Proposition 14.9. *(1) For any linear map $f\colon E \to F$, we have*

$$f^{**} = f.$$

(2) For any two linear maps $f, g\colon E \to F$ and any scalar $\lambda \in \mathbb{R}$:

$$\begin{aligned}(f+g)^* &= f^* + g^*\\ (\lambda f)^* &= \overline{\lambda} f^*.\end{aligned}$$

(3) If E, F, G are Hermitian spaces with respective inner products $\langle -,- \rangle_1, \langle -,- \rangle_2$, and $\langle -,- \rangle_3$, and if $f\colon E \to F$ and $g\colon F \to G$ are two linear maps, then

$$(g \circ f)^* = f^* \circ g^*.$$

As in the Euclidean case, a linear map $f\colon E \to E$ (where E is a finite-dimensional Hermitian space) is *self-adjoint* if $f = f^*$. The map f is *positive semidefinite* iff

$$\langle f(x), x \rangle \geq 0 \quad \text{all } x \in E;$$

positive definite iff

$$\langle f(x), x \rangle > 0 \quad \text{all } x \in E,\ x \neq 0.$$

An interesting corollary of Proposition 14.3 is that a positive semidefinite linear map must be self-adjoint. In fact, we can prove a slightly more general result.

Proposition 14.10. *Given any finite-dimensional Hermitian space E with Hermitian product $\langle -,- \rangle$, for any linear map $f\colon E \to E$, if $\langle f(x), x \rangle \in \mathbb{R}$ for all $x \in E$, then f is self-adjoint. In particular, any positive semidefinite linear map $f\colon E \to E$ is self-adjoint.*

Proof. Since $\langle f(x), x \rangle \in \mathbb{R}$ for all $x \in E$, we have

$$\begin{aligned}\langle f(x), x \rangle &= \overline{\langle f(x), x \rangle}\\ &= \langle x, f(x) \rangle\\ &= \langle f^*(x), x \rangle,\end{aligned}$$

so we have

$$\langle (f - f^*)(x), x \rangle = 0 \quad \text{all } x \in E,$$

and Proposition 14.3 implies that $f - f^* = 0$. ☐

Beware that Proposition 14.10 is false if E is a real Euclidean space.

As in the Euclidean case, Theorem 14.6 can be used to show that any Hermitian space of finite dimension has an orthonormal basis. The proof is unchanged.

Proposition 14.11. *Given any nontrivial Hermitian space E of finite dimension $n \geq 1$, there is an orthonormal basis $(u_1, \ldots, u_n)$ for E.*