The existence of the isomorphism $\flat\colon \overline{E} \to E^*$ is crucial to the existence of adjoint maps. Indeed, Theorem 14.6 allows us to define the adjoint of a linear map on a Hermitian space. Let E be a Hermitian space of finite dimension n, and let $f\colon E \to E$ be a linear map. For every $u \in E$, the map

$$v \mapsto \overline{u \cdot f(v)}$$

is clearly a linear form in E^*, and by Theorem 14.6, there is a unique vector in E denoted by $f^*(u)$, such that

$$\overline{f^*(u) \cdot v} = \overline{u \cdot f(v)},$$

that is,

$$f^*(u) \cdot v = u \cdot f(v), \quad \text{for every } v \in E.$$

The following proposition shows that the map f^* is linear.

Proposition 14.8. *Given a Hermitian space E of finite dimension, for every linear map $f\colon E \to E$ there is a unique linear map $f^*\colon E \to E$ such that*

$$f^*(u) \cdot v = u \cdot f(v), \quad \textit{for all } u, v \in E.$$

Proof. Careful inspection of the proof of Proposition 12.8 reveals that it applies unchanged. The only potential problem is in proving that $f^*(\lambda u) = \lambda f^*(u)$, but everything takes place in the first argument of the Hermitian product, and there, we have linearity. ☐

Definition 14.6. Given a Hermitian space E of finite dimension, for every linear map $f\colon E \to E$, the unique linear map $f^*\colon E \to E$ such that

$$f^*(u) \cdot v = u \cdot f(v), \quad \text{for all } u, v \in E$$

given by Proposition 14.8 is called the *adjoint of f (w.r.t. to the Hermitian product)*.

The fact that

$$v \cdot u = \overline{u \cdot v}$$

implies that the adjoint f^* of f is also characterized by

$$f(u) \cdot v = u \cdot f^*(v),$$

for all $u, v \in E$.

Given two Hermitian spaces E and F, where the Hermitian product on E is denoted by $\langle -, - \rangle_1$ and the Hermitian product on F is denoted by $\langle -, - \rangle_2$, given any linear map $f\colon E \to F$, it is immediately verified that the proof of Proposition 14.8 can be adapted to show that there is a unique linear map $f^*\colon F \to E$ such that

$$\langle f(u), v \rangle_2 = \langle u, f^*(v) \rangle_1$$

for all $u \in E$ and all $v \in F$. The linear map f^* is also called the *adjoint* of f.

As in the Euclidean case, the following properties immediately follow from the definition of the adjoint map.