Note that

$$E_2 \coprod E_1 = \{\{\langle 2, v\rangle, \langle 1, u\rangle\} \mid v \in E_2,\ u \in E_1\} = E_1 \coprod E_2.$$

Thus, every member $\{\langle 1, u\rangle, \langle 2, v\rangle\}$ of $E_1 \coprod E_2$ can be viewed as an *unordered pair* consisting of the two vectors u and v, tagged with the index 1 and 2, respectively.

Remark: In fact, $E_1 \coprod E_2$ is just the product $\prod_{i\in\{1,2\}} E_i$ of the family $(E_i)_{i\in\{1,2\}}$.

This is not to be confused with the cartesian product $E_1 \times E_2$. The vector space $E_1 \times E_2$ is the set of all ordered pairs $\langle u, v\rangle$, where $u \in E_1$, and $v \in E_2$, with addition and multiplication by a scalar defined such that

$$\begin{aligned} \langle u_1, v_1\rangle + \langle u_2, v_2\rangle &= \langle u_1 + u_2, v_1 + v_2\rangle, \\ \lambda\langle u, v\rangle &= \langle \lambda u, \lambda v\rangle. \end{aligned}$$

There is a bijection between $\prod_{i\in\{1,2\}} E_i$ and $E_1 \times E_2$, but as we just saw, elements of $\prod_{i\in\{1,2\}} E_i$ are certain sets. The product $E_1 \times \cdots \times E_n$ of any number of vector spaces can also be defined. We will do this shortly.

The following property holds.

Proposition 6.1. *Given any two vector spaces, E_1 and E_2, the set $E_1 \coprod E_2$ is a vector space. For every pair of linear maps, $f\colon E_1 \to G$ and $g\colon E_2 \to G$, there is a unique linear map, $f + g\colon E_1 \coprod E_2 \to G$, such that $(f + g) \circ in_1 = f$ and $(f + g) \circ in_2 = g$, as in the following diagram:*

$$\begin{tikzcd} E_1 \arrow[d, "in_1"'] \arrow[dr, "f"] & \\ E_1 \coprod E_2 \arrow[r, "f+g"] & G \\ E_2 \arrow[u, "in_2"] \arrow[ur, "g"'] & \end{tikzcd}$$

Proof. Define

$$(f + g)(\{\langle 1, u\rangle, \langle 2, v\rangle\}) = f(u) + g(v),$$

for every $u \in E_1$ and $v \in E_2$. It is immediately verified that $f + g$ is the unique linear map with the required properties. $\square$

We already noted that $E_1 \coprod E_2$ is in bijection with $E_1 \times E_2$. If we define the *projections* $\pi_1\colon E_1 \coprod E_2 \to E_1$ and $\pi_2\colon E_1 \coprod E_2 \to E_2$, such that

$$\pi_1(\{\langle 1, u\rangle, \langle 2, v\rangle\}) = u,$$

and

$$\pi_2(\{\langle 1, u\rangle, \langle 2, v\rangle\}) = v,$$

we have the following proposition.