Chapter 7

Determinants

In this chapter all vector spaces are defined over an arbitrary field K. For the sake of concreteness, the reader may safely assume that $K = \mathbb{R}$.

7.1 Permutations, Signature of a Permutation

This chapter contains a review of determinants and their use in linear algebra. We begin with permutations and the signature of a permutation. Next, we define multilinear maps and alternating multilinear maps. Determinants are introduced as alternating multilinear maps taking the value 1 on the unit matrix (following Emil Artin). It is then shown how to compute a determinant using the Laplace expansion formula, and the connection with the usual definition is made. It is shown how determinants can be used to invert matrices and to solve (at least in theory!) systems of linear equations (the Cramer formulae). The determinant of a linear map is defined. We conclude by defining the characteristic polynomial of a matrix (and of a linear map) and by proving the celebrated Cayley-Hamilton theorem which states that every matrix is a "zero" of its characteristic polynomial (we give two proofs; one computational, the other one more conceptual).

Determinants can be defined in several ways. For example, determinants can be defined in a fancy way in terms of the exterior algebra (or alternating algebra) of a vector space. We will follow a more algorithmic approach due to Emil Artin. No matter which approach is followed, we need a few preliminaries about permutations on a finite set. We need to show that every permutation on n elements is a product of transpositions, and that the parity of the number of transpositions involved is an invariant of the permutation. Let $[n] = \{1, 2 \ldots, n\}$, where $n \in \mathbb{N}$, and $n > 0$.

Definition 7.1. A *permutation on n elements* is a bijection $\pi \colon [n] \to [n]$. When $n = 1$, the only function from [1] to [1] is the constant map: $1 \mapsto 1$. Thus, we will assume that $n \geq 2$. A *transposition* is a permutation $\tau \colon [n] \to [n]$ such that, for some $i < j$ (with $1 \leq i < j \leq n$), $\tau(i) = j$, $\tau(j) = i$, and $\tau(k) = k$, for all $k \in [n] - \{i, j\}$. In other words, a transposition exchanges two distinct elements $i, j \in [n]$. A *cyclic permutation of order k (or k-cycle)* is a