Proof. We proceed by induction on m. When $m = 0$, the family $(u_1, \ldots, u_m)$ is empty, and the proposition holds trivially. For the induction step, we have a linearly independent family $(u_1, \ldots, u_m, u_{m+1})$. Consider the linearly independent family $(u_1, \ldots, u_m)$. By the induction hypothesis, $m \leq n$, and there is a replacement of m of the vectors v_j by $(u_1, \ldots, u_m)$, such that after renaming some of the indices of the vs, the families $(u_1, \ldots, u_m, v_{m+1}, \ldots, v_n)$ and $(v_1, \ldots, v_n)$ generate the same subspace of E. The vector u_{m+1} can also be expressed as a linear combination of $(v_1, \ldots, v_n)$, and since $(u_1, \ldots, u_m, v_{m+1}, \ldots, v_n)$ and $(v_1, \ldots, v_n)$ generate the same subspace, u_{m+1} can be expressed as a linear combination of $(u_1, \ldots, u_m, v_{m+1}, \ldots, v_n)$, say

$$u_{m+1} = \sum_{i=1}^{m} \lambda_i u_i + \sum_{j=m+1}^{n} \lambda_j v_j.$$

We claim that $\lambda_j \neq 0$ for some j with $m + 1 \leq j \leq n$, which implies that $m + 1 \leq n$.

Otherwise, we would have

$$u_{m+1} = \sum_{i=1}^{m} \lambda_i u_i,$$

a nontrivial linear dependence of the u_i, which is impossible since $(u_1, \ldots, u_{m+1})$ are linearly independent.

Therefore, $m + 1 \leq n$, and after renaming indices if necessary, we may assume that $\lambda_{m+1} \neq 0$, so we get

$$v_{m+1} = -\sum_{i=1}^{m} (\lambda_{m+1}^{-1} \lambda_i) u_i - \lambda_{m+1}^{-1} u_{m+1} - \sum_{j=m+2}^{n} (\lambda_{m+1}^{-1} \lambda_j) v_j.$$

Observe that the families $(u_1, \ldots, u_m, v_{m+1}, \ldots, v_n)$ and $(u_1, \ldots, u_{m+1}, v_{m+2}, \ldots, v_n)$ generate the same subspace, since u_{m+1} is a linear combination of $(u_1, \ldots, u_m, v_{m+1}, \ldots, v_n)$ and v_{m+1} is a linear combination of $(u_1, \ldots, u_{m+1}, v_{m+2}, \ldots, v_n)$. Since $(u_1, \ldots, u_m, v_{m+1}, \ldots, v_n)$ and $(v_1, \ldots, v_n)$ generate the same subspace, we conclude that $(u_1, \ldots, u_{m+1}, v_{m+2}, \ldots, v_n)$ and and $(v_1, \ldots, v_n)$ generate the same subspace, which concludes the induction hypothesis. ☐

Here is an example illustrating the replacement lemma. Consider sequences (u_1, u_2, u_3) and $(v_1, v_2, v_3, v_4, v_5)$, where (u_1, u_2, u_3) is a linearly independent family and with the u_is expressed in terms of the v_js as follows:

$$\begin{aligned}
u_1 &= v_4 + v_5 \\
u_2 &= v_3 + v_4 - v_5 \\
u_3 &= v_1 + v_2 + v_3.
\end{aligned}$$

From the first equation we get

$$v_4 = u_1 - v_5,$$