3.4 Linear Independence, Subspaces

One of the most useful properties of vector spaces is that they possess bases. What this means is that in every vector space E, there is some set of vectors, $\{e_1, \ldots, e_n\}$, such that *every* vector $v \in E$ can be written as a linear combination,

$$v = \lambda_1 e_1 + \cdots + \lambda_n e_n,$$

of the e_i, for some scalars, $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Furthermore, the n-tuple, $(\lambda_1, \ldots, \lambda_n)$, as above is unique.

This description is fine when E has a finite basis, $\{e_1, \ldots, e_n\}$, but this is not always the case! For example, the vector space of real polynomials, $\mathbb{R}[X]$, does not have a finite basis but instead it has an infinite basis, namely

$$1,\ X,\ X^2,\ \ldots, X^n,\ \ldots$$

One might wonder if it is possible for a vector space to have bases of different sizes, or even to have a finite basis as well as an infinite basis. We will see later on that this is not possible; all bases of a vector space have the same number of elements (cardinality), which is called the *dimension* of the space. However, we have the following problem: If a vector space has an infinite basis, $\{e_1, e_2, \ldots, \}$, how do we define linear combinations? Do we allow linear combinations

$$\lambda_1 e_1 + \lambda_2 e_2 + \cdots$$

with infinitely many nonzero coefficients?

If we allow linear combinations with infinitely many nonzero coefficients, then we have to make sense of these sums and this can only be done reasonably if we define such a sum as the limit of the sequence of vectors, $s_1, s_2, \ldots, s_n, \ldots$, with $s_1 = \lambda_1 e_1$ and

$$s_{n+1} = s_n + \lambda_{n+1} e_{n+1}.$$

But then, how do we define such limits? Well, we have to define some topology on our space, by means of a norm, a metric or some other mechanism. This can indeed be done and this is what Banach spaces and Hilbert spaces are all about but this seems to require a lot of machinery.

A way to avoid limits is to restrict our attention to linear combinations involving only *finitely many* vectors. We may have an infinite supply of vectors but we only form linear combinations involving finitely many nonzero coefficients. Technically, this can be done by introducing *families of finite support*. This gives us the ability to manipulate families of scalars indexed by some fixed infinite set and yet to be treat these families as if they were finite.

With these motivations in mind, given a set A, recall that an *I-indexed family* $(a_i)_{i \in I}$ of elements of A (for short, a *family*) is a function $a\colon I \to A$, or equivalently a set of pairs $\{(i, a_i) \mid i \in I\}$. We agree that when $I = \emptyset$, $(a_i)_{i \in I} = \emptyset$. A family $(a_i)_{i \in I}$ is finite if I is finite.