of H, and the $n \times n$ upper Hessenberg matrix H_n obtained by deleting the last row of $\widetilde{H}_n$,

$$H_n = \begin{pmatrix} h_{11} & h_{12} & h_{13} & \cdots & h_{1n} \\ h_{21} & h_{22} & h_{23} & \cdots & h_{2n} \\ 0 & h_{32} & h_{33} & \cdots & h_{3n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & h_{nn-1} & h_{nn} \end{pmatrix}.$$

If we denote by U_n the $m \times n$ matrix consisting of the first n columns of U, denoted $u_1, \ldots, u_n$, then the matrix consisting of the first n columns of the matrix $UH = AU$ can be expressed as

$$AU_n = U_{n+1}\widetilde{H}_n. \tag{$*_1$}$$

It follows that the nth column of this matrix can be expressed as

$$Au_n = h_{1n}u_1 + \cdots + h_{nn}u_n + h_{n+1n}u_{n+1}. \tag{$*_2$}$$

Since $(u_1, \ldots, u_n)$ form an orthonormal basis, we deduce from $(*_2)$ that

$$\langle u_j, Au_n \rangle = u_j^* A u_n = h_{jn}, \quad j = 1, \ldots, n. \tag{$*_3$}$$

Equations $(*_2)$ and $(*_3)$ show that U_{n+1} and $\widetilde{H}_n$ can be computed iteratively using the following algorithm due to Arnoldi, known as *Arnoldi iteration*:

Given an arbitrary nonzero vector $b \in \mathbb{C}^m$, let $u_1 = b/\|b\|$;
for $n = 1, 2, 3, \ldots$ **do**
  $z := Au_n$;
  for $j = 1$ **to** n **do**
    $h_{jn} := u_j^* z$;
    $z := z - h_{jn}u_j$
  endfor
  $h_{n+1n} := \|z\|$;
  if $h_{n+1n} = 0$ **quit**
  $u_{n+1} = z/h_{n+1n}$

When $h_{n+1n} = 0$, we say that we have a *breakdown* of the Arnoldi iteration.

Arnoldi iteration is an algorithm for producing the $n \times n$ Hessenberg submatrix H_n of the full Hessenberg matrix H consisting of its first n rows and n columns (the first n columns of U are also produced), not using Householder matrices.

As long as $h_{j+1j} \neq 0$ for $j = 1, \ldots, n$, Equation $(*_2)$ shows by an easy induction that u_{n+1} belong to the span of $(b, Ab, \ldots, A^n b)$, and obviously Au_n belongs to the span of $(u_1, \ldots, u_{n+1})$, and thus the following spaces are identical:

$$\mathrm{Span}(b, Ab, \ldots, A^n b) = \mathrm{Span}(u_1, \ldots, u_{n+1}).$$