We also have the commutative diagram.

$$\begin{tikzcd}
E \times F \arrow[r, "\iota_\otimes"] \arrow[d, "(f' \circ f) \times (g' \circ g)"'] & E \otimes F \arrow[d, "(f' \circ f) \otimes (g' \circ g)"] \\
E'' \times F'' \arrow[r, "\iota''_\otimes"'] & E'' \otimes F''.
\end{tikzcd}$$

Since we immediately verify that

$$(f' \circ f) \times (g' \circ g) = (f' \times g') \circ (f \times g),$$

by uniqueness of the map between $E \otimes F$ and $E'' \otimes F''$ in the above diagram, we conclude that

$$(f' \circ f) \otimes (g' \circ g) = (f' \otimes g') \circ (f \otimes g),$$

as claimed. $\square$

The above formula $(*)$ yields the following useful fact.

Proposition 33.11. *If $f\colon E \to E'$ and $g\colon F \to F'$ are isomorphims, then $f \otimes g\colon E \otimes F \to E' \otimes F'$ is also an isomorphism.*

Proof. If $f^{-1}\colon E' \to E$ is the inverse of $f\colon E \to E'$ and $g^{-1}\colon F' \to F$ is the inverse of $g\colon F \to F'$, then $f^{-1} \otimes g^{-1}\colon E' \otimes F' \to E \otimes F$ is the inverse of $f \otimes g\colon E \otimes F \to E' \otimes F'$, which is shown as follows:

$$\begin{aligned}
(f \otimes g) \circ (f^{-1} \otimes g^{-1}) &= (f \circ f^{-1}) \otimes (g \circ g^{-1}) \\
&= \mathrm{id}_{E'} \otimes \mathrm{id}_{F'} \\
&= \mathrm{id}_{E' \otimes F'},
\end{aligned}$$

and

$$\begin{aligned}
(f^{-1} \otimes g^{-1}) \circ (f \otimes g) &= (f^{-1} \circ f) \otimes (g^{-1} \circ g) \\
&= \mathrm{id}_{E} \otimes \mathrm{id}_{F} \\
&= \mathrm{id}_{E \otimes F}.
\end{aligned}$$

Therefore, $f \otimes g\colon E \otimes F \to E' \otimes F'$ is an isomorphism. $\square$

The generalization to the tensor product $f_1 \otimes \cdots \otimes f_n$ of $n \geq 3$ linear maps $f_i\colon E_i \to F_i$ is immediate, and left to the reader.