- Computing the permanent is a #P-perfect problem (L. Valiant).
- Permanents count the number of SDRs of sequences of subsets of a given set.

7.10 Further Readings

Thorough expositions of the material covered in Chapter 3–6 and 7 can be found in Strang [170, 169], Lax [113], Lang [109], Artin [7], Mac Lane and Birkhoff [118], Hoffman and Kunze [94], Dummit and Foote [54], Bourbaki [25, 26], Van Der Waerden [179], Serre [156], Horn and Johnson [95], and Bertin [15]. These notions of linear algebra are nicely put to use in classical geometry, see Berger [11, 12], Tisseron [175] and Dieudonné [49].

7.11 Problems

Problem 7.1. Prove that every transposition can be written as a product of basic transpositions.

Problem 7.2. (1) Given two vectors in $\mathbb{R}^2$ of coordinates (c_1-a_1, c_2-a_2) and (b_1-a_1, b_2-a_2), prove that they are linearly dependent iff

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

(2) Given three vectors in $\mathbb{R}^3$ of coordinates $(d_1-a_1, d_2-a_2, d_3-a_3)$, $(c_1-a_1, c_2-a_2, c_3-a_3)$, and $(b_1-a_1, b_2-a_2, b_3-a_3)$, prove that they are linearly dependent iff

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0.$$

Problem 7.3. Let A be the $(m+n)\times(m+n)$ block matrix (over any field K) given by

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_4 \end{pmatrix},$$

where A_1 is an $m\times m$ matrix, A_2 is an $m\times n$ matrix, and A_4 is an $n\times n$ matrix. Prove that $\det(A) = \det(A_1)\det(A_4)$.

Use the above result to prove that if A is an upper triangular $n\times n$ matrix, then $\det(A) = a_{11}a_{22}\cdots a_{nn}$.