(2) B is a maximal linearly independent family of E.

(3) B is a minimal generating family of E.

Proof. We will first prove the equivalence of (1) and (2). Assume (1). Since B is a basis, it is a linearly independent family. We claim that B is a maximal linearly independent family. If B is not a maximal linearly independent family, then there is some vector $w \in E$ such that the family B' obtained by adding w to B is linearly independent. However, since B is a basis of E, the vector w can be expressed as a linear combination of vectors in B, contradicting the fact that B' is linearly independent.

Conversely, assume (2). We claim that B spans E. If B does not span E, then there is some vector $w \in E$ which is not a linear combination of vectors in B. By Lemma 3.6, the family B' obtained by adding w to B is linearly independent. Since B is a proper subfamily of B', this contradicts the assumption that B is a maximal linearly independent family. Therefore, B must span E, and since B is also linearly independent, it is a basis of E.

Now we will prove the equivalence of (1) and (3). Again, assume (1). Since B is a basis, it is a generating family of E. We claim that B is a minimal generating family. If B is not a minimal generating family, then there is a proper subfamily B' of B that spans E. Then, every $w \in B - B'$ can be expressed as a linear combination of vectors from B', contradicting the fact that B is linearly independent.

Conversely, assume (3). We claim that B is linearly independent. If B is not linearly independent, then some vector $w \in B$ can be expressed as a linear combination of vectors in $B' = B - \{w\}$. Since B generates E, the family B' also generates E, but B' is a proper subfamily of B, contradicting the minimality of B. Since B spans E and is linearly independent, it is a basis of E. $\square$

The second key result of linear algebra is that *for any two bases $(u_i)_{i\in I}$ and $(v_j)_{j\in J}$ of a vector space E, the index sets I and J have the same cardinality.* In particular, if E has a finite basis of n elements, every basis of E has n elements, and the integer n is called the *dimension* of the vector space E.

To prove the second key result, we can use the following *replacement lemma* due to Steinitz. This result shows the relationship between finite linearly independent families and finite families of generators of a vector space. We begin with a version of the lemma which is a bit informal, but easier to understand than the precise and more formal formulation given in Proposition 3.10. The technical difficulty has to do with the fact that some of the indices need to be renamed.

Proposition 3.9. *(Replacement lemma, version 1) Given a vector space E, let $(u_1, \ldots, u_m)$ be any finite linearly independent family in E, and let $(v_1, \ldots, v_n)$ be any finite family such that every u_i is a linear combination of $(v_1, \ldots, v_n)$. Then we must have $m \leq n$, and there is a replacement of m of the vectors v_j by $(u_1, \ldots, u_m)$, such that after renaming some of the indices of the v_js, the families $(u_1, \ldots, u_m, v_{m+1}, \ldots, v_n)$ and $(v_1, \ldots, v_n)$ generate the same subspace of E.*