Definition 37.27. A topological space, E, is *arcwise connected* if for any two points, $a, b \in E$, there is an arc, $\gamma\colon [0,1] \to E$, joining a and b, i.e., such that $\gamma(0) = a$ and $\gamma(1) = b$. A topological space, E, is *locally arcwise connected* if for every $a \in E$, for every neighborhood, V, of a, there is an arcwise connected neighborhood, U, of a such that $U \subseteq V$. See Figure 37.26.

The space $\mathbb{R}^n$ is locally arcwise connected, since for any open ball, any two points in this ball are joined by a line segment. Manifolds and surfaces are also locally arcwise connected. Proposition 37.18 also applies to arcwise connectedness (this is a simple exercise). The following theorem is crucial to the theory of manifolds and surfaces:

Theorem 37.23. *If a topological space, E, is arcwise connected, then it is connected. If a topological space, E, is connected and locally arcwise connected, then E is arcwise connected.*

Proof. First, assume that E is arcwise connected. Pick any point, a, in E. Since E is arcwise connected, for every $b \in E$, there is a path, $\gamma_b\colon [0,1] \to E$, from a to b and so,

$$E = \bigcup_{b \in E} \gamma_b([0,1])$$

a union of connected subsets all containing a. By Lemma 37.19, E is connected.

Now assume that E is connected and locally arcwise connected. For any point $a \in E$, let F_a be the set of all points, b, such that there is an arc, $\gamma_b\colon [0,1] \to E$, from a to b. Clearly, F_a contains a. We show that F_a is both open and closed. For any $b \in F_a$, since E is locally arcwise connected, there is an arcwise connected neighborhood U containing b (because E is a neighborhood of b). Thus, b can be joined to every point $c \in U$ by an arc, and since by the definition of F_a, there is an arc from a to b, the composition of these two arcs yields an arc from a to c, which shows that $c \in F_a$. But then $U \subseteq F_a$ and thus, F_a is open. See Figure 37.27 (*i.*). Now assume that b is in the complement of F_a. As in the previous case, there is some arcwise connected neighborhood U containing b. Thus, every point $c \in U$ can be joined to b by an arc. If there was an arc joining a to c, we would get an arc from a to b, contradicting the fact that b is in the complement of F_a. Thus, every point $c \in U$ is in the complement of F_a, which shows that U is contained in the complement of F_a, and thus, that the the complement of F_a is open. See Figure 37.27 (*ii.*). Consequently, we have shown that F_a is both open and closed and since it is nonempty, we must have $E = F_a$, which shows that E is arcwise connected. $\square$

If E is locally arcwise connected, the above argument shows that the connected components of E are arcwise connected.

It is not true that a connected space is arcwise connected. For example, the space consisting of the graph of the function

$$f(x) = \sin(1/x),$$

where $x > 0$, together with the portion of the y-axis, for which $-1 \leq y \leq 1$, is connected, but not arcwise connected. See Figure 37.25.