The above determinant is called the *resultant of f and g*.

Note that the matrix of the resultant is an $(n+m) \times (n+m)$ matrix, with the first row (involving the a_is) occurring n times, each time shifted over to the right by one column, and the $(n+1)$th row (involving the b_js) occurring m times, each time shifted over to the right by one column.

Hint. Express the matrix of T over some suitable basis.

(4) Compute the resultant in the following three cases:

(a) $m = n = 1$, and write $f(X) = aX + b$ and $g(X) = cX + d$.

(b) $m = 1$ and $n \geq 2$ arbitrary.

(c) $f(X) = aX^2 + bX + c$ and $g(X) = 2aX + b$.

(5) Compute the resultant of $f(X) = X^3 + pX + q$ and $g(X) = 3X^2 + p$, and

$$\begin{aligned} f(X) &= a_0X^2 + a_1X + a_2 \\ g(X) &= b_0X^2 + b_1X + b_2. \end{aligned}$$

In the second case, you should get

$$4R(f, g) = (2a_0b_2 - a_1b_1 + 2a_2b_0)^2 - (4a_0a_2 - a_1^2)(4b_0b_2 - b_1^2).$$

Problem 7.9. Let A, B, C, D be $n \times n$ real or complex matrices.

(1) Prove that if A is invertible and if $AC = CA$, then

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - CB).$$

(2) Prove that if H is an $n \times n$ Hadamard matrix $(n \geq 2)$, then $|\det(H)| = n^{n/2}$.

(3) Prove that if H is an $n \times n$ Hadamard matrix $(n \geq 2)$, then

$$\det \begin{pmatrix} H & H \\ H & -H \end{pmatrix} = (2n)^n.$$

Problem 7.10. Compute the product of the following determinants

$$\begin{vmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{vmatrix} \begin{vmatrix} x & -y & -z & -t \\ y & x & -t & z \\ z & t & x & -y \\ t & -z & y & x \end{vmatrix}$$

to prove the following identity (due to Euler):

$$\begin{aligned} (a^2 + b^2 + c^2 + d^2)(x^2 + y^2 + z^2 + t^2) &= (ax + by + cz + dt)^2 + (ay - bx + ct - dz)^2 \\ &\quad + (az - bt - cx + dy)^2 + (at + bz - cy - dx)^2. \end{aligned}$$