Chapter 6

Direct Sums

In this chapter all vector spaces are defined over an arbitrary field K. For the sake of concreteness, the reader may safely assume that $K = \mathbb{R}$.

6.1 Sums, Direct Sums, Direct Products

There are some useful ways of forming new vector spaces from older ones, in particular, direct products and direct sums. Regarding direct sums, there is a subtle point, which is that if we attempt to define the direct sum $E \coprod F$ of two vector spaces using the cartesian product $E \times F$, we don't quite get the right notion because elements of $E \times F$ are ordered pairs, but we want $E \coprod F = F \coprod E$. Thus, we want to think of the elements of $E \coprod F$ as unordrered pairs of elements. It is possible to do so by considering the direct sum of a *family* $(E_i)_{i\in\{1,2\}}$, and more generally of a family $(E_i)_{i\in I}$. For simplicity, we begin by considering the case where $I = \{1, 2\}$.

Definition 6.1. Given a family $(E_i)_{i\in\{1,2\}}$ of two vector spaces, we define the *(external) direct sum* $E_1 \coprod E_2$ (or *coproduct*) of the family $(E_i)_{i\in\{1,2\}}$ as the set

$$E_1 \coprod E_2 = \{\{\langle 1, u\rangle, \langle 2, v\rangle\} \mid u \in E_1,\ v \in E_2\},$$

with addition

$$\{\langle 1, u_1\rangle, \langle 2, v_1\rangle\} + \{\langle 1, u_2\rangle, \langle 2, v_2\rangle\} = \{\langle 1, u_1 + u_2\rangle, \langle 2, v_1 + v_2\rangle\},$$

and scalar multiplication

$$\lambda\{\langle 1, u\rangle, \langle 2, v\rangle\} = \{\langle 1, \lambda u\rangle, \langle 2, \lambda v\rangle\}.$$

We define the *injections* $in_1 \colon E_1 \to E_1 \coprod E_2$ and $in_2 \colon E_2 \to E_1 \coprod E_2$ as the linear maps defined such that,

$$in_1(u) = \{\langle 1, u\rangle, \langle 2, 0\rangle\},$$

and

$$in_2(v) = \{\langle 1, 0\rangle, \langle 2, v\rangle\}.$$