For any integer $n \in \mathbb{Z}$, we define g^n by

$$g^n = \begin{cases} g^n & \text{if } n \geq 0 \\ (g^{-1})^{(-n)} & \text{if } n < 0. \end{cases}$$

The following properties are easily verified:

$$\begin{aligned} g^i \cdot g^j &= g^{i+j} \\ (g^i)^{-1} &= g^{-i} \\ g^i \cdot g^j &= g^j \cdot g^i, \end{aligned}$$

for all $i, j \in \mathbb{Z}$.

Define the subset $\langle g \rangle$ of G by

$$\langle g \rangle = \{g^n \mid n \in \mathbb{Z}\}.$$

The following proposition is left as an exercise.

Proposition 2.14. *Given a group G, for any element $g \in G$, the set $\langle g \rangle$ is the smallest abelian subgroup of G containing g.*

Definition 2.13. A group G is *cyclic* iff there is some element $g \in G$ such that $G = \langle g \rangle$. An element $g \in G$ with this property is called a *generator* of G.

The Klein four group V of Example 2.2 is abelian, but not cyclic. This is because V has four elements, but all the elements different from the identity have order 2.

Cyclic groups are quotients of $\mathbb{Z}$. For this, we use a basic property of $\mathbb{Z}$. Recall that for any $n \in \mathbb{Z}$, we let $n\mathbb{Z}$ denote the set of multiples of n,

$$n\mathbb{Z} = \{nk \mid k \in \mathbb{Z}\}.$$

Proposition 2.15. *Every subgroup H of $\mathbb{Z}$ is of the form $H = n\mathbb{Z}$ for some $n \in \mathbb{N}$.*

Proof. If H is the trivial group $\{0\}$, then let $n = 0$. If H is nontrivial, for any nonzero element $m \in H$, we also have $-m \in H$ and either m or $-m$ is positive, so let n be the smallest positive integer in H. By Proposition 2.14, $n\mathbb{Z}$ is the smallest subgroup of H containing n. For any $m \in H$ with $m \neq 0$, we can write

$$m = nq + r, \quad \text{with} \quad 0 \leq r < n.$$

Now, since $n\mathbb{Z} \subseteq H$, we have $nq \in H$, and since $m \in H$, we get $r = m - nq \in H$. However, $0 \leq r < n$, contradicting the minimality of n, so $r = 0$, and $H = n\mathbb{Z}$. $\square$