(2) *If Ω is convex (open), if the functions φ_i $(1 \leq i \leq m)$ and J are convex and differentiable at the point $u \in U$, if the constraints are qualified, and if $u \in U$ is a minimum of Problem (P), then there exists some vector $\lambda \in \mathbb{R}^m_+$ such that the pair $(u, \lambda) \in \Omega \times \mathbb{R}^m_+$ is a saddle point of the Lagrangian L.*

Proof. (1) Since (u, λ) is a saddle point of L we have $\sup_{\mu \in \mathbb{R}^m_+} L(u, \mu) = L(u, \lambda)$ which implies that $L(u, \mu) \leq L(u, \lambda)$ for all $\mu \in \mathbb{R}^m_+$, which means that

$$J(u) + \sum_{i=1}^m \mu_i \varphi_i(u) \leq J(u) + \sum_{i=1}^m \lambda_i \varphi_i(u),$$

that is,

$$\sum_{i=1}^m (\mu_i - \lambda_i)\varphi_i(u) \leq 0 \quad \text{for all } \mu \in \mathbb{R}^m_+.$$

If we let each μ_i be large enough, then $\mu_i - \lambda_i > 0$, and if we had $\varphi_i(u) > 0$, then the term $(\mu_i - \lambda_i)\varphi_i(u)$ could be made arbitrarily large and positive, so we conclude that $\varphi_i(u) \leq 0$ for $i = 1, \ldots, m$, and consequently, $u \in U$. For $\mu = 0$, we conclude that $\sum_{i=1}^m \lambda_i \varphi_i(u) \geq 0$. However, since $\lambda_i \geq 0$ and $\varphi_i(u) \leq 0$, (since $u \in U$), we have $\sum_{i=1}^m \lambda_i \varphi_i(u) \leq 0$. Combining these two inequalities shows that

$$\sum_{i=1}^m \lambda_i \varphi_i(u) = 0. \tag{$*_1$}$$

This shows that $J(u) = L(u, \lambda)$. Since the inequality $L(u, \lambda) \leq L(v, \lambda)$ is

$$J(u) + \sum_{i=1}^m \lambda_i \varphi_i(u) \leq J(v) + \sum_{i=1}^m \lambda_i \varphi_i(v),$$

by $(*_1)$ we obtain

$$\begin{aligned} J(u) &\leq J(v) + \sum_{i=1}^m \lambda_i \varphi_i(v) && \text{for all } v \in \Omega \\ &\leq J(v) && \text{for all } v \in U \ (\text{since } \varphi_i(v) \leq 0 \text{ and } \lambda_i \geq 0), \end{aligned}$$

which shows that u is a minimum of J on U.

(2) The hypotheses required to apply Theorem 50.6(1) are satisfied. Consequently if $u \in U$ is a solution of Problem (P), then there exists some vector $\lambda \in \mathbb{R}^m_+$ such that the KKT conditions hold:

$$J'(u) + \sum_{i=1}^m \lambda_i (\varphi'_i)_u = 0 \quad \text{and} \quad \sum_{i=1}^m \lambda_i \varphi_i(u) = 0.$$