and that the value $\epsilon(0)$ plays absolutely no role in this definition. The condition for f to be differentiable at a amounts to the fact that

$$\lim_{h\mapsto 0} \frac{\|f(a+h) - f(a) - L(h)\|}{\|h\|} = 0$$

as $h \neq 0$ approaches 0, when $a + h \in A$. However, it does no harm to assume that $\epsilon(0) = 0$, and we will assume this from now on.

Again, we note that the derivative $\mathrm{D}f(a)$ of f at a provides an affine approximation of f, locally around a.

Remarks:

(1) Since the notion of limit is purely topological, the existence and value of a derivative is independent of the choice of norms in E and F, as long as they are equivalent norms.

(2) If $h\colon (-a, a) \to \mathbb{R}$ is a real-valued function defined on some open interval containing 0, we say that h *is* $o(t)$ *for* $t \to 0$, and we write $h(t) = o(t)$, if

$$\lim_{t\mapsto 0,\, t\neq 0} \frac{h(t)}{t} = 0.$$

With this notation (the *little o notation*), the function f is differentiable at a iff

$$f(a + h) - f(a) - L(h) = o(\|h\|),$$

which is also written as

$$f(a + h) = f(a) + L(h) + o(\|h\|).$$

The following proposition shows that our new definition is consistent with the definition of the directional derivative and that *the continuous linear map* L *is unique*, if it exists.

Proposition 39.1. *Let E and F be two normed affine spaces, let A be a nonempty open subset of E, and let $f\colon A \to F$ be any function. For any $a \in A$, if $\mathrm{D}f(a)$ is defined, then f is continuous at a and f has a directional derivative $\mathrm{D}_u f(a)$ for every $u \neq 0$ in $\overrightarrow{E}$, and furthermore,*

$$\mathrm{D}_u f(a) = \mathrm{D}f(a)(u).$$

Proof. If $L = \mathrm{D}f(a)$ exists, then for any nonzero vector $u \in \overrightarrow{E}$, because A is open, for any $t \in \mathbb{R} - \{0\}$ (or $t \in \mathbb{C} - \{0\}$) small enough, $a + tu \in A$, so

$$\begin{aligned} f(a + tu) &= f(a) + L(tu) + \epsilon(tu)\|tu\| \\ &= f(a) + tL(u) + |t|\epsilon(tu)\|u\| \end{aligned}$$