$$X - \mu = \begin{pmatrix} 0 & -1 \\ 2 & 0 \\ -2 & 1 \end{pmatrix},$$

and

$$\Sigma = \frac{1}{2} \begin{pmatrix} 0 & 2 & -2 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 2 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}.$$

Remark: The factor $\frac{1}{n-1}$ is irrelevant for our purposes and can be ignored.

Example 23.9. Here is the matrix $X - \mu$ in the case of our bearded mathematicians: since

$$\mu_1 = 1828.4, \quad \mu_2 = 5.6,$$

we get the following centered data set.

Name	year	length
Carl Friedrich Gauss	−51.4	−5.6
Camille Jordan	9.6	6.4
Adrien-Marie Legendre	−76.4	−5.6
Bernhard Riemann	−2.4	9.4
David Hilbert	33.6	−3.6
Henri Poincaré	25.6	−0.6
Emmy Noether	53.6	−5.6
Karl Weierstrass	13.4	−5.6
Eugenio Beltrami	6.6	−3.6
Hermann Schwarz	14.6	14.4

See Figure 23.3.

We can think of the vector C_j as representing the features of X in the direction e_j (the jth canonical basis vector in $\mathbb{R}^d$, namely $e_j = (0, \ldots, 1, \ldots 0)$, with a 1 in the jth position).

If $v \in \mathbb{R}^d$ is a unit vector, we wish to consider the projection of the data points $X_1, \ldots, X_n$ onto the line spanned by v. Recall from Euclidean geometry that if $x \in \mathbb{R}^d$ is any vector and $v \in \mathbb{R}^d$ is a unit vector, the projection of x onto the line spanned by v is

$$\langle x, v \rangle v.$$

Thus, with respect to the basis v, the projection of x has coordinate $\langle x, v \rangle$. If x is represented by a row vector and v by a column vector, then

$$\langle x, v \rangle = xv.$$