is constant on disjoint "connected components" and which takes possibly distinct values on disjoint components. This can be stated in terms of the concept of a locally constant function.

Definition 37.22. Given two topological spaces X, Y, a function $f\colon X \to Y$ is *locally constant* if for every $x \in X$, there is an open set $U \subseteq X$ such that $x \in U$ and f is constant on U.

We claim that a locally constant function is continuous. In fact, we will prove that $f^{-1}(V)$ is open for every subset, $V \subseteq Y$ (not just for an open set V). It is enough to show that $f^{-1}(y)$ is open for every $y \in Y$, since for every subset $V \subseteq Y$,

$$f^{-1}(V) = \bigcup_{y \in V} f^{-1}(y),$$

and open sets are closed under arbitrary unions. However, either $f^{-1}(y) = \emptyset$ if $y \in Y - f(X)$ or f is constant on $U = f^{-1}(y)$ if $y \in f(X)$ (with value y), and since f is locally constant, for every $x \in U$, there is some open set, $W \subseteq X$, such that $x \in W$ and f is constant on W, which implies that $f(w) = y$ for all $w \in W$ and thus, that $W \subseteq U$, showing that U is a union of open sets and thus, is open. The following proposition shows that a space is connected iff every locally constant function is constant:

Proposition 37.17. *A topological space is connected iff every locally constant function is constant. See Figure 37.23.*

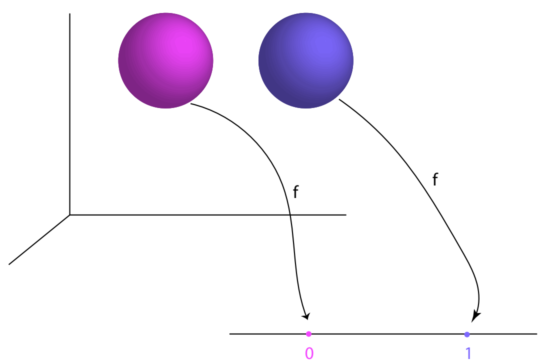

Figure 37.23: An example of a locally constant, but not constant, real-valued function f over the disconnected set consisting of the disjoint union of the two solid balls. On the pink ball, f is 0, while on the purple ball, f is 1.

Proof. First, assume that X is connected. Let $f\colon X \to Y$ be a locally constant function to some space Y and assume that f is not constant. Pick any $y \in f(X)$. Since f is not constant, $U_1 = f^{-1}(y) \neq X$, and of course, $U_1 \neq \emptyset$. We proved just before Proposition