and thus we must have

$$\lambda(u)w + \lambda(u)\overrightarrow{f}(u) = \mu w + g(u), \tag{$*_1$}$$

for some $\lambda(u) \neq 0$.

If $\mathrm{Ker}\,\overrightarrow{f} = \overrightarrow{E}$, the linear map $\overrightarrow{f}$ is the null map, and since we are requiring that the restriction of $\widetilde{f}$ to $\mathbf{P}(\overrightarrow{E})$ be equal to $\mathbf{P}(\overrightarrow{f})$, the linear map g must also be the null map on $\overrightarrow{E}$. Thus, $\widetilde{f}$ is unique, and the restriction of $\widetilde{f}$ to $\mathbf{P}(\overrightarrow{E})$ is the partial map undefined everywhere.

If $\overrightarrow{E} - \mathrm{Ker}\,\overrightarrow{f} \neq \emptyset$, by taking a basis of $\mathrm{Im}\,\overrightarrow{f}$ and some inverse image of this basis, we obtain a basis B of a subspace $\overrightarrow{G}$ of $\overrightarrow{E}$ such that $\overrightarrow{E} = \mathrm{Ker}\,\overrightarrow{f} \oplus \overrightarrow{G}$. Since $\overrightarrow{E} = \mathrm{Ker}\,\overrightarrow{f} \oplus \overrightarrow{G}$ where $\dim(\overrightarrow{G}) \geq 1$, for any $x \in \mathrm{Ker}\,\overrightarrow{f}$ and any nonnull vector $y \in \overrightarrow{G}$, we have

$$\begin{aligned}
\lambda(x)w &= \mu w + g(x),\\
\lambda(y)w + \lambda(y)\overrightarrow{f}(y) &= \mu w + g(y),
\end{aligned}$$

and

$$\lambda(x+y)w + \lambda(x+y)\overrightarrow{f}(x+y) = \mu w + g(x+y),$$

which by linearity yields

$$(\lambda(x+y) - \lambda(x) - \lambda(y) + \mu)w + (\lambda(x+y) - \lambda(y))\overrightarrow{f}(y) = 0.$$

Since $F = Kw \oplus H$ and $\overrightarrow{f}\colon \overrightarrow{E} \to H$, we must have $\lambda(x+y) = \lambda(y)$ and $\lambda(x) = \mu$. Then the equation

$$\lambda(x)w = \mu w + g(x)$$

yields $\mu w = \mu w + g(x)$, shows that g vanishes on $\mathrm{Ker}\,\overrightarrow{f}$.

If $\dim(\overrightarrow{G}) = 1$ then by $(*_1)$, for any $y \in \overrightarrow{G}$ we have

$$\lambda(y)w + \lambda(y)\overrightarrow{f}(y) = \mu w + g(y),$$

and for any $\nu \neq 0$ we have

$$\lambda(\nu y)w + \lambda(\nu y)\overrightarrow{f}(\nu y) = \mu w + g(\nu y),$$

which by linearity yields

$$(\lambda(\nu y) - \nu\lambda(y) - \mu + \nu\mu)w + (\nu\lambda(\nu y) - \nu\lambda(y))\overrightarrow{f}(y) = 0.$$

Since $F = Kw \oplus H$, $\overrightarrow{f}\colon \overrightarrow{E} \to H$, and $\nu \neq 0$, we must have $\lambda(\nu y) = \lambda(y)$. Then we must also have $(\lambda(y) - \mu)(1 - \nu) = 0$.