Since

$$X_{k+1} - X_k = (g'_{X_k})^{-1}(C - X_k^2),$$

deduce that if $X_k \neq C^2$, then $X_k - X_{k+1}$ is SPD.

Open problem: Does Theorem 41.1 apply for some suitable r, M, β?

(7) Prove that if C and X_0 commute, provided that the equation $X_k Z + Z X_k = C$ has a unique solution for all k, then X_k and C commute for all k and Z is given by

$$Z = \frac{1}{2} X_k^{-1} C = \frac{1}{2} C X_k^{-1}.$$

Deduce that

$$X_{k+1} = \frac{1}{2}(X_k + X_k^{-1} C) = \frac{1}{2}(X_k + C X_k^{-1}).$$

This is the matrix analog of the formula given in Problem 41.1(1).

Prove that if C and X_0 have positive eigenvalues and C and X_0 commute, then X_{k+1} has positive eigenvalues for all $k \geq 0$ and thus the sequence (X_k) is defined.

Hint. Because X_k and C commute, X_k^{-1} and C commute, and obviously X_k and X_k^{-1} commute. By Proposition 31.9, X_k, X_k^{-1}, and C are triangulable in a common basis, so there is some orthogonal matrix P and some upper-triangular matrices T_1, T_2 such that

$$X_k = P T_1 P^\top, \quad X_k^{-1} = P T_1^{-1} P^\top, \quad C = P T_2 P^\top.$$

It follows that

$$X_{k+1} = \frac{1}{2} P (T_1 + T_1^{-1} T_2) P^\top.$$

Also recall that the diagonal entries of an upper-triangular matrix are the eigenvalues of that matrix.

We conjecture that if C has positive eigenvalues, then the Newton sequence converges starting with any X_0 of the form $X_0 = \mu I_n$, with $\mu > 0$.

(8) Implement the above method in `Matlab` (there is a command `kron(A, B)` to form the Kronecker product of A and B). Test your program on diagonalizable and nondiagonalizable matrices, including

$$W = \begin{pmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 5 & 4 & 1 & 1 \\ 4 & 5 & 1 & 1 \\ 1 & 1 & 4 & 2 \\ 1 & 1 & 2 & 4 \end{pmatrix},$$

and

$$A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 0.01 & 0 & 0 \\ -1 & -1 & 100 & 100 \\ -1 & -1 & -100 & 100 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$