See Figure 6.1.

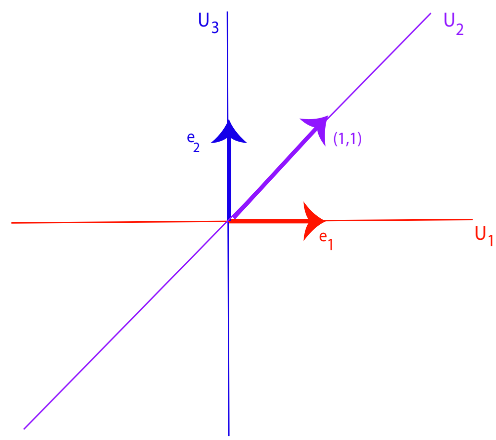

Figure 6.1: The linear subspaces U_1, U_2, and U_3 illustrated as lines in $\mathbb{R}^2$.

As in the case of a sum, $U_1 \oplus U_2 = U_2 \oplus U_1$. *Observe that when the map a is injective, then it is a linear isomorphism between* $U_1 \times \cdots \times U_p$ *and* $U_1 \oplus \cdots \oplus U_p$. The difference is that $U_1 \times \cdots \times U_p$ is defined even if the spaces U_i are not assumed to be subspaces of some common space.

If E is a direct sum $E = U_1 \oplus \cdots \oplus U_p$, since any p nonzero vectors $u_1, \ldots, u_p$ with $u_i \in U_i$ are linearly independent, if we pick a basis $(u_k)_{k \in I_j}$ in U_j for $j = 1, \ldots, p$, then $(u_i)_{i \in I}$ with $I = I_1 \cup \cdots \cup I_p$ is a basis of E. Intuitively, E is split into p independent subspaces.

Conversely, given a basis $(u_i)_{i \in I}$ of E, if we partition the index set I as $I = I_1 \cup \cdots \cup I_p$, then each subfamily $(u_k)_{k \in I_j}$ spans some subspace U_j of E, and it is immediately verified that we have a direct sum

$$E = U_1 \oplus \cdots \oplus U_p.$$

Definition 6.4. Let $f \colon E \to E$ be a linear map. For any subspace U of E, if $f(U) \subseteq U$ we say that U is *invariant under* f.

Assume that E is finite-dimensional, a direct sum $E = U_1 \oplus \cdots \oplus U_p$, and that each U_j is invariant under f. If we pick a basis $(u_i)_{i \in I}$ as above with $I = I_1 \cup \cdots \cup I_p$ and with each $(u_k)_{k \in I_j}$ a basis of U_j, since each U_j is invariant under f, the image $f(u_k)$ of every basis vector u_k with $k \in I_j$ belongs to U_j, so the matrix A representing f over the basis $(u_i)_{i \in I}$ is a *block diagonal* matrix of the form

$$A = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_p \end{pmatrix},$$