For example, if $n = 5$ and $m = 3$, we have

$$\lambda_1 \leq \mu_1 \leq \lambda_3$$
$$\lambda_2 \leq \mu_2 \leq \lambda_4$$
$$\lambda_3 \leq \mu_3 \leq \lambda_5.$$

**Proposition 17.25.** *Let $A$ be an $n \times n$ symmetric matrix, $R$ be an $n \times m$ matrix such that $R^\top R = I$ (with $m \leq n$), and let $B = R^\top AR$ (an $m \times m$ matrix). The following properties hold:*

*(a) The eigenvalues of $B$ interlace the eigenvalues of $A$.*

*(b) If $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ are the eigenvalues of $A$ and $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_m$ are the eigenvalues of $B$, and if $\lambda_i = \mu_i$, then there is an eigenvector $v$ of $B$ with eigenvalue $\mu_i$ such that $Rv$ is an eigenvector of $A$ with eigenvalue $\lambda_i$.*

*Proof.* (a) Let $(u_1, \ldots, u_n)$ be an orthonormal basis of eigenvectors for $A$, and let $(v_1, \ldots, v_m)$ be an orthonormal basis of eigenvectors for $B$. Let $U_j$ be the subspace spanned by $(u_1, \ldots, u_j)$ and let $V_j$ be the subspace spanned by $(v_1, \ldots, v_j)$. For any $i$, the subspace $V_i$ has dimension $i$ and the subspace $R^\top U_{i-1}$ has dimension at most $i - 1$. Therefore, there is some nonzero vector $v \in V_i \cap (R^\top U_{i-1})^\perp$, and since

$$v^\top R^\top u_j = (Rv)^\top u_j = 0, \quad j = 1, \ldots, i - 1,$$

we have $Rv \in (U_{i-1})^\perp$. By Proposition 17.24 and using the fact that $R^\top R = I$, we have

$$\lambda_i \leq \frac{(Rv)^\top ARv}{(Rv)^\top Rv} = \frac{v^\top Bv}{v^\top v}.$$

On the other hand, by Proposition 17.23,

$$\mu_i = \max_{x \neq 0, x \in \{v_{i+1}, \ldots, v_n\}^\perp} \frac{x^\top Bx}{x^\top x} = \max_{x \neq 0, x \in \{v_1, \ldots, v_i\}} \frac{x^\top Bx}{x^\top x},$$

so

$$\frac{w^\top Bw}{w^\top w} \leq \mu_i \quad \text{for all } w \in V_i,$$

and since $v \in V_i$, we have

$$\lambda_i \leq \frac{v^\top Bv}{v^\top v} \leq \mu_i, \quad i = 1, \ldots, m.$$

We can apply the same argument to the symmetric matrices $-A$ and $-B$, to conclude that

$$-\lambda_{n-m+i} \leq -\mu_i,$$