29.4 Adjoint of a Linear Map

Let E_1 and E_2 be two K-vector spaces, and let $\varphi_1\colon E_1 \times E_1 \to K$ be a sesquilinear form on E_1 and $\varphi_2\colon E_2 \times E_2 \to K$ be a sesquilinear form on E_2. It is also possible to deal with the more general situation where we have four vector spaces E_1, F_1, E_2, F_2 and two sesquilinear forms $\varphi_1\colon E_1 \times F_1 \to K$ and $\varphi_2\colon E_2 \times F_2 \to K$, but we will leave this generalization as an exercise. We also assume that l_{φ_1} and r_{φ_1} are bijective, which implies that that φ_1 is nondegenerate. This is automatic if the space E_1 is finite dimensional and φ_1 is nondegenerate.

Given any linear map $f\colon E_1 \to E_2$, for any fixed $u \in E_2$, we can consider the linear form in E_1^* given by
$$x \mapsto \varphi_2(f(x), u), \quad x \in E_1.$$
Since $r_{\varphi_1}\colon \overline{E_1} \to E_1^*$ is bijective, there is a unique $y \in E_1$ (because the vector spaces E_1 and $\overline{E_1}$ only differ by scalar multiplication), so that
$$\varphi_2(f(x), u) = \varphi_1(x, y), \quad \text{for all } x \in E_1.$$
If we denote this unique $y \in E_1$ by $f^{*l}(u)$, then we have
$$\varphi_2(f(x), u) = \varphi_1(x, f^{*l}(u)), \quad \text{for all } x \in E_1, \text{ and all } u \in E_2.$$
Thus, we get a function $f^{*l}\colon E_2 \to E_1$. We claim that this function is a linear map. For any $v_1, v_2 \in E_2$, we have
$$\begin{aligned}
\varphi_2(f(x), v_1 + v_2) &= \varphi_2(f(x), v_1) + \varphi_2(f(x), v_2) \\
&= \varphi_1(x, f^{*l}(v_1)) + \varphi_1(x, f^{*l}(v_2)) \\
&= \varphi_1(x, f^{*l}(v_1) + f^{*l}(v_2)) \\
&= \varphi_1(x, f^{*l}(v_1 + v_2)),
\end{aligned}$$
for all $x \in E_1$. Since r_{φ_1} is injective, we conclude that
$$f^{*l}(v_1 + v_2) = f^{*l}(v_1) + f^{*l}(v_2).$$
For any $\lambda \in K$, we have
$$\begin{aligned}
\varphi_2(f(x), \lambda v) &= \overline{\lambda}\varphi_2(f(x), v) \\
&= \overline{\lambda}\varphi_1(x, f^{*l}(v)) \\
&= \varphi_1(x, \lambda f^{*l}(v)) \\
&= \varphi_1(x, f^{*l}(\lambda v)),
\end{aligned}$$
for all $x \in E_1$. Since r_{φ_1} is injective, we conclude that
$$f^{*l}(\lambda v) = \lambda f^{*l}(v).$$