The first main result is Theorem 31.6 which states that if $f\colon E \to E$ is a linear map on a finite-dimensional space E, then f is diagonalizable iff its minimal polynomial m is of the form

$$m = (X - \lambda_1) \cdots (X - \lambda_k),$$

where $\lambda_1, \ldots, \lambda_k$ are distinct elements of K.

One of the technical tools used to prove this result is the notion of *f-conductor*; see Definition 31.2. As a corollary of Theorem 31.6 we obtain results about finite commuting families of diagonalizable or triangulable linear maps.

If $f\colon E \to E$ is a linear map and $\lambda \in K$ is an eigenvalue of f, recall that the eigenspace E_λ associated with λ is the kernel of the linear map $\lambda \mathrm{id} - f$. If all the eigenvalues $\lambda_1 \ldots, \lambda_k$ of f are in K and if f is diagonalizable, then

$$E = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k},$$

but in general there are not enough eigenvectors to span E. A remedy is to generalize the notion of eigenvector and look for (nonzero) vectors u (called generalized eigenvectors) such that

$$(\lambda \mathrm{id} - f)^r(u) = 0, \quad \text{for some } r \geq 1.$$

Then, it turns out that if the minimal polynomial of f is of the form

$$m = (X - \lambda_1)^{r_1} \cdots (X - \lambda_k)^{r_k},$$

then $r = r_i$ does the job for λ_i; that is, if we let

$$W_i = \mathrm{Ker}\,(\lambda_i \mathrm{id} - f)^{r_i},$$

then

$$E = W_1 \oplus \cdots \oplus W_k.$$

The above facts are parts of the *primary decomposition theorem* (Theorem 31.11). It is a special case of a more general result involving the factorization of the minimal polynomial m into its irreducible monic factors; see Theorem 31.10.

Theorem 31.11 implies that every linear map f that has all its eigenvalues in K can be written as $f = D + N$, where D is diagonalizable and N is nilpotent (which means that $N^r = 0$ for some positive integer r). Furthermore D and N commute and are unique. This is the *Jordan decomposition*, Theorem 31.12.

The Jordan decomposition suggests taking a closer look at nilpotent maps. We prove that for any nilpotent linear map $f\colon E \to E$ on a finite-dimensional vector space E of dimension n over a field K, there is a basis of E such that the matrix N of f is of the form

$$N = \begin{pmatrix} 0 & \nu_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \nu_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \nu_n \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$