Theorem 16.4. *Let $r\colon \mathbf{SU}(2) \to \mathbf{SO}(3)$ be the homomorphism of Definition 16.5. For every unit quaternion*

$$q = \begin{pmatrix} a+ib & c+id \\ -(c-id) & a-ib \end{pmatrix},$$

we have $r_q = I_3$ iff $u = (b,c,d) = 0$ iff $|a| = 1$. If $u \neq 0$, then either $a = 0$ and r_q is a rotation by π around the axis of rotation determined by the vector $u = (b,c,d)$, or $0 < |a| < 1$ and r_q is the rotation around the axis of rotation determined by the vector $u = (b,c,d)$ and the angle of rotation $\theta \neq \pi$ with $0 < \theta < 2\pi$, is given by

$$\tan(\theta/2) = \frac{\|u\|}{a}.$$

Here we are assuming that a basis (w_1, w_2) has been chosen in the plane orthogonal to $u = (b,c,d)$ such that (w_1, w_2, u) is positively oriented, that is, $\det(w_1, w_2, u) > 0$ (where w_1, w_2, u are expressed over the canonical basis (e_1, e_2, e_3), which is chosen to define positive orientation).

Remark: Under the orientation defined above, we have

$$\cos(\theta/2) = a, \quad 0 < \theta < 2\pi.$$

Note that the condition $0 < \theta < 2\pi$ implies that θ is uniquely determined by the above equation. This is not the case if we choose π such that $-\pi < \theta < \pi$ since both θ and $-\theta$ satisfy the equation, and this shows why the condition $0 < \theta < 2\pi$ is preferable. If $0 < a < 1$, then $0 < \theta < \pi$, and if $-1 < a < 0$, then $\pi < \theta < 2\pi$. In the second case, r_q is also the rotation of axis $-u$ and of angle $-(2\pi - \theta) = \theta - 2\pi$ with $0 < 2\pi - \theta < \pi$, but this time the orientation of the plane orthogonal to $-u = (b,c,d)$ is the opposite orientation from before. This orientation is given by (w_2, w_1), so that $(w_2, w_1, -u)$ has positive orientation. Since the quaternions q and $-q$ define the same rotation, we may assume that $a > 0$, in which case $0 < \theta < \pi$, but we have to remember that if $a < 0$ and if we pick $-q$ instead of q, the vector defining the axis of rotation becomes $-u$, which amounts to flipping the orientation of the plane orthogonal to the axis of rotation.

The map r is surjective, but this is not obvious. We will return to this point after finding the matrix representing r_q explicitly.

16.3 Matrix Representation of the Rotation r_q

Given a unit quaternion q of the form

$$q = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$$