Chapter 35

Introduction to Modules; Modules over a PID

35.1 Modules over a Commutative Ring

In this chapter we introduce modules over a commutative ring (with unity). After a quick overview of fundamental concepts such as free modules, torsion modules, and some basic results about them, we focus on finitely generated modules over a PID and we prove the structure theorems for this class of modules (invariant factors and elementary divisors). Our main goal is not to give a comprehensive exposition of modules, but instead to apply the structure theorem to the $K[X]$-module E_f defined by a linear map f acting on a finite-dimensional vector space E, and to obtain several normal forms for f, including the rational canonical form.

A module is the generalization of a vector space E over a field K obtained replacing the field K by a commutative ring A (with unity 1). Although formally the definition is the same, the fact that some nonzero elements of A are not invertible has some serious consequences. For example, it is possible that $\lambda \cdot u = 0$ for some nonzero $\lambda \in A$ and some nonzero $u \in E$, and a module may no longer have a basis.

For the sake of completeness, we give the definition of a module, although it is the same as Definition 3.1 with the field K replaced by a ring A. In this chapter, *all rings under consideration are assumed to be commutative and to have an identity element* 1.

Definition 35.1. Given a ring A, a *(left) module over* A (or A*-module*) is a set M (of vectors) together with two operations $+\colon M \times M \to M$ (called *vector addition*),[1] and $\cdot\colon A \times M \to M$ (called *scalar multiplication*) satisfying the following conditions for all $\alpha, \beta \in A$ and all $u, v \in M$;

(M0) M is an abelian group w.r.t. $+$, with identity element 0;

[1] The symbol $+$ is overloaded, since it denotes both addition in the ring A and addition of vectors in M. It is usually clear from the context which $+$ is intended.