Theorem 29.48. *(Witt–Sharpened Version) Let E be a finite-dimensional space equipped with a nondegenerate symmetric bilinear forms φ. For any subspace U of E, every linear injective metric map f from U into E extends to an isometry g of E with a prescribed value ± 1 of $\det(g)$ iff*

$$\dim(U) + \dim(\mathrm{rad}(U)) < \dim(E) = n.$$

If

$$\dim(U) + \dim(\mathrm{rad}(U)) = \dim(E) = n,$$

and $\det(f) = -1$, then there is no $g \in \mathbf{SO}(\varphi)$ extending f.

Proof. If g_1 and g_2 are two extensions of f such that $\det(g_1)\det(g_2) = -1$, then $h = g_1^{-1} \circ g_2$ is an isometry such that $\det(h) = -1$, and h leaves every vector of U fixed. Conversely, if h is an isometry such that $\det(h) = -1$, and $h(u) = u$ for all $u \in U$, then for any extesnion g_1 of f, the map $g_2 = h \circ g_1$ is another extension of f such that $\det(g_2) = -\det(g_1)$. Therefore, we need to show that a map h as above exists.

If $\dim(U) + \dim(\mathrm{rad}(U)) < \dim(E)$, consider the nondegenerate completion $\overline{U}$ of U given by Proposition 29.32. We know that $\dim(\overline{U}) = \dim(U) + \dim(\mathrm{rad}(U)) < n$, and since $\overline{U}$ is nondegenerate, we have

$$E = \overline{U} \overset{\perp}{\oplus} \overline{U}^{\perp},$$

with $\overline{U}^{\perp} \neq (0)$. Pick any isometry τ of $\overline{U}^{\perp}$ such that $\det(\tau) = -1$, and extend it to an isometry h of E whose restriction to $\overline{U}$ is the identity.

If $\dim(U) + \dim(\mathrm{rad}(U)) = \dim(E) = n$, then $U = V \overset{\perp}{\oplus} W$ with $V = \mathrm{rad}(U)$ and since $\dim(\overline{U}) = \dim(U) + \dim(\mathrm{rad}(U)) = n$, we have

$$E = \overline{U} = (V \oplus V') \overset{\perp}{\oplus} W,$$

where $V \oplus V' = \mathrm{Ar}_{2r} = W^{\perp}$ is an Artinian space. Any isometry h of E which is the identity on U and with $\det(h) = -1$ is the identity on W, and thus it must map $W^{\perp} = \mathrm{Ar}_{2r} = V \oplus V'$ into itself, and the restriction h' of h to Ar_{2r} has $\det(h') = -1$. However, h' is the identity on $V = \mathrm{rad}(U)$, a totally isotropic subspace of Ar_{2r} of dimension r, and by Proposition 29.42, we have $\det(h') = +1$, a contradiction. $\square$

It can be shown that the center of $\mathbf{O}(\varphi)$ is $\{\mathrm{id}, -\mathrm{id}\}$. For further properties of orthogonal groups, see Grove [83], Jacobson [98], Taylor [174], and Artin [6].