and defined on generators by

$$\Big(\bigwedge^p f\Big)(u_1 \wedge \cdots \wedge u_p) = f(u_1) \wedge \cdots \wedge f(u_p).$$

Combining $\bigwedge^p$ and duality, we get a linear map $\bigwedge^p f^\top \colon \bigwedge^p F^* \to \bigwedge^p E^*$ defined on generators by

$$\Big(\bigwedge^p f^\top\Big)(v_1^* \wedge \cdots \wedge v_p^*) = f^\top(v_1^*) \wedge \cdots \wedge f^\top(v_p^*).$$

Proposition 34.11. *If $f \colon E \to F$ is any linear map between two finite-dimensional vector spaces E and F, then*

$$\mu\Big(\Big(\bigwedge^p f^\top\Big)(\omega)\Big)(u_1, \ldots, u_p) = \mu(\omega)(f(u_1), \ldots, f(u_p)), \qquad \omega \in \bigwedge^p F^*,\ \ u_1, \ldots, u_p \in E.$$

Proof. It is enough to prove the formula on generators. By definition of μ, we have

$$\begin{aligned}
\mu\Big(\Big(\bigwedge^p f^\top\Big)(v_1^* \wedge \cdots \wedge v_p^*)\Big)(u_1, \ldots, u_p) &= \mu(f^\top(v_1^*) \wedge \cdots \wedge f^\top(v_p^*))(u_1, \ldots, u_p) \\
&= \det(f^\top(v_j^*)(u_i)) \\
&= \det(v_j^*(f(u_i))) \\
&= \mu(v_1^* \wedge \cdots \wedge v_p^*)(f(u_1), \ldots, f(u_p)),
\end{aligned}$$

as claimed. ∎

Remark: The map $\bigwedge^p f^\top$ is often denoted f^*, although this is an ambiguous notation since p is dropped. Proposition 34.11 gives us the behavior of $\bigwedge^p f^\top$ under the identification of $\bigwedge^p E^*$ and $\mathrm{Alt}^p(E; K)$ *via* the isomorphism μ.

As in the case of symmetric powers, the map from E^n to $\bigwedge^n(E)$ given by $(u_1, \ldots, u_n) \mapsto u_1 \wedge \cdots \wedge u_n$ yields a surjection $\pi \colon E^{\otimes n} \to \bigwedge^n(E)$. Now this map has some section, so there is some injection $\eta \colon \bigwedge^n(E) \to E^{\otimes n}$ with $\pi \circ \eta = \mathrm{id}$. As we saw in Proposition 34.10 there is a canonical isomorphism

$$(\bigwedge^n(E))^* \cong \bigwedge^n(E^*)$$

for any field K, even of positive characteristic. However, if our field K has characteristic 0, then there is a special section having a natural definition involving an antisymmetrization process.

Recall, from Section 33.10 that we have a left action of the symmetric group $\mathfrak{S}_n$ on $E^{\otimes n}$. The tensors $z \in E^{\otimes n}$ such that

$$\sigma \cdot z = \mathrm{sgn}(\sigma)\, z, \quad \text{for all} \quad \sigma \in \mathfrak{S}_n$$

are called *antisymmetrized* tensors. We define the map $\eta \colon E^n \to E^{\otimes n}$ by

$$\eta(u_1, \ldots, u_n) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma)\, u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)}.^1$$

[1] It is the division by $n!$ that requires the field to have characteristic zero.