Definition 26.6. Given a projective space $\mathbf{P}(E)$, for any two distinct hyperplanes $\mathbf{P}(H)$ and $\mathbf{P}(H')$, for any point $c \in \mathbf{P}(E)$ neither in $\mathbf{P}(H)$ nor in $\mathbf{P}(H')$, the *projection (or perspectivity) of center c between* $\mathbf{P}(H)$ *and* $\mathbf{P}(H')$ is the map $f\colon \mathbf{P}(H) \to \mathbf{P}(H')$ defined such that for every $a \in \mathbf{P}(H)$, the point $f(a)$ is the intersection of the line $\langle c, a\rangle$ through c and a with $\mathbf{P}(H')$.

Let us verify that f is well–defined and a bijective projective transformation. Since the hyperplanes $\mathbf{P}(H)$ and $\mathbf{P}(H')$ are distinct, the hyperplanes H and H' in E are distinct, and since c is neither in $\mathbf{P}(H)$ nor in $\mathbf{P}(H')$, letting $c = p(u)$ for some nonnull vector $u \in E$, then $u \notin H$ and $u \notin H'$, and thus $E = H \oplus Ku = H' \oplus Ku$. If $\pi\colon E \to H'$ is the linear map (projection onto H' parallel to u) defined such that

$$\pi(w + \lambda u) = w,$$

for all $w \in H'$ and all $\lambda \in K$, since $E = H \oplus Ku = H' \oplus Ku$, the restriction $g\colon H \to H'$ of $\pi\colon E \to H'$ to H is a linear bijection between H and H', and clearly $f = \mathbf{P}(g)$, which shows that f is a projectivity.

Remark: Going back to the linear map $\pi\colon E \to H'$ (projection onto H' parallel to u), note that $\mathbf{P}(\pi)\colon \mathbf{P}(E) \to \mathbf{P}(H')$ is also a projective map, but it is not injective, and thus only a partial map. More generally, given a direct sum $E = V \oplus W$, the projection $\pi\colon E \to V$ onto V parallel to W induces a projective map $\mathbf{P}(\pi)\colon \mathbf{P}(E) \to \mathbf{P}(V)$, and given another direct sum $E = U \oplus W$, the restriction of π to U induces a perspectivity f between $\mathbf{P}(U)$ and $\mathbf{P}(V)$. Geometrically, f is defined as follows: Given any point $a \in \mathbf{P}(U)$, if $\langle \mathbf{P}(W), a\rangle$ is the smallest projective subspace containing $\mathbf{P}(W)$ and a, the point $f(a)$ is the intersection of $\langle \mathbf{P}(W), a\rangle$ with $\mathbf{P}(V)$.

Figure 26.11 illustrates a projection f of center c between two projective lines Δ and Δ' (in the real projective plane).

If we consider three distinct points d_1, d_2, d_3 on Δ and their images d_1', d_2', d_3' on Δ' under the projection f, then ratios are not preserved, that is,

$$\frac{\overrightarrow{d_3d_1}}{\overrightarrow{d_3d_2}} \neq \frac{\overrightarrow{d_3'd_1'}}{\overrightarrow{d_3'd_2'}}.$$

However, if we consider four distinct points d_1, d_2, d_3, d_4 on Δ and their images d_1', d_2', d_3', d_4' on Δ' under the projection f, we will show later that we have the following preservation of the so-called “cross-ratio”

$$\frac{\overrightarrow{d_3d_1}}{\overrightarrow{d_3d_2}}\Bigg/\frac{\overrightarrow{d_4d_1}}{\overrightarrow{d_4d_2}} = \frac{\overrightarrow{d_3'd_1'}}{\overrightarrow{d_3'd_2'}}\Bigg/\frac{\overrightarrow{d_4'd_1'}}{\overrightarrow{d_4'd_2'}}.$$

Cross-ratios and projections play an important role in geometry (for some very elegant illustrations of this fact, see Sidler [161]).