## 34.8 Testing Decomposability ⊛

We are now ready to tackle the problem of finding criteria for decomposability. Such criteria will use the left hook. Once again, in this section *all vector spaces are assumed to have finite dimension*. But before stating our criteria, we need a few preliminary results.

**Proposition 34.25.** *Given $z \in \bigwedge^p E$ with $z \neq 0$, the smallest vector space $W \subseteq E$ such that $z \in \bigwedge^p W$ is generated by the vectors of the form*

$$u^* \lrcorner z, \qquad \text{with } u^* \in \textstyle\bigwedge^{p-1} E^*.$$

*Proof.* First let $W$ be any subspace such that $z \in \bigwedge^p(W)$ and let $(e_1, \ldots, e_r, e_{r+1}, \ldots, e_n)$ be a basis of $E$ such that $(e_1, \ldots, e_r)$ is a basis of $W$. Then, $u^* = \sum_I \lambda_I e_I^*$, where $I \subseteq \{1, \ldots, n\}$ and $|I| = p - 1$, and $z = \sum_J \mu_J e_J$, where $J \subseteq \{1, \ldots, r\}$ and $|J| = p \leq r$. It follows immediately from the formula of Proposition 34.18 (4), namely

$$e_I^* \lrcorner e_J = \rho_{J-I,J} e_{J-I},$$

that $u^* \lrcorner z \in W$, since $J - I \subseteq \{1, \ldots, r\}$.

Next we prove that if $W$ is the smallest subspace of $E$ such that $z \in \bigwedge^p(W)$, then $W$ is generated by the vectors of the form $u^* \lrcorner z$, where $u^* \in \bigwedge^{p-1} E^*$. Suppose not. Then the vectors $u^* \lrcorner z$ with $u^* \in \bigwedge^{p-1} E^*$ span a proper subspace $U$ of $W$. We prove that for every subspace $W'$ of $W$ with $\dim(W') = \dim(W) - 1 = r - 1$, it is not possible that $u^* \lrcorner z \in W'$ for all $u^* \in \bigwedge^{p-1} E^*$. But then, as $U$ is a proper subspace of $W$, it is contained in some subspace $W'$ with $\dim(W') = r - 1$, and we have a contradiction.

Let $w \in W - W'$ and pick a basis of $W$ formed by a basis $(e_1, \ldots, e_{r-1})$ of $W'$ and $w$. Any $z \in \bigwedge^p(W)$ can be written as $z = z' + w \wedge z''$, where $z' \in \bigwedge^p W'$ and $z'' \in \bigwedge^{p-1} W'$, and since $W$ is the smallest subspace containing $z$, we have $z'' \neq 0$. Consequently, if we write $z'' = \sum_I \lambda_I e_I$ in terms of the basis $(e_1, \ldots, e_{r-1})$ of $W'$, there is some $e_I$, with $I \subseteq \{1, \ldots, r - 1\}$ and $|I| = p - 1$, so that the coefficient $\lambda_I$ is nonzero. Now, using any basis of $E$ containing $(e_1, \ldots, e_{r-1}, w)$, by Proposition 34.18 (4), we see that

$$e_I^* \lrcorner (w \wedge e_I) = \lambda w, \qquad \lambda = \pm 1.$$

It follows that

$$e_I^* \lrcorner z = e_I^* \lrcorner (z' + w \wedge z'') = e_I^* \lrcorner z' + e_I^* \lrcorner (w \wedge z'') = e_I^* \lrcorner z' + \lambda\lambda_I w,$$

with $e_I^* \lrcorner z' \in W'$, which shows that $e_I^* \lrcorner z \notin W'$. Therefore, $W$ is indeed generated by the vectors of the form $u^* \lrcorner z$, where $u^* \in \bigwedge^{p-1} E^*$. ☐

To help understand Proposition 34.25, let $E$ be the vector space with basis $\{e_1, e_2, e_3, e_4\}$ and $z = e_1 \wedge e_2 + e_2 \wedge e_3$. Note that $z \in \bigwedge^2 E$. To find the smallest vector space $W \subseteq E$