Appendix C

Zorn's Lemma; Some Applications

C.1 Statement of Zorn's Lemma

Zorn's lemma is a particularly useful form of the axiom of choice, especially for algebraic applications. Readers who want to learn more about Zorn's lemma and its applications to algebra should consult either Lang [109], Appendix 2, §2 (pp. 878-884) and Chapter III, §5 (pp. 139-140), or Artin [7], Appendix §1 (pp. 588-589). For the logical ramifications of Zorn's lemma and its equivalence with the axiom of choice, one should consult Schwartz [150], (Vol. 1), Chapter I, §6, or a text on set theory such as Enderton [56], Suppes [173], or Kuratowski and Mostowski [108].

Given a set, S, a *partial order, $\leq$, on S* is a binary relation on S (i.e., $\leq \subseteq S \times S$) which is

(1) *reflexive*, i.e., $x \leq x$, for all $x \in S$,

(2) *transitive*, i.e, if $x \leq y$ and $y \leq z$, then $x \leq z$, for all $x, y, z \in S$, and

(3) *antisymmetric*, i.e, if $x \leq y$ and $y \leq x$, then $x = y$, for all $x, y \in S$.

A pair $(S, \leq)$, where $\leq$ is a partial order on S, is called a *partially ordered set* or *poset*. Given a poset, $(S, \leq)$, a subset, C, of S is *totally ordered* or a *chain* if for every pair of elements $x, y \in C$, either $x \leq y$ or $y \leq x$. The empty set is trivially a chain. A subset, P, (empty or not) of S is *bounded* if there is some $b \in S$ so that $x \leq b$ for all $x \in P$. Observe that the empty subset of S is bounded if and only if S is nonempty. A *maximal element of P* is an element, $m \in P$, so that $m \leq x$ implies that $m = x$, for all $x \in P$. Zorn's lemma can be stated as follows:

Lemma C.1. *Given a partially ordered set, $(S, \leq)$, if every chain is bounded, then S has a maximal element.*

Proof. See any of Schwartz [150], Enderton [56], Suppes [173], or Kuratowski and Mostowski [108]. $\square$