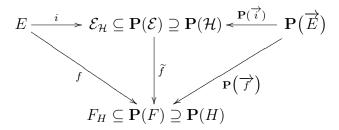
is an affine bijection between E_H and the affine hyperplane w+H in E, where $w \in E-H$ is any fixed vector. Choosing w as an origin in E_H , we know that $\widehat{E}_H = H + Kw$, and since $E = H \oplus Kw$, it is obvious how to define a linear bijection between $\widehat{E}_H = H + Kw$ and $E = H \oplus Kw$. As a consequence the projective spaces \widehat{E}_H and $\mathbf{P}(E)$ are isomorphic, i.e., there is a projectivity between them.

Proposition 26.17. Given any affine space (E, \overrightarrow{E}) , for every projective space $\mathbf{P}(F)$ (where F is some vector space), every hyperplane H in F, and every map $f: E \to \mathbf{P}(F)$ such that $f(E) \subseteq F_H$ and f is affine $(F_H$ being viewed as an affine patch), there is a unique projective map $\widetilde{f}: \widetilde{E} \to \mathbf{P}(F)$ such that

$$f = \widetilde{f} \circ i \quad and \quad \mathbf{P}(\overrightarrow{f}) = \widetilde{f} \circ \mathbf{P}(\overrightarrow{i}),$$

(where $\overrightarrow{i}: \overrightarrow{E} \to \overrightarrow{E}$ and $\overrightarrow{f}: \overrightarrow{E} \to H$ are the linear maps associated with the affine maps $i: E \to \widetilde{E}$ and $f: E \to \mathbf{P}(F)$), as in the following diagram:



Proof. The existence of \widetilde{f} is a consequence of Proposition 25.6, where we observe that \widehat{F}_H is isomorphic to F. Just take the projective map $\mathbf{P}(\widehat{f}) \colon \widetilde{E} \to \mathbf{P}(F)$, where $\widehat{f} \colon \widehat{E} \to F$ is the unique linear map extending f. It remains to prove its uniqueness.

As explained in the proof of Proposition 26.16, the affine patch F_H is affinely isomorphic to some affine hyperplane of the form w+H for some $w \in F-H$. If we pick any $a \in E$, since by hypothesis $f(a) \in F_H$, we may assume that $w \in F-H$ is chosen so that f(a) = [w], and we have $F = Kw \oplus H$. Since $f: E \to F_H$ is affine, for any $a \in E$ and any $u \in E$, we have

$$f(a+u) = f(a) + \overrightarrow{f}(u) = w + \overrightarrow{f}(u),$$

where $\overrightarrow{f}: \overrightarrow{E} \to H$ is a linear map, and where f(a) is viewed as the vector w.

Assume that $\widetilde{f} \colon \widetilde{E} \to \mathbf{P}(F)$ exists with the desired property. Then there is some linear map $g \colon \widehat{E} \to F$ such that $\widetilde{f} = \mathbf{P}(g)$. Our goal is to prove that $g = \mu \widehat{f}$ for some nonzero $\mu \in K$. First, we prove that g vanishes on Ker f.

Since $f = \widetilde{f} \circ i$, we must have f(a) = [w] = [g(a)], and thus $g(a) = \mu w$, for some $\mu \neq 0$. Also, for every $u \in \overrightarrow{E}$,

$$f(a+u) = [w] + \overrightarrow{f}(u) = [w+\overrightarrow{f}(u)] = [g(a+u)]$$
$$= [g(a) + g(u)] = [\mu w + g(u)],$$