Next, we turn to *graph drawings* (Chapter 21). Graph drawing is a very attractive application of so-called spectral techniques, which is a fancy way of saying that that eigenvalues and eigenvectors of the graph Laplacian are used. Furthermore, it turns out that graph clustering using normalized cuts can be cast as a certain type of graph drawing.

Given an undirected graph $G = (V, E)$, with $|V| = m$, we would like to draw G in $\mathbb{R}^n$ for n (much) smaller than m. The idea is to assign a point $\rho(v_i)$ in $\mathbb{R}^n$ to the vertex $v_i \in V$, for every $v_i \in V$, and to draw a line segment between the points $\rho(v_i)$ and $\rho(v_j)$. Thus, a *graph drawing* is a function $\rho\colon V \to \mathbb{R}^n$.

We define the *matrix of a graph drawing ρ (in $\mathbb{R}^n$)* as a $m \times n$ matrix R whose ith row consists of the row vector $\rho(v_i)$ corresponding to the point representing v_i in $\mathbb{R}^n$. Typically, we want $n < m$; in fact n should be much smaller than m.

Since there are infinitely many graph drawings, it is desirable to have some criterion to decide which graph is better than another. Inspired by a physical model in which the edges are springs, it is natural to consider a representation to be better if it requires the springs to be less extended. We can formalize this by defining the *energy* of a drawing R by

$$\mathcal{E}(R) = \sum_{\{v_i, v_j\} \in E} \|\rho(v_i) - \rho(v_j)\|^2,$$

where $\rho(v_i)$ is the ith row of R and $\|\rho(v_i) - \rho(v_j)\|^2$ is the square of the Euclidean length of the line segment joining $\rho(v_i)$ and $\rho(v_j)$.

Then "good drawings" are drawings that minimize the energy function $\mathcal{E}$. Of course, the trivial representation corresponding to the zero matrix is optimum, so we need to impose extra constraints to rule out the trivial solution.

We can consider the more general situation where the springs are not necessarily identical. This can be modeled by a symmetric weight (or stiffness) matrix $W = (w_{ij})$, with $w_{ij} \geq 0$. In this case, our energy function becomes

$$\mathcal{E}(R) = \sum_{\{v_i, v_j\} \in E} w_{ij} \|\rho(v_i) - \rho(v_j)\|^2.$$

Following Godsil and Royle [77], we prove that

$$\mathcal{E}(R) = \mathrm{tr}(R^\top L R),$$

where

$$L = D - W,$$

is the familiar unnormalized Laplacian matrix associated with W, and where D is the degree matrix associated with W.

It can be shown that there is no loss in generality in assuming that the columns of R are pairwise orthogonal and that they have unit length. Such a matrix satisfies the equation