because $g \circ s = \mathrm{id}_G$, which shows that $u = s(v) = 0$. Thus, $F = \operatorname{Ker} g \oplus \operatorname{Im} s$, and since by assumption, $\operatorname{Im} f = \operatorname{Ker} g$, we have $F = \operatorname{Im} f \oplus \operatorname{Im} s$. But then, since f and s are injective, $f + s \colon E \oplus G \to F$ is an isomorphism. The proof of (b) is very similar. ☐

Note that we can choose a retraction $r \colon F \to E$ so that $\operatorname{Ker} r = \operatorname{Im} s$, since
$F = \operatorname{Ker} g \oplus \operatorname{Im} s = \operatorname{Im} f \oplus \operatorname{Im} s$ and f is injective so we can set $r \equiv 0$ on $\operatorname{Im} s$.

Given a sequence of linear maps $E \xrightarrow{f} F \xrightarrow{g} G$, when $\operatorname{Im} f = \operatorname{Ker} g$, we say that the sequence $E \xrightarrow{f} F \xrightarrow{g} G$ is *exact at F*. If in addition to being exact at F, f is injective and g is surjective, we say that we have a *short exact sequence*, and this is denoted as

$$0 \longrightarrow E \xrightarrow{f} F \xrightarrow{g} G \longrightarrow 0.$$

The property of a short exact sequence given by Proposition 6.15 is often described by saying that $0 \longrightarrow E \xrightarrow{f} F \xrightarrow{g} G \longrightarrow 0$ is a (short) *split exact sequence*.

As a corollary of Proposition 6.15, we have the following result which shows that given a linear map $f \colon E \to F$, its domain E is the direct sum of its kernel $\operatorname{Ker} f$ with some isomorphic copy of its image $\operatorname{Im} f$.

Theorem 6.16. *(Rank-nullity theorem) Let E and F be vector spaces, and let $f \colon E \to F$ be a linear map. Then, E is isomorphic to $\operatorname{Ker} f \oplus \operatorname{Im} f$, and thus,*

$$\dim(E) = \dim(\operatorname{Ker} f) + \dim(\operatorname{Im} f) = \dim(\operatorname{Ker} f) + \operatorname{rk}(f).$$

See Figure 6.3.

Proof. Consider

$$\operatorname{Ker} f \xrightarrow{i} E \xrightarrow{f'} \operatorname{Im} f,$$

where $\operatorname{Ker} f \xrightarrow{i} E$ is the inclusion map, and $E \xrightarrow{f'} \operatorname{Im} f$ is the surjection associated with $E \xrightarrow{f} F$. Then, we apply Proposition 6.15 to any section $\operatorname{Im} f \xrightarrow{s} E$ of f' to get an isomorphism between E and $\operatorname{Ker} f \oplus \operatorname{Im} f$, and Proposition 6.7, to get $\dim(E) = \dim(\operatorname{Ker} f) + \dim(\operatorname{Im} f)$. ☐

Definition 6.10. The dimension $\dim(\operatorname{Ker} f)$ of the kernel of a linear map f is called the *nullity* of f.

We now derive some important results using Theorem 6.16.

Proposition 6.17. *Given a vector space E, if U and V are any two subspaces of E, then*

$$\dim(U) + \dim(V) = \dim(U + V) + \dim(U \cap V),$$

an equation known as Grassmann's relation.