Proof. If $f\colon E \to F$ is injective, then it has a retraction $r\colon F \to E$ such that $r \circ f = \mathrm{id}_E$, and if $f\colon E \to F$ is surjective, then it has a section $s\colon F \to E$ such that $f \circ s = \mathrm{id}_F$. Now if $f\colon E \to F$ is injective, then we have

$$(r \circ f)^\top = f^\top \circ r^\top = \mathrm{id}_{E^*},$$

which implies that $f^\top$ is surjective, and if f is surjective, then we have

$$(f \circ s)^\top = s^\top \circ f^\top = \mathrm{id}_{F^*},$$

which implies that $f^\top$ is injective. $\square$

The following proposition gives a natural interpretation of the dual $(E/U)^*$ of a quotient space E/U.

Proposition 11.9. *For any subspace U of a vector space E, if $p\colon E \to E/U$ is the canonical surjection onto E/U, then $p^\top$ is injective and*

$$\mathrm{Im}(p^\top) = U^0 = (\mathrm{Ker}\,(p))^0.$$

Therefore, $p^\top$ is a linear isomorphism between $(E/U)^*$ and U^0.

Proof. Since p is surjective, by Proposition 11.8, the map $p^\top$ is injective. Obviously, $U = \mathrm{Ker}\,(p)$. Observe that $\mathrm{Im}(p^\top)$ consists of all linear forms $\psi \in E^*$ such that $\psi = \varphi \circ p$ for some $\varphi \in (E/U)^*$, and since $\mathrm{Ker}\,(p) = U$, we have $U \subseteq \mathrm{Ker}\,(\psi)$. Conversely for any linear form $\psi \in E^*$, if $U \subseteq \mathrm{Ker}\,(\psi)$, then ψ factors through E/U as $\psi = \overline{\psi} \circ p$ as shown in the following commutative diagram

$$\begin{tikzcd} E \arrow[r, "p"] \arrow[dr, "\psi"'] & E/U \arrow[d, "\overline{\psi}"] \\ & K, \end{tikzcd}$$

where $\overline{\psi}\colon E/U \to K$ is given by

$$\overline{\psi}(\overline{v}) = \psi(v), \quad v \in E,$$

where $\overline{v} \in E/U$ denotes the equivalence class of $v \in E$. The map $\overline{\psi}$ does not depend on the representative chosen in the equivalence class $\overline{v}$, since if $\overline{v'} = \overline{v}$, that is $v' - v = u \in U$, then $\psi(v') = \psi(v + u) = \psi(v) + \psi(u) = \psi(v) + 0 = \psi(v)$. Therefore, we have

$$\begin{aligned} \mathrm{Im}(p^\top) &= \{\varphi \circ p \mid \varphi \in (E/U)^*\} \\ &= \{\psi\colon E \to K \mid U \subseteq \mathrm{Ker}\,(\psi)\} \\ &= U^0, \end{aligned}$$

which proves our result. $\square$