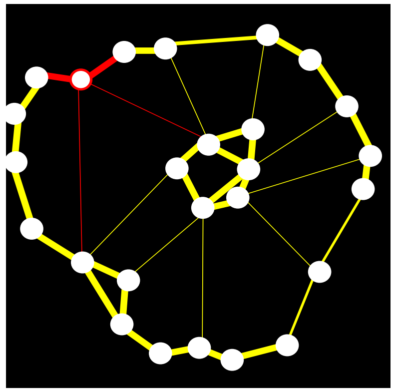

Figure 20.1: Degree of a node.

graph Laplacian L of a directed graph G in an "old-fashion" way, by showing that for any orientation of a graph G,

$$BB^\top = D - A = L$$

is an invariant. We also define the (unnormalized) *graph Laplacian* L of a weighted graph $G = (V, W)$ as $L = D - W$. We show that the notion of incidence matrix can be generalized to weighted graphs in a simple way. For any graph G^σ obtained by orienting the underlying graph of a weighted graph $G = (V, W)$, there is an incidence matrix B^σ such that

$$B^\sigma (B^\sigma)^\top = D - W = L.$$

We also prove that

$$x^\top L x = \frac{1}{2} \sum_{i,j=1}^{m} w_{i\,j} (x_i - x_j)^2 \quad \text{for all } x \in \mathbb{R}^m.$$

Consequently, $x^\top L x$ does not depend on the diagonal entries in W, and if $w_{i\,j} \geq 0$ for all $i, j \in \{1, \ldots, m\}$, then L is positive semidefinite. Then if W consists of nonnegative entries, the eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m$ of L are real and nonnegative, and there is an orthonormal basis of eigenvectors of L. We show that the number of connected components of the graph $G = (V, W)$ is equal to the dimension of the kernel of L, which is also equal to the dimension of the kernel of the transpose $(B^\sigma)^\top$ of any incidence matrix B^σ obtained by orienting the underlying graph of G.

We also define the normalized graph Laplacians L_{sym} and L_{rw}, given by

$$\begin{aligned} L_{\mathrm{sym}} &= D^{-1/2} L D^{-1/2} = I - D^{-1/2} W D^{-1/2} \\ L_{\mathrm{rw}} &= D^{-1} L = I - D^{-1} W, \end{aligned}$$

and prove some simple properties relating the eigenvalues and the eigenvectors of L, L_{sym} and L_{rw}. These normalized graph Laplacians show up when dealing with normalized cuts.