Chapter 17

Spectral Theorems in Euclidean and Hermitian Spaces

17.1 Introduction

The goal of this chapter is to show that there are nice normal forms for symmetric matrices, skew-symmetric matrices, orthogonal matrices, and normal matrices. The spectral theorem for symmetric matrices states that symmetric matrices have real eigenvalues and that they can be diagonalized over an orthonormal basis. The spectral theorem for Hermitian matrices states that Hermitian matrices also have real eigenvalues and that they can be diagonalized over a complex orthonormal basis. Normal real matrices can be block diagonalized over an orthonormal basis with blocks having size at most two and there are refinements of this normal form for skew-symmetric and orthogonal matrices.

The spectral result for real symmetric matrices can be used to prove two characterizations of the eigenvalues of a symmetric matrix in terms of the *Rayleigh ratio*. The first characterization is the *Rayleigh–Ritz theorem* and the second one is the *Courant–Fischer theorem*. Both results are used in optimization theory and to obtain results about perturbing the eigenvalues of a symmetric matrix.

In this chapter all vector spaces are finite-dimensional real or complex vector spaces.

17.2 Normal Linear Maps: Eigenvalues and Eigenvectors

We begin by studying normal maps, to understand the structure of their eigenvalues and eigenvectors. This section and the next three were inspired by Lang [109], Artin [7], Mac Lane and Birkhoff [118], Berger [11], and Bertin [15].