Proof. We have

$$\varphi(ghg^{-1}) = \varphi(g)\varphi(h)\varphi(g^{-1}) = \varphi(g)e'\varphi(g)^{-1} = \varphi(g)\varphi(g)^{-1} = e',$$

for all $h \in H = \text{Ker } \varphi$ and all $g \in G$. Thus, by definition of $H = \text{Ker } \varphi$, we have $gHg^{-1} \subseteq H$. ☐

Definition 2.10. For any group G, a subgroup N of G is a *normal subgroup* of G iff

$$gNg^{-1} = N, \quad \text{for all } g \in G.$$

This is denoted by $N \lhd G$.

Proposition 2.11 shows that the kernel Ker φ of a homomorphism $\varphi\colon G \to G'$ is a normal subgroup of G.

Observe that if G is abelian, then *every* subgroup of G is normal.

Consider Example 2.2. Let $R \in \mathbf{SO}(2)$ and $A \in \mathbf{SL}(2, \mathbb{R})$ be the matrices

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then

$$A^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

and we have

$$ARA^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix},$$

and clearly $ARA^{-1} \notin \mathbf{SO}(2)$. Therefore $\mathbf{SO}(2)$ is not a normal subgroup of $\mathbf{SL}(2, \mathbb{R})$. The same counter-example shows that $\mathbf{O}(2)$ is not a normal subgroup of $\mathbf{GL}(2, \mathbb{R})$.

Let $R \in \mathbf{SO}(2)$ and $Q \in \mathbf{SO}(3)$ be the matrices

$$R = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then

$$Q^{-1} = Q^{\top} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$