A lower-triangular matrix with diagonal entries equal to 1 is called a *unit lower-triangular* matrix. Given an $n \times n$ matrix $A = (a_{i\,j})$, for any k with $1 \leq k \leq n$, let $A(1 : k, 1 : k)$ denote the submatrix of A whose entries are $a_{i\,j}$, where $1 \leq i, j \leq k$.[1] For example, if A is the 5×5 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix},$$

then

$$A(1 : 3, 1 : 3) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Proposition 8.2. *Let A be an invertible $n \times n$-matrix. Then A has an LU-factorization $A = LU$ iff every matrix $A(1 : k, 1 : k)$ is invertible for $k = 1, \ldots, n$. Furthermore, when A has an LU-factorization, we have*

$$\det(A(1 : k, 1 : k)) = \pi_1 \pi_2 \cdots \pi_k, \quad k = 1, \ldots, n,$$

where π_k is the pivot obtained after $k-1$ elimination steps. Therefore, the kth pivot is given by

$$\pi_k = \begin{cases} a_{11} = \det(A(1 : 1, 1 : 1)) & \text{if } k = 1 \\ \dfrac{\det(A(1 : k, 1 : k))}{\det(A(1 : k-1, 1 : k-1))} & \text{if } k = 2, \ldots, n. \end{cases}$$

Proof. First assume that $A = LU$ is an LU-factorization of A. We can write

$$A = \begin{pmatrix} A(1 : k, 1 : k) & A_2 \\ A_3 & A_4 \end{pmatrix} = \begin{pmatrix} L_1 & 0 \\ L_3 & L_4 \end{pmatrix} \begin{pmatrix} U_1 & U_2 \\ 0 & U_4 \end{pmatrix} = \begin{pmatrix} L_1 U_1 & L_1 U_2 \\ L_3 U_1 & L_3 U_2 + L_4 U_4 \end{pmatrix},$$

where L_1, L_4 are unit lower-triangular and U_1, U_4 are upper-triangular. (Note, $A(1 : k, 1 : k)$, L_1, and U_1 are $k \times k$ matrices; A_2 and U_2 are $k \times (n-k)$ matrices; A_3 and L_3 are $(n-k) \times k$ matrices; A_4, L_4, and U_4 are $(n-k) \times (n-k)$ matrices.) Thus,

$$A(1 : k, 1 : k) = L_1 U_1,$$

and since U is invertible, U_1 is also invertible (the determinant of U is the product of the diagonal entries in U, which is the product of the diagonal entries in U_1 and U_4). As L_1 is invertible (since its diagonal entries are equal to 1), we see that $A(1 : k, 1 : k)$ is invertible for $k = 1, \ldots, n$.

Conversely, assume that $A(1 : k, 1 : k)$ is invertible for $k = 1, \ldots, n$. We just need to show that Gaussian elimination does not need pivoting. We prove by induction on k that the kth step does not need pivoting.

[1] We are using `Matlab`'s notation.