(4) *Strictly convex quadratic function*: $f(x) = \frac{1}{2}x^\top Ax$, where A is an $n \times n$ symmetric positive definite matrix, with $\mathrm{dom}(f) = \mathbb{R}^n$. The function $x \mapsto y^\top x - \frac{1}{2}x^\top Ax$ has a unique maximum when is gradient is zero, namely

$$y = Ax.$$

Substituting for $x = A^{-1}y$ in $y^\top x - \frac{1}{2}x^\top Ax$, we obtain

$$y^\top A^{-1}y - \frac{1}{2}y^\top A^{-1}y = -\frac{1}{2}y^\top A^{-1}y,$$

so

$$f^*(y) = -\frac{1}{2}y^\top A^{-1}y$$

with $\mathrm{dom}(f^*) = \mathbb{R}^n$.

(5) *Log-determinant*: $f(X) = \log\det(X^{-1})$, where X is an $n\times n$ symmetric positive definite matrix. Then

$$f(Y) = \log\det((-Y)^{-1}) - n,$$

where Y is an $n \times n$ symmetric negative definite matrix; see Boyd and Vandenberghe; see [29], Section 3.3.1, Example 3.23.

(6) *Norm on* $\mathbb{R}^n$: $f(x) = \|x\|$ for any norm $\| \; \|$ on $\mathbb{R}^n$, with $\mathrm{dom}(f) = \mathbb{R}^n$. Recall from Section 14.7 that the dual norm $\| \; \|^D$ of the norm $\| \; \|$ (with respect to the canonical inner product $x \cdot y = y^\top x$ on $\mathbb{R}^n$ is given by

$$\|y\|^D = \sup_{\|x\|=1} |y^\top x|,$$

and that

$$|y^\top x| \leq \|x\| \, \|y\|^D .$$

We have

$$\begin{aligned} f^*(y) &= \sup_{x\in\mathbb{R}^n} (y^\top x - \|x\|) \\ &= \sup_{x\in\mathbb{R}^n, x\neq 0} \left(y^\top \frac{x}{\|x\|} - 1 \right) \|x\| \\ &\leq \sup_{x\in\mathbb{R}^n, x\neq 0} (\|y\|^D - 1) \, \|x\| , \end{aligned}$$

so if $\|y\|^D > 1$ this last term goes to $+\infty$, but if $\|y\|^D \leq 1$, then its maximum is 0. Therefore,

$$f^*(y) = \|y\|^* = \begin{cases} 0 & \text{if } \|y\|^D \leq 1 \\ +\infty & \text{otherwise.} \end{cases}$$