39.2 Properties of Derivatives

Proposition 39.3. *Given two normed affine spaces E and F, if $f\colon E \to F$ is a constant function, then $\mathrm{D}f(a) = 0$, for every $a \in E$. If $f\colon E \to F$ is a continuous affine map, then $\mathrm{D}f(a) = \overrightarrow{f}$, for every $a \in E$, the linear map associated with f.*

Proof. Straightforward. □

Proposition 39.4. *Given a normed affine space E and a normed vector space F, for any two functions $f, g\colon E \to F$, for every $a \in E$, if $\mathrm{D}f(a)$ and $\mathrm{D}g(a)$ exist, then $\mathrm{D}(f+g)(a)$ and $\mathrm{D}(\lambda f)(a)$ exist, and*

$$\begin{aligned} \mathrm{D}(f+g)(a) &= \mathrm{D}f(a) + \mathrm{D}g(a), \\ \mathrm{D}(\lambda f)(a) &= \lambda \mathrm{D}f(a). \end{aligned}$$

Proof. Straightforward. □

Given two normed vector spaces $(E_1, \| \, \|_1)$ and $(E_2, \| \, \|_2)$, there are three natural and equivalent norms that can be used to make $E_1 \times E_2$ into a normed vector space:

1. $\|(u_1, u_2)\|_1 = \|u_1\|_1 + \|u_2\|_2$.
2. $\|(u_1, u_2)\|_2 = (\|u_1\|_1^2 + \|u_2\|_2^2)^{1/2}$.
3. $\|(u_1, u_2)\|_\infty = \max(\|u_1\|_1, \|u_2\|_2)$.

We usually pick the first norm. If E_1, E_2, and F are three normed vector spaces, recall that a bilinear map $f\colon E_1 \times E_2 \to F$ is *continuous* iff there is some constant $C \geq 0$ such that

$$\|f(u_1, u_2)\| \leq C \|u_1\|_1 \|u_2\|_2 \quad \text{for all } u_1 \in E_1 \text{ and all } u_2 \in E_2.$$

Proposition 39.5. *Given three normed vector spaces E_1, E_2, and F, for any continuous bilinear map $f\colon E_1 \times E_2 \to F$, for every $(a, b) \in E_1 \times E_2$, $\mathrm{D}f(a, b)$ exists, and for every $u \in E_1$ and $v \in E_2$,*

$$\mathrm{D}f(a, b)(u, v) = f(u, b) + f(a, v).$$

Proof. Since f is bilinear, a simple computation implies that

$$\begin{aligned} f((a, b) + (u, v)) - f(a, b) - (f(u, b) + f(a, v)) &= f(a + u, b + v) - f(a, b) - f(u, b) - f(a, v) \\ &= f(a + u, b) + f(a + u, v) - f(a, b) - f(u, b) - f(a, v) \\ &= f(a, b) + f(u, b) + f(a, v) + f(u, v) - f(a, b) - f(u, b) - f(a, v) \\ &= f(u, v). \end{aligned}$$