then

$$\sum_{i\in I}\lambda_i v_i = \sum_{i\in I}\lambda_i f(u_i) = f\left(\sum_{i\in I}\lambda_i u_i\right) = 0,$$

and $\lambda_i = 0$ for all $i \in I$ because $(v_i)_{i\in I}$ is linearly independent, which means that $x = 0$. Therefore, $\operatorname{Ker} f = (0)$, which implies that f is injective. The part where f is surjective is left as a simple exercise. $\square$

Figure 3.11 provides an illustration of Proposition 3.18 when $E = \mathbb{R}^3$ and $V = \mathbb{R}^2$

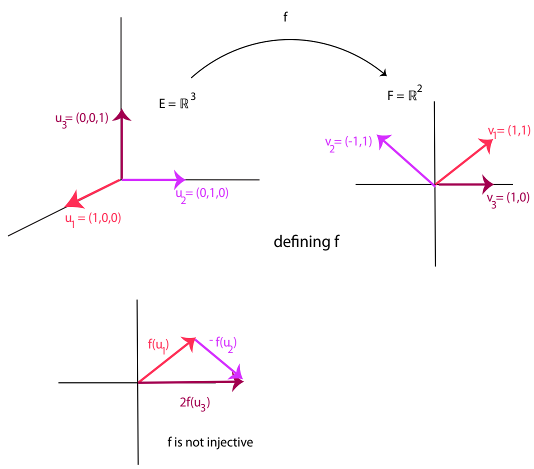

Figure 3.11: Given $u_1 = (1,0,0)$, $u_2 = (0,1,0)$, $u_3 = (0,0,1)$ and $v_1 = (1,1)$, $v_2 = (-1,1)$, $v_3 = (1,0)$, define the unique linear map $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ by $f(u_1) = v_1$, $f(u_2) = v_2$, and $f(u_3) = v_3$. This map is surjective but not injective since $f(u_1 - u_2) = f(u_1) - f(u_2) = (1,1) - (-1,1) = (2,0) = 2f(u_3) = f(2u_3)$.

By the second part of Proposition 3.18, an injective linear map $f\colon E \to F$ sends a basis $(u_i)_{i\in I}$ to a linearly independent family $(f(u_i))_{i\in I}$ of F, which is also a basis when f is bijective. Also, when E and F have the same finite dimension n, $(u_i)_{i\in I}$ is a basis of E, and $f\colon E \to F$ is injective, then $(f(u_i))_{i\in I}$ is a basis of F (by Proposition 3.8).

We can now show that the vector space $K^{(I)}$ of Definition 3.11 has a universal property that amounts to saying that $K^{(I)}$ is the vector space freely generated by I. Recall that $\iota\colon I \to K^{(I)}$, such that $\iota(i) = e_i$ for every $i \in I$, is an injection from I to $K^{(I)}$.

Proposition 3.19. *Given any set I, for any vector space F, and for any function $f\colon I \to F$, there is a unique linear map $\overline{f}\colon K^{(I)} \to F$, such that*

$$f = \overline{f} \circ \iota,$$