Problem 23.2. Let X be an $m \times n$ real matrix. For any strictly positive constant $K > 0$, the matrix $X^\top X + KI_n$ is invertible. Prove that the limit of the matrix $(X^\top X + KI_n)^{-1} X^\top$ when K goes to zero is equal to the pseudo-inverse X^+ of X.

Problem 23.3. Use `Matlab` to find the pseudo-inverse of the 8×6 matrix

$$A = \begin{pmatrix} 64 & 2 & 3 & 61 & 60 & 6 \\ 9 & 55 & 54 & 12 & 13 & 51 \\ 17 & 47 & 46 & 20 & 21 & 43 \\ 40 & 26 & 27 & 37 & 36 & 30 \\ 32 & 34 & 35 & 29 & 28 & 38 \\ 41 & 23 & 22 & 44 & 45 & 19 \\ 49 & 15 & 14 & 52 & 53 & 11 \\ 8 & 58 & 59 & 5 & 4 & 62 \end{pmatrix}.$$

Observe that the sums of the columns are all equal to 260. Let b be the vector of dimension 8 whose coordinates are all equal to 256. Find the solution x^+ of the system $Ax = b$.

Problem 23.4. The purpose of this problem is to show that Proposition 23.9 (the Eckart–Young theorem) also holds for the Frobenius norm. This problem is adapted from Strang [171], Section I.9.

Suppose the $m \times n$ matrix B of rank at most k minimizes $\|A - B\|_F$. Start with an SVD of B,

$$B = V \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} U^\top,$$

where D is a diagonal $k \times k$ matrix. We can write

$$A = V \begin{pmatrix} L + E + R & F \\ G & H \end{pmatrix} U^\top,$$

where L is strictly lower triangular in the first k rows, E is diagonal, and R is strictly upper triangular, and let

$$C = V \begin{pmatrix} L + D + R & F \\ 0 & 0 \end{pmatrix} U^\top,$$

which clearly has rank $\leq k$.

(1) Prove that

$$\|A - B\|_F^2 = \|A - C\|_F^2 + \|L\|_F^2 + \|R\|_F^2 + \|F\|_F^2 .$$

Since $\|A - B\|_F$ is minimal, show that $L = R = F = 0$.

Similarly, show that $G = 0$.