Chapter 43

Schur Complements and Applications

Schur complements arise naturally in the process of inverting block matrices of the form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

and in characterizing when symmetric versions of these matrices are positive definite or positive semidefinite. These characterizations come up in various quadratic optimization problems; see Boyd and Vandenberghe [29], especially Appendix B. In the most general case, pseudo-inverses are also needed.

In this chapter we introduce Schur complements and describe several interesting ways in which they are used. Along the way we provide some details and proofs of some results from Appendix A.5 (especially Section A.5.5) of Boyd and Vandenberghe [29].

43.1 Schur Complements

Let M be an $n \times n$ matrix written as a 2×2 block matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A is a $p \times p$ matrix and D is a $q \times q$ matrix, with $n = p + q$ (so B is a $p \times q$ matrix and C is a $q \times p$ matrix). We can try to solve the linear system

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix},$$

that is,

$$\begin{aligned} Ax + By &= c, \\ Cx + Dy &= d, \end{aligned}$$