and thus $\widehat{f}(u) = \overrightarrow{f}(u)$ for all $u \in \overrightarrow{E}$. Then we have

$$\widehat{f}(u \mathbin{\widehat{+}} \lambda a) = \lambda f(a) + \overrightarrow{f}(u),$$

which proves the uniqueness of $\widehat{f}$. On the other hand, the map $\widehat{f}$ defined as above is clearly a linear map extending f.

When $\lambda \neq 0$, we have

$$\widehat{f}(u \mathbin{\widehat{+}} \lambda a) = \widehat{f}(\lambda(a + \lambda^{-1}u)) = \lambda \widehat{f}(a + \lambda^{-1}u) = \lambda f(a + \lambda^{-1}u).$$

$\square$

Proposition 25.5 shows that $\langle \widehat{E}, j, \omega \rangle$, is a homogenization of $(E, \overrightarrow{E})$. As a corollary, we obtain the following proposition.

Proposition 25.6. *Given two affine spaces E and F and an affine map $f \colon E \to F$, there is a unique linear map $\widehat{f} \colon \widehat{E} \to \widehat{F}$ extending f, as in the diagram below,*

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "j"'] & F \arrow[d, "j"] \\ \widehat{E} \arrow[r, "\widehat{f}"'] & \widehat{F} \end{tikzcd}$$

such that

$$\widehat{f}(u \mathbin{\widehat{+}} \lambda a) = \overrightarrow{f}(u) \mathbin{\widehat{+}} \lambda f(a),$$

for all $a \in E$, all $u \in \overrightarrow{E}$, and all $\lambda \in \mathbb{R}$, where $\overrightarrow{f}$ is the linear map associated with f. In particular, when $\lambda \neq 0$, we have

$$\widehat{f}(u \mathbin{\widehat{+}} \lambda a) = \lambda f(a + \lambda^{-1}u).$$

Proof. Consider the vector space $\widehat{F}$ and the affine map $j \circ f \colon E \to \widehat{F}$. By Proposition 25.5, there is a unique linear map $\widehat{f} \colon \widehat{E} \to \widehat{F}$ extending $j \circ f$, and thus extending f. $\square$

Note that $\widehat{f} \colon \widehat{E} \to \widehat{F}$ has the property that $\widehat{f}(\overrightarrow{E}) \subseteq \overrightarrow{F}$. More generally, since

$$\widehat{f}(u \mathbin{\widehat{+}} \lambda a) = \overrightarrow{f}(u) \mathbin{\widehat{+}} \lambda f(a),$$

the linear map $\widehat{f}$ is weight-preserving. Also observe that we recover f from $\widehat{f}$, by letting $\lambda = 1$ in $\widehat{f}(u \mathbin{\widehat{+}} \lambda a) = \lambda f(a + \lambda^{-1}u)$, that is, we have

$$f(a + u) = \widehat{f}(u \mathbin{\widehat{+}} a).$$