is positive semidefinite, the pseudo-inverse A^+ of A is given by

$$A^+ = U^\top \begin{pmatrix} \Sigma_r^{-1} & 0 \\ 0 & 0 \end{pmatrix} U,$$

and since

$$f(x) = g(y) = \frac{1}{2} y^\top \Sigma_r y - y^\top c,$$

by Proposition 42.4 the minimum of g is achieved iff $y^* = \Sigma_r^{-1} c$. Since $f(x)$ is independent of z, we can choose $z = 0$, and since $d = 0$, for x^* given by

$$Ux^* = \begin{pmatrix} \Sigma_r^{-1} c \\ 0 \end{pmatrix} \quad \text{and} \quad Ub = \begin{pmatrix} c \\ 0 \end{pmatrix},$$

we deduce that

$$x^* = U^\top \begin{pmatrix} \Sigma_r^{-1} c \\ 0 \end{pmatrix} = U^\top \begin{pmatrix} \Sigma_r^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c \\ 0 \end{pmatrix} = U^\top \begin{pmatrix} \Sigma_r^{-1} & 0 \\ 0 & 0 \end{pmatrix} Ub = A^+ b, \tag{$*$}$$

and the minimum value of f is

$$f(x^*) = \frac{1}{2}(A^+ b)^\top A A^+ b - b^\top A^+ b = \frac{1}{2} b^\top A^+ A A^+ b - b^\top A^+ b = -\frac{1}{2} b^\top A^+ b,$$

since A^+ is symmetric and $A^+ A A^+ = A^+$. For any $x \in \mathbb{R}^n$ of the form

$$x = A^+ b + U^\top \begin{pmatrix} 0 \\ z \end{pmatrix}, \quad z \in \mathbb{R}^{n-r},$$

since

$$x = A^+ b + U^\top \begin{pmatrix} 0 \\ z \end{pmatrix} = U^\top \begin{pmatrix} \Sigma_r^{-1} c \\ 0 \end{pmatrix} + U^\top \begin{pmatrix} 0 \\ z \end{pmatrix} = U^\top \begin{pmatrix} \Sigma_r^{-1} c \\ z \end{pmatrix},$$

and since $f(x)$ is independent of z (because $f(x) = g(y)$), we have

$$f(x) = f(x^*) = -\frac{1}{2} b^\top A^+ b. \qquad \square$$

The problem of minimizing the function

$$f(x) = \frac{1}{2} x^\top A x - x^\top b$$

in the case where we add either linear constraints of the form $C^\top x = 0$ or affine constraints of the form $C^\top x = t$ (where $t \in \mathbb{R}^m$ and $t \neq 0$) where C is an $n \times m$ matrix can be reduced to the unconstrained case using a QR-decomposition of C. Let us show how to do this for linear constraints of the form $C^\top x = 0$.