Problem 41.6. A method for computing the nth root $x^{1/n}$ of a positive real number $x \in \mathbb{R}$, with $n \in \mathbb{N}$ a positive integer $n \geq 2$, proceeds as follows: define the sequence (x_k), where x_0 is any chosen positive real, and

$$x_{k+1} = \frac{1}{n}\left((n-1)x_k + \frac{x}{x_k^{n-1}}\right), \quad k \geq 0.$$

(1) Implement the above method in `Matlab` and test it for various input values of x, x_0, and of $n \geq 2$, by running successively your program for $m = 2, 3, \ldots, 100$ iterations. Have your program plot the points (i, x_i) to watch how quickly the sequence converges.

Experiment with various choices of x_0. One of these choices should be $x_0 = x$. Compare your answers with the result of applying the of `Matlab` function $x \mapsto x^{1/n}$.

In some case, when x_0 is small, the number of iterations has to be at least 1000. Exhibit this behavior.

Problem 41.7. Refer to Problem 41.6 for the definition of the sequence (x_k).

(1) Define the *relative error* ϵ_k as

$$\epsilon_k = \frac{x_k}{x^{1/n}} - 1, \quad k \geq 0.$$

Prove that

$$\epsilon_{k+1} = \frac{x^{(1-1/n)}}{n x_k^{n-1}}\left(\frac{(n-1)x_k^n}{x} - \frac{n x_k^{n-1}}{x^{(1-1/n)}} + 1\right),$$

and then that

$$\epsilon_{k+1} = \frac{1}{n(\epsilon_k+1)^{n-1}}\left(\epsilon_k(\epsilon_k+1)^{n-2}((n-1)\epsilon_k + (n-2)) + 1 - (\epsilon_k+1)^{n-2}\right),$$

for all $k \geq 0$.

(2) Since

$$\epsilon_k + 1 = \frac{x_k}{x^{1/n}},$$

and since we assumed $x_0, x > 0$, we have $\epsilon_0 + 1 > 0$. We would like to prove that

$$\epsilon_k \geq 0, \quad \text{for all} \quad k \geq 1.$$

For this consider the variations of the function f given by

$$f(u) = (n-1)u^n - n x^{1/n} u^{n-1} + x,$$

for $u \in \mathbb{R}$.

Use the above to prove that $f(u) \geq 0$ for all $u \geq 0$. Conclude that

$$\epsilon_k \geq 0, \quad \text{for all} \quad k \geq 1.$$