to infinity. It can also be shown that Pappus's theorem implies Desargues's theorem. Many results of projective or affine geometry can be obtained using the method of "sending points to infinity."

We now discuss briefly the notion of cross-ratio, since it is a major concept of projective geometry.

26.10 The Cross-Ratio

Recall that affine maps preserve the ratio of three collinear points. In general, projective maps do not preserve the ratio of three collinear points. However, bijective projective maps preserve the "ratio of ratios" of any four collinear points (three of which are distinct). Such ratios are called *cross-ratios* (in French, "birapport"). There are several ways of introducing cross-ratios, but since we already have Proposition 26.5 at our disposal, we can circumvent some of the tedious calculations needed if other approaches are chosen.

Given a field K, say $K = \mathbb{R}$, recall that the projective line $\mathbb{P}^1_K$ consists of all equivalence classes $[x, y]$ of pairs $(x, y) \in K^2$ such that $(x, y) \neq (0, 0)$, under the equivalence relation $\sim$ defined such that

$$(x_1, y_1) \sim (x_2, y_2) \quad \text{iff} \quad x_2 = \lambda x_1 \quad \text{and} \quad y_2 = \lambda y_1,$$

for some $\lambda \in K - \{0\}$. Letting $\infty = [1, 0]$, the projective line $\mathbb{P}^1_K$ is in bijection with $K \cup \{\infty\}$. Furthermore, letting $0 = [0, 1]$ and $1 = [1, 1]$, the triple $(\infty, 0, 1)$ forms a projective frame for $\mathbb{P}^1_K$. Using this projective frame and Proposition 26.5, we define the cross-ratio of four collinear points as follows.

Definition 26.8. Given a projective line $\Delta = \mathbf{P}(D)$ over a field K, for any sequence (a, b, c, d) of four points in Δ, where a, b, c are distinct (i.e., (a, b, c) is a projective frame), the *cross-ratio* $[a, b, c, d]$ is defined as the element $h(d) \in \mathbb{P}^1_K$, where $h\colon \Delta \to \mathbb{P}^1_K$ is the unique projectivity such that $h(a) = \infty$, $h(b) = 0$, and $h(c) = 1$ (which exists by Proposition 26.5, since (a, b, c) is a projective frame for Δ and $(\infty, 0, 1)$ is a projective frame for $\mathbb{P}^1_K$). For any projective space $\mathbf{P}(E)$ (of dimension ≥ 2) over a field K and any sequence (a, b, c, d) of four collinear points in $\mathbf{P}(E)$, where a, b, c are distinct, the cross-ratio $[a, b, c, d]$ is defined using the projective line Δ that the points a, b, c, d define. For any affine space E and any sequence (a, b, c, d) of four collinear points in E, where a, b, c are distinct, the cross-ratio $[a, b, c, d]$ is defined by considering E as embedded in $\widetilde{E}$.

It should be noted that the definition of the cross-ratio $[a, b, c, d]$ depends on the order of the points. Thus, there could be $24 = 4!$ different possible values depending on the permutation of $\{a, b, c, d\}$. In fact, there are at most 6 distinct values. Also, note that $[a, b, c, d] = \infty$ iff $d = a$, $[a, b, c, d] = 0$ iff $d = b$, and $[a, b, c, d] = 1$ iff $d = c$. Thus, $[a, b, c, d] \in K - \{0, 1\}$ iff $d \notin \{a, b, c\}$.