by substituting in the Equation $(**)$ we get

$$\left(\nu - \frac{p_f + q_f}{p+q}\right)\eta = \frac{p_f - q_f}{p+q} b + \frac{1}{p+q} w^\top \left(\sum_{i \in K_\mu} v_j - \sum_{i \in K_\lambda} u_i\right) + \begin{pmatrix}\lambda^\top & \mu^\top\end{pmatrix} X^\top X \begin{pmatrix}\lambda \\ \mu\end{pmatrix}.$$

We also know that $w^\top u_{i_0} - b = \eta$ and $-w^\top v_{j_0} + b = \eta$ for some i_0 and some j_0. In the first case $b = -\eta + w^\top u_{i_0}$, and by substituting b in the above equation we get the equation

$$\begin{aligned}\left(\nu - \frac{p_f + q_f}{p+q}\right)\eta = -\frac{p_f - q_f}{p+q}\eta + \frac{p_f - q_f}{p+q} w^\top u_{i_0} + \frac{1}{p+q} w^\top \left(\sum_{i \in K_\mu} v_j - \sum_{i \in K_\lambda} u_i\right) \\ + \begin{pmatrix}\lambda^\top & \mu^\top\end{pmatrix} X^\top X \begin{pmatrix}\lambda \\ \mu\end{pmatrix},\end{aligned}$$

that is,

$$\begin{aligned}\left(\nu - \frac{2q_f}{p+q}\right)\eta = \frac{p_f - q_f}{p+q} w^\top u_{i_0} + \frac{1}{p+q} w^\top \left(\sum_{i \in K_\mu} v_j - \sum_{i \in K_\lambda} u_i\right) \\ + \begin{pmatrix}\lambda^\top & \mu^\top\end{pmatrix} X^\top X \begin{pmatrix}\lambda \\ \mu\end{pmatrix}.\end{aligned}$$

In the second case $b = \eta + w^\top v_{j_0}$, and we get the equation

$$\begin{aligned}\left(\nu - \frac{p_f + q_f}{p+q}\right)\eta = \frac{p_f - q_f}{p+q}\eta + \frac{p_f - q_f}{p+q} w^\top v_{j_0} + \frac{1}{p+q} w^\top \left(\sum_{i \in K_\mu} v_j - \sum_{i \in K_\lambda} u_i\right) \\ + \begin{pmatrix}\lambda^\top & \mu^\top\end{pmatrix} X^\top X \begin{pmatrix}\lambda \\ \mu\end{pmatrix},\end{aligned}$$

that is,

$$\begin{aligned}\left(\nu - \frac{2p_f}{p+q}\right)\eta = \frac{p_f - q_f}{p+q} w^\top v_{j_0} + \frac{1}{p+q} w^\top \left(\sum_{i \in K_\mu} v_j - \sum_{i \in K_\lambda} u_i\right) \\ + \begin{pmatrix}\lambda^\top & \mu^\top\end{pmatrix} X^\top X \begin{pmatrix}\lambda \\ \mu\end{pmatrix}.\end{aligned}$$

We need to choose ν such that $2p_f/(p+q) - \nu \neq 0$ and $2q_f/(p+q) - \nu \neq 0$. Since by Proposition 54.1, we have $\max\{2p_f/(p+q), 2q_f/(p+q)\} \leq \nu$, it suffices to pick ν such that $\max\{2p_f/(p+q), 2q_f/(p+q)\} < \nu$. If this condition is satisfied we can solve for η, and then we find b from either $b = -\eta + w^\top u_{i_0}$ or $b = \eta + w^\top v_{j_0}$. □

Remark: Of course the hypotheses of the proposition imply that $w^\top u_{i_0} - b = \eta$ and $-w^\top v_{j_0} + b = \eta$ for some i_0 and some j_0. Thus we can also compute b and η using the formulae

$$\begin{aligned} b &= \frac{w^\top (u_{i_0} + v_{j_0})}{2} \\ \eta &= \frac{w^\top (u_{i_0} - v_{j_0})}{2}. \end{aligned}$$