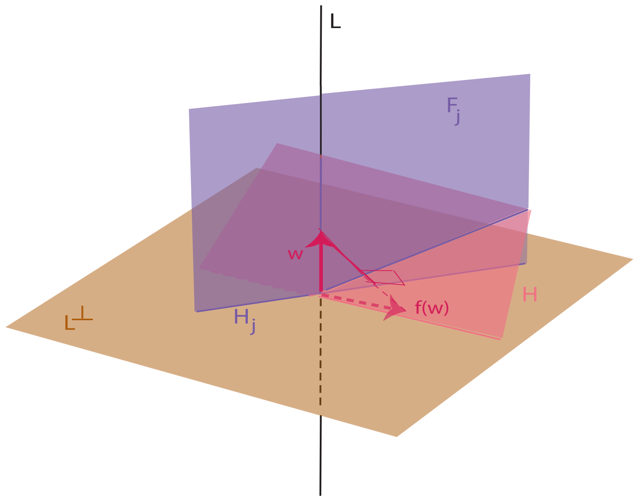

Figure 27.4: An isometry f as a composition of reflections when 1 is not an eigenvalue of f. Note that the pink plane H is perpendicular to $f(w) - w$.

to $n-1$ of reflections, and when n is even, every improper orthogonal transformation is the product of an odd number less than or equal to $n-1$ of reflections.

In particular, for $n = 3$, every rotation is the product of two reflections about planes. When n is odd, we can say more about improper isometries. Indeed, when n is odd, every improper isometry admits the eigenvalue -1. This is because if E is a Euclidean space of finite dimension and $f\colon E \to E$ is an isometry, because $\|f(u)\| = \|u\|$ for every $u \in E$, if λ is any eigenvalue of f and u is an eigenvector associated with λ, then

$$\|f(u)\| = \|\lambda u\| = |\lambda|\|u\| = \|u\|,$$

which implies $|\lambda| = 1$, since $u \neq 0$. Thus, the real eigenvalues of an isometry are either $+1$ or -1. However, it is well known that polynomials of odd degree always have some real root. As a consequence, the characteristic polynomial $\det(f - \lambda \mathrm{id})$ of f has some real root, which is either $+1$ or -1. Since f is an improper isometry, $\det(f) = -1$, and since $\det(f)$ is the product of the eigenvalues, the real roots cannot all be $+1$, and thus -1 is an eigenvalue of f. Going back to the proof of Theorem 27.1, since -1 is an eigenvalue of f, there is some nonnull eigenvector w such that $f(w) = -w$. Using the second part of the proof, we see that the hyperplane H orthogonal to $f(w) - w = -2w$ is in fact orthogonal to w, and thus f is the product of $k \leq n$ reflections about hyperplanes $H, F_1, \ldots, F_{k-1}$ such that $F_i = H_i \oplus L$, where L is a line orthogonal to H, and the H_i are hyperplanes in $H = L^{\perp}$ orthogonal to L. However, k must be odd, and so $k-1$ is even, and thus the composition of the reflections about $F_1, \ldots, F_{k-1}$ is a rotation. Thus, when n is odd, an improper isometry is the composition of a reflection about a hyperplane H with a rotation consisting of reflections about hyperplanes $F_1, \ldots, F_{k-1}$ containing a line, L, orthogonal to