of the symmetric (respectively Hermitian) tridiagonal matrix H_n applies, but there are more methods for finding the eigenvalues of symmetric (respectively Hermitian) tridiagonal matrices. Also theorems about error estimates exist. The version of Lanczos iteration given above may run into problems in floating point arithmetic. What happens is that the vectors u_j may lose the property of being orthogonal, so it may be necessary to reorthogonalize them. For more on all this, see Demmel [48] (Chapter 7, in particular Section 7.2-7.4). The version of GMRES using Lanczos iteration is called MINRES.

We close our brief survey of methods for computing the eigenvalues and the eigenvectors of a matrix with a quick discussion of two methods known as power methods.

18.7 Power Methods

Let A be an $m \times m$ complex or real matrix. There are two power methods, both of which yield one eigenvalue and one eigenvector associated with this vector:

(1) *Power iteration.*

(2) *Inverse (power) iteration.*

Power iteration only works if the matrix A has an eigenvalue λ of largest modulus, which means that if $\lambda_1, \ldots, \lambda_m$ are the eigenvalues of A, then

$$|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_m| \geq 0.$$

In particular, if A is a real matrix, then λ_1 must be real (since otherwise there are two complex conjugate eigenvalues of the same largest modulus). If the above condition is satisfied, then power iteration yields λ_1 and some eigenvector associated with it. The method is simple enough:

Pick some initial unit vector x^0 and compute the following sequence (x^k), where

$$x^{k+1} = \frac{Ax^k}{\|Ax^k\|}, \quad k \geq 0.$$

We would expect that (x^k) converges to an eigenvector associated with λ_1, but this is not quite correct. The following results are proven in Serre [156] (Section 13.5.1). First assume that $\lambda_1 \neq 0$.

We have

$$\lim_{k \mapsto \infty} \left\|Ax^k\right\| = |\lambda_1|.$$

If A is a complex matrix which has a unique complex eigenvalue λ_1 of largest modulus, then

$$v = \lim_{k \mapsto \infty} \left(\frac{\overline{\lambda_1}}{|\lambda_1|} \right)^k x^k$$