Proposition 29.39. *Let φ be a nondegenerate symmetric bilinear form on a vector space E. For any two nonzero vectors $u, v \in E$, if $\varphi(u,u) = \varphi(v,v)$ and $v - u$ is nonisotropic, then the hyperplane reflection $\tau_H = \tau_{v-u}$ maps u to v, with $H = (K(v-u))^{\perp}$.*

Proof. Since $v - u$ is not isotropic, $\varphi(v-u, v-u) \neq 0$, and we have

$$\begin{aligned}
\tau_{v-u}(u) &= u - 2\frac{\varphi(u, v-u)}{\varphi(v-u, v-u)}(v-u) \\
&= u - 2\frac{\varphi(u,v) - \varphi(u,u)}{\varphi(v,v) - 2\varphi(u,v) + \varphi(u,u)}(v-u) \\
&= u - \frac{2(\varphi(u,v) - \varphi(u,u))}{2(\varphi(u,u) - 2\varphi(u,v))}(v-u) \\
&= v,
\end{aligned}$$

which proves the proposition. ☐

We can now obtain a cheap version of the Cartan–Dieudonné theorem.

Theorem 29.40. *(Cartan–Dieudonné, weak form) Let φ be a nondegenerate symmetric bilinear form on a K-vector space E of dimension n ($\mathrm{char}(K) \neq 2$). Then, every isometry $f \in \mathbf{O}(\varphi)$ with $f \neq \mathrm{id}$ is the composition of at most $2n - 1$ hyperplane reflections.*

Proof. We proceed by induction on n. For $n = 0$, this is trivial (since $\mathbf{O}(\varphi) = \{\mathrm{id}\}$).

Next, assume that $n \geq 1$. Since φ is nondegenerate, we know that there is some nonisotropic vector $u \in E$. There are three cases.

Case 1. $f(u) = u$.

Since φ is nondegenrate and u is nonisotropic, the hyperplane $H = (Ku)^{\perp}$ is nondegenerate, $E = H \overset{\perp}{\oplus} Ku$, and since $f(u) = u$, we must have $f(H) = H$. The restriction f' of of f to H is an isometry of H. By the induction hypothesis, we can write

$$f' = \tau'_k \circ \cdots \circ \tau'_1,$$

where τ_i is some hyperplane reflection about a hyperplane L_i in H, with $k \leq 2n - 3$. We can extend each τ'_i to a reflection τ_i about the hyperplane $L_i \overset{\perp}{\oplus} Ku$ so that $\tau_i(u) = u$, and clearly,

$$f = \tau_k \circ \cdots \circ \tau_1.$$

Case 2. $f(u) = -u$.

If τ is the hyperplane reflection about the hyperplane $H = (Ku)^{\perp}$, then $g = \tau \circ f$ is an isometry of E such that $g(u) = u$, and we are back to Case (1). Since $\tau^2 = 1$ We obtain

$$f = \tau \circ \tau_k \circ \cdots \circ \tau_1$$