and $u \cdot u = 1$, so we get

$$\nu = -\frac{\sin\lambda\Omega}{\sin\Omega}\cos\theta\sin\theta\cos\varphi\, u + \cos\lambda\Omega\sin\theta\, u + \frac{\sin\lambda\Omega}{\sin\Omega}\cos^2\theta\sin\varphi\, v + \frac{\sin\lambda\Omega}{\sin\Omega}\sin^2\theta\sin\varphi\, v - \frac{\sin\lambda\Omega}{\sin\Omega}\sin^2\theta\sin\varphi(u\cdot v)u.$$

Using

$$\sin\theta\sin\varphi(u\cdot v) = \cos\Omega - \cos\theta\cos\varphi,$$

we get

$$\begin{aligned}
\nu &= -\frac{\sin\lambda\Omega}{\sin\Omega}\cos\theta\sin\theta\cos\varphi\, u + \cos\lambda\Omega\sin\theta\, u + \frac{\sin\lambda\Omega}{\sin\Omega}\sin\varphi\, v \\
&\quad - \frac{\sin\lambda\Omega}{\sin\Omega}\sin\theta(\cos\Omega - \cos\theta\cos\varphi)u \\
&= \cos\lambda\Omega\sin\theta\, u + \frac{\sin\lambda\Omega}{\sin\Omega}\sin\varphi\, v - \frac{\sin\lambda\Omega}{\sin\Omega}\sin\theta\cos\Omega\, u \\
&= \frac{(\cos\lambda\Omega\sin\Omega - \sin\lambda\Omega\cos\Omega)}{\sin\Omega}\sin\theta\, u + \frac{\sin\lambda\Omega}{\sin\Omega}\sin\varphi\, v \\
&= \frac{\sin(1-\lambda)\Omega}{\sin\Omega}\sin\theta\, u + \frac{\sin\lambda\Omega}{\sin\Omega}\sin\varphi\, v.
\end{aligned}$$

Putting the scalar part and the vector part together, we obtain

$$\begin{aligned}
q_1(q_1^{-1}q_2)^\lambda &= \left[\frac{\sin(1-\lambda)\Omega}{\sin\Omega}\cos\theta + \frac{\sin\lambda\Omega}{\sin\Omega}\cos\varphi,\ \frac{\sin(1-\lambda)\Omega}{\sin\Omega}\sin\theta\, u + \frac{\sin\lambda\Omega}{\sin\Omega}\sin\varphi\, v\right], \\
&= \frac{\sin(1-\lambda)\Omega}{\sin\Omega}[\cos\theta,\ \sin\theta\, u] + \frac{\sin\lambda\Omega}{\sin\Omega}[\cos\varphi,\ \sin\varphi\, v].
\end{aligned}$$

This yields the celebrated *slerp interpolation formula*

$$Z(\lambda) = q_1(q_1^{-1}q_2)^\lambda = \frac{\sin(1-\lambda)\Omega}{\sin\Omega}q_1 + \frac{\sin\lambda\Omega}{\sin\Omega}q_2,$$

with

$$\cos\Omega = \cos\theta\cos\varphi + \sin\theta\sin\varphi(u\cdot v).$$

16.7 Nonexistence of a "Nice" Section from SO(3) to SU(2)

We conclude by discussing the problem of a consistent choice of sign for the quaternion q representing a rotation $R = \rho_q \in \mathbf{SO}(3)$. We are looking for a "nice" section $s\colon \mathbf{SO}(3) \to \mathbf{SU}(2)$, that is, a function s satisfying the condition

$$\rho\circ s = \mathrm{id},$$

where ρ is the surjective homomorphism $\rho\colon \mathbf{SU}(2) \to \mathbf{SO}(3)$.