1. For any $\beta \neq 0$, if β is an eigenvalue of $\mathcal{L}_\omega$, then
$$\frac{\beta+\omega-1}{\beta^{1/2}\omega}, \quad -\frac{\beta+\omega-1}{\beta^{1/2}\omega}$$
are eigenvalues of J.

2. For every $\alpha \neq 0$, if α and $-\alpha$ are eigenvalues of J, then $\mu_+(\alpha,\omega)$ and $\mu_-(\alpha,\omega)$ are eigenvalues of $\mathcal{L}_\omega$, where $\mu_+(\alpha,\omega)$ and $\mu_-(\alpha,\omega)$ are the squares of the roots of the equation
$$\lambda^2 - \alpha\omega\lambda + \omega - 1 = 0.$$

It follows that
$$\rho(\mathcal{L}_\omega) = \max_{\alpha \mid p_J(\alpha)=0} \{\max(|\mu_+(\alpha,\omega)|, |\mu_-(\alpha,\omega)|)\},$$
and since we are assuming that J has real roots, we are led to study the function
$$M(\alpha,\omega) = \max\{|\mu_+(\alpha,\omega)|, |\mu_-(\alpha,\omega)|\},$$
where $\alpha \in \mathbb{R}$ and $\omega \in (0,2)$. Actually, because $M(-\alpha,\omega) = M(\alpha,\omega)$, it is only necessary to consider the case where $\alpha \geq 0$.

Note that for $\alpha \neq 0$, the roots of the equation
$$\lambda^2 - \alpha\omega\lambda + \omega - 1 = 0.$$
are
$$\frac{\alpha\omega \pm \sqrt{\alpha^2\omega^2 - 4\omega + 4}}{2}.$$
In turn, this leads to consider the roots of the equation
$$\omega^2\alpha^2 - 4\omega + 4 = 0,$$
which are
$$\frac{2(1 \pm \sqrt{1-\alpha^2})}{\alpha^2},$$
for $\alpha \neq 0$. Since we have
$$\frac{2(1+\sqrt{1-\alpha^2})}{\alpha^2} = \frac{2(1+\sqrt{1-\alpha^2})(1-\sqrt{1-\alpha^2})}{\alpha^2(1-\sqrt{1-\alpha^2})} = \frac{2}{1-\sqrt{1-\alpha^2}}$$
and
$$\frac{2(1-\sqrt{1-\alpha^2})}{\alpha^2} = \frac{2(1+\sqrt{1-\alpha^2})(1-\sqrt{1-\alpha^2})}{\alpha^2(1+\sqrt{1-\alpha^2})} = \frac{2}{1+\sqrt{1-\alpha^2}},$$
these roots are
$$\omega_0(\alpha) = \frac{2}{1+\sqrt{1-\alpha^2}}, \quad \omega_1(\alpha) = \frac{2}{1-\sqrt{1-\alpha^2}}.$$