$g_{ij} = \langle e_i, e_j \rangle$, and (g^{ij}) is its inverse, then for every vector $u = \sum_{j=1}^n u^j e_j \in E$ and every one-form $\omega = \sum_{i=1}^n \omega_i e_i^* \in E^*$, we have

$$u^\flat = \sum_{i=1}^n \omega_i e_i^*, \quad \textit{with} \quad \omega_i = \sum_{j=1}^n g_{ij} u^j,$$

and

$$\omega^\sharp = \sum_{j=1}^n (\omega^\sharp)^j e_j, \quad \textit{with} \quad (\omega^\sharp)^i = \sum_{j=1}^n g^{ij} \omega_j.$$

Proof. For every $u = \sum_{j=1}^n u^j e_j$, since $u^\flat(v) = \langle u, v \rangle$ for all $v \in E$, we have

$$u^\flat(e_i) = \langle u, e_i \rangle = \left\langle \sum_{j=1}^n u^j e_j, e_i \right\rangle = \sum_{j=1}^n u^j \langle e_j, e_i \rangle = \sum_{j=1}^n g_{ij} u^j,$$

so we get

$$u^\flat = \sum_{i=1}^n \omega_i e_i^*, \quad \text{with} \quad \omega_i = \sum_{j=1}^n g_{ij} u^j.$$

If we write $\omega \in E^*$ as $\omega = \sum_{i=1}^n \omega_i e_i^*$ and $\omega^\sharp \in E$ as $\omega^\sharp = \sum_{j=1}^n (\omega^\sharp)^j e_j$, since

$$\omega_i = \omega(e_i) = \langle \omega^\sharp, e_i \rangle = \sum_{j=1}^n (\omega^\sharp)^j g_{ij}, \qquad 1 \leq i \leq n,$$

we get

$$(\omega^\sharp)^i = \sum_{j=1}^n g^{ij} \omega_j,$$

where (g^{ij}) is the inverse of the matrix (g_{ij}). ☐

The map $\flat$ has the effect of lowering (flattening!) indices, and the map $\sharp$ has the effect of raising (sharpening!) indices.

Here is an explicit example of Proposition 33.2. Let (e_1, e_2) be a basis of E such that

$$\langle e_1, e_1 \rangle = 1, \qquad \langle e_1, e_2 \rangle = 2, \qquad \langle e_2, e_2 \rangle = 5.$$

Then

$$g = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}, \qquad g^{-1} = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}.$$

Set $u = u^1 e_1 + u^2 e_2$ and observe that

$$\begin{aligned} u^\flat(e_1) &= \langle u^1 e_1 + u^2 e_2, e_1 \rangle = \langle e_1, e_1 \rangle u^1 + \langle e_2, e_1 \rangle u^2 = g_{11} u^1 + g_{12} u^2 = u^1 + 2u^2 \\ u^\flat(e_2) &= \langle u^1 e_1 + u^2 e_2, e_2 \rangle = \langle e_1, e_2 \rangle u^1 + \langle e_2, e_2 \rangle u^2 = g_{21} u^1 + g_{22} u^2 = 2u^1 + 5u^2, \end{aligned}$$