$U^{\perp}$. Using the remark before the theorem and Proposition 29.37, we can find a transvection τ mapping W''_{i+1} onto W'_{i+1} and leaving every vector in U fixed. Then, $\tau \circ g_i$ maps $(u_1, v_1, \ldots, u_{i+1}, v_{i+1})$ to $(u'_1, v'_1, \ldots, u'_{i+1}, v'_{i+1})$, establishing the induction step.

For the second statement, since we already proved that every transvection has a determinant equal to $+1$, this also holds for any composition of transvections in G, and since $G = \mathbf{Sp}(2m, K)$, we are done. $\square$

It can also be shown that the center of $\mathbf{Sp}(2m, K)$ is reduced to the subgroup $\{\mathrm{id}, -\mathrm{id}\}$. The *projective symplectic group* $\mathbf{PSp}(2m, K)$ is the quotient group $\mathbf{PSp}(2m, K)/\{\mathrm{id}, -\mathrm{id}\}$. All symplectic projective groups are simple, except $\mathbf{PSp}(2, \mathbb{F}_2)$, $\mathbf{PSp}(2, \mathbb{F}_3)$, and $\mathbf{PSp}(4, \mathbb{F}_2)$, see Grove [83].

The orders of the symplectic groups over finite fields can be determined. For details, see Artin [6], Jacobson [98] and Grove [83].

An interesting property of symplectic spaces is that the determinant of a skew-symmetric matrix B is the square of some polynomial $\mathrm{Pf}(B)$ called the *Pfaffian*; see Jacobson [98] and Artin [6]. We leave considerations of the Pfaffian to the exercises.

We now take a look at the orthogonal groups.

29.9 Orthogonal Groups and the Cartan–Dieudonné Theorem

In this section we are dealing with a nondegenerate symmetric bilinear from φ over a finite-dimensional vector space E of dimension n over a field of characateristic not equal to 2. Recall that the orthogonal group $\mathbf{O}(\varphi)$ is the group of isometries of φ; that is, the group of linear maps $f\colon E \to E$ such that

$$\varphi(f(u), f(v)) = \varphi(u, v) \quad \text{for all } u, v \in E.$$

The elements of $\mathbf{O}(\varphi)$ are also called *orthogonal transformations*. If M is the matrix of φ in any basis, then a matrix A represents an orthogonal transformation iff

$$A^{\top} M A = M.$$

Since φ is nondegenerate, M is invertible, so we see that $\det(A) = \pm 1$. The subgroup

$$\mathbf{SO}(\varphi) = \{f \in \mathbf{O}(\varphi) \mid \det(f) = 1\}$$

is called the *special orthogonal group* (of φ), and its members are called *rotations* (or *proper orthogonal transformations*). Isometries $f \in \mathbf{O}(\varphi)$ such that $\det(f) = -1$ are called *improper orthogonal transformations*, or sometimes *reversions*.