

Chapter 37

Topology

37.1 Metric Spaces and Normed Vector Spaces

This chapter contains a review of basic topological concepts. First metric spaces are defined. Next normed vector spaces are defined. Closed and open sets are defined, and their basic properties are stated. The general concept of a topological space is defined. The closure and the interior of a subset are defined. The subspace topology and the product topology are defined. Continuous maps and homeomorphisms are defined. Limits of sequences are defined. Continuous linear maps and multilinear maps are defined and studied briefly. The chapter ends with the definition of a normed affine space.

Most spaces considered in this book have a topological structure given by a metric or a norm, and we first review these notions. We begin with metric spaces. Recall that $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$.

Definition 37.1. A *metric space* is a set E together with a function $d: E \times E \rightarrow \mathbb{R}_+$, called a *metric*, or *distance*, assigning a nonnegative real number $d(x, y)$ to any two points $x, y \in E$, and satisfying the following conditions for all $x, y, z \in E$:

$$(D1) \quad d(x, y) = d(y, x). \quad (\text{symmetry})$$

$$(D2) \quad d(x, y) \geq 0, \text{ and } d(x, y) = 0 \text{ iff } x = y. \quad (\text{positivity})$$

$$(D3) \quad d(x, z) \leq d(x, y) + d(y, z). \quad (\text{triangle inequality})$$

Geometrically, Condition (D3) expresses the fact that in a triangle with vertices x, y, z , the length of any side is bounded by the sum of the lengths of the other two sides. From (D3), we immediately get

$$|d(x, y) - d(y, z)| \leq d(x, z).$$

Let us give some examples of metric spaces. Recall that the *absolute value* $|x|$ of a real number $x \in \mathbb{R}$ is defined such that $|x| = x$ if $x \geq 0$, $|x| = -x$ if $x < 0$, and for a complex number $x = a + ib$, by $|x| = \sqrt{a^2 + b^2}$.