Example 30.4. Considering the polynomial $f(X) = X^2 - 1$, both $+1$ and -1 are zeros of $f(X)$. Over the field of reals, the polynomial $g(X) = X^2 + 1$ has no zeros. Over the field $\mathbb{C}$ of complex numbers, $g(X) = X^2 + 1$ has two roots i and $-i$, the square roots of -1, which are "imaginary numbers."

We have the following basic proposition showing the relationship between polynomial division and roots.

Proposition 30.19. *Let $f \in A[X]$ be any polynomial and $\alpha \in A$ any element of A. If the result of dividing f by $X - \alpha$ is $f = (X - \alpha)q + r$, then $r = 0$ iff $f(\alpha) = 0$, i.e., α is a root of f iff $r = 0$.*

Proof. We have $f = (X - \alpha)q + r$, with $\deg(r) < 1 = \deg(X - \alpha)$. Thus, r is a constant in K, and since $f(\alpha) = (\alpha - \alpha)q(\alpha) + r$, we get $f(\alpha) = r$, and the proposition is trivial. $\square$

We now consider the issue of multiplicity of a root.

Proposition 30.20. *Let $f \in A[X]$ be any nonnull polynomial and $h \geq 0$ any integer. The following conditions are equivalent.*

(1) f is divisible by $(X - \alpha)^h$ but not by $(X - \alpha)^{h+1}$.

(2) There is some $g \in A[X]$ such that $f = (X - \alpha)^h g$ and $g(\alpha) \neq 0$.

Proof. Assume (1). Then, we have $f = (X - \alpha)^h g$ for some $g \in A[X]$. If we had $g(\alpha) = 0$, by Proposition 30.19, g would be divisible by $(X - \alpha)$, and then f would be divisible by $(X - \alpha)^{h+1}$, contradicting (1).

Assume (2), that is, $f = (X - \alpha)^h g$ and $g(\alpha) \neq 0$. If f is divisible by $(X - \alpha)^{h+1}$, then we have $f = (X - \alpha)^{h+1} g_1$, for some $g_1 \in A[X]$. Then, we have

$$(X - \alpha)^h g = (X - \alpha)^{h+1} g_1,$$

and thus

$$(X - \alpha)^h (g - (X - \alpha) g_1) = 0,$$

and since the leading coefficient of $(X - \alpha)^h$ is 1 (show this by induction), by Proposition 30.1, $(X - \alpha)^h$ is not a zero divisor, and we get $g - (X - \alpha)g_1 = 0$, i.e., $g = (X - \alpha)g_1$, and so $g(\alpha) = 0$, contrary to the hypothesis. $\square$

As a consequence of Proposition 30.20, for every nonnull polynomial $f \in A[X]$ and every $\alpha \in A$, there is a unique integer $h \geq 0$ such that f is divisible by $(X - \alpha)^h$ but not by $(X - \alpha)^{h+1}$. Indeed, since f is divisible by $(X - \alpha)^h$, we have $h \leq \deg(f)$. When $h = 0$, α is not a root of f, i.e., $f(\alpha) \neq 0$. The interesting case is when α is a root of f.