Proposition 28.7. *Let E be a nontrivial Hermitian space of dimension n. Given any orthonormal basis $(e_1, \ldots, e_n)$, for any n-tuple of vectors $(v_1, \ldots, v_n)$, there is a sequence of $n-1$ isometries $h_1, \ldots, h_{n-1}$, such that h_i is a hyperplane reflection or the identity, and if $(r_1, \ldots, r_n)$ are the vectors given by*

$$r_j = h_{n-1} \circ \cdots \circ h_2 \circ h_1(v_j) \quad 1 \leq j \leq n,$$

then every r_j is a linear combination of the vectors $(e_1, \ldots, e_j)$, $(1 \leq j \leq n)$. Equivalently, the matrix R whose columns are the components of the r_j over the basis $(e_1, \ldots, e_n)$ is an upper triangular matrix. Furthermore, if we allow one more isometry h_n of the form

$$h_n = \rho_{e_n, \varphi_n} \circ \cdots \circ \rho_{e_1, \varphi_1}$$

after $h_1, \ldots, h_{n-1}$, we can ensure that the diagonal entries of R are nonnegative.

Proof. The proof is very similar to the proof of Proposition 13.3, but it needs to be modified a little bit since Proposition 28.1 is weaker than Proposition 13.2. We explain how to modify the induction step, leaving the base case and the rest of the proof as an exercise.

As in the proof of Proposition 13.3, the vectors $(e_1, \ldots, e_k)$ form a basis for the subspace denoted as U_k', the vectors $(e_{k+1}, \ldots, e_n)$ form a basis for the subspace denoted as U_k'', the subspaces U_k' and U_k'' are orthogonal, and $E = U_k' \oplus U_k''$. Let

$$u_{k+1} = h_k \circ \cdots \circ h_2 \circ h_1(v_{k+1}).$$

We can write

$$u_{k+1} = u_{k+1}' + u_{k+1}'',$$

where $u_{k+1}' \in U_k'$ and $u_{k+1}'' \in U_k''$. Let

$$r_{k+1,k+1} = \left\| u_{k+1}'' \right\|, \quad \text{and} \quad e^{i\theta_{k+1}} |u_{k+1}'' \cdot e_{k+1}| = u_{k+1}'' \cdot e_{k+1}.$$

If $u_{k+1}'' = e^{i\theta_{k+1}} r_{k+1,k+1}\, e_{k+1}$, we let $h_{k+1} = \mathrm{id}$. Otherwise, by Proposition 28.1, there is a unique hyperplane reflection h_{k+1} such that

$$h_{k+1}(u_{k+1}'') = e^{i\theta_{k+1}} r_{k+1,k+1}\, e_{k+1},$$

where h_{k+1} is the reflection about the hyperplane H_{k+1} orthogonal to the vector

$$w_{k+1} = r_{k+1,k+1}\, e_{k+1} - e^{-i\theta_{k+1}} u_{k+1}''.$$

At the end of the induction, we have a triangular matrix R, but the diagonal entries $e^{i\theta_j} r_{j,j}$ of R may be complex. Letting

$$h_{n+1} = \rho_{e_n, -\theta_n} \circ \cdots \circ \rho_{e_1, -\theta_1},$$

we observe that the diagonal entries of the matrix of vectors

$$r_j' = h_{n+1} \circ h_n \circ \cdots \circ h_2 \circ h_1(v_j)$$

is triangular with nonnegative entries. ∎