31.6 Nilpotent Linear Maps and Jordan Form

This section is devoted to a normal form for nilpotent maps. We follow Godement's exposition [76]. Let $f\colon E \to E$ be a nilpotent linear map on a finite-dimensional vector space over a field K, and assume that f is not the zero map. There is a smallest positive integer $r \geq 1$ such $f^r \neq 0$ and $f^{r+1} = 0$. Clearly, the polynomial X^{r+1} annihilates f, and it is the minimal polynomial of f since $f^r \neq 0$. It follows that $r + 1 \leq n = \dim(E)$. Let us define the subspaces N_i by

$$N_i = \mathrm{Ker}\,(f^i), \quad i \geq 0.$$

Note that $N_0 = (0)$, $N_1 = \mathrm{Ker}\,(f)$, and $N_{r+1} = E$. Also, it is obvious that

$$N_i \subseteq N_{i+1}, \quad i \geq 0.$$

Proposition 31.13. *Given a nilpotent linear map f with $f^r \neq 0$ and $f^{r+1} = 0$ as above, the inclusions in the following sequence are strict:*

$$(0) = N_0 \subset N_1 \subset \cdots \subset N_r \subset N_{r+1} = E.$$

Proof. We proceed by contradiction. Assume that $N_i = N_{i+1}$ for some i with $0 \leq i \leq r$. Since $f^{r+1} = 0$, for every $u \in E$, we have

$$0 = f^{r+1}(u) = f^{i+1}(f^{r-i}(u)),$$

which shows that $f^{r-i}(u) \in N_{i+1}$. Since $N_i = N_{i+1}$, we get $f^{r-i}(u) \in N_i$, and thus $f^r(u) = 0$. Since this holds for all $u \in E$, we see that $f^r = 0$, a contradiction. ◻

Proposition 31.14. *Given a nilpotent linear map f with $f^r \neq 0$ and $f^{r+1} = 0$, for any integer i with $1 \leq i \leq r$, for any subspace U of E, if $U \cap N_i = (0)$, then $f(U) \cap N_{i-1} = (0)$, and the restriction of f to U is an isomorphism onto $f(U)$.*

Proof. Pick $v \in f(U) \cap N_{i-1}$. We have $v = f(u)$ for some $u \in U$ and $f^{i-1}(v) = 0$, which means that $f^i(u) = 0$. Then $u \in U \cap N_i$, so $u = 0$ since $U \cap N_i = (0)$, and $v = f(u) = 0$. Therefore, $f(U) \cap N_{i-1} = (0)$. The restriction of f to U is obviously surjective on $f(U)$. Suppose that $f(u) = 0$ for some $u \in U$. Then $u \in U \cap N_1 \subseteq U \cap N_i = (0)$ (since $i \geq 1$), so $u = 0$, which proves that f is also injective on U. ◻

Proposition 31.15. *Given a nilpotent linear map f with $f^r \neq 0$ and $f^{r+1} = 0$, there exists a sequence of subspace $U_1, \ldots, U_{r+1}$ of E with the following properties:*

(1) $N_i = N_{i-1} \oplus U_i$, for $i = 1, \ldots, r + 1$.

(2) We have $f(U_i) \subseteq U_{i-1}$, and the restriction of f to U_i is an injection, for $i = 2, \ldots, r+1$.

See Figure 31.2.