is $\mathbb{R}^m_+$, where

$$L(v, \mu) = J(v) + \sum_{i=1}^{m} \mu_i \varphi_i(v)$$

is the Lagrangian of our problem. Now the projection p_+ from $\mathbb{R}^m$ to $\mathbb{R}^m_+$ is very simple, namely

$$(p_+(\lambda))_i = \max\{\lambda_i, 0\}, \quad 1 \leq i \leq m.$$

It follows that the projection-gradient method should be applicable to the *Dual Problem (D):*

$$\begin{aligned} &\text{maximize} && G(\mu) \\ &\text{subject to} && \mu \in \mathbb{R}^m_+. \end{aligned}$$

If the hypotheses of Theorem 50.17 hold, then a solution λ of the Dual Program (D) yields a solution u_λ of the primal problem.

Uzawa's method is essentially the gradient method with fixed stepsize applied to the Dual Problem (D). However, it is designed to yield a solution of the primal problem.

Uzawa's method:

Given an arbitrary initial vector $\lambda^0 \in \mathbb{R}^m_+$, two sequences $(\lambda^k)_{k\geq 0}$ and $(u^k)_{k\geq 0}$ are constructed, with $\lambda^k \in \mathbb{R}^m_+$ and $u^k \in V$.

Assuming that $\lambda^0, \lambda^1, \ldots, \lambda^k$ are known, u^k and λ^{k+1} are determined as follows:

u^k is the unique solution of the minimization problem, find $u^k \in V$ such that

$$\text{(UZ)} \quad \begin{cases} J(u^k) + \displaystyle\sum_{i=1}^{m} \lambda_i^k \varphi_i(u^k) = \inf_{v \in V} \left(J(v) + \displaystyle\sum_{i=1}^{m} \lambda_i^k \varphi_i(v) \right); \text{ and} \\ \qquad\qquad \lambda_i^{k+1} = \max\{\lambda_i^k + \rho\varphi_i(u^k), 0\}, \quad 1 \leq i \leq m, \end{cases}$$

where $\rho > 0$ is a suitably chosen parameter.

Recall that in the proof of Theorem 50.17 we showed $(*_{\text{deriv}})$, namely

$$G'_{\lambda^k}(\xi) = \langle \nabla G_{\lambda^k}, \xi \rangle = \sum_{i=1}^{m} \xi_i \varphi_i(u^k),$$

which means that $(\nabla G_{\lambda^k})_i = \varphi_i(u^k)$. Then the second equation in (UZ) corresponds to the gradient-projection step

$$\lambda^{k+1} = p_+(\lambda^k + \rho \nabla G_{\lambda^k}).$$

Note that because the problem is a maximization problem we use a positive sign instead of a negative sign. Uzawa's method is indeed a gradient method.

Basically, Uzawa's method replaces a constrained optimization problem by a sequence of unconstrained optimization problems involving the Lagrangian of the (primal) problem.