where λ is the column vector $\lambda = (\lambda_1, \ldots, \lambda_n)$. Since A has an inverse A^{-1}, by multiplying both sides of the equation $A\lambda = 0$ by A^{-1} we obtain

$$A^{-1}A\lambda = I_n\lambda = \lambda = A^{-1}0 = 0,$$

which shows that the columns $(A^1, \ldots, A^n)$ are linearly independent.

Conversely, assume that the columns $(A^1, \ldots, A^n)$ are linearly independent. Since the vector space $E = K^n$ has dimension n, the vectors $(v_1, \ldots, v_n) = (A^1, \ldots, A^n)$ form a basis of K^n. By definition, the matrix A is defined by expressing each vector $v_j = A^j$ as the linear combination $A^j = \sum_{i=1}^n a_{ij}e_i$, where $(e_1, \ldots, e_n)$ is the canonical basis of K^n, and since $(v_1, \ldots, v_n)$ is a basis, by Proposition 4.3, the matrix A is invertible. $\square$

Proposition 4.3 suggests the following definition.

Definition 4.3. Given a vector space E of dimension n, for any two bases $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ of E, let $P = (a_{i\,j})$ be the invertible matrix defined such that

$$v_j = \sum_{i=1}^n a_{i\,j} u_i,$$

which is also the matrix of the identity $\mathrm{id}\colon E \to E$ with respect to the bases $(v_1, \ldots, v_n)$ and $(u_1, \ldots, u_n)$, *in that order*. Indeed, we express each $\mathrm{id}(v_j) = v_j$ over the basis $(u_1, \ldots, u_n)$. The coefficients $a_{1j}, a_{2j}, \ldots, a_{nj}$ of v_j over the basis $(u_1, \ldots, u_n)$ form the jth column of the matrix P shown below:

$$\begin{array}{c} \\ u_1 \\ u_2 \\ \vdots \\ u_n \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & \ldots & v_n \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \end{array}.$$

The matrix P is called the *change of basis matrix from* $(u_1, \ldots, u_n)$ *to* $(v_1, \ldots, v_n)$.

Clearly, the change of basis matrix from $(v_1, \ldots, v_n)$ to $(u_1, \ldots, u_n)$ is P^{-1}. Since $P = (a_{i\,j})$ is the matrix of the identity $\mathrm{id}\colon E \to E$ with respect to the bases $(v_1, \ldots, v_n)$ and $(u_1, \ldots, u_n)$, given any vector $x \in E$, if $x = x_1u_1 + \cdots + x_nu_n$ over the basis $(u_1, \ldots, u_n)$ and $x = x'_1v_1 + \cdots + x'_nv_n$ over the basis $(v_1, \ldots, v_n)$, from Proposition 4.2, we have

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{1\,1} & \ldots & a_{1\,n} \\ \vdots & \ddots & \vdots \\ a_{n\,1} & \ldots & a_{n\,n} \end{pmatrix} \begin{pmatrix} x'_1 \\ \vdots \\ x'_n \end{pmatrix},$$