3.7 Linear Maps

Now that we understand vector spaces and how to generate them, we would like to be able to transform one vector space E into another vector space F. A function between two vector spaces that preserves the vector space structure is called a homomorphism of vector spaces, or *linear map*. Linear maps formalize the concept of linearity of a function.

Keep in mind that linear maps, which are transformations of space, are usually far more important than the spaces themselves.

In the rest of this section, we assume that all vector spaces are over a given field K (say $\mathbb{R}$).

Definition 3.18. Given two vector spaces E and F, a *linear map* between E and F is a function $f\colon E \to F$ satisfying the following two conditions:

$$\begin{aligned} f(x+y) &= f(x) + f(y) && \text{for all } x, y \in E; \\ f(\lambda x) &= \lambda f(x) && \text{for all } \lambda \in K,\ x \in E. \end{aligned}$$

Setting $x = y = 0$ in the first identity, we get $f(0) = 0$. *The basic property of linear maps is that they transform linear combinations into linear combinations.* Given any finite family $(u_i)_{i \in I}$ of vectors in E, given any family $(\lambda_i)_{i \in I}$ of scalars in K, we have

$$f\left(\sum_{i \in I} \lambda_i u_i\right) = \sum_{i \in I} \lambda_i f(u_i).$$

The above identity is shown by induction on $|I|$ using the properties of Definition 3.18.

Example 3.6.

1. The map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined such that

   $$\begin{aligned} x' &= x - y \\ y' &= x + y \end{aligned}$$

   is a linear map. The reader should check that it is the composition of a rotation by $\pi/4$ with a magnification of ratio $\sqrt{2}$.

2. For any vector space E, the *identity map* $\mathrm{id}\colon E \to E$ given by

   $$\mathrm{id}(u) = u \quad \text{for all } u \in E$$

   is a linear map. When we want to be more precise, we write id_E instead of id.