Remark: As we noted, the hypothesis of Zorn's lemma implies that S is nonempty (since the empty set must be bounded). A partially ordered set such that every chain is bounded is sometimes called *inductive*.

We now give some applications of Zorn's lemma.

C.2 Proof of the Existence of a Basis in a Vector Space

Using Zorn's lemma, we can prove that Theorem 3.7 holds for arbitrary vector spaces, and not just for finitely generated vector spaces, as promised in Chapter 3.

Theorem C.2. *Given any family, $S = (u_i)_{i\in I}$, generating a vector space E and any linearly independent subfamily, $L = (u_j)_{j\in J}$, of S (where $J \subseteq I$), there is a basis, B, of E such that $L \subseteq B \subseteq S$.*

Proof. Consider the set $\mathcal{L}$ of linearly independent families, B, such that $L \subseteq B \subseteq S$. Since $L \in \mathcal{L}$, this set is nonempty. We claim that $\mathcal{L}$ is inductive. Consider any chain, $(B_l)_{l\in\Lambda}$, of linearly independent families B_l in $\mathcal{L}$, and look at $B = \bigcup_{l\in\Lambda} B_l$. The family B is of the form $B = (v_h)_{h\in H}$, for some index set H, and it must be linearly independent. Indeed, if this was not true, there would be some family $(\lambda_h)_{h\in H}$ of scalars, of finite support, so that

$$\sum_{h\in H} \lambda_h v_h = 0,$$

where not all λ_h are zero. Since $B = \bigcup_{l\in\Lambda} B_l$ and only finitely many λ_h are nonzero, there is a finite subset, F, of Λ, so that $v_h \in B_{f_h}$ iff $\lambda_h \neq 0$. But $(B_l)_{l\in\Lambda}$ is a chain, and if we let $f = \max\{f_h \mid f_h \in F\}$, then $v_h \in B_f$, for all v_h for which $\lambda_h \neq 0$. Thus,

$$\sum_{h\in H} \lambda_h v_h = 0$$

would be a nontrivial linear dependency among vectors from B_f, a contradiction. Therefore, $B \in \mathcal{L}$, and since B is obviously an upper bound for the B_l's, we have proved that $\mathcal{L}$ is inductive. By Zorn's lemma (Lemma C.1), the set $\mathcal{L}$ has some maximal element, say $B = (u_h)_{h\in H}$. The rest of the proof is the same as in the proof of Theorem 3.7, but we repeat it for the reader's convenience. We claim that B generates E. Indeed, if B does not generate E, then there is some $u_p \in S$ that is not a linear combination of vectors in B (since S generates E), with $p \notin H$. Then, by Lemma 3.6, the family $B' = (u_h)_{h\in H\cup\{p\}}$ is linearly independent, and since $L \subseteq B \subset B' \subseteq S$, this contradicts the maximality of B. Thus, B is a basis of E such that $L \subseteq B \subseteq S$. $\square$

Another important application of Zorn's lemma is the existence of maximal ideals.