called *modes* (or *normal modes*). Complete solutions of the problem are series obtained by combining the normal modes, and they are of the form

$$u(x,t) = \sum_{k=1}^{\infty} \sin\left(\frac{k\pi x}{L}\right)\left(A_k \cos\left(\frac{k\pi ct}{L}\right) + B_k \sin\left(\frac{k\pi ct}{L}\right)\right),$$

where the coefficients A_k, B_k are determined from the Fourier series of $u_{i,0}$ and $u_{i,1}$.

We now consider discrete approximations of our problem. As before, consider a finite dimensional subspace V_a of V and assume that we have approximations $u_{a,0}$ and $u_{a,1}$ of $u_{i,0}$ and $u_{i,1}$. If we pick a basis $(w_1, \ldots, w_n)$ of V_a, then we can write our unknown function $u(x,t)$ as

$$u(x,t) = u_1(t)w_1 + \cdots + u_n(t)w_n,$$

where $u_1, \ldots, u_n$ are functions of t. Then, if we write $\mathbf{u} = (u_1, \ldots, u_n)$, the discrete version of our problem is

$$\begin{aligned} A\frac{d^2\mathbf{u}}{dt^2} + K\mathbf{u} &= 0, \\ u(x,0) &= u_{a,0}(x), \quad 0 \leq x \leq L, \\ \frac{\partial u}{\partial t}(x,0) &= u_{a,1}(x), \quad 0 \leq x \leq L, \end{aligned}$$

where $A = (\langle w_i, w_j\rangle)$ and $K = (a(w_i, w_j))$ are two symmetric matrices, called the *mass matrix* and the *stiffness matrix*, respectively. In fact, because a and the inner product $\langle -, -\rangle$ are positive definite, these matrices are also positive definite.

We have made some progress since we now have a system of ODE's, and we can solve it by analogy with the scalar case. So, we look for solutions of the form $\mathbf{U}\cos\omega t$ (or $\mathbf{U}\sin\omega t$), where $\mathbf{U}$ is an n-dimensional vector. We find that we should have

$$(K - \omega^2 A)\mathbf{U}\cos\omega t = 0,$$

which implies that ω must be a solution of the equation

$$K\mathbf{U} = \omega^2 A\mathbf{U}.$$

Thus, we have to find some λ such that

$$K\mathbf{U} = \lambda A\mathbf{U},$$

a problem known as a *generalized eigenvalue problem*, since the ordinary eigenvalue problem for K is

$$K\mathbf{U} = \lambda\mathbf{U}.$$