However, since f is symmetric, we have $f_\otimes(z) = 0$ for every $z \in C$. Thus, we get an induced linear map $h\colon (E^{\otimes n})/C \to F$ making the following diagram commute.

$$\begin{tikzcd} & E^{\otimes n} \arrow[dr, "p"] \arrow[dd, "f_\otimes"] & \\ E^n \arrow[ur, "\varphi_0"] \arrow[dr, "f"'] & & (E^{\otimes n})/C \arrow[dl, "h"] \\ & F & \end{tikzcd}$$

If we define $h([z]) = f_\otimes(z)$ for every $z \in E^{\otimes n}$, where $[z]$ is the equivalence class in $(E^{\otimes n})/C$ of $z \in E^{\otimes n}$, the above diagram shows that $f = h \circ p \circ \varphi_0 = h \circ \varphi$. We now prove the uniqueness of h. For any linear map $f_\odot\colon (E^{\otimes n})/C \to F$ such that $f = f_\odot \circ \varphi$, since $\varphi(u_1, \ldots, u_n) = u_1 \odot \cdots \odot u_n$ and the vectors $u_1 \odot \cdots \odot u_n$ generate $(E^{\otimes n})/C$, the map $f_\odot$ is uniquely defined by

$$f_\odot(u_1 \odot \cdots \odot u_n) = f(u_1, \ldots, u_n).$$

Since $f = h \circ \varphi$, the map h is unique, and we let $f_\odot = h$. Thus, $\mathrm{S}^n(E) = (E^{\otimes n})/C$ and φ constitute a symmetric n-th tensor power of E. $\square$

The map φ from E^n to $\mathrm{S}^n(E)$ is often denoted $\iota_\odot$, so that

$$\iota_\odot(u_1, \ldots, u_n) = u_1 \odot \cdots \odot u_n.$$

Again, the actual construction is not important. What is important is that the symmetric n-th power has the universal mapping property with respect to symmetric multilinear maps.

Remark: The notation $\odot$ for the commutative multiplication of symmetric tensor powers is not standard. Another notation commonly used is $\cdot$. We often abbreviate "symmetric tensor power" as "symmetric power." The symmetric power $\mathrm{S}^n(E)$ is also denoted $\mathrm{Sym}^n E$ but we prefer to use the notation Sym to denote spaces of symmetric multilinear maps. To be consistent with the use of $\odot$, we could have used the notation $\bigodot^n E$. Clearly, $\mathrm{S}^1(E) \cong E$ and it is convenient to set $\mathrm{S}^0(E) = K$.

The fact that the map $\varphi\colon E^n \to \mathrm{S}^n(E)$ is symmetric and multilinear can also be expressed as follows:

$$\begin{aligned} u_1 \odot \cdots \odot (v_i + w_i) \odot \cdots \odot u_n &= (u_1 \odot \cdots \odot v_i \odot \cdots \odot u_n) + (u_1 \odot \cdots \odot w_i \odot \cdots \odot u_n), \\ u_1 \odot \cdots \odot (\lambda u_i) \odot \cdots \odot u_n &= \lambda(u_1 \odot \cdots \odot u_i \odot \cdots \odot u_n), \\ u_{\sigma(1)} \odot \cdots \odot u_{\sigma(n)} &= u_1 \odot \cdots \odot u_n, \end{aligned}$$

for all permutations $\sigma \in \mathfrak{S}_n$.

The last identity shows that the "operation" $\odot$ is commutative. This allows us to view the symmetric tensor $u_1 \odot \cdots \odot u_n$ as an object called a multiset.