and φ is surjective. Note that M is isomorphic to $A^{(I)}/\mathrm{Im}(\psi)$. In such a situation we say that we have an *exact sequence* and this is denoted by the diagram

$$A^{(J)} \xrightarrow{\psi} A^{(I)} \xrightarrow{\varphi} M \longrightarrow 0.$$

Definition 35.6. Given an A-module M, a *presentation* of M is an exact sequence

$$A^{(J)} \xrightarrow{\psi} A^{(I)} \xrightarrow{\varphi} M \longrightarrow 0$$

which means that

1. $\mathrm{Im}(\psi) = \mathrm{Ker}\,(\varphi)$.
2. φ is surjective.

Consequently, M is isomorphic to $A^{(I)}/\mathrm{Im}(\psi)$. If I and J are both finite, we say that this is a *finite presentation* of M.

Observe that in the case of a finite presentation, I and J are finite, and if $|J| = n$ and $|I| = m$, then ψ is a linear map $\psi\colon A^n \to A^m$, so it is given by some $m \times n$ matrix R with coefficients in A called the *presentation matrix* of M. Every column R^j of R may thought of as a relation

$$a_{j1}e_1 + \cdots + a_{jm}e_m = 0$$

among the generators $e_1, \ldots, e_m$ of A^m, so we have n relations among these generators. Also the images of $e_1, \ldots, e_m$ in M are generators of M, so we can think of the above relations as relations among the generators of M.

The submodule of A^m spanned by the columns of R is *the set of relations* of M, and the columns of R are called a *complete set of relations* for M. The vectors $e_1, \ldots, e_m$ are called a set of *generators* for M. We may also say that the generators $e_1, \ldots, e_m$ and the relations $R^1, \ldots, R^n$ (the columns of R) are a (finite) presentation of the module M. The *module M presented by R is isomorphic to A^m/RA^n*, where we denote by RA^n the image of A^n by the linear map defined by R.

For example, the $\mathbb{Z}$-module presented by the 1×1 matrix $R = (5)$ is the quotient, $\mathbb{Z}/5\mathbb{Z}$, of $\mathbb{Z}$ by the submodule $5\mathbb{Z}$ corresponding to the single relation

$$5e_1 = 0.$$

But $\mathbb{Z}/5\mathbb{Z}$ has other presentations. For example, if we consider the matrix of relations

$$R = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix},$$