The next theorem holds in general, but the proof is more sophisticated for vector spaces that do not have a finite set of generators. *Thus, in this chapter, we only prove the theorem for finitely generated vector spaces.*

Theorem 3.7. *Given any finite family $S = (u_i)_{i \in I}$ generating a vector space E and any linearly independent subfamily $L = (u_j)_{j \in J}$ of S (where $J \subseteq I$), there is a basis B of E such that $L \subseteq B \subseteq S$.*

Proof. Consider the set of linearly independent families B such that $L \subseteq B \subseteq S$. Since this set is nonempty and finite, it has some maximal element (that is, a subfamily $B = (u_h)_{h \in H}$ of S with $H \subseteq I$ of maximum cardinality), say $B = (u_h)_{h \in H}$. We claim that B generates E. Indeed, if B does not generate E, then there is some $u_p \in S$ that is not a linear combination of vectors in B (since S generates E), with $p \notin H$. Then by Lemma 3.6, the family $B' = (u_h)_{h \in H \cup \{p\}}$ is linearly independent, and since $L \subseteq B \subset B' \subseteq S$, this contradicts the maximality of B. Thus, B is a basis of E such that $L \subseteq B \subseteq S$. $\square$

Remark: Theorem 3.7 also holds for vector spaces that are not finitely generated. In this case, the problem is to guarantee the existence of a maximal linearly independent family B such that $L \subseteq B \subseteq S$. The existence of such a maximal family can be shown using Zorn's lemma, see Appendix C and the references given there.

A situation where the full generality of Theorem 3.7 is needed is the case of the vector space $\mathbb{R}$ over the field of coefficients $\mathbb{Q}$. The numbers 1 and $\sqrt{2}$ are linearly independent over $\mathbb{Q}$, so according to Theorem 3.7, the linearly independent family $L = (1, \sqrt{2})$ can be extended to a basis B of $\mathbb{R}$. Since $\mathbb{R}$ is uncountable and $\mathbb{Q}$ is countable, such a basis must be uncountable!

The notion of a basis can also be defined in terms of the notion of maximal linearly independent family and minimal generating family.

Definition 3.7. Let $(v_i)_{i \in I}$ be a family of vectors in a vector space E. We say that $(v_i)_{i \in I}$ a *maximal linearly independent family of E* if it is linearly independent, and if for any vector $w \in E$, the family $(v_i)_{i \in I} \cup_k \{w\}$ obtained by adding w to the family $(v_i)_{i \in I}$ is linearly dependent. We say that $(v_i)_{i \in I}$ a *minimal generating family of E* if it spans E, and if for any index $p \in I$, the family $(v_i)_{i \in I - \{p\}}$ obtained by removing v_p from the family $(v_i)_{i \in I}$ does not span E.

The following proposition giving useful properties characterizing a basis is an immediate consequence of Lemma 3.6.

Proposition 3.8. *Given a vector space E, for any family $B = (v_i)_{i \in I}$ of vectors of E, the following properties are equivalent:*

(1) B is a basis of E.