The identity $(*)$ proved in Proposition 33.10 shows that if $g\colon N \to P$ is another linear map, then

$$\tau_G(g) \circ \tau_G(f) = (g \otimes \mathrm{id}_G) \circ (f \otimes \mathrm{id}_G) = (g \circ f) \otimes (\mathrm{id}_G \circ \mathrm{id}_G) = (g \circ f) \otimes \mathrm{id}_G = \tau_G(g \circ f).$$

Clearly, $\tau_G(0) = 0$, and a direct computation on generators also shows that

$$\tau_G(\mathrm{id}_M) = (\mathrm{id}_M \otimes \mathrm{id}_G) = \mathrm{id}_{M \otimes G},$$

and that if $f'\colon M \to N$ is another linear map, then

$$\tau_G(f + f') = \tau_G(f) + \tau_G(f').$$

In fancy terms, τ_G is a functor. Now, if $E \oplus F$ is a direct sum, it is a standard fact of linear algebra that if $\pi_E\colon E \oplus F \to E$ and $\pi_F\colon E \oplus F \to F$ are the projection maps, then

$$\pi_E \circ \pi_E = \pi_E \qquad \pi_F \circ \pi_F = \pi_F \qquad \pi_E \circ \pi_F = 0 \qquad \pi_F \circ \pi_E = 0 \qquad \pi_E + \pi_F = \mathrm{id}_{E \oplus F}.$$

If we apply τ_G to these identites, we get

$$\begin{array}{lll} \tau_G(\pi_E) \circ \tau_G(\pi_E) = \tau_G(\pi_E) & \tau_G(\pi_F) \circ \tau_G(\pi_F) = \tau_G(\pi_F) & \\ \tau_G(\pi_E) \circ \tau_G(\pi_F) = 0 & \tau_G(\pi_F) \circ \tau_G(\pi_E) = 0 & \tau_G(\pi_E) + \tau_G(\pi_F) = \mathrm{id}_{(E \oplus F) \otimes G}. \end{array}$$

Observe that $\tau_G(\pi_E) = \pi_E \otimes \mathrm{id}_G$ is a map from $(E \oplus F) \otimes G$ onto $E \otimes G$ and that $\tau_G(\pi_F) = \pi_F \otimes \mathrm{id}_G$ is a map from $(E \oplus F) \otimes G$ onto $F \otimes G$, and by linear algebra, the above equations mean that we have a direct sum

$$(E \otimes G) \oplus (F \otimes G) \cong (E \oplus F) \otimes G.$$

(4) We have the linear map $\epsilon\colon E \to K \otimes E$ given by

$$\epsilon(u) = 1 \otimes u, \quad \text{for all } u \in E.$$

The map $(\lambda, u) \mapsto \lambda u$ from $K \times E$ to E is bilinear, so it induces a unique linear map $\eta\colon K \otimes E \to E$ making the following diagram commute

$$\begin{tikzcd} K \times E \arrow[r, "\iota_\otimes"] \arrow[dr] & K \otimes E \arrow[d, "\eta"] \\ & E, \end{tikzcd}$$

such that $\eta(\lambda \otimes u) = \lambda u$, for all $\lambda \in K$ and all $u \in E$. We have

$$(\eta \circ \epsilon)(u) = \eta(1 \otimes u) = 1u = u,$$

and

$$(\epsilon \circ \eta)(\lambda \otimes u) = \epsilon(\lambda u) = 1 \otimes (\lambda u) = \lambda(1 \otimes u) = \lambda \otimes u,$$

which shows that both $\epsilon \circ \eta$ and $\eta \circ \epsilon$ are the identity, so ϵ and η are isomorphisms. $\square$