where $u'_{k+1} \in U'_k$ and $u''_{k+1} \in U''_k$. Let

$$r_{k+1,k+1} = \left\| u''_{k+1} \right\|, \quad \text{and} \quad e^{i\theta_{k+1}} |u''_{k+1} \cdot e_{k+1}| = u''_{k+1} \cdot e_{k+1}.$$

If $u''_{k+1} = e^{i\theta_{k+1}} r_{k+1,k+1}\, e_{k+1}$, we let $h_{k+1} = \mathrm{id}$. Otherwise, by Proposition 14.19(1) (with $u = u''_{k+1}$ and $v = r_{k+1,k+1}\, e_{k+1}$), there is a unique hyperplane reflection h_{k+1} such that

$$h_{k+1}(u''_{k+1}) = e^{i\theta_{k+1}} r_{k+1,k+1}\, e_{k+1},$$

where h_{k+1} is the reflection about the hyperplane H_{k+1} orthogonal to the vector

$$w_{k+1} = r_{k+1,k+1}\, e_{k+1} - e^{-i\theta_{k+1}} u''_{k+1}.$$

At the end of the induction, we have a triangular matrix R, but the diagonal entries $e^{i\theta_j} r_{j,j}$ of R may be complex. Letting

$$h_n = \rho_{e_n,\,-\theta_n} \circ \cdots \circ \rho_{e_1,-\theta_1},$$

we observe that the diagonal entries of the matrix of vectors

$$r'_j = h_n \circ h_{n-1} \circ \cdots \circ h_2 \circ h_1(v_j)$$

is triangular with nonnegative entries. $\square$

Remark: For numerical stability, it is preferable to use $w_{k+1} = r_{k+1,k+1}\, e_{k+1} + e^{-i\theta_{k+1}} u''_{k+1}$ instead of $w_{k+1} = r_{k+1,k+1}\, e_{k+1} - e^{-i\theta_{k+1}} u''_{k+1}$. The effect of that choice is that the diagonal entries in R will be of the form $-e^{i\theta_j} r_{j,j} = e^{i(\theta_j+\pi)} r_{j,j}$. Of course, we can make these entries nonegative by applying

$$h_n = \rho_{e_n,\,\pi-\theta_n} \circ \cdots \circ \rho_{e_1,\pi-\theta_1}$$

after h_{n-1}.

As in the Euclidean case, Proposition 14.20 immediately implies the QR-decomposition for arbitrary complex $n \times n$-matrices, where Q is now unitary (see Kincaid and Cheney [102] and Ciarlet [41]).

Proposition 14.21. *For every complex $n \times n$-matrix A, there is a sequence $H_1, \ldots, H_{n-1}$ of matrices, where each H_i is either a Householder matrix or the identity, and an upper triangular matrix R, such that*

$$R = H_{n-1} \cdots H_2 H_1 A.$$

As a corollary, there is a pair of matrices Q, R, where Q is unitary and R is upper triangular, such that $A = QR$ (a QR-decomposition of A). Furthermore, R can be chosen so that its diagonal entries are nonnegative. This can be achieved by a diagonal matrix D with entries such that $|d_{ii}| = 1$ for $i = 1, \ldots, n$, and we have $A = \widetilde{Q}\widetilde{R}$ with

$$\widetilde{Q} = H_1 \cdots H_{n-1} D, \quad \widetilde{R} = D^* R,$$

where $\widetilde{R}$ is upper triangular and has nonnegative diagonal entries.

Proof. It is essentially identical to the proof of Proposition 13.4, and we leave the details as an exercise. For the last statement, observe that $h_n \circ \cdots \circ h_1$ is also an isometry. $\square$