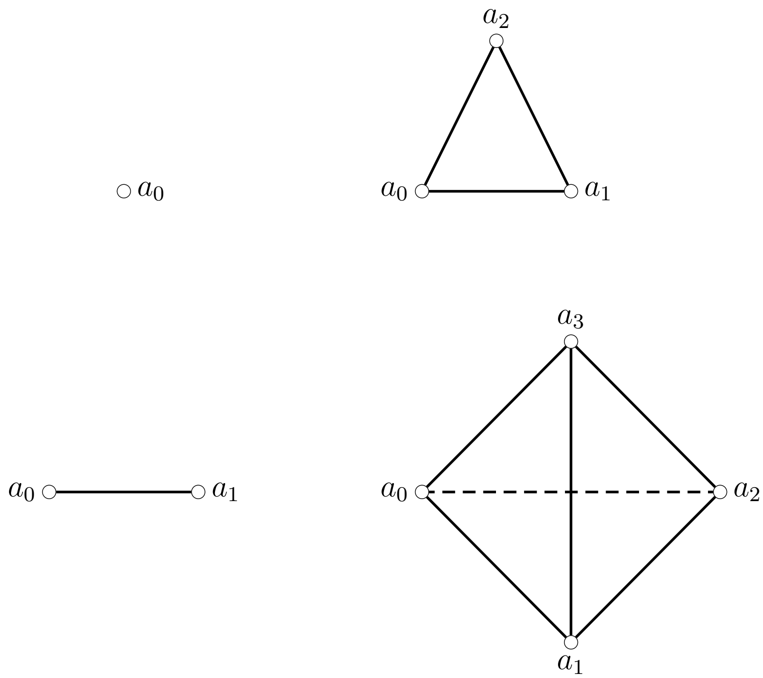

Figure 24.16: Examples of affine frames and their convex hulls.

Even though Proposition 24.6 is rather dull, it is one of the key ingredients in the proof of beautiful and deep theorems about convex sets, such as Carathéodory's theorem, Radon's theorem, and Helly's theorem.

A family of two points (a, b) in E is affinely independent iff $\overrightarrow{ab} \neq 0$, iff $a \neq b$. If $a \neq b$, the affine subspace generated by a and b is the set of all points $(1-\lambda)a + \lambda b$, which is the unique line passing through a and b. A family of three points (a, b, c) in E is affinely independent iff $\overrightarrow{ab}$ and $\overrightarrow{ac}$ are linearly independent, which means that a, b, and c are not on the same line (they are not collinear). In this case, the affine subspace generated by (a, b, c) is the set of all points $(1 - \lambda - \mu)a + \lambda b + \mu c$, which is the unique plane containing a, b, and c. A family of four points (a, b, c, d) in E is affinely independent iff $\overrightarrow{ab}$, $\overrightarrow{ac}$, and $\overrightarrow{ad}$ are linearly independent, which means that a, b, c, and d are not in the same plane (they are not coplanar). In this case, a, b, c, and d are the vertices of a tetrahedron. Figure 24.16 shows affine frames and their convex hulls for $|I| = 0, 1, 2, 3$.

Given $n+1$ affinely independent points $(a_0, \ldots, a_n)$ in E, we can consider the set of points $\lambda_0 a_0 + \cdots + \lambda_n a_n$, where $\lambda_0 + \cdots + \lambda_n = 1$ and $\lambda_i \geq 0$ ($\lambda_i \in \mathbb{R}$). Such affine combinations are called *convex combinations*. This set is called the *convex hull* of $(a_0, \ldots, a_n)$ (or *n-simplex spanned by* $(a_0, \ldots, a_n)$). When $n = 1$, we get the segment between a_0 and a_1, including a_0 and a_1. When $n = 2$, we get the interior of the triangle whose vertices are a_0, a_1, a_2, including boundary points (the edges). When $n = 3$, we get the interior of the tetrahedron