**Proposition 37.4.** *Given a topological space $(E, \mathcal{O})$, given any subset $A$ of $E$, the closure $\overline{A}$ of $A$ is the set of all points $x \in E$ such that for every open set $U$ containing $x$, then $U \cap A \neq \emptyset$. See Figure 37.10.*

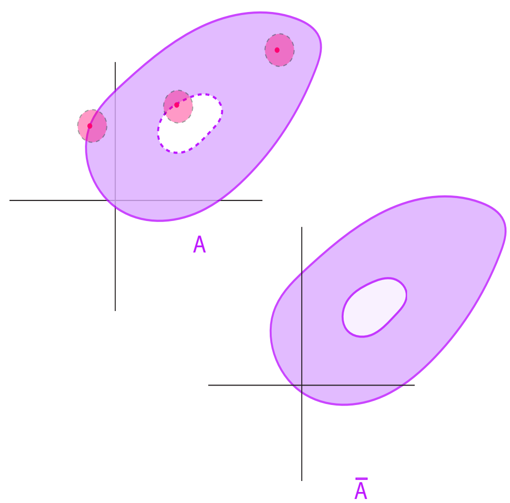


Figure 37.10: The topological space $(E, \mathcal{O})$ is $\mathbb{R}^2$ with topology induced by the Euclidean metric. The purple subset $A$ is illustrated with three red points, each in its closure since the open ball centered at each point has nontrivial intersection with $A$.

*Proof.* If $A = \emptyset$, since $\emptyset$ is closed, the proposition holds trivially. Thus, assume that $A \neq \emptyset$. First assume that $x \in \overline{A}$. Let $U$ be any open set such that $x \in U$. If $U \cap A = \emptyset$, since $U$ is open, then $E - U$ is a closed set containing $A$, and since $\overline{A}$ is the intersection of all closed sets containing $A$, we must have $x \in E - U$, which is impossible. Conversely, assume that $x \in E$ is a point such that for every open set $U$ containing $x$, then $U \cap A \neq \emptyset$. Let $F$ be any closed subset containing $A$. If $x \notin F$, since $F$ is closed, then $U = E - F$ is an open set such that $x \in U$, and $U \cap A = \emptyset$, a contradiction. Thus, we have $x \in F$ for every closed set containing $A$, that is, $x \in \overline{A}$. ∎

Often it is necessary to consider a subset $A$ of a topological space $E$, and to view the subset $A$ as a topological space. The following proposition shows how to define a topology on a subset.

**Proposition 37.5.** *Given a topological space $(E, \mathcal{O})$, given any subset $A$ of $E$, let*

$$\mathcal{U} = \{U \cap A \mid U \in \mathcal{O}\}$$

*be the family of all subsets of $A$ obtained as the intersection of any open set in $\mathcal{O}$ with $A$. The following properties hold.*