In terms of the basis (x_1, x_2, x_3), the map $f(x, y, z) = (x + y + z, y + z, z)$ has the Jordan block matrix representation $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ since

$$\begin{aligned} f(x_1) &= f(1,0,0) = (1,0,0) = x_1 \\ f(x_2) &= f(1,1,0) = (2,1,0) = x_1 + x_2 \\ f(x_3) &= f(1,0,1) = (2,1,1) = x_2 + x_3. \end{aligned}$$

Combining Theorem 36.15 and Proposition 36.16, we obtain a strong version of the Jordan form.

Theorem 36.17. *(Jordan Canonical Form) Let E be finite-dimensional K-vector space. The following properties are equivalent:*

(1) The eigenvalues of f all belong to K.

(2) There is a basis of E in which the matrix of f is upper (or lower) triangular.

(3) There exist a basis of E in which the matrix A of f is Jordan matrix. Furthermore, the number of Jordan blocks $J_r(\lambda)$ appearing in A, for fixed r and λ, is uniquely determined by f.

Proof. The implication (1) $\Longrightarrow$ (3) follows from Theorem 36.15 and Proposition 36.16. The implications (3) $\Longrightarrow$ (2) and (2) $\Longrightarrow$ (1) are trivial. $\square$

Compared to Theorem 31.17, the new ingredient is the uniqueness assertion in (3), which is not so easy to prove.

Observe that the minimal polynomial of f is the least common multiple of the polynomials $(X - \lambda)^r$ associated with the Jordan blocks $J_r(\lambda)$ appearing in A, and the characteristic polynomial of A is the product of these polynomials.

We now return to the problem of computing effectively the similarity invariants of a matrix M. By Proposition 36.11, this is equivalent to computing the invariant factors of $XI - M$. In principle, this can be done using Proposition 35.35. A procedure to do this effectively for the ring $A = K[X]$ is to convert $XI - M$ to its Smith normal form. This will also yield the rational canonical form for M.

36.5 The Smith Normal Form

The Smith normal form is the special case of Proposition 35.35 applied to the PID $K[X]$ where K is a field, but it also says that the matrices P and Q are products of elementary matrices. It turns out that such a result holds for any Euclidean ring, and the proof is basically the same.