Can we define a multiplication $\mathrm{Sym}^m(E;K) \times \mathrm{Sym}^n(E;K) \longrightarrow \mathrm{Sym}^{m+n}(E;K)$ directly on symmetric multilinear forms, so that the following diagram commutes?

$$\begin{tikzcd} \mathrm{S}^m(E^*) \times \mathrm{S}^n(E^*) \arrow[r, "\odot"] \arrow[d, "\mu_m \times \mu_n"] & \mathrm{S}^{m+n}(E^*) \arrow[d, "\mu_{m+n}"] \\ \mathrm{Sym}^m(E;K) \times \mathrm{Sym}^n(E;K) \arrow[r, "\cdot"] & \mathrm{Sym}^{m+n}(E;K) \end{tikzcd}$$

The answer is *yes*! The solution is to define this multiplication such that for $f \in \mathrm{Sym}^m(E;K)$ and $g \in \mathrm{Sym}^n(E;K)$,

$$(f \cdot g)(u_1, \ldots, u_{m+n}) = \sum_{\sigma \in \mathrm{shuffle}(m,n)} f(u_{\sigma(1)}, \ldots, u_{\sigma(m)}) g(u_{\sigma(m+1)}, \ldots, u_{\sigma(m+n)}), \qquad (*)$$

where shuffle(m,n) consists of all (m,n)-"shuffles;" that is, permutations σ of $\{1, \ldots m+n\}$ such that $\sigma(1) < \cdots < \sigma(m)$ and $\sigma(m+1) < \cdots < \sigma(m+n)$. Observe that a (m,n)-shuffle is completely determined by the sequence $\sigma(1) < \cdots < \sigma(m)$.

For example, suppose $m = 2$ and $n = 1$. Given $v_1^*, v_2^*, v_3^* \in E^*$, the multiplication structure on $\mathrm{S}(E^*)$ implies that $(v_1^* \odot v_2^*) \cdot v_3^* = v_1^* \odot v_2^* \odot v_3^* \in \mathrm{S}^3(E^*)$. Furthermore, for $u_1, u_2, u_3, \in E$,

$$\begin{aligned} \mu_3(v_1^* \odot v_2^* \odot v_3^*)(u_1, u_2, u_3) &= \sum_{\sigma \in \mathfrak{S}_3} v_{\sigma(1)}^*(u_1) v_{\sigma(2)}^*(u_2) v_{\sigma(3)}^*(u_3) \\ &= v_1^*(u_1)v_2^*(u_2)v_3^*(u_3) + v_1^*(u_1)v_3^*(u_2)v_2^*(u_3) \\ &\quad + v_2^*(u_1)v_1^*(u_2)v_3^*(u_3) + v_2^*(u_1)v_3^*(u_2)v_1^*(u_3) \\ &\quad + v_3^*(u_1)v_1^*(u_2)v_2^*(u_3) + v_3^*(u_1)v_2^*(u_2)v_1^*(u_3). \end{aligned}$$

Now the $(2,1)$- shuffles of $\{1,2,3\}$ are the following three permutations, namely

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

If $f \cong \mu_2(v_1^* \odot v_2^*)$ and $g \cong \mu_1(v_3^*)$, then $(*)$ implies that

$$\begin{aligned} (f \cdot g)(u_1, u_2, u_3) &= \sum_{\sigma \in \mathrm{shuffle}(2,1)} f(u_{\sigma(1)}, u_{\sigma_(2)}) g(u_{\sigma(3)}) \\ &= f(u_1, u_2)g(u_3) + f(u_1, u_3)g(u_2) + f(u_2, u_3)g(u_1) \\ &= \mu_2(v_1^* \odot v_2^*)(u_1, u_2)\mu_1(v_3^*)(u_3) + \mu_2(v_1^* \odot v_2^*)(u_1, u_3)\mu_1(v_3^*)(u_2) \\ &\quad + \mu_2(v_1^* \odot v_2^*)(u_2, u_3)\mu_1(v_3^*)(u_1) \\ &= (v_1^*(u_1)v_2^*(u_2) + v_2^*(u_1)v_1^*(u_2))v_3^*(u_3) \\ &\quad + (v_1^*(u_1)v_2^*(u_3) + v_2^*(u_1)v_1^*(u_3))v_3^*(u_2) \\ &\quad + (v_1^*(u_2)v_2^*(u_3) + v_2^*(u_2)v_1^*(u_3))v_3^*(u_1) \\ &= \mu_3(v_1^* \odot v_2^* \odot v_3^*)(u_1, u_2, u_3). \end{aligned}$$