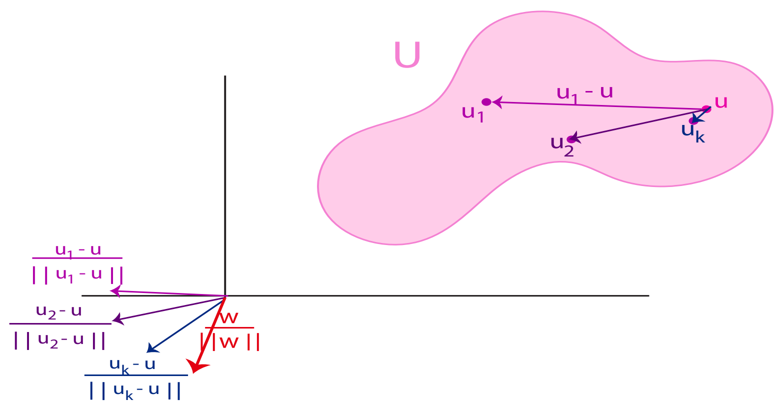

Figure 50.2: Let U be the pink region in $\mathbb{R}^2$ with fuchsia point $u \in U$. For any sequence $(u_k)_{k\geq 0}$ of points in U which converges to u, form the chords $u_k - u$ and take the limit to construct the red vector w.

exists, then there is a sequence $(u_k)_{k\geq 0}$ of vectors in U converging to u as in Definition 50.2, with $u_k = \gamma(t_k)$ for some sequence $(t_k)_{k\geq 0}$ of reals $t_k > 0$ such that $\lim_{k\mapsto\infty} t_k = 0$, so that

$$u_k - u = t_k\gamma'(0) + t_k\epsilon_k, \quad \lim_{k\mapsto\infty} \epsilon_k = 0,$$

and we get

$$\lim_{k\mapsto\infty} \frac{u_k - u}{\|u_k - u\|} = \frac{\gamma'(0)}{\|\gamma'(0)\|}.$$

For an illustration of this paragraph in $\mathbb{R}^2$, see Figure 50.3.

Example 50.1. In $V = \mathbb{R}^2$, let φ_1 and φ_2 be given by

$$\begin{aligned}\varphi_1(u_1, u_2) &= -u_1 - u_2\\ \varphi_2(u_1, u_2) &= u_1(u_1^2 + u_2^2) - (u_1^2 - u_2^2),\end{aligned}$$

and let

$$U = \{(u_1, u_2) \in \mathbb{R}^2 \mid \varphi_1(u_1, u_2) \leq 0,\ \varphi_2(u_1, u_2) \leq 0\}.$$

The region U is shown in Figure 50.4 and is bounded by the curve given by the equation $\varphi_1(u_1, u_2) = 0$, that is, $-u_1 - u_2 = 0$, the line of slope -1 through the origin, and the curve given by the equation $u_1(u_1^2 + u_2^2) - (u_1^2 - u_2^2) = 0$, a nodal cubic through the origin. We obtain a parametric definition of this curve by letting $u_2 = tu_1$, and we find that

$$u_1(t) = \frac{u_1^2(t) - u_2^2(t)}{u_1^2(t) + u_2^2(t)} = \frac{1 - t^2}{1 + t^2}, \quad u_2(t) = \frac{t(1 - t^2)}{1 + t^2}.$$

The tangent vector at t is given by $(u_1'(t), u_2'(t))$ with

$$u_1'(t) = \frac{-2t(1 + t^2) - (1 - t^2)2t}{(1 + t^2)^2} = \frac{-4t}{(1 + t^2)^2}$$