since $\langle v, w\rangle = 0$. Therefore,

$$f(W^{\perp}) \subseteq W^{\perp}.$$

Clearly, the restriction of f to $W^{\perp}$ is self-adjoint, and we conclude by applying the induction hypothesis to $W^{\perp}$ (whose dimension is $n-1$). $\square$

We now come back to normal linear maps. One of the key points in the proof of Theorem 17.8 is that we found a subspace W with the property that $f(W) \subseteq W$ implies that $f(W^{\perp}) \subseteq W^{\perp}$. In general, this does not happen, *but normal maps satisfy a stronger property which ensures that such a subspace exists*.

The following proposition provides a condition that will allow us to show that a normal linear map can be diagonalized. It actually holds for any linear map. We found the inspiration for this proposition in Berger [11].

Proposition 17.9. *Given a Hermitian space E, for any linear map $f\colon E \to E$ and any subspace W of E, if $f(W) \subseteq W$, then $f^*\big(W^{\perp}\big) \subseteq W^{\perp}$. Consequently, if $f(W) \subseteq W$ and $f^*(W) \subseteq W$, then $f\big(W^{\perp}\big) \subseteq W^{\perp}$ and $f^*\big(W^{\perp}\big) \subseteq W^{\perp}$.*

Proof. If $u \in W^{\perp}$, then

$$\langle w, u\rangle = 0 \quad \text{for all } w \in W.$$

However,

$$\langle f(w), u\rangle = \langle w, f^*(u)\rangle,$$

and $f(W) \subseteq W$ implies that $f(w) \in W$. Since $u \in W^{\perp}$, we get

$$0 = \langle f(w), u\rangle = \langle w, f^*(u)\rangle,$$

which shows that $\langle w, f^*(u)\rangle = 0$ for all $w \in W$, that is, $f^*(u) \in W^{\perp}$. Therefore, we have $f^*(W^{\perp}) \subseteq W^{\perp}$.

We just proved that if $f(W) \subseteq W$, then $f^*\big(W^{\perp}\big) \subseteq W^{\perp}$. If we also have $f^*(W) \subseteq W$, then by applying the above fact to f^*, we get $f^{**}(W^{\perp}) \subseteq W^{\perp}$, and since $f^{**} = f$, this is just $f(W^{\perp}) \subseteq W^{\perp}$, which proves the second statement of the proposition. $\square$

It is clear that the above proposition also holds for Euclidean spaces.

Although we are ready to prove that for every normal linear map f (over a Hermitian space) there is an orthonormal basis of eigenvectors (see Theorem 17.13 below), we now return to real Euclidean spaces.

Proposition 17.10. *If $f\colon E \to E$ is a linear map and $w = u + iv$ is an eigenvector of $f_{\mathbb{C}}\colon E_{\mathbb{C}} \to E_{\mathbb{C}}$ for the eigenvalue $z = \lambda + i\mu$, where $u, v \in E$ and $\lambda, \mu \in \mathbb{R}$, then*

$$f(u) = \lambda u - \mu v \quad \text{and} \quad f(v) = \mu u + \lambda v. \tag{$*$}$$

As a consequence,

$$f_{\mathbb{C}}(u - iv) = f(u) - if(v) = (\lambda - i\mu)(u - iv),$$

which shows that $\overline{w} = u - iv$ is an eigenvector of $f_{\mathbb{C}}$ for $\overline{z} = \lambda - i\mu$.