Example 22.3. The matrix
$$A = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$$
of Example 22.1 has positive eigenvalues $(1, 1)$, but its symmetric part
$$H(A) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$
is *not* positive definite, since its eigenvalues are $-1, 3$.

Beware that if A is a complex skew-Hermitian matrix, which means that $A^* = -A$, then
$$(u^* A u)^* = -u^* A u,$$
but this only implies that *the real part of* $u^* A u$ *is zero*. So for any arbitrary complex square matrix A, in general,
$$u^* A u \neq u^* H(A) u,$$
where $H(A) = (1/2)(A + A^*)$.

If $f \colon E \to F$ is any linear map, we just showed that $f^* \circ f$ and $f \circ f^*$ are positive semidefinite self-adjoint linear maps. This fact has the remarkable consequence that every linear map has two important decompositions:

1. The polar form.

2. The singular value decomposition (SVD).

The wonderful thing about the singular value decomposition is that there exist two orthonormal bases $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_m)$ such that, with respect to these bases, f is a diagonal matrix consisting of the singular values of f or 0. Thus, in some sense, f can always be diagonalized with respect to *two* orthonormal bases. The SVD is also a useful tool for solving overdetermined linear systems in the least squares sense and for data analysis, as we show later on.

First we show some useful relationships between the kernels and the images of f, f^*, $f^* \circ f$, and $f \circ f^*$. Recall that if $f \colon E \to F$ is a linear map, the *image* $\mathrm{Im}\, f$ *of* f is the subspace $f(E)$ of F, and the *rank of* f is the dimension $\dim(\mathrm{Im}\, f)$ of its image. Also recall that (Theorem 6.16)
$$\dim(\mathrm{Ker}\, f) + \dim(\mathrm{Im}\, f) = \dim(E),$$
and that (Propositions 12.11 and 14.13) for every subspace W of E,
$$\dim(W) + \dim(W^{\perp}) = \dim(E).$$