It is instructive to characterize when a 2×2 real matrix A is symmetric positive definite. Write

$$A = \begin{pmatrix} a & c \\ c & b \end{pmatrix}.$$

Then we have

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & c \\ c & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + 2cxy + by^2.$$

If the above expression is strictly positive for all nonzero vectors $\binom{x}{y}$, then for $x = 1, y = 0$ we get $a > 0$ and for $x = 0, y = 1$ we get $b > 0$. Then we can write

$$\begin{aligned} ax^2 + 2cxy + by^2 &= \left(\sqrt{a}x + \frac{c}{\sqrt{a}}y\right)^2 + by^2 - \frac{c^2}{a}y^2 \\ &= \left(\sqrt{a}x + \frac{c}{\sqrt{a}}y\right)^2 + \frac{1}{a}\left(ab - c^2\right)y^2. \end{aligned} \tag{$\dagger$}$$

Since $a > 0$, if $ab - c^2 \leq 0$, then we can choose $y > 0$ so that the second term is negative or zero, and we can set $x = -(c/a)y$ to make the first term zero, in which case $ax^2 + 2cxy + by^2 \leq 0$, so we must have $ab - c^2 > 0$.

Conversely, if $a > 0, b > 0$ and $ab > c^2$, then for any $(x, y) \neq (0, 0)$, if $y = 0$, then $x \neq 0$ and the first term of ($\dagger$) is positive, and if $y \neq 0$, then the second term of ($\dagger$) is positive. Therefore, the matrix A is symmetric positive definite iff

$$a > 0,\ b > 0,\ ab > c^2. \tag{$*$}$$

Note that $ab - c^2 = \det(A)$, so the third condition says that $\det(A) > 0$.

Observe that the condition $b > 0$ is redundant, since if $a > 0$ and $ab > c^2$, then we must have $b > 0$ (and similarly $b > 0$ and $ab > c^2$ implies that $a > 0$).

We can try to visualize the space of 2×2 real symmetric positive definite matrices in $\mathbb{R}^3$, by viewing (a, b, c) as the coordinates along the x, y, z axes. Then the locus determined by the strict inequalities in ($*$) corresponds to the region on the side of the cone of equation $xy = z^2$ that does not contain the origin and for which $x > 0$ and $y > 0$. For $z = \delta$ fixed, the equation $xy = \delta^2$ define a hyperbola in the plane $z = \delta$. The cone of equation $xy = z^2$ consists of the lines through the origin that touch the hyperbola $xy = 1$ in the plane $z = 1$. We only consider the branch of this hyperbola for which $x > 0$ and $y > 0$. See Figure 8.6.

It is not hard to show that the inverse of a real symmetric positive definite matrix is also real symmetric positive definite, but the product of two real symmetric positive definite matrices may *not* be symmetric positive definite, as the following example shows:

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1\sqrt{2} \\ -1/\sqrt{2} & 3/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 & 2/\sqrt{2} \\ -1/\sqrt{2} & 5/\sqrt{2} \end{pmatrix}.$$