Remark: If A and b are perturbed simultaneously, so that we get the "perturbed" system

$$(A + \Delta A)(x + \Delta x) = b + \Delta b,$$

it can be shown that if $\|\Delta A\| < 1/\left\|A^{-1}\right\|$ (and $b \neq 0$), then

$$\frac{\|\Delta x\|}{\|x\|} \leq \frac{\mathrm{cond}(A)}{1 - \|A^{-1}\| \, \|\Delta A\|} \left(\frac{\|\Delta A\|}{\|A\|} + \frac{\|\Delta b\|}{\|b\|} \right);$$

see Demmel [48], Section 2.2 and Horn and Johnson [95], Section 5.8.

We now list some properties of condition numbers and figure out what $\mathrm{cond}(A)$ is in the case of the spectral norm (the matrix norm induced by $\| \, \|_2$). First, we need to introduce a very important factorization of matrices, the *singular value decomposition*, for short, *SVD*.

It can be shown (see Section 22.2) that given any $n \times n$ matrix $A \in \mathrm{M}_n(\mathbb{C})$, there exist two unitary matrices U and V, and a *real* diagonal matrix $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$, with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$, such that

$$A = V \Sigma U^*.$$

Definition 9.11. Given a complex $n \times n$ matrix A, a triple (U, V, Σ) such that $A = V\Sigma U^*$, where U and V are $n \times n$ unitary matrices and $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ is a diagonal matrix of real numbers $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$, is called a *singular decomposition* (for short *SVD*) of A. If A is a real matrix, then U and V are orthogonal matrices The nonnegative numbers $\sigma_1, \ldots, \sigma_n$ are called the *singular values* of A.

The factorization $A = V\Sigma U^*$ implies that

$$A^*A = U\Sigma^2 U^* \quad \text{and} \quad AA^* = V\Sigma^2 V^*,$$

which shows that $\sigma_1^2, \ldots, \sigma_n^2$ are the eigenvalues of *both* A^*A and AA^*, that the columns of U are corresponding eigenvectors for A^*A, and that the columns of V are corresponding eigenvectors for AA^*.

Since σ_1^2 is the largest eigenvalue of A^*A (and AA^*), note that $\sqrt{\rho(A^*A)} = \sqrt{\rho(AA^*)} = \sigma_1$.

Corollary 9.15. *The spectral norm $\|A\|_2$ of a matrix A is equal to the largest singular value of A. Equivalently, the spectral norm $\|A\|_2$ of a matrix A is equal to the ℓ^∞-norm of its vector of singular values,*

$$\|A\|_2 = \max_{1 \leq i \leq n} \sigma_i = \|(\sigma_1, \ldots, \sigma_n)\|_\infty .$$

Since the Frobenius norm of a matrix A is defined by $\|A\|_F = \sqrt{\mathrm{tr}(A^*A)}$ and since

$$\mathrm{tr}(A^*A) = \sigma_1^2 + \cdots + \sigma_n^2$$

where $\sigma_1^2, \ldots, \sigma_n^2$ are the eigenvalues of A^*A, we see that

$$\|A\|_F = (\sigma_1^2 + \cdots + \sigma_n^2)^{1/2} = \|(\sigma_1, \ldots, \sigma_n)\|_2 .$$