Otherwise, $w^* = 0$, and u^* is a feasible solution of $(P2)$. Since $(u^*, 0_m)$ is a basic feasible solution of $(\widehat{P})$ the columns corresponding to nonzero components of u^* are linearly independent. Some of the coordinates of u^* could be equal to 0, but since A has rank m we can add columns of A to obtain a basis K^* associated with u^*, and u^* is indeed a basic feasible solution of $(P2)$.

Running the simplex algorithm on the Linear Program $\widehat{P}$ to obtain an initial feasible solution (u_0, K_0) of the linear program $(P2)$ is called *Phase I* of the simplex algorithm. Running the simplex algorithm on the Linear Program $(P2)$ with some initial feasible solution (u_0, K_0) is called *Phase II* of the simplex algorithm. If a feasible solution of the Linear Program $(P2)$ is readily available then Phase I is skipped. Sometimes, at the end of Phase I, an optimal solution of $(P2)$ is already obtained.

In summary, we proved the following fact worth recording.

Proposition 46.1. *For any Linear Program $(P2)$*

$$\begin{aligned} &\text{maximize} && cx \\ &\text{subject to} && Ax = b \ \textit{and}\ x \geq 0, \end{aligned}$$

in standard form, where A is an $m \times n$ matrix of rank m and $b \geq 0$, consider the Linear Program $(\widehat{P})$ in standard form

$$\begin{aligned} &\text{maximize} && -(x_{n+1} + \cdots + x_{n+m}) \\ &\text{subject to} && \widehat{A}\,\widehat{x} = b \ \textit{and}\ \widehat{x} \geq 0. \end{aligned}$$

The simplex algorithm with a pivot rule that prevents cycling started on the basic feasible solution $\widehat{x} = (0_n, b)$ of $(\widehat{P})$ terminates with an optimal solution (u^*, w^*).

(1) If $w^* \neq 0$, then $\mathcal{P}(A, b) = \emptyset$, that is, the Linear Program $(P2)$ has no feasible solution.

(2) If $w^* = 0$, then $\mathcal{P}(A, b) \neq \emptyset$, and u^* is a basic feasible solution of $(P2)$ associated with some basis K.

Proposition 46.1 shows that determining whether the polyhedron $\mathcal{P}(A, b)$ defined by a system of equations $Ax = b$ and inequalities $x \geq 0$ is nonempty is decidable. This decision procedure uses a fail-safe version of the simplex algorithm (that prevents cycling), and the proof that it always terminates and returns an answer is nontrivial.

46.3 How to Perform a Pivoting Step Efficiently

We now discuss briefly how to perform the computation of (u^+, K^+) from a basic feasible solution (u, K).

In order to avoid applying permutation matrices it is preferable to allow a basis K to be a sequence of indices, possibly out of order. Thus, for any $m \times n$ matrix A (with $m \leq n$)