As a corollary of Proposition 29.1, we have the following characterization of a nondegenerate bilinear map. The proof is left as an exercise.

Proposition 29.2. *Given a bilinear map $\varphi\colon E \times F \to K$, if E and F have the same finite dimension, then the following properties are equivalent:*

(1) The map l_φ is injective.

(2) The map l_φ is surjective.

(3) The map r_φ is injective.

(4) The map r_φ is surjective.

(5) The bilinear form φ is nondegenerate.

Observe that in terms of the canonical pairing between E^* and E given by

$$\langle f, u\rangle = f(u), \quad f \in E^*, u \in E,$$

(and the canonical pairing between F^* and F), we have

$$\varphi(u, v) = \langle l_\varphi(u), v\rangle = \langle r_\varphi(v), u\rangle \quad u \in E, v \in F.$$

Proposition 29.3. *Given a bilinear map $\varphi\colon E \times F \to K$, if φ is nondegenerate and E and F are finite-dimensional, then $\dim(E) = \dim(F) = n$, and for every basis $(e_1, \ldots, e_n)$ of E, there is a basis $(f_1, \ldots, f_n)$ of F such that $\varphi(e_i, f_j) = \delta_{ij}$, for all $i, j = 1, \ldots, n$.*

Proof. Since φ is nondegenerate, by Proposition 29.1 we have $\dim(E) = \dim(F) = n$, and by Proposition 29.2, the linear map r_φ is bijective. Then, if $(e_1^*, \ldots, e_n^*)$ is the dual basis (in E^*) of the basis $(e_1, \ldots, e_n)$, the vectors $(f_1, \ldots, f_n)$ given by $f_i = r_\varphi^{-1}(e_i^*)$ form a basis of F, and we have

$$\varphi(e_i, f_j) = \langle r_\varphi(f_j), e_i\rangle = \langle e_i^*, e_j\rangle = \delta_{ij},$$

as claimed. ∎

If $E = F$ and φ is symmetric, then we have the following interesting result.

Theorem 29.4. *Given any bilinear form $\varphi\colon E \times E \to K$ with $\dim(E) = n$, if φ is symmetric (possibly degenerate) and K does not have characteristic 2, then there is a basis $(e_1, \ldots, e_n)$ of E such that $\varphi(e_i, e_j) = 0$, for all $i \neq j$.*

Proof. We proceed by induction on $n \geq 0$, following a proof due to Chevalley. The base case $n = 0$ is trivial. For the induction step, assume that $n \geq 1$ and that the induction hypothesis holds for all vector spaces of dimension $n - 1$. If $\varphi(u, v) = 0$ for all $u, v \in E$, then the statement holds trivially. Otherwise, since K does not have characteristic 2, equation

$$2\varphi(u, v) = \varphi(u + v, u + v) - \varphi(u, u) - \varphi(v, v) \tag{$*$}$$