(2) The formula of Taylor–Maclaurin shows that for all $u + w \in B$, we have

$$J(u + w) = J(u) + \frac{1}{2}\mathrm{D}^2 J(v)(w, w) \geq J(u),$$

for some $v \in (u, u+w)$ (recall that $(u, u+w) = \{(1-\lambda)(u+w)+\lambda(u+w) \mid 0 < \lambda < 1\}$). $\square$

There are no converses of the two assertions of Theorem 40.6. However, there is a condition on $\mathrm{D}^2 J(u)$ that implies the condition of Part (1). Since this condition is easier to state when $E = \mathbb{R}^n$, we begin with this case.

Recall that a $n \times n$ symmetric matrix A is *positive definite* if $x^\top Ax > 0$ for all $x \in \mathbb{R}^n - \{0\}$. In particular, A must be invertible.

Proposition 40.7. *For any symmetric matrix A, if A is positive definite, then there is some $\alpha > 0$ such that*

$$x^\top Ax \geq \alpha \left\|x\right\|^2 \quad \textit{for all } x \in \mathbb{R}^n.$$

Proof. Pick any norm in $\mathbb{R}^n$ (recall that all norms on $\mathbb{R}^n$ are equivalent). Since the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n \mid \left\|x\right\| = 1\}$ is compact and since the function $f(x) = x^\top Ax$ is never zero on S^{n-1}, the function f has a minimum $\alpha > 0$ on S^{n-1}. Using the usual trick that $x = \left\|x\right\| (x/\left\|x\right\|)$ for every nonzero vector $x \in \mathbb{R}^n$ and the fact that the inequality of the proposition is trivial for $x = 0$, from

$$x^\top Ax \geq \alpha \quad \text{for all } x \text{ with } \left\|x\right\| = 1,$$

we get

$$x^\top Ax \geq \alpha \left\|x\right\|^2 \quad \text{for all } x \in \mathbb{R}^n,$$

as claimed. $\square$

We can combine Theorem 40.6 and Proposition 40.7 to obtain a useful sufficient condition for the existence of a strict local minimum. First let us introduce some terminology.

Definition 40.6. Given a function $J \colon \Omega \to \mathbb{R}$ as before, say that a point $u \in \Omega$ is a *nondegenerate critical point* if $dJ(u) = 0$ and if the Hessian matrix $\nabla^2 J(u)$ is invertible.

Proposition 40.8. *Let $J \colon \Omega \to \mathbb{R}$ be a function defined on some open subset $\Omega \subseteq \mathbb{R}^n$. If J is differentiable in Ω and if some point $u \in \Omega$ is a nondegenerate critical point such that $\nabla^2 J(u)$ is positive definite, then J has a strict local minimum at u.*

Remark: It is possible to generalize Proposition 40.8 to infinite-dimensional spaces by finding a suitable generalization of the notion of a nondegenerate critical point. Firstly, we assume that E is a Banach space (a complete normed vector space). Then we define the dual E' of E as the set of continuous linear forms on E, so that $E' = \mathcal{L}(E; \mathbb{R})$. Following Lang, we use the notation E' for the space of continuous linear forms to avoid confusion