composition of at most $n+2$ affine reflections. When $n \geq 2$, the identity is the composition of any reflection with itself.

Proof. First, we use Theorem 27.10. If f has a fixed point Ω, we choose Ω as an origin and work in the vector space E_Ω. Since f behaves as a linear isometry, the result follows from Theorem 27.1. More specifically, we can write $\overrightarrow{f} = \overrightarrow{s_k} \circ \cdots \circ \overrightarrow{s_1}$ for $k \leq n$ hyperplane reflections $\overrightarrow{s_i}$. We define the affine reflections s_i such that

$$s_i(a) = \Omega + \overrightarrow{s_i}(\overrightarrow{\Omega a})$$

for all $a \in E$, and we note that $f = s_k \circ \cdots \circ s_1$, since

$$f(a) = \Omega + \overrightarrow{s_k} \circ \cdots \circ \overrightarrow{s_1}(\overrightarrow{\Omega a})$$

for all $a \in E$. If f has no fixed point, then $f = t \circ g$ for some affine isometry g that has a fixed point Ω and some translation $t = t_\tau$, with $\overrightarrow{f}(\tau) = \tau$. By the argument just given, we can write $g = s_k \circ \cdots \circ s_1$ for some affine reflections (at most n). However, by Lemma 27.9, the translation $t = t_\tau$ can be achieved by two affine reflections about parallel hyperplanes, and thus $f = s_{k+2} \circ \cdots \circ s_1$, for some affine reflections (at most $n+2$). $\square$

When $n \geq 3$, we can also characterize the affine isometries in $\mathbf{SE}(n)$ in terms of affine flips. Remarkably, not only we can do without translations, but we can even bound the number of affine flips by n.

Theorem 27.12. *Let E be a Euclidean affine space of dimension $n \geq 3$. Every affine rigid motion $f \in \mathbf{SE}(E)$ is the composition of an even number of affine flips $f = f_{2k} \circ \cdots \circ f_1$, where $2k \leq n$.*

Proof. As in the proof of Theorem 27.11, we distinguish between the two cases where f has some fixed point or not. If f has a fixed point Ω, we apply Theorem 27.5. More specifically, we can write $\overrightarrow{f} = \overrightarrow{f_{2k}} \circ \cdots \circ \overrightarrow{f_1}$ for some flips $\overrightarrow{f_i}$. We define the affine flips f_i such that

$$f_i(a) = \Omega + \overrightarrow{f_i}(\overrightarrow{\Omega a})$$

for all $a \in E$, and we note that $f = f_{2k} \circ \cdots \circ f_1$, since

$$f(a) = \Omega + \overrightarrow{f_{2k}} \circ \cdots \circ \overrightarrow{f_1}(\overrightarrow{\Omega a})$$

for all $a \in E$.

If f does not have a fixed point, as in the proof of Theorem 27.11, we get

$$f = t_\tau \circ f_{2k} \circ \cdots \circ f_1,$$

for some affine flips f_i. We need to get rid of the translation. However, $\overrightarrow{f}(\tau) = \tau$, and by the second part of Theorem 27.5, we can assume that $\tau \in \overrightarrow{F_{2k}}^{\perp}$, where $\overrightarrow{F_{2k}}$ is the direction