It should be observed that the conditions of Theorem 41.1 are typically quite stringent. It can be shown that Theorem 41.1 applies to the function f of Example 41.1 given by $f(x) = x^2 - \alpha$ with $\alpha > 0$, for any $x_0 > 0$ such that

$$\frac{6}{7}\alpha \leq x_0^2 \leq \frac{6}{5}\alpha,$$

with $\beta = 2/5$, $r = (1/6)x_0$, $M = 3/(5x_0)$. Such values of x_0 are quite close to $\sqrt{\alpha}$.

If we assume that we already know that some element $a \in \Omega$ is a zero of f, the next theorem gives sufficient conditions for a special version of a generalized Newton method to converge. For this special method the linear isomorphisms $A_k(x)$ are independent of $x \in \Omega$.

Theorem 41.2. *Let X be a Banach space and let $f \colon \Omega \to Y$ be differentiable on the open subset $\Omega \subseteq X$. If $a \in \Omega$ is a point such that $f(a) = 0$, if $f'(a)$ is a linear isomorphism, and if there is some λ with $0 < \lambda < 1/2$ such that*

$$\sup_{k \geq 0} \|A_k - f'(a)\|_{\mathcal{L}(X;Y)} \leq \frac{\lambda}{\|(f'(a))^{-1}\|_{\mathcal{L}(Y;X)}},$$

then there is a closed ball B of center a such that for every $x_0 \in B$, the sequence (x_k) defined by

$$x_{k+1} = x_k - A_k^{-1}(f(x_k)), \quad k \geq 0,$$

is entirely contained within B and converges to a, which is the only zero of f in B. Furthermore, the convergence is geometric, which means that

$$\|x_k - a\| \leq \beta^k \|x_0 - a\|,$$

for some $\beta < 1$.

A proof of Theorem 41.2 can be found in Ciarlet [41] (Section 7.5).

For the sake of completeness, we state a version of the Newton–Kantorovich theorem which corresponds to the case where $A_k(x) = f'(x)$. In this instance, a stronger result can be obtained especially regarding upper bounds, and we state a version due to Gragg and Tapia which appears in Problem 7.5-4 of Ciarlet [41].

Theorem 41.3. *(Newton–Kantorovich) Let X be a Banach space, and let $f \colon \Omega \to Y$ be differentiable on the open subset $\Omega \subseteq X$. Assume that there exist three positive constants λ, μ, ν and a point $x_0 \in \Omega$ such that*

$$0 < \lambda\mu\nu \leq \frac{1}{2},$$