and

$$\begin{aligned} **(e_{i_1} \wedge \cdots \wedge e_{i_k}) &= \mathrm{sign}(i_1, \ldots i_k, j_1, \ldots, j_{n-k})\ *(e_{j_1} \wedge \cdots \wedge e_{j_{n-k}}) \\ &= \mathrm{sign}(i_1, \ldots i_k, j_1, \ldots, j_{n-k})\, \mathrm{sign}(j_1, \ldots, j_{n-k}, i_1, \ldots i_k)\, e_{i_1} \wedge \cdots \wedge e_{i_k}. \end{aligned}$$

It is easy to see that

$$\mathrm{sign}(i_1, \ldots i_k, j_1, \ldots, j_{n-k})\, \mathrm{sign}(j_1, \ldots, j_{n-k}, i_1, \ldots i_k) = (-1)^{k(n-k)},$$

which yields

$$**(e_{i_1} \wedge \cdots \wedge e_{i_k}) = (-1)^{k(n-k)}\, e_{i_1} \wedge \cdots \wedge e_{i_k},$$

as claimed.

(ii) These identities are easily checked on basis elements; see Jost [101], Chapter 2, Lemma 2.1.1. In particular let

$$x = e_{i_1} \wedge \cdots \wedge e_{i_k}, \qquad y = e_{i_j} \wedge \cdots \wedge e_{i_j}, \qquad x, y \in \bigwedge^k V,$$

where $(e_i)_{i=1}^n$ is an orthonormal basis of V. If $x \neq y$, $\langle x, y\rangle_\wedge = 0$ since there is some e_{i_p} of x not equal to any e_{j_q} of y by the orthonormality of the basis, this means the p^{th} row of $(\langle e_{i_l}, e_{j_s}\rangle)$ consists entirely of zeroes. Also $x \neq y$ implies that $y \wedge *x = 0$ since

$$*x = \mathrm{sign}(i_1, \ldots i_k, l_1, \ldots, l_{n-k}) e_{l_1} \wedge \cdots \wedge e_{l_{n-k}},$$

where e_{l_s} is the same as some e_p in y. A similar argument shows that if $x \neq y$, $x \wedge *y = 0$. So now assume $x = y$. Then

$$\begin{aligned} *(e_{i_1} \wedge \cdots \wedge e_{i_k} \wedge *(e_{i_1} \wedge \cdots \wedge e_{i_k})) &= *(e_1 \wedge e_2 \cdots \wedge e_n) \\ &= 1 = \langle x, x\rangle_\wedge. \end{aligned}$$

□

It is possible to express $*(1)$ in terms of any basis (not necessarily orthonormal) of V.

Proposition 34.17. *If V is any finite-dimensional oriented vector space, for any basis $(v_!, \ldots, v_n)$ of V, we have*

$$*(1) = \frac{1}{\sqrt{\det(\langle v_i, v_j\rangle)}}\, v_1 \wedge \cdots \wedge v_n.$$

Proof. If $(e_1, \ldots, e_n)$ is an orthonormal basis of V and $(v_1, \ldots, v_n)$ is any other basis of V, then

$$\langle v_1 \wedge \cdots \wedge v_n, v_1 \wedge \cdots \wedge v_n\rangle_\wedge = \det(\langle v_i, v_j\rangle),$$

and since

$$v_1 \wedge \cdots \wedge v_n = \det(A)\, e_1 \wedge \cdots \wedge e_n$$