In any case, we know from Proposition 51.2 and Proposition 51.3 that the minimum set of f is convex, and closed iff f is closed.

Subdifferentials provide the first criterion for deciding whether a vector $x \in \mathbb{R}^n$ belongs to the minimum set of f. Indeed, the very definition of a subgradient says that $x \in \mathbb{R}^n$ belongs to the minimum set of f iff $0 \in \partial f(x)$. Using Proposition 51.16, we obtain the following result.

Proposition 51.34. *Let f be a proper convex function over $\mathbb{R}^n$. A vector $x \in \mathbb{R}^n$ belongs to the minimum set of f iff*

$$0 \in \partial f(x)$$

iff $f(x)$ is finite and

$$f'(x; y) \geq 0 \quad \textit{for all } y \in \mathbb{R}^n.$$

Of course, if f is differentiable at x, then $\partial f(x) = \{\nabla f_x\}$, and we obtain the well-known condition $\nabla f_x = 0$.

There are many ways of expressing the conditions of Proposition 51.34, and the minimum set of f can even be characterized in terms of the conjugate function f^*. The notion of direction of recession plays a key role.

Definition 51.20. Let $f \colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be any function. A *direction of recession* of f is any non-zero vector $u \in \mathbb{R}^n$ such that for every $x \in \mathrm{dom}(f)$, the function $\lambda \mapsto f(x + \lambda u)$ is nonincreasing (this means that for all $\lambda_1, \lambda_2 \in \mathbb{R}$, if $\lambda_1 < \lambda_2$, then $x + \lambda_1 u \in \mathrm{dom}(f)$, $x + \lambda_2 u \in \mathrm{dom}(f)$, and $f(x + \lambda_2 u) \leq f(x + \lambda_1 u)$).

Example 51.12. Consider the function $f \colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = 2x + y^2$. Since

$$f(x + \lambda u, y + \lambda v) = 2(x + \lambda u) + (y + \lambda v)^2 = 2x + y^2 + 2(u + yv)\lambda + v^2\lambda^2,$$

if $v \neq 0$, we see that the above quadratic function of λ increases for $\lambda \geq -(u + yv)/v^2$. If $v = 0$, then the function $\lambda \mapsto 2x + y^2 + 2u\lambda$ decreases to $-\infty$ when λ goes to $+\infty$ if $u < 0$, so all vectors $(-u, 0)$ with $u > 0$ are directions of recession. See Figure 51.25.

The function $f(x, y) = 2x + x^2 + y^2$ does not have any direction of recession, because

$$f(x + \lambda u, y + \lambda v) = 2x + x^2 + y^2 + 2(u + ux + yv)\lambda + (u^2 + v^2)\lambda^2,$$

and since $(u, v) \neq (0, 0)$, we have $u^2 + v^2 > 0$, so as a function of λ, the above quadratic function increases for $\lambda \geq -(u + ux + yv)/(u^2 + v^2)$. See Figure 51.25.

In fact, the above example is typical. For any symmetric positive definite $n \times n$ matrix A and any vector $b \in \mathbb{R}^n$, the quadratic strictly convex function q given by $q(x) = x^\top Ax + b^\top x$ has no directions of recession. For any $u \in \mathbb{R}^n$, with $u \neq 0$, we have

$$\begin{aligned} q(x + \lambda u) &= (x + \lambda u)^\top A(x + \lambda u) + b^\top(x + \lambda u) \\ &= x^\top Ax + b^\top x + (2x^\top Au + b^\top u)\lambda + (u^\top Au)\lambda^2. \end{aligned}$$