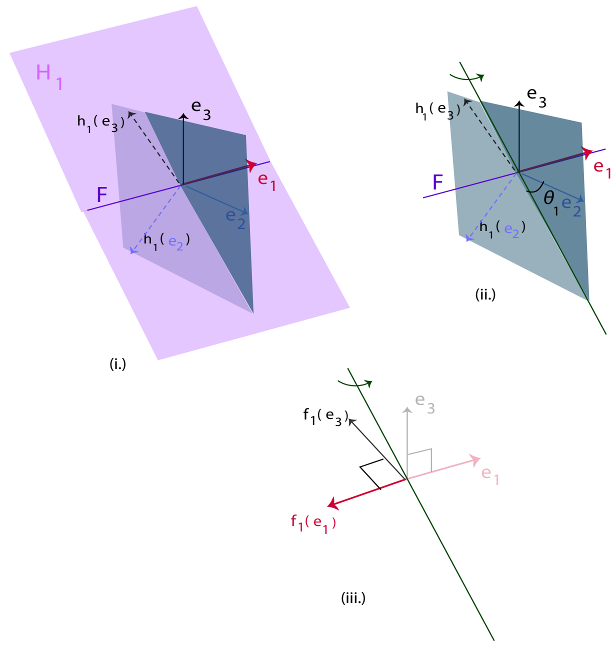

Figure 27.7: The conversion of the hyperplane reflection h_1 into the flip or 180° rotation around the green axis in the e_2e_3-plane. The green axis corresponds to the restriction of the eigenspace associated with eigenvalue 1.

Using Lemma 27.4 and the Cartan–Dieudonné theorem, we obtain the following characterization of rotations when $n \geq 3$.

Theorem 27.5. *Let E be a Euclidean space of dimension $n \geq 3$. Every rotation $f \in \mathbf{SO}(E)$ is the composition of an even number of flips $f = f_{2k} \circ \cdots \circ f_1$, where $2k \leq n$. Furthermore, if $u \neq 0$ is invariant under f (i.e., $u \in \mathrm{Ker}\,(f - \mathrm{id})$), we can pick the last flip f_{2k} such that $u \in F_{2k}^{\perp}$, where F_{2k} is the subspace of dimension $n-2$ determining f_{2k}.*

Proof. By Theorem 27.1, the rotation f can be expressed as an even number of hyperplane reflections $f = s_{2k} \circ s_{2k-1} \circ \cdots \circ s_2 \circ s_1$, with $2k \leq n$. By Lemma 27.4, every composition of two reflections $s_{2i} \circ s_{2i-1}$ can be replaced by the composition of two flips $f_{2i} \circ f_{2i-1}$ $(1 \leq i \leq k)$, which yields $f = f_{2k} \circ \cdots \circ f_1$, where $2k \leq n$.

Assume that $f(u) = u$, with $u \neq 0$. We have already made the remark that in the case where 1 is an eigenvalue of f, the proof of Theorem 27.1 shows that the reflections s_i can be chosen so that $s_i(u) = u$. In particular, if each reflection s_i is a reflection about the hyperplane H_i, we have $u \in H_{2k-1} \cap H_{2k}$. Letting $F = H_{2k-1} \cap H_{2k}$, pick an orthonormal basis $(e_1, \ldots, e_{n-3}, e_{n-2})$ of F, where

$$e_{n-2} = \frac{u}{\|u\|}.$$