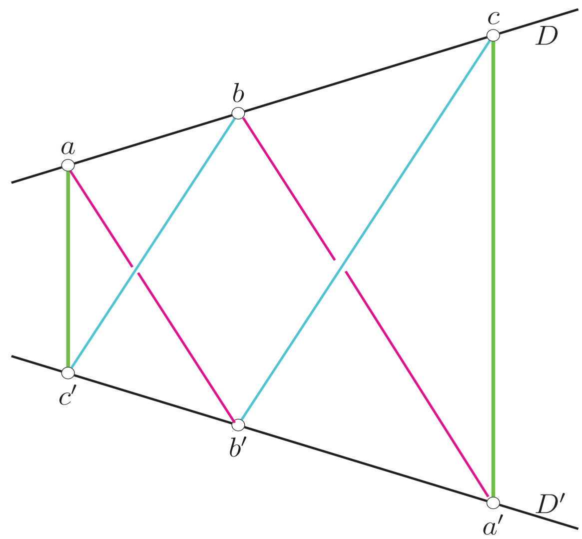

Figure 24.23: Pappus's theorem (affine version).

parallel, the points a, b, a', b' are coplanar. Thus, either $\langle a, a' \rangle$ and $\langle b, b' \rangle$ are parallel, or they have some intersection d. We consider the second case where they intersect, leaving the other case as an easy exercise. Let f be the dilatation of center d such that $f(a) = a'$. By Proposition 24.11, we get $f(b) = b'$. If $f(c) = c''$, again by Proposition 24.11 twice, the lines $\langle b, c \rangle$ and $\langle b', c'' \rangle$ are parallel, and the lines $\langle a, c \rangle$ and $\langle a', c'' \rangle$ are parallel. From this it follows that $c'' = c'$. Indeed, recall that $\langle b, c \rangle$ and $\langle b', c' \rangle$ are parallel, and similarly $\langle a, c \rangle$ and $\langle a', c' \rangle$ are parallel. Thus, the lines $\langle b', c'' \rangle$ and $\langle b', c' \rangle$ are identical, and similarly the lines $\langle a', c'' \rangle$ and $\langle a', c' \rangle$ are identical. Since $\overrightarrow{a'c'}$ and $\overrightarrow{b'c'}$ are linearly independent, these lines have a unique intersection, which must be $c'' = c'$.

The direction where it is assumed that the lines $\langle a, a' \rangle$, $\langle b, b' \rangle$ and $\langle c, c' \rangle$, are either parallel or concurrent is left as an exercise (in fact, the proof is quite similar). $\square$

Desargues's theorem is illustrated in Figure 24.24.

There is a fancier version of Desargues's theorem, but it is easier to prove it using projective geometry. It should be noted that in axiomatic presentations of projective geometry, Desargues's theorem is related to the associativity of the ground field K (in the present case, $K = \mathbb{R}$). Also, Desargues's theorem yields a geometric characterization of the affine dilatations. An affine dilatation f on an affine space E is a bijection that maps every line D to a line $f(D)$ parallel to D. We leave the proof as an exercise.