and if we let

$$\begin{aligned}\rho^- &= \frac{1-\sqrt{1-2\lambda\mu\nu}}{\mu\nu}\\ \rho^+ &= \frac{1+\sqrt{1-2\lambda\mu\nu}}{\mu\nu}\\ B &= \{x \in X \mid \|x - x_0\| < \rho^-\}\\ \Omega^+ &= \{x \in \Omega \mid \|x - x_0\| < \rho^+\},\end{aligned}$$

then $\overline{B} \subseteq \Omega$, $f'(x_0)$ *is an isomorphism of* $\mathcal{L}(X; Y)$, *and*

$$\begin{aligned}\left\|(f'(x_0))^{-1}\right\| &\leq \mu,\\ \left\|(f'(x_0))^{-1} f(x_0)\right\| &\leq \lambda,\\ \sup_{x,y\in\Omega^+} \|f'(x) - f'(y)\| &\leq \nu \|x - y\|.\end{aligned}$$

Then $f'(x)$ *is isomorphism of* $\mathcal{L}(X; Y)$ *for all* $x \in B$, *and the sequence defined by*

$$x_{k+1} = x_k - (f'(x_k))^{-1}(f(x_k)), \quad k \geq 0$$

is entirely contained within the ball B *and converges to a zero* a *of* f *which is the only zero of* f *in* Ω^+. *Finally, if we write* $\theta = \rho^-/\rho^+$, *then we have the following bounds:*

$$\begin{aligned}\|x_k - a\| &\leq \frac{2\sqrt{1-2\lambda\mu\nu}}{\lambda\mu\nu}\frac{\theta^{2k}}{1-\theta^{2k}}\|x_1 - x_0\| && \text{if } \lambda\mu\nu < \frac{1}{2}\\ \|x_k - a\| &\leq \frac{\|x_1 - x_0\|}{2^{k-1}} && \text{if } \lambda\mu\nu = \frac{1}{2},\end{aligned}$$

and

$$\frac{2\|x_{k+1} - x_k\|}{1 + \sqrt{(1 + 4\theta^{2k}(1+\theta^{2k})^{-2})}} \leq \|x_k - a\| \leq \theta^{2k-1}\|x_k - x_{k-1}\|.$$

We can now specialize Theorems 41.1 and 41.2 to the search of zeros of the derivative $J' \colon \Omega \to E'$, of a function $J \colon \Omega \to \mathbb{R}$, with $\Omega \subseteq E$. The second derivative J'' of J is a continuous bilinear form $J'' \colon E \times E \to \mathbb{R}$, but is is convenient to view it as a linear map in $\mathcal{L}(E, E')$; the continuous linear form $J''(u)$ is given by $J''(u)(v) = J''(u, v)$. In our next theorem, which follows immediately from Theorem 41.1, we assume that the $A_k(x)$ are isomorphisms in $\mathcal{L}(E, E')$.

Theorem 41.4. *Let* E *be a Banach space, let* $J \colon \Omega \to \mathbb{R}$ *be twice differentiable on the open subset* $\Omega \subseteq E$, *and assume that there are constants* $r, M, \beta > 0$ *such that if we let*

$$B = \{x \in E \mid \|x - x_0\| \leq r\} \subseteq \Omega,$$

then