and by adding and subtracting these identities, we get the parallelogram law and the equation

$$\langle u, v\rangle = \frac{1}{4}(\|u+v\|^2 - \|u-v\|^2),$$

which allows us to recover $\langle -, -\rangle$ from the norm.

Conversely, if $\| \; \|$ is a norm satisfying the parallelogram law, and if it comes from an inner product, then this inner product must be given by

$$\langle u, v\rangle = \frac{1}{4}(\|u+v\|^2 - \|u-v\|^2).$$

We need to prove that the above form is indeed symmetric and bilinear.

Symmetry holds because $\|u-v\| = \|-(u-v)\| = \|v-u\|$. Let us prove additivity in the variable u. By the parallelogram law, we have

$$2(\|x+z\|^2 + \|y\|^2) = \|x+y+z\|^2 + \|x-y+z\|^2$$

which yields

$$\begin{aligned}
\|x+y+z\|^2 &= 2(\|x+z\|^2 + \|y\|^2) - \|x-y+z\|^2 \\
\|x+y+z\|^2 &= 2(\|y+z\|^2 + \|x\|^2) - \|y-x+z\|^2,
\end{aligned}$$

where the second formula is obtained by swapping x and y. Then by adding up these equations, we get

$$\|x+y+z\|^2 = \|x\|^2 + \|y\|^2 + \|x+z\|^2 + \|y+z\|^2 - \frac{1}{2}\|x-y+z\|^2 - \frac{1}{2}\|y-x+z\|^2.$$

Replacing z by $-z$ in the above equation, we get

$$\|x+y-z\|^2 = \|x\|^2 + \|y\|^2 + \|x-z\|^2 + \|y-z\|^2 - \frac{1}{2}\|x-y-z\|^2 - \frac{1}{2}\|y-x-z\|^2,$$

Since $\|x-y+z\| = \|-(x-y+z)\| = \|y-x-z\|$ and $\|y-x+z\| = \|-(y-x+z)\| = \|x-y-z\|$, by subtracting the last two equations, we get

$$\begin{aligned}
\langle x+y, z\rangle &= \frac{1}{4}(\|x+y+z\|^2 - \|x+y-z\|^2) \\
&= \frac{1}{4}(\|x+z\|^2 - \|x-z\|^2) + \frac{1}{4}(\|y+z\|^2 - \|y-z\|^2) \\
&= \langle x, z\rangle + \langle y, z\rangle,
\end{aligned}$$

as desired.

Proving that

$$\langle \lambda x, y\rangle = \lambda\langle x, y\rangle \quad \text{for all } \lambda \in \mathbb{R}$$