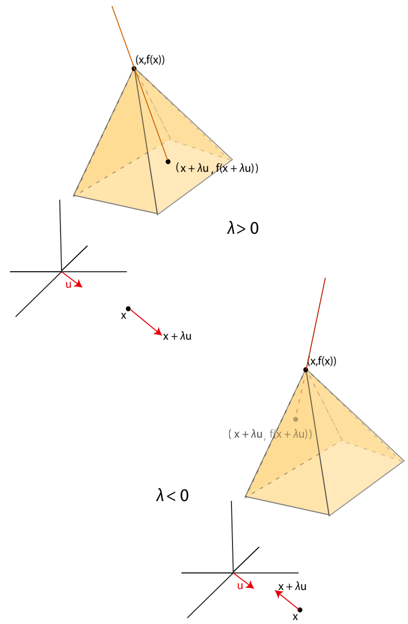

Figure 51.19: Let $f\colon \mathbb{R}^2 \to \mathbb{R}\cup\{-\infty, +\infty\}$ be the function whose graph (in $\mathbb{R}^3$) is the surface of the peach pyramid. The top figure illustrates that $f'(x; u)$ is the slope of the slanted burnt orange line, while the bottom figure depicts the line associated with $\lim_{\lambda\uparrow 0} \frac{f(x+\lambda u)-f(x)}{\lambda}$.

so *the (two-sided) directional derivative* $\mathrm{D}_u f(x)$ *exists iff* $-f'(x; -u) = f'(x; u)$. Also, if f is differentiable at x, then

$$f'(x; u) = \langle \nabla f_x, u\rangle, \quad \text{for all } u \in \mathbb{R}^n,$$

where ∇f_x is the gradient of f at x. Here is the first remarkable result.

Proposition 51.15. *Let $f\colon \mathbb{R}^n \to \mathbb{R}\cup\{-\infty, +\infty\}$ be a convex function. For any $x \in \mathbb{R}^n$, if $f(x)$ is finite, then the function*

$$\lambda \mapsto \frac{f(x+\lambda u) - f(x)}{\lambda}$$

is a nondecreasing function of $\lambda > 0$, so that $f'(x; u)$ exists for any $u \in \mathbb{R}^n$, and

$$f'(x; u) = \inf_{\lambda>0} \frac{f(x+\lambda u) - f(x)}{\lambda}.$$

Furthermore, $f'(x; u)$ is a positively homogeneous convex function of u (which means that $f'(x; \alpha u) = \alpha f'(x; u)$ for all $\alpha \in \mathbb{R}$ with $\alpha > 0$ and all $u \in \mathbb{R}^n$), $f'(x; 0) = 0$, and

$$-f'(x; -u) \leq f'(x; u) \quad \textit{for all } u \in \mathbb{R}^n$$