and we set

$$\sigma(u) = 0$$

if u is a unit.

We can't divide anymore, but we can find gcd's and use Bezout to mimic division. The key ingredient is this: for any two nonzero elements $a, b \in A$, if a does not divide b then let $d \neq 0$ be a gcd of a and b. By Bezout, there exist $x, y \in A$ such that

$$ax + by = d.$$

We can also write $a = td$ and $b = -sd$, for some $s, t \in A$, so that $tdx - sdy = d$, which implies that

$$tx - sy = 1,$$

since A is an integral domain. Observe that

$$\begin{pmatrix} t & -s \\ -y & x \end{pmatrix} \begin{pmatrix} x & s \\ y & t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which shows that both matrices on the left of the equation are invertible, and so is the transpose of the second one,

$$\begin{pmatrix} x & y \\ s & t \end{pmatrix}$$

(they all have determinant 1). We also have

$$as + bt = tds - sdt = 0,$$

so

$$\begin{pmatrix} x & y \\ s & t \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} d \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} x & s \\ y & t \end{pmatrix} = \begin{pmatrix} d & 0 \end{pmatrix}.$$

Because a does not divide b, their gcd d has strictly fewer prime factors than a, so

$$\sigma(d) < \sigma(a).$$

Using matrices of the form

$$\begin{pmatrix} x & y & 0 & 0 & \cdots & 0 \\ s & t & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 \end{pmatrix}$$

with $xt - ys = 1$, we can modify Steps 2a and Step 2b to obtain the following theorem.