39.6 Second-Order and Higher-Order Derivatives

Given two normed affine spaces E and F, and some open subset A of E, if $\mathrm{D}f(a)$ is defined for every $a \in A$, then we have a mapping $\mathrm{D}f\colon A \to \mathcal{L}(\overrightarrow{E}; \overrightarrow{F})$. Since $\mathcal{L}(\overrightarrow{E}; \overrightarrow{F})$ is a normed vector space, if $\mathrm{D}f$ exists on an open subset U of A containing a, we can consider taking the derivative of $\mathrm{D}f$ at some $a \in A$.

Definition 39.12. Given a function $f\colon A \to F$ defined on some open subset A of E such that $\mathrm{D}f(a)$ is defined for every $a \in A$, If $\mathrm{D}(\mathrm{D}f)(a)$ exists for every $a \in A$, we get a mapping $\mathrm{D}^2 f\colon A \to \mathcal{L}(\overrightarrow{E}; \mathcal{L}(\overrightarrow{E}; \overrightarrow{F}))$ called the *second derivative of f on A*, where $\mathrm{D}^2 f(a) = \mathrm{D}(\mathrm{D}f)(a)$, for every $a \in A$.

As in the case of the first derivative $\mathrm{D}f_a$ where $\mathrm{D}f_a(u) = \mathrm{D}_u f(a)$, where $\mathrm{D}_u f(a)$ is the directional derivative of f at a in the direction u, it would be useful to express $\mathrm{D}^2 f(a)(u)(v)$ in terms of two directional derivatives. This can indeed be done. If $\mathrm{D}^2 f(a)$ exists, then for every $u \in \overrightarrow{E}$,

$$\mathrm{D}^2 f(a)(u) = \mathrm{D}(\mathrm{D}f)(a)(u) = \mathrm{D}_u(\mathrm{D}f)(a) \in \mathcal{L}(\overrightarrow{E}; \overrightarrow{F}).$$

We have the following result.

Proposition 39.19. *If $\mathrm{D}^2 f(a)$ exists, then $\mathrm{D}_u(\mathrm{D}_v f)(a)$ exists and*

$$\mathrm{D}^2 f(a)(u)(v) = \mathrm{D}_u(\mathrm{D}_v f)(a), \quad \textit{for all } u, v \in \overrightarrow{E}.$$

Proof. Recall from Proposition 37.61, that the map app from $\mathcal{L}(\overrightarrow{E}; \overrightarrow{F}) \times \overrightarrow{E}$ to $\overrightarrow{F}$, defined such that for every $L \in \mathcal{L}(\overrightarrow{E}; \overrightarrow{F})$, for every $v \in \overrightarrow{E}$,

$$\mathrm{app}(L, v) = L(v),$$

is a continuous bilinear map. Thus, in particular, given a fixed $v \in \overrightarrow{E}$, the linear map $\mathrm{app}_v\colon \mathcal{L}(\overrightarrow{E}; \overrightarrow{F}) \to \overrightarrow{F}$, defined such that $\mathrm{app}_v(L) = L(v)$, is a continuous map.

Also recall from Proposition 39.7, that if $h\colon A \to G$ is a function such that $\mathrm{D}h(a)$ exits, and $k\colon G \to H$ is a continuous linear map, then, $\mathrm{D}(k \circ h)(a)$ exists, and

$$k(\mathrm{D}h(a)(u)) = \mathrm{D}(k \circ h)(a)(u),$$

that is,

$$k(\mathrm{D}_u h(a)) = \mathrm{D}_u(k \circ h)(a),$$

Applying these two facts to $h = \mathrm{D}f$, and to $k = \mathrm{app}_v$, we have

$$\mathrm{D}_u(\mathrm{D}f)(a)(v) = \mathrm{D}_u(\mathrm{app}_v \circ \mathrm{D}f)(a).$$

But $(\mathrm{app}_v \circ \mathrm{D}f)(x) = \mathrm{D}f(x)(v) = \mathrm{D}_v f(x)$, for every $x \in A$, that is, $\mathrm{app}_v \circ \mathrm{D}f = \mathrm{D}_v f$ on A. So, we have

$$\mathrm{D}_u(\mathrm{D}f)(a)(v) = \mathrm{D}_u(\mathrm{D}_v f)(a),$$