then using associativity and commutativity several times (more rigorously, using induction on $i_1 - 1$), we get

$$\begin{aligned}\left(a_{i_1} + \left(\sum_{i=1}^{i_1-1} a_i\right)\right) + \left(\sum_{i=i_1+1}^{p} a_i\right) &= \left(\sum_{i=1}^{i_1-1} a_i\right) + a_{i_1} + \left(\sum_{i=i_1+1}^{p} a_i\right)\\ &= \sum_{i=1}^{p} a_i,\end{aligned}$$

as claimed.

The cases where $i_1 = 1$ or $i_1 = p$ are treated similarly, but in a simpler manner since either $P = ()$ or $Q = ()$ (where $()$ denotes the empty sequence). $\square$

Having done all this, we can now make sense of sums of the form $\sum_{i\in I} a_i$, for any finite indexed set $I$ and any family $a = (a_i)_{i\in I}$ of elements in $A$, where $A$ is a set equipped with a binary operation $+$ which is associative and commutative.

Indeed, since $I$ is finite, it is in bijection with the set $\{1, \ldots, n\}$ for some $n \in \mathbb{N}$, and any total ordering $\preceq$ on $I$ corresponds to a permutation $I_{\preceq}$ of $\{1, \ldots, n\}$ (where we identify a permutation with its image). For any total ordering $\preceq$ on $I$, we define $\sum_{i\in I,\preceq} a_i$ as

$$\sum_{i\in I,\preceq} a_i = \sum_{j\in I_{\preceq}} a_j.$$

Then for any other total ordering $\preceq'$ on $I$, we have

$$\sum_{i\in I,\preceq'} a_i = \sum_{j\in I_{\preceq'}} a_j,$$

and since $I_{\preceq}$ and $I_{\preceq'}$ are different permutations of $\{1, \ldots, n\}$, by Proposition 3.3, we have

$$\sum_{j\in I_{\preceq}} a_j = \sum_{j\in I_{\preceq'}} a_j.$$

Therefore, the sum $\sum_{i\in I,\preceq} a_i$ does not depend on the total ordering on $I$. We define *the* sum $\sum_{i\in I} a_i$ as the common value $\sum_{i\in I,\preceq} a_i$ for all total orderings $\preceq$ of $I$.

Here are some examples with $A = \mathbb{R}$:

1. If $I = \{1, 2, 3\}$, $a = \{(1, 2), (2, -3), (3, \sqrt{2})\}$, then $\sum_{i\in I} a_i = 2 - 3 + \sqrt{2} = -1 + \sqrt{2}$.
2. If $I = \{2, 5, 7\}$, $a = \{(2, 2), (5, -3), (7, \sqrt{2})\}$, then $\sum_{i\in I} a_i = 2 - 3 + \sqrt{2} = -1 + \sqrt{2}$.
3. If $I = \{r, g, b\}$, $a = \{(r, 2), (g, -3), (b, 1)\}$, then $\sum_{i\in I} a_i = 2 - 3 + 1 = 0$.