Chapter 26

Basics of Projective Geometry

Think geometrically, prove algebraically.
—John Tate

26.1 Why Projective Spaces?

For a novice, projective geometry usually appears to be a bit odd, and it is not obvious to motivate why its introduction is inevitable and in fact fruitful. One of the main motivations arises from algebraic geometry.

The main goal of algebraic geometry is to study the properties of geometric objects, such as curves and surfaces, defined implicitly in terms of algebraic equations. For instance, the equation

$$x^2 + y^2 - 1 = 0$$

defines a circle in $\mathbb{R}^2$. More generally, we can consider the curves defined by general equations

$$ax^2 + by^2 + cxy + dx + ey + f = 0$$

of degree 2, known as *conics*. It is then natural to ask whether it is possible to classify these curves according to their generic geometric shape. This is indeed possible. Except for so-called singular cases, we get ellipses, parabolas, and hyperbolas. The same question can be asked for surfaces defined by quadratic equations, known as *quadrics*, and again, a classification is possible. However, these classifications are a bit artificial. For example, an ellipse and a hyperbola differ by the fact that a hyperbola has points at infinity, and yet, their geometric properties are identical, provided that points at infinity are handled properly.

Another important problem is the study of intersection of geometric objects (defined algebraically). For example, given two curves C_1 and C_2 of degree m and n, respectively, what is the number of intersection points of C_1 and C_2? (by degree of the curve we mean the total degree of the defining polynomial).