Chapter 27

The Cartan–Dieudonné Theorem

In this chapter the structure of the orthogonal group is studied in more depth. In particular, we prove that every isometry in $\mathbf{O}(n)$ is the composition of at most n reflections about hyperplanes (for $n \geq 2$, see Theorem 27.1). This important result is a special case of the "Cartan–Dieudonné theorem" (Cartan [33], Dieudonné [50]). We also prove that every rotation in $\mathbf{SO}(n)$ is the composition of at most n flips (for $n \geq 3$).

Affine isometries are defined, and their fixed points are investigated. First, we characterize the set of fixed points of an affine map. Then we show that the Cartan–Dieudonné theorem can be generalized to affine isometries: Every rigid motion in $\mathbf{Is}(n)$ is the composition of at most n affine reflections if it has a fixed point, or else of at most $n+2$ affine reflections. We prove that every rigid motion in $\mathbf{SE}(n)$ is the composition of at most n affine flips (for $n \geq 3$).

27.1 The Cartan–Dieudonné Theorem for Linear Isometries

The fact that the group $\mathbf{O}(n)$ of linear isometries is generated by the reflections is a special case of a theorem known as the Cartan–Dieudonné theorem. Elie Cartan proved a version of this theorem early in the twentieth century. A proof can be found in his book on spinors [33], which appeared in 1937 (Chapter I, Section 10, pages 10–12). Cartan's version applies to nondegenerate quadratic forms over $\mathbb{R}$ or $\mathbb{C}$. The theorem was generalized to quadratic forms over arbitrary fields by Dieudonné [50]. One should also consult Emil Artin's book [6], which contains an in-depth study of the orthogonal group and another proof of the Cartan–Dieudonné theorem.

Theorem 27.1. *Let E be a Euclidean space of dimension $n \geq 1$. Every isometry $f \in \mathbf{O}(E)$ that is not the identity is the composition of at most n reflections. When $n \geq 2$, the identity is the composition of any reflection with itself.*