and by Proposition A.4 (1), we have

$$g(f\,((\lambda_k)_{k\in K})) = (\lambda_k)_{k\in K},$$

and thus $g \circ f = \mathrm{id}$ and $f \circ g = \mathrm{id}$. By Proposition A.4 (2), the linear map g is an isometry. Therefore, f is a linear bijection and an isometry between $\ell^2(K)$ and E, with inverse g. $\square$

Remark: The surjectivity of the map $g\colon E \to \ell^2(K)$ is known as the *Riesz–Fischer* theorem.

Having done all this hard work, we sketch how these results apply to Fourier series. Again we refer the readers to Rudin [140] or Lang [111, 112] for a comprehensive exposition.

Let $\mathcal{C}(T)$ denote the set of all periodic continuous functions $f\colon [-\pi,\pi] \to \mathbb{C}$ with period 2π. There is a Hilbert space $\mathrm{L}^2(T)$ containing $\mathcal{C}(T)$ and such that $\mathcal{C}(T)$ is dense in $\mathrm{L}^2(T)$, whose inner product is given by

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)\overline{g(x)}dx.$$

The Hilbert space $\mathrm{L}^2(T)$ is the space of *Lebesgue square-integrable periodic functions* (of period 2π).

It turns out that the family $(e^{ikx})_{k\in\mathbb{Z}}$ is a total orthogonal family in $\mathrm{L}^2(T)$, because it is already dense in $\mathcal{C}(T)$ (for instance, see Rudin [140]). Then the Riesz–Fischer theorem says that for every family $(c_k)_{k\in\mathbb{Z}}$ of complex numbers such that

$$\sum_{k\in\mathbb{Z}} |c_k|^2 < \infty,$$

there is a unique function $f \in \mathrm{L}^2(T)$ such that f is equal to its Fourier series

$$f(x) = \sum_{k\in\mathbb{Z}} c_k e^{ikx},$$

where the Fourier coefficients c_k of f are given by the formula

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-ikt}dt.$$

The Parseval theorem says that

$$\sum_{k=-\infty}^{+\infty} c_k\overline{d_k} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\overline{g(t)}dt$$

for all $f, g \in \mathrm{L}^2(T)$, where c_k and d_k are the Fourier coefficients of f and g.