The definition of an operator norm also implies that

$$\|I\|_{\mathrm{op}} = 1.$$

The above shows that the Frobenius norm is not a subordinate matrix norm for $n \geq 2$ (why?).

If $\| \;\|$ is a vector norm on $\mathbb{C}^n$, the operator norm $\| \;\|_{\mathrm{op}}$ that it induces applies to matrices in $\mathrm{M}_n(\mathbb{C})$. If we are careful to denote vectors and matrices so that no confusion arises, for example, by using lower case letters for vectors and upper case letters for matrices, it should be clear that $\|A\|_{\mathrm{op}}$ is the operator norm of the matrix A and that $\|x\|$ is the vector norm of x. Consequently, following common practice to alleviate notation, we will drop the subscript "op" and simply write $\|A\|$ instead of $\|A\|_{\mathrm{op}}$.

The notion of subordinate norm can be slightly generalized.

Definition 9.8. If $K = \mathbb{R}$ or $K = \mathbb{C}$, for any norm $\| \;\|$ on $\mathrm{M}_{m,n}(K)$, and for any two norms $\| \;\|_a$ on K^n and $\| \;\|_b$ on K^m, we say that the norm $\| \;\|$ is *subordinate* to the norms $\| \;\|_a$ and $\| \;\|_b$ if

$$\|Ax\|_b \leq \|A\| \, \|x\|_a \quad \text{for all } A \in \mathrm{M}_{m,n}(K) \text{ and all } x \in K^n.$$

Remark: For any norm $\| \;\|$ on $\mathbb{C}^n$, we can define the function $\| \;\|_{\mathbb{R}}$ on $\mathrm{M}_n(\mathbb{R})$ by

$$\|A\|_{\mathbb{R}} = \sup_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \frac{\|Ax\|}{\|x\|} = \sup_{\substack{x \in \mathbb{R}^n \\ \|x\| = 1}} \|Ax\|.$$

The function $A \mapsto \|A\|_{\mathbb{R}}$ is a matrix norm on $\mathrm{M}_n(\mathbb{R})$, and

$$\|A\|_{\mathbb{R}} \leq \|A\|,$$

for all real matrices $A \in \mathrm{M}_n(\mathbb{R})$. However, it is possible to construct vector norms $\| \;\|$ on $\mathbb{C}^n$ and *real* matrices A such that

$$\|A\|_{\mathbb{R}} < \|A\|.$$

In order to avoid this kind of difficulties, we define subordinate matrix norms over $\mathrm{M}_n(\mathbb{C})$. Luckily, it turns out that $\|A\|_{\mathbb{R}} = \|A\|$ for the vector norms, $\| \;\|_1$, $\| \;\|_2$, and $\| \;\|_\infty$.

We now prove Proposition 9.6 for real matrix norms.

Proposition 9.9. *For any matrix norm $\| \;\|$ on $\mathrm{M}_n(\mathbb{R})$ and for any square $n \times n$ matrix $A \in \mathrm{M}_n(\mathbb{R})$, we have*

$$\rho(A) \leq \|A\|.$$

Proof. We follow the proof in Denis Serre's book [156]. If A is a real matrix, the problem is that the eigenvectors associated with the eigenvalue of maximum modulus may be complex. We use a trick based on the fact that for every matrix A (real or complex),

$$\rho(A^k) = (\rho(A))^k,$$