decomposition theorem, and Hodge's theorem about the isomorphism between the de Rham cohomology groups and the spaces of harmonic forms.

To understand all this, one needs to learn about differential forms, which turn out to be certain kinds of skew-symmetric (also called alternating) tensors.

If one's only goal is to define differential forms, then it is possible to take some short cuts and to avoid introducing the general notion of a tensor. However, tensors that are not necessarily skew-symmetric arise naturally, such as the curvature tensor, and in the theory of vector bundles, general tensor products are needed.

Consequently, we made the (perhaps painful) decision to provide a fairly detailed exposition of tensors, starting with arbitrary tensors, and then specializing to symmetric and alternating tensors. In particular, we explain rather carefully the process of taking the dual of a tensor (of all three flavors).

We refrained from following the approach in which a tensor is defined as a multilinear map defined on a product of dual spaces, because it seems very artificial and confusing (certainly to us). This approach relies on duality results that only hold in finite dimension, and consequently unecessarily restricts the theory of tensors to finite dimensional spaces. We also feel that it is important to begin with a coordinate-free approach. Bases can be chosen for computations, but tensor algebra should not be reduced to raising or lowering indices.

Readers who feel that they are familiar with tensors should probably skip this chapter and the next. They can come back to them "by need."

We begin by defining tensor products of vector spaces over a field and then we investigate some basic properties of these tensors, in particular the existence of bases and duality. After this we investigate special kinds of tensors, namely symmetric tensors and skew-symmetric tensors. Tensor products of modules over a commutative ring with identity will be discussed very briefly. They show up naturally when we consider the space of sections of a tensor product of vector bundles.

Given a linear map $f\colon E \to F$ (where $E$ and $F$ are two vector spaces over a field $K$), we know that if we have a basis $(u_i)_{i\in I}$ for $E$, then $f$ is completely determined by its values $f(u_i)$ on the basis vectors. For a multilinear map $f\colon E^n \to F$, we don't know if there is such a nice property but it would certainly be very useful.

In many respects tensor products allow us to define multilinear maps in terms of their action on a suitable basis. The crucial idea is to *linearize*, that is, to create a new vector space $E^{\otimes n}$ such that the multilinear map $f\colon E^n \to F$ is turned into a *linear map* $f_{\otimes}\colon E^{\otimes n} \to F$ which is equivalent to $f$ in a strong sense. If in addition, $f$ is symmetric, then we can define a symmetric tensor power $\mathrm{Sym}^n(E)$, and every symmetric multilinear map $f\colon E^n \to F$ is turned into a *linear map* $f_{\odot}\colon \mathrm{Sym}^n(E) \to F$ which is equivalent to $f$ in a strong sense. Similarly, if $f$ is alternating, then we can define a skew-symmetric tensor power $\bigwedge^n(E)$, and every alternating multilinear map is turned into a *linear map* $f_{\wedge}\colon \bigwedge^n(E) \to F$ which is equivalent to $f$ in a strong sense.