33.3 Bases of Tensor Products

We showed that $E_1 \otimes \cdots \otimes E_n$ is generated by the vectors of the form $u_1 \otimes \cdots \otimes u_n$. However, these vectors are not linearly independent. This situation can be fixed when considering bases.

To explain the idea of the proof, consider the case when we have two spaces E and F both of dimension 3. Given a basis (e_1, e_2, e_3) of E and a basis (f_1, f_2, f_3) of F, we would like to prove that

$$e_1 \otimes f_1, \quad e_1 \otimes f_2, \quad e_1 \otimes f_3, \quad e_2 \otimes f_1, \quad e_2 \otimes f_2, \quad e_2 \otimes f_3, \quad e_3 \otimes f_1, \quad e_3 \otimes f_2, \quad e_3 \otimes f_3$$

are linearly independent. To prove this, it suffices to show that for any vector space G, if $w_{11}, w_{12}, w_{13}, w_{21}, w_{22}, w_{23}, w_{31}, w_{32}, w_{33}$ are any vectors in G, then there is a bilinear map $h\colon E \times F \to G$ such that

$$h(e_i, e_j) = w_{ij}, \quad 1 \leq i, j \leq 3.$$

Because h yields a unique linear map $h_\otimes\colon E \otimes F \to G$ such that

$$h_\otimes(e_i \otimes e_j) = w_{ij}, \quad 1 \leq i, j \leq 3,$$

and by Proposition 33.4, the vectors

$$e_1 \otimes f_1, \quad e_1 \otimes f_2, \quad e_1 \otimes f_3, \quad e_2 \otimes f_1, \quad e_2 \otimes f_2, \quad e_2 \otimes f_3, \quad e_3 \otimes f_1, \quad e_3 \otimes f_2, \quad e_3 \otimes f_3$$

are linearly independent. This suggests understanding how a bilinear function $f\colon E \times F \to G$ is expressed in terms of its values $f(e_i, f_j)$ on the basis vectors (e_1, e_2, e_3) and (f_1, f_2, f_3), and this can be done easily. Using bilinearity we obtain

$$\begin{aligned} f(u_1e_1 + u_2e_2 + u_3e_3, v_1f_1 + v_2f_2 + v_3f_3) &= u_1v_1f(e_1, f_1) + u_1v_2f(e_1, f_2) + u_1v_3f(e_1, f_3) \\ &\quad + u_2v_1f(e_2, f_1) + u_2v_2f(e_2, f_2) + u_2v_3f(e_2, f_3) \\ &\quad + u_3v_1f(e_3, f_1) + u_3v_2f(e_3, f_2) + u_3v_3f(e_3, f_3). \end{aligned}$$

Therefore, given $w_{11}, w_{12}, w_{13}, w_{21}, w_{22}, w_{23}, w_{31}, w_{32}, w_{33} \in G$, the function h given by

$$\begin{aligned} h(u_1e_1 + u_2e_2 + u_3e_3, v_1f_1 + v_2f_2 + v_3f_3) &= u_1v_1w_{11} + u_1v_2w_{12} + u_1v_3w_{13} \\ &\quad + u_2v_1w_{21} + u_2v_2w_{22} + u_2v_3w_{23} \\ &\quad + u_3v_1w_{31} + u_3v_2w_{33} + u_3v_3w_{33} \end{aligned}$$

is clearly bilinear, and by construction $h(e_i, f_j) = w_{ij}$, so it does the job.

The generalization of this argument to any number of vector spaces of any dimension (even infinite) is straightforward.

Proposition 33.12. *Given $n \geq 2$ vector spaces $E_1, \ldots, E_n$, if $(u_i^k)_{i \in I_k}$ is a basis for E_k, $1 \leq k \leq n$, then the family of vectors*

$$(u_{i_1}^1 \otimes \cdots \otimes u_{i_n}^n)_{(i_1, \ldots, i_n) \in I_1 \times \ldots \times I_n}$$

is a basis of the tensor product $E_1 \otimes \cdots \otimes E_n$.