However,

$$\dim(F) + \dim(G) + \dim(H) = i + j + n - k + 1$$

and $i + j = k + n$, so we have

$$\dim(F \cap G \cap H) \geq i + j + n - k + 1 - 2n = k + n + n - k + 1 - 2n = 1,$$

which shows that $F \cap G \cap H \neq (0)$. Then for any unit vector $z \in F \cap G \cap H \neq (0)$, we have

$$\lambda_k(A + B) \leq z^\top (A + B)z, \quad \lambda_i(A) \geq z^\top Az, \quad \lambda_j(B) \geq z^\top Bz,$$

establishing the desired inequality $\lambda_k(A + B) \leq \lambda_i(A) + \lambda_j(B)$. □

In the special case $i = j = k$, we obtain

$$\lambda_1(A) + \lambda_1(B) \leq \lambda_1(A + B), \quad \lambda_n(A + B) \leq \lambda_n(A) + \lambda_n(B).$$

It follows that λ_1 (as a function) is concave, while λ_n (as a function) is convex.

If $i = k$ and $j = 1$, we obtain

$$\lambda_k(A) + \lambda_1(B) \leq \lambda_k(A + B),$$

and if $i = k$ and $j = n$, we obtain

$$\lambda_k(A + B) \leq \lambda_k(A) + \lambda_n(B),$$

and combining them, we get

$$\lambda_k(A) + \lambda_1(B) \leq \lambda_k(A + B) \leq \lambda_k(A) + \lambda_n(B).$$

In particular, if B is positive semidefinite, since its eigenvalues are nonnegative, we obtain the following inequality known as the *monotonicity theorem* for symmetric (or Hermitian) matrices: if A and B are symmetric (or Hermitian) and B is positive semidefinite, then

$$\lambda_k(A) \leq \lambda_k(A + B) \quad k = 1, \ldots, n.$$

The reader is referred to Horn and Johnson [95] (Chapters 4 and 7) for a very complete treatment of matrix inequalities and interlacing results, and also to Lax [113] and Serre [156].

17.8 Summary

The main concepts and results of this chapter are listed below:

- *Normal* linear maps, *self-adjoint* linear maps, *skew-self-adjoint* linear maps, and *orthogonal* linear maps.