It is important to note that when *both* the constraints, the domain of definition $\Omega$, *and* the objective function $J$ are *convex*, if the KKT conditions hold for some $u \in U$ and some $\lambda \in \mathbb{R}^m_+$, then Theorem 50.6 implies that $J$ has a (global) minimum at $u$ with respect to $U$, *independently* of any assumption on the qualification of the constraints.

The above theorem suggests introducing the function $L\colon \Omega \times \mathbb{R}^m_+ \to \mathbb{R}$ given by

$$L(v, \lambda) = J(v) + \sum_{i=1}^{m} \lambda_i \varphi_i(v),$$

with $\lambda = (\lambda_1, \ldots, \lambda_m)$. The function $L$ is called the *Lagrangian* of the *Minimization Problem* $(P)$:

$$\begin{aligned} &\text{minimize} && J(v) \\ &\text{subject to} && \varphi_i(v) \leq 0, \quad i = 1, \ldots, m. \end{aligned}$$

The KKT conditions of Theorem 50.6 imply that for any $u \in U$, if the vector $\lambda = (\lambda_1, \ldots, \lambda_m)$ is known and if $u$ is a minimum of $J$ on $U$, then

$$\begin{aligned} \frac{\partial L}{\partial u}(u) &= 0 \\ J(u) &= L(u, \lambda). \end{aligned}$$

The Lagrangian technique "absorbs" the constraints into the new objective function $L$ and reduces the problem of finding a constrained minimum of the function $J$, to the problem of finding an unconstrained minimum of the function $L(v, \lambda)$. This is the main point of Lagrangian duality which will be treated in the next section.

A case that arises often in practice is the case where the constraints $\varphi_i$ are affine. If so, the $m$ constraints $a_i x \leq b_i$ can be expressed in matrix form as $Ax \leq b$, where $A$ is an $m \times n$ matrix whose $i$th row is the row vector $a_i$. The KKT conditions of Theorem 50.6 yield the following corollary.

**Proposition 50.7.** *If $U$ is given by*

$$U = \{x \in \Omega \mid Ax \leq b\},$$

*where $\Omega$ is an open convex subset of $\mathbb{R}^n$ and $A$ is an $m \times n$ matrix, and if $J$ is differentiable at $u$ and $J$ has a local minimum at $u$, then there exist some vector $\lambda \in \mathbb{R}^m$, such that*

$$\begin{aligned} &\nabla J_u + A^\top \lambda = 0 \\ &\lambda_i \geq 0 \;\; \text{and} \;\; \text{if } a_i u < b_i, \text{ then } \lambda_i = 0, \; i = 1, \ldots, m. \end{aligned}$$

*If the function $J$ is convex, then the above conditions are also sufficient for $J$ to have a minimum at $u \in U$.*