Next, assume $n > 1$. If $p = 1$, then $I_{k_1} = I$ and the formula is trivial, so assume that $p \geq 2$ and write $J = (k_2, \ldots, k_p)$. There are two cases.

Case 1. The sequence I_{k_1} has a single element, say β, which is the first element of I. In this case, write C for the sequence obtained from I by deleting its first element β. By definition,

$$\sum_{\alpha\in I} a_\alpha = a_\beta + \Bigl(\sum_{\alpha\in C} a_\alpha\Bigr),$$

and

$$\sum_{k\in K}\Bigl(\sum_{\alpha\in I_k} a_\alpha\Bigr) = a_\beta + \Bigl(\sum_{j\in J}\Bigl(\sum_{\alpha\in I_j} a_\alpha\Bigr)\Bigr).$$

Since $|C| = n-1$, by the induction hypothesis, we have

$$\Bigl(\sum_{\alpha\in C} a_\alpha\Bigr) = \sum_{j\in J}\Bigl(\sum_{\alpha\in I_j} a_\alpha\Bigr),$$

which yields our identity.

Case 2. The sequence I_{k_1} has at least two elements. In this case, let β be the first element of I (and thus of I_{k_1}), let I' be the sequence obtained from I by deleting its first element β, let I'_{k_1} be the sequence obtained from I_{k_1} by deleting its first element β, and let $I'_{k_i} = I_{k_i}$ for $i = 2, \ldots, p$. Recall that $J = (k_2, \ldots, k_p)$ and $K = (k_1, \ldots, k_p)$. The sequence I' has $n-1$ elements, so by the induction hypothesis applied to I' and the I'_{k_i}, we get

$$\sum_{\alpha\in I'} a_\alpha = \sum_{k\in K}\Bigl(\sum_{\alpha\in I'_k} a_\alpha\Bigr) = \Bigl(\sum_{\alpha\in I'_{k_1}} a_\alpha\Bigr) + \Bigl(\sum_{j\in J}\Bigl(\sum_{\alpha\in I_j} a_\alpha\Bigr)\Bigr).$$

If we add the lefthand side to a_β, by definition we get

$$\sum_{\alpha\in I} a_\alpha.$$

If we add the righthand side to a_β, using associativity and the definition of an indexed sum, we get

$$\begin{aligned}
a_\beta + \Bigl(\Bigl(\sum_{\alpha\in I'_{k_1}} a_\alpha\Bigr) + \Bigl(\sum_{j\in J}\Bigl(\sum_{\alpha\in I_j} a_\alpha\Bigr)\Bigr)\Bigr) &= \Bigl(a_\beta + \Bigl(\sum_{\alpha\in I'_{k_1}} a_\alpha\Bigr)\Bigr) + \Bigl(\sum_{j\in J}\Bigl(\sum_{\alpha\in I_j} a_\alpha\Bigr)\Bigr)\\
&= \Bigl(\sum_{\alpha\in I_{k_1}} a_\alpha\Bigr) + \Bigl(\sum_{j\in J}\Bigl(\sum_{\alpha\in I_j} a_\alpha\Bigr)\Bigr)\\
&= \sum_{k\in K}\Bigl(\sum_{\alpha\in I_k} a_\alpha\Bigr),
\end{aligned}$$

as claimed. $\square$