If we assume by induction that matrix A_k is real (with $k = 2\ell+1, \ell \geq 0$), then the matrix $S = A_k^2 - 2(\Re\sigma_k)A_k + |\sigma_k|^2 I$ is also real, and since $Q_k Q_{k+1}$ is unitary and $R_{k+1}R_k$ is upper triangular, we see that

$$S = Q_k Q_{k+1} R_{k+1} R_k$$

is a QR-factorization of the real matrix S, thus $Q_k Q_{k+1}$ and $R_{k+1}R_k$ can be chosen to be real matrices, in which case $(Q_k Q_{k+1})^*$ is also real, and thus

$$A_{k+2} = Q_{k+1}^* Q_k^* A_k Q_k Q_{k+1} = (Q_k Q_{k+1})^* A_k Q_k Q_{k+1}$$

is real. Consequently, if $A_1 = A$ is real, then $A_{2\ell+1}$ is real for all $\ell \geq 0$.

The strategy that consists in picking σ_k and $\overline{\sigma}_k$ as the complex conjugate eigenvalues of the corner block

$$\begin{pmatrix} (H_k)_{n-1n-1} & (H_k)_{n-1n} \\ (H_k)_{nn-1} & (H_k)_{nn} \end{pmatrix}$$

is called the *Francis shift* (here we are assuming that A has be reduced to upper Hessenberg form).

It should be noted that there are matrices for which neither a shift by $(H_k)_{nn}$ nor the Francis shift works. For instance, the permutation matrix

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

has eigenvalues $e^{i2\pi/3}, e^{i4\pi/3}, +1$, and neither of the above shifts apply to the matrix

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

However, a shift by 1 does work. There are other kinds of matrices for which the QR algorithm does not converge. Demmel gives the example of matrices of the form

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & h & 0 \\ 0 & -h & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

where h is small.

Algorithms implementing the QR algorithm with shifts and double shifts perform "exceptional" shifts every 10 shifts. Despite the fact that the QR algorithm has been perfected since the 1960's, it is still an open problem to find a shift strategy that ensures convergence of all matrices.

Implicit shifting is based on a result known as the *implicit Q theorem*. This theorem says that if A is reduced to upper Hessenberg form as $A = UHU^*$ and if H is unreduced