Problem 22.9. Let A be a real $n \times n$ matrix.

(1) Assume A is invertible. Prove that if $A = Q_1S_1 = Q_2S_2$ are two polar decompositions of A, then $Q_1 = Q_2$ and $S_1 = S_2$.

Hint. $A^\top A = S_1^2 = S_2^2$, with S_1 and S_2 symmetric positive definite. Then use Problem 17.7.

(2) Now assume that A is singular. Prove that if $A = Q_1S_1 = Q_2S_2$ are two polar decompositions of A, then $S_1 = S_2$, but Q_1 may not be equal to Q_2.

Problem 22.10. (1) Let A be any invertible (real) $n \times n$ matrix. Prove that for every SVD, $A = VDU^\top$ of A, the product $VU^\top$ is the same (i.e., if $V_1DU_1^\top = V_2DU_2^\top$, then $V_1U_1^\top = V_2U_2^\top$). What does $VU^\top$ have to do with the polar form of A?

(2) Given any invertible (real) $n \times n$ matrix, A, prove that there is a unique orthogonal matrix, $Q \in \mathbf{O}(n)$, such that $\|A - Q\|_F$ is minimal (under the Frobenius norm). In fact, prove that $Q = VU^\top$, where $A = VDU^\top$ is an SVD of A. Moreover, if $\det(A) > 0$, show that $Q \in \mathbf{SO}(n)$.

What can you say if A is singular (i.e., non-invertible)?

Problem 22.11. (1) Prove that for any $n \times n$ matrix A and any orthogonal matrix Q, we have

$$\max\{\mathrm{tr}(QA) \mid Q \in \mathbf{O}(n)\} = \sigma_1 + \cdots + \sigma_n,$$

where $\sigma_1 \geq \cdots \geq \sigma_n$ are the singular values of A. Furthermore, this maximum is achieved by $Q = UV^\top$, where $A = V\Sigma U^\top$ is any SVD for A.

(2) By applying the above result with $A = X^\top Z$ and $Q = R$, deduce the following result: for any two fixed $n \times k$ matrices X and Z, the minimum of the set

$$\{\|X - ZR\|_F \mid R \in \mathbf{O}(k)\}$$

is achieved by $R = UV^\top$ for any SVD decomposition $V\Sigma U^\top = X^\top Z$ of $X^\top Z$.

Remark: The problem of finding an orthogonal matrix R such that ZR comes as close as possible to X is called the *orthogonal Procrustes problem*; see Strang [171] (Section IV.9) for the history of this problem.