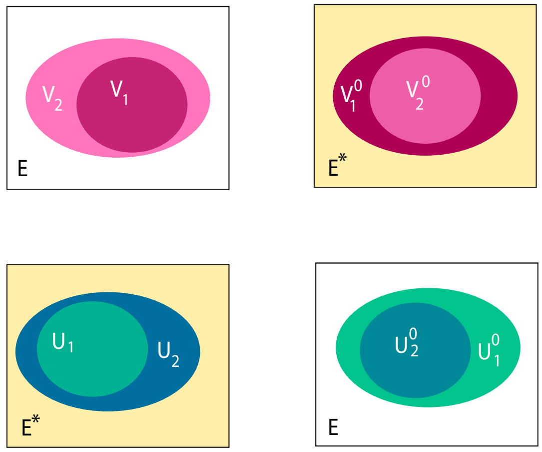


Figure 11.2: The top pair of figures schematically illustrates the relation if $V_1 \subseteq V_2 \subseteq E$, then $V_2^0 \subseteq V_1^0 \subseteq E^*$, while the bottom pair of figures illustrates the relationship if $U_1 \subseteq U_2 \subseteq E^*$, then $U_2^0 \subseteq U_1^0 \subseteq E$.

**Example 11.2.** Let $E = \mathrm{M}_2(\mathbb{R})$, the space of real $2 \times 2$ matrices, and let $V$ be the subspace of $\mathrm{M}_2(\mathbb{R})$ spanned by the matrices

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

We check immediately that the subspace $V$ consists of all matrices of the form

$$\begin{pmatrix} b & a \\ a & c \end{pmatrix},$$

that is, all symmetric matrices. The matrices

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

in $V$ satisfy the equation

$$a_{12} - a_{21} = 0,$$

and all scalar multiples of these equations, so $V^0$ is the subspace of $E^*$ spanned by the linear form given by $u^*(a_{11}, a_{12}, a_{21}, a_{22}) = a_{12} - a_{21}$. By the duality theorem (Theorem 11.4) we have

$$\dim(V^0) = \dim(E) - \dim(V) = 4 - 3 = 1.$$

**Example 11.3.** The above example generalizes to $E = \mathrm{M}_n(\mathbb{R})$ for any $n \geq 1$, but this time, consider the space $U$ of linear forms asserting that a matrix $A$ is symmetric; these are the linear forms spanned by the $n(n-1)/2$ equations

$$a_{ij} - a_{ji} = 0, \quad 1 \leq i < j \leq n;$$