(2) Consider the $n \times n$ symmetric matrices $S^{i,j}$ defined for all i, j with $1 \leq i < j \leq n$ and $n \geq 3$, such that the only nonzero entries are

$$\begin{aligned} S^{i,j}(i,j) &= 1 \\ S^{i,j}(i,i) &= 0 \\ S^{i,j}(j,i) &= 1 \\ S^{i,j}(j,j) &= 0 \\ S^{i,j}(k,k) &= 1, \quad 1 \leq k \leq n, k \neq i, j, \end{aligned}$$

and if $i + 2 \leq j$ then $S^{i,j}(i+1, i+1) = -1$, else if $i > 1$ and $j = i + 1$ then $S^{i,j}(1,1) = -1$, and if $i = 1$ and $j = 2$, then $S^{i,j}(3,3) = -1$.

For example,

$$S^{i,j} = \begin{pmatrix} 1 & & & & & & & & & & \\ & \ddots & & & & & & & & & \\ & & 1 & & & & & & & & \\ & & & 0 & 0 & \cdots & 0 & 1 & & & \\ & & & 0 & -1 & \cdots & 0 & 0 & & & \\ & & & \vdots & \vdots & \ddots & \vdots & \vdots & & & \\ & & & 0 & 0 & \cdots & 1 & 0 & & & \\ & & & 1 & 0 & \cdots & 0 & 0 & & & \\ & & & & & & & & 1 & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & 1 \end{pmatrix}.$$

Note that $S^{i,j}$ has a single diagonal entry equal to -1. Prove that the $S^{i,j}$ are rotations matrices.

Use Problem 3.15 together with the $S^{i,j}$ to form a basis of the $n \times n$ symmetric matrices.

(3) Prove that if $n \geq 3$, the set of all linear combinations of matrices in $\mathbf{SO}(n)$ is the space $\mathrm{M}_n(\mathbb{R})$ of all $n \times n$ matrices.

Prove that if $n \geq 3$ and if a matrix $A \in \mathrm{M}_n(\mathbb{R})$ commutes with all rotations matrices, then A commutes with all matrices in $\mathrm{M}_n(\mathbb{R})$.

What happens for $n = 2$?

Problem 12.13. (1) Let H be the affine hyperplane in $\mathbb{R}^n$ given by the equation

$$a_1 x_1 + \cdots + a_n x_n = c,$$

with $a_i \neq 0$ for some $i, 1 \leq i \leq n$. The linear hyperplane H_0 parallel to H is given by the equation

$$a_1 x_1 + \cdots + a_n x_n = 0,$$