linear combination $\alpha = \sum \lambda_{i_1,\ldots,i_n} (e^1_{i_1})^* \otimes \cdots \otimes (e^n_{i_n})^*$, for some unique $\lambda_{i_1,\ldots,i_n} \in K$. If we choose $\beta = e^1_{i_1} \otimes \cdots \otimes e^n_{i_n}$, then we get

$$\begin{aligned} 0 = \langle \alpha, e^1_{i_1} \otimes \cdots \otimes e^n_{i_n} \rangle &= \left\langle \sum \lambda_{i_1,\ldots,i_n} (e^1_{i_1})^* \otimes \cdots \otimes (e^n_{i_n})^*, e^1_{i_1} \otimes \cdots \otimes e^n_{i_n} \right\rangle \\ &= \sum \lambda_{i_1,\ldots,i_n} \langle (e^1_{i_1})^* \otimes \cdots \otimes (e^n_{i_n})^*, e^1_{i_1} \otimes \cdots \otimes e^n_{i_n} \rangle \\ &= \lambda_{i_1,\ldots,i_n}. \end{aligned}$$

Therefore, $\alpha = 0$,

Conversely, given any $\beta \in E_1 \otimes \cdots \otimes E_n$, assume that $\langle \alpha, \beta \rangle = 0$, for all $\alpha \in E^*_1 \otimes \cdots \otimes E^*_n$. The vector β is a finite linear combination $\beta = \sum \lambda_{i_1,\ldots,i_n} e^1_{i_1} \otimes \cdots \otimes e^n_{i_n}$, for some unique $\lambda_{i_1,\ldots,i_n} \in K$. If we choose $\alpha = (e^1_{i_1})^* \otimes \cdots \otimes (e^n_{i_n})^*$, then we get

$$\begin{aligned} 0 = \langle (e^1_{i_1})^* \otimes \cdots \otimes (e^n_{i_n})^*, \beta \rangle &= \left\langle (e^1_{i_1})^* \otimes \cdots \otimes (e^n_{i_n})^*, \sum \lambda_{i_1,\ldots,i_n} e^1_{i_1} \otimes \cdots \otimes e^n_{i_n} \right\rangle \\ &= \sum \lambda_{i_1,\ldots,i_n} \langle (e^1_{i_1})^* \otimes \cdots \otimes (e^n_{i_n})^*, e^1_{i_1} \otimes \cdots \otimes e^n_{i_n} \rangle \\ &= \lambda_{i_1,\ldots,i_n}. \end{aligned}$$

Therefore, $\beta = 0$. ☐

By Proposition 33.1,[1] we have a canonical isomorphism

$$(E_1 \otimes \cdots \otimes E_n)^* \cong E^*_1 \otimes \cdots \otimes E^*_n.$$

Here is our main proposition about duality of tensor products.

Proposition 33.16. *We have canonical isomorphisms*

$$(E_1 \otimes \cdots \otimes E_n)^* \cong E^*_1 \otimes \cdots \otimes E^*_n,$$

and

$$\mu \colon E^*_1 \otimes \cdots \otimes E^*_n \cong \mathrm{Hom}(E_1, \ldots, E_n; K).$$

Proof. The second isomorphism follows from the isomorphism $(E_1 \otimes \cdots \otimes E_n)^* \cong E^*_1 \otimes \cdots \otimes E^*_n$ together with the isomorphism $\mathrm{Hom}(E_1, \ldots, E_n; K) \cong (E_1 \otimes \cdots \otimes E_n)^*$ given by Proposition 33.8. ☐

Remarks:

1. The isomorphism $\mu \colon E^*_1 \otimes \cdots \otimes E^*_n \cong \mathrm{Hom}(E_1, \ldots, E_n; K)$ can be described explicitly as the linear extension to $E^*_1 \otimes \cdots \otimes E^*_n$ of the map given by

$$\mu(v^*_1 \otimes \cdots \otimes v^*_n)(u_1 \ldots, u_n) = v^*_1(u_1) \cdots v^*_n(u_n).$$

[1] This is where the assumption that our spaces are finite-dimensional is used.