When E and F are finite dimensional with $\dim(E) = n$ and $\dim(F) = m$, if $m \geq n$, then f is an immersion iff the Jacobian matrix, $J(f)(a)$, has full rank n for all $a \in E$ and if $n \geq m$, then f is a submersion iff the Jacobian matrix, $J(f)(a)$, has full rank m for all $a \in E$.

Example 39.8. For example, $f\colon \mathbb{R} \to \mathbb{R}^2$ defined by $f(t) = (\cos(t), \sin(t))$ is an immersion since $J(f)(t) = \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}$ has rank 1 for all t. On the other hand, $f\colon \mathbb{R} \to \mathbb{R}^2$ defined by $f(t) = (t^2, t^2)$ is not an immersion since $J(f)(t) = \begin{pmatrix} 2t \\ 2t \end{pmatrix}$ vanishes at $t = 0$. See Figure 39.6. An example of a submersion is given by the projection map $f\colon \mathbb{R}^2 \to \mathbb{R}$, where $f(x, y) = x$, since $J(f)(x, y) = \begin{pmatrix} 1 & 0 \end{pmatrix}$.

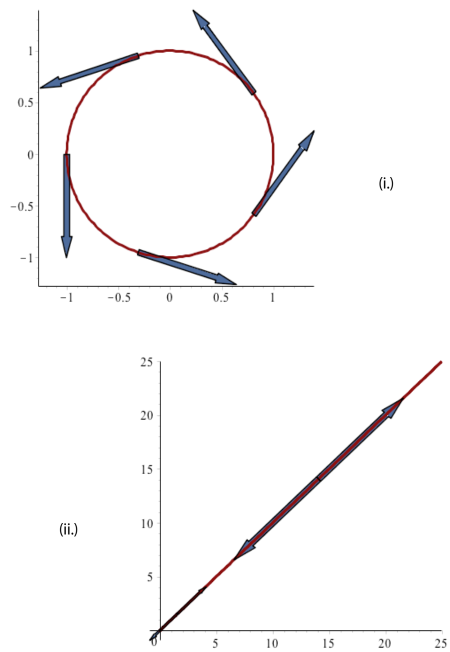

Figure 39.6: Figure $(i.)$ is the immersion of $\mathbb{R}$ into $\mathbb{R}^2$ given by $f(t) = (\cos(t), \sin(t))$. Figure $(ii.)$, the parametric curve $f(t) = (t^2, t^2)$, is not an immersion since the tangent vanishes at the origin.

The following results can be shown.

Proposition 39.16. *Let A be an open subset of $\mathbb{R}^n$, and let $f\colon A \to \mathbb{R}^m$ be a function. For every $a \in A$, $f\colon A \to \mathbb{R}^m$ is a submersion at a iff there exists an open subset U of A containing a, an open subset $W \subseteq \mathbb{R}^{n-m}$, and a diffeomorphism $\varphi\colon U \to f(U) \times W$, such that,*

$$f = \pi_1 \circ \varphi,$$