If $g\colon \mathbb{R}^2 \to \mathbb{R}$ is the function given by $g(x, y) = x^2 - y^2$, then $g'_{(x,y)} = (2x \quad -2y)$, so $g'_{(0,0)} = (0 \;\; 0)$, yet near $(0, 0)$ the function g takes negative and positive values. See Figure 40.2.

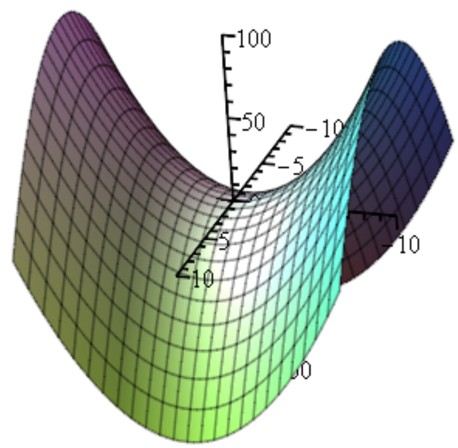

Figure 40.2: The graph of $g(x, y) = x^2 - y^2$. Note that $(0, 0)$ is a saddle point and not a local extremum.

It is very important to note that the hypothesis that Ω *is open* is crucial for the validity of Proposition 40.1.

For example, if J is the identity function on $\mathbb{R}$ and $U = [0, 1]$, a closed subset, then $J'(x) = 1$ for all $x \in [0, 1]$, even though J has a minimum at $x = 0$ and a maximum at $x = 1$.

In many practical situations, we need to look for local extrema of a function J *under additional constraints*. This situation can be formalized conveniently as follows. We have a function $J\colon \Omega \to \mathbb{R}$ defined on some open subset Ω of a normed vector space, but we also have some subset U of Ω, and we are looking for the local extrema of J *with respect to the set* U.

The elements $u \in U$ are often called *feasible solutions* of the optimization problem consisting in finding the local extrema of some objective function J with respect to some subset U of Ω defined by a set of constraints. Note that in most cases, U is *not* open. In fact, U is usually closed.

Definition 40.3. If $J\colon \Omega \to \mathbb{R}$ is a real-valued function defined on some open subset Ω of a normed vector space E and if U is some subset of Ω, we say that J has a *local minimum* (or *relative minimum*) at the point $u \in U$ *with respect to* U if there is some open subset $W \subseteq \Omega$ containing u such that

$$J(u) \leq J(w) \quad \text{for all } w \in U \cap W.$$