is a group under matrix multiplication, with identity element the identity matrix I_n; we have $Q^{-1} = Q^\top$. This group is called the *orthogonal group* and is usually denoted by $\mathbf{O}(n)$.

8. The set of $n \times n$ invertible matrices Q with real coefficients such that

$$QQ^\top = Q^\top Q = I_n \quad \text{and} \quad \det(Q) = 1$$

is a group under matrix multiplication, with identity element the identity matrix I_n; as in (6), we have $Q^{-1} = Q^\top$. This group is called the *special orthogonal group* or *rotation group* and is usually denoted by $\mathbf{SO}(n)$.

The groups in (5)–(8) are nonabelian for $n \geq 2$, except for $\mathbf{SO}(2)$ which is abelian (but $\mathbf{O}(2)$ is not abelian).

It is customary to denote the operation of an abelian group G by $+$, in which case the inverse a^{-1} of an element $a \in G$ is denoted by $-a$.

The identity element of a group is *unique*. In fact, we can prove a more general fact:

Proposition 2.1. *For any binary operation $\cdot\colon M \times M \to M$, if $e' \in M$ is a left identity and if $e'' \in M$ is a right identity, which means that*

$$e' \cdot a = a \quad \text{for all} \quad a \in M \tag{G2l}$$

and

$$a \cdot e'' = a \quad \text{for all} \quad a \in M, \tag{G2r}$$

then $e' = e''$.

Proof. If we let $a = e''$ in equation (G2l), we get

$$e' \cdot e'' = e'',$$

and if we let $a = e'$ in equation (G2r), we get

$$e' \cdot e'' = e',$$

and thus

$$e' = e' \cdot e'' = e'',$$

as claimed. $\square$

Proposition 2.1 implies that the identity element of a monoid is unique, and since every group is a monoid, the identity element of a group is unique. Furthermore, every element in a group has a *unique inverse*. This is a consequence of a slightly more general fact: