where $\mathbb{Z}/2\mathbb{Z}$ is viewed as a $\mathbb{Z}$-modules, but $(1,0)$ and $(0,1)$ are not linearly independent, since

$$2(1,0)+2(0,1)=(0,0).$$

A useful fact is that every module is a quotient of some free module. Indeed, if M is an A-module, pick any spanning set I for M (such a set exists, for example, $I=M$), and consider the unique homomorphism $\varphi\colon A^{(I)}\to M$ extending the identity function from I to itself. Then we have an isomorphism $A^{(I)}/\mathrm{Ker}\,(\varphi)\approx M$.

In particular, if M is finitely generated, we can pick I to be a finite set of generators, in which case we get an isomorphism $A^n/\mathrm{Ker}\,(\varphi)\approx M$, for some natural number n. A finitely generated module is sometimes called a module of *finite type*.

The case $n=1$ is of particular interest. A module M is said to be *cyclic* if it is generated by a single element. In this case $M=Ax$, for some $x\in M$. We have the linear map $m_x\colon A\to M$ given by $a\mapsto ax$ for every $a\in A$, and it is obviously surjective since $M=Ax$. Since the kernel $\mathfrak{a}=\mathrm{Ker}\,(m_x)$ of m_x is an ideal in A, we get an isomorphism $A/\mathfrak{a}\approx Ax$. Conversely, for any ideal $\mathfrak{a}$ of A, if $M=A/\mathfrak{a}$, we see that M is generated by the image x of 1 in M, so M is a cyclic module.

The ideal $\mathfrak{a}=\mathrm{Ker}\,(m_x)$ is the set of all $a\in A$ such that $ax=0$. This is called the *annihilator* of x, and it is the special case of the following more general situation.

Definition 35.5. If M is any A-module, for any subset S of M, the set of all $a\in A$ such that $ax=0$ for all $x\in S$ is called the *annihilator* of S, and it is denoted by $\mathrm{Ann}(S)$. If $S=\{x\}$, we write $\mathrm{Ann}(x)$ instead of $\mathrm{Ann}(\{x\})$. A nonzero element $x\in M$ is called a *torsion element* iff $\mathrm{Ann}(x)\neq(0)$. The set consisting of all torsion elements in M and 0 is denoted by M_{tor}.

It is immediately verified that $\mathrm{Ann}(S)$ is an ideal of A, and by definition,

$$M_{\mathrm{tor}}=\{x\in M\mid(\exists a\in A,\ a\neq 0)(ax=0)\}.$$

If a ring has zero divisors, then the set of all torsion elements in an A-module M may not be a submodule of M. For example, if $M=A=\mathbb{Z}/6\mathbb{Z}$, then $M_{\mathrm{tor}}=\{2,3,4\}$, but $3+4=1$ is not a torsion element. Also, a free module may not be torsion-free because there may be torsion elements, as the example of $\mathbb{Z}/6\mathbb{Z}$ as a free module over itself shows.

However, if A is an integral domain, then a free module is torsion-free and M_{tor} is a submodule of M. (Recall that an integral domain is commutative).

Proposition 35.3. *If A is an integral domain, then for any A-module M, the set M_{tor} of torsion elements in M is a submodule of M.*

Proof. If $x,y\in M$ are torsion elements $(x,y\neq 0)$, then there exist some nonzero elements $a,b\in A$ such that $ax=0$ and $by=0$. Since A is an integral domain, $ab\neq 0$, and then for all $\lambda,\mu\in A$, we have

$$ab(\lambda x+\mu y)=b\lambda ax+a\mu by=0.$$