*Proof.* By Theorem 28.2, $f \in \mathbf{SU}(n)$ can be written as a composition

$$\rho_{u_n,\theta_n} \circ \cdots \circ \rho_{u_1,\theta_1},$$

where $(u_1, \ldots, u_n)$ is an orthonormal basis of eigenvectors. Since $f$ is a rotation, $\det(f) = 1$, and this implies that $\theta_1 + \cdots + \theta_n = 0$. By Proposition 28.4,

$$f = h_{u_n - u_{n-1}} \circ h_{u_n - e^{-i(\theta_1+\cdots+\theta_{n-1})} u_{n-1}} \circ \cdots \circ h_{u_2 - u_1} \circ h_{u_2 - e^{-i\theta_1} u_1},$$

a composition of $2n-2$ hyperplane reflections. In general, if $f \in \mathbf{U}(n)$, by the remark after Theorem 28.2, $f$ can be written as $f = \rho_\theta \circ g$, where $g \in \mathbf{SU}(n)$ is a rotation, and $\rho_\theta$ is a Hermitian reflection. We conclude by applying what we just proved to $g$. $\square$

As a corollary of Theorem 28.5, the following interesting result can be shown (this is not hard, do it!). First, recall that a linear map $f \colon E \to E$ is *self-adjoint* (or *Hermitian*) iff $f = f^*$. Then, the subgroup of $\mathbf{U}(n)$ generated by the Hermitian isometries is equal to the group

$$\mathbf{SU}(n)^{\pm} = \{f \in \mathbf{U}(n) \mid \det(f) = \pm 1\}.$$

Equivalently, $\mathbf{SU}(n)^{\pm}$ is equal to the subgroup of $\mathbf{U}(n)$ generated by the hyperplane reflections.

This problem had been left open by Dieudonné in [49]. Evidently, it was settled since the publication of the third edition of the book [49].

Inspection of the proof of Proposition 27.4 reveals that this Proposition also holds for Hermitian spaces. Thus, when $n \geq 3$, the composition of any two hyperplane reflections is equal to the composition of two flips. As a consequence, a version of Theorem 27.5 holds for rotations in a Hermitian space of dimension at least 3.

**Theorem 28.6.** *Let $E$ be a Hermitan space of dimension $n \geq 3$. Every rotation $f \in \mathbf{SU}(E)$ is the composition of an even number of flips $f = f_{2k} \circ \cdots \circ f_1$, where $k \leq n-1$. Furthermore, if $u \neq 0$ is invariant under $f$ (i.e. $u \in \mathrm{Ker}\,(f - \mathrm{id})$), we can pick the last flip $f_{2k}$ such that $u \in F_{2k}^{\perp}$, where $F_{2k}$ is the subspace of dimension $n-2$ determining $f_{2k}$.*

*Proof.* It is identical to that of Theorem 27.5, except that it uses Theorem 28.5 instead of Theorem 27.1. The second part of the Proposition also holds, because if $u \neq 0$ is an eigenvector of $f$ for 1, then $u$ is one of the vectors in the orthonormal basis of eigenvectors used in 28.2. The details are left as an exercise. $\square$

We now show that the $QR$-decomposition in terms of (complex) Householder matrices holds for complex matrices. We need the version of Proposition 28.1 and a trick at the end of the argument, but the proof is basically unchanged.