(a) Either the constraints φ_i are affine for all $i = 1, \ldots, m$ and $U \neq \emptyset$, or

(b) There is some vector $v \in \Omega$ such that the following conditions hold for $i = 1, \ldots, m$:

   (i) $\varphi_i(v) \leq 0$.

   (ii) If φ_i is not affine, then $\varphi_i(v) < 0$.

The above qualification conditions are known as *Slater's conditions*.

Condition (b)(i) also implies that U has nonempty relative interior. If Ω is convex, then U is also convex. This is because for all $u, v \in \Omega$, if $u \in U$ and $v \in U$, that is $\varphi_i(u) \leq 0$ and $\varphi_i(v) \leq 0$ for $i = 1, \ldots, m$, since the functions φ_i are convex, for all $\theta \in [0, 1]$ we have

$$\begin{aligned} \varphi_i((1-\theta)u + \theta v) &\leq (1-\theta)\varphi_i(u) + \theta\varphi_i(v) && \text{since } \varphi_i \text{ is convex} \\ &\leq 0 && \text{since } 1-\theta \geq 0, \theta \geq 0, \varphi_i(u) \leq 0, \varphi_i(v) \leq 0, \end{aligned}$$

and any intersection of convex sets is convex.

It is important to observe that a *nonaffine equality constraint* $\varphi_i(u) = 0$ is *never* qualified.

Indeed, $\varphi_i(u) = 0$ is equivalent to $\varphi_i(u) \leq 0$ and $-\varphi_i(u) \leq 0$, so if these constraints are qualified and if φ_i is not affine then there is some nonzero vector $v \in \Omega$ such that both $\varphi_i(v) < 0$ and $-\varphi_i(v) < 0$, which is impossible. For this reason, *equality constraints are often assumed to be affine*.

The following theorem yields a more flexible version of Theorem 50.5 for constraints given by convex functions. If in addition, the function J is also *convex*, then the KKT conditions are also a *sufficient* condition for a local minimum.

Theorem 50.6. *Let $\varphi_i \colon \Omega \to \mathbb{R}$ be m convex constraints defined on some open convex subset Ω of a finite-dimensional Euclidean vector space V (more generally, a real Hilbert space V), let $J \colon \Omega \to \mathbb{R}$ be some function, let U be given by*

$$U = \{x \in \Omega \mid \varphi_i(x) \leq 0, \ \ 1 \leq i \leq m\},$$

and let $u \in U$ be any point such that the functions φ_i and J are differentiable at u.

(1) If J has a local minimum at u with respect to U, and if the constraints are qualified, then there exist some scalars $\lambda_i(u) \in \mathbb{R}$, such that the KKT condition hold:

$$J'_u + \sum_{i=1}^{m} \lambda_i(u)(\varphi'_i)_u = 0$$

and

$$\sum_{i=1}^{m} \lambda_i(u)\varphi_i(u) = 0, \quad \lambda_i(u) \geq 0, \quad i = 1, \ldots, m.$$