Proposition 29.27. *If Φ is any nondegenerate quadratic form (over a field of characteristic $\neq 2$) such that there is some nonzero vector $x \in E$ with $\Phi(x) = 0$, then for every $\alpha \in K$, there is some $y \in E$ such that $\Phi(y) = \alpha$.*

Proof. Since by hypothesis there is some nonzero vector $u \in E$ with $\Phi(u) = 0$, by Proposition 29.26 there is another isotropic vector v such that u and v are linearly independent and such that (after rescaling) $\varphi(u, v) = 1$. Then for any $\alpha \in K$, check that

$$\Phi\left(u + \frac{\alpha}{2} v\right) = \alpha,$$

as desired. $\square$

Remark: Some authors refer to the above plane as a *hyperbolic plane*. Berger (and others) point out that this terminology is undesirable because the notion of hyperbolic plane already exists in differential geometry and refers to a very different object.

We leave it as an exercice to figure out that the group of isometries of the Artinian plane, the set of all 2×2 matrices A such that

$$A^\top \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

consists of all matrices of the form

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{pmatrix}, \quad \lambda \in K - \{0\}.$$

In particular, if $K = \mathbb{R}$, then this group denoted $\mathbf{O}(1, 1)$ has four connected components.

We now turn to the Witt decomposition.

29.7 Witt Decomposition

From now on, $\varphi \colon E \times E \to K$ is an ϵ-Hermitian form. The following assumption will be needed:

Property (T). For every $u \in E$, there is some $\alpha \in K$ such that $\varphi(u, u) = \alpha + \epsilon\overline{\alpha}$.

Property (T) is always satisfied if φ is alternating, or if K is of characteristic $\neq 2$ and $\epsilon = \pm 1$, with $\alpha = \frac{1}{2}\varphi(u, u)$.

The following (bizarre) technical lemma will be needed.

Lemma 29.28. *Let φ be an ϵ-Hermitian form on E and assume that φ satisfies property (T). For any totally isotropic subspace $U \neq (0)$ of E, for every $x \in E$ not orthogonal to U, and for every $\alpha \in K$, there is some $y \in U$ so that*

$$\varphi(x + y, x + y) = \alpha + \epsilon\overline{\alpha}.$$