The steepest edge rule is one of the most popular. The idea is to maximize the ratio

$$\frac{c(u^+ - u)}{\|u^+ - u\|}.$$

The random edge rule picks the index $j^+ \notin K$ of the entering basis vector uniformly at random among all eligible indices.

Let us now return to the issue of the initialization of the simplex algorithm. We use the Linear Program $(\widehat{P})$ introduced during the proof of Theorem 45.7.

Consider a Linear Program $(P2)$

$$\begin{aligned} &\text{maximize} \quad cx \\ &\text{subject to} \quad Ax = b \text{ and } x \geq 0, \end{aligned}$$

in standard form where A is an $m \times n$ matrix of rank m.

First, observe that since the constraints are equations, we can ensure that $b \geq 0$, because every equation $a_i x = b_i$ where $b_i < 0$ can be replaced by $-a_i x = -b_i$. The next step is to introduce the Linear Program $(\widehat{P})$ in standard form

$$\begin{aligned} &\text{maximize} \quad -(x_{n+1} + \cdots + x_{n+m}) \\ &\text{subject to} \quad \widehat{A}\,\widehat{x} = b \text{ and } \widehat{x} \geq 0, \end{aligned}$$

where $\widehat{A}$ and $\widehat{x}$ are given by

$$\widehat{A} = \begin{pmatrix} A & I_m \end{pmatrix}, \quad \widehat{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_{n+m} \end{pmatrix}.$$

Since we assumed that $b \geq 0$, the vector $\widehat{x} = (0_n, b)$ is a feasible solution of $(\widehat{P})$, in fact a basic feasible solutions since the matrix associated with the indices $n+1, \ldots, n+m$ is the identity matrix I_m. Furthermore, since $x_i \geq 0$ for all i, the objective function $-(x_{n+1} + \cdots + x_{n+m})$ is bounded above by 0.

If we execute the simplex algorithm with a pivot rule that prevents cycling, starting with the basic feasible solution $(0_n, d)$, since the objective function is bounded by 0, the simplex algorithm terminates with an optimal solution given by some basic feasible solution, say (u^*, w^*), with $u^* \in \mathbb{R}^n$ and $w^* \in \mathbb{R}^m$.

As in the proof of Theorem 45.7, for every feasible solution $u \in \mathcal{P}(A, b)$, the vector $(u, 0_m)$ is an optimal solution of $(\widehat{P})$. Therefore, if $w^* \neq 0$, then $\mathcal{P}(A, b) = \emptyset$, since otherwise for every feasible solution $u \in \mathcal{P}(A, b)$ the vector $(u, 0_m)$ would yield a value of the objective function $-(x_{n+1} + \cdots + x_{n+m})$ equal to 0, but (u^*, w^*) yields a strictly negative value since $w^* \neq 0$.