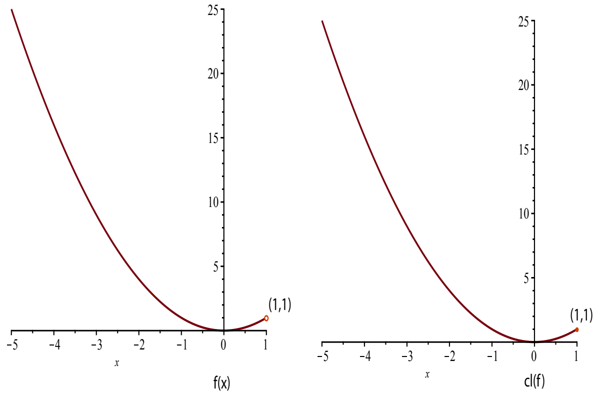

Figure 51.5: The proper convex function of Example 51.3 and its closure. These two functions only differ at the relative boundary point of $\mathrm{dom}(f)$, namely $x = 1$.

f has the constant value $-\infty$ on C, and so it can be considered to be continuous on C. Thus we are led to consider proper functions.

Definition 51.10. Given a proper convex function f, for any subset $S \subseteq \mathrm{dom}(f)$, we say that f is *continuous relative to* S if the restriction of f to S is continuous, with S endowed with the subspace topology.

The following result is proven in Rockafellar [138] (Theorem 10.1).

Proposition 51.7. *If f is a proper convex function, then f is continuous on any convex relatively open subset C ($\mathbf{relint}(C) = C$) contained in its effective domain $\mathrm{dom}(f)$, in particular relative to $\mathbf{relint}(\mathrm{dom}(f))$.*

As a corollary, any convex function f which is finite on $\mathbb{R}^n$ is continuous.

The behavior of a convex function at relative boundary points of the effective domain can be tricky. Here is an example due to Rockafellar [138] illustrating the problems.

Example 51.4. Consider the proper convex function (on $\mathbb{R}^2$) given by

$$f(x, y) = \begin{cases} y^2/(2x) & \text{if } x > 0 \\ 0 & \text{if } x = 0, y = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

We have

$$\mathrm{dom}(f) = \{(x, y) \in \mathbb{R}^2 \mid x > 0\} \cup \{(0, 0)\}.$$

See Figure 51.6.

The function f is continuous on the open right half-plane $\{(x, y) \in \mathbb{R}^2 \mid x > 0\}$, but not at $(0, 0)$. The limit of $f(x, y)$ when (x, y) approaches $(0, 0)$ on the parabola of equation