   (b) If $E = E_{i,j;\beta}$, multiply the ith column of P' by the matrix $\beta(A)$ obtained by substituting the matrix A for X, and then subtract the resulting vector from column j.

   (c) If $E = E_{i,\lambda}$ where $\lambda \in K$, then multiply the ith column of P' by λ^{-1}.

2. When step (1) terminates, the first $n-m$ columns of P' are zero and the last m are linearly independent. For $i = 1, \ldots, m$, multiply the $(n-m+i)$th column w_i of P' successively by I, A^1, A^2, A^{n_i-1}, where n_i is the degree of the polynomial q_i (appearing in D), and form the $n \times n$ matrix P consisting of the vectors

$$w_1, Aw_1, \ldots, A^{n_1-1}w_1, w_2, Aw_2, \ldots, A^{n_2-1}w_2, \ldots, w_m, Aw_m, \ldots, A^{n_m-1}w_m.$$

   Then, $P^{-1}AP$ is the canonical rational form of A.

Here is an example taken from Dummit and Foote [54] (Chapter 12, Section 12.2). Let A be the matrix

$$A = \begin{pmatrix} 1 & 2 & -4 & 4 \\ 2 & -1 & 4 & -8 \\ 1 & 0 & 1 & -2 \\ 0 & 1 & -2 & 3 \end{pmatrix}.$$

One should check that the following sequence of row and column operations produces the Smith normal form D of $XI - A$:

$$\begin{array}{llllll} \text{row } P(1,3) & \text{row } E_{1,-1} & \text{row } E_{2,1;2} & \text{row } E_{3,1;-(X-1)} & \text{column } E_{1,3;X-1} & \text{column } E_{1,4;2} \\ \text{row } P(2,4) & \text{row} E_{2,-1} & \text{row } E_{3,2;2} & \text{row } E_{4,2;-(X+1)} & \text{column } E_{2,3;2} & \text{column } E_{2,4;X-3}, \end{array}$$

with

$$D = \begin{pmatrix} 1 & 0 & 0 & \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (X-1)^2 & 0 \\ 0 & 0 & 0 & (X-1)^2 \end{pmatrix}.$$

Then, applying Step 1 of the above algorithm, we get the sequence of column operations:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{P(1,3)} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{E_{1,-1}} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{E_{2,1,-2}}$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ -2 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{E_{3,1,A-I}} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{P(2,4)} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \xrightarrow{E_{2,-1}}$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \xrightarrow{E_{3,2,-2}} \begin{pmatrix} 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \xrightarrow{E_{4,2;A+I}} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = P'.$$