We define the homomorphism $\varphi\colon A \to A/\mathfrak{a} \times A/\mathfrak{b}$ by

$$\varphi(x) = (\overline{x}_{\mathfrak{a}}, \overline{x}_{\mathfrak{b}}),$$

where $\overline{x}_{\mathfrak{a}}$ is the equivalence class of x modulo $\mathfrak{a}$ (resp. $\overline{x}_{\mathfrak{b}}$ is the equivalence class of x modulo $\mathfrak{b}$). Recall that the ideal $\mathfrak{a}$ defines the equivalence relation $\equiv_{\mathfrak{a}}$ on A given by

$$x \equiv_{\mathfrak{a}} y \quad \text{iff} \quad x - y \in \mathfrak{a},$$

and that $A/\mathfrak{a}$ is the quotient ring of equivalence classes $\overline{x}_{\mathfrak{a}}$, where $x \in A$, and similarly for $A/\mathfrak{b}$. Sometimes, we also write $x \equiv y \pmod{\mathfrak{a}}$ for $x \equiv_{\mathfrak{a}} y$.

Clearly, the kernel of the homomorphism φ is $\mathfrak{a} \cap \mathfrak{b}$. If we assume that $\mathfrak{a} + \mathfrak{b} = A$, then $\mathrm{Ker}\,(\varphi) = \mathfrak{a} \cap \mathfrak{b} = \mathfrak{a}\mathfrak{b}$, and because φ has a constant value on the equivalence classes modulo $\mathfrak{a}\mathfrak{b}$, the map φ induces a quotient homomorphism

$$\theta\colon A/\mathfrak{a}\mathfrak{b} \to A/\mathfrak{a} \times A/\mathfrak{b}.$$

Because $\mathrm{Ker}\,(\varphi) = \mathfrak{a}\mathfrak{b}$, the homomorphism θ is injective. The Chinese Remainder Theorem says that θ is an isomorphism.

Theorem 32.14. *Given a commutative ring A, let $\mathfrak{a}$ and $\mathfrak{b}$ be any two ideals of A such that $\mathfrak{a} + \mathfrak{b} = A$. Then, the homomorphism $\theta\colon A/\mathfrak{a}\mathfrak{b} \to A/\mathfrak{a} \times A/\mathfrak{b}$ is an isomorphism.*

Proof. We already showed that θ is injective, so we need to prove that θ is surjective. We need to prove that for any $y, z \in A$, there is some $x \in A$ such that

$$\begin{aligned} x &\equiv y \pmod{\mathfrak{a}} \\ x &\equiv z \pmod{\mathfrak{b}}. \end{aligned}$$

Since $\mathfrak{a} + \mathfrak{b} = A$, there exist some $a \in \mathfrak{a}$ and some $b \in \mathfrak{b}$ such that

$$a + b = 1.$$

If we let

$$x = az + by,$$

then we have

$$x \equiv_{\mathfrak{a}} by \equiv_{\mathfrak{a}} (1 - a)y \equiv_{\mathfrak{a}} y - ay \equiv_{\mathfrak{a}} y,$$

and similarly

$$x \equiv_{\mathfrak{b}} az \equiv_{\mathfrak{b}} (1 - b)z \equiv_{\mathfrak{b}} z - bz \equiv_{\mathfrak{b}} z,$$

which shows that $x = az + by$ works. ∎

Theorem 32.14 can be generalized to any (finite) number of ideals.