which in turn implies that

$$u^\flat = \omega_1 e_1^* + \omega_2 e_2^* = u^\flat(e_1)e_1^* + u^\flat(e_2)e_2^* = (u^1 + 2u^2)e_1^* + (2u^1 + 5u^2)e_2^*.$$

Given $\omega = \omega_1 e_1^* + \omega_2 e_2^*$, we calculate $\omega^\sharp = (\omega^\sharp)^1 e_1 + (\omega^\sharp)^2 e_2$ from the following two linear equalities:

$$\begin{aligned}
\omega_1 = \omega(e_1) &= \langle \omega^\sharp, e_1\rangle = \langle (\omega^\sharp)^1 e_1 + (\omega^\sharp)^2 e_2, e_1\rangle \\
&= \langle e_1, e_1\rangle(\omega^\sharp)^1 + \langle e_2, e_1\rangle(\omega^\sharp)^2 = (\omega^\sharp)^1 + 2(\omega^\sharp)^2 = g_{11}(\omega^\sharp)^1 + g_{12}(\omega^\sharp)^2 \\
\omega_2 = \omega(e_2) &= \langle \omega^\sharp, e_2\rangle = \langle (\omega^\sharp)^1 e_1 + (\omega^\sharp)^2 e_2, e_2\rangle \\
&= \langle e_1, e_2\rangle(\omega^\sharp)^1 + \langle e_2, e_2\rangle(\omega^\sharp)^2 = 2(\omega^\sharp)^1 + 5(\omega^\sharp)^2 = g_{21}(\omega^\sharp)^1 + g_{22}(\omega^\sharp)^2.
\end{aligned}$$

These equalities are concisely written as

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} (\omega^\sharp)^1 \\ (\omega^\sharp)^2 \end{pmatrix} = g \begin{pmatrix} (\omega^\sharp)^1 \\ (\omega^\sharp)^2 \end{pmatrix}.$$

Then

$$\begin{pmatrix} (\omega^\sharp)^1 \\ (\omega^\sharp)^2 \end{pmatrix} = g^{-1} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix},$$

which in turn implies

$$(\omega^\sharp)^1 = 5\omega_1 - 2\omega_2, \qquad (\omega^\sharp)^2 = -2\omega_1 + \omega_2,$$

i.e.

$$\omega^\sharp = (5\omega_1 - 2\omega_2)e_1 + (-2\omega_1 + \omega_2)e_2.$$

The inner product $\langle -, - \rangle$ on E induces an inner product on E^* denoted $\langle -, - \rangle_{E^*}$, and given by

$$\langle \omega_1, \omega_2 \rangle_{E^*} = \langle \omega_1^\sharp, \omega_2^\sharp \rangle, \quad \text{for all } \omega_1, \omega_2 \in E^*.$$

Then we have

$$\langle u^\flat, v^\flat \rangle_{E^*} = \langle (u^\flat)^\sharp, (v^\flat)^\sharp \rangle = \langle u, v \rangle \quad \text{for all} \quad u, v \in E.$$

If $(e_1, \ldots, e_n)$ is a basis of E and $g_{ij} = \langle e_i, e_j \rangle$, as

$$(e_i^*)^\sharp = \sum_{k=1}^n g^{ik} e_k,$$

an easy computation shows that

$$\langle e_i^*, e_j^* \rangle_{E^*} = \langle (e_i^*)^\sharp, (e_j^*)^\sharp \rangle = g^{ij};$$