Note there are no constraints on diagonal entries, and half of the equations

$$a_{ij} - a_{ji} = 0, \quad 1 \leq i \neq j \leq n$$

are redundant. It is easy to check that the equations (linear forms) for which $i < j$ are linearly independent. To be more precise, let U be the space of linear forms in E^* spanned by the linear forms

$$u^*_{ij}(a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{n1}, \ldots, a_{nn}) = a_{ij} - a_{ji}, \quad 1 \leq i < j \leq n.$$

The dimension of U is $n(n-1)/2$. Then the set U^0 of common solutions of these equations is the space $\mathbf{S}(n)$ of symmetric matrices. By the duality theorem (Theorem 11.4), this space has dimension

$$\frac{n(n+1)}{2} = n^2 - \frac{n(n-1)}{2}.$$

We leave it as an exercise to find a basis of $\mathbf{S}(n)$.

Example 11.4. If $E = \mathrm{M}_n(\mathbb{R})$, consider the subspace U of linear forms in E^* spanned by the linear forms

$$\begin{aligned}
u^*_{ij}(a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{n1}, \ldots, a_{nn}) &= a_{ij} + a_{ji}, \quad 1 \leq i < j \leq n \\
u^*_{ii}(a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{n1}, \ldots, a_{nn}) &= a_{ii}, \quad 1 \leq i \leq n.
\end{aligned}$$

It is easy to see that these linear forms are linearly independent, so $\dim(U) = n(n+1)/2$. The space U^0 of matrices $A \in \mathrm{M}_n(\mathbb{R})$ satifying all of the above equations is clearly the space $\mathbf{Skew}(n)$ of skew-symmetric matrices. By the duality theorem (Theorem 11.4), the dimension of U^0 is

$$\frac{n(n-1)}{2} = n^2 - \frac{n(n+1)}{2}.$$

We leave it as an exercise to find a basis of $\mathbf{Skew}(n)$.

Example 11.5. For yet another example with $E = \mathrm{M}_n(\mathbb{R})$, for any $A \in \mathrm{M}_n(\mathbb{R})$, consider the linear form in E^* given by

$$\mathrm{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn},$$

called the *trace* of A. The subspace U^0 of E consisting of all matrices A such that $\mathrm{tr}(A) = 0$ is a space of dimension $n^2 - 1$. We leave it as an exercise to find a basis of this space.

The dimension equations

$$\begin{aligned}
\dim(V) + \dim(V^0) &= \dim(E) \\
\dim(U) + \dim(U^0) &= \dim(E)
\end{aligned}$$

are always true (if E is finite-dimensional). This is part of the duality theorem (Theorem 11.4).