Proof. Given any $x, y \in \operatorname{Im} f$, there are some $u, v \in E$ such that $x = f(u)$ and $y = f(v)$, and for all $\lambda, \mu \in K$, we have

$$f(\lambda u + \mu v) = \lambda f(u) + \mu f(v) = \lambda x + \mu y,$$

and thus, $\lambda x + \mu y \in \operatorname{Im} f$, showing that $\operatorname{Im} f$ is a subspace of F.

Given any $x, y \in \operatorname{Ker} f$, we have $f(x) = 0$ and $f(y) = 0$, and thus,

$$f(\lambda x + \mu y) = \lambda f(x) + \mu f(y) = 0,$$

that is, $\lambda x + \mu y \in \operatorname{Ker} f$, showing that $\operatorname{Ker} f$ is a subspace of E.

First, assume that $\operatorname{Ker} f = (0)$. We need to prove that $f(x) = f(y)$ implies that $x = y$. However, if $f(x) = f(y)$, then $f(x) - f(y) = 0$, and by linearity of f we get $f(x - y) = 0$. Because $\operatorname{Ker} f = (0)$, we must have $x - y = 0$, that is $x = y$, so f is injective. Conversely, assume that f is injective. If $x \in \operatorname{Ker} f$, that is $f(x) = 0$, since $f(0) = 0$ we have $f(x) = f(0)$, and by injectivity, $x = 0$, which proves that $\operatorname{Ker} f = (0)$. Therefore, f is injective iff $\operatorname{Ker} f = (0)$. $\square$

Since by Proposition 3.17, the image $\operatorname{Im} f$ of a linear map f is a subspace of F, we can define the *rank* $\operatorname{rk}(f)$ of f as the dimension of $\operatorname{Im} f$.

Definition 3.20. Given a linear map $f \colon E \to F$, the *rank* $\operatorname{rk}(f)$ of f is the dimension of the image $\operatorname{Im} f$ of f.

A fundamental property of bases in a vector space is that they allow the definition of linear maps as unique homomorphic extensions, as shown in the following proposition.

Proposition 3.18. *Given any two vector spaces E and F, given any basis $(u_i)_{i\in I}$ of E, given any other family of vectors $(v_i)_{i\in I}$ in F, there is a unique linear map $f \colon E \to F$ such that $f(u_i) = v_i$ for all $i \in I$. Furthermore, f is injective iff $(v_i)_{i\in I}$ is linearly independent, and f is surjective iff $(v_i)_{i\in I}$ generates F.*

Proof. If such a linear map $f \colon E \to F$ exists, since $(u_i)_{i\in I}$ is a basis of E, every vector $x \in E$ can written uniquely as a linear combination

$$x = \sum_{i\in I} x_i u_i,$$

and by linearity, we must have

$$f(x) = \sum_{i\in I} x_i f(u_i) = \sum_{i\in I} x_i v_i.$$

Define the function $f \colon E \to F$, by letting

$$f(x) = \sum_{i\in I} x_i v_i, \quad x = \sum_{i\in I} x_i u_i, \tag{$\dagger$}$$