since $\det(e_1, \ldots, e_n) = 1$. Now letting

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = B \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix},$$

we get

$$\det(v_1, \ldots, v_n) = \det(B),$$

and since

$$\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = A \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix},$$

we get

$$\det(w_1, \ldots, w_n) = \det(A) \det(v_1, \ldots, v_n) = \det(A) \det(B). \qquad \square$$

It should be noted that all the results of this section, up to now, also hold when K is a commutative ring and not necessarily a field. We can now characterize when an $n \times n$-matrix A is invertible in terms of its determinant $\det(A)$.

7.4 Inverse Matrices and Determinants

In the next two sections, K is a commutative ring and when needed a field.

Definition 7.7. Let K be a commutative ring. Given a matrix $A \in \mathrm{M}_n(K)$, let $\widetilde{A} = (b_{i\,j})$ be the matrix defined such that

$$b_{i\,j} = (-1)^{i+j} \det(A_{j\,i}),$$

the cofactor of $a_{j\,i}$. The matrix $\widetilde{A}$ is called the *adjugate* of A, and each matrix $A_{j\,i}$ is called a *minor* of the matrix A.

For example, if

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -2 & -2 \\ 3 & 3 & -3 \end{pmatrix},$$