where r_i is the smallest integer, such that, $\pi^{r_i}(i) = i$ and $2 \leq r_i \leq n$. If π is not the identity, then it can be written in a unique way (up to the order) as a composition $\pi = \sigma_1 \circ \ldots \circ \sigma_s$ of cyclic permutations with disjoint domains, where s is the number of orbits with at least two elements. Every permutation $\pi\colon [n] \to [n]$ can be written as a nonempty composition of transpositions.

Proof. Consider the relation R_π defined on $[n]$ as follows: $iR_\pi j$ iff there is some $k \geq 1$ such that $j = \pi^k(i)$. We claim that R_π is an equivalence relation. Transitivity is obvious. We claim that for every $i \in [n]$, there is some least r $(1 \leq r \leq n)$ such that $\pi^r(i) = i$.

Indeed, consider the following sequence of $n + 1$ elements:

$$\langle i, \pi(i), \pi^2(i), \ldots, \pi^n(i) \rangle.$$

Since $[n]$ only has n distinct elements, there are some h, k with $0 \leq h < k \leq n$ such that

$$\pi^h(i) = \pi^k(i),$$

and since π is a bijection, this implies $\pi^{k-h}(i) = i$, where $0 \leq k - h \leq n$. Thus, we proved that there is some integer $m \geq 1$ such that $\pi^m(i) = i$, so there is such a smallest integer r.

Consequently, R_π is reflexive. It is symmetric, since if $j = \pi^k(i)$, letting r be the least $r \geq 1$ such that $\pi^r(i) = i$, then

$$i = \pi^{kr}(i) = \pi^{k(r-1)}(\pi^k(i)) = \pi^{k(r-1)}(j).$$

Now, for every $i \in [n]$, the equivalence class (orbit) of i is a subset of $[n]$, either the singleton $\{i\}$ or a set of the form

$$J = \{i, \pi(i), \pi^2(i), \ldots, \pi^{r_i - 1}(i)\},$$

where r_i is the smallest integer such that $\pi^{r_i}(i) = i$ and $2 \leq r_i \leq n$, and in the second case, the restriction of π to J induces a cyclic permutation σ_i, and $\pi = \sigma_1 \circ \ldots \circ \sigma_s$, where s is the number of equivalence classes having at least two elements.

For the second part of the proposition, we proceed by induction on n. If $n = 2$, there are exactly two permutations on $[2]$, the transposition τ exchanging 1 and 2, and the identity. However, $\mathrm{id}_2 = \tau^2$. Now, let $n \geq 3$. If $\pi(n) = n$, since by the induction hypothesis, the restriction of π to $[n-1]$ can be written as a product of transpositions, π itself can be written as a product of transpositions. If $\pi(n) = k \neq n$, letting τ be the transposition such that $\tau(n) = k$ and $\tau(k) = n$, it is clear that $\tau \circ \pi$ leaves n invariant, and by the induction hypothesis, we have $\tau \circ \pi = \tau_m \circ \ldots \circ \tau_1$ for some transpositions, and thus

$$\pi = \tau \circ \tau_m \circ \ldots \circ \tau_1,$$

a product of transpositions (since $\tau \circ \tau = \mathrm{id}_n$). $\square$