Conversely, if $u \in \operatorname{Ker}\left(\frac{1}{2}(\mathrm{id} - f)\right)$, then $f(u) = u$, so

$$\frac{1}{2}(\mathrm{id} + f)(u) = \frac{1}{2}(u + u) = u,$$

and thus

$$\operatorname{Ker}\left(\frac{1}{2}(\mathrm{id} - f)\right) \subseteq \operatorname{Im}\left(\frac{1}{2}(\mathrm{id} + f)\right).$$

Therefore,

$$U^+ = \operatorname{Ker}\left(\frac{1}{2}(\mathrm{id} - f)\right) = \operatorname{Im}\left(\frac{1}{2}(\mathrm{id} + f)\right),$$

and so, $f(u) = u$ on U^+ and $f(u) = -u$ on U^-. $\square$

We now assume that $K = \mathbb{C}$. The involutions of E that are unitary transformations are characterized as follows.

Proposition 14.24. *Let $f \in \mathbf{GL}(E)$ be an involution. The following properties are equivalent:*

(a) The map f is unitary; that is, $f \in \mathbf{U}(E)$.

(b) The subspaces $U^- = \operatorname{Im}(\frac{1}{2}(\mathrm{id} - f))$ and $U^+ = \operatorname{Im}(\frac{1}{2}(\mathrm{id} + f))$ are orthogonal.

Furthermore, if E is finite-dimensional, then (a) and (b) are equivalent to (c) below:

(c) The map is self-adjoint; that is, $f = f^*$.

Proof. If f is unitary, then from $\langle f(u), f(v)\rangle = \langle u, v\rangle$ for all $u, v \in E$, we see that if $u \in U^+$ and $v \in U^-$, we get

$$\langle u, v\rangle = \langle f(u), f(v)\rangle = \langle u, -v\rangle = -\langle u, v\rangle,$$

so $2\langle u, v\rangle = 0$, which implies $\langle u, v\rangle = 0$, that is, U^+ and U^- are orthogonal. Thus, (a) implies (b).

Conversely, if (b) holds, since $f(u) = u$ on U^+ and $f(u) = -u$ on U^-, we see that $\langle f(u), f(v)\rangle = \langle u, v\rangle$ if $u, v \in U^+$ or if $u, v \in U^-$. Since $E = U^+ \oplus U^-$ and since U^+ and U^- are orthogonal, we also have $\langle f(u), f(v)\rangle = \langle u, v\rangle$ for all $u, v \in E$, and (b) implies (a).

If E is finite-dimensional, the adjoint f^* of f exists, and we know that $f^{-1} = f^*$. Since f is an involution, $f^2 = \mathrm{id}$, which implies that $f^* = f^{-1} = f$. $\square$

A unitary involution is the identity on $U^+ = \operatorname{Im}(\frac{1}{2}(\mathrm{id} + f))$, and $f(v) = -v$ for all $v \in U^- = \operatorname{Im}(\frac{1}{2}(\mathrm{id} - f))$. Furthermore, E is an orthogonal direct sum $E = U^+ \oplus U^-$. We say that f is an *orthogonal reflection* about U^+. In the special case where U^+ is a hyperplane, we say that f is a *hyperplane reflection*. We already studied hyperplane reflections in the Euclidean case; see Chapter 13.