and an easy induction yields

$$\lambda_i^k = \frac{\|\nabla J_{u_k}\|^2}{\|\nabla J_{u_i}\|^2}, \quad 0 \leq i \leq k-1.$$

Consequently, using $(*_3)$ we have

$$\begin{aligned} d_k &= \sum_{i=0}^{k-1} \frac{\|\nabla J_{u_k}\|^2}{\|\nabla J_{u_i}\|^2} \nabla J_{u_i} + \nabla J_{u_k} \\ &= \nabla J_{u_k} + \frac{\|\nabla J_{u_k}\|^2}{\left\|\nabla J_{u_{k-1}}\right\|^2} \left(\sum_{i=0}^{k-2} \frac{\left\|\nabla J_{u_{k-1}}\right\|^2}{\|\nabla J_{u_i}\|^2} \nabla J_{u_i} + \nabla J_{u_{k-1}} \right) \\ &= \nabla J_{u_k} + \frac{\|\nabla J_{u_k}\|^2}{\left\|\nabla J_{u_{k-1}}\right\|^2} d_{k-1}, \end{aligned}$$

which concludes the proof. $\square$

It remains to compute ρ_k, which is the solution of the line search

$$J(u_k - \rho_k d_k) = \inf_{\rho \in \mathbb{R}} J(u_k - \rho d_k).$$

Since J is a quadratic functional, a basic computation left to the reader shows that the function to be minimized is

$$\rho \mapsto \frac{\rho^2}{2} \langle A d_k, d_k \rangle - \rho \langle \nabla J_{u_k}, d_k \rangle + J(u_k),$$

whose mininum is obtained when its derivative is zero, that is,

$$\rho_k = \frac{\langle \nabla J_{u_k}, d_k \rangle}{\langle A d_k, d_k \rangle}. \tag{$*_5$}$$

In summary, the conjugate gradient method finds the minimum u of the elliptic quadratic functional

$$J(v) = \frac{1}{2} \langle Av, v \rangle - \langle b, v \rangle$$

by computing the sequence of vectors $u_1, d_1, \ldots, u_{k-1}, d_{k-1}, u_k$, starting from any vector u_0, with

$$d_0 = \nabla J_{u_0}.$$

If $\nabla J_{u_0} = 0$, then the algorithm terminates with $u = u_0$. Otherwise, for $k \geq 0$, assuming that $\nabla J_{u_i} \neq 0$ for $i = 1, \ldots, k$, compute

$$(*_6) \quad \begin{cases} \rho_k = \dfrac{\langle \nabla J_{u_k}, d_k \rangle}{\langle A d_k, d_k \rangle} \\ u_{k+1} = u_k - \rho_k d_k \\ d_{k+1} = \nabla J_{u_{k+1}} + \dfrac{\left\|\nabla J_{u_{k+1}}\right\|^2}{\|\nabla J_{u_k}\|^2} d_k. \end{cases}$$