which can be written in matrix form as

$$\begin{pmatrix} A^{-1} & B \\ B^\top & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} b \\ f \end{pmatrix}.$$

We shall prove in Proposition 42.3 below that our constrained minimization problem has a unique solution actually given by the above system.

Note that the matrix of this system is symmetric. We solve it as follows. Eliminating x from the first equation

$$A^{-1}x + B\lambda = b,$$

we get

$$x = A(b - B\lambda),$$

and substituting into the second equation, we get

$$B^\top A(b - B\lambda) = f,$$

that is,

$$B^\top AB\lambda = B^\top Ab - f.$$

However, by a previous remark, since A is symmetric positive definite and the columns of B are linearly independent, $B^\top AB$ is symmetric positive definite, and thus invertible. Thus we obtain the solution

$$\lambda = (B^\top AB)^{-1}(B^\top Ab - f), \qquad x = A(b - B\lambda).$$

Note that this way of solving the system requires solving for the Lagrange multipliers first.

Letting $e = b - B\lambda$, we also note that the system

$$\begin{pmatrix} A^{-1} & B \\ B^\top & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} b \\ f \end{pmatrix}$$

is equivalent to the system

$$\begin{aligned} e &= b - B\lambda, \\ x &= Ae, \\ B^\top x &= f. \end{aligned}$$

The latter system is called the *equilibrium equations* by Strang [169]. Indeed, Strang shows that the equilibrium equations of many physical systems can be put in the above form. This includes spring-mass systems, electrical networks and trusses, which are structures built from elastic bars. In each case, x, e, b, A, λ, f, and $K = B^\top AB$ have a physical interpretation. The matrix $K = B^\top AB$ is usually called the *stiffness matrix*. Again, the reader is referred to Strang [169].