for all $f \in E^*$ and all $v \in E$. Recall that given a subset $V$ of $E$ (respectively a subset $U$ of $E^*$), the *orthogonal* $V^0$ *of* $V$ is the subspace of $E^*$ defined such that

$$V^0 = \{f \in E^* \mid \langle f, v\rangle = 0, \text{ for every } v \in V\},$$

and that the *orthogonal* $U^0$ *of* $U$ is the subspace of $E$ defined such that

$$U^0 = \{v \in E \mid \langle f, v\rangle = 0, \text{ for every } f \in U\} = \bigcap_{f\in U} \operatorname{Ker} f.$$

Then, by Theorem 11.4 (since $E$ and $E^*$ have the same finite dimension $n+1$), $U = U^{00}$, $V = V^{00}$, and the maps

$$V \mapsto V^0 \quad \text{and} \quad U \mapsto U^0$$

are inverse bijections, where $V$ is a subspace of $E$, and $U$ is a subspace of $E^*$.

These maps set up a *duality* between subspaces of $E$ and subspaces of $E^*$. Furthermore, we know that $U$ has dimension $k$ iff $U^0$ has dimension $n+1-k$, and similarly for $V$ and $V^0$.

Since a linear system $P = \mathbf{P}(U)$ of hyperplanes in $\mathcal{H}(E)$ corresponds to a subspace $U$ of $E^*$, and since

$$U^0 = \bigcap_{f\in U} \operatorname{Ker} f$$

is the intersection of all the hyperplanes defined by nonnull linear forms in $U$, we can view a linear system $P = \mathbf{P}(U) = \mathbf{P}(U^{00})$ in $\mathcal{H}(E)$ as the family of hyperplanes in $\mathbf{P}(E)$ containing $\mathbf{P}(U^0)$.

In view of the identification of $\mathbf{P}(E^*)$ with the set $\mathcal{H}(E)$ of hyperplanes in $\mathbf{P}(E)$, by passing to projective spaces, the above bijection between the set of subspaces of $E$ and the set of subspaces of $E^*$ yields a bijection between the set of projective subspaces of $\mathbf{P}(E)$ and the set of linear systems in $\mathcal{H}(E)$ (or equivalently, the set of projective subspaces of $\mathbf{P}(E^*)$) called *duality*. Recall that a point of $\mathcal{H}(E)$ is a hyperplane in $\mathbf{P}(E)$.

More specifically, assuming that $E$ has dimension $n+1$, so that $\mathbf{P}(E)$ has dimension $n$, if $Q = \mathbf{P}(V)$ is any projective subspace of $\mathbf{P}(E)$ (where $V$ is any subspace of $E$) and if $P = \mathbf{P}(U)$ is any linear system in $\mathcal{H}(E)$ (where $U$ is any subspace of $E^*$), we get a subspace $Q^0$ of $\mathcal{H}(E)$ defined by

$$Q^0 = \{\mathbf{P}(H) \mid Q \subseteq \mathbf{P}(H),\ \mathbf{P}(H) \text{ a hyperplane in } \mathcal{H}(E)\},$$

and a subspace $P^0$ of $\mathbf{P}(E)$ defined by

$$P^0 = \bigcap\{\mathbf{P}(H) \mid \mathbf{P}(H) \in P,\ \mathbf{P}(H) \text{ a hyperplane in } \mathcal{H}(E)\}.$$

We have $P = P^{00}$ and $Q = Q^{00}$. Since $Q^0$ is determined by $\mathbf{P}(V^0)$, if $Q = \mathbf{P}(V)$ has dimension $k$ (i.e., if $V$ has dimension $k+1$), then $Q^0$ has dimension $n-k-1$ (since $V$ has dimension $k+1$ and $\dim(E) = n+1$, then $V^0$ has dimension $n+1-(k+1) = n-k$). Thus,

$$\dim(Q) + \dim(Q^0) = n-1,$$