*Proof.* If $y = A^+Au$, then

$$y = A^+Au = U \begin{pmatrix} I_r & 0 \\ 0 & 0_{n-r} \end{pmatrix} U^\top u = U \begin{pmatrix} z \\ 0 \end{pmatrix},$$

for some $z \in \mathbb{R}^r$. Conversely, if $U^\top y = \left(\begin{smallmatrix} z \\ 0 \end{smallmatrix}\right)$, then $y = U\left(\begin{smallmatrix} z \\ 0 \end{smallmatrix}\right)$, and so

$$\begin{aligned} A^+AU \begin{pmatrix} z \\ 0 \end{pmatrix} &= U \begin{pmatrix} I_r & 0 \\ 0 & 0_{n-r} \end{pmatrix} U^\top U \begin{pmatrix} z \\ 0 \end{pmatrix} \\ &= U \begin{pmatrix} I_r & 0 \\ 0 & 0_{n-r} \end{pmatrix} \begin{pmatrix} z \\ 0 \end{pmatrix} \\ &= U \begin{pmatrix} z \\ 0 \end{pmatrix} = y, \end{aligned}$$

which shows that $y \in \mathrm{range}(A^+A)$. $\square$

Analogous results hold for complex matrices, but in this case, $V$ and $U$ are unitary matrices and $AA^+$ and $A^+A$ are Hermitian orthogonal projections.

If $A$ is a normal matrix, which means that $AA^\top = A^\top A$, then there is an intimate relationship between SVD's of $A$ and block diagonalizations of $A$. As a consequence, the pseudo-inverse of a normal matrix $A$ can be obtained directly from a block diagonalization of $A$.

If $A$ is a (real) normal matrix, then we know from Theorem 17.18 that $A$ can be block diagonalized with respect to an orthogonal matrix $U$ as

$$A = U\Lambda U^\top,$$

where $\Lambda$ is the (real) block diagonal matrix

$$\Lambda = \mathrm{diag}(B_1, \ldots, B_n),$$

consisting either of $2 \times 2$ blocks of the form

$$B_j = \begin{pmatrix} \lambda_j & -\mu_j \\ \mu_j & \lambda_j \end{pmatrix}$$

with $\mu_j \neq 0$, or of one-dimensional blocks $B_k = (\lambda_k)$. Then we have the following proposition:

**Proposition 23.7.** *For any (real) normal matrix $A$ and any block diagonalization $A = U\Lambda U^\top$ of $A$ as above, the pseudo-inverse of $A$ is given by*

$$A^+ = U\Lambda^+U^\top,$$

*where $\Lambda^+$ is the pseudo-inverse of $\Lambda$. Furthermore, if*

$$\Lambda = \begin{pmatrix} \Lambda_r & 0 \\ 0 & 0 \end{pmatrix},$$