Prove that $\varphi(e_1) = e_2$ (where e_i is the identity element of G_i) and that

$$\varphi(a^{-1}) = (\varphi(a))^{-1}, \quad a \in G_1.$$

(3) Let S^1 be the unit circle, that is

$$S^1 = \{e^{i\theta} = \cos\theta + i\sin\theta \mid 0 \leq \theta < 2\pi\},$$

and let φ be the function given by

$$\varphi \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} = (a, c, e^{ib}).$$

Prove that φ is a surjective function onto $G = \mathbb{R} \times \mathbb{R} \times S^1$, and that if we define multiplication on this set by

$$(x_1, y_1, u_1) \cdot (x_2, y_2, u_2) = (x_1 + x_2, y_1 + y_2, e^{ix_1 y_2} u_1 u_2),$$

then G is a group and φ is a group homomorphism from H onto G.

(4) The *kernel* of a homomorphism $\varphi\colon G_1 \to G_2$ is defined as

$$\mathrm{Ker}\,(\varphi) = \{a \in G_1 \mid \varphi(a) = e_2\}.$$

Find explicitly the kernel of φ and show that it is a subgroup of H.

Problem 3.2. For any $m \in \mathbb{Z}$ with $m > 0$, the subset $m\mathbb{Z} = \{mk \mid k \in \mathbb{Z}\}$ is an abelian subgroup of $\mathbb{Z}$. Check this.

(1) Give a group isomorphism (an invertible homomorphism) from $m\mathbb{Z}$ to $\mathbb{Z}$.

(2) Check that the inclusion map $i\colon m\mathbb{Z} \to \mathbb{Z}$ given by $i(mk) = mk$ is a group homomorphism. Prove that if $m \geq 2$ then there is no group homomorphism $p\colon \mathbb{Z} \to m\mathbb{Z}$ such that $p \circ i = \mathrm{id}$.

Remark: The above shows that abelian groups fail to have some of the properties of vector spaces. We will show later that a linear map satisfying the condition $p \circ i = \mathrm{id}$ always exists.

Problem 3.3. Let $E = \mathbb{R} \times \mathbb{R}$, and define the addition operation

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1, +y_2), \quad x_1, x_2, y_1, y_2 \in \mathbb{R},$$

and the multiplication operation $\cdot\colon \mathbb{R} \times E \to E$ by

$$\lambda \cdot (x, y) = (\lambda x, y), \quad \lambda, x, y \in \mathbb{R}.$$

Show that E with the above operations $+$ and $\cdot$ is not a vector space. Which of the axioms is violated?