lemma. For a geometric interpretation of supporting hyperplane see Figure 14.1. This result is a consequence of the *Hahn–Banach theorem*; see Gallier [72]. We give the proof in the case where E is a real Euclidean space. Some minor modifications have to be made when dealing with complex vector spaces and are left as an exercise.

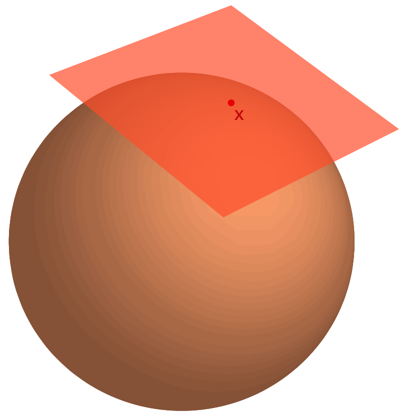

Figure 14.1: The orange tangent plane is a supporting hyperplane to the unit ball in $\mathbb{R}^3$ since this ball is entirely contained in "one side" of the tangent plane.

Since the unit ball $B = \{z \in E \mid \|z\| \leq 1\}$ is closed and convex, the Minkowski lemma says for every x such that $\|x\| = 1$, there is an affine map g of the form

$$g(z) = \langle z, w \rangle - \langle x, w \rangle$$

with $\|w\| = 1$, such that $g(x) = 0$ and $g(z) \leq 0$ for all z such that $\|z\| \leq 1$. Then it is clear that

$$\sup_{\|z\|=1} \langle z, w \rangle = \langle x, w \rangle,$$

and so

$$\|w\|^D = \langle x, w \rangle.$$

It follows that

$$\|x\|^{DD} \geq \langle w/\|w\|^D, x \rangle = \frac{\langle x, w \rangle}{\|w\|^D} = 1 = \|x\|$$

for all x such that $\|x\| = 1$. By homogeneity, this is true for all $y \in E$, which completes the proof in the real case. When E is a complex vector space, we have to view the unit ball B as a closed convex set in $\mathbb{R}^{2n}$ and we use the fact that there is real affine map of the form

$$g(z) = \Re\langle z, w \rangle - \Re\langle x, w \rangle$$

such that $g(x) = 0$ and $g(z) \leq 0$ for all z with $\|z\| = 1$, so that $\|w\|^D = \Re\langle x, w \rangle$. $\square$

More details on dual norms and unitarily invariant norms can be found in Horn and Johnson [95] (Chapters 5 and 7).