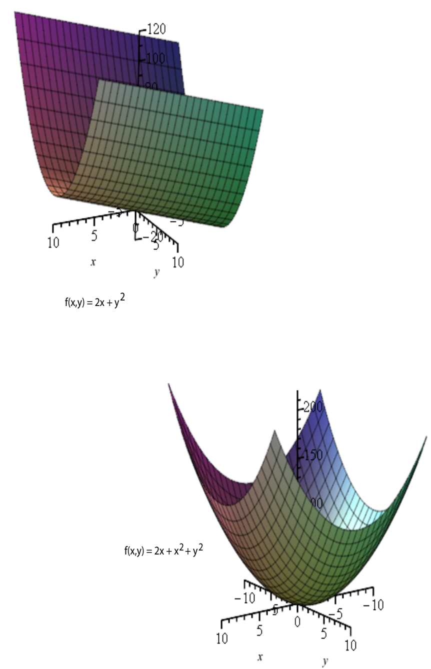

Figure 51.25: The graphs of the two functions discussed in Example 51.12. The graph of $f(x, y) = 2x + y^2$ slopes "downward" along the negative x-axis, reflecting the fact that $(-u, 0)$ is a direction of recession.

Since $u \neq 0$ and A is SPD, we have $u^\top Au > 0$, and the above quadratic function increases for $\lambda \geq -(2x^\top Au + b^\top u)/(2u^\top Au)$.

The above fact yields an important trick of convex optimization. If f is any proper closed and convex function, then for any quadratic strictly convex function q, the function $h = f + q$ is a proper and closed strictly convex function that has a minimum which is attained for a *unique* vector. This trick is at the core of the method of augmented Lagrangians, and in particular ADMM. Surprisingly, a rigorous proof requires the deep theorem below.

One should be careful not to conclude hastily that if a convex function is proper and closed, then $\mathrm{dom}(f)$ and $\mathrm{Im}(f)$ are also closed. Also, a closed and proper convex function may not attain its minimum. For example, the function $f\colon \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ given by

$$f(x) = \begin{cases} \dfrac{1}{x} & \text{if } x > 0 \\ +\infty & \text{if } x \leq 0 \end{cases}$$

is a proper, closed and convex function, but $\mathrm{dom}(f) = (0, +\infty)$ and $\mathrm{Im}(f) = (0, +\infty)$. Note that $\inf f = 0$ is not attained. The problem is that f has 1 has a direction of recession as evidenced by the graph provided in Figure 51.26.

The following theorem is proven in Rockafellar [138] (Theorem 27.1).