We also have the following proposition that gives a sufficient condition implying that η and b can be found in terms of an optimal solution (λ, μ) of the dual.

Proposition 54.4. *If $(w, b, \eta, \epsilon, \xi)$ is an optimal solution of Problem $(\mathrm{SVM}_{s2'})$ with $w \neq 0$ and $\eta > 0$, if*

$$\max\{2p_f/(p+q), 2q_f/(p+q)\} < \nu < \min\{2p/(p+q), 2q/(p+q)\},$$

then η and b can always be determined from an optimal solution (λ, μ) of the dual in terms of a single support vector.

Proof. By Theorem 54.3 some u_{i_0} and some v_{j_0} is a support vector. As we already explained, Problem $(\mathrm{SVM}_{s2'})$ satisfies the conditions for having a zero duality gap. Therefore, for optimal solutions we have

$$L(w, \epsilon, \xi, b, \eta, \lambda, \mu, \alpha, \beta) = G(\lambda, \mu, \alpha, \beta),$$

which means that

$$\frac{1}{2}w^\top w - \nu\eta + \frac{1}{p+q}\left(\sum_{i=1}^{p}\epsilon_i + \sum_{j=1}^{q}\xi_j\right) = -\frac{1}{2}\begin{pmatrix}\lambda^\top & \mu^\top\end{pmatrix} X^\top X \begin{pmatrix}\lambda \\ \mu\end{pmatrix},$$

and since

$$w = -X\begin{pmatrix}\lambda \\ \mu\end{pmatrix},$$

we get

$$\frac{1}{p+q}\left(\sum_{i=1}^{p}\epsilon_i + \sum_{j=1}^{q}\xi_j\right) = \nu\eta - \begin{pmatrix}\lambda^\top & \mu^\top\end{pmatrix} X^\top X \begin{pmatrix}\lambda \\ \mu\end{pmatrix}. \tag{$*$}$$

Let $K_\lambda = \{i \in \{1, \ldots, p\} \mid \lambda_i = K_s\}$ and $K_\mu = \{j \in \{1, \ldots, q\} \mid \mu_j = K_s\}$. By definition, $p_f = |K_\lambda|$ and $q_f = |K_\mu|$ (here we assuming that $K_s = 1/(p+q)$). By complementary slackness the following equations are active:

$$\begin{aligned} w^\top u_i - b &= \eta - \epsilon_i & \quad & i \in K_\lambda \\ -w^\top v_j + b &= \eta - \xi_j & & j \in K_\mu. \end{aligned}$$

But $(*)$ can be written as

$$\frac{1}{p+q}\left(\sum_{i\in K_\lambda}\epsilon_i + \sum_{j\in K_\mu}\xi_j\right) = \nu\eta - \begin{pmatrix}\lambda^\top & \mu^\top\end{pmatrix} X^\top X \begin{pmatrix}\lambda \\ \mu\end{pmatrix}, \tag{$**$}$$

and since

$$\begin{aligned} \epsilon_i &= \eta - w^\top u_i + b & \quad & i \in K_\lambda \\ \xi_j &= \eta + w^\top v_j - b & & j \in K_\mu, \end{aligned}$$