(1) $\kappa(x, y) = \kappa_1(x, y) + \kappa_2(x, y)$.

(2) $\kappa(x, y) = a\kappa_1(x, y)$.

(3) $\kappa(x, y) = f(x)\overline{f(y)}$.

(4) $\kappa(x, y) = \kappa_3(\psi(x), \psi(y))$.

(5) If B is a symmetric positive semidefinite $n \times n$ matrix, then the map $\kappa\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by

$$\kappa(x, y) = x^\top B y$$

is a positive definite kernel.

Proof. (1) For every finite subset $S = \{x_1, \ldots, x_p\}$ of X, if K_1 is the $p \times p$ matrix

$$K_1 = (\kappa_1(x_k, x_j))_{1 \leq j,k \leq p}$$

and if if K_2 is the $p \times p$ matrix

$$K_2 = (\kappa_2(x_k, x_j))_{1 \leq j,k \leq p},$$

then for any $u \in \mathbb{C}^p$, we have

$$u^*(K_1 + K_2)u = u^* K_1 u + u^* K_2 u \geq 0,$$

since $u^* K_1 u \geq 0$ and $u^* K_2 u \geq 0$ because κ_2 and κ_2 are positive definite kernels, which means that K_1 and K_2 are positive semidefinite.

(2) We have

$$u^*(aK_1)u = au^* K_1 u \geq 0,$$

since $a > 0$ and $u^* K_1 u \geq 0$.

(3) For every finite subset $S = \{x_1, \ldots, x_p\}$ of X, if K is the $p \times p$ matrix

$$K = (\kappa(x_k, x_j))_{1 \leq j,k \leq p} = (\overline{f(x_k)} f(x_j))_{1 \leq j,k \leq p}$$

then we have

$$u^* K u = u^\top K^\top \overline{u} = \sum_{j,k=1}^{p} \kappa(x_j, x_k) u_j \overline{u_k} = \sum_{j,k=1}^{p} u_j f(x_j) \overline{u_k f(x_k)} = \left| \sum_{j=1}^{p} u_j f(x_j) \right|^2 \geq 0.$$

(4) For every finite subset $S = \{x_1, \ldots, x_p\}$ of X, the $p \times p$ matrix K given by

$$K = (\kappa(x_k, x_j))_{1 \leq j,k \leq p} = (\kappa_3(\psi(x_k), \psi(x_j)))_{1 \leq j,k \leq p}$$

is symmetric positive semidefinite since κ_3 is a positive definite kernel.