Proof. By property (T), we have $\varphi(x, x) = \beta + \epsilon\overline{\beta}$ for some $\beta \in K$. For any $y \in U$, since φ is ϵ-Hermitian, $\varphi(y, x) = \epsilon\overline{\varphi(x, y)}$, and since U is totally isotropic $\varphi(y, y) = 0$, so we have

$$\begin{aligned}\varphi(x + y, x + y) &= \varphi(x, x) + \varphi(x, y) + \varphi(y, x) + \varphi(y, y)\\ &= \beta + \epsilon\overline{\beta} + \varphi(x, y) + \epsilon\overline{\varphi(x, y)}\\ &= \beta + \varphi(x, y) + \epsilon\overline{(\beta + \varphi(x, y)}.\end{aligned}$$

Since x is not orthogonal to U, the function $y \mapsto \varphi(x, y) + \beta$ is not the constant function. Consequently, this function takes the value α for some $y \in U$, which proves the lemma. $\square$

Definition 29.18. Let φ be an ϵ-Hermitian form on E. A *weak Witt decomposition* of E is a triple (U, U', W), such that

(i) $E = U \oplus U' \oplus W$ (a direct sum).

(ii) U and U' are totally isotropic.

(iii) W is nondegenerate and orthogonal to $U \oplus U'$.

We say that a weak Witt decomposition (U, U', W) is *nontrivial* if $U \neq (0)$ and $U' \neq (0)$. Furthermore, if E is finite-dimensional, then $\dim(U) = \dim(U')$ and in a suitable basis, the matrix representing φ is of the form

$$\begin{pmatrix} 0 & A & 0\\ \epsilon\overline{A} & 0 & 0\\ 0 & 0 & B \end{pmatrix}$$

We say that φ is a *neutral form* if it is nondegenerate, E is finite-dimensional, and if $W = (0)$. In this case, the matrix B is missing.

A Witt decomposition for which W has no nonzero isotropic vectors (W is anisotropic) is called a *Witt decomposition*.

Observe that if Φ is nondegenerate, then we have the trivial weak Witt decomposition obtained by letting $U = U' = (0)$ and $W = E$. Thus a weak Witt decomposition is informative only if E is not anisotropic (there is some nonzero isotropic vector, *i.e.* some $u \neq 0$ such that $\Phi(u) = 0$), in which case the most informative nontrivial weak Witt decompositions are those for which W is anisotropic and U and U' are as big as possible.

Sometimes, we use the notation $U_1 \overset{\perp}{\oplus} U_2$ to indicate that in a direct sum $U_1 \oplus U_2$, the subspaces U_1 and U_2 are orthogonal. Then, in Definition 29.18, we can write that $E = (U \oplus U') \overset{\perp}{\oplus} W$.

The first step in showing the existence of a Witt decomposition is this.