Proposition 11.17. *Given any $m \times n$ matrix A over a field K (typically $K = \mathbb{R}$ or $K = \mathbb{C}$), the rank of A is the maximum natural number r such that there is an $r \times r$ submatrix B of A obtained by selecting r rows and r columns of A, such that* $\det(B) \neq 0$.

This is not a very efficient way of finding the rank of a matrix. We will see that there are better ways using various decompositions such as LU, QR, or SVD.

11.7 Properties of the Double Transpose

First we have the following property showing the naturality of the eval map.

Proposition 11.18. *For any linear map $f\colon E \to F$, we have*

$$f^{\top\top} \circ \mathrm{eval}_E = \mathrm{eval}_F \circ f,$$

or equivalently the following diagram commutes:

$$\begin{tikzcd} E^{**} \arrow[r, "f^{\top\top}"] & F^{**} \\ E \arrow[u, "\mathrm{eval}_E"] \arrow[r, "f"'] & F. \arrow[u, "\mathrm{eval}_F"'] \end{tikzcd}$$

Proof. For every $u \in E$ and every $\varphi \in F^*$, we have

$$\begin{aligned} (f^{\top\top} \circ \mathrm{eval}_E)(u)(\varphi) &= \langle f^{\top\top}(\mathrm{eval}_E(u)), \varphi\rangle \\ &= \langle \mathrm{eval}_E(u), f^{\top}(\varphi)\rangle \\ &= \langle f^{\top}(\varphi), u\rangle \\ &= \langle \varphi, f(u)\rangle \\ &= \langle \mathrm{eval}_F(f(u)), \varphi\rangle \\ &= \langle (\mathrm{eval}_F \circ f)(u), \varphi\rangle \\ &= (\mathrm{eval}_F \circ f)(u)(\varphi), \end{aligned}$$

which proves that $f^{\top\top} \circ \mathrm{eval}_E = \mathrm{eval}_F \circ f$, as claimed. ∎

If E and F are finite-dimensional, then eval_E and eval_F are isomorphisms, so Proposition 11.18 shows that

$$f^{\top\top} = \mathrm{eval}_F \circ f \circ \mathrm{eval}_E^{-1}. \tag{$*$}$$

The above equation is often interpreted as follows: if we identify E with its bidual E^{**} and F with its bidual F^{**}, then $f^{\top\top} = f$. This is an abuse of notation; the rigorous statement is $(*)$.

As a corollary of Proposition 11.18, we obtain the following result.