Since J is differentiable at u, we have

$$0 \leq J(u_k) - J(u) = J'_u(u_k - u) + \|u_k - u\| \, \epsilon_k, \tag{$*$}$$

for some sequence $(\epsilon_k)_{k\geq 0}$ such that $\lim_{k\mapsto\infty} \epsilon_k = 0$. Since J'_u is linear and continuous, and since

$$u_k - u = \|u_k - u\| \frac{w}{\|w\|} + \|u_k - u\| \, \delta_k, \quad \lim_{k\mapsto\infty} \delta_k = 0, \;\; w \neq 0,$$

$(*)$ implies that

$$0 \leq \frac{\|u_k - u\|}{\|w\|} \, (J'_u(w) + \eta_k),$$

with

$$\eta_k = \|w\| \, (J'_u(\delta_k) + \epsilon_k).$$

Since J'_u is continuous, we have $\lim_{k\mapsto\infty} \eta_k = 0$. But then $J'_u(w) \geq 0$, since if $J'_u(w) < 0$, then for k large enough the expression $J'_u(w) + \eta_k$ would be negative, and since $u_k \neq u$, the expression $(\|u_k - u\| \, / \, \|w\|)(J'_u(w) + \eta_k)$ would also be negative, a contradiction. $\square$

From now on we assume that U is defined by a set of inequalities, that is

$$U = \{x \in \Omega \mid \varphi_i(x) \leq 0, \;\; 1 \leq i \leq m\},$$

where the functions $\varphi_i \colon \Omega \to \mathbb{R}$ are continuous (and usually differentiable). As we explained earlier, an equality constraint $\varphi_i(x) = 0$ is treated as the conjunction of the two inequalities $\varphi_i(x) \leq 0$ and $-\varphi_i(x) \leq 0$. Later on we will see that when the functions φ_i are convex, since $-\varphi_i$ is not necessarily convex, it is desirable to treat equality constraints separately, but for the time being we won't.

50.2 Active Constraints and Qualified Constraints

Our next goal is find sufficient conditions for the cone $C(u)$ to be convex, for any $u \in U$. For this we assume that the functions φ_i are differentiable at u. It turns out that the constraints φ_i that matter are those for which $\varphi_i(u) = 0$, namely the constraints that are tight, or as we say, active.

Definition 50.3. Given m functions $\varphi_i \colon \Omega \to \mathbb{R}$ defined on some open subset Ω of some vector space V, let U be the set defined by

$$U = \{x \in \Omega \mid \varphi_i(x) \leq 0, \;\; 1 \leq i \leq m\}.$$

For any $u \in U$, a constraint φ_i is said to be *active* at u if $\varphi_i(u) = 0$, else *inactive* at u if $\varphi_i(u) < 0$.

If a constraint φ_i is active at u, this corresponds to u being on a piece of the boundary of U determined by some of the equations $\varphi_i(u) = 0$; see Figure 50.6.