12.11 Problems

Problem 12.1. E be a vector space of dimension 2, and let (e_1, e_2) be a basis of E. Prove that if $a > 0$ and $b^2 - ac < 0$, then the bilinear form defined such that

$$\varphi(x_1 e_1 + y_1 e_2,\, x_2 e_1 + y_2 e_2) = a x_1 x_2 + b(x_1 y_2 + x_2 y_1) + c y_1 y_2$$

is a Euclidean inner product.

Problem 12.2. Let $\mathcal{C}[a, b]$ denote the set of continuous functions $f\colon [a, b] \to \mathbb{R}$. Given any two functions $f, g \in \mathcal{C}[a, b]$, let

$$\langle f, g\rangle = \int_a^b f(t)g(t)dt.$$

Prove that the above bilinear form is indeed a Euclidean inner product.

Problem 12.3. Consider the inner product

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(t)g(t)dt$$

of Problem 12.2 on the vector space $\mathcal{C}[-\pi, \pi]$. Prove that

$$\langle \sin px, \sin qx\rangle = \begin{cases} \pi & \text{if } p = q,\ p, q \geq 1, \\ 0 & \text{if } p \neq q,\ p, q \geq 1, \end{cases}$$

$$\langle \cos px, \cos qx\rangle = \begin{cases} \pi & \text{if } p = q,\ p, q \geq 1, \\ 0 & \text{if } p \neq q,\ p, q \geq 0, \end{cases}$$

$$\langle \sin px, \cos qx\rangle = 0,$$

for all $p \geq 1$ and $q \geq 0$, and $\langle 1, 1\rangle = \int_{-\pi}^{\pi} dx = 2\pi$.

Problem 12.4. Prove that the following matrix is orthogonal and skew-symmetric:

$$M = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{pmatrix}.$$

Problem 12.5. Let E and F be two finite Euclidean spaces, let $(u_1, \ldots, u_n)$ be a basis of E, and let $(v_1, \ldots, v_m)$ be a basis of F. For any linear map $f\colon E \to F$, if A is the matrix of f w.r.t. the basis $(u_1, \ldots, u_n)$ and B is the matrix of f^* w.r.t. the basis $(v_1, \ldots, v_m)$, if G_1 is the Gram matrix of the inner product on E (w.r.t. $(u_1, \ldots, u_n)$) and if G_2 is the Gram matrix of the inner product on F (w.r.t. $(v_1, \ldots, v_m)$), then

$$B = G_1^{-1} A^{\top} G_2.$$