Since $\psi(u) = 0$ iff $u \in U$, and $\psi(v) \geq 0$ for all $v \in \mathbb{R}^n$, we have $J_\epsilon(u) = J(u)$, and since u_ϵ is the minimizer of J_ϵ we have $J_\epsilon(u_\epsilon) \leq J_\epsilon(u)$, so we obtain

$$J(u_\epsilon) \leq J(u_\epsilon) + \frac{1}{\epsilon}\,\psi(u_\epsilon) = J_\epsilon(u_\epsilon) \leq J_\epsilon(u) = J(u),$$

that is,

$$J_\epsilon(u_\epsilon) \leq J(u). \tag{$*_1$}$$

Since J is coercive, the family $(u_\epsilon)_{\epsilon>0}$ is bounded. By compactness (since we are in $\mathbb{R}^n$), there exists a subsequence $(u_{\epsilon(i)})_{i\geq 0}$ with $\lim_{i\mapsto\infty} \epsilon(i) = 0$ and some element $u' \in \mathbb{R}^n$ such that

$$\lim_{i\mapsto\infty} u_{\epsilon(i)} = u'.$$

From the inequality $J(u_\epsilon) \leq J(u)$ proven in $(*_1)$ and the continuity of J, we deduce that

$$J(u') = \lim_{i\mapsto\infty} J(u_{\epsilon(i)}) \leq J(u). \tag{$*_2$}$$

By definition of $J_\epsilon(u_\epsilon)$ and $(*_1)$, we have

$$0 \leq \psi(u_{\epsilon(i)}) \leq \epsilon(i)(J(u) - J(u_{\epsilon(i)})),$$

and since the sequence $(u_{\epsilon(i)})_{i\geq 0}$ converges, the numbers $J(u) - J(u_{\epsilon(i)})$ are bounded independently of i. Consequently, since $\lim_{i\mapsto\infty} \epsilon(i) = 0$ and since the function ψ is continuous, we have

$$0 = \lim_{i\mapsto\infty} \psi(u_{\epsilon(i)}) = \psi(u'),$$

which shows that $u' \in U$. Since by $(*_2)$ we have $J(u') \leq J(u)$, and since both $u, u' \in U$ and u is the unique minimizer of J over U we must have $u' = u$. Therfore u' is the unique minimizer of J over U. But then the whole family $(u_\epsilon)_{\epsilon>0}$ converges to u since we can use the same argument as above for *every* subsequence of $(u_\epsilon)_{\epsilon>0}$. ☐

Note that a convex function $\psi\colon \mathbb{R}^n \to \mathbb{R}$ is automatically continuous, so the assumption of continuity is redundant.

As an application of Proposition 49.19, if U is given by

$$U = \{v \in \mathbb{R}^n \mid \varphi_i(v) \leq 0,\ i = 1, \ldots, m\},$$

where the functions $\varphi_i\colon \mathbb{R}^n \to \mathbb{R}$ are convex, we can take ψ to be the function given by

$$\psi(v) = \sum_{i=1}^m \max\{\varphi_i(v), 0\}.$$