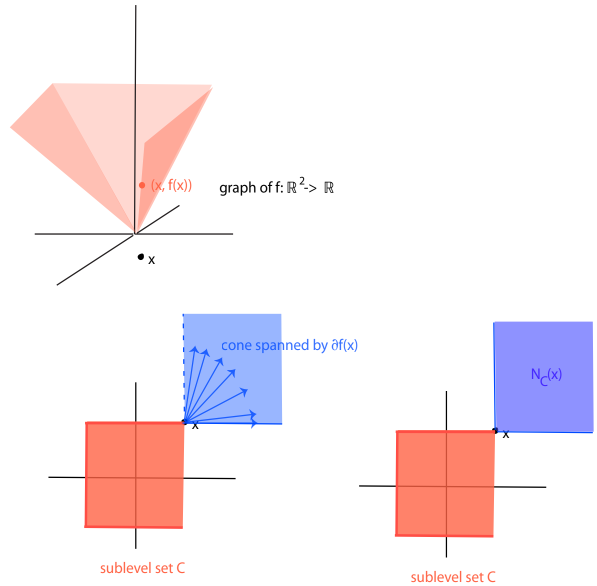

Figure 51.22: Let f be the proper convex function whose graph in $\mathbb{R}^3$ is the peach polyhedral surface. The sublevel set $C = \{z \in \mathbb{R}^2 \mid f(z) \leq f(x)\}$ is the orange square which is closed on three sides. Then the normal cone $N_C(x)$ is the closure of the convex cone spanned by $\partial f(x)$.

is nonempty, closed and bounded. If

$$\alpha = \sup_{y \in \partial f(S)} \|y\|_2 < +\infty,$$

then f is Lipschitizan on S, and we have

$$\begin{aligned} f'(x; z) &\leq \alpha \|z\|_2 && \text{for all } x \in S \text{ and all } z \in \mathbb{R}^n \\ |f(y) - f(x)| &\leq \alpha \|y - z\|_2 && \text{for all } x, y \in S. \end{aligned}$$

Proposition 51.24 is proven in Rockafellar [138] (Theorem 24.7).

The subdifferentials of a proper convex function f and its conjugate f^* are closely related. First, we have the following proposition from Rockafellar [138] (Theorem 12.2).

Proposition 51.27. *Let f be convex function on $\mathbb{R}^n$. The conjugate function f^* of f is a closed and convex function, proper iff f is proper. Furthermore, $(\mathrm{cl}(f))^* = f^*$, and $f^{**} = \mathrm{cl}(f)$.*

As a corollary of Proposition 51.27, it can be shown that

$$f^*(y) = \sup_{x \in \mathbf{relint}(\mathrm{dom}(f))} (\langle x, y \rangle - f(x)).$$

The following result is proven in Rockafellar [138] (Theorem 23.5).