29.2 Sesquilinear Forms

In order to accomodate Hermitian forms, we assume that some involutive automorphism, $\lambda \mapsto \overline{\lambda}$, of the field K is given. This automorphism of K satisfies the following properties:

$$\begin{aligned}\overline{(\lambda + \mu)} &= \overline{\lambda} + \overline{\mu}\\ \overline{(\lambda\mu)} &= \overline{\lambda}\,\overline{\mu}\\ \overline{\overline{\lambda}} &= \lambda.\end{aligned}$$

Since any field automorphism maps the multiplicative unit 1 to itself, we have $\overline{1} = 1$.

If the automorphism $\lambda \mapsto \overline{\lambda}$ is the identity, then we are in the standard situation of a bilinear form. When $K = \mathbb{C}$ (the complex numbers), then we usually pick the automorphism of $\mathbb{C}$ to be *conjugation*; namely, the map

$$a + ib \mapsto a - ib.$$

Definition 29.8. Given two vector spaces E and F over a field K with an involutive automorphism $\lambda \mapsto \overline{\lambda}$, a map $\varphi\colon E \times F \to K$ is a (right) *sesquilinear form* iff the following conditions hold: For all $u, u_1, u_2 \in E$, all $v, v_1, v_2 \in F$, for all $\lambda, \mu \in K$, we have

$$\begin{aligned}\varphi(u_1 + u_2, v) &= \varphi(u_1, v) + \varphi(u_2, v)\\ \varphi(u, v_1 + v_2) &= \varphi(u, v_1) + \varphi(u, v_2)\\ \varphi(\lambda u, v) &= \lambda\varphi(u, v)\\ \varphi(u, \mu v) &= \overline{\mu}\varphi(u, v).\end{aligned}$$

Again, $\varphi(0, v) = \varphi(u, 0) = 0$. If $E = F$, then we have

$$\begin{aligned}\varphi(\lambda u + \mu v, \lambda u + \mu v) &= \lambda\varphi(u, \lambda u + \mu v) + \mu\varphi(v, \lambda u + \mu v)\\ &= \lambda\overline{\lambda}\varphi(u, u) + \lambda\overline{\mu}\varphi(u, v) + \overline{\lambda}\mu\varphi(v, u) + \mu\overline{\mu}\varphi(v, v).\end{aligned}$$

If we let $\lambda = \mu = 1$ and then $\lambda = 1, \mu = -1$, we get

$$\begin{aligned}\varphi(u + v, u + v) &= \varphi(u, u) + \varphi(u, v) + \varphi(v, u) + \varphi(v, v)\\ \varphi(u - v, u - v) &= \varphi(u, u) - \varphi(u, v) - \varphi(v, u) + \varphi(v, v),\end{aligned}$$

so by subtraction, we get

$$2(\varphi(u, v) + \varphi(v, u)) = \varphi(u + v, u + v) - \varphi(u - v, u - v) \quad \text{for } u, v \in E.$$

If we replace v by λv (with $\lambda \neq 0$), we get

$$2(\overline{\lambda}\varphi(u, v) + \lambda\varphi(v, u)) = \varphi(u + \lambda v, u + \lambda v) - \varphi(u - \lambda v, u - \lambda v),$$