We call the vector space $K^{(I)}$ the vector space *freely generated* by the set I.

Problem 12.17. (Some pitfalls of infinite dimension) Let E be the vector space freely generated by the set of natural numbers, $\mathbb{N} = \{0, 1, 2, \ldots\}$, and let $(e_0, e_1, e_2, \ldots, e_n, \ldots)$ be its canonical basis. We define the function φ such that

$$\varphi(e_i, e_j) = \begin{cases} \delta_{ij} & \text{if } i, j \geq 1, \\ 1 & \text{if } i = j = 0, \\ 1/2^j & \text{if } i = 0, j \geq 1, \\ 1/2^i & \text{if } i \geq 1, j = 0, \end{cases}$$

and we extend φ by bilinearity to a function $\varphi\colon E \times E \to K$. This means that if $u = \sum_{i \in \mathbb{N}} \lambda_i e_i$ and $v = \sum_{j \in \mathbb{N}} \mu_j e_j$, then

$$\varphi\left(\sum_{i \in \mathbb{N}} \lambda_i e_i, \sum_{j \in \mathbb{N}} \mu_j e_j\right) = \sum_{i,j \in \mathbb{N}} \lambda_i \mu_j \varphi(e_i, e_j),$$

but remember that $\lambda_i \neq 0$ and $\mu_j \neq 0$ *only for finitely many indices* i, j.

(1) Prove that φ is positive definite, so that it is an inner product on E.

What would happen if we changed $1/2^j$ to 1 (or any constant)?

(2) Let H be the subspace of E spanned by the family $(e_i)_{i \geq 1}$, a hyperplane in E. Find $H^{\perp}$ and $H^{\perp\perp}$, and prove that

$$H \neq H^{\perp\perp}.$$

(3) Let U be the subspace of E spanned by the family $(e_{2i})_{i \geq 1}$, and let V be the subspace of E spanned by the family $(e_{2i-1})_{i \geq 1}$. Prove that

$$\begin{aligned} U^{\perp} &= V \\ V^{\perp} &= U \\ U^{\perp\perp} &= U \\ V^{\perp\perp} &= V, \end{aligned}$$

yet

$$(U \cap V)^{\perp} \neq U^{\perp} + V^{\perp}$$

and

$$(U + V)^{\perp\perp} \neq U + V.$$

If W is the subspace spanned by e_0 and e_1, prove that

$$(W \cap H)^{\perp} \neq W^{\perp} + H^{\perp}.$$