Proof. In this case the inner product on $\mathrm{M}_n(\mathbb{C})$ is the Frobenius inner product $\langle A, B\rangle = \mathrm{tr}(B^*A)$, and the dual norm of the spectral norm is given by

$$\|A\|_2^D = \sup\{|\mathrm{tr}(A^*B)| \mid \|B\|_2 = 1\}.$$

If we factor A using an SVD as $A = V\Sigma U^*$, where U and V are unitary and Σ is a diagonal matrix whose r nonzero entries are the singular values $\sigma_1 > \cdots > \sigma_r > 0$, where r is the rank of A, then

$$|\mathrm{tr}(A^*B)| = |\mathrm{tr}(U\Sigma V^*B)| = |\mathrm{tr}(\Sigma V^*BU)|,$$

so if we pick $B = VU^*$, a unitary matrix such that $\|B\|_2 = 1$, we get

$$|\mathrm{tr}(A^*B)| = \mathrm{tr}(\Sigma) = \sigma_1 + \cdots + \sigma_r,$$

and thus

$$\|A\|_2^D \geq \sigma_1 + \cdots + \sigma_r.$$

Since $\|B\|_2 = 1$ and U and V are unitary, by Proposition 9.10 we have $\|V^*BU\|_2 = \|B\|_2 = 1$. If $Z = V^*BU$, by definition of the operator norm

$$1 = \|Z\|_2 = \sup\{\|Zx\|_2 \mid \|x\|_2 = 1\},$$

so by picking x to be the canonical vector e_j, we see that $\|Z^j\|_2 \leq 1$ where Z^j is the jth column of Z, so $|z_{jj}| \leq 1$, and since

$$|\mathrm{tr}(\Sigma V^*BU)| = |\mathrm{tr}(\Sigma Z)| = \left|\sum_{j=1}^r \sigma_j z_{jj}\right| \leq \sum_{j=1}^r \sigma_j |z_{jj}| \leq \sum_{j=1}^r \sigma_j,$$

and we conclude that

$$|\mathrm{tr}(\Sigma V^*BU)| \leq \sum_{j=1}^r \sigma_j.$$

The above implies that

$$\|A\|_2^D \leq \sigma_1 + \cdots + \sigma_r,$$

and since we also have $\|A\|_2^D \geq \sigma_1 + \cdots + \sigma_r$, we conclude that

$$\|A\|_2^D = \sigma_1 + \cdots + \sigma_r,$$

proving our proposition. $\square$

Definition 14.15. Given any complex matrix $n \times n$ matrix A of rank r, its *nuclear norm* (or *trace norm*) is given by

$$\|A\|_N = \sigma_1 + \cdots + \sigma_r.$$