or equivalently by

$$\|f\| = \inf\{\lambda \in \mathbb{R} \mid \|f(x)\| \leq \lambda \|x\|, \text{ for all } x \in E\}.$$

Here because E may be infinite-dimensional, sup can't be replaced by max and inf can't be replaced by min. It is not hard to show that the map $f \mapsto \|f\|$ is a norm on $\mathcal{L}(E; F)$ satisfying the property

$$\|f(x)\| \leq \|f\| \|x\|$$

for all $x \in E$, and that if $f \in \mathcal{L}(E; F)$ and $g \in \mathcal{L}(F; G)$, then

$$\|g \circ f\| \leq \|g\| \|f\|.$$

Operator norms play an important role in functional analysis, especially when the spaces E and F are *complete*.

9.4 Inequalities Involving Subordinate Norms

In this section we discuss two technical inequalities which will be needed for certain proofs in the last three sections of this chapter. First we prove a proposition which will be needed when we deal with the condition number of a matrix.

Proposition 9.11. *Let $\| \ \|$ be any matrix norm, and let $B \in \mathrm{M}_n(\mathbb{C})$ such that $\|B\| < 1$.*

(1) If $\| \ \|$ is a subordinate matrix norm, then the matrix $I + B$ is invertible and

$$\left\|(I + B)^{-1}\right\| \leq \frac{1}{1 - \|B\|}.$$

(2) If a matrix of the form $I + B$ is singular, then $\|B\| \geq 1$ for every matrix norm (not necessarily subordinate).

Proof. (1) Observe that $(I + B)u = 0$ implies $Bu = -u$, so

$$\|u\| = \|Bu\|.$$

Recall that

$$\|Bu\| \leq \|B\| \|u\|$$

for every subordinate norm. Since $\|B\| < 1$, if $u \neq 0$, then

$$\|Bu\| < \|u\|,$$

which contradicts $\|u\| = \|Bu\|$. Therefore, we must have $u = 0$, which proves that $I + B$ is injective, and thus bijective, i.e., invertible. Then we have

$$(I + B)^{-1} + B(I + B)^{-1} = (I + B)(I + B)^{-1} = I,$$