

**Definition 37.8.** A topological space  $(E, \mathcal{O})$  is said to satisfy the *Hausdorff separation axiom* (or  *$T_2$ -separation axiom*) if for any two distinct points  $a \neq b$  in  $E$ , there exist two open sets  $U_a$  and  $U_b$  such that,  $a \in U_a$ ,  $b \in U_b$ , and  $U_a \cap U_b = \emptyset$ . When the  $T_2$ -separation axiom is satisfied, we also say that  $(E, \mathcal{O})$  is a *Hausdorff space*.

As shown by Proposition 37.1, any metric space is a topological Hausdorff space, the family of open sets being in fact the family of arbitrary unions of open balls. Similarly, any normed vector space is a topological Hausdorff space, the family of open sets being the family of arbitrary unions of open balls. The topology  $\mathcal{O}$  consisting of all subsets of  $E$  is called the *discrete topology*.

**Remark:** Most (if not all) spaces used in analysis are Hausdorff spaces. Intuitively, the Hausdorff separation axiom says that there are enough “small” open sets. Without this axiom, some counter-intuitive behaviors may arise. For example, a sequence may have more than one limit point (or a compact set may not be closed). Nevertheless, non-Hausdorff topological spaces arise naturally in algebraic geometry. But even there, some substitute for separation is used.

One of the reasons why topological spaces are important is that the definition of a topology only involves a certain family  $\mathcal{O}$  of sets, and not **how** such family is generated from a metric or a norm. For example, different metrics or different norms can define the same family of open sets. Many topological properties only depend on the family  $\mathcal{O}$  and not on the specific metric or norm. But the fact that a topology is definable from a metric or a norm is important, because it usually implies nice properties of a space. All our examples will be spaces whose topology is defined by a metric or a norm.

**Definition 37.9.** A topological space  $(E, \mathcal{O})$  is *metrizable* if there is a distance on  $E$  defining the topology  $\mathcal{O}$ .

Note that in a metric space  $(E, d)$ , the metric  $d$  is *explicitly given*. However, in general, the topology of a metrizable space  $(E, \mathcal{O})$  is not specified by an explicitly given metric, but *some metric* defining the topology  $\mathcal{O}$  exists. Obviously, a metrizable topological space must be Hausdorff. Actually, a stronger separation property holds, a metrizable space is normal; see Definition 37.30.

**Remark:** By taking complements we can state properties of the closed sets dual to those of Definition 37.7. Thus,  $\emptyset$  and  $E$  are closed sets, and the closed sets are closed under finite unions and arbitrary intersections.

It is also worth noting that the Hausdorff separation axiom implies that for every  $a \in E$ , the set  $\{a\}$  is closed. Indeed, if  $x \in E - \{a\}$ , then  $x \neq a$ , and so there exist open sets  $U_a$  and  $U_x$  such that  $a \in U_a$ ,  $x \in U_x$ , and  $U_a \cap U_x = \emptyset$ . See Figure 37.6. Thus, for every  $x \in E - \{a\}$ , there is an open set  $U_x$  containing  $x$  and contained in  $E - \{a\}$ , showing by (O3) that  $E - \{a\}$  is open, and thus that the set  $\{a\}$  is closed.