assume that we have

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = A^\top \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}.$$

Then,

$$f(v_1, \ldots, v_n) = \Big(\sum_{\pi \in \mathfrak{S}_n} \epsilon(\pi) a_{\pi(1)\,1} \cdots a_{\pi(n)\,n}\Big) f(u_1, \ldots, u_n),$$

where the sum ranges over all permutations π *on* $\{1, \ldots, n\}$.

Proof. Expanding $f(v_1, \ldots, v_n)$ by multilinearity, we get a sum of terms of the form

$$a_{\pi(1)\,1} \cdots a_{\pi(n)\,n} f(u_{\pi(1)}, \ldots, u_{\pi(n)}),$$

for all possible functions $\pi\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$. However, because f is alternating, only the terms for which π is a permutation are nonzero. By Proposition 7.1, every permutation π is a product of transpositions, and by Proposition 7.2, the parity $\epsilon(\pi)$ of the number of transpositions only depends on π. Then applying Proposition 7.3 (3) to each transposition in π, we get

$$a_{\pi(1)\,1} \cdots a_{\pi(n)\,n} f(u_{\pi(1)}, \ldots, u_{\pi(n)}) = \epsilon(\pi) a_{\pi(1)\,1} \cdots a_{\pi(n)\,n} f(u_1, \ldots, u_n).$$

Thus, we get the expression of the lemma. $\square$

For the case of $n = 2$, the proof details of Lemma 7.4 become

$$\begin{aligned}
f(v_1, v_2) &= f(a_{11}u_1 + a_{21}u_2, a_{12}u_1 + a_{22}u_2) \\
&= f(a_{11}u_1 + a_{21}u_2, a_{12}u_1) + f(a_{11}u_1 + a_{21}u_2, a_{22}u_2) \\
&= f(a_{11}u_1, a_{12}u_1) + f(a_{21}u_2, a_{12}u_1) + f(a_{11}u_a, a_{22}u_2) + f(a_{21}u_2, a_{22}u_2) \\
&= a_{11}a_{12}f(u_1, u_1) + a_{21}a_{12}f(u_2, u_1) + a_{11}a_{22}f(u_1, u_2) + a_{21}a_{22}f(u_2, u_2) \\
&= a_{21}a_{12}f(u_2, u_1)a_{11}a_{22}f(u_1, u_2) \\
&= (a_{11}a_{22} - a_{12}a_{22})\, f(u_1, u_2).
\end{aligned}$$

Hopefully the reader will recognize the quantity $a_{11}a_{22} - a_{12}a_{22}$. It is the determinant of the 2×2 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

This is no accident. The quantity

$$\det(A) = \sum_{\pi \in \mathfrak{S}_n} \epsilon(\pi) a_{\pi(1)\,1} \cdots a_{\pi(n)\,n}$$