of *rational functions* over F, although the terminology is a bit misleading, since elements of $F(X_1, \ldots, X_n)$ only define functions when the dominator is nonnull.

We now have the following crucial lemma which shows that if a polynomial $f(X)$ is reducible over $F[X]$ where F is the fraction field of A, then $f(X)$ is already reducible over $A[X]$.

Lemma 32.8. *Let A be a UFD and let F be the fraction field of A. For any nonnull polynomial $f(X) \in A[X]$ of degree m, if $f(X)$ is not the product of two polynomials of degree strictly smaller than m, then $f(X)$ is irreducible in $F[X]$.*

Proof. Assume that $f(X)$ is reducible in $F[X]$ and that $f(X)$ is neither null nor a unit. Then,

$$f(X) = G(X)H(X),$$

where $G(X), H(X) \in F[X]$ are polynomials of degree $p, q \geq 1$. Let a be the product of the denominators of the coefficients of $G(X)$, and b the product of the denominators of the coefficients of $H(X)$. Then, $a, b \neq 0$, $g_1(X) = aG(X) \in A[X]$ has degree $p \geq 1$, $h_1(X) = bH(X) \in A[X]$ has degree $q \geq 1$, and

$$abf(X) = g_1(X)h_1(X).$$

Let $c = ab$. If c is a unit, then $f(X)$ is also reducible in $A[X]$. Otherwise, $c = c_1 \cdots c_n$, where $c_i \in A$ is irreducible. We now use induction on n to prove that

$$f(X) = g(X)h(X),$$

for some polynomials $g(X) \in A[X]$ of degree $p \geq 1$ and $h(X) \in A[X]$ of degree $q \geq 1$.

If $n = 1$, since $c = c_1$ is irreducible, by Lemma 32.5, either c divides $g_1(X)$ or c divides $h_1(X)$. Say that c divides $g_1(X)$, the other case being similar. Then, $g_1(X) = cg(X)$ for some $g(X) \in A[X]$ of degree $p \geq 1$, and since $A[X]$ is an integral ring, we get

$$f(X) = g(X)h_1(X),$$

showing that $f(X)$ is reducible in $A[X]$. If $n > 1$, since

$$c_1 \cdots c_n f(X) = g_1(X)h_1(X),$$

c_1 divides $g_1(X)h_1(X)$, and as above, either c_1 divides $g_1(X)$ or c divides $h_1(X)$. In either case, we get

$$c_2 \cdots c_n f(X) = g_2(X)h_2(X)$$

for some polynomials $g_2(X) \in A[X]$ of degree $p \geq 1$ and $h_2(X) \in A[X]$ of degree $q \geq 1$. By the induction hypothesis, we get

$$f(X) = g(X)h(X),$$

for some polynomials $g(X) \in A[X]$ of degree $p \geq 1$ and $h(X) \in A[X]$ of degree $q \geq 1$, showing that $f(X)$ is reducible in $A[X]$. □