then by expanding according to the first row, we have

$$D(A) = a_{1\,1}\begin{vmatrix} a_{2\,2} & a_{2\,3} \\ a_{3\,2} & a_{3\,3} \end{vmatrix} - a_{1\,2}\begin{vmatrix} a_{2\,1} & a_{2\,3} \\ a_{3\,1} & a_{3\,3} \end{vmatrix} + a_{1\,3}\begin{vmatrix} a_{2\,1} & a_{2\,2} \\ a_{3\,1} & a_{3\,2} \end{vmatrix},$$

that is,

$$D(A) = a_{1\,1}(a_{2\,2}a_{3\,3} - a_{3\,2}a_{2\,3}) - a_{1\,2}(a_{2\,1}a_{3\,3} - a_{3\,1}a_{2\,3}) + a_{1\,3}(a_{2\,1}a_{3\,2} - a_{3\,1}a_{2\,2}),$$

which gives the explicit formula

$$D(A) = a_{1\,1}a_{2\,2}a_{3\,3} + a_{2\,1}a_{3\,2}a_{1\,3} + a_{3\,1}a_{1\,2}a_{2\,3} - a_{1\,1}a_{3\,2}a_{2\,3} - a_{2\,1}a_{1\,2}a_{3\,3} - a_{3\,1}a_{2\,2}a_{1\,3}.$$

We now show that each $D \in \mathcal{D}_n$ is a determinant (map).

Lemma 7.5. *For every $n \geq 1$, for every $D \in \mathcal{D}_n$ as defined in Definition 7.6, D is an alternating multilinear map such that $D(I_n) = 1$.*

Proof. By induction on n, it is obvious that $D(I_n) = 1$. Let us now prove that D is multilinear. Let us show that D is linear in each column. Consider any Column k. Since

$$D(A) = (-1)^{i+1}a_{i\,1}D(A_{i\,1}) + \cdots + (-1)^{i+j}a_{i\,j}D(A_{i\,j}) + \cdots + (-1)^{i+n}a_{i\,n}D(A_{i\,n}),$$

if $j \neq k$, then by induction, $D(A_{i\,j})$ is linear in Column k, and $a_{i\,j}$ does not belong to Column k, so $(-1)^{i+j}a_{i\,j}D(A_{i\,j})$ is linear in Column k. If $j = k$, then $D(A_{i\,j})$ does not depend on Column $k = j$, since $A_{i\,j}$ is obtained from A by deleting Row i and Column $j = k$, and $a_{i\,j}$ belongs to Column $j = k$. Thus, $(-1)^{i+j}a_{i\,j}D(A_{i\,j})$ is linear in Column k. Consequently, in all cases, $(-1)^{i+j}a_{i\,j}D(A_{i\,j})$ is linear in Column k, and thus, $D(A)$ is linear in Column k.

Let us now prove that D is alternating. Assume that two adjacent columns of A are equal, say $A^k = A^{k+1}$. Assume that $j \neq k$ and $j \neq k+1$. Then the matrix $A_{i\,j}$ has two identical adjacent columns, and by the induction hypothesis, $D(A_{i\,j}) = 0$. The remaining terms of $D(A)$ are

$$(-1)^{i+k}a_{i\,k}D(A_{i\,k}) + (-1)^{i+k+1}a_{i\,k+1}D(A_{i\,k+1}).$$

However, the two matrices $A_{i\,k}$ and $A_{i\,k+1}$ are equal, since we are assuming that Columns k and $k+1$ of A are identical and $A_{i\,k}$ is obtained from A by deleting Row i and Column k while $A_{i\,k+1}$ is obtained from A by deleting Row i and Column $k+1$. Similarly, $a_{i\,k} = a_{i\,k+1}$, since Columns k and $k+1$ of A are equal. But then,

$$(-1)^{i+k}a_{i\,k}D(A_{i\,k}) + (-1)^{i+k+1}a_{i\,k+1}D(A_{i\,k+1}) = (-1)^{i+k}a_{i\,k}D(A_{i\,k}) - (-1)^{i+k}a_{i\,k}D(A_{i\,k}) = 0.$$

This shows that D is alternating and completes the proof. ∎

Lemma 7.5 shows the existence of determinants. We now prove their uniqueness.