and

$$u + \lambda 1 \mapsto \langle \Omega_1 + \lambda^{-1} u, \lambda\rangle \mapsto u + \lambda 1,$$

and since $\widehat{\Omega}$ is the identity on $\overrightarrow{E}$, we have shown that $\widehat{\Omega} \circ \widehat{\Omega}^{-1} = \mathrm{id}$, and $\widehat{\Omega}^{-1} \circ \widehat{\Omega} = \mathrm{id}$. This shows that $\widehat{\Omega}$ is a bijection. $\square$

Figure 25.4 illustrates the embedding of the affine space E into the vector space $\mathcal{F}$, when E is an affine plane.

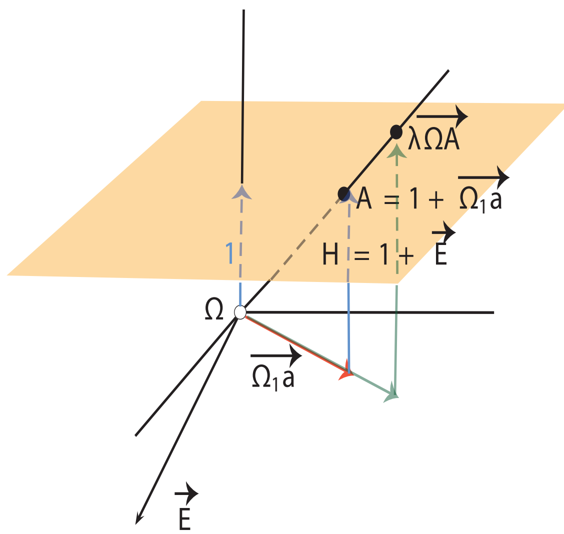

Figure 25.4: Embedding an affine space $(E, \overrightarrow{E})$ into a vector space $\mathcal{F}$.

Proposition 25.4 gives a nice interpretation of the sum operation $\widehat{+}$ of $\widehat{E}$. Given two weighted points $\langle a_1, \lambda_1\rangle$ and $\langle a_2, \lambda_2\rangle$, we have

$$\langle a_1, \lambda_1\rangle \,\widehat{+}\, \langle a_2, \lambda_2\rangle = \widehat{\Omega}^{-1}(\widehat{\Omega}(\langle a_1, \lambda_1\rangle) + \widehat{\Omega}(\langle a_2, \lambda_2\rangle)).$$

The operation $\widehat{\Omega}(\langle a_1, \lambda_1\rangle) + \widehat{\Omega}(\langle a_2, \lambda_2\rangle)$ has a simple geometric interpretation. If $\lambda_1 + \lambda_2 \neq 0$, then find the points M_1 and M_2 on the lines passing through the origin Ω of $\mathcal{F}$ and the points $A_1 = \widehat{\Omega}(a_1)$ and $A_2 = \widehat{\Omega}(a_2)$ in the hyperplane H, such that $\overrightarrow{\Omega M_1} = \lambda_1 \overrightarrow{\Omega A_1}$ and $\overrightarrow{\Omega M_2} = \lambda_2 \overrightarrow{\Omega A_2}$, add the vectors $\overrightarrow{\Omega M_1}$ and $\overrightarrow{\Omega M_2}$, getting a point N such that $\overrightarrow{\Omega N} = \overrightarrow{\Omega M_1} + \overrightarrow{\Omega M_2}$, and consider the intersection G of the line passing through Ω and N with the hyperplane H. Then, G is the barycenter of A_1 and A_2 assigned the weights $\lambda_1/(\lambda_1 + \lambda_2)$ and $\lambda_2/(\lambda_1 + \lambda_2)$, and if $g = \widehat{\Omega}^{-1}(\overrightarrow{\Omega G})$, then $\widehat{\Omega}^{-1}(\overrightarrow{\Omega N}) = \langle g, \lambda_1 + \lambda_2\rangle$. See Figure 25.5.

Instead of adding the vectors $\overrightarrow{\Omega M_1}$ and $\overrightarrow{\Omega M_2}$, we can take the middle N' of the segment M_1M_2, and G is the intersection of the line passing through Ω and N' with the hyperplane H as illustrated in Figure 25.5.