The case of a real function suggests the following method for finding the zeros of a function $f \colon \Omega \to Y$, with $\Omega \subseteq X$: given a starting point $x_0 \in \Omega$, the sequence (x_k) is defined by

$$x_{k+1} = x_k - (f'(x_k))^{-1}(f(x_k)) \tag{$*$}$$

for all $k \geq 0$.

For the above to make sense, it must be ensured that

(1) All the points x_k remain within Ω.

(2) The function f is differentiable within Ω.

(3) The derivative $f'(x)$ is a bijection from X to Y for all $x \in \Omega$.

These are rather demanding conditions but there are sufficient conditions that guarantee that they are met. Another practical issue is that it may be very costly to compute $(f'(x_k))^{-1}$ at every iteration step. In the next section we investigate generalizations of Newton's method which address the issues that we just discussed.

41.2 Generalizations of Newton's Method

Suppose that $f \colon \Omega \to \mathbb{R}^n$ is given by n functions $f_i \colon \Omega \to \mathbb{R}$, where $\Omega \subseteq \mathbb{R}^n$. In this case, finding a zero a of f is equivalent to solving the system

$$\begin{aligned} f_1(a_1 \ldots, a_n) &= 0 \\ f_2(a_1 \ldots, a_n) &= 0 \\ &\vdots \\ f_n(a_1 \ldots, a_n) &= 0. \end{aligned}$$

In the standard Newton method, the iteration step is given by $(*)$, namely

$$x_{k+1} = x_k - (f'(x_k))^{-1}(f(x_k)),$$

and if we define Δx_k as $\Delta x_k = x_{k+1} - x_k$, we see that $\Delta x_k = -(f'(x_k))^{-1}(f(x_k))$, so Δx_k is obtained by solving the equation

$$f'(x_k)\Delta x_k = -f(x_k),$$

and then we set $x_{k+1} = x_k + \Delta x_k$.

The generalization is as follows.

Variant 1. A single iteration of Newton's method consists in solving the linear system

$$(J(f)(x_k))\Delta x_k = -f(x_k),$$