As a consequence of $(*_{11})$, we obtain

$$\begin{aligned} L(u_\lambda, \mu) &= J(u_\lambda) + \sum_{i=1}^m \mu_i \varphi_i(u_\lambda) \\ &\leq J(u_\lambda) + \sum_{i=1}^m \lambda_i \varphi_i(u_\lambda) = L(u_\lambda, \lambda), \end{aligned}$$

for all $\mu \in \mathbb{R}^m_+$, that is,

$$L(u_\lambda, \mu) \leq L(u_\lambda, \lambda), \quad \text{for all } \mu \in \mathbb{R}^m_+, \tag{$*_{12}$}$$

which implies the second inequality

$$\sup_{\mu \in \mathbb{R}^m_+} L(u_\mu, \mu) = L(u_\lambda, \lambda)$$

stating that (u_λ, λ) is a saddle point. Therefore, (u_λ, λ) is a saddle point of L, as claimed.

(2) The hypotheses are exactly those required by Theorem 50.15(2), thus there is some $\lambda \in \mathbb{R}^m_+$ such that (u, λ) is a saddle point of the Lagrangian L, and by Theorem 50.15(1) we have $J(u) = L(u, \lambda)$. By Proposition 50.14, we have

$$J(u) = L(u, \lambda) = \inf_{v \in \Omega} L(v, \lambda) = \sup_{\mu \in \mathbb{R}^m_+} \inf_{v \in \Omega} L(v, \mu),$$

which can be rewritten as

$$J(u) = G(\lambda) = \sup_{\mu \in \mathbb{R}^m_+} G(\mu).$$

In other words, Problem (D) has a solution, and $J(u) = G(\lambda)$. □

Remark: Note that Theorem 50.17(2) could have already be obtained as a consequence of Theorem 50.15(2), but the dual function G was not yet defined. If (u, λ) is a saddle point of the Lagrangian L (defined on $\Omega \times \mathbb{R}^m_+$), then by Proposition 50.14, the vector λ is a solution of Problem (D). Conversely, under the hypotheses of Part (1) of Theorem 50.17, if λ is a solution of Problem (D), then (u_λ, λ) is a saddle point of L. *Consequently, under the above hypotheses, the set of solutions of the Dual Problem (D) coincide with the set of second arguments λ of the saddle points (u, λ) of L.* In some sense, this result is the "dual" of the result stated in Theorem 50.15, namely that the set of solutions of Problem (P) coincides with set of first arguments u of the saddle points (u, λ) of L.

Informally, in Theorem 50.17(1), the hypotheses say that if $G(\mu)$ can be "computed nicely," in the sense that there is a unique minimizer u_μ of $L(v, \mu)$ (with $v \in \Omega$) such that $G(\mu) = L(u_\mu, \mu)$, and if a maximizer λ of $G(\mu)$ (with $\mu \in \mathbb{R}^m_+$) can be determined, then u_λ yields the minimum value of J, that is, $p^* = J(u_\lambda)$. If the constraints are qualified and if the functions J and φ_i are convex and differentiable, then since the KKT conditions hold, the duality gap is zero; that is,

$$G(\lambda) = L(u_\lambda, \lambda) = J(u_\lambda).$$