb{R}^n$ given by

$$U_3 = \left\{ v \in \mathbb{R}^n \mid \sum_{i=1}^{n} (\varphi_i(1) - \varphi_i(0)) v_i = 0 \right\}.$$

This time show that the necessary condition for having a minimum on $U_3$ yields the equation $1 + \lambda(1 - 1) = 0$. Conclude that $J$ does not have a minimum on $U_3$.