14.9 Problems

Problem 14.1. Let $(E, \langle -, - \rangle)$ be a Hermitian space of finite dimension. Prove that if $f \colon E \to E$ is a self-adjoint linear map (that is, $f^* = f$), then $\langle f(x), x \rangle \in \mathbb{R}$ for all $x \in E$.

Problem 14.2. Prove the polarization identities of Proposition 14.1.

Problem 14.3. Let E be a real Euclidean space. Give an example of a nonzero linear map $f \colon E \to E$ such that $\langle f(u), u \rangle = 0$ for all $u \in E$.

Problem 14.4. Prove Proposition 14.9.

Problem 14.5. (1) Prove that every matrix in $\mathbf{SU}(2)$ is of the form

$$A = \begin{pmatrix} a + ib & c + id \\ -c + id & a - ib \end{pmatrix}, \quad a^2 + b^2 + c^2 + d^2 = 1,\ a, b, c, d \in \mathbb{R},$$

(2) Prove that the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

all belong to $\mathbf{SU}(2)$ and are linearly independent over $\mathbb{C}$.

(3) Prove that the linear span of $\mathbf{SU}(2)$ over $\mathbb{C}$ is the complex vector space $\mathrm{M}_2(\mathbb{C})$ of all complex 2×2 matrices.

Problem 14.6. The purpose of this problem is to prove that the linear span of $\mathbf{SU}(n)$ over $\mathbb{C}$ is $\mathrm{M}_n(\mathbb{C})$ for all $n \geq 3$. One way to prove this result is to adapt the method of Problem 12.12, so please review this problem.

Every complex matrix $A \in \mathrm{M}_n(\mathbb{C})$ can be written as

$$A = \frac{A + A^*}{2} + \frac{A - A^*}{2}$$

where the first matrix is Hermitian and the second matrix is skew-Hermitian. Observe that if $A = (z_{ij})$ is a Hermitian matrix, that is $A^* = A$, then $z_{ji} = \overline{z}_{ij}$, so if $z_{ij} = a_{ij} + ib_{ij}$ with $a_{ij}, b_{ij} \in \mathbb{R}$, then $a_{ij} = a_{ji}$ and $b_{ij} = -b_{ji}$. On the other hand, if $A = (z_{ij})$ is a skew-Hermitian matrix, that is $A^* = -A$, then $z_{ji} = -\overline{z}_{ij}$, so $a_{ij} = -a_{ji}$ and $b_{ij} = b_{ji}$.

The Hermitian and the skew-Hermitian matrices do not form complex vector spaces because they are not closed under multiplication by a complex number, but we can get around this problem by treating the real part and the complex part of these matrices separately and using multiplication by reals.