(2) *The space E is the union of a countable family of compact subsets of E.*

(3) *The space E is separable.*

Proof. We show (1) implies (2), (2) implies (3), and (3) implies (1). Obviously, (1) implies (2) since the $\overline{U_n}$ are compact.

If (2) holds, then $E = \bigcup_{n\geq 0} K_n$, for some compact subsets K_n. By Proposition 37.38, each compact subset K_n is separable, so let S_n be a countable dense subset of K_n, Then $S = \bigcup_{n\geq 0} S_n$ is a countable dense subset of E, since

$$E = \bigcup_{n\geq 0} K_n \subseteq \bigcup_{n\geq 0} \overline{S_n} \subseteq \overline{S} \subseteq E.$$

Consequently (3) holds.

If (3) holds, let $S = \{s_n\}$ be a countable dense subset of E. By Proposition 37.37, the space E has a countable basis $\mathcal{B}$ of open sets O_n. Since E is locally compact, for every $x \in E$, there is some compact neighborhood W_x containing x, and by Proposition 37.8, there some index $n(x)$ such that $x \in O_{n(x)} \subseteq W_x$. Since W_x is a compact neighborhood, we deduce that $\overline{O_{n(x)}}$ is compact. Consequently, there is a subfamily of $\mathcal{B}$ consisting of open subsets O_i such that $\overline{O_i}$ is compact, which is a countable basis for the topology of E, so we may assume that we restrict our attention to this basis. We define the sequence $(U_n)_{n\geq 1}$ of open subsets of E by induction as follows: Set $U_1 = O_1$, and let

$$U_{n+1} = O_{n+1} \cup V_r(\overline{U_n}),$$

where $r > 0$ is chosen so that $\overline{V_r(\overline{U_n})}$ is compact, which is possible by Proposition 37.35. We immediately check that the U_n satisfy (1) of Proposition 37.40. □

It can also be shown that if E is a locally compact space that has a countable basis, then E_ω also has a countable basis (and in fact, is metrizable).

We also have the following property.

Proposition 37.41. *Given a second-countable topological space E, every open cover $(U_i)_{i\in I}$, of E contains some countable subcover.*

Proof. Let $(O_n)_{n\geq 0}$ be a countable basis for the topology. Then all sets O_n contained in some U_i can be arranged into a countable subsequence, $(\Omega_m)_{m\geq 0}$, of $(O_n)_{n\geq 0}$ and for every Ω_m, there is some U_{i_m} such that $\Omega_m \subseteq U_{i_m}$. Furthermore, every U_i is some union of sets Ω_j, and thus, every $a \in E$ belongs to some Ω_j, which shows that $(\Omega_m)_{m\geq 0}$ is a countable open subcover of $(U_i)_{i\in I}$. □

As an immediate corollary of Proposition 37.41, a locally connected second-countable space has countably many connected components.