17.5 Normal and Other Special Matrices

First we consider real matrices. Recall the following definitions.

Definition 17.3. Given a real $m \times n$ matrix A, the *transpose $A^\top$ of A* is the $n \times m$ matrix $A^\top = (a^\top_{i\,j})$ defined such that

$$a^\top_{i\,j} = a_{j\,i}$$

for all i, j, $1 \leq i \leq m$, $1 \leq j \leq n$. A real $n \times n$ matrix A is

- *normal* if
$$A\,A^\top = A^\top A,$$

- *symmetric* if
$$A^\top = A,$$

- *skew-symmetric* if
$$A^\top = -A,$$

- *orthogonal* if
$$A\,A^\top = A^\top A = I_n.$$

Recall from Proposition 12.14 that when E is a Euclidean space and $(e_1, \ldots, e_n)$ is an orthonormal basis for E, if A is the matrix of a linear map $f\colon E \to E$ w.r.t. the basis $(e_1, \ldots, e_n)$, then $A^\top$ is the matrix of the adjoint f^* of f. Consequently, a normal linear map has a normal matrix, a self-adjoint linear map has a symmetric matrix, a skew-self-adjoint linear map has a skew-symmetric matrix, and an orthogonal linear map has an orthogonal matrix.

Furthermore, if $(u_1, \ldots, u_n)$ is another orthonormal basis for E and P is the change of basis matrix whose columns are the components of the u_i w.r.t. the basis $(e_1, \ldots, e_n)$, then P is orthogonal, and for any linear map $f\colon E \to E$, if A is the matrix of f w.r.t $(e_1, \ldots, e_n)$ and B is the matrix of f w.r.t. $(u_1, \ldots, u_n)$, then

$$B = P^\top AP.$$

As a consequence, Theorems 17.12 and 17.14–17.16 can be restated as follows.