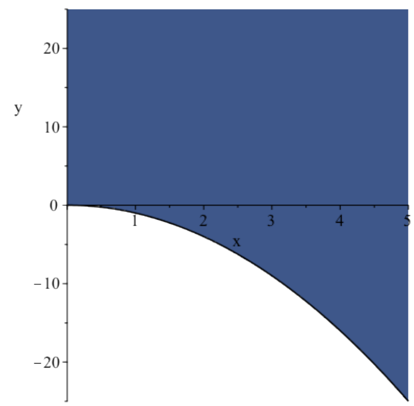

Figure 51.3: The epigraph of the concave function $f(x) = -x^2$ if $x \geq 0$ and $+\infty$ otherwise.

(1) A function $f\colon C \to \mathbb{R}^n \cup \{+\infty\}$ *is convex on* C *iff*

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y)$$

for all $x, y \in C$ *and all* λ *such that* $0 < \lambda < 1$.

(2) A function $f\colon \mathbb{R}^n \to \mathbb{R}^n \cup \{-\infty, +\infty\}$ *is convex iff*

$$f((1-\lambda)x + \lambda y) < (1-\lambda)\alpha + \lambda\beta$$

for all $\alpha, \beta \in \mathbb{R}$, *all* $x, y \in \mathbb{R}^n$ *such that* $f(x) < \alpha$ *and* $f(y) < \beta$, *and all* λ *such that* $0 < \lambda < 1$.

The "good" convex functions that we would like to deal with are defined below.

Definition 51.5. A convex function $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$ is *proper*[1] if its epigraph is nonempty and does not contain any vertical line. Equivalently, a convex function f is proper if $f(x) > -\infty$ for all $x \in \mathbb{R}^n$ and $f(x) < +\infty$ for some $x \in \mathbb{R}^n$. A convex function which is not proper is called an *improper function*.

Observe that a convex function f is proper iff $\mathrm{dom}(f) \neq \emptyset$ and if the restriction of f to $\mathrm{dom}(f)$ is a finite function.

It is immediately verified that a set C is convex iff its indicator function I_C is convex, and clearly, the indicator function of a convex set is proper.

The important object of study is the set of proper functions, but improper functions can't be avoided.

[1]This terminology is unfortunate because it clashes with the notion of a proper function from topology, which has to do with the preservation of compact subsets under inverse images.