17.4 Self-Adjoint, Skew-Self-Adjoint, and Orthogonal Linear Maps

We begin with self-adjoint maps.

Theorem 17.14. *Given a Euclidean space E of dimension n, for every self-adjoint linear map $f\colon E \to E$, there is an orthonormal basis $(e_1, \ldots, e_n)$ of eigenvectors of f such that the matrix of f w.r.t. this basis is a diagonal matrix*

$$\begin{pmatrix} \lambda_1 & & \ldots & \\ & \lambda_2 & \ldots & \\ \vdots & \vdots & \ddots & \vdots \\ & & \ldots & \lambda_n \end{pmatrix},$$

where $\lambda_i \in \mathbb{R}$.

Proof. We already proved this; see Theorem 17.8. However, it is instructive to give a more direct method not involving the complexification of $\langle -, - \rangle$ and Proposition 17.5.

Since $\mathbb{C}$ is algebraically closed, $f_{\mathbb{C}}$ has some eigenvalue $\lambda + i\mu$, and let $u + iv$ be some eigenvector of $f_{\mathbb{C}}$ for $\lambda + i\mu$, where $\lambda, \mu \in \mathbb{R}$ and $u, v \in E$. We saw in the proof of Proposition 17.10 that

$$f(u) = \lambda u - \mu v \quad \text{and} \quad f(v) = \mu u + \lambda v.$$

Since $f = f^*$,

$$\langle f(u), v \rangle = \langle u, f(v) \rangle$$

for all $u, v \in E$. Applying this to

$$f(u) = \lambda u - \mu v \quad \text{and} \quad f(v) = \mu u + \lambda v,$$

we get

$$\langle f(u), v \rangle = \langle \lambda u - \mu v, v \rangle = \lambda \langle u, v \rangle - \mu \langle v, v \rangle$$

and

$$\langle u, f(v) \rangle = \langle u, \mu u + \lambda v \rangle = \mu \langle u, u \rangle + \lambda \langle u, v \rangle,$$

and thus we get

$$\lambda \langle u, v \rangle - \mu \langle v, v \rangle = \mu \langle u, u \rangle + \lambda \langle u, v \rangle,$$

that is,

$$\mu(\langle u, u \rangle + \langle v, v \rangle) = 0,$$

which implies $\mu = 0$, since either $u \neq 0$ or $v \neq 0$. Therefore, λ is a real eigenvalue of f.

Now going back to the proof of Theorem 17.12, only the case where $\mu = 0$ applies, and the induction shows that all the blocks are one-dimensional. $\square$