We also review the concept of limit of a sequence. Given any set $E$, a *sequence* is any function $x\colon \mathbb{N} \to E$, usually denoted by $(x_n)_{n\in\mathbb{N}}$, or $(x_n)_{n\geq 0}$, or even by $(x_n)$.

**Definition 37.19.** Given a topological space $(E, \mathcal{O})$, we say that *a sequence* $(x_n)_{n\in\mathbb{N}}$ *converges to some* $a \in E$ if for every open set $U$ containing $a$, there is some $n_0 \geq 0$, such that, $x_n \in U$, for all $n \geq n_0$. We also say that *a is a limit of* $(x_n)_{n\in\mathbb{N}}$. See Figure 37.20.

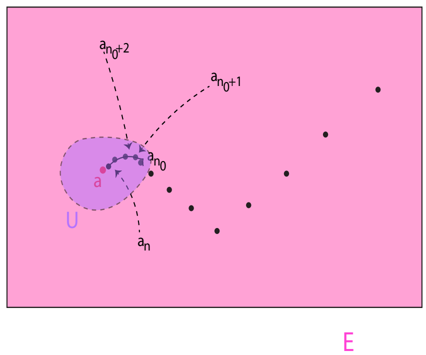


Figure 37.20: A schematic illustration of Definition 37.19.

When $E$ is a metric space with metric $d$, it is easy to show that this is equivalent to the fact that,

for every $\epsilon > 0$, there is some $n_0 \geq 0$, such that, $d(x_n, a) \leq \epsilon$, for all $n \geq n_0$.

When $E$ is a normed vector space with norm $\| \;\|$, it is easy to show that this is equivalent to the fact that,

for every $\epsilon > 0$, there is some $n_0 \geq 0$, such that, $\|x_n - a\| \leq \epsilon$, for all $n \geq n_0$.

The following proposition shows the importance of the Hausdorff separation axiom.

**Proposition 37.12.** *Given a topological space* $(E, \mathcal{O})$*, if the Hausdorff separation axiom holds, then every sequence has at most one limit.*

*Proof.* Left as an exercise. ☐

It is worth noting that the notion of limit is topological, in the sense that a sequence converge to a limit $b$ iff it converges to the same limit $b$ in any equivalent metric (and similarly for equivalent norms).

If $E$ is a metric space and if $A$ is a subset of $E$, there is a convenient way of showing that a point $x \in E$ belongs to the closure $\overline{A}$ of $A$ in terms of sequences.

**Proposition 37.13.** *Given any metric space* $(E, d)$*, for any subset* $A$ *of* $E$ *and any point* $x \in E$*, we have* $x \in \overline{A}$ *iff there is a sequence* $(a_n)$ *of points* $a_n \in A$ *converging to* $x$*.*