its eigenvalues must be positive. We will have to learn about the spectral theorem for symmetric matrices to establish this criterion (see Proposition 22.3).

Proposition 8.11 also holds for complex Hermitian positive definite matrices, where in (d), the factorization $LDL^\top$ is replaced by LDL^*.

For more on the stability analysis and efficient implementation methods of Gaussian elimination, LU-factoring and Cholesky factoring, see Demmel [48], Trefethen and Bau [176], Ciarlet [41], Golub and Van Loan [80], Meyer [125], Strang [169, 170], and Kincaid and Cheney [102].

8.10 Reduced Row Echelon Form (RREF)

Gaussian elimination described in Section 8.2 can also be applied to rectangular matrices. This yields a method for determining whether a system $Ax = b$ is solvable and a description of all the solutions when the system is solvable, for any rectangular $m \times n$ matrix A.

It turns out that the discussion is simpler if we rescale all pivots to be 1, and for this we need a third kind of elementary matrix. For any $\lambda \neq 0$, let $E_{i,\lambda}$ be the $n \times n$ diagonal matrix

$$E_{i,\lambda} = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \lambda & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix},$$

with $(E_{i,\lambda})_{ii} = \lambda$ $(1 \leq i \leq n)$. Note that $E_{i,\lambda}$ is also given by

$$E_{i,\lambda} = I + (\lambda - 1)e_{i\,i},$$

and that $E_{i,\lambda}$ is invertible with

$$E_{i,\lambda}^{-1} = E_{i,\lambda^{-1}}.$$

Now after $k-1$ elimination steps, if the bottom portion

$$(a_{kk}^{(k)}, a_{k+1k}^{(k)}, \ldots, a_{mk}^{(k)})$$

of the kth column of the current matrix A_k is nonzero so that a pivot π_k can be chosen, after a permutation of rows if necessary, we also divide row k by π_k to obtain the pivot 1, and not only do we zero all the entries $i = k+1, \ldots, m$ in column k, but also all the entries $i = 1, \ldots, k-1$, so that the only nonzero entry in column k is a 1 in row k. These row operations are achieved by multiplication on the left by elementary matrices.

If $a_{kk}^{(k)} = a_{k+1k}^{(k)} = \cdots = a_{mk}^{(k)} = 0$, we move on to column $k+1$.