An obvious induction shows that

$$A_{k+1} = Q_k^* \cdots Q_1^* A_1 Q_1 \cdots Q_k = P_k^* A P_k,$$

that is

$$A_{k+1} = P_k^* A P_k. \tag{$*_2$}$$

Therefore, A_{k+1} and A are similar, so they have the same eigenvalues.

The basic QR iteration method consists in computing the sequence of matrices A_k, and in the ideal situation, to expect that A_k "converges" to an upper triangular matrix, more precisely that the part of A_k below the main diagonal goes to zero, and the diagonal entries converge to the eigenvalues of A.

This ideal situation is only achieved in rather special cases. For one thing, if A is unitary (or orthogonal in the real case), since in the QR decomposition we have $R = I$, we get $A_2 = IQ = Q = A_1$, so the method does *not* make any progress. Also, if A is a real matrix, since the A_k are also real, if A has complex eigenvalues, then the part of A_k below the main diagonal can't go to zero. Generally, the method runs into troubles whenever A has distinct eigenvalues with the same modulus.

The convergence of the sequence (A_k) is only known under some fairly restrictive hypotheses. Even under such hypotheses, this is not really genuine convergence. Indeed, it can be shown that the part of A_k below the main diagonal goes to zero, and the diagonal entries converge to the eigenvalues of A, but the part of A_k above the diagonal *may not converge*. However, for the purpose of finding the eigenvalues of A, this does not matter.

The following convergence result is proven in Ciarlet [41] (Chapter 6, Theorem 6.3.10 and Serre [156] (Chapter 13, Theorem 13.2). It is rarely applicable in practice, except for symmetric (or Hermitian) positive definite matrices, as we will see shortly.

Theorem 18.1. *Suppose the (complex) $n \times n$ matrix A is invertible, diagonalizable, and that its eigenvalues $\lambda_1, \ldots, \lambda_n$ have different moduli, so that*

$$|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n| > 0.$$

If $A = P\Lambda P^{-1}$, where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, and if P^{-1} has an LU-factorization, then the strictly lower-triangular part of A_k converges to zero, and the diagonal of A_k converges to Λ.

Proof. We reproduce the proof in Ciarlet [41] (Chapter 6, Theorem 6.3.10). The strategy is to study the asymptotic behavior of the matrices $P_k = Q_1 Q_2 \cdots Q_k$. For this, it turns out that we need to consider the powers A^k.

Step 1. Let $\mathcal{R}_k = R_k \cdots R_2 R_1$. We claim that

$$A^k = (Q_1 Q_2 \cdots Q_k)(R_k \cdots R_2 R_1) = P_k \mathcal{R}_k. \tag{$*_3$}$$