and

$$\{j_1, \ldots, j_q, \ldots j_k\},$$

where $i_p = i$ and $j_q = j$, since

$$\tau(\pi(\pi^{-1}(i_p))) = \tau(i_p) = \tau(i) = j = j_q$$

and

$$\tau(\pi(\pi^{-1}(j_q))) = \tau(j_q) = \tau(j) = i = i_p,$$

we see that the classes J_l and J_m merge into a single class, and thus, the number of classes associated with $\tau \circ \pi$ is $r - 1$, and $\epsilon(\tau \circ \pi) = (-1)^{n-r+1} = -(-1)^{n-r} = -\epsilon(\pi)$.

Now, let $\pi = \tau_m \circ \ldots \circ \tau_1$ be any product of transpositions. By the first part of the proposition, we have

$$\epsilon(\pi) = (-1)^{m-1}\epsilon(\tau_1) = (-1)^{m-1}(-1) = (-1)^m,$$

since $\epsilon(\tau_1) = -1$ for a transposition. $\square$

Remark: When $\pi = \mathrm{id}_n$ is the identity permutation, since we agreed that the composition of 0 transpositions is the identity, it it still correct that $(-1)^0 = \epsilon(\mathrm{id}) = +1$. From the proposition, it is immediate that $\epsilon(\pi' \circ \pi) = \epsilon(\pi')\epsilon(\pi)$. In particular, since $\pi^{-1} \circ \pi = \mathrm{id}_n$, we get $\epsilon(\pi^{-1}) = \epsilon(\pi)$.

We can now proceed with the definition of determinants.

7.2 Alternating Multilinear Maps

First we define multilinear maps, symmetric multilinear maps, and alternating multilinear maps.

Remark: Most of the definitions and results presented in this section also hold when K is a commutative ring and when we consider modules over K (free modules, when bases are needed).

Let $E_1, \ldots, E_n$, and F, be vector spaces over a field K, where $n \geq 1$.

Definition 7.3. A function $f\colon E_1 \times \ldots \times E_n \to F$ is a *multilinear map (or an n-linear map)* if it is linear in each argument, holding the others fixed. More explicitly, for every i, $1 \leq i \leq n$, for all $x_1 \in E_1, \ldots, x_{i-1} \in E_{i-1}$, $x_{i+1} \in E_{i+1}, \ldots, x_n \in E_n$, for all $x, y \in E_i$, for all $\lambda \in K$,

$$\begin{aligned} f(x_1, \ldots, x_{i-1}, x + y, x_{i+1}, \ldots, x_n) &= f(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n) \\ &\quad + f(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n), \\ f(x_1, \ldots, x_{i-1}, \lambda x, x_{i+1}, \ldots, x_n) &= \lambda f(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n). \end{aligned}$$