In a Hilbert space, the dual space V' is the set of all continuous linear forms $\omega\colon V \to \mathbb{R}$, and the existence of the isomorphism $\sharp$ between V' and V is given by the Riesz representation theorem; see Proposition 48.9. This theorem allows a generalization of the notion of gradient. Indeed, if $f\colon V \to \mathbb{R}$ is a function defined on the Hilbert space V and if f is differentiable at some point $u \in V$, then by definition, the derivative $df_u\colon V \to \mathbb{R}$ is a continuous linear form, so by the Riesz representation theorem (Proposition 48.9) there is a unique vector, denoted $\nabla f_u \in V$, such that

$$df_u(v) = \langle v, \nabla f_u\rangle \quad \text{for all } v \in V.$$

Definition 49.3. The unique vector ∇f_u such that

$$df_u(v) = \langle v, \nabla f_u\rangle \quad \text{for all } v \in V$$

is called the *gradient* of f at u.

Similarly, since the second derivative $\mathrm{D}^2 f_u\colon V \to V'$ of f induces a continuous symmetric billinear form from $V \times V$ to $\mathbb{R}$, by Proposition 48.10, there is a unique continuous self-adjoint linear map $\nabla^2 f_u\colon V \to V$ such that

$$\mathrm{D}^2 f_u(v, w) = \langle \nabla^2 f_u(v), w\rangle \quad \text{for all } v, w \in V.$$

The map $\nabla^2 f_u$ is a generalization of the *Hessian*.

The next theorem is a rather general result about the existence of minima of convex functions defined on convex domains. The proof is quite involved and can be omitted upon first reading.

Theorem 49.2. *Let U be a nonempty, convex, closed subset of a separable Hilbert space V, and let $J\colon V \to \mathbb{R}$ be a convex, differentiable function which is coercive if U is unbounded. Then there is a least one element $u \in V$ such that*

$$u \in U \quad \textit{and} \quad J(u) = \inf_{v \in U} J(v).$$

Proof. As in the proof of Proposition 49.1, since the function J is coercive, we may assume that U is bounded and convex (however, if V infinite dimensional, then U is not compact in general). The proof proceeds in four steps.

Step 1. Consider a *minimizing sequence* $(u_k)_{k \geq 0}$, namely a sequence of elements $u_k \in V$ such that

$$u_k \in U \quad \text{for all } k \geq 0, \qquad \lim_{k \mapsto \infty} J(u_k) = \inf_{v \in U} J(v).$$

At this stage, it is possible that $\inf_{v \in U} J(v) = -\infty$, but we will see that this is actually impossible. However, since U is bounded, the sequence $(u_k)_{k \geq 0}$ is bounded. Our goal is to prove that there is some subsequence of $(w_\ell)_{\ell \geq 0}$ of $(u_k)_{k \geq 0}$ that converges weakly.

Since the sequence $(u_k)_{k \geq 0}$ is bounded there is some constant $C > 0$ such that $\|u_k\| \leq C$ for all $k \geq 0$. Then by the Cauchy–Schwarz inequality, for every $v \in V$ we have

$$|\langle v, u_k\rangle| \leq \|v\| \, \|u_k\| \leq C \, \|v\| \,,$$