Remark: The tensor spaces, $T^{r,s}(V)$ are also denoted $T^r_s(V)$. A tensor $\alpha \in T^{r,s}(V)$ is said to be *contravariant* in the first r arguments and *covariant* in the last s arguments. This terminology refers to the way tensors behave under coordinate changes. Given a basis $(e_1, \ldots, e_n)$ of V, if $(e_1^*, \ldots, e_n^*)$ denotes the dual basis, then every tensor $\alpha \in T^{r,s}(V)$ is given by an expression of the form

$$\alpha = \sum_{\substack{i_1,\ldots,i_r\\ j_1,\ldots,j_s}} a^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^*_{j_1} \otimes \cdots \otimes e^*_{j_s}.$$

The tradition in classical tensor notation is to use lower indices on vectors and upper indices on linear forms and in accordance to *Einstein summation convention* (or *Einstein notation*) the position of the indices on the coefficients is reversed. *Einstein summation convention* (already encountered in Section 33.1) is to assume that a summation is performed for all values of every index that appears simultaneously once as an upper index and once as a lower index. According to this convention, the tensor α above is written

$$\alpha = a^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s}.$$

An older view of tensors is that they are multidimensional arrays of coefficients,

$$\left(a^{i_1,\ldots,i_r}_{j_1,\ldots,j_s}\right),$$

subject to the rules for changes of bases.

Another operation on general tensors, contraction, is useful in differential geometry.

Definition 33.14. For all $r, s \geq 1$, the *contraction* $c_{i,j} \colon T^{r,s}(V) \to T^{r-1,s-1}(V)$, with $1 \leq i \leq r$ and $1 \leq j \leq s$, is the linear map defined on generators by

$$\begin{aligned} & c_{i,j}(u_1 \otimes \cdots \otimes u_r \otimes v_1^* \otimes \cdots \otimes v_s^*) \\ & \qquad = v_j^*(u_i)\, u_1 \otimes \cdots \otimes \widehat{u_i} \otimes \cdots \otimes u_r \otimes v_1^* \otimes \cdots \otimes \widehat{v_j^*} \otimes \cdots \otimes v_s^*, \end{aligned}$$

where the hat over an argument means that it should be omitted.

Let us figure our what is $c_{1,1} \colon T^{1,1}(V) \to \mathbb{R}$, that is $c_{1,1} \colon V \otimes V^* \to \mathbb{R}$. If $(e_1, \ldots, e_n)$ is a basis of V and $(e_1^*, \ldots, e_n^*)$ is the dual basis, by Proposition 33.17 every $h \in V \otimes V^* \cong \mathrm{Hom}(V, V)$ can be expressed as

$$h = \sum_{i,j=1}^n a_{ij}\, e_i \otimes e_j^*.$$

As

$$c_{1,1}(e_i \otimes e_j^*) = \delta_{i,j},$$

we get

$$c_{1,1}(h) = \sum_{i=1}^n a_{ii} = \mathrm{tr}(h),$$