Proposition 14.16. *Given any $n \times n$ complex matrix A, if A is invertible, then there is a unitary matrix U and an upper triangular matrix R with positive diagonal entries such that $A = UR$.*

The proof is absolutely the same as in the real case!

Remark: If A is invertible and if $A = U_1R_1 = U_2R_2$ are two QR-decompositions for A, then

$$R_1R_2^{-1} = U_1^*U_2.$$

Then it is easy to show that there is a diagonal matrix D with diagonal entries such that $|d_{ii}| = 1$ for $i = 1, \ldots, n$, and $U_2 = U_1D$, $R_2 = D^*R_1$.

We have the following version of the Hadamard inequality for complex matrices. The proof is essentially the same as in the Euclidean case but it uses Proposition 14.16 instead of Proposition 12.16.

Proposition 14.17. *(Hadamard) For any complex $n \times n$ matrix $A = (a_{ij})$, we have*

$$|\det(A)| \leq \prod_{i=1}^{n}\left(\sum_{j=1}^{n}|a_{ij}|^2\right)^{1/2} \quad \textit{and} \quad |\det(A)| \leq \prod_{j=1}^{n}\left(\sum_{i=1}^{n}|a_{ij}|^2\right)^{1/2}.$$

Moreover, equality holds iff either A has orthogonal rows in the left inequality or orthogonal columns in the right inequality.

We also have the following version of Proposition 12.18 for Hermitian matrices. The proof of Proposition 12.18 goes through because the Cholesky decomposition for a Hermitian positive definite A matrix holds in the form $A = B^*B$, where B is upper triangular with positive diagonal entries. The details are left to the reader.

Proposition 14.18. *(Hadamard) For any complex $n \times n$ matrix $A = (a_{ij})$, if A is Hermitian positive semidefinite, then we have*

$$\det(A) \leq \prod_{i=1}^{n} a_{ii}.$$

Moreover, if A is positive definite, then equality holds iff A is a diagonal matrix.

14.5 Hermitian Reflections and QR-Decomposition

If A is an $n \times n$ complex singular matrix, there is some (not necessarily unique) QR-decomposition $A = QR$ with Q a unitary matrix which is a product of Householder reflections and R an upper triangular matrix, but the proof is more involved. One way to proceed is to generalize the notion of hyperplane reflection. This is not really surprising since in the Hermitian case there are improper isometries whose determinant can be any unit complex number. Hyperplane reflections are generalized as follows.