A single point is projectively independent. Two points a, b are projectively independent if $a \neq b$. Two distinct points define a (unique) projective line. Three points a, b, c are projectively independent if they are distinct, and neither belongs to the projective line defined by the other two. Three projectively independent points define a (unique) projective plane.

A closer look at projective subspaces will show some of the advantages of projective geometry: In considering intersection properties, there are no exceptions due to parallelism, as in affine spaces.

Let E be a nontrivial vector space. Given any nontrivial subset S of E, the subset S defines a subset $U = p(S - \{0\})$ of the projective space $\mathbf{P}(E)$, and if $\langle S \rangle$ denotes the subspace of E spanned by S, it is immediately verified that $\mathbf{P}(\langle S \rangle)$ is the intersection of all projective subspaces containing U, and this projective subspace is denoted by $\langle U \rangle$. Then $n \geq 2$ point $a_1, \ldots, a_n \in \mathbf{P}(E)$ are projectively independent iff for all $i = 1, \ldots, n$ the point a_i does not belong to the projective subspace $\langle a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \rangle$ spanned by $\{a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n\}$.

Given any subspaces M and N of E, recall from Proposition 24.15 that we have the *Grassmann relation*

$$\dim(M) + \dim(N) = \dim(M + N) + \dim(M \cap N).$$

Then the following proposition is easily shown.

Proposition 26.1. *Given a projective space $\mathbf{P}(E)$, for any two projective subspaces U, V of $\mathbf{P}(E)$, we have*

$$\dim(U) + \dim(V) = \dim(\langle U \cup V \rangle) + \dim(U \cap V).$$

Furthermore, if $\dim(U) + \dim(V) \geq \dim(\mathbf{P}(E))$, then $U \cap V$ is nonempty and if $\dim(\mathbf{P}(E)) = n$, then:

(i) The intersection of any n hyperplanes is nonempty.

(ii) For every hyperplane H and every point $a \notin H$, every line D containing a intersects H in a unique point.

(iii) In a projective plane, every two distinct lines intersect in a unique point.

As a corollary, in 3D projective space ($\dim(\mathbf{P}(E)) = 3$), for every plane H, every line not contained in H intersects H in a unique point.

It is often useful to deal with projective hyperplanes in terms of nonnull linear forms and equations. Recall that the map

$$[f]_\sim \mapsto \operatorname{Ker} f$$