Definition 34.6. An *orientation* of a vector space V of dimension n is given by the choice of some basis $(e_1, \ldots, e_n)$. We say that a basis $(u_1, \ldots, u_n)$ of V is *positively oriented* iff $\det(u_1, \ldots, u_n) > 0$ (where $\det(u_1, \ldots, u_n)$ denotes the determinant of the matrix whose jth column consists of the coordinates of u_j over the basis $(e_1, \ldots, e_n)$), otherwise it is *negatively oriented*. An *oriented vector space* is a vector space V together with an orientation of V.

If V is oriented by the basis $(e_1, \ldots, e_n)$, then V^* is oriented by the dual basis $(e_1^*, \ldots, e_n^*)$. If σ is any permutation of $\{1, \ldots, n\}$, then the basis $(e_{\sigma(1)}, \ldots, e_{\sigma(n)})$ has positive orientation iff the signature $\mathrm{sgn}(\sigma)$ of the permutation σ is even.

If V is an oriented vector space of dimension n, then we can define a linear isomorphism

$$* \colon \bigwedge^k V \to \bigwedge^{n-k} V,$$

called the *Hodge $*$-operator*. The existence of this operator is guaranteed by the following proposition.

Proposition 34.15. *Let V be any oriented Euclidean vector space whose orientation is given by some chosen orthonormal basis $(e_1, \ldots, e_n)$. For any alternating tensor $\alpha \in \bigwedge^k V$, there is a unique alternating tensor $*\alpha \in \bigwedge^{n-k} V$ such that*

$$\alpha \wedge \beta = \langle *\alpha, \beta \rangle_\wedge \, e_1 \wedge \cdots \wedge e_n$$

for all $\beta \in \bigwedge^{n-k} V$. The alternating tensor $*\alpha$ is independent of the choice of the positive orthonormal basis $(e_1, \ldots, e_n)$.

Proof. Since $\bigwedge^n V$ has dimension 1, the alternating tensor $e_1 \wedge \cdots \wedge e_n$ is a basis of $\bigwedge^n V$. It follows that for any fixed $\alpha \in \bigwedge^k V$, the linear map λ_α from $\bigwedge^{n-k} V$ to $\bigwedge^n V$ given by

$$\lambda_\alpha(\beta) = \alpha \wedge \beta$$

is of the form

$$\lambda_\alpha(\beta) = f_\alpha(\beta) \, e_1 \wedge \cdots \wedge e_n$$

for some linear form $f_\alpha \in \left(\bigwedge^{n-k} V\right)^*$. But then, by the duality induced by the inner product $\langle -, - \rangle$ on $\bigwedge^{n-k} V$, there is a unique vector $*\alpha \in \bigwedge^{n-k} V$ such that

$$f_\lambda(\beta) = \langle *\alpha, \beta \rangle_\wedge \quad \text{for all } \beta \in \bigwedge^{n-k} V,$$

which implies that

$$\alpha \wedge \beta = \lambda_\alpha(\beta) = f_\alpha(\beta) \, e_1 \wedge \cdots \wedge e_n = \langle *\alpha, \beta \rangle_\wedge \, e_1 \wedge \cdots \wedge e_n,$$

as claimed. If $(e_1', \ldots, e_n')$ is any other positively oriented orthonormal basis, by Proposition 34.2, $e_1' \wedge \cdots \wedge e_n' = \det(P) \, e_1 \wedge \cdots \wedge e_n = e_1 \wedge \cdots \wedge e_n$, since $\det(P) = 1$ where P is the change of basis from $(e_1, \ldots, e_n)$ to $(e_1', \ldots, e_n')$ and both bases are positively oriented. $\square$