Proof. Compute $\langle f(x+y), x+y\rangle$ and $\langle f(x-y), x-y\rangle$:

$$\begin{aligned}\langle f(x+y), x+y\rangle &= \langle f(x), x\rangle + \langle f(x), y\rangle + \langle f(y), x\rangle + \langle y, y\rangle \\ \langle f(x-y), x-y\rangle &= \langle f(x), x\rangle - \langle f(x), y\rangle - \langle f(y), x\rangle + \langle y, y\rangle;\end{aligned}$$

then subtract the second equation from the first to obtain

$$\langle f(x+y), x+y\rangle - \langle f(x-y), x-y\rangle = 2(\langle f(x), y\rangle + \langle f(y), x\rangle).$$

If $\langle f(u), u\rangle = 0$ for all $u \in E$, we get

$$\langle f(x), y\rangle + \langle f(y), x\rangle = 0 \quad \text{for all } x, y \in E.$$

Then the above equation also holds if we replace x by ix, and we obtain

$$i\langle f(x), y\rangle - i\langle f(y), x\rangle = 0, \quad \text{for all } x, y \in E,$$

so we have

$$\begin{aligned}\langle f(x), y\rangle + \langle f(y), x\rangle &= 0 \\ \langle f(x), y\rangle - \langle f(y), x\rangle &= 0,\end{aligned}$$

which implies that $\langle f(x), y\rangle = 0$ for all $x, y \in E$. Since $\langle -, -\rangle$ is positive definite, we have $f(x) = 0$ for all $x \in E$; that is, $f = 0$. □

One should be careful not to apply Proposition 14.3 to a linear map on a real Euclidean space because it is false! The reader should find a counterexample.

The Cauchy–Schwarz inequality and the Minkowski inequalities extend to pre-Hilbert spaces and to Hermitian spaces.

Proposition 14.4. *Let $\langle E, \varphi\rangle$ be a pre-Hilbert space with associated quadratic form Φ. For all $u, v \in E$, we have the* Cauchy–Schwarz inequality

$$|\varphi(u, v)| \leq \sqrt{\Phi(u)}\sqrt{\Phi(v)}.$$

Furthermore, if $\langle E, \varphi\rangle$ is a Hermitian space, the equality holds iff u and v are linearly dependent.

We also have the Minkowski inequality

$$\sqrt{\Phi(u+v)} \leq \sqrt{\Phi(u)} + \sqrt{\Phi(v)}.$$

Furthermore, if $\langle E, \varphi\rangle$ is a Hermitian space, the equality holds iff u and v are linearly dependent, where in addition, if $u \neq 0$ and $v \neq 0$, then $u = \lambda v$ for some real λ such that $\lambda > 0$.