Problem 12.15. (1) Find two symmetric matrices, A and B, such that AB is not symmetric.

(2) Find two matrices A and B such that

$$e^A e^B \neq e^{A+B}.$$

Hint. Try

$$A = \pi \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \pi \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

and use the Rodrigues formula.

(3) Find some square matrices A, B such that $AB \neq BA$, yet

$$e^A e^B = e^{A+B}.$$

Hint. Look for 2×2 matrices with zero trace and use Problem 9.15.

Problem 12.16. Given a field K and any nonempty set I, let $K^{(I)}$ be the subset of the cartesian product K^I consisting of all functions $\lambda\colon I \to K$ with *finite support*, which means that $\lambda(i) = 0$ for all but finitely many $i \in I$. We usually denote the function defined by λ as $(\lambda_i)_{i\in I}$, and call is a *family indexed by* I. We define addition and multiplication by a scalar as follows:

$$(\lambda_i)_{i\in I} + (\mu_i)_{i\in I} = (\lambda_i + \mu_i)_{i\in I},$$

and

$$\alpha \cdot (\mu_i)_{i\in I} = (\alpha\mu_i)_{i\in I}.$$

(1) Check that $K^{(I)}$ is a vector space.

(2) If I is any nonempty subset, for any $i \in I$, we denote by e_i the family $(e_j)_{j\in I}$ defined so that

$$e_j = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i. \end{cases}$$

Prove that the family $(e_i)_{i\in I}$ is linearly independent and spans $K^{(I)}$, so that it is a basis of $K^{(I)}$ called the *canonical basis* of $K^{(I)}$. When I is finite, say of cardinality n, then prove that $K^{(I)}$ is isomorphic to K^n.

(3) The function $\iota\colon I \to K^{(I)}$, such that $\iota(i) = e_i$ for every $i \in I$, is clearly an injection.

For any other vector space F, for any function $f\colon I \to F$, prove that there is a *unique linear map* $\overline{f}\colon K^{(I)} \to F$, such that

$$f = \overline{f} \circ \iota,$$

as in the following commutative diagram:

$$\begin{tikzcd} I \arrow[r, "\iota"] \arrow[dr, "f"'] & K^{(I)} \arrow[d, "\overline{f}"] \\ & F \end{tikzcd}.$$