and similarly the matrix $B_{\mathcal{Q}}$ representing $f_{\mathcal{Q}}$ over $\mathcal{E}$ is

$$B_{\mathcal{Q}} = \begin{pmatrix} \lambda_1 q_1^x & \lambda_2 q_2^x & \lambda_3 q_3^x \\ \lambda_1 q_1^y & \lambda_2 q_2^y & \lambda_3 q_3^y \\ \lambda_1 q_1^z & \lambda_2 q_2^z & \lambda_3 q_3^z \end{pmatrix} = \begin{pmatrix} q_1^x & q_2^x & q_3^x \\ q_1^y & q_2^y & q_3^y \\ q_1^z & q_2^z & q_3^z \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

and we have

$$A_{\mathcal{E}} = B_{\mathcal{Q}} B_{\mathcal{P}}^{-1}.$$

Therefore, we have

$$A_{\mathcal{E}} = \begin{pmatrix} q_1^x & q_2^x & q_3^x \\ q_1^y & q_2^y & q_3^y \\ q_1^z & q_2^z & q_3^z \end{pmatrix} \begin{pmatrix} \frac{\lambda_1}{\alpha_1} & 0 & 0 \\ 0 & \frac{\lambda_2}{\alpha_2} & 0 \\ 0 & 0 & \frac{\lambda_3}{\alpha_3} \end{pmatrix} \begin{pmatrix} p_1^x & p_2^x & p_3^x \\ p_1^y & p_2^y & p_3^y \\ p_1^z & p_2^z & p_3^z \end{pmatrix}^{-1},$$

as claimed ☐

The above method generalizes immediately to any dimension (and any field K). If $([p_1], \ldots, [p_{n+1}], [p_{n+2}])$ and $[(q_1], \ldots, [q_{n+1}], [q_{n+2}])$ are any two projective frames in a projective space $\mathbb{P}(E)$ where E is a K-vector space of dimension $n+1$, then $(p_1, \ldots, p_{n+1})$ is a basis of E denoted by $\mathcal{P}$ and $(q_1, \ldots, q_{n+1})$ is a basis of E denoted $\mathcal{Q}$, and we can write

$$\begin{aligned} p_{n+2} &= \alpha_1 p_1 + \cdots + \alpha_{n+1} p_{n+1} \\ q_{n+2} &= \lambda_1 q_1 + \cdots + \lambda_{n+1} q_{n+1} \end{aligned}$$

for some unique $\alpha_i, \lambda_i \in K$ such that $\alpha_i \neq 0$ and $\lambda_i \neq 0$ for $i = 1, \ldots, n+1$. If we assume that $E = K^{n+1}$, then the canonical basis is $\mathcal{E} = (e_1, \ldots, e_{n+1})$.

If we express the coordinates of the q_j over the basis $\mathcal{P}$ by

$$q_j = (x_j^1, \ldots, x_j^n, x_j^{n+1}), \quad j = 1, \ldots, n+2,$$

then we have the following proposition.

Proposition 26.10. *With respect to the basis $\mathcal{P} = (p_1, \ldots, p_{n+1})$, the matrix $A_{\mathcal{P}}$ of the unique homography h of $\mathbb{P}(E)$ where E is a K-vector space of dimension $n+1$, mapping the projective frame $([p_1], \ldots, [p_{n+1}], [p_{n+2}])$ to the projective frame $[(q_1], \ldots, [q_{n+1}], [q_{n+2}])$ is given by*

$$A_{\mathcal{P}} = \begin{pmatrix} x_1^1 & \ldots & x_n^1 & x_{n+1}^1 \\ \vdots & \ddots & \vdots & \vdots \\ x_1^n & \ldots & x_n^n & x_{n+1}^n \\ x_1^{n+1} & \ldots & x_n^{n+1} & x_{n+1}^{n+1} \end{pmatrix} \begin{pmatrix} \frac{\lambda_1}{\alpha_1} & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & \frac{\lambda_n}{\alpha_n} & 0 \\ 0 & \ldots & 0 & \frac{\lambda_{n+1}}{\alpha_{n+1}} \end{pmatrix}.$$