Proof. If $\sum_{k=0}^{\infty} u_k$ is absolutely convergent, then we prove that the sequence (S_m) is a Cauchy sequence; that is, for every $\epsilon > 0$, there is some $p > 0$ such that for all $n \geq m \geq p$,

$$\|S_n - S_m\| \leq \epsilon.$$

Observe that

$$\|S_n - S_m\| = \|u_{m+1} + \cdots + u_n\| \leq \|u_{m+1}\| + \cdots + \|u_n\|,$$

and since the sequence $\sum_{k=0}^{\infty} \|u_k\|$ converges, it satisfies Cauchy's criterion. Thus, the sequence (S_m) also satisfies Cauchy's criterion, and since E is a complete vector space, the sequence (S_m) converges. $\square$

Remark: It can be shown that if $(E, \| \|)$ is a normed vector space such that every absolutely convergent series is also convergent, then E must be complete (see Schwartz [150]).

An important corollary of absolute convergence is that if the terms in series $\sum_{k=0}^{\infty} u_k$ are rearranged, then the resulting series is still absolutely convergent and has the *same sum*. More precisely, let σ be any permutation (bijection) of the natural numbers. The series $\sum_{k=0}^{\infty} u_{\sigma(k)}$ is called a *rearrangement* of the original series. The following result can be shown (see Schwartz [150]).

Proposition 9.19. *Assume $(E, \| \|)$ is a normed vector space. If a series $\sum_{k=0}^{\infty} u_k$ is convergent as well as absolutely convergent, then for every permutation σ of $\mathbb{N}$, the series $\sum_{k=0}^{\infty} u_{\sigma(k)}$ is convergent and absolutely convergent, and its sum is equal to the sum of the original series:*

$$\sum_{k=0}^{\infty} u_{\sigma(k)} = \sum_{k=0}^{\infty} u_k.$$

In particular, if $(E, \| \|)$ is a complete normed vector space, then Proposition 9.19 holds.

We now apply Proposition 9.18 to the matrix exponential.

9.8 The Matrix Exponential

Proposition 9.20. *For any $n \times n$ real or complex matrix A, the series*

$$\sum_{k=0}^{\infty} \frac{A^k}{k!}$$

converges absolutely for any operator norm on $\mathrm{M}_n(\mathbb{C})$ (or $\mathrm{M}_n(\mathbb{R})$).