The following proposition is almost obvious, but very important. It shows that projectivities between projective lines are characterized by the preservation of the cross-ratio of any four points (three of which are distinct).

Proposition 26.20. *Given any two projective lines Δ and Δ', for any sequence (a, b, c, d) of points in Δ and any sequence (a', b', c', d') of points in Δ', if a, b, c are distinct and a', b', c' are distinct, there is a unique projectivity $f\colon \Delta \to \Delta'$ such that $f(a) = a'$, $f(b) = b'$, $f(c) = c'$, and $f(d) = d'$ iff $[a, b, c, d] = [a', b', c', d']$.*

Proof. First, assume that $f\colon \Delta \to \Delta'$ is a projectivity such that $f(a) = a'$, $f(b) = b'$, $f(c) = c'$, and $f(d) = d'$. Let $h\colon \Delta \to \mathbb{P}^1_K$ be the unique projectivity such that $h(a) = \infty$, $h(b) = 0$, and $h(c) = 1$, and let $h'\colon \Delta' \to \mathbb{P}^1_K$ be the unique projectivity such that $h'(a') = \infty$, $h'(b') = 0$, and $h'(c') = 1$. By definition, $[a, b, c, d] = h(d)$ and $[a', b', c', d'] = h'(d')$. However, $h' \circ f\colon \Delta \to \mathbb{P}^1_K$ is a projectivity such that $(h' \circ f)(a) = \infty$, $(h' \circ f)(b) = 0$, and $(h' \circ f)(c) = 1$, and by the uniqueness of h, we get $h = h' \circ f$. But then, $[a, b, c, d] = h(d) = h'(f(d)) = h'(d') = [a', b', c', d']$.

Conversely, assume that $[a, b, c, d] = [a', b', c', d']$. Since (a, b, c) and (a', b', c') are projective frames, by Proposition 26.5, there is a unique projectivity $g\colon \Delta \to \Delta'$ such that $g(a) = a'$, $g(b) = b'$, and $g(c) = c'$. Now, $h' \circ g\colon \Delta \to \mathbb{P}^1_K$ is a projectivity such that $(h' \circ g)(a) = \infty$, $(h' \circ g)(b) = 0$, and $(h' \circ g)(c) = 1$, and thus, $h = h' \circ g$. However, $h'(d') = [a', b', c', d'] = [a, b, c, d] = h(d) = h'(g(d))$, and since h' is injective, we get $d' = g(d)$. $\square$

As a corollary of Proposition 26.20, given any three distinct points a, b, c on a projective line Δ, for every $\lambda \in \mathbb{P}^1_K$ there is a unique point $d \in \Delta$ such that $[a, b, c, d] = \lambda$.

In order to compute explicitly the cross-ratio, we show the following easy proposition.

Proposition 26.21. *Given any projective line $\Delta = \mathbf{P}(D)$, for any three distinct points a, b, c in Δ, if $a = p(u)$, $b = p(v)$, and $c = p(u + v)$, where (u, v) is a basis of D, and for any $[\lambda, \mu]_\sim \in \mathbb{P}^1_K$ and any point $d \in \Delta$, we have*

$$d = p(\lambda u + \mu v) \quad \textit{iff} \quad [a, b, c, d] = [\lambda, \mu]_\sim.$$

Proof. If (e_1, e_2) is the basis of K^2 such that $e_1 = (1, 0)$ and $e_2 = (0, 1)$, it is obvious that $p(e_1) = \infty$, $p(e_2) = 0$, and $p(e_1 + e_2) = 1$. Let $f\colon D \to K^2$ be the bijective linear map such that $f(u) = e_1$ and $f(v) = e_2$. Then $f(u + v) = e_1 + e_2$, and thus f induces the unique projectivity $\mathbf{P}(f)\colon \mathbf{P}(D) \to \mathbb{P}^1_K$ such that $\mathbf{P}(f)(a) = \infty$, $\mathbf{P}(f)(b) = 0$, and $\mathbf{P}(f)(c) = 1$. Then

$$\mathbf{P}(f)(p(\lambda u + \mu v)) = [f(\lambda u + \mu v)]_\sim = [\lambda e_1 + \mu e_2]_\sim = [\lambda, \mu]_\sim,$$

that is,

$$d = p(\lambda u + \mu v) \quad \text{iff} \quad [a, b, c, d] = [\lambda, \mu]_\sim,$$

as claimed. $\square$