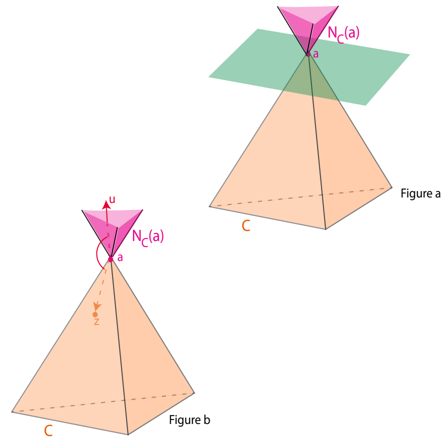

Figure 51.11: Let C be the solid peach tetrahedron in $\mathbb{R}^3$. The small upside-down magenta tetrahedron is the translate of $N_C(a)$. Figure (a) shows that the normal cone is separated from C by the horizontal green supporting hyperplane. Figure (b) shows that any vector $u \in N_C(a)$ does not make an acute angle with a line segment in C emanating from a.

Assume that $f(x)$ is finite. Observe that the subgradient inequality says that 0 is a subgradient at x iff f has a global minimum at x. In this case, the hyperplane $\mathcal{H}$ (in $\mathbb{R}^{n+1}$) defined by the affine form $\omega(y, \alpha) = f(x) - \alpha$ is a horizontal supporting hyperplane to the epigraph $\mathbf{epi}(f)$ at $(x, f(x))$. If $u \in \partial f(x)$ and $u \neq 0$, then $(*_{\text{subgrad}})$ says that the hyperplane induced by the affine form $z \mapsto \langle z - x, u\rangle + f(x)$ as in Proposition 51.9 is a nonvertical supporting hyperplane $\mathcal{H}$ (in $\mathbb{R}^{n+1}$) to the epigraph $\mathbf{epi}(f)$ at $(x, f(x))$. The vector $(u, -1) \in \mathbb{R}^{n+1}$ is normal to the hyperplane $\mathcal{H}$. See Figure 51.13.

Indeed, if $u \neq 0$, the hyperplane $\mathcal{H}$ is given by

$$\mathcal{H} = \{(y, \alpha) \in \mathbb{R}^{n+1} \mid \omega(y, \alpha) = 0\}$$

with $\omega(y, \alpha) = \langle y - x, u\rangle + f(x) - \alpha$, so $\omega(x, f(x)) = 0$, which means that $(x, f(x)) \in \mathcal{H}$. Also, for any $(z, \beta) \in \mathbf{epi}(f)$, by $(*_{\text{subgrad}})$, we have

$$\omega(z, \beta) = \langle z - x, u\rangle + f(x) - \beta \leq f(z) - \beta \leq 0,$$

since $(z, \beta) \in \mathbf{epi}(f)$, so $\mathbf{epi}(f) \subseteq \mathcal{H}_-$, and $\mathcal{H}$ is a nonvertical supporting hyperplane (in $\mathbb{R}^{n+1}$) to the epigraph $\mathbf{epi}(f)$ at $(x, f(x))$. Since

$$\omega(y, \alpha) = \langle y - x, u\rangle + f(x) - \alpha = \langle (y - x, \alpha), (u, -1)\rangle + f(x),$$

the vector $(u, -1)$ is indeed normal to the hyperplane $\mathcal{H}$.

The above facts are important and recorded as the following proposition.

Proposition 51.10. *For every $x \in \mathbb{R}^n$, if $f(x)$ is finite, then f is subdifferentiable at x if and only if there is a nonvertical supporting hyperplane (in $\mathbb{R}^{n+1}$) to the epigraph* $\mathbf{epi}(f)$ *at*