procedure to calculate $Q = (Q^1 Q^2 Q^3)$. By definition

$$A^1 = Q'^1 = Q^1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

and since $A^2 = \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}$, we discover that

$$Q'^2 = A^2 - (A^2 \cdot Q^1)Q^1 = \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}.$$

Hence, $Q^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. Finally,

$$Q'^3 = A_3 - (A^3 \cdot Q^1)Q^1 - (A^3 \cdot Q^2)Q^2 = \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix},$$

which implies that $Q^3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. According to Proposition 12.16, in order to determine R we need to calculate

$$\begin{aligned} r_{11} &= \left\| Q'^1 \right\| = 1 & r_{12} &= A^2 \cdot Q^1 = 1 & r_{13} &= A^3 \cdot Q^1 = 1 \\ & & r_{22} &= \left\| Q'^2 \right\| = 4 & r_{23} &= A_3 \cdot Q^2 = 1 \\ & & & & r_{33} &= \left\| Q'^3 \right\| = 5. \end{aligned}$$

In summary, we have found that the QR-decomposition of $A = \begin{pmatrix} 0 & 0 & 5 \\ 0 & 4 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ is

$$Q = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad R = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 5 \end{pmatrix}.$$

Example 12.14. Another example of QR-decomposition is

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 1/\sqrt{2} & \sqrt{2} \\ 0 & 1/\sqrt{2} & \sqrt{2} \\ 0 & 0 & 1 \end{pmatrix}.$$