and it is easy to see that they are linearly independent. Therefore, the space U of linear forms in E^* spanned by the above linear forms (equations) has dimension $n-1$, and the space U^0 of matrices satisfying all these equations has dimension n^2-n+1. It is not so obvious to find a basis for this space.

We will now pin down the relationship between a vector space E and its bidual E^{**}.

11.4 The Bidual and Canonical Pairings

Proposition 11.5. *Let E be a vector space. The following properties hold:*

(a) The linear map $\mathrm{eval}_E\colon E \to E^{**}$ *defined such that*

$$\mathrm{eval}_E(v) = \mathrm{eval}_v \quad \textit{for all } v \in E,$$

that is, $\mathrm{eval}_E(v)(u^*) = \langle u^*, v\rangle = u^*(v)$ *for every $u^* \in E^*$, is injective.*

(b) When E is of finite dimension n, the linear map $\mathrm{eval}_E\colon E \to E^{**}$ *is an isomorphism (called the canonical isomorphism).*

Proof. (a) Let $(u_i)_{i\in I}$ be a basis of E, and let $v = \sum_{i\in I} v_i u_i$. If $\mathrm{eval}_E(v) = 0$, then in particular $\mathrm{eval}_E(v)(u_i^*) = 0$ for all u_i^*, and since

$$\mathrm{eval}_E(v)(u_i^*) = \langle u_i^*, v\rangle = v_i,$$

we have $v_i = 0$ for all $i \in I$, that is, $v = 0$, showing that $\mathrm{eval}_E\colon E \to E^{**}$ is injective.

If E is of finite dimension n, by Theorem 11.4, for every basis $(u_1, \ldots, u_n)$, the family $(u_1^*, \ldots, u_n^*)$ is a basis of the dual space E^*, and thus the family $(u_1^{**}, \ldots, u_n^{**})$ is a basis of the bidual E^{**}. This shows that $\dim(E) = \dim(E^{**}) = n$, and since by Part (a), we know that $\mathrm{eval}_E\colon E \to E^{**}$ is injective, in fact, $\mathrm{eval}_E\colon E \to E^{**}$ is bijective (by Proposition 6.19). $\square$

When a vector space E has infinite dimension, E and its bidual E^{**} are never isomorphic.

When E is of finite dimension and $(u_1, \ldots, u_n)$ is a basis of E, in view of the canonical isomorphism $\mathrm{eval}_E\colon E \to E^{**}$, the basis $(u_1^{**}, \ldots, u_n^{**})$ of the bidual is *identified* with $(u_1, \ldots, u_n)$.

Proposition 11.5 can be reformulated very fruitfully in terms of pairings, a remarkably useful concept discovered by Pontrjagin in 1931 (adapted from E. Artin [6], Chapter 1). Given two vector spaces E and F over a field K, we say that a function $\varphi\colon E \times F \to K$ is *bilinear* if for every $v \in V$, the map $u \mapsto \varphi(u, v)$ (from E to K) is linear, and for every $u \in E$, the map $v \mapsto \varphi(u, v)$ (from F to K) is linear.

Definition 11.4. Given two vector spaces E and F over K, a *pairing between E and F* is a bilinear map $\varphi\colon E \times F \to K$. Such a pairing is *nondegenerate* iff