14.6 Orthogonal Projections and Involutions

In this section we begin by assuming that the field K is not a field of characteristic 2. Recall that a linear map $f\colon E \to E$ is an *involution* iff $f^2 = \mathrm{id}$, and is *idempotent* iff $f^2 = f$. We know from Proposition 6.9 that if f is idempotent, then

$$E = \mathrm{Im}(f) \oplus \mathrm{Ker}\,(f),$$

and that the restriction of f to its image is the identity. For this reason, a linear idempotent map is called a *projection*. The connection between involutions and projections is given by the following simple proposition.

Proposition 14.22. *For any linear map $f\colon E \to E$, we have $f^2 = \mathrm{id}$ iff $\frac{1}{2}(\mathrm{id} - f)$ is a projection iff $\frac{1}{2}(\mathrm{id} + f)$ is a projection; in this case, f is equal to the difference of the two projections $\frac{1}{2}(\mathrm{id} + f)$ and $\frac{1}{2}(\mathrm{id} - f)$.*

Proof. We have

$$\left(\frac{1}{2}(\mathrm{id} - f)\right)^2 = \frac{1}{4}(\mathrm{id} - 2f + f^2)$$

so

$$\left(\frac{1}{2}(\mathrm{id} - f)\right)^2 = \frac{1}{2}(\mathrm{id} - f) \quad \text{iff} \quad f^2 = \mathrm{id}.$$

We also have

$$\left(\frac{1}{2}(\mathrm{id} + f)\right)^2 = \frac{1}{4}(\mathrm{id} + 2f + f^2),$$

so

$$\left(\frac{1}{2}(\mathrm{id} + f)\right)^2 = \frac{1}{2}(\mathrm{id} + f) \quad \text{iff} \quad f^2 = \mathrm{id}.$$

Obviously, $f = \frac{1}{2}(\mathrm{id} + f) - \frac{1}{2}(\mathrm{id} - f)$. ∎

Proposition 14.23. *For any linear map $f\colon E \to E$, let $U^+ = \mathrm{Ker}\,(\frac{1}{2}(\mathrm{id} - f))$ and let $U^- = \mathrm{Im}(\frac{1}{2}(\mathrm{id} - f))$. If $f^2 = \mathrm{id}$, then*

$$U^+ = \mathrm{Ker}\left(\frac{1}{2}(\mathrm{id} - f)\right) = \mathrm{Im}\left(\frac{1}{2}(\mathrm{id} + f)\right),$$

and so, $f(u) = u$ on U^+ and $f(u) = -u$ on U^-.

Proof. If $f^2 = \mathrm{id}$, then

$$(\mathrm{id} - f) \circ (\mathrm{id} + f) = \mathrm{id} - f^2 = \mathrm{id} - \mathrm{id} = 0,$$

which implies that

$$\mathrm{Im}\left(\frac{1}{2}(\mathrm{id} + f)\right) \subseteq \mathrm{Ker}\left(\frac{1}{2}(\mathrm{id} - f)\right).$$