can't be diagonalized. Sometimes a matrix fails to be diagonalizable because its eigenvalues do not belong to the field of coefficients, such as

$$A_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

whose eigenvalues are $\pm i$. This is not a serious problem because A_2 can be diagonalized over the complex numbers. However, A_1 is a "fatal" case! Indeed, its eigenvalues are both 1 and the problem is that A_1 does not have enough eigenvectors to span E.

The next best thing is that there is a basis with respect to which f is represented by an *upper triangular* matrix. In this case we say that f can be *triangularized*, or that f is *triangularizable*. As we will see in Section 15.2, if all the eigenvalues of f belong to the field of coefficients K, then f can be triangularized. In particular, this is the case if $K = \mathbb{C}$.

Now an alternative to triangularization is to consider the representation of f with respect to *two* bases $(e_1, \ldots, e_n)$ and $(f_1, \ldots, f_n)$, rather than a single basis. In this case, if $K = \mathbb{R}$ or $K = \mathbb{C}$, it turns out that we can even pick these bases to be *orthonormal*, and we get a diagonal matrix Σ with *nonnegative entries*, such that

$$f(e_i) = \sigma_i f_i, \quad 1 \leq i \leq n.$$

The nonzero σ_i's are the *singular values* of f, and the corresponding representation is the *singular value decomposition*, or *SVD*. The SVD plays a very important role in applications, and will be considered in detail in Chapter 22.

In this section we focus on the possibility of diagonalizing a linear map, and we introduce the relevant concepts to do so. Given a vector space E over a field K, let id denote the identity map on E.

The notion of eigenvalue of a linear map $f\colon E \to E$ defined on an infinite-dimensional space E is quite subtle because it cannot be defined in terms of eigenvectors as in the finite-dimensional case. The problem is that the map $\lambda\,\mathrm{id} - f$ (with $\lambda \in \mathbb{C}$) could be noninvertible (because it is not surjective) and yet injective. In finite dimension this cannot happen, so until further notice we *assume that E is of finite dimension n*.

Definition 15.1. Given any vector space E of finite dimension n and any linear map $f\colon E \to E$, a scalar $\lambda \in K$ is called an *eigenvalue, or proper value, or characteristic value of f* if there is some *nonzero* vector $u \in E$ such that

$$f(u) = \lambda u.$$

Equivalently, λ is an eigenvalue of f if $\mathrm{Ker}\,(\lambda\,\mathrm{id} - f)$ is nontrivial (i.e., $\mathrm{Ker}\,(\lambda\,\mathrm{id} - f) \neq \{0\}$) iff $\lambda\,\mathrm{id} - f$ is *not* invertible (this is where the fact that E is finite-dimensional is used; a linear map from E to itself is injective iff it is invertible). A vector $u \in E$ is called an *eigenvector, or proper vector, or characteristic vector of f* if $u \neq 0$ and if there is some $\lambda \in K$ such that

$$f(u) = \lambda u;$$