is the *null set*, or *kernel,* of the affine map $f\colon \mathbb{A}^m \to \mathbb{R}$, in the sense that

$$H = f^{-1}(0) = \{x \in \mathbb{A}^m \mid f(x) = 0\},$$

where $x = (x_1, \ldots, x_m)$.

Thus, it is interesting to consider *affine forms*, which are just affine maps $f\colon E \to \mathbb{R}$ from an affine space to $\mathbb{R}$. Unlike linear forms $f^*$, for which $\operatorname{Ker} f^*$ is never empty (since it always contains the vector 0), it is possible that $f^{-1}(0) = \emptyset$ for an affine form $f$. Given an affine map $f\colon E \to \mathbb{R}$, we also denote $f^{-1}(0)$ by $\operatorname{Ker} f$, and we call it the *kernel* of $f$. Recall that an (affine) hyperplane is an affine subspace of codimension 1. The relationship between affine hyperplanes and affine forms is given by the following proposition.

**Proposition 24.14.** *Let $E$ be an affine space. The following properties hold:*

(a) *Given any nonconstant affine form $f\colon E \to \mathbb{R}$, its kernel $H = \operatorname{Ker} f$ is a hyperplane.*

(b) *For any hyperplane $H$ in $E$, there is a nonconstant affine form $f\colon E \to \mathbb{R}$ such that $H = \operatorname{Ker} f$. For any other affine form $g\colon E \to \mathbb{R}$ such that $H = \operatorname{Ker} g$, there is some $\lambda \in \mathbb{R}$ such that $g = \lambda f$ (with $\lambda \neq 0$).*

(c) *Given any hyperplane $H$ in $E$ and any (nonconstant) affine form $f\colon E \to \mathbb{R}$ such that $H = \operatorname{Ker} f$, every hyperplane $H'$ parallel to $H$ is defined by a nonconstant affine form $g$ such that $g(a) = f(a) - \lambda$, for all $a \in E$ and some $\lambda \in \mathbb{R}$.*

*Proof.* The proof is straightforward, and is omitted. It is also given in Gallier [70]. □

When $E$ is of dimension $n$, given an affine frame $(a_0, (u_1, \ldots, u_n))$ of $E$ with origin $a_0$, recall from Definition 24.5 that every point of $E$ can be expressed uniquely as $x = a_0 + x_1u_1 + \cdots + x_nu_n$, where $(x_1, \ldots, x_n)$ are the *coordinates* of $x$ with respect to the affine frame $(a_0, (u_1, \ldots, u_n))$.

Also recall that every linear form $f^*$ is such that $f^*(x) = \lambda_1x_1 + \cdots + \lambda_nx_n$, for every $x = x_1u_1 + \cdots + x_nu_n$ and some $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Since an affine form $f\colon E \to \mathbb{R}$ satisfies the property $f(a_0 + x) = f(a_0) + \overrightarrow{f}(x)$, denoting $f(a_0 + x)$ by $f(x_1, \ldots, x_n)$, we see that we have

$$f(x_1, \ldots, x_n) = \lambda_1x_1 + \cdots + \lambda_nx_n + \mu,$$

where $\mu = f(a_0) \in \mathbb{R}$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Thus, a hyperplane is the set of points whose coordinates $(x_1, \ldots, x_n)$ satisfy the (affine) equation

$$\lambda_1x_1 + \cdots + \lambda_nx_n + \mu = 0.$$