Show that $g_1(e_1) = e_1$,

$$g_1 \circ f \circ h_1(e_1) = u_1' + r_{2,1}e_2,$$

and that

$$\langle e_1, g_1 \circ f \circ h_1(e_j)\rangle = 0$$

for all $j, 2 \leq j \leq n$. At the end of this stage, show that $g_1 \circ f \circ h_1$ has a matrix such that all entries on its first row except perhaps the first are zero, and that all entries on the first column, except perhaps the first two, are zero.

Assume by induction that some isometries $g_1, \ldots, g_k$ and $h_1, \ldots, h_k$ have been found, either reflections or the identity, and such that

$$f_k = g_k \circ \cdots \circ g_1 \circ f \circ h_1 \cdots \circ h_k$$

has a matrix which is lower bidiagonal up to and including row and column k, where $1 \leq k \leq n-2$.

Let

$$v_{k+1} = f_k^*(e_{k+1}) = v_{k+1}' + v_{k+1}'',$$

where $v_{k+1}' \in U_k'$ and $v_{k+1}'' \in U_k''$, and let $r_{k+1,\,k+1} = \left\| v_{k+1}'' \right\|$. Find an isometry h_{k+1} (reflection or id) such that

$$h_{k+1}(v_{k+1}'') = r_{k+1,\,k+1}e_{k+1}.$$

Show that if h_{k+1} is a reflection, then $U_k' \subseteq H_{k+1}$, where H_{k+1} is the hyperplane defining the reflection h_{k+1}. Deduce that $h_{k+1}(v_{k+1}') = v_{k+1}'$, and that

$$h_{k+1}(f_k^*(e_{k+1})) = v_{k+1}' + r_{k+1,\,k+1}e_{k+1}.$$

Observe that $h_{k+1}(f_k^*(e_{k+1})) \in U_{k+1}'$, so that

$$\langle h_{k+1}(f_k^*(e_{k+1})), e_j\rangle = 0$$

for all j, $k+2 \leq j \leq n$, and thus,

$$\langle e_{k+1}, f_k \circ h_{k+1}(e_j)\rangle = 0$$

for all j, $k+2 \leq j \leq n$.

Next let

$$u_{k+1} = f_k \circ h_{k+1}(e_{k+1}) = u_{k+1}' + u_{k+1}'',$$

where $u_{k+1}' \in U_{k+1}'$ and $u_{k+1}'' \in U_{k+1}''$, and let $r_{k+2,\,k+1} = \left\| u_{k+1}'' \right\|$. Find an isometry g_{k+1} (reflection or id) such that

$$g_{k+1}(u_{k+1}'') = r_{k+2,\,k+1}e_{k+2}.$$

Show that if g_{k+1} is a reflection, then $U_{k+1}' \subseteq G_{k+1}$, where G_{k+1} is the hyperplane defining the reflection g_{k+1}. Deduce that $g_{k+1}(e_i) = e_i$ for all i, $1 \leq i \leq k+1$, and that

$$g_{k+1} \circ f_k \circ h_{k+1}(e_{k+1}) = u_{k+1}' + r_{k+2,\,k+1}e_{k+2}.$$