In particular, if A is a real matrix and if A is *skew-symmetric*, then

$$x^\top Ax = 0.$$

Thus, for any real matrix (symmetric or not),

$$x^\top Ax = x^\top H(A)x,$$

where $H(A) = (A + A^\top)/2$, the symmetric part of A.

There are situations in which it is necessary to add linear constraints to the problem of maximizing a quadratic function on the sphere. This problem was completely solved by Golub [78] (1973). The problem is the following: given an $n \times n$ real symmetric matrix A and an $n \times p$ matrix C,

$$\begin{aligned} &\text{minimize} && x^\top Ax \\ &\text{subject to} && x^\top x = 1,\ C^\top x = 0,\ x \in \mathbb{R}^n. \end{aligned}$$

As in Section 42.2, Golub shows that the linear constraint $C^\top x = 0$ can be eliminated as follows: if we use a QR decomposition of C, by permuting the columns, we may assume that

$$C = Q^\top \begin{pmatrix} R & S \\ 0 & 0 \end{pmatrix} \Pi,$$

where Q is an orthogonal $n \times n$ matrix, R is an $r \times r$ invertible upper triangular matrix, and S is an $r \times (p - r)$ matrix (assuming C has rank r). If we let

$$x = Q^\top \begin{pmatrix} y \\ z \end{pmatrix},$$

where $y \in \mathbb{R}^r$ and $z \in \mathbb{R}^{n-r}$, then $C^\top x = 0$ becomes

$$\Pi^\top \begin{pmatrix} R^\top & 0 \\ S^\top & 0 \end{pmatrix} Qx = \Pi^\top \begin{pmatrix} R^\top & 0 \\ S^\top & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = 0,$$

which implies $y = 0$, and every solution of $C^\top x = 0$ is of the form

$$x = Q^\top \begin{pmatrix} 0 \\ z \end{pmatrix}.$$

Our original problem becomes

$$\begin{aligned} &\text{minimize} && (y^\top\ z^\top)QAQ^\top \begin{pmatrix} y \\ z \end{pmatrix} \\ &\text{subject to} && z^\top z = 1,\ z \in \mathbb{R}^{n-r}, \\ & && y = 0,\ y \in \mathbb{R}^r. \end{aligned}$$