and since φ is a homomorphism $(\varphi(g_1))^{-1} = \varphi(g_1^{-1})$, so

$$e' = (\varphi(g_1))^{-1}\varphi(g_2) = \varphi(g_1^{-1})\varphi(g_2) = \varphi(g_1^{-1}g_2).$$

This shows that $g_1^{-1}g_2 \in \mathrm{Ker}\ \varphi$, but since $\mathrm{Ker}\ \varphi = \{e\}$ we have $g_1^{-1}g_2 = e$, and thus $g_2 = g_1$, proving that φ is injective. $\square$

Definition 2.9. We say that a group homomorphism $\varphi\colon G \to G'$ is an *isomorphism* if there is a homomorphism $\psi\colon G' \to G$, so that

$$\psi \circ \varphi = \mathrm{id}_G \quad \text{and} \quad \varphi \circ \psi = \mathrm{id}_{G'}. \tag{$\dagger$}$$

If φ is an isomorphism we say that the groups G and G' are *isomorphic*. When $G' = G$, a group isomorphism is called an *automorphism*.

The reasoning used in the proof of Proposition 2.2 shows that if a a group homomorphism $\varphi\colon G \to G'$ is an isomorphism, then the homomorphism $\psi\colon G' \to G$ satisfying Condition ($\dagger$) is unique. This homomorphism is denoted φ^{-1}.

The left translations L_g and the right translations R_g are automorphisms of G.

Suppose $\varphi\colon G \to G'$ is a bijective homomorphism, and let φ^{-1} be the inverse of φ (as a function). Then for all $a, b \in G$, we have

$$\varphi(\varphi^{-1}(a)\varphi^{-1}(b)) = \varphi(\varphi^{-1}(a))\varphi(\varphi^{-1}(b)) = ab,$$

and so

$$\varphi^{-1}(ab) = \varphi^{-1}(a)\varphi^{-1}(b),$$

which proves that φ^{-1} is a homomorphism. Therefore, we proved the following fact.

Proposition 2.10. *A bijective group homomorphism $\varphi\colon G \to G'$ is an isomorphism.*

Observe that the property

$$gH = Hg, \quad \text{for all } g \in G. \tag{$*$}$$

is equivalent by multiplication on the right by g^{-1} to

$$gHg^{-1} = H, \quad \text{for all } g \in G,$$

and the above is equivalent to

$$gHg^{-1} \subseteq H, \quad \text{for all } g \in G. \tag{$**$}$$

This is because $gHg^{-1} \subseteq H$ implies $H \subseteq g^{-1}Hg$, and this for all $g \in G$.

Proposition 2.11. *Let $\varphi\colon G \to G'$ be a group homomorphism. Then $H = \mathrm{Ker}\ \varphi$ satisfies Property ($**$), and thus Property ($*$).*