Proposition 11.13. *If $f\colon E \to F$ is any linear map, then the following identities hold:*

$$\begin{aligned}\operatorname{Im} f^\top &= (\operatorname{Ker}(f))^0\\ \operatorname{Ker}(f^\top) &= (\operatorname{Im} f)^0\\ \operatorname{Im} f &= (\operatorname{Ker}(f^\top)^0\\ \operatorname{Ker}(f) &= (\operatorname{Im} f^\top)^0.\end{aligned}$$

Proof. The equation $\operatorname{Ker}(f^\top) = (\operatorname{Im} f)^0$ has already been proven in Proposition 11.11.

By the duality theorem $(\operatorname{Ker}(f))^{00} = \operatorname{Ker}(f)$, so from $\operatorname{Im} f^\top = (\operatorname{Ker}(f))^0$ we get $\operatorname{Ker}(f) = (\operatorname{Im} f^\top)^0$. Similarly, $(\operatorname{Im} f)^{00} = \operatorname{Im} f$, so from $\operatorname{Ker}(f^\top) = (\operatorname{Im} f)^0$ we get $\operatorname{Im} f = (\operatorname{Ker}(f^\top)^0$. Therefore, what is left to be proven is that $\operatorname{Im} f^\top = (\operatorname{Ker}(f))^0$.

Let $p\colon E \to E/\operatorname{Ker}(f)$ be the canonical surjection, $\overline{f}\colon E/\operatorname{Ker}(f) \to \operatorname{Im} f$ be the isomorphism induced by f, and $j\colon \operatorname{Im} f \to F$ be the inclusion map. Then, we have

$$f = j \circ \overline{f} \circ p,$$

which implies that

$$f^\top = p^\top \circ \overline{f}^\top \circ j^\top.$$

Since p is surjective, $p^\top$ is injective, since j is injective, $j^\top$ is surjective, and since $\overline{f}$ is bijective, $\overline{f}^\top$ is also bijective. It follows that $(E/\operatorname{Ker}(f))^* = \operatorname{Im}(\overline{f}^\top \circ j^\top)$, and we have

$$\operatorname{Im} f^\top = \operatorname{Im} p^\top.$$

Since $p\colon E \to E/\operatorname{Ker}(f)$ is the canonical surjection, by Proposition 11.9 applied to $U = \operatorname{Ker}(f)$, we get

$$\operatorname{Im} f^\top = \operatorname{Im} p^\top = (\operatorname{Ker}(f))^0,$$

as claimed. $\square$

In summary, the equation

$$\operatorname{Im} f^\top = (\operatorname{Ker}(f))^0$$

applies in any dimension, and it implies that

$$\operatorname{Ker}(f) = (\operatorname{Im} f^\top)^0.$$

The following proposition shows the relationship between the matrix representing a linear map $f\colon E \to F$ and the matrix representing its transpose $f^\top\colon F^* \to E^*$.

Proposition 11.14. *Let E and F be two vector spaces, and let $(u_1, \ldots, u_n)$ be a basis for E and $(v_1, \ldots, v_m)$ be a basis for F. Given any linear map $f\colon E \to F$, if $M(f)$ is the $m \times n$-matrix representing f w.r.t. the bases $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_m)$, then the $n \times m$-matrix $M(f^\top)$ representing $f^\top\colon F^* \to E^*$ w.r.t. the dual bases $(v_1^*, \ldots, v_m^*)$ and $(u_1^*, \ldots, u_n^*)$ is the transpose $M(f)^\top$ of $M(f)$.*