Prove that
$$B_0^3(t) + B_1^3(t) + B_2^3(t) + B_3^3(t) = 1.$$

(3) Prove that the Bernstein polynomials of degree 2 are linearly independent, and that the Bernstein polynomials of degree 3 are linearly independent.

Problem 4.4. Recall that the *binomial coefficient* $\binom{m}{k}$ is given by
$$\binom{m}{k} = \frac{m!}{k!(m-k)!},$$
with $0 \leq k \leq m$.

For any $m \geq 1$, we have the $m+1$ *Bernstein polynomials* of degree m given by
$$B_k^m(t) = \binom{m}{k}(1-t)^{m-k}t^k, \quad 0 \leq k \leq m.$$

(1) Prove that
$$B_k^m(t) = \sum_{j=k}^{m} (-1)^{j-k} \binom{m}{j}\binom{j}{k} t^j. \tag{$*$}$$
Use the above to prove that $B_0^m(t), \ldots, B_m^m(t)$ are linearly independent.

(2) Prove that
$$B_0^m(t) + \cdots + B_m^m(t) = 1.$$

(3) What can you say about the symmetries of the $(m+1) \times (m+1)$ matrix expressing $B_0^m, \ldots, B_m^m$ in terms of the basis $1, t, \ldots, t^m$?

Prove your claim (beware that in equation $(*)$ the coefficient of t^j in B_k^m is the entry on the $(k+1)$th row of the $(j+1)$th column, since $0 \leq k, j \leq m$. Make appropriate modifications to the indices).

What can you say about the sum of the entries on each row of the above matrix? What about the sum of the entries on each column?

(4) The purpose of this question is to express the t^i in terms of the Bernstein polynomials $B_0^m(t), \ldots, B_m^m(t)$, with $0 \leq i \leq m$.

First, prove that
$$t^i = \sum_{j=0}^{m-i} t^i B_j^{m-i}(t), \quad 0 \leq i \leq m.$$
Then prove that
$$\binom{m}{i}\binom{m-i}{j} = \binom{m}{i+j}\binom{i+j}{i}.$$