j_0 must exist). To improve numerical stability we average over the following sets of indices. Let I_λ and I_μ be the set of indices given by

$$\begin{aligned} I_\lambda &= \{i \in \{1, \ldots, p\} \mid \lambda_i > 0\} \\ I_\mu &= \{j \in \{1, \ldots, q\} \mid \mu_j > 0\}, \end{aligned}$$

and let $p_m = |I_\lambda|$ and $q_m = |I_\mu|$. We obtain the formula

$$b = \left(w^\top \left(\left(\sum_{i \in I_\lambda} u_i \right) / p_m + \left(\sum_{j \in I_\mu} v_j \right) / q_m \right) + \left(\sum_{i \in I_\lambda} \epsilon_i \right) / p_m - \left(\sum_{j \in I_\mu} \xi_j \right) / q_m \right) / 2.$$

We now prove that for a fixed K_s, the solution to Problem (SVM$_{s4}$) is unique and independent of the value of ν.

Theorem 54.8. *For K_s and ν fixed, if Problem* (SVM$_{s4}$) *succeeds, then it has a unique solution. If Problem* (SVM$_{s4}$) *succeeds and returns $(\lambda, \mu, \eta, w, b)$ for the value ν and $(\lambda^\kappa, \mu^\kappa, \eta^\kappa,$ w^κ, $b^\kappa)$ for the value $\kappa\nu$ with $\kappa > 0$, then*

$$\lambda^\kappa = \kappa\lambda, \;\; \mu^\kappa = \kappa\mu, \;\; \eta^\kappa = \kappa\eta, \;\; w^\kappa = \kappa w, \;\; b^\kappa = \kappa b.$$

As a consequence, $\delta = \eta / \|w\| = \eta^\kappa / \|w^\kappa\| = \delta^\kappa$, and the hyperplanes $H_{w,b}, H_{w,b+\eta}$ and $H_{w,b-\eta}$ are independent of ν.

Proof. We already observed that for an optimal solution with $\eta > 0$, we have $\gamma = 0$. This means that (λ, μ) is a solution of the problem

$$\begin{aligned} &\text{minimize} \quad \frac{1}{2} \begin{pmatrix} \lambda^\top & \mu^\top \end{pmatrix} \left(X^\top X + \frac{1}{2K_s} I_{p+q} \right) \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \\ &\text{subject to} \\ &\qquad \sum_{i=1}^p \lambda_i - \sum_{j=1}^q \mu_j = 0 \\ &\qquad \sum_{i=1}^p \lambda_i + \sum_{j=1}^q \mu_j = \nu \\ &\qquad \lambda_i \geq 0, \quad i = 1, \ldots, p \\ &\qquad \mu_j \geq 0, \quad j = 1, \ldots, q. \end{aligned}$$

Since $K_s > 0$ and $X^\top X$ is symmetric positive semidefinite, the matrix $P = X^\top X + \frac{1}{2K_s} I_{p+q}$ is symmetric positive definite. Let $\Omega = \mathbb{R}^{p+q}$ and let U be the convex set given by

$$U = \left\{ \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \in \mathbb{R}_+^{p+q} \;\middle|\; \begin{pmatrix} \mathbf{1}_p^\top & -\mathbf{1}_q^\top \\ \mathbf{1}_p^\top & \mathbf{1}_q^\top \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} 0 \\ (p+q)K_s\nu \end{pmatrix} \right\}.$$