Definition 51.1. Let $C \subseteq \mathbb{R}^n$ be any subset of $\mathbb{R}^n$. The *indicator function* I_C of C is the function given by

$$I_C(u) = \begin{cases} 0 & \text{if } u \in C \\ +\infty & \text{if } u \notin C. \end{cases}$$

The indicator function I_C is a variant of the characteristic function χ_C of the set C (defined such that $\chi_C(u) = 1$ if $u \in C$ and $\chi_C(u) = 0$ if $u \notin C$). Rockafellar denotes the indicator function I_C by $\delta(-|C)$; that is, $\delta(u|C) = I_C(u)$; see Rockafellar [138], Page 28.

Given a partial function $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$, by setting $f(u) = +\infty$ if $u \notin \mathrm{dom}(f)$, we convert the partial function f into a total function with values in $\mathbb{R} \cup \{-\infty, +\infty\}$. Still, one has to remember that such functions are really partial functions, but $-\infty$ and $+\infty$ play different roles. The value $f(x) = -\infty$ indicates that computing $f(x)$ using a *minimization procedure did not terminate*, but $f(x) = +\infty$ means that the *function f is really undefined at x*.

The definition of a convex function $f\colon S \to \mathbb{R} \cup \{-\infty, +\infty\}$ needs to be slightly modified to accommodate the infinite values $\pm\infty$. The cleanest definition uses the notion of epigraph.

A remarkable and very useful fact is that the optimization problem

$$\begin{aligned} &\text{minimize} \quad J(u) \\ &\text{subject to} \quad u \in C, \end{aligned}$$

where C is a closed convex set in $\mathbb{R}^n$ and J is a convex function can be rewritten in term of the indicator function I_C of C, as

$$\begin{aligned} &\text{minimize} \quad J(u) + I_C(z) \\ &\text{subject to} \quad u - z = 0. \end{aligned}$$

But $J(u) + I_C(z)$ is not differentiable, even if J is, which forces us to deal with convex functions which are not differentiable

Convex functions are not necessarily differentiable, but if a convex function f has a finite value $f(u)$ at u (which means that $f(u) \in \mathbb{R}$), then it has a one-sided directional derivative at u. Another crucial notion is the notion of subgradient, which is a substitute for the notion of gradient when the function f is not differentiable at u.

In Section 51.1, we introduce extended real-valued functions, which are functions that may also take the values $\pm\infty$. In particular, we define proper convex functions, and the closure of a convex function. Subgradients and subdifferentials are defined in Section 51.2. We discuss some properties of subgradients in Section 51.3 and Section 51.4. In particular, we relate subgradients to one-sided directional derivatives. In Section 51.5, we discuss the problem of finding the minimum of a proper convex function and give some criteria in terms of subdifferentials. In Section 51.6, we sketch the generalization of the results presented in Chapter 50 about the Lagrangian framework to programs allowing an objective function and