Remark: We already proved in Proposition 29.13 that if U is finite-dimensional, then $\mathrm{codim}(U^{\perp}) = \dim(U)$ and $U^{\perp\perp} = U$, but it doesn't hurt to give another proof. Observe that (i) implies that

$$\dim(U) + \dim(\mathrm{rad}(U)) \leq \dim(E).$$

We can now proceed with the Witt decomposition, but before that, we quickly take care of the structure theorem for alternating bilinear forms (the case where $\varphi(u, u) = 0$ for all $u \in E$). For an alternating bilinear form, the space E is totally isotropic. For example in dimension 2, the matrix

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

defines the alternating form given by

$$\varphi((x_1, y_1), (x_2, y_2)) = x_1 y_2 - x_2 y_1.$$

This case is surprisingly general.

Proposition 29.23. *Let $\varphi\colon E \times E \to K$ be an alternating bilinear form on E. If $u, v \in E$ are two (nonzero) vectors such that $\varphi(u, v) = \lambda \neq 0$, then u and v are linearly independent. If we let $u_1 = \lambda^{-1} u$ and $v_1 = v$, then $\varphi(u_1, v_1) = 1$, and the restriction of φ to the plane spanned by u_1 and v_1 is represented by the matrix*

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Proof. If u and v were linearly dependent, as $u, v \neq 0$, we could write $v = \mu u$ for some $\mu \neq 0$, but then, since φ is alternating, we would have

$$\lambda = \varphi(u, v) = \varphi(u, \mu u) = \mu \varphi(u, u) = 0,$$

contradicting the fact that $\lambda \neq 0$. The rest is obvious. $\square$

Proposition 29.23 yields a plane spanned by two vectors u_1, v_1 such that $\varphi(u_1, u_1) = \varphi(v_1, v_1) = 0$ and $\varphi(u_1, v_1) = 1$. Such a plane is called a *hyperbolic plane*. If E is finite-dimensional, we obtain the following theorem.

Theorem 29.24. *Let $\varphi\colon E \times E \to K$ be an alternating bilinear form on a space E of finite dimension n. Then, there is a direct sum decomposition of E into pairwise orthogonal subspaces*

$$E = W_1 \oplus \cdots \oplus W_r \oplus \mathrm{rad}(E),$$

where each W_i is a hyperbolic plane and $\mathrm{rad}(E) = E^{\perp}$. *Therefore, there is a basis of E of the form*

$$(u_1, v_1, \ldots, u_r, v_r, w_1, \ldots, w_{n-2r}),$$