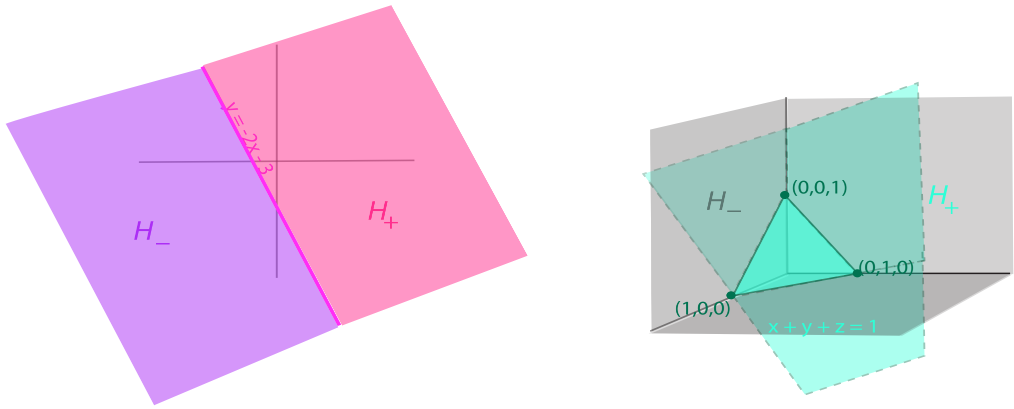

i. ii.

Figure 44.2: Figure i. illustrates the hyperplane $H(\varphi)$ for $\varphi(x, y) = 2x + y + 3$, while Figure ii. illustrates the hyperplane $H(\varphi)$ for $\varphi(x, y, z) = x + y + z - 1$.

with $\varphi(x, y, z) = x + y + z - 1$; this affine form defines the plane given by the equation $x + y + z = 1$, which is the plane through the points $(0, 0, 1), (0, 1, 0)$, and $(1, 0, 0)$. Both of these hyperplanes are illustrated in Figure 44.2.

Definition 44.8. For any two vector $x, y \in \mathbb{R}^n$ with $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ we write $x \leq y$ iff $x_i \leq y_i$ for $i = 1, \ldots, n$, and $x \geq y$ iff $y \leq x$. In particular $x \geq 0$ iff $x_i \geq 0$ for $i = 1, \ldots, n$.

Certain special types of convex sets called cones and $\mathcal{H}$-polyhedra play an important role. The set of feasible solutions of a linear program is an $\mathcal{H}$-polyhedron, and cones play a crucial role in the proof of Proposition 45.1 and in the Farkas–Minkowski proposition (Proposition 47.2).

44.3 Cones, Polyhedral Cones, and $\mathcal{H}$-Polyhedra

Cones and polyhedral cones are defined as follows.

Definition 44.9. Given a nonempty subset $S \subseteq \mathbb{R}^n$, the *cone* $C = \text{cone}(S)$ spanned by S is the convex set

$$\text{cone}(S) = \left\{ \sum_{i=1}^{k} \lambda_i u_i,\ u_i \in S, \lambda_i \in \mathbb{R},\ \lambda_i \geq 0 \right\},$$

of positive combinations of vectors from S. If S consists of a finite set of vectors, the cone $C = \text{cone}(S)$ is called a *polyhedral cone*. Figure 44.3 illustrates a polyhedral cone.