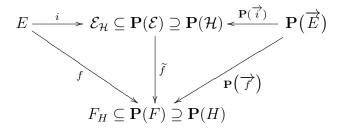
affine patch), there is a unique projective map $\widetilde{f} \colon \mathbf{P}(\mathcal{E}) \to \mathbf{P}(F)$ such that

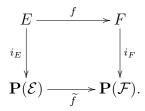
$$f = \widetilde{f} \circ i$$
 and $\mathbf{P}(\overrightarrow{f}) = \widetilde{f} \circ \mathbf{P}(\overrightarrow{i})$

(where $\overrightarrow{i}: \overrightarrow{E} \to \mathcal{H}$ and $\overrightarrow{f}: \overrightarrow{E} \to H$ are the linear maps associated with the affine maps $i: E \to \mathbf{P}(\mathcal{E})$ and $f: E \to \mathbf{P}(F)$), as in the following diagram:



The points of $\mathbf{P}(\mathcal{E})$ in $\mathbf{P}(\mathcal{H})$ are called *points at infinity*, and the projective hyperplane $\mathbf{P}(\mathcal{H})$ is called the *hyperplane at infinity*. We will also denote the point $[u]_{\sim}$ of $\mathbf{P}(\mathcal{H})$ (where $u \neq 0$) by u_{∞} . As usual, objects defined by a universal property are unique up to isomorphism. We leave the proof as an exercise.

The importance of the notion of projective completion stems from the fact that every affine map $f: E \to F$ extends in a unique way to a projective map $\widetilde{f}: \mathbf{P}(\mathcal{E}) \to \mathbf{P}(\mathcal{F})$, where $\langle \mathbf{P}(\mathcal{E}), \mathbf{P}(\mathcal{H}_E), i_E \rangle$ is a projective completion of E and $\langle \mathbf{P}(\mathcal{F}), \mathbf{P}(\mathcal{H}_F), i_F \rangle$ is a projective completion of F, provided that the restriction of \widetilde{f} to $\mathbf{P}(\overrightarrow{E})$ agrees with $\mathbf{P}(\overrightarrow{f})$, as illustrated in the following commutative diagram:



We will now show that $\langle \widetilde{E}, \mathbf{P}(\overrightarrow{E}), i \rangle$ is the projective completion of E, where $i \colon E \to \widetilde{E}$ is the injection of E into $\widetilde{E} = E \cup \mathbf{P}(\overrightarrow{E})$. For example, if $E = \mathbb{A}^1_K$ is an affine line, its projective completion $\widetilde{\mathbb{A}^1_K}$ is isomorphic to the projective line $\mathbf{P}(K^2)$, and they both can be identified with $\mathbb{A}^1_K \cup \{\infty\}$, the result of adding a point at infinity (∞) to \mathbb{A}^1_K . In general, the projective completion $\widetilde{\mathbb{A}^m_K}$ of the affine space \mathbb{A}^m_K is isomorphic to $\mathbb{P}(K^{m+1})$. Thus, $\widetilde{\mathbb{A}^m}$ is isomorphic to \mathbb{RP}^m , and $\widetilde{\mathbb{A}^m_K}$ is isomorphic to \mathbb{CP}^m .

First, let us observe that if E is a vector space and H is a hyperplane in E, then the homogenization \widehat{E}_H of the affine patch E_H (the complement of the projective hyperplane $\mathbf{P}(H)$ in $\mathbf{P}(E)$) is isomorphic to E. The proof is rather simple and uses the fact that there