Problem 49.3. Let A be a real $n \times n$ symmetric positive definite matrix and let $b \in \mathbb{R}^n$.

(1) Prove that if we apply the steepest descent method (for the Euclidean norm) to

$$J(v) = \frac{1}{2}\langle Av, v\rangle - \langle b, v\rangle,$$

and if we define the norm $\|v\|_A$ by

$$\|v\|_A = \langle Av, v\rangle^{1/2},$$

we get the inequality

$$\|u_{k+1} - u\|_A^2 \leq \|u_k - u\|_A^2 \left(1 - \frac{\|A(u_k - u)\|_2^4}{\|u_k - u\|_A^2 \|A(u_k - u)\|_A^2}\right).$$

(2) Using a diagonalization of A, where the eigenvalues of A are denoted $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, prove that

$$\|u_{k+1} - u\|_A \leq \frac{\mathrm{cond}_2(A) - 1}{\mathrm{cond}_2(A) + 1} \|u_k - u\|_A,$$

where $\mathrm{cond}_2(A) = \lambda_n/\lambda_1$, and thus

$$\|u_k - u\|_A \leq \left(\frac{\mathrm{cond}_2(A) - 1}{\mathrm{cond}_2(A) + 1}\right)^k \|u_0 - u\|_A.$$

(3) Prove that when $\mathrm{cond}_2(A) = 1$, then $A = I$ and the method converges in one step.

Problem 49.4. Prove that the method of Polak–Ribière converges if $J\colon \mathbb{R}^n \to \mathbb{R}$ is elliptic and a C^2 function.

Problem 49.5. Prove that the backtracking line search method described in Section 49.5 has the property that for t small enough the condition $J(u_k + td_k) \leq J(u_k) + \alpha t\langle \nabla J_{u_k}, d_k\rangle$ will hold and the search will stop. Prove that the exit inequality $J(u_k + td_k) \leq J(u_k) + \alpha t\langle \nabla J_{u_k}, d_k\rangle$ holds for all $t \in (0, t_0]$, for some $t_0 > 0$, so the backtracking line search stops with a step length ρ_k that satisfies $\rho_k = 1$ or $\rho_k \in (\beta t_0, t_0]$.

Problem 49.6. Let $d_{\mathrm{nsd},k}$ and $d_{\mathrm{sd},k}$ be the normalized and unnormalized descent directions of the steepest descent method for an arbitrary norm (see Section 49.8). Prove that

$$\begin{aligned}
\langle \nabla J_{u_k}, d_{\mathrm{nsd},k}\rangle &= -\|\nabla J_{u_k}\|^D \\
\langle \nabla J_{u_k}, d_{\mathrm{sd},k}\rangle &= -(\|\nabla J_{u_k}\|^D)^2 \\
d_{\mathrm{sd},k} &= \arg\min_v \left(\langle \nabla J_{u_k}, v\rangle + \frac{1}{2}\|v\|^2\right).
\end{aligned}$$