Proof. We begin with a preliminary result. Let $A(\mu)$ be a tridiagonal matrix by block of the form

$$A(\mu) = \begin{pmatrix} A_1 & \mu^{-1}C_1 & 0 & 0 & \cdots & 0 \\ \mu B_1 & A_2 & \mu^{-1}C_2 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mu B_{p-2} & A_{p-1} & \mu^{-1}C_{p-1} \\ 0 & \cdots & \cdots & 0 & \mu B_{p-1} & A_p \end{pmatrix},$$

then

$$\det(A(\mu)) = \det(A(1)), \quad \mu \neq 0.$$

To prove this fact, form the block diagonal matrix

$$P(\mu) = \mathrm{diag}(\mu I_1, \mu^2 I_2, \ldots, \mu^p I_p),$$

where I_j is the identity matrix of the same dimension as the block A_j. Then it is easy to see that

$$A(\mu) = P(\mu)A(1)P(\mu)^{-1},$$

and thus,

$$\det(A(\mu)) = \det(P(\mu)A(1)P(\mu)^{-1}) = \det(A(1)).$$

Since the Jacobi matrix is $J = D^{-1}(E + F)$, the eigenvalues of J are the zeros of the characteristic polynomial

$$p_J(\lambda) = \det(\lambda I - D^{-1}(E + F)),$$

and thus, they are also the zeros of the polynomial

$$q_J(\lambda) = \det(\lambda D - E - F) = \det(D)p_J(\lambda).$$

Similarly, since the Gauss–Seidel matrix is $\mathcal{L}_1 = (D - E)^{-1}F$, the zeros of the characteristic polynomial

$$p_{\mathcal{L}_1}(\lambda) = \det(\lambda I - (D - E)^{-1}F)$$

are also the zeros of the polynomial

$$q_{\mathcal{L}_1}(\lambda) = \det(\lambda D - \lambda E - F) = \det(D - E)p_{\mathcal{L}_1}(\lambda).$$

Since $A = D - E - F$ is tridiagonal (or tridiagonal by blocks), $\lambda^2 D - \lambda^2 E - F$ is also tridiagonal (or tridiagonal by blocks), and by using our preliminary result with $\mu = \lambda \neq 0$ starting with the matrix $\lambda^2 D - \lambda E - \lambda F$, we get

$$\lambda^n q_J(\lambda) = \det(\lambda^2 D - \lambda E - \lambda F) = \det(\lambda^2 D - \lambda^2 E - F) = q_{\mathcal{L}_1}(\lambda^2).$$

By continuity, the above equation also holds for $\lambda = 0$. But then we deduce that: