Example 50.3. Let J, φ_1 and φ_2 be the functions defined on $\mathbb{R}$ by

$$\begin{aligned}J(x) &= x\\ \varphi_1(x) &= -x\\ \varphi_2(x) &= x-1.\end{aligned}$$

In this case

$$U = \{x \in \mathbb{R} \mid -x \leq 0,\ x-1 \leq 0\} = [0,1].$$

Since the constraints are affine, they are automatically qualified for any $u \in [0,1]$. The system of equations and inequalities shown above becomes

$$\begin{aligned}1 - \lambda_1 + \lambda_2 &= 0\\ -\lambda_1 x + \lambda_2(x-1) &= 0\\ -x &\leq 0\\ x-1 &\leq 0\\ \lambda_1, \lambda_2 &\geq 0.\end{aligned}$$

The first equality implies that $\lambda_1 = 1 + \lambda_2$. The second equality then becomes

$$-(1+\lambda_2)x + \lambda_2(x-1) = 0,$$

which implies that $\lambda_2 = -x$. Since $0 \leq x \leq 1$, or equivalently $-1 \leq -x \leq 0$, and $\lambda_2 \geq 0$, we conclude that $\lambda_2 = 0$ and $\lambda_1 = 1$ is the solution associated with $x = 0$, the minimum of $J(x) = x$ over $[0,1]$. Observe that the case $x = 1$ corresponds to the maximum and not a minimum of $J(x) = x$ over $[0,1]$.

Remark: Unless the linear forms $(\varphi'_i)_u$ for $i \in I(u)$ are linearly independent, the $\lambda_i(u)$ are generally not unique. Also, if $I(u) = \emptyset$, then the KKT conditions reduce to $J'_u = 0$. This is not surprising because in this case u belongs to the relative interior of U.

If the constraints are all affine equality constraints, then the KKT conditions are a bit simpler. We will consider this case shortly.

The conditions for the qualification of nonaffine constraints are hard (if not impossible) to use in practice, because they depend on $u \in U$ and on the derivatives $(\varphi'_i)_u$. Thus it is desirable to find simpler conditions. Fortunately, this is possible if the nonaffine functions φ_i are *convex*.

Definition 50.6. Let $U \subseteq \Omega \subseteq V$ be given by

$$U = \{x \in \Omega \mid \varphi_i(x) \leq 0,\ 1 \leq i \leq m\},$$

where Ω is an open subset of the Euclidean vector space V. If the functions $\varphi_i \colon \Omega \to \mathbb{R}$ are convex, we say that the constraints are *qualified* if the following conditions hold: