Therefore, M_{tor} is a submodule of M. $\square$

The module M_{tor} is called the *torsion submodule* of M. If $M_{\text{tor}} = (0)$, then we say that M is *torsion-free*, and if $M = M_{\text{tor}}$, then we say that M is a *torsion module*.

If M is not finitely generated, then it is possible that $M_{\text{tor}} \neq 0$, yet the annihilator of M_{tor} is reduced to 0. For example, let take the $\mathbb{Z}$-module

$$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \times \cdots \times \mathbb{Z}/p\mathbb{Z} \times \cdots ,$$

where p ranges over the set of primes. Call this module M and the set of primes P. Observe that M is generated by $\{\alpha_p\}_{p\in P}$, where α_p is the tuple whose only nonzero entry is $\overline{1}_p$, the generator of $\mathbb{Z}/p\mathbb{Z}$, i.e.,

$$\alpha_p = (\overline{0}, \overline{0}, \overline{0}, \cdots , \overline{1}_p, \overline{0}, \cdots), \qquad \mathbb{Z}/p\mathbb{Z} = \{n \cdot \overline{1}_p\}_{n=0}^{p-1}.$$

In other words, M is not finitely generated. Furthermore, since $p \cdot \overline{1}_p = \overline{0}$, we have $\{\alpha_p\}_{p\in P} \subset M_{\text{tor}}$. However, because p ranges over all primes, the only possible nonzero annihilator of $\{\alpha_p\}_{p\in P}$ would be the product of all the primes. Hence $\text{Ann}(\{\alpha_p\}_{p\in P}) = (0)$. Because of the subset containment, we conclude that $\text{Ann}(M_{\text{tor}}) = (0)$.

However, if M is finitely generated, it is *not* possible that $M_{\text{tor}} \neq 0$, yet the annihilator of M_{tor} is reduced to 0, since if $x_1, \ldots, x_n$ generate M and if $a_1, \ldots, a_n$ annihilate $x_1, \ldots, x_n$, then $a_1 \cdots a_n$ annihilates every element of M.

Proposition 35.4. *If A is an integral domain, then for any A-module M, the quotient module M/M_{tor} is torsion free.*

Proof. Let $\overline{x}$ be an element of M/M_{tor} and assume that $a\overline{x} = 0$ for some $a \neq 0$ in A. This means that $ax \in M_{\text{tor}}$, so there is some $b \neq 0$ in A such that $bax = 0$. Since $a, b \neq 0$ and A is an integral domain, $ba \neq 0$, so $x \in M_{\text{tor}}$, which means that $\overline{x} = 0$. $\square$

If A is an integral domain and if F is a free A-module with basis $(u_1, \ldots, u_n)$, then F can be embedded in a K-vector space F_K isomorphic to K^n, where $K = \text{Frac}(A)$ is the fraction field of A. Similarly, any submodule M of F is embedded into a subspace M_K of F_K. Note that any linearly independent vectors $(u_1, \ldots, u_m)$ in the A-module M remain linearly independent in the vector space M_K, because any linear dependence over K is of the form

$$\frac{a_1}{b_1} u_1 + \cdots + \frac{a_m}{b_m} u_m = 0$$

for some $a_i, b_i \in A$, with $b_1 \cdots b_m \neq 0$, so if we multiply by $b_1 \cdots b_m \neq 0$, we get a linear dependence in the A-module M. Then we see that the maximum number of linearly independent vectors in the A-module M is at most n. The maximum number of linearly independent vectors in a finitely generated submodule of a free module (over an integral domain) is called the *rank* of the module M. If $(u_1, \ldots, u_m)$ are linearly independent where