(3) Prove that if $n = 2$, then

$$0 \leq \epsilon_{k+1} = \frac{\epsilon_k^2}{2(\epsilon_k + 1)}, \quad \text{for all} \quad k \geq 0,$$

else if $n \geq 3$, then

$$0 \leq \epsilon_{k+1} \leq \frac{(n-1)}{n}\epsilon_k, \quad \text{for all} \quad k \geq 1.$$

Prove that the sequence (x_k) converges to $x^{1/n}$ for every initial value $x_0 > 0$.

(4) When $n = 2$, we saw in Problem 41.7(3) that

$$0 \leq \epsilon_{k+1} = \frac{\epsilon_k^2}{2(\epsilon_k + 1)}, \quad \text{for all} \quad k \geq 0.$$

For $n = 3$, prove that

$$\epsilon_{k+1} = \frac{2\epsilon_k^2(3/2 + \epsilon_k)}{3(\epsilon_k + 1)^2}, \quad \text{for all} \quad k \geq 0,$$

and for $n = 4$, prove that

$$\epsilon_{k+1} = \frac{3\epsilon_k^2}{4(\epsilon_k + 1)^3}\left(2 + (8/3)\epsilon_k + \epsilon_k^2\right), \quad \text{for all} \quad k \geq 0.$$

Let μ_3 and μ_4 be the functions given by

$$\begin{aligned} \mu_3(a) &= \frac{3}{2} + a \\ \mu_4(a) &= 2 + \frac{8}{3}a + a^2, \end{aligned}$$

so that if $n = 3$, then

$$\epsilon_{k+1} = \frac{2\epsilon_k^2\mu_3(\epsilon_k)}{3(\epsilon_k + 1)^2}, \quad \text{for all} \quad k \geq 0,$$

and if $n = 4$, then

$$\epsilon_{k+1} = \frac{3\epsilon_k^2\mu_4(\epsilon_k)}{4(\epsilon_k + 1)^3}, \quad \text{for all} \quad k \geq 0.$$

Prove that

$$a\mu_3(a) \leq (a+1)^2 - 1, \quad \text{for all} \quad a \geq 0,$$

and

$$a\mu_4(a) \leq (a+1)^3 - 1, \quad \text{for all} \quad a \geq 0.$$

Let $\eta_{3,k} = \mu_3(\epsilon_1)\epsilon_k$ when $n = 3$, and $\eta_{4,k} = \mu_4(\epsilon_1)\epsilon_k$ when $n = 4$. Prove that

$$\eta_{3,k+1} \leq \frac{2}{3}\eta_{3,k}^2, \quad \text{for all} \quad k \geq 1,$$