and setting $a = x$, the point x is the unique point such that

$$\sum_{i \in I} \lambda_i \overrightarrow{xa_i} = 0.$$

In physical terms, the barycenter is the *center of mass* of the family of weighted points $((a_i, \lambda_i))_{i \in I}$ (where the masses have been normalized, so that $\sum_{i \in I} \lambda_i = 1$, and negative masses are allowed).

Remarks:

(1) Since the barycenter of a family $((a_i, \lambda_i))_{i \in I}$ of weighted points is defined for families $(\lambda_i)_{i \in I}$ of scalars with finite support (and such that $\sum_{i \in I} \lambda_i = 1$), we might as well assume that I is finite. Then, for all $m \geq 2$, it is easy to prove that the barycenter of m weighted points can be obtained by repeated computations of barycenters of two weighted points.

(2) This result still holds, provided that the field K has at least three distinct elements, but the proof is trickier!

(3) When $\sum_{i \in I} \lambda_i = 0$, the vector $\sum_{i \in I} \lambda_i \overrightarrow{aa_i}$ does not depend on the point a, and we may denote it by $\sum_{i \in I} \lambda_i a_i$. This observation will be used to define a vector space in which linear combinations of both points and vectors make sense, regardless of the value of $\sum_{i \in I} \lambda_i$.

Figure 24.11 illustrates the geometric construction of the barycenters g_1 and g_2 of the weighted points $\left(a, \frac{1}{4}\right)$, $\left(b, \frac{1}{4}\right)$, and $\left(c, \frac{1}{2}\right)$, and $(a, -1)$, $(b, 1)$, and $(c, 1)$.

The point g_1 can be constructed geometrically as the middle of the segment joining c to the middle $\frac{1}{2}a + \frac{1}{2}b$ of the segment (a, b), since

$$g_1 = \frac{1}{2}\left(\frac{1}{2}a + \frac{1}{2}b\right) + \frac{1}{2}c.$$

The point g_2 can be constructed geometrically as the point such that the middle $\frac{1}{2}b + \frac{1}{2}c$ of the segment (b, c) is the middle of the segment (a, g_2), since

$$g_2 = -a + 2\left(\frac{1}{2}b + \frac{1}{2}c\right).$$

Later on, we will see that a polynomial curve can be defined as a set of barycenters of a fixed number of points. For example, let (a, b, c, d) be a sequence of points in $\mathbb{A}^2$. Observe that

$$(1-t)^3 + 3t(1-t)^2 + 3t^2(1-t) + t^3 = 1,$$