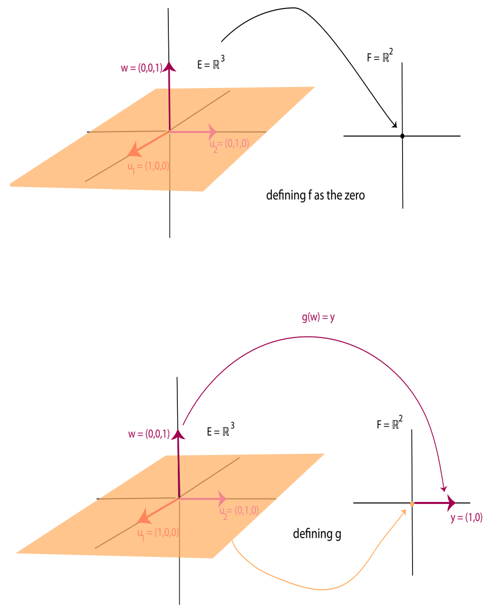

Figure 3.12: Let $E = \mathbb{R}^3$ and $F = \mathbb{R}^2$. The vectors $u_1 = (1, 0, 0)$, $u_2 = (0, 1, 0)$ do not generate $\mathbb{R}^3$ since both the zero map and the map g, where $g(0, 0, 1) = (1, 0)$, send the peach xy-plane to the origin.

map $g\colon E \to F$ such that $g(w) = y$, and $g(e_j) = 0$ for all $j \in (I_0 \cup J) - \{j_0\}$. By definition of the basis $(e_j)_{j \in I_0 \cup J}$ of E, we have $g(u_i) = 0$ for all $i \in I$, and since $f \neq g$, this contradicts the fact that there is at most one such map. See Figure 3.12.

(2) If the family $(u_i)_{i\in I}$ is linearly independent, then by Theorem 3.11, $(u_i)_{i\in I}$ can be extended to a basis of E, and the conclusion follows by Proposition 3.18. Conversely, assume that $(u_i)_{i\in I}$ is linearly dependent. Then there is some family $(\lambda_i)_{i\in I}$ of scalars (not all zero) such that

$$\sum_{i\in I} \lambda_i u_i = 0.$$

By the assumption, for any nonzero vector $y \in F$, for every $i \in I$, there is some linear map $f_i\colon E \to F$, such that $f_i(u_i) = y$, and $f_i(u_j) = 0$, for $j \in I - \{i\}$. Then we would get

$$0 = f_i\Big(\sum_{i\in I} \lambda_i u_i\Big) = \sum_{i\in I} \lambda_i f_i(u_i) = \lambda_i y,$$

and since $y \neq 0$, this implies $\lambda_i = 0$ for every $i \in I$. Thus, $(u_i)_{i\in I}$ is linearly independent. $\square$

Given vector spaces E, F, and G, and linear maps $f\colon E \to F$ and $g\colon F \to G$, it is easily verified that the composition $g \circ f\colon E \to G$ of f and g is a linear map.