4. Let d be any positive integer. If d is not divisible by any integer of the form m^2, with $m \in \mathbb{N}$ and $m \geq 2$, then we say that d is *square-free*. For example, $d = 1, 2, 3, 5, 6, 7, 10$ are square-free, but $4, 8, 9, 12$ are not square-free. If d is any square-free integer and if $d \geq 2$, then the set of real numbers

$$\mathbb{Z}[\sqrt{d}] = \{a + b\sqrt{d} \in \mathbb{R} \mid a, b \in \mathbb{Z}\}$$

is a commutative a ring. If $z = a + b\sqrt{d} \in \mathbb{Z}[\sqrt{d}]$, we write $\overline{z} = a - b\sqrt{d}$. Note that $z\overline{z} = a^2 - db^2$.

5. Similarly, if $d \geq 1$ is a positive square-free integer, then the set of complex numbers

$$\mathbb{Z}[\sqrt{-d}] = \{a + ib\sqrt{d} \in \mathbb{C} \mid a, b \in \mathbb{Z}\}$$

is a commutative ring. If $z = a + ib\sqrt{d} \in \mathbb{Z}[\sqrt{-d}]$, we write $\overline{z} = a - ib\sqrt{d}$. Note that $z\overline{z} = a^2 + db^2$. The case where $d = 1$ is a famous example that was investigated by Gauss, and $\mathbb{Z}[\sqrt{-1}]$, also denoted $\mathbb{Z}[i]$, is called the ring of *Gaussian integers*.

6. The group of $n \times n$ matrices $\mathrm{M}_n(\mathbb{R})$ is a ring under matrix multiplication. However, it is not a commutative ring.

7. The group $\mathcal{C}(a, b)$ of continuous functions $f\colon (a, b) \to \mathbb{R}$ is a ring under the operation $f \cdot g$ defined such that

$$(f \cdot g)(x) = f(x)g(x)$$

for all $x \in (a, b)$.

Definition 2.17. Given a ring A, for any element $a \in A$, if there is some element $b \in A$ such that $b \neq 0$ and $ab = 0$, then we say that a is a *zero divisor*. A ring A is an *integral domain* (or an *entire ring*) if $0 \neq 1$, A is commutative, and $ab = 0$ implies that $a = 0$ or $b = 0$, for all $a, b \in A$. In other words, an integral domain is a nontrivial commutative ring with no zero divisors besides 0.

Example 2.7.

1. The rings $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$, are integral domains.

2. The ring $\mathbb{R}[X]$ of polynomials in one variable with real coefficients is an integral domain.

3. For any positive integer, $n \in \mathbb{N}$, we have the ring $\mathbb{Z}/n\mathbb{Z}$. Observe that if n is composite, then this ring has zero-divisors. For example, if $n = 4$, then we have

$$2 \cdot 2 \equiv 0 \pmod{4}.$$

The reader should prove that $\mathbb{Z}/n\mathbb{Z}$ is an integral domain iff n is prime (use Proposition 2.17).