**Remarks:**

(1) If $E$ is a Hilbert space and $(u_k)_{k\in K}$ is a total orthogonal family in $E$, there is a simpler argument to prove that $u = 0$ if $\langle u, u_k\rangle = 0$ for all $k \in K$ based on the continuity of $\langle -, -\rangle$. The argument is to prove that the assumption implies that $\langle v, u\rangle = 0$ for all $v \in E$. Since $\langle -, -\rangle$ is positive definite, this implies that $u = 0$. By continuity of $\langle -, -\rangle$, for every $\epsilon > 0$, there is some $\eta > 0$ such that for every finite subset $I$ of $K$, for every family $(\lambda_i)_{i\in I}$, for every $v \in E$,

$$\left|\langle v, u\rangle - \left\langle \sum_{i\in I} \lambda_i u_i, u\right\rangle\right| < \epsilon$$

whenever

$$\left\| v - \sum_{i\in I} \lambda_i u_i \right\| < \eta.$$

Since $(u_k)_{k\in K}$ is dense in $E$, for every $v \in E$, there is some finite subset $I$ of $K$ and some family $(\lambda_i)_{i\in I}$ such that

$$\left\| v - \sum_{i\in I} \lambda_i u_i \right\| < \eta,$$

and since by assumption, $\left\langle \sum_{i\in I} \lambda_i u_i, u\right\rangle = 0$, we get

$$|\langle v, u\rangle| < \epsilon.$$

Since this holds for every $\epsilon > 0$, we must have $\langle v, u\rangle = 0$

(2) If $V$ is any nonempty subset of $E$, the kind of argument used in the previous remark can be used to prove that $V^{\perp}$ is closed (even if $V$ is not), and that $V^{\perp\perp}$ is the closure of $V$.

We will now prove that every Hilbert space has some Hilbert basis. This requires using a fundamental theorem from set theory known as *Zorn's lemma*, which we quickly review.

Given any set $X$ with a partial ordering $\leq$, recall that a nonempty subset $C$ of $X$ is a *chain* if it is totally ordered (i.e., for all $x, y \in C$, either $x \leq y$ or $y \leq x$). A nonempty subset $Y$ of $X$ is *bounded* iff there is some $b \in X$ such that $y \leq b$ for all $y \in Y$. Some $m \in X$ is *maximal* iff for every $x \in X$, $m \leq x$ implies that $x = m$. We can now state Zorn's lemma. For more details, see Rudin [140], Lang [109], or Artin [7].

**Proposition A.6.** *(Zorn's lemma) Given any nonempty partially ordered set $X$, if every (nonempty) chain in $X$ is bounded, then $X$ has some maximal element.*