Also observe that for an optimal solution, we have

$$\begin{aligned}\frac{1}{2}\left\|y - Xw\right\|_2^2 + w^\top X^\top (y - Xw) &= \frac{1}{2}\left\|y\right\|^2 - w^\top X^\top y + \frac{1}{2} w^\top X^\top X w + w^\top X^\top y - w^\top X^\top X w \\ &= \frac{1}{2}\left(\left\|y\right\|_2^2 - \left\|Xw\right\|_2^2\right) \\ &= \frac{1}{2}\left(\left\|y\right\|_2^2 - \left\|y - \lambda\right\|_2^2\right) = G(\lambda).\end{aligned}$$

Since the objective function is convex and the constaints are qualified, by Theorem 50.19(2) the duality gap is zero, so for optimal solutions of the primal and the dual, $G(\lambda) = L(\xi, w, \epsilon)$, that is

$$\frac{1}{2}\left\|y - Xw\right\|_2^2 + w^\top X^\top (y - Xw) = \frac{1}{2}\left\|\xi\right\|_2^2 + \tau\left\|w\right\|_1 = \frac{1}{2}\left\|y - Xw\right\|_2^2 + \tau\left\|w\right\|_1,$$

which yields the equation

$$w^\top X^\top (y - Xw) = \tau\left\|w\right\|_1. \tag{$**_1$}$$

The above is the inner product of w and $X^\top(y - Xw)$, so whenever $w_i \neq 0$, since $\left\|w\right\|_1 = |w_1| + \cdots + |w_n|$, in view of $(*)$, we must have $(X^\top(y - Xw))_i = \tau \mathrm{sgn}(w_i)$. If

$$S = \{i \in \{1, \ldots, n\} \mid w_i \neq 0\}, \tag{$\dagger$}$$

if X_S denotes the matrix consisting of the columns of X indexed by S, and if w_S denotes the vector consisting of the nonzero components of w, then we have

$$X_S^\top (y - X_S w_S) = \tau \mathrm{sgn}(w_S). \tag{$**_2$}$$

We also have

$$\left\|X_{\overline{S}}^\top (y - X_S w_S)\right\|_\infty \leq \tau, \tag{$**_3$}$$

where $\overline{S}$ is the complement of S.

Equation $(**_2)$ yields

$$X_S^\top X_S w_S = X_S^\top y - \tau \mathrm{sgn}(w_S),$$

so if $X_S^\top X_S$ is invertible (which will be the case if the columns of X are linearly independent), we get

$$w_S = (X_S^\top X_S)^{-1}(X_S^\top y - \tau \mathrm{sgn}(w_S)). \tag{$**_4$}$$

In theory, if we know the support of w and the signs of its components, then w_S is determined, but in practice this is useless since the problem is to find the support and the sign of the solution.