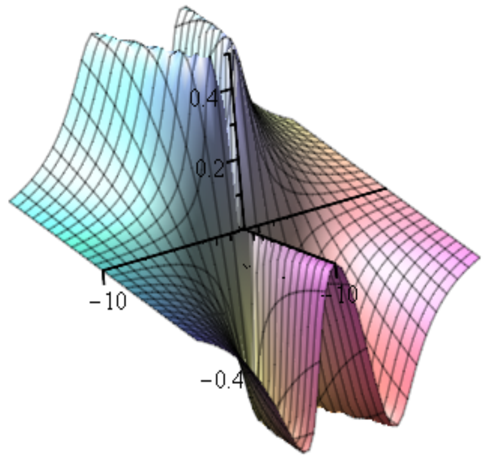


Figure 39.2: The graph of the function from Example 39.1. Note that $f$ is not continuous at $(0,0)$, despite the existence of $\mathrm{D}_u f(0,0)$ for all $u \neq 0$.

For any $u \neq 0$, letting $u = \begin{pmatrix} h \\ k \end{pmatrix}$, we have

$$\frac{f(0+tu) - f(0)}{t} = \frac{h^2 k}{t^2 h^4 + k^2},$$

so that

$$\mathrm{D}_u f(0,0) = \begin{cases} \frac{h^2}{k} & \text{if } k \neq 0 \\ 0 & \text{if } k = 0. \end{cases}$$

Thus, $\mathrm{D}_u f(0,0)$ exists for all $u \neq 0$.

On the other hand, if $\mathrm{D}f(0,0)$ existed, it would be a linear map $\mathrm{D}f(0,0)\colon \mathbb{R}^2 \to \mathbb{R}$ represented by a row matrix $(\alpha\ \beta)$, and we would have $\mathrm{D}_u f(0,0) = \mathrm{D}f(0,0)(u) = \alpha h + \beta k$, but the explicit formula for $\mathrm{D}_u f(0,0)$ is not linear. As a matter of fact, the function $f$ is not continuous at $(0,0)$. For example, on the parabola $y = x^2$, $f(x,y) = \frac{1}{2}$, and when we approach the origin on this parabola, the limit is $\frac{1}{2}$, but $f(0,0) = 0$.

To avoid the problems arising with directional derivatives we introduce a more uniform notion.

Given two normed spaces $E$ and $F$, recall that a linear map $f\colon E \to F$ is *continuous* iff there is some constant $C \geq 0$ such that

$$\|f(u)\| \leq C\,\|u\| \quad \text{for all } u \in E.$$