where R is an orthogonal matrix and Σ is a diagonal matrix

$$\Sigma = \mathrm{diag}(\lambda_1, \ldots, \lambda_s, 0, \ldots, 0),$$

where s is the rank of P and $\lambda_1 \geq \cdots \geq \lambda_s > 0$. Then $v^\top P v = 0$ is equivalent to

$$v^\top R^\top \Sigma R v = 0,$$

equivalently

$$(Rv)^\top \Sigma R v = 0.$$

If we write $Rv = y$, then we have

$$0 = (Rv)^\top \Sigma R v = y^\top \Sigma y = \sum_{i=1}^{s} \lambda_i y_i^2,$$

and since $\lambda_i > 0$ for $i = 1, \ldots, s$, this implies that $y_i = 0$ for $i = 1, \ldots, s$. Consequently, $\Sigma y = \Sigma R v = 0$, and so $Pv = R^\top \Sigma R v = 0$, as claimed. Since $v \neq 0$, the vector $(v, 0)$ is a nontrivial solution of Equations $(*)$, a contradiction of the invertibility assumption of the KKT-matrix.

Observe that we proved that $Av = 0$ and $Pv = 0$ iff $Av = 0$ and $v^\top P v = 0$, so we easily obtain the fact that Condition (2) is equivalent to the invertibility of the KKT-matrix. Parts (3) and (4) are left as an exercise. $\square$

In particular, if P is positive definite, then Proposition 50.11(4) applies, as we already know from Proposition 42.3. In this case, we can solve for x by elimination. We get

$$x = -P^{-1}(A^\top \lambda + q), \quad \text{where} \quad \lambda = -(AP^{-1}A^\top)^{-1}(b + AP^{-1}q).$$

In practice, we do not invert P and $AP^{-1}A^\top$. Instead, we solve the linear systems

$$\begin{aligned} Pz &= q \\ PE &= A^\top \\ (AE)\lambda &= -(b + Az) \\ Px &= -(A^\top \lambda + q). \end{aligned}$$

Observe that $(AP^{-1}A^\top)^{-1}$ is the Schur complement of P in the KKT matrix.

Since the KKT-matrix is symmetric, if it is invertible, we can convert it to $LDL^\top$ form using Proposition 8.6. This method is only practical when the problem is small or when A and P are sparse.

If the KKT-matrix is invertible but P is not, then we can use a trick involving Proposition 50.11. We find a symmetric positive semidefinite matrix Q such that $P + A^\top Q A$ is symmetric