where Λ_r has rank r, then

$$\Lambda^+ = \begin{pmatrix} \Lambda_r^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof. Assume that $B_1, \ldots, B_p$ are 2×2 blocks and that $\lambda_{2p+1}, \ldots, \lambda_n$ are the scalar entries. We know that the numbers $\lambda_j \pm i\mu_j$, and the λ_{2p+k} are the eigenvalues of A. Let $\rho_{2j-1} = \rho_{2j} = \sqrt{\lambda_j^2 + \mu_j^2} = \sqrt{\det(B_i)}$ for $j = 1, \ldots, p$, $\rho_j = |\lambda_j|$ for $j = 2p+1, \ldots, r$. Multiplying U by a suitable permutation matrix, we may assume that the blocks of Λ are ordered so that $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_r > 0$. Then it is easy to see that

$$AA^\top = A^\top A = U\Lambda U^\top U\Lambda^\top U^\top = U\Lambda\Lambda^\top U^\top,$$

with

$$\Lambda\Lambda^\top = \mathrm{diag}(\rho_1^2, \ldots, \rho_r^2, 0, \ldots, 0),$$

so $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_r > 0$ are the singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ of A. Define the diagonal matrix

$$\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_r, 0, \ldots, 0),$$

where $r = \mathrm{rank}(A)$, $\sigma_1 \geq \cdots \geq \sigma_r > 0$ and the block diagonal matrix Θ defined such that the block B_i in Λ is replaced by the block $\sigma^{-1}B_i$ where $\sigma = \sqrt{\det(B_i)}$, the nonzero scalar λ_j is replaced $\lambda_j/|\lambda_j|$, and a diagonal zero is replaced by 1. Observe that Θ is an orthogonal matrix and

$$\Lambda = \Theta\Sigma.$$

But then we can write

$$A = U\Lambda U^\top = U\Theta\Sigma U^\top,$$

and we if let $V = U\Theta$, since U is orthogonal and Θ is also orthogonal, V is also orthogonal and $A = V\Sigma U^\top$ *is an SVD for A*. Now we get

$$A^+ = U\Sigma^+ V^\top = U\Sigma^+\Theta^\top U^\top.$$

However, since Θ is an orthogonal matrix, $\Theta^\top = \Theta^{-1}$, and a simple calculation shows that

$$\Sigma^+\Theta^\top = \Sigma^+\Theta^{-1} = \Lambda^+,$$

which yields the formula

$$A^+ = U\Lambda^+ U^\top.$$

Also observe that Λ_r is invertible and

$$\Lambda^+ = \begin{pmatrix} \Lambda_r^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore, the pseudo-inverse of a normal matrix can be computed directly from any block diagonalization of A, as claimed. $\square$