Fortunately, because A is SPD, we can reduce this generalized eigenvalue problem to a standard eigenvalue problem. A good way to do so is to use a Cholesky decomposition of A as

$$A = LL^\top,$$

where L is a lower triangular matrix (see Theorem 8.10). Because A is SPD, it is invertible, so L is also invertible, and

$$K\mathbf{U} = \lambda A\mathbf{U} = \lambda LL^\top\mathbf{U}$$

yields

$$L^{-1}K\mathbf{U} = \lambda L^\top\mathbf{U},$$

which can also be written as

$$L^{-1}K(L^\top)^{-1}L^\top\mathbf{U} = \lambda L^\top\mathbf{U}.$$

Then, if we make the change of variable

$$\mathbf{Y} = L^\top\mathbf{U},$$

using the fact $(L^\top)^{-1} = (L^{-1})^\top$, the above equation is equivalent to

$$L^{-1}K(L^{-1})^\top\mathbf{Y} = \lambda\mathbf{Y},$$

a standard eigenvalue problem for the matrix $\widehat{K} = L^{-1}K(L^{-1})^\top$. Furthermore, we know from Section 8.8 that since K is SPD and L^{-1} is invertible, the matrix $\widehat{K} = L^{-1}K(L^{-1})^\top$ is also SPD.

Consequently, $\widehat{K}$ has positive real eigenvalues $(\omega_1^2, \ldots, \omega_n^2)$ (not necessarily distinct) and it can be diagonalized with respect to an orthonormal basis of eigenvectors, say $\mathbf{Y}^1, \ldots, \mathbf{Y}^n$. Then, since $\mathbf{Y} = L^\top\mathbf{U}$, the vectors

$$\mathbf{U}^i = (L^\top)^{-1}\mathbf{Y}^i, \quad i = 1, \ldots, n,$$

are linearly independent and are solutions of the generalized eigenvalue problem; that is,

$$K\mathbf{U}^i = \omega_i^2 A\mathbf{U}^i, \quad i = 1, \ldots, n.$$

More is true. Because the vectors $\mathbf{Y}^1, \ldots, \mathbf{Y}^n$ are orthonormal, and because $\mathbf{Y}^i = L^\top\mathbf{U}^i$, from

$$(\mathbf{Y}^i)^\top\mathbf{Y}^j = \delta_{ij},$$

we get

$$(\mathbf{U}^i)^\top LL^\top\mathbf{U}^j = \delta_{ij}, \quad 1 \leq i, j \leq n,$$

and since $A = LL^\top$, this yields

$$(\mathbf{U}^i)^\top A\mathbf{U}^j = \delta_{ij}, \quad 1 \leq i, j \leq n.$$