3. The map $D\colon \mathbb{R}[X] \to \mathbb{R}[X]$ defined such that

$$D(f(X)) = f'(X),$$

where $f'(X)$ is the derivative of the polynomial $f(X)$, is a linear map.

4. The map $\Phi\colon \mathcal{C}([a,b]) \to \mathbb{R}$ given by

$$\Phi(f) = \int_a^b f(t)dt,$$

where $\mathcal{C}([a,b])$ is the set of continuous functions defined on the interval $[a,b]$, is a linear map.

5. The function $\langle -,- \rangle\colon \mathcal{C}([a,b]) \times \mathcal{C}([a,b]) \to \mathbb{R}$ given by

$$\langle f, g\rangle = \int_a^b f(t)g(t)dt,$$

is linear in each of the variable $f$, $g$. It also satisfies the properties $\langle f, g\rangle = \langle g, f\rangle$ and $\langle f, f\rangle = 0$ iff $f = 0$. It is an example of an *inner product*.

**Definition 3.19.** Given a linear map $f\colon E \to F$, we define its *image (or range)* $\operatorname{Im} f = f(E)$, as the set

$$\operatorname{Im} f = \{y \in F \mid (\exists x \in E)(y = f(x))\},$$

and its *Kernel (or nullspace)* $\operatorname{Ker} f = f^{-1}(0)$, as the set

$$\operatorname{Ker} f = \{x \in E \mid f(x) = 0\}.$$

The derivative map $D\colon \mathbb{R}[X] \to \mathbb{R}[X]$ from Example 3.6(3) has kernel the constant polynomials, so $\operatorname{Ker} D = \mathbb{R}$. If we consider the second derivative $D \circ D\colon \mathbb{R}[X] \to \mathbb{R}[X]$, then the kernel of $D \circ D$ consists of all polynomials of degree $\leq 1$. The image of $D\colon \mathbb{R}[X] \to \mathbb{R}[X]$ is actually $\mathbb{R}[X]$ itself, because every polynomial $P(X) = a_0X^n + \cdots + a_{n-1}X + a_n$ of degree $n$ is the derivative of the polynomial $Q(X)$ of degree $n+1$ given by

$$Q(X) = a_0\frac{X^{n+1}}{n+1} + \cdots + a_{n-1}\frac{X^2}{2} + a_nX.$$

On the other hand, if we consider the restriction of $D$ to the vector space $\mathbb{R}[X]_n$ of polynomials of degree $\leq n$, then the kernel of $D$ is still $\mathbb{R}$, but the image of $D$ is the $\mathbb{R}[X]_{n-1}$, the vector space of polynomials of degree $\leq n-1$.

**Proposition 3.17.** *Given a linear map $f\colon E \to F$, the set $\operatorname{Im} f$ is a subspace of $F$ and the set $\operatorname{Ker} f$ is a subspace of $E$. The linear map $f\colon E \to F$ is injective iff $\operatorname{Ker} f = (0)$ (where $(0)$ is the trivial subspace $\{0\}$).*