Chapter 31

Annihilating Polynomials and the Primary Decomposition

In this chapter all vector spaces are defined over an arbitrary field K.

In Section 7.7 we explained that if $f\colon E \to E$ is a linear map on a K-vector space E, then for any polynomial $p(X) = a_0X^d + a_1X^{d-1} + \cdots + a_d$ with coefficients in the field K, we can define the *linear map* $p(f)\colon E \to E$ by

$$p(f) = a_0f^d + a_1f^{d-1} + \cdots + a_d\mathrm{id},$$

where $f^k = f \circ \cdots \circ f$, the k-fold composition of f with itself. Note that

$$p(f)(u) = a_0f^d(u) + a_1f^{d-1}(u) + \cdots + a_du,$$

for every vector $u \in E$. Then we showed that if E is finite-dimensional and if $\chi_f(X) = \det(XI - f)$ is the characteristic polynomial of f, by the Cayley–Hamilton theorem, we have

$$\chi_f(f) = 0.$$

This fact suggests looking at the set of all polynomials $p(X)$ such that

$$p(f) = 0.$$

Such polynomials are called *annihilating polynomials* of f, the set of all these polynomials, denoted $\mathrm{Ann}(f)$, is called the *annihilator* of f, and the Cayley-Hamilton theorem shows that it is nontrivial since it contains a polynomial of positive degree. It turns out that $\mathrm{Ann}(f)$ contains a polynomial m_f of smallest degree that generates $\mathrm{Ann}(f)$, and this polynomial divides the characteristic polynomial. Furthermore, the polynomial m_f encapsulates a lot of information about f, in particular whether f can be diagonalized. One of the main reasons for this is that a scalar $\lambda \in K$ is a zero of the minimal polynomial m_f if and only if λ is an eigenvalue of f.