Chapter 25

Embedding an Affine Space in a Vector Space

25.1 The "Hat Construction," or Homogenizing

For all practical purposes, most geometric objects, including curves and surfaces, live in affine spaces. A disadvantage of the affine world is that points and vectors live in disjoint universes. It is often more convenient, at least mathematically, to deal with linear objects (vector spaces, linear combinations, linear maps), rather than affine objects (affine spaces, affine combinations, affine maps). Actually, it would also be advantageous if we could manipulate points and vectors as if they lived in a common universe, using perhaps an extra bit of information to distinguish between them if necessary.

Such a "homogenization" (or "hat construction") can be achieved. As a matter of fact, such a homogenization of an affine space and its associated vector space will be very useful to define and manipulate rational curves and surfaces. Indeed, the hat construction yields a canonical construction of the projective completion of an affine space. It also leads to a very elegant method for obtaining the various formulae giving the derivatives of a polynomial curve, or the directional derivatives of polynomial surfaces. However, these formulae are not needed here. Thus we omit this topic, referring the readers to Gallier [70].

This chapter proceeds as follows. First, the construction of a vector space $\widehat{E}$ in which both E and $\overrightarrow{E}$ are embedded as (affine) hyperplanes is described. It is shown how affine frames in E become bases in $\widehat{E}$. It turns out that $\widehat{E}$ is characterized by a universality property: Affine maps to vector spaces extend uniquely to linear maps. As a consequence, affine maps between affine spaces E and F extend to linear maps between $\widehat{E}$ and $\widehat{F}$.

Let us first explain how to distinguish between points and vectors practically, using what amounts to a "hacking trick." Then, we will show that such a procedure can be put on firm mathematical grounds.

Assume that we consider the real affine space E of dimension 3, and that we have some