Proposition 20.5. *Let $G = (V, W)$ be a weighted graph. The number c of connected components $K_1, \ldots, K_c$ of the underlying graph of G is equal to the dimension of the nullspace of L, which is equal to the multiplicity of the eigenvalue 0. Furthermore, the nullspace of L has a basis consisting of indicator vectors of the connected components of G, that is, vectors $(f_1, \ldots, f_m)$ such that $f_j = 1$ iff $v_j \in K_i$ and $f_j = 0$ otherwise.*

Proof. Since $L = BB^\top$ for the incidence matrix B associated with any oriented graph obtained from G, and since L and $B^\top$ have the same nullspace, by Proposition 20.1, the dimension of the nullspace of L is equal to the number c of connected components of G and the indicator vectors of the connected components of G form a basis of $\mathrm{Ker}\,(L)$. ☐

Proposition 20.5 implies that if the underlying graph of G is connected, then the second eigenvalue λ_2 of L is strictly positive.

Remarkably, the eigenvalue λ_2 contains a lot of information about the graph G (assuming that $G = (V, E)$ is an undirected graph). This was first discovered by Fiedler in 1973, and for this reason, λ_2 is often referred to as the *Fiedler number*. For more on the properties of the Fiedler number, see Godsil and Royle [77] (Chapter 13) and Chung [39]. More generally, the spectrum $(0, \lambda_2, \ldots, \lambda_m)$ of L contains a lot of information about the combinatorial structure of the graph G. Leverage of this information is the object of *spectral graph theory*.

20.3 Normalized Laplacian Matrices of Graphs

It turns out that normalized variants of the graph Laplacian are needed, especially in applications to graph clustering. These variants make sense only if G has no isolated vertices.

Definition 20.18. Given a weighted graph $G = (V, W)$, a vertex $u \in V$ is *isolated* if it is not incident to any other vertex. This means that every row of W contains some strictly positive entry.

If G has no isolated vertices, then the degree matrix D contains positive entries, so it is invertible and $D^{-1/2}$ makes sense; namely

$$D^{-1/2} = \mathrm{diag}(d_1^{-1/2}, \ldots, d_m^{-1/2}),$$

and similarly for any real exponent α.

Definition 20.19. Given any weighted directed graph $G = (V, W)$ with no isolated vertex and with $V = \{v_1, \ldots, v_m\}$, the *(normalized) graph Laplacians L_{sym} and L_{rw} of G* are defined by

$$\begin{aligned} L_{\mathrm{sym}} &= D^{-1/2} L D^{-1/2} = I - D^{-1/2} W D^{-1/2} \\ L_{\mathrm{rw}} &= D^{-1} L = I - D^{-1} W. \end{aligned}$$