Theorem 10.6. *If $A = D - E - F$ is Hermitian positive definite, and if $0 < \omega < 2$, then the relaxation method converges. This also holds for a block decomposition of A.*

Proof. Recall that for the relaxation method, $A = M - N$ with

$$M = \frac{D}{\omega} - E$$

$$N = \frac{1-\omega}{\omega} D + F,$$

and because $D^* = D$, $E^* = F$ (since A is Hermitian) and $\omega \neq 0$ is real, we have

$$M^* + N = \frac{D^*}{\omega} - E^* + \frac{1-\omega}{\omega} D + F = \frac{2-\omega}{\omega} D.$$

If D consists of the diagonal entries of A, then we know from Section 8.8 that these entries are all positive, and since $\omega \in (0,2)$, we see that the matrix $((2-\omega)/\omega)D$ is positive definite. If D consists of diagonal blocks of A, because A is positive, definite, by choosing vectors z obtained by picking a nonzero vector for each block of D and padding with zeros, we see that each block of D is positive definite, and thus D itself is positive definite. Therefore, in all cases, $M^* + N$ is positive definite, and we conclude by using Proposition 10.5. $\square$

Remark: What if we allow the parameter ω to be a nonzero complex number $\omega \in \mathbb{C}$? In this case, we get

$$M^* + N = \frac{D^*}{\overline{\omega}} - E^* + \frac{1-\omega}{\omega} D + F = \left(\frac{1}{\omega} + \frac{1}{\overline{\omega}} - 1\right) D.$$

But,

$$\frac{1}{\omega} + \frac{1}{\overline{\omega}} - 1 = \frac{\omega + \overline{\omega} - \omega\overline{\omega}}{\omega\overline{\omega}} = \frac{1 - (\omega - 1)(\overline{\omega} - 1)}{|\omega|^2} = \frac{1 - |\omega - 1|^2}{|\omega|^2},$$

so the relaxation method also converges for $\omega \in \mathbb{C}$, provided that

$$|\omega - 1| < 1.$$

This condition reduces to $0 < \omega < 2$ if ω is real.

Unfortunately, Theorem 10.6 does not apply to Jacobi's method, but in special cases, Proposition 10.5 can be used to prove its convergence. On the positive side, if a matrix is strictly column (or row) diagonally dominant, then it can be shown that the method of Jacobi and the method of Gauss–Seidel both converge. The relaxation method also converges if $\omega \in (0,1]$, but this is not a very useful result because the speed-up of convergence usually occurs for $\omega > 1$.

We now prove that, without *any* assumption on $A = D - E - F$, other than the fact that A and D are invertible, in order for the relaxation method to converge, we must have $\omega \in (0,2)$.