algorithm and its variants. A particularly nice feature of the tableau formalism is that the update of a tableau can be performed using elementary row operations identical to the operations used during the reduction of a matrix to row reduced echelon form (rref). What differs is the criterion for the choice of the pivot.

However, we do not discuss other methods such as the ellipsoid method or interior points methods. For these more algorithmic issues, we refer the reader to standard texts on linear programming. In our opinion, one of the clearest (and among the most concise!) is Matousek and Gardner [123]; Chvatal [40] and Schrijver [148] are classics. Papadimitriou and Steiglitz [134] offers a very crisp presentation in the broader context of combinatorial optimization, and Bertsimas and Tsitsiklis [21] and Vanderbei [181] are very complete.

Linear programming has to do with maximizing a linear cost function $c_1x_1 + \cdots + c_nx_n$ with respect to $m$ "linear" inequalities of the form

$$a_{i1}x_1 + \cdots + a_{in}x_n \leq b_i.$$

These constraints can be put together into an $m \times n$ matrix $A = (a_{ij})$, and written more concisely as

$$Ax \leq b.$$

For technical reasons that will appear clearer later on, it is often preferable to add the nonnegativity constaints $x_i \geq 0$ for $i = 1, \ldots, n$. We write $x \geq 0$. It is easy to show that every linear program is equivalent to another one satisfying the constraints $x \geq 0$, at the expense of adding new variables that are also constrained to be nonnegative. Let $\mathcal{P}(A, b)$ be the set of *feasible solutions* of our linear program given by

$$\mathcal{P}(A, b) = \{x \in \mathbb{R}^n \mid Ax \leq b,\ x \geq 0\}.$$

Then there are two basic questions:

(1) Is $\mathcal{P}(A, b)$ nonempty, that is, does our linear program have a chance to have a solution?

(2) Does the objective function $c_1x_1 + \cdots + c_nx_n$ have a maximum value on $\mathcal{P}(A, b)$?

The answer to both questions can be **no**. But if $\mathcal{P}(A, b)$ is nonempty and if the objective function is bounded above (on $\mathcal{P}(A, b)$), then it can be shown that the maximum of $c_1x_1 + \cdots + c_nx_n$ is achieved by some $x \in \mathcal{P}(A, b)$. Such a solution is called an *optimal solution*. Perhaps surprisingly, this result is not so easy to prove (unless one has the simplex method at his disposal). We will prove this result in full detail (see Proposition 45.1).

The reason why linear constraints are so important is that the domain of potential optimal solutions $\mathcal{P}(A, b)$ is *convex*. In fact, $\mathcal{P}(A, b)$ is a convex polyhedron which is the intersection of half-spaces cut out by affine hyperplanes. The objective function being linear is convex, and this is also a crucial fact. Thus, we are led to study convex sets, in particular those that arise from solutions of inequalities defined by affine forms, but also convex cones.