we also have $f(x_{k-1}) = -A_{k-1}(x_\ell)(x_k - x_{k-1})$, so we have

$$f(x_k) = f(x_k) - f(x_{k-1}) - A_{k-1}(x_\ell)(x_k - x_{k-1}),$$

and as in the base case, applying the mean value theorem (Proposition 39.12) to the function $x \mapsto f(x) - A_{k-1}(x_\ell)(x)$, by (2), we obtain

$$\|f(x_k)\| \leq \sup_{x \in B} \|f'(x) - A_{k-1}(x_\ell)\| \, \|x_k - x_{k-1}\| \leq \frac{\beta}{M} \|x_k - x_{k-1}\|,$$

proving (c) for k.

Step 2. Prove that f has a zero in B.

To do this we prove that (x_k) is a Cauchy sequence. This is because, using $(*_2)$, we have

$$\begin{aligned} \|x_{k+j} - x_k\| &\leq \sum_{i=0}^{j-1} \|x_{k+i+1} - x_{k+i}\| \leq \beta^k \left(\sum_{i=0}^{j-1} \beta^i \right) \|x_1 - x_0\| \\ &\leq \frac{\beta^k}{1-\beta} \|x_1 - x_0\|, \end{aligned}$$

for all $k \geq 0$ and all $j \geq 0$, proving that (x_k) is a Cauchy sequence. Since B is a closed ball in a complete normed vector space, B is complete and the Cauchy sequence (x_k) converges to a limit $a \in B$. Since f is continuous on Ω (because it is differentiable), by (c) we obtain

$$\|f(a)\| = \lim_{k \mapsto \infty} \|f(x_k)\| \leq \frac{\beta}{M} \lim_{k \mapsto \infty} \|x_k - x_{k-1}\| = 0,$$

which yields $f(a) = 0$.

Since

$$\|x_{k+j} - x_k\| \leq \frac{\beta^k}{1-\beta} \|x_1 - x_0\|,$$

if we let j tend to infinity, we obtain the inequality

$$\|x_k - a\| = \|a - x_k\| \leq \frac{\beta^k}{1-\beta} \|x_1 - x_0\|,$$

which is the last statement of the theorem.

Step 3. Prove that f has a unique zero in B.

Suppose $f(a) = f(b) = 0$ with $a, b \in B$. Since $A_0^{-1}(x_0)(A_0(x_0)(b-a)) = b - a$, we have

$$b - a = -A_0^{-1}(x_0)(f(b) - f(a) - A_0(x_0)(b-a)),$$

which by (1) and (2) and the mean value theorem implies that

$$\|b - a\| \leq \left\|A_0^{-1}(x_0)\right\| \sup_{x \in B} \|f'(x) - A_0(x_0)\| \, \|b - a\| \leq \beta \|b - a\|.$$

Since $0 < \beta < 1$, the inequality $\|b - a\| \leq \beta \|b - a\|$ is only possible if $a = b$. $\square$