If u and v are nonzero vectors then the Cauchy–Schwarz inequality implies that

$$-1 \leq \frac{u \cdot v}{\|u\| \, \|v\|} \leq +1.$$

Then there is a unique $\theta \in [0, \pi]$ such that

$$\cos \theta = \frac{u \cdot v}{\|u\| \, \|v\|}.$$

We have $u = v$ iff $\theta = 0$ and $u = -v$ iff $\theta = \pi$. For $0 < \theta < \pi$, the vectors u and v are linearly independent and there is an orientation of the plane spanned by u and v such that θ is the angle between u and v. See Problem 12.8 for the precise notion of orientation. If u is a unit vector (which means that $\|u\| = 1$), then the vector

$$(\|v\| \cos \theta) u = (u \cdot v) u = (v \cdot u) u$$

is called the *orthogonal projection* of v onto the space spanned by u.

Remark: One might wonder if every norm on a vector space is induced by some Euclidean inner product. In general this is false, but remarkably, there is a simple necessary and sufficient condition, which is that the norm must satisfy the *parallelogram law*:

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2).$$

See Figure 12.1.

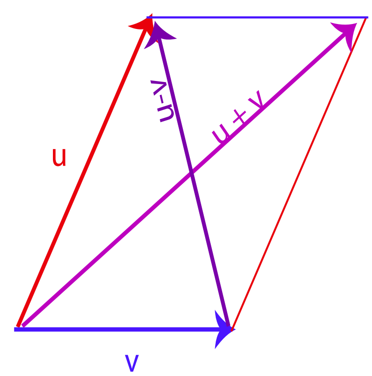

Figure 12.1: The parallelogram law states that the sum of the lengths of the diagonals of the parallelogram determined by vectors u and v equals the sum of all the sides.

If $\langle -, - \rangle$ is an inner product, then we have

$$\begin{aligned} \|u + v\|^2 &= \|u\|^2 + \|v\|^2 + 2\langle u, v \rangle \\ \|u - v\|^2 &= \|u\|^2 + \|v\|^2 - 2\langle u, v \rangle, \end{aligned}$$