Step 3. Prove that the inner product defined by (†) is positive definite.

For any finite subset $\{f_1, \ldots, f_n\}$ of H_0 and any $z \in \mathbb{C}^n$, we have

$$\sum_{j,k=1}^{n} \langle f_j, f_k \rangle z_j \overline{z_k} = \left\langle \sum_{j=1}^{n} z_j f_j, \sum_{j=1}^{n} z_j f_j \right\rangle \geq 0,$$

which shows that the map $(f, g) \mapsto \langle f, g \rangle$ from $H_0 \times H_0$ to $\mathbb{C}$ is a positive definite kernel.

Observe that for all $f \in H_0$ and all $x \in X$, (†) implies that

$$\langle f, \kappa_x \rangle = \sum_{j=1}^{k} \alpha_j \kappa(x_j, x) = f(x),$$

a property known as the *reproducing property*. The above implies that

$$\langle \kappa_x, \kappa_y \rangle = \kappa(x, y). \tag{$**$}$$

By Proposition 53.4 applied to the positive definite kernel $(f, g) \mapsto \langle f, g \rangle$, we have

$$|\langle f, \kappa_x \rangle|^2 \leq \langle f, f \rangle \langle \kappa_x, \kappa_x \rangle,$$

that is,

$$|f(x)|^2 \leq \langle f, f \rangle \kappa(x, x),$$

so $\langle f, f \rangle = 0$ implies that $f(x) = 0$ for all $x \in X$, which means that $\langle -, - \rangle$ as defined by (†) is positive definite. Therefore, $\langle -, - \rangle$ is a Hermitian inner product on H_0, and by ($**$) and since $\varphi(x) = \kappa_x$, we have

$$\kappa(x, y) = \langle \varphi(x), \varphi(y) \rangle, \qquad \text{for all } x, y \in X.$$

Step 4. Define the embedding η.

Let H be the Hilbert space which is the completion of H_0, so that H_0 is dense in H. The map $\eta\colon H \to \mathbb{C}^X$ given by

$$\eta(f)(x) = \langle f, \kappa_x \rangle$$

is obviously linear, and it is injective because the family $(\kappa_x)_{x \in X}$ spans H_0 which is dense in H, thus it is also dense in H, so if $\langle f, \kappa_x \rangle = 0$ for all $x \in X$, then $f = 0$. □

Corollary 53.9. *If we identify a function $f \in H$ with the function $\eta(f)$, then we have the reproducing property*

$$\langle f, \kappa_x \rangle = f(x), \qquad \textit{for all } f \in H \textit{ and all } x \in X.$$

If X is finite, then $\mathbb{C}^X$ is finite-dimensional. If X is a separable topological space and if κ is continuous, then it can be shown that H is a separable Hilbert space.