Proof. These are proved in Bourbaki [25] (Chapter III, §11, Section 11), but the proofs of (3) and (4) are very concise. We elaborate on the proofs of (2) and (4), the proof of (3) being similar.

In (2) if $H \cap L \neq \emptyset$, then $e_H \wedge e_L$ contains some vector twice and so $e_H \wedge e_L = 0$. Otherwise, $e_H \wedge e_L$ consists of

$$e_{h_1} \wedge \cdots \wedge e_{h_p} \wedge e_{\ell_1} \wedge \cdots \wedge e_{\ell_q},$$

and to order the sequence of indices in increasing order we need to transpose any two indices (h_i, ℓ_j) corresponding to an inversion, which yields $\rho_{H,L} e_{H \cup L}$.

Let us now consider (4). We have $|L| = p + q$ and $|H| = p$, and the q-vector $e_H^* \lrcorner e_L$ is characterized by

$$\langle v^*, e_H^* \lrcorner e_L \rangle = \langle v^* \wedge e_H^*, e_L \rangle$$

for all $v^* \in \bigwedge^q E^*$. There are two cases.

Case 1: $H \not\subseteq L$. If so, no matter what $v^* \in \bigwedge^q E^*$ is, since H contains some index h not in L, the hth row $(e_h^*(e_{\ell_1}), \ldots, e_h^*(e_{\ell_{p+q}}))$ of the determinant $\langle v^* \wedge e_H^*, e_L \rangle$ must be zero, so $\langle v^* \wedge e_H^*, e_L \rangle = 0$ for all $v^* \in \bigwedge^q E^*$, and since the pairing is nongenerate, we must have $e_H^* \lrcorner e_L = 0$.

Case 2: $H \subseteq L$. In this case, for $v^* = e_{L-H}^*$, by (2) we have

$$\langle e_{L-H}^*, e_H^* \lrcorner e_L \rangle = \langle e_{L-H}^* \wedge e_H^*, e_L \rangle = \langle \rho_{L-H,H} e_L^*, e_L \rangle = \rho_{L-H,H},$$

which yields

$$\langle e_{L-H}^*, e_H^* \lrcorner e_L \rangle = \rho_{L-H,H}.$$

The q-vector $e_H^* \lrcorner e_L$ can be written as a linear combination $e_H^* \lrcorner e_L = \sum_J \lambda_J e_J$ with $|J| = q$ so

$$\langle e_{L-H}^*, e_H^* \lrcorner e_L \rangle = \sum_J \lambda_J \langle e_{L-H}^*, e_J \rangle.$$

By definition of the pairing, $\langle e_{L-H}^*, e_J \rangle = 0$ unless $J = L - H$, which means that

$$\langle e_{L-H}^*, e_H^* \lrcorner e_L \rangle = \lambda_{L-H} \langle e_{L-H}^*, e_{L-H} \rangle = \lambda_{L-H},$$

so $\lambda_{L-H} = \rho_{L-H,H}$, as claimed. $\square$

Using Proposition 34.18, we have the

Proposition 34.19. *For the left hook*

$$\lrcorner : E \times \bigwedge^{q+1} E^* \longrightarrow \bigwedge^{q} E^*,$$

for every $u \in E$, $x^* \in \bigwedge^{q+1-s} E^*$, and $y^* \in \bigwedge^s E^*$, we have

$$u \lrcorner (x^* \wedge y^*) = (-1)^s (u \lrcorner x^*) \wedge y^* + x^* \wedge (u \lrcorner y^*).$$