we obtain

$$tB = t \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix} = 3tI_3 + \begin{pmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix}$$

and so

$$e^{tB} = \begin{pmatrix} e^{3t} & 0 & 0 \\ 0 & e^{3t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix} \begin{pmatrix} 1 & t & (1/2)t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{3t} & te^{3t} & (1/2)t^2e^{3t} \\ 0 & e^{3t} & te^{3t} \\ 0 & 0 & e^{3t} \end{pmatrix}.$$

The columns of e^{tB} form a basis of the space of solutions of the system of linear differential equations

$$\begin{aligned} \frac{dY_1}{dt} &= 3Y_1 + Y_2 \\ \frac{dY_2}{dt} &= 3Y_2 + Y_3 \\ \frac{dY_3}{dt} &= 3Y_3, \end{aligned}$$

in matrix form,

$$\begin{pmatrix} \frac{dY_1}{dt} \\ \frac{dY_2}{dt} \\ \frac{dY_3}{dt} \end{pmatrix} = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}.$$

Explicitly, the general solution of the above system is

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = c_1 \begin{pmatrix} e^{3t} \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} te^{3t} \\ e^{3t} \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} (1/2)t^2e^{3t} \\ te^{3t} \\ e^{3t} \end{pmatrix},$$

with $c_1, c_2, c_3 \in \mathbb{R}$. Solving systems of first-order linear differential equations is discussed in Artin [7] and more extensively in Hirsh and Smale [93].

31.7 Summary

The main concepts and results of this chapter are listed below:

- Ideals, principal ideals, greatest common divisors.
- Monic polynomial, irreducible polynomial, relatively prime polynomials.
- Annihilator of a linear map.
- Minimal polynomial of a linear map.