Proposition 26.28. *Given any two lines D_1, D_2 in a real Euclidean plane $(E, \overrightarrow{E})$, letting D_I and D_J be the isotropic lines in $\widetilde{E}_{\mathbb{C}}$ joining the intersection point $D_1 \cap D_2$ of D_1 and D_2 to the circular points I and J, if θ is the angle of the two lines D_1, D_2, we have*

$$[D_1, D_2, D_I, D_J] = e^{i2\theta},$$

known as Laguerre's formula, and independently of the orientation of the plane, we have

$$\theta = \frac{1}{2}|\log_U([D_1, D_2, D_I, D_J])|,$$

known as Cayley's formula.

In particular, note that $\theta = \pi/2$ iff $[D_1, D_2, D_I, D_J] = -1$, that is, if (D_1, D_2, D_I, D_J) forms a harmonic division. Thus, two lines D_1 and D_2 are orthogonal iff they form a harmonic division with D_I and D_J.

The above considerations show that it is not necessary to assume that $(E, \overrightarrow{E})$ is a real Euclidean plane to define the angle of two lines and orthogonality. Instead, it is enough to assume that two complex conjugate points I, J on the line H at infinity are given. We say that $\langle I, J\rangle$ provides a *similarity structure* on $\widetilde{E}_{\mathbb{C}}$. Note in passing that a circle can be defined as a conic in $\widetilde{E}_{\mathbb{C}}$ that contains the circular points I, J. Indeed, the equation of a conic is of the form

$$ax^2 + by^2 + cxy + dxz + eyz + fz^2 = 0.$$

If this conic contains the circular points $I = (1, -i, 0)$ and $J = (1, i, 0)$, we get the two equations

$$\begin{aligned} a - b - ic &= 0, \\ a - b + ic &= 0, \end{aligned}$$

from which we get $2ic = 0$ and $a = b$, that is, $c = 0$ and $a = b$. The resulting equation

$$ax^2 + ay^2 + dxz + eyz + fz^2 = 0$$

is indeed that of a circle.

Instead of using the function $\log_U \colon (U - \{-1\}) \to]-\pi, \pi[$ as logarithm, one may use the complex logarithm function $\log \colon \mathbb{C}^* \to B$, where $\mathbb{C}^* = \mathbb{C} - \{0\}$ and

$$B = \{x + iy \mid x, y \in \mathbb{R}, \ -\pi < y \leq \pi\}.$$

Indeed, the restriction of the complex exponential function $z \mapsto e^z$ to B is bijective, and thus, log is well-defined on C^* (note that log is a homeomorphism from $\mathbb{C} - \{x \mid x \in \mathbb{R}, \ x \leq 0\}$ onto $\{x + iy \mid x, y \in \mathbb{R}, \ -\pi < y < \pi\}$, the interior of B). Then Cayley's formula reads as

$$\theta = \frac{1}{2i}\log([D_1, D_2, D_I, D_J]),$$