Chapter 4

Matrices and Linear Maps

In this chapter, all vector spaces are defined over an arbitrary field K. For the sake of concreteness, the reader may safely assume that $K = \mathbb{R}$.

4.1 Representation of Linear Maps by Matrices

Proposition 3.18 shows that given two vector spaces E and F and a basis $(u_j)_{j\in J}$ of E, every linear map $f\colon E \to F$ is uniquely determined by the family $(f(u_j))_{j\in J}$ of the images under f of the vectors in the basis $(u_j)_{j\in J}$.

If we also have a basis $(v_i)_{i\in I}$ of F, then every vector $f(u_j)$ can be written in a unique way as

$$f(u_j) = \sum_{i\in I} a_{i\,j} v_i,$$

where $j \in J$, for a family of scalars $(a_{i\,j})_{i\in I}$. Thus, with respect to the two bases $(u_j)_{j\in J}$ of E and $(v_i)_{i\in I}$ of F, the linear map f is completely determined by a "$I \times J$-matrix" $M(f) = (a_{i\,j})_{(i,j)\in I\times J}$.

Remark: Note that we intentionally assigned the index set J to the basis $(u_j)_{j\in J}$ of E, and the index set I to the basis $(v_i)_{i\in I}$ of F, so that the rows of the matrix $M(f)$ associated with $f\colon E \to F$ are indexed by I, and the columns of the matrix $M(f)$ are indexed by J. Obviously, this causes a mildly unpleasant reversal. If we had considered the bases $(u_i)_{i\in I}$ of E and $(v_j)_{j\in J}$ of F, we would obtain a $J \times I$-matrix $M(f) = (a_{j\,i})_{(j,i)\in J\times I}$. No matter what we do, there will be a reversal! We decided to stick to the bases $(u_j)_{j\in J}$ of E and $(v_i)_{i\in I}$ of F, so that we get an $I \times J$-matrix $M(f)$, knowing that we may occasionally suffer from this decision!

When I and J are finite, and say, when $|I| = m$ and $|J| = n$, the linear map f is determined by the matrix $M(f)$ whose entries in the j-th column are the components of the