(1)
$$\sup_{k\geq 0}\sup_{x\in B}\left\|A_k^{-1}(x)\right\|_{\mathcal{L}(E';E)} \leq M,$$

(2) $\beta < 1$ *and*
$$\sup_{k\geq 0}\sup_{x,x'\in B}\|J''(x) - A_k(x')\|_{\mathcal{L}(E;E')} \leq \frac{\beta}{M}$$

(3)
$$\|J'(x_0)\| \leq \frac{r}{M}(1-\beta).$$

Then the sequence (x_k) *defined by*
$$x_{k+1} = x_k - A_k^{-1}(x_\ell)(J'(x_k)), \quad 0 \leq \ell \leq k$$
is entirely contained within B *and converges to a zero* a *of* J'*, which is the only zero of* J' *in* B*. Furthermore, the convergence is geometric, which means that*
$$\|x_k - a\| \leq \frac{\|x_1 - x_0\|}{1-\beta}\beta^k.$$

In the next theorem, which follows immediately from Theorem 41.2, we assume that the $A_k(x)$ are isomorphisms in $\mathcal{L}(E, E')$ that are independent of $x \in \Omega$.

Theorem 41.5. *Let* E *be a Banach space and let* $J\colon \Omega \to \mathbb{R}$ *be twice differentiable on the open subset* $\Omega \subseteq E$*. If* $a \in \Omega$ *is a point such that* $J'(a) = 0$*, if* $J''(a)$ *is a linear isomorphism, and if there is some* λ *with* $0 < \lambda < 1/2$ *such that*
$$\sup_{k\geq 0}\|A_k - J''(a)\|_{\mathcal{L}(E;E')} \leq \frac{\lambda}{\|(J''(a))^{-1}\|_{\mathcal{L}(E';E)}},$$
then there is a closed ball B *of center* a *such that for every* $x_0 \in B$*, the sequence* (x_k) *defined by*
$$x_{k+1} = x_k - A_k^{-1}(J'(x_k)), \quad k \geq 0,$$
is entirely contained within B *and converges to* a*, which is the only zero of* J' *in* B*. Furthermore, the convergence is geometric, which means that*
$$\|x_k - a\| \leq \beta^k \|x_0 - a\|,$$
for some $\beta < 1$*.*

When $E = \mathbb{R}^n$, the Newton method given by Theorem 41.4 yields an iteration step of the form
$$x_{k+1} = x_k - A_k^{-1}(x_\ell)\nabla J(x_k), \quad 0 \leq \ell \leq k,$$