As a consequence, when I is finite, say of size $p = \dim(E)$, the dimension of $\mathrm{S}^m(E)$ is the number of finite multisets $(j_1, \ldots, j_p)$, such that $j_1 + \cdots + j_p = m$, $j_k \geq 0$. We leave as an exercise to show that this number is $\binom{p+m-1}{m}$. Thus, if $\dim(E) = p$, then the dimension of $\mathrm{S}^m(E)$ is $\binom{p+m-1}{m}$. Compare with the dimension of $E^{\otimes m}$, which is p^m. In particular, when $p = 2$, the dimension of $\mathrm{S}^m(E)$ is $m + 1$. This can also be seen directly.

Remark: The number $\binom{p+m-1}{m}$ is also the number of homogeneous monomials

$$X_1^{j_1} \cdots X_p^{j_p}$$

of total degree m in p variables (we have $j_1 + \cdots + j_p = m$). This is not a coincidence! Given a vector space E and a basis $(e_i)_{i \in I}$ for E, Proposition 33.28 shows that every symmetric tensor $z \in \mathrm{S}^m(E)$ can be written in a unique way as

$$z = \sum_{\substack{M \in \mathbb{N}^{(I)} \\ \sum_{i \in I} M(i) = m \\ \{i_1, \ldots, i_k\} = \mathrm{dom}(M)}} \lambda_M \, e_{i_1}^{\odot M(i_1)} \odot \cdots \odot e_{i_k}^{\odot M(i_k)},$$

for some unique family of scalars $\lambda_M \in K$, all zero except for a finite number.

This looks like a homogeneous polynomial of total degree m, where the monomials of total degree m are the symmetric tensors

$$e_{i_1}^{\odot M(i_1)} \odot \cdots \odot e_{i_k}^{\odot M(i_k)}$$

in the "indeterminates" e_i, where $i \in I$ (recall that $M(i_1) + \cdots + M(i_k) = m$) and implies that polynomials can be defined in terms of symmetric tensors.

33.9 Some Useful Isomorphisms for Symmetric Powers

We can show the following property of the symmetric tensor product, using the proof technique of Proposition 33.13 (3).

Proposition 33.29. *We have the following isomorphism:*

$$\mathrm{S}^n(E \oplus F) \cong \bigoplus_{k=0}^{n} \mathrm{S}^k(E) \otimes \mathrm{S}^{n-k}(F).$$

33.10 Duality for Symmetric Powers

In this section all vector spaces are assumed to have *finite dimension over a field of characteristic zero*. We define a nondegenerate pairing $\mathrm{S}^n(E^*) \times \mathrm{S}^n(E) \longrightarrow K$ as follows: Consider the multilinear map

$$(E^*)^n \times E^n \longrightarrow K$$