showing that the *old* coordinates (x_i) of x (over $(u_1, \ldots, u_n)$) are expressed in terms of the *new* coordinates (x'_i) of x (over $(v_1, \ldots, v_n)$). This fact may seem wrong, but it is correct as we can reassure ourselves by doing the following computation. Suppose that $n = 2$, so that

$$\begin{aligned} v_1 &= a_{11}u_1 + a_{21}u_2 \\ v_2 &= a_{12}u_1 + a_{22}u_2, \end{aligned}$$

and our matrix is

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

The same vector x is written as

$$x = x_1u_1 + x_2u_2 = x'_1v_1 + x'_2v_2,$$

so by substituting the expressions for v_1 and v_2 as linear combinations of u_1 and u_2, we obtain

$$\begin{aligned} x_1u_1 + x_2u_2 &= x'_1v_1 + x'_2v_2 \\ &= x'_1(a_{11}u_1 + a_{21}u_2) + x'_2(a_{12}u_1 + a_{22}u_2) \\ &= (a_{11}x'_1 + a_{12}x'_2)u_1 + (a_{21}x'_1 + a_{22}x'_2)u_2, \end{aligned}$$

and since u_1 and u_2 are linearly independent, we must have

$$\begin{aligned} x_1 &= a_{11}x'_1 + a_{12}x'_2 \\ x_2 &= a_{21}x'_1 + a_{22}x'_2, \end{aligned}$$

namely

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix},$$

as claimed.

If the vectors $u_1, \ldots, u_n$ and the vectors $v_1, \ldots, v_n$ are vectors in K^n, then we can form the $n \times n$ matrix $U = (u_1 \ \cdots \ u_n)$ whose columns are $u_1, \ldots, u_n$ and the $n \times n$ matrix $V = (v_1 \ \cdots \ v_n)$ whose columns are $v_1, \ldots, v_n$. Then we can express the change of basis P from $(u_1, \ldots, u_n)$ to $(v_1, \ldots, v_n)$ in terms of U and V. Indeeed, the equation

$$v_j = \sum_{i=1}^{n} a_{ij}u_i$$

can be expressed in matrix form as

$$v_j = UA^j,$$