does not depend on the choice of $\Omega \in F$. If we identify E to $\overrightarrow{E}$ by choosing any origin Ω in F, we note that g is identified with the symmetry with respect to $\overrightarrow{F}$ and parallel to $\overrightarrow{G}$. Thus, the map g is an affine isometry, and it is called the *affine orthogonal symmetry about* F. Since

$$g(a) = \Omega + \overrightarrow{\Omega a} - 2p_{\overrightarrow{G}}(\overrightarrow{\Omega a})$$

for all $\Omega \in F$ and for all $a \in E$, we note that the linear map $\overrightarrow{g}$ associated with g is the (linear) symmetry about the subspace $\overrightarrow{F}$ (the direction of F), and parallel to $\overrightarrow{G}$ (the direction of G).

Remark: The map $p\colon E \to F$ such that $p(a) = a - q(a)$, or equivalently

$$\overrightarrow{ap(a)} = -q(a) = -p_{\overrightarrow{G}}(\overrightarrow{\Omega a}),$$

is also independent of $\Omega \in F$, and it is called the *affine orthogonal projection onto* F.

The following amusing lemma shows the extra power afforded by affine orthogonal symmetries: Translations are subsumed! Given two parallel affine subspaces F_1 and F_2 in E, letting $\overrightarrow{F}$ be the common direction of F_1 and F_2 and $\overrightarrow{G} = \overrightarrow{F}^{\perp}$ be its orthogonal complement, for any $a \in F_1$, the affine subspace $a + \overrightarrow{G}$ intersects F_2 in a single point b (see Lemma 24.16). We define the *distance between* F_1 *and* F_2 as $\|\overrightarrow{ab}\|$. It is easily seen that the distance between F_1 and F_2 is independent of the choice of a in F_1, and that it is the minimum of $\|\overrightarrow{xy}\|$ for all $x \in F_1$ and all $y \in F_2$.

Proposition 27.9. *Given any affine space E, if $f\colon E \to E$ and $g\colon E \to E$ are affine orthogonal symmetries about parallel affine subspaces F_1 and F_2, then $g \circ f$ is a translation defined by the vector $2\overrightarrow{ab}$, where $\overrightarrow{ab}$ is any vector perpendicular to the common direction $\overrightarrow{F}$ of F_1 and F_2 such that $\|\overrightarrow{ab}\|$ is the distance between F_1 and F_2, with $a \in F_1$ and $b \in F_2$. Conversely, every translation by a vector τ is obtained as the composition of two affine orthogonal symmetries about parallel affine subspaces F_1 and F_2 whose common direction is orthogonal to $\tau = \overrightarrow{ab}$, for some $a \in F_1$ and some $b \in F_2$ such that the distance between F_1 and F_2 is $\|\overrightarrow{ab}\|/2$.*

Proof. We observed earlier that the linear maps $\overrightarrow{f}$ and $\overrightarrow{g}$ associated with f and g are the linear reflections about the directions of F_1 and F_2. However, F_1 and F_2 have the same direction, and so $\overrightarrow{f} = \overrightarrow{g}$. Since $\overrightarrow{g \circ f} = \overrightarrow{g} \circ \overrightarrow{f}$ and since $\overrightarrow{f} \circ \overrightarrow{g} = \overrightarrow{f} \circ \overrightarrow{f} = \mathrm{id}$, because every reflection is an involution, we have $\overrightarrow{g \circ f} = \mathrm{id}$, proving that $g \circ f$ is a translation. If we pick $a \in F_1$, then $g \circ f(a) = g(a)$, the affine reflection of $a \in F_1$ about F_2, and it is easily checked that $g \circ f$ is the translation by the vector $\tau = \overrightarrow{ag(a)}$ whose norm is twice the distance between F_1 and F_2. The second part of the lemma is left as an easy exercise. $\square$