and that

$$\begin{vmatrix} u_k & x_k \\ v_k & y_k \end{vmatrix} = -1 - \frac{7}{3}k - \frac{23}{12}k^2 - \frac{2}{3}k^3 - \frac{1}{12}k^4.$$

As a consequence, prove that amazingly

$$d_n = D_{n-2} = -\frac{1}{12}n^2(n^2-1).$$

(6) Prove that the characteristic polynomial of A is indeed

$$P_A(\lambda) = \lambda^{n-2}\left(\lambda^2 - n^2\lambda - \frac{1}{12}n^2(n^2-1)\right).$$

Use the above to show that the two nonzero eigenvalues of A are

$$\lambda = \frac{n}{2}\left(n \pm \frac{\sqrt{3}}{3}\sqrt{4n^2-1}\right).$$

The negative eigenvalue λ_1 can also be expressed as

$$\lambda_1 = n^2\frac{(3-2\sqrt{3})}{6}\sqrt{1-\frac{1}{4n^2}}.$$

Use this expression to explain the following phenomenon: if we add any number greater than or equal to $(2/25)n^2$ to every diagonal entry of A we get an invertible matrix. What about $0.077351n^2$? Try it!

Problem 15.16. Let A be a symmetric tridiagonal $n \times n$-matrix

$$A = \begin{pmatrix} b_1 & c_1 & & & & \\ c_1 & b_2 & c_2 & & & \\ & c_2 & b_3 & c_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & c_{n-2} & b_{n-1} & c_{n-1} \\ & & & & c_{n-1} & b_n \end{pmatrix},$$

where it is assumed that $c_i \neq 0$ for all i, $1 \leq i \leq n-1$, and let A_k be the $k \times k$-submatrix consisting of the first k rows and columns of A, $1 \leq k \leq n$. We define the polynomials $P_k(x)$ as follows: $(0 \leq k \leq n)$.

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= b_1 - x, \\ P_k(x) &= (b_k - x)P_{k-1}(x) - c_{k-1}^2 P_{k-2}(x), \end{aligned}$$