Corollary 51.36. *Let f be a closed proper convex function on $\mathbb{R}^n$. Then the minimal set of f is a non-empty bounded set iff f has no directions of recession. In particular, if f has no directions of recession, then the minimum* $\inf f$ *of f is finite and attained for some $x \in \mathbb{R}^n$.*

Theorem 51.14 implies the following result which is very important for the design of optimization procedures.

Proposition 51.37. *Let f be a proper and closed convex function over $\mathbb{R}^n$. The function h given by $h(x) = f(x) + q(x)$ obtained by adding any strictly convex quadratic function q of the form $q(x) = x^\top Ax + b^\top x$ (where A is symmetric positive definite) is a proper closed strictly convex function such that* $\inf h$ *is finite, and there is a unique $x^* \in \mathbb{R}^n$ such that h attains its minimum in x^* (that is, $h(x^*) = \inf h$).*

Proof. By Theorem 51.14 there is some affine form φ given by $\varphi(x) = c^\top x + \alpha$ (with $\alpha \in \mathbb{R}$) such that $f(x) \geq \varphi(x)$ for all $x \in \mathbb{R}^n$. Then we have

$$h(x) = f(x) + q(x) \geq x^\top Ax + (b^\top + c^\top)x + \alpha \quad \text{for all } x \in \mathbb{R}^n.$$

Since A is symmetric positive definite, by Example 51.12, the quadratic function Q given by $Q(x) = x^\top Ax + (b^\top + c^\top)x + \alpha$ has no directions of recession. Since $h(x) \geq Q(x)$ for all $x \in \mathbb{R}^n$, we claim that h has no directions of recession. Otherwise, there would be some nonzero vector u, such that the function $\lambda \mapsto h(x + \lambda u)$ is nonincreasing for all $x \in \mathrm{dom}(h)$, so $h(x + \lambda u) \leq \beta$ for some β for all λ. But we showed that for λ large enough, the function $\lambda \mapsto Q(x + \lambda u)$ increases like λ^2, so for λ large enough, we will have $Q(x + \lambda u) > \beta$, contradicting the fact that h majorizes Q. By Corollary 51.36, h has a finite minimum x^* which is attained.

If f and g are proper convex functions and if g is strictly convex, then $f + g$ is a proper function. For all $x, y \in \mathbb{R}^n$, for any λ such that $0 < \lambda < 1$, since f is convex and g is strictly convex, we have

$$\begin{aligned} f((1-\lambda)x + \lambda y) &\leq (1-\lambda)f(x) + \lambda f(y) \\ g((1-\lambda)x + \lambda y) &< (1-\lambda)g(x) + \lambda g(y), \end{aligned}$$

so we deduce that

$$f((1-\lambda)x + \lambda y) + g((1-\lambda)x + \lambda y) < ((1-\lambda)(f(x) + g(x)) + \lambda(f(x) + g(x))),$$

which shows that $f + g$ is strictly convex. Then, as $f + q$ is strictly convex, it has a unique minimum at x^*. ∎

We now come back to the problem of minimizing a proper convex function h over a nonempty convex subset C . Here is a nice characterization.

Proposition 51.38. *Let h be a proper convex function on $\mathbb{R}^n$, and let C be a nonempty convex subset of $\mathbb{R}^n$.*