Use the above facts to prove that

$$t^i = \sum_{j=0}^{m-i} \frac{\binom{i+j}{i}}{\binom{m}{i}} B^m_{i+j}(t).$$

Conclude that the Bernstein polynomials $B^m_0(t), \ldots, B^m_m(t)$ form a basis of the vector space of polynomials of degree $\leq m$.

Compute the matrix expressing $1, t, t^2$ in terms of $B^2_0(t), B^2_1(t), B^2_2(t)$, and the matrix expressing $1, t, t^2, t^3$ in terms of $B^3_0(t), B^3_1(t), B^3_2(t), B^3_3(t)$.

You should find

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1/2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1/3 & 2/3 & 1 \\ 0 & 0 & 1/3 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(5) A *polynomial curve $C(t)$ of degree m* in the plane is the set of points $C(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ given by two polynomials of degree $\leq m$,

$$\begin{aligned} x(t) &= \alpha_0 t^{m_1} + \alpha_1 t^{m_1-1} + \cdots + \alpha_{m_1} \\ y(t) &= \beta_0 t^{m_2} + \beta_1 t^{m_2-1} + \cdots + \beta_{m_2}, \end{aligned}$$

with $1 \leq m_1, m_2 \leq m$ and $\alpha_0, \beta_0 \neq 0$.

Prove that there exist $m+1$ points $b_0, \ldots, b_m \in \mathbb{R}^2$ so that

$$C(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = B^m_0(t) b_0 + B^m_1(t) b_1 + \cdots + B^m_m(t) b_m$$

for all $t \in \mathbb{R}$, with $C(0) = b_0$ and $C(1) = b_m$. Are the points $b_1, \ldots, b_{m-1}$ generally on the curve?

We say that the curve C is a *Bézier curve* and $(b_0, \ldots, b_m)$ is the list of *control points* of the curve (control points need not be distinct).

Remark: Because $B^m_0(t) + \cdots + B^m_m(t) = 1$ and $B^m_i(t) \geq 0$ when $t \in [0, 1]$, the curve segment $C[0, 1]$ corresponding to $t \in [0, 1]$ belongs to the convex hull of the control points. This is an important property of Bézier curves which is used in geometric modeling to find the intersection of curve segments. Bézier curves play an important role in computer graphics and geometric modeling, but also in robotics because they can be used to model the trajectories of moving objects.