so the feasibility equations become

$$\begin{aligned} Ax &= b \\ Px + q + A^\top \lambda &= 0, \end{aligned}$$

which in matrix form become

$$\begin{pmatrix} P & A^\top \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} -q \\ b \end{pmatrix}. \tag{KKT-eq}$$

The matrix of the linear system is usually called the *KKT-matrix*. Observe that the KKT matrix was already encountered in Proposition 42.3 with a different notation; there we had $P = A^{-1}$, $A = B^\top$, $q = b$, and $b = f$.

If the KKT matrix is invertible, then its unique solution (x^*, λ^*) yields a unique minimum x^* of Problem (P). If the KKT matrix is singular but the System (KKT-eq) is solvable, then *any solution* (x^*, λ^*) yields a minimum x^* of Problem (P).

Proposition 50.10. *If the System (KKT-eq) is not solvable, then Program (P) is unbounded below.*

Proof. We use the fact shown in Section 30.7, that a linear system $Bx = c$ has no solution iff there is some y that $B^\top y = 0$ and $y^\top c \neq 0$. By changing y to $-y$ if necessary, we may assume that $y^\top c > 0$. We apply this fact to the linear system (KKT-eq), so B is the KKT-matrix, which is symmetric, and we obtain the condition that there exist $v \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^m$ such that

$$Pv + A^\top \lambda = 0, \quad Av = 0, \quad -q^\top v + b^\top \lambda > 0.$$

Since the $m \times n$ matrix A has rank m and $b \in \mathbb{R}^m$, the system $Ax = b$, is solvable, so for any feasible x_0 (which means that $Ax_0 = b$), since $Av = 0$, the vector $x = x_0 + tv$ is also a feasible solution for all $t \in \mathbb{R}$. Using the fact that $Pv = -A^\top \lambda$, $v^\top P = -\lambda^\top A$, $Av = 0$, $x_0^\top A^\top = b^\top$, and P is symmetric, we have

$$\begin{aligned} J(x_0 + tv) &= J(x_0) + (v^\top P x_0 + q^\top v)t + (1/2)(v^\top P v)t^2 \\ &= J(x_0) + (x_0^\top P v + q^\top v)t - (1/2)(\lambda^\top A v)t^2 \\ &= J(x_0) + (-x_0^\top A^\top \lambda + q^\top v)t \\ &= J(x_0) - (b^\top \lambda - q^\top v)t, \end{aligned}$$

and since $-q^\top v + b^\top \lambda > 0$, the above expression goes to $-\infty$ when t goes to $+\infty$. ∎

It is obviously important to have criteria to decide whether the KKT-matrix is invertible. There are indeed such criteria, as pointed in Boyd and Vandenberghe [29] (Chapter 10, Exercise 10.1).