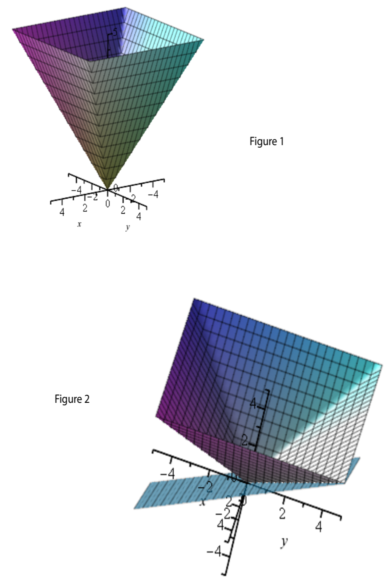

Figure 51.15: Figure (1) shows the graph in $\mathbb{R}^3$ of $f(x,y) = \|(x,y)\|_\infty = \sup\{|x|, |y|\}$. Figure (2) shows the supporting hyperplane with normal $(\frac{1}{2}, \frac{1}{2}, -1)$, where $(\frac{1}{2}, \frac{1}{2}) \in \partial f(0)$.

Theorem 51.11. *(Minkowski) Let C be a nonempty convex set in $\mathbb{R}^n$. For any point $a \in C - \mathbf{relint}(C)$, there is a supporting hyperplane H to C at a.*

Theorem 51.11 is proven in Rockafellar [138] (Theorem 11.6). See also Berger [11] (Proposition 11.5.2). The proof is not as simple as one might expect, and is based on a geometric version of the Hahn–Banach theorem.

In order to prove Theorem 51.14 below we need two technical propositions.

Proposition 51.12. *Let C be any nonempty convex set in $\mathbb{R}^n$. For any $x \in \mathbf{relint}(C)$ and any $y \in \overline{C}$, we have $(1-\lambda)x + \lambda y \in \mathbf{relint}(C)$ for all λ such that $0 \leq \lambda < 1$. In other words, the line segment from x to y including x and excluding y lies entirely within $\mathbf{relint}(C)$.*

Proposition 51.12 is proven in Rockafellar [138] (Theorem 6.1). The proof is not difficult but quite technical.

Proposition 51.13. *For any proper convex function f on $\mathbb{R}^n$, we have*

$$\mathbf{relint}(\mathbf{epi}(f)) = \{(x,\mu) \in \mathbb{R}^{n+1} \mid x \in \mathbf{relint}(\mathrm{dom}(f)),\ f(x) < \mu\}.$$

Proof. Proposition 51.13 is proven in Rockafellar [138] (Lemma 7.3). By working in the affine hull of $\mathbf{epi}(f)$, the statement of Proposition 51.13 is equivalent to

$$\mathrm{int}(\mathbf{epi}(f)) = \{(x,\mu) \in \mathbb{R}^{m+1} \mid x \in \mathrm{int}(\mathrm{dom}(f)),\ f(x) < \mu\},$$