the matrices L_{sym} and L_{rw} are similar, which implies that they have the same spectrum. In fact, since $D^{1/2}$ is invertible,

$$L_{\mathrm{rw}}u = D^{-1}Lu = \lambda u$$

iff

$$D^{-1/2}Lu = \lambda D^{1/2}u$$

iff

$$D^{-1/2}LD^{-1/2}D^{1/2}u = L_{\mathrm{sym}}D^{1/2}u = \lambda D^{1/2}u,$$

which shows that a vector $u \neq 0$ is an eigenvector of L_{rw} for λ iff $D^{1/2}u$ is an eigenvector of L_{sym} for λ.

(3) We already know that L and L_{sym} are positive semidefinite.

(4) Since $D^{-1/2}$ is invertible, we have

$$Lu = \lambda Du$$

iff

$$D^{-1/2}Lu = \lambda D^{1/2}u$$

iff

$$D^{-1/2}LD^{-1/2}D^{1/2}u = L_{\mathrm{sym}}D^{1/2}u = \lambda D^{1/2}u,$$

which shows that a vector $u \neq 0$ is a solution of the generalized eigenvalue problem $Lu = \lambda Du$ iff $D^{1/2}u$ is an eigenvector of L_{sym} for the eigenvalue λ. The second part of the statement follows from (2).

(5) Since D^{-1} is invertible, we have $Lu = 0$ iff $D^{-1}Lu = L_{\mathrm{rw}}u = 0$. Similarly, since $D^{-1/2}$ is invertible, we have $Lu = 0$ iff $D^{-1/2}LD^{-1/2}D^{1/2}u = 0$ iff $D^{1/2}u \in \mathrm{Ker}\,(L_{\mathrm{sym}})$.

(6) Since $L\mathbf{1} = 0$, we get $L_{\mathrm{rw}}\mathbf{1} = D^{-1}L\mathbf{1} = 0$. That $D^{1/2}\mathbf{1}$ is in the nullspace of L_{sym} follows from (2). Properties (7)–(10) are proven in Chung [39] (Chapter 1). $\square$

The eigenvalues the matrices L_{sym} and L_{rw} from Example 20.1 are

$$0,\ 7257,\ 1.1667,\ 1.5,\ 1.6076.$$

On the other hand, the eigenvalues of the unnormalized Laplacian for G_1 are

$$0,\ 1.5858,\ 3,\ 4.4142,\ 5.$$

Remark: Observe that although the matrices L_{sym} and L_{rw} have the same spectrum, the matrix L_{rw} is generally not symmetric, whereas L_{sym} is symmetric.

A version of Proposition 20.5 also holds for the graph Laplacians L_{sym} and L_{rw}. This follows easily from the fact that Proposition 20.1 applies to the underlying graph of a weighted graph. The proof is left as an exercise.