Chapter 41

Newton's Method and Its Generalizations

In Chapter 40 we investigated the problem of determining when a function $J\colon \Omega \to \mathbb{R}$ defined on some open subset Ω of a normed vector space E has a local extremum. Proposition 40.1 gives a necessary condition when J is differentiable: if J has a local extremum at $u \in \Omega$, then we must have

$$J'(u) = 0.$$

Thus we are led to the problem of finding the zeros of the derivative

$$J'\colon \Omega \to E',$$

where $E' = \mathcal{L}(E; \mathbb{R})$ is the set of linear continuous functions from E to $\mathbb{R}$; that is, the *dual* of E, as defined in the remark after Proposition 40.8.

This leads us to consider the problem in a more general form, namely, given a function $f\colon \Omega \to Y$ from an open subset Ω of a normed vector space X to a normed vector space Y, find

(i) Sufficient conditions which guarantee the *existence of a zero* of the function f; that is, an element $a \in \Omega$ such that $f(a) = 0$.

(ii) An *algorithm* for approximating such an a, that is, a sequence (x_k) of points of Ω whose limit is a.

In this chapter we discuss Newton's method and some of it generalizations to give (partial) answers to Problems (i) and (i).

41.1 Newton's Method for Real Functions of a Real Argument

When $X = Y = \mathbb{R}$, we can use *Newton's method* to find a zero of a function $f\colon \Omega \to \mathbb{R}$. We pick some initial element $x_0 \in \mathbb{R}$ "close enough" to a zero a of f, and we define the sequence