is positive semidefinite. Thus, we want to know when the function

$$f(x, y) = (x^\top \ y^\top) \begin{pmatrix} A & B \\ B^\top & C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x^\top A x + 2x^\top B y + y^\top C y$$

has a minimum with respect to both x and y. If we hold y constant, Proposition 42.5 implies that $f(x, y)$ has a minimum iff $A \succeq 0$ and $(I - AA^+)By = 0$, and then the minimum value is

$$f(x^*, y) = -y^\top B^\top A^+ B y + y^\top C y = y^\top (C - B^\top A^+ B) y.$$

Since we want $f(x, y)$ to be uniformly bounded from below for all x, y, we must have $(I - AA^+)B = 0$. Now $f(x^*, y)$ has a minimum iff $C - B^\top A^+ B \succeq 0$. Therefore, we have established that $f(x, y)$ has a minimum over all x, y iff

$$A \succeq 0, \quad (I - AA^+)B = 0, \quad C - B^\top A^+ B \succeq 0.$$

Similar reasoning applies if we first minimize with respect to y and then with respect to x, but this time, the Schur complement $A - BC^+B^\top$ of C is involved. Putting all these facts together, we get our main result:

Theorem 43.5. *Given any symmetric matrix*

$$M = \begin{pmatrix} A & B \\ B^\top & C \end{pmatrix}$$

the following conditions are equivalent:

(1) $M \succeq 0$ *(M is positive semidefinite).*

(2) $A \succeq 0, \quad (I - AA^+)B = 0, \quad C - B^\top A^+ B \succeq 0.$

(3) $C \succeq 0, \quad (I - CC^+)B^\top = 0, \quad A - BC^+B^\top \succeq 0.$

If $M \succeq 0$ as in Theorem 43.5, then it is easy to check that we have the following factorizations (using the fact that $A^+AA^+ = A^+$ and $C^+CC^+ = C^+$):

$$\begin{pmatrix} A & B \\ B^\top & C \end{pmatrix} = \begin{pmatrix} I & BC^+ \\ 0 & I \end{pmatrix} \begin{pmatrix} A - BC^+B^\top & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} I & 0 \\ C^+B^\top & I \end{pmatrix}$$

and

$$\begin{pmatrix} A & B \\ B^\top & C \end{pmatrix} = \begin{pmatrix} I & 0 \\ B^\top A^+ & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & C - B^\top A^+ B \end{pmatrix} \begin{pmatrix} I & A^+B \\ 0 & I \end{pmatrix}.$$