40.5 Problems

Problem 40.1. Find the extrema of the function $J(v_1, v_2) = v_2^2$ on the subset U given by

$$U = \{(v_1, v_2) \in \mathbb{R}^2 \mid v_1^2 + v_2^2 - 1 = 0\}.$$

Problem 40.2. Find the extrema of the function $J(v_1, v_2) = v_1 + (v_2 - 1)^2$ on the subset U given by

$$U = \{(v_1, v_2) \in \mathbb{R}^2 \mid v_1^2 = 0\}.$$

Problem 40.3. Let A be an $n \times n$ real symmetric matrix, B an $n \times n$ symmetric positive definite matrix, and let $b \in \mathbb{R}^n$.

(1) Prove that a necessary condition for the function J given by

$$J(v) = \frac{1}{2} v^\top A v - b^\top v$$

to have an extremum at $u \in U$, with U defined by

$$U = \{v \in \mathbb{R}^n \mid v^\top B v = 1\},$$

is that there is some $\lambda \in \mathbb{R}$ such that

$$Au - b = \lambda B u.$$

(2) Prove that there is a symmetric positive definite matrix S such that $B = S^2$. Prove that if $b = 0$, then λ is an eigenvalue of the symmetric matrix $S^{-1} A S^{-1}$.

(3) Prove that for all $(u, \lambda) \in U \times \mathbb{R}$, if $Au - b = \lambda B u$, then

$$J(v) - J(u) = \frac{1}{2}(v - u)^\top (A - \lambda B)(v - u)$$

for all $v \in U$. Deduce that without additional assumptions, it is not possible to conclude that u is an extremum of J on U.

Problem 40.4. Let E be a normed vector space, and let U be a subset of E such that for all $u, v \in U$, we have $(1/2)(u + v) \in U$.

(1) Prove that if U is closed, then U is convex.

Hint. Every real $\theta \in (0, 1)$ can be written as

$$\theta = \sum_{n \geq 1} \alpha_n 2^{-n},$$

with $\alpha_n \in \{0, 1\}$.

(2) Does the result in (1) hold if U is not closed?