Proposition 35.39. *Given a ring homomomorphism $\rho\colon A \to B$ and given any A-module M, the map $\varphi\colon M \to \rho_*(\rho^*(M))$ given by $\varphi(x) = 1 \otimes_A x$ is A-linear and $\varphi(M)$ spans the B-module $\rho^*(M)$. For every B-module N, and for every A-linear map $f\colon M \to \rho_*(N)$, there is a unique B-linear map*

$$\overline{f}\colon \rho^*(M) \to N$$

such that

$$\rho_*(\overline{f}) \circ \varphi = f$$

as in the following commutative diagram

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[dr, "f"'] & \rho_*(\rho^*(M)) \arrow[d, "\rho_*(\overline{f})"] \\ & \rho_*(N) \end{tikzcd}$$

or equivalently,

$$\overline{f}(1 \otimes_A x) = f(x), \quad \text{for all } x \in M.$$

As a consequence of Proposition 35.39, we obtain the following result.

Proposition 35.40. *Given a ring homomomorphism $\rho\colon A \to B$, for any two A-modules M and N, for every A-linear map $f\colon M \to N$, there is a unique B-linear map $\rho^*(f)\colon \rho^*(M) \to \rho^*(N)$ (also denoted $\overline{f}$) given by*

$$\rho^*(f) = \mathrm{id}_B \otimes f,$$

such that the following diagam commutes:

$$\begin{tikzcd} M \arrow[r, "\varphi_M"] \arrow[d, "f"'] & \rho_*(\rho^*(M)) \arrow[d, "\rho_*(\rho^*(f))"] \\ N \arrow[r, "\varphi_N"'] & \rho_*(\rho^*(N)) \end{tikzcd}$$

Proof. Apply Proposition 35.40 to the A-linear map $\varphi_N \circ f$. $\square$

If S spans the module M, it is clear that $\varphi(S)$ spans $\rho^*(M)$. In particular, if M is finitely generated, so if $\rho^*(M)$. Bases of M also extend to bases of $\rho^*(M)$.

Proposition 35.41. *Given a ring homomomorphism $\rho\colon A \to B$, for any A-module M, if $(u_1, \ldots, u_n)$ is a basis of M, then $(\varphi(u_1), \ldots, \varphi(u_n))$ is a basis of $\rho^*(M)$, where φ is the A-linear map $\varphi\colon M \to \rho_*(\rho^*(M))$ given by $\varphi(x) = 1 \otimes_A x$. Furthermore, if ρ is injective, then so is φ.*

Proof. The first assertion follows immediately from Proposition 35.13, since it asserts that every element z of $\rho^*(M) = \rho_*(B) \otimes_A M$ can be written in a unique way as

$$z = b_1 \otimes u_1 + \cdots + b_n \otimes u_n = b_1(1 \otimes u_1) + \cdots + b_n(1 \otimes u_n),$$