5.8 Problems

Problem 5.1. (Haar extravaganza) Consider the matrix

$$W_{3,3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

(1) Show that given any vector $c = (c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8)$, the result $W_{3,3}c$ of applying $W_{3,3}$ to c is

$$W_{3,3}c = (c_1 + c_5, c_1 - c_5, c_2 + c_6, c_2 - c_6, c_3 + c_7, c_3 - c_7, c_4 + c_8, c_4 - c_8),$$

the last step in reconstructing a vector from its Haar coefficients.

(2) Prove that the inverse of $W_{3,3}$ is $(1/2)W_{3,3}^{\top}$. Prove that the columns and the rows of $W_{3,3}$ are orthogonal.

(3) Let $W_{3,2}$ and $W_{3,1}$ be the following matrices:

$$W_{3,2} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad W_{3,1} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Show that given any vector $c = (c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8)$, the result $W_{3,2}c$ of applying $W_{3,2}$ to c is

$$W_{3,2}c = (c_1 + c_3, c_1 - c_3, c_2 + c_4, c_2 - c_4, c_5, c_6, c_7, c_8),$$

the second step in reconstructing a vector from its Haar coefficients, and the result $W_{3,1}c$ of applying $W_{3,1}$ to c is

$$W_{3,1}c = (c_1 + c_2, c_1 - c_2, c_3, c_4, c_5, c_6, c_7, c_8),$$

the first step in reconstructing a vector from its Haar coefficients.

Conclude that

$$W_{3,3}W_{3,2}W_{3,1} = W_3,$$