Proposition 29.21. *Given an ϵ-Hermitian form $\varphi\colon E\times E\to K$ on E, if φ is nondegenerate and if U is a finite-dimensional subspace of E, then* $\mathrm{rad}(U)=\mathrm{rad}(U^{\perp})$, *and the following conditions are equivalent:*

(i) U *is nondegenerate.*

(ii) $U^{\perp}$ *is nondegenerate.*

(iii) $E=U\oplus U^{\perp}$.

Proof. By definition, $\mathrm{rad}(U^{\perp})=U^{\perp}\cap U^{\perp\perp}$, and since φ is nondegenerate and U is finite-dimensional, $U^{\perp\perp}=U$, so $\mathrm{rad}(U^{\perp})=U^{\perp}\cap U^{\perp\perp}=U\cap U^{\perp}=\mathrm{rad}(U)$.

By Proposition 29.20, (i) implies (iii). If $E=U\oplus U^{\perp}$, then $\mathrm{rad}(U)=U\cap U^{\perp}=(0)$, so U is nondegenerate and (iii) implies (i). Since $\mathrm{rad}(U^{\perp})=\mathrm{rad}(U)$, (iii) also implies (ii). Now, if $U^{\perp}$ is nondegenerate, we have $U^{\perp}\cap U^{\perp\perp}=(0)$, and since $U\subseteq U^{\perp\perp}$, we get

$$U\cap U^{\perp}\subseteq U^{\perp\perp}\cap U^{\perp}=(0),$$

which shows that U is nondegenerate, proving the implication (ii) $\Longrightarrow$ (i). $\square$

If E is finite-dimensional, we have the following results.

Proposition 29.22. *Given an ϵ-Hermitian form $\varphi\colon E\times E\to K$ on a finite-dimensional space E, if φ is nondegenerate, then for every subspace U of E we have*

(i) $\dim(U)+\dim(U^{\perp})=\dim(E)$.

(ii) $U^{\perp\perp}=U$.

Proof. (i) Since φ is nondegenerate and E is finite-dimensional, the semilinear map $l_{\varphi}\colon E\to E^*$ is bijective. By transposition, the inclusion $i\colon U\to E$ yields a surjection $r\colon E^*\to U^*$ (with $r(f)=f\circ i$ for every $f\in E^*$; the map $f\circ i$ is the restriction of the linear form f to U). It follows that the semilinear map $r\circ l_{\varphi}\colon E\to U^*$ given by

$$(r\circ l_{\varphi})(x)(u)=\overline{\varphi(x,u)}\quad x\in E,u\in U$$

is surjective, and its kernel is $U^{\perp}$. Thus, we have

$$\dim(U^*)+\dim(U^{\perp})=\dim(E),$$

and since $\dim(U)=\dim(U^*)$ because U is finite-dimensional, we get

$$\dim(U)+\dim(U^{\perp})=\dim(U^*)+\dim(U^{\perp})=\dim(E).$$

(ii) Applying the above formula to $U^{\perp}$, we deduce that $\dim(U)=\dim(U^{\perp\perp})$. Since $U\subseteq U^{\perp\perp}$, we must have $U^{\perp\perp}=U$. $\square$