Proof. There are two cases.

Case 1. $\varphi(u, v) \neq 0$.

In this case, $u \neq v$, since $\varphi(u, u) = 0$. Let us look for a symplectic transvection of the form $\tau_{v-u,\lambda}$. We want

$$v = u + \lambda\varphi(v - u, u)(v - u) = u + \lambda\varphi(v, u)(v - u),$$

which yields

$$(\lambda\varphi(v, u) - 1)(v - u) = 0.$$

Since $\varphi(u, v) \neq 0$ and $\varphi(v, u) = -\varphi(u, v)$, we can pick $\lambda = \varphi(v, u)^{-1}$ and $\tau_{v-u,\lambda}$ maps u to v.

Case 2. $\varphi(u, v) = 0$.

If $u = v$, use $\tau_{u,0} = \mathrm{id}$. Now, assume $u \neq v$. We claim that it is possible to pick some $w \in E$ such that $\varphi(u, w) \neq 0$ and $\varphi(v, w) \neq 0$. Indeed, if $(Ku)^{\perp} = (Kv)^{\perp}$, then pick any nonzero vector w not in the hyperplane $(Ku)^{\perp}$. Othwerwise, $(Ku)^{\perp}$ and $(Kv)^{\perp}$ are two distinct hyperplanes, so neither is contained in the other (they have the same dimension), so pick any nonzero vector w_1 such that $w_1 \in (Ku)^{\perp}$ and $w_1 \notin (Kv)^{\perp}$, and pick any nonzero vector w_2 such that $w_2 \in (Kv)^{\perp}$ and $w_2 \notin (Ku)^{\perp}$. If we let $w = w_1 + w_2$, then $\varphi(u, w) = \varphi(u, w_2) \neq 0$, and $\varphi(v, w) = \varphi(v, w_1) \neq 0$. From case 1, we have some symplectic transvection τ_{w-u,λ_1} such that $\tau_{w-u,\lambda_1}(u) = w$, and some symplectic transvection τ_{v-w,λ_2} such that $\tau_{v-w,\lambda_2}(w) = v$, so the composition $\tau_{v-w,\lambda_2} \circ \tau_{w-u,\lambda_1}$ maps u to v. ☐

Next, we would like to extend Proposition 29.36 to two hyperbolic planes W_1 and W_2.

Proposition 29.37. *Given any two hyperbolic planes W_1 and W_2 given by bases (u_1, v_1) and (u_2, v_2) (with $\varphi(u_i, u_i) = \varphi(v_i, v_i) = 0$ and $\varphi(u_i, v_i) = 1$, for $i = 1, 2$), there is a symplectic map f such that $f(u_1) = u_2$, $f(v_1) = v_2$, and f is the composition of at most four symplectic transvections.*

Proof. From Proposition 29.36, we can map u_1 to u_2, using a map f which is the composition of at most two symplectic transvections. Say $v_3 = f(v_1)$. We claim that there is a map g such that $g(u_2) = u_2$ and $g(v_3) = v_2$, and g is the composition of at most two symplectic transvections. If so, $g \circ f$ maps the pair (u_1, v_1) to the pair (u_2, v_2), and $g \circ f$ consists of at most four symplectic transvections. Thus, we need to prove the following claim:

Claim. If (u, v) and (u, v') are hyperbolic bases determining two hyperbolic planes, then there is a symplectic map g such that $g(u) = u$, $g(v) = v'$, and g is the composition of at most two symplectic transvections. There are two case.

Case 1. $\varphi(v, v') \neq 0$.

In this case, there is a symplectic transvection $\tau_{v'-v,\lambda}$ such that $\tau_{v'-v,\lambda}(v) = v'$. We also have

$$\varphi(u, v' - v) = \varphi(u, v') - \varphi(u, v) = 1 - 1 = 0.$$