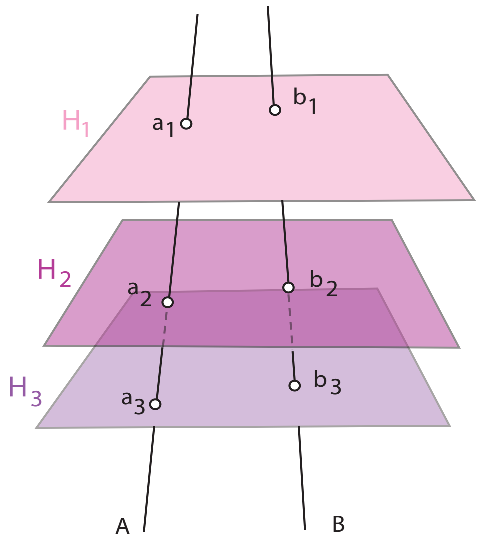

Figure 24.21: The theorem of Thales.

$(u_1, \ldots, u_m)$ has unit volume (see Berger [11], Section 9.12), we see that affine bijections preserve the ratio of volumes of parallelotopes. In fact, this ratio is independent of the choice of the parallelotopes of unit volume. In particular, the affine bijections $f \in \mathbf{GA}(E)$ such that $\det\big(\overrightarrow{f}\big) = 1$ preserve volumes. These affine maps form a subgroup $\mathbf{SA}(E)$ of $\mathbf{GA}(E)$ called the *special affine group of E*. We now take a glimpse at affine geometry.

24.9 Affine Geometry: A Glimpse

In this section we state and prove three fundamental results of affine geometry. Roughly speaking, affine geometry is the study of properties invariant under affine bijections. We now prove one of the oldest and most basic results of affine geometry, the theorem of Thales.

Proposition 24.10. *Given any affine space E, if H_1, H_2, H_3 are any three distinct parallel hyperplanes, and A and B are any two lines not parallel to H_i, letting $a_i = H_i \cap A$ and $b_i = H_i \cap B$, then the following ratios are equal:*

$$\frac{\overrightarrow{a_1a_3}}{\overrightarrow{a_1a_2}} = \frac{\overrightarrow{b_1b_3}}{\overrightarrow{b_1b_2}} = \rho.$$

Conversely, for any point d on the line A, if $\frac{\overrightarrow{a_1d}}{\overrightarrow{a_1a_2}} = \rho$, then $d = a_3$.

Proof. Figure 24.21 illustrates the theorem of Thales. We sketch a proof, leaving the details as an exercise. Since H_1, H_2, H_3 are parallel, they have the same direction $\overrightarrow{H}$, a hyperplane