35.3 Tensor Products of Modules over a Commutative Ring

It is possible to define tensor products of modules over a ring, just as in Section 33.2, and the results of this section continue to hold. The results of Section 33.4 also continue to hold since they are based on the universal mapping property. However, the results of Section 33.3 on bases generally fail, except for free modules. Similarly, the results of Section 33.5 on duality generally fail. Tensor algebras can be defined for modules, as in Section 33.6. Symmetric tensor and alternating tensors can be defined for modules but again, results involving bases generally fail.

Tensor products of modules have some unexpected properties. For example, if p and q are relatively prime integers, then

$$\mathbb{Z}/p\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/q\mathbb{Z} = (0).$$

This is because, by Bezout's identity, there are $a, b \in \mathbb{Z}$ such that

$$ap + bq = 1,$$

so, for all $x \in \mathbb{Z}/p\mathbb{Z}$ and all $y \in \mathbb{Z}/q\mathbb{Z}$, we have

$$\begin{aligned} x \otimes y &= ap(x \otimes y) + bq(x \otimes y) \\ &= a(px \otimes y) + b(x \otimes qy) \\ &= a(0 \otimes y) + b(x \otimes 0) \\ &= 0. \end{aligned}$$

It is possible to salvage certain properties of tensor products holding for vector spaces by restricting the class of modules under consideration. For example, *projective modules* have a pretty good behavior w.r.t. tensor products.

A free A-module F, is a module that has a basis (*i.e.*, there is a family, $(e_i)_{i\in I}$, of linearly independent vectors in F that span F). Projective modules have many equivalent characterizations. Here is one that is best suited for our needs:

Definition 35.7. An A-module, P, is *projective* if it is a summand of a free module, that is, if there is a free A-module, F, and some A-module, Q, so that

$$F = P \oplus Q.$$

Given any A-module, M, we let $M^* = \mathrm{Hom}_A(M, A)$ be its *dual*. We have the following proposition:

Proposition 35.11. *For any finitely-generated projective A-modules, P, and any A-module, Q, we have the isomorphisms:*

$$\begin{aligned} P^{**} &\cong P \\ \mathrm{Hom}_A(P, Q) &\cong P^* \otimes_A Q. \end{aligned}$$