for some $a \in A$. However, there must be some n such that $a \in (a_n)$, and thus,

$$(a_n) \subseteq (a) \subseteq (a_n),$$

and the chain stabilizes at (a_n).

As a consequence, there are maximal ideals in $\mathcal{S}$. Let (a) be a maximal ideal in $\mathcal{S}$. Then, for any ideal (d) such that

$$(a) \subset (d) \quad \text{and} \quad (a) \neq (d),$$

we must have $d \notin \mathcal{S}$, since otherwise (a) would not be a maximal ideal in $\mathcal{S}$. Observe that a is not irreducible, since $(a) \in \mathcal{S}$, and thus,

$$a = bc$$

for some $b, c \in A$, where neither b nor c is a unit. Then,

$$(a) \subseteq (b) \quad \text{and} \quad (a) \subseteq (c).$$

If $(a) = (b)$, then $b = au$ for some $u \in A$, and then

$$a = auc,$$

so that

$$1 = uc,$$

since A is an integral domain, and thus, c is a unit, a contradiction. Thus, $(a) \neq (b)$, and similarly, $(a) \neq (c)$. But then, by a previous observation $b \notin \mathcal{S}$ and $c \notin \mathcal{S}$, and since a and b are not units, both b and c factor as products of irreducible elements and so does $a = bc$, a contradiction. This implies that $\mathcal{S} = \emptyset$, so every nonnull element that is a not a unit can be factored as a product of irreducible elements

To prove the uniqueness of factorizations, we use Proposition 32.2. Assume that a is irreducible and that a divides bc. If a does not divide b, by a previous remark, a and b are relatively prime, and by Proposition 32.11, there are some $x, y \in A$ such that

$$ax + by = 1.$$

Thus,

$$acx + bcy = c,$$

and since a divides bc, we see that a must divide c, as desired. $\square$

Thus, we get another justification of the fact that $\mathbb{Z}$ is a UFD and that if K is a field, then $K[X]$ is a UFD.

It should also be noted that in a UFD, gcd's of nonnull elements always exist. Indeed, this is trivial if a or b is a unit, and otherwise, we can write

$$a = p_1 \cdots p_m \quad \text{and} \quad b = q_1 \cdots q_n$$