A direct proof of the Farkas–Minkowski proposition not involving Proposition 47.1 is given at the end of this section.

**Remark:** There is a generalization of the Farkas–Minkowski proposition applying to infinite dimensional real Hilbert spaces; see Theorem 48.12 (or Ciarlet [41], Chapter 9).

Proposition 47.2 implies our first version of Farkas' lemma.

**Proposition 47.3.** *(Farkas Lemma, Version I) Let $A$ be an $m \times n$ matrix and let $b \in \mathbb{R}^m$ be any vector. The linear system $Ax = b$ has no solution $x \geq 0$ iff there is some nonzero linear form $y \in (\mathbb{R}^m)^*$ such that $yA \geq 0_n^\top$ and $yb < 0$.*

*Proof.* First assume that there is some nonzero linear form $y \in (\mathbb{R}^m)^*$ such that $yA \geq 0$ and $yb < 0$. If $x \geq 0$ is a solution of $Ax = b$, then we get

$$yAx = yb,$$

but if $yA \geq 0$ and $x \geq 0$, then $yAx \geq 0$, and yet by hypothesis $yb < 0$, a contradiction.

Next assume that $Ax = b$ has no solution $x \geq 0$. This means that $b$ does not belong to the polyhedral cone $C = \text{cone}(\{A^1, \ldots, A^n\})$ spanned by the columns of $A$. By Proposition 47.2, there is a nonzero linear form $y \in (\mathbb{R}^m)^*$ such that

1. $yA^j \geq 0$ for $j = 1, \ldots, n$.

2. $yb < 0$,

which says that $yA \geq 0_n^\top$ and $yb < 0$. $\square$

Next consider the solvability of a system of inequalities of the form $Ax \leq b$ and $x \geq 0$.

**Proposition 47.4.** *(Farkas Lemma, Version II) Let $A$ be an $m \times n$ matrix and let $b \in \mathbb{R}^m$ be any vector. The system of inequalities $Ax \leq b$ has no solution $x \geq 0$ iff there is some nonzero linear form $y \in (\mathbb{R}^m)^*$ such that $y \geq 0_m^\top$, $yA \geq 0_n^\top$ and $yb < 0$.*

*Proof.* We use the trick of linear programming which consists of adding "slack variables" $z_i$ to convert inequalities $a_i x \leq b_i$ into equations $a_i x + z_i = b_i$ with $z_i \geq 0$ already discussed just before Definition 44.9. If we let $z = (z_1, \ldots, z_m)$, it is obvious that the system $Ax \leq b$ has a solution $x \geq 0$ iff the equation

$$\begin{pmatrix} A & I_m \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} = b$$

has a solution $\begin{pmatrix} x \\ z \end{pmatrix}$ with $x \geq 0$ and $z \geq 0$. Now by Farkas I, the above system has no solution with with $x \geq 0$ and $z \geq 0$ iff there is some nonzero linear form $y \in (\mathbb{R}^m)^*$ such that

$$y \begin{pmatrix} A & I_m \end{pmatrix} \geq 0_{n+m}^\top$$

and $yb < 0$, that is, $yA \geq 0_n^\top$, $y \geq 0_m^\top$ and $yb < 0$. $\square$