If $(e_1, \ldots, e_m)$ is a basis of E, then for every basis element $(e_{i_1}^*)^{\odot n_1} \odot \cdots \odot (e_{i_k}^*)^{\odot n_k}$ of $\mathrm{S}^n(E^*)$, with $n_1 + \cdots + n_k = n$, we have

$$\mu((e_{i_1}^*)^{\odot n_1} \odot \cdots \odot (e_{i_k}^*)^{\odot n_k})(\underbrace{e_{i_1}, \ldots, e_{i_1}}_{n_1} \ldots, \underbrace{e_{i_k}, \ldots, e_{i_k}}_{n_k}) = n_1! \cdots n_k!,$$

If the field K has positive characteristic, then it is possible that $n_1! \cdots n_k! = 0$, and this is why we required K to be of characteristic 0 in order for Proposition 33.30 to hold.

2. The canonical isomorphism of Proposition 33.30 holds under more general conditions. Namely, that K is a commutative algebra with identity over $\mathbb{Q}$, and that the E is a finitely-generated projective K-module (see Definition 35.7). See Bourbaki, [25] (Chapter III, §11, Section 5, Proposition 8).

The map from E^n to $\mathrm{S}^n(E)$ given by $(u_1, \ldots, u_n) \mapsto u_1 \odot \cdots \odot u_n$ yields a surjection $\pi\colon E^{\otimes n} \to \mathrm{S}^n(E)$. Because we are dealing with vector spaces, this map has some section; that is, there is some injection $\eta\colon \mathrm{S}^n(E) \to E^{\otimes n}$ with $\pi \circ \eta = \mathrm{id}$. Since our field K has characteristic 0, there is a special section having a natural definition involving a symmetrization process defined as follows: For every permutation σ, we have the map $r_\sigma\colon E^n \to E^{\otimes n}$ given by

$$r_\sigma(u_1, \ldots, u_n) = u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)}.$$

As r_σ is clearly multilinear, r_σ extends to a linear map $(r_\sigma)_\otimes\colon E^{\otimes n} \to E^{\otimes n}$ making the following diagram commute

$$\begin{tikzcd} E^n \arrow[r, "\iota_\otimes"] \arrow[dr, "r_\sigma"'] & E^{\otimes n} \arrow[d, "(r_\sigma)_\otimes"] \\ & E^{\otimes n}, \end{tikzcd}$$

and we get a map $\mathfrak{S}_n \times E^{\otimes n} \longrightarrow E^{\otimes n}$, namely

$$\sigma \cdot z = (r_\sigma)_\otimes(z).$$

It is immediately checked that this is a left action of the symmetric group $\mathfrak{S}_n$ on $E^{\otimes n}$, and the tensors $z \in E^{\otimes n}$ such that

$$\sigma \cdot z = z, \quad \text{for all} \quad \sigma \in \mathfrak{S}_n$$

are called *symmetrized* tensors.

We define the map $\eta\colon E^n \to E^{\otimes n}$ by

$$\eta(u_1, \ldots, u_n) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \sigma \cdot (u_1 \otimes \cdots \otimes u_n) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)}.$$