If $\lambda_l = 0$ for all $l \in L$, we have

$$\sum_{i\in(I-\{p\})} \lambda_i u_i - u_p = 0,$$

contradicting the fact that $(u_i)_{i\in I}$ is linearly independent. Thus, $\lambda_l \neq 0$ for some $l \in L$, say $l = q$. Since $\lambda_q \neq 0$, we have

$$v_{\rho(q)} = \sum_{i\in(I-\{p\})} (-\lambda_q^{-1}\lambda_i)u_i + \lambda_q^{-1}u_p + \sum_{l\in(L-\{q\})} (-\lambda_q^{-1}\lambda_l)v_{\rho(l)}. \tag{2}$$

We claim that the families $(u_i)_{i\in(I-\{p\})} \cup (v_{\rho(l)})_{l\in L}$ and $(u_i)_{i\in I} \cup (v_{\rho(l)})_{l\in(L-\{q\})}$ generate the same subset of $E$. Indeed, the second family is obtained from the first by replacing $v_{\rho(q)}$ by $u_p$, and vice-versa, and $u_p$ is a linear combination of $(u_i)_{i\in(I-\{p\})} \cup (v_{\rho(l)})_{l\in L}$, by (1), and $v_{\rho(q)}$ is a linear combination of $(u_i)_{i\in I} \cup (v_{\rho(l)})_{l\in(L-\{q\})}$, by (2). Thus, the families $(u_i)_{i\in I} \cup (v_{\rho(l)})_{l\in(L-\{q\})}$ and $(v_j)_{j\in J}$ generate the same subspace of $E$, and the proposition holds for $L - \{q\}$ and the restriction of the injection $\rho\colon L \to J$ to $L - \{q\}$, since $L \cap I = \emptyset$ and $|L| = n - m$ imply that $(L - \{q\}) \cap I = \emptyset$ and $|L - \{q\}| = n - (m + 1)$. $\square$

The idea is that $m$ of the vectors $v_j$ can be *replaced* by the linearly independent $u_i$s in such a way that the same subspace is still generated. The purpose of the function $\rho\colon L \to J$ is to pick $n - m$ elements $j_1, \ldots, j_{n-m}$ of $J$ and to relabel them $l_1, \ldots, l_{n-m}$ in such a way that these new indices do not clash with the indices in $I$; this way, the vectors $v_{j_1}, \ldots, v_{j_{n-m}}$ who "survive" (i.e. are not replaced) are relabeled $v_{l_1}, \ldots, v_{l_{n-m}}$, and the other $m$ vectors $v_j$ with $j \in J - \{j_1, \ldots, j_{n-m}\}$ are replaced by the $u_i$. The index set of this new family is $I \cup L$.

Actually, one can prove that Proposition 3.10 implies Theorem 3.7 when the vector space is finitely generated. Putting Theorem 3.7 and Proposition 3.10 together, we obtain the following fundamental theorem.

**Theorem 3.11.** *Let $E$ be a finitely generated vector space. Any family $(u_i)_{i\in I}$ generating $E$ contains a subfamily $(u_j)_{j\in J}$ which is a basis of $E$. Any linearly independent family $(u_i)_{i\in I}$ can be extended to a family $(u_j)_{j\in J}$ which is a basis of $E$ (with $I \subseteq J$). Furthermore, for every two bases $(u_i)_{i\in I}$ and $(v_j)_{j\in J}$ of $E$, we have $|I| = |J| = n$ for some fixed integer $n \geq 0$.*

*Proof.* The first part follows immediately by applying Theorem 3.7 with $L = \emptyset$ and $S = (u_i)_{i\in I}$. For the second part, consider the family $S' = (u_i)_{i\in I} \cup (v_h)_{h\in H}$, where $(v_h)_{h\in H}$ is any finitely generated family generating $E$, and with $I \cap H = \emptyset$. Then apply Theorem 3.7 to $L = (u_i)_{i\in I}$ and to $S'$. For the last statement, assume that $(u_i)_{i\in I}$ and $(v_j)_{j\in J}$ are bases of $E$. Since $(u_i)_{i\in I}$ is linearly independent and $(v_j)_{j\in J}$ spans $E$, Proposition 3.10 implies that $|I| \leq |J|$. A symmetric argument yields $|J| \leq |I|$. $\square$

**Remark:** Theorem 3.11 also holds for vector spaces that are not finitely generated. This can be shown as follows. Let $(u_i)_{i\in I}$ be a basis of $E$, let $(v_j)_{j\in J}$ be a generating family of $E$,