Theorem 32.15. *(Chinese Remainder Theorem) Given a commutative ring A, let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ be any $n \geq 2$ ideals of A such that $\mathfrak{a}_i + \mathfrak{a}_j = A$ for all $i \neq j$. Then, the homomorphism $\theta\colon A/\mathfrak{a}_1 \cdots \mathfrak{a}_n \to A/\mathfrak{a}_1 \times \cdots \times A/\mathfrak{a}_n$ is an isomorphism.*

Proof. The map $\theta\colon A/\mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_n \to A/\mathfrak{a}_1 \times \cdots \times A/\mathfrak{a}_n$ is induced by the homomorphism $\varphi\colon A \to A/\mathfrak{a}_1 \times \cdots \times A/\mathfrak{a}_n$ given by

$$\varphi(x) = (\overline{x}_{\mathfrak{a}_1}, \ldots, \overline{x}_{\mathfrak{a}_n}).$$

Clearly, $\mathrm{Ker}\,(\varphi) = \mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_n$, so θ is well-defined and injective. We need to prove that

$$\mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_n = \mathfrak{a}_1 \cdots \mathfrak{a}_n$$

and that θ is surjective. We proceed by induction. The case $n = 2$ is Theorem 32.14. By induction, assume that

$$\mathfrak{a}_2 \cap \cdots \cap \mathfrak{a}_n = \mathfrak{a}_2 \cdots \mathfrak{a}_n.$$

We claim that

$$\mathfrak{a}_1 + \mathfrak{a}_2 \cdots \mathfrak{a}_n = A.$$

Indeed, since $\mathfrak{a}_1 + \mathfrak{a}_i = A$ for $i = 2, \ldots, n$, there exist some $a_i \in \mathfrak{a}_1$ and some $b_i \in \mathfrak{a}_i$ such that

$$a_i + b_i = 1, \quad i = 2, \ldots, n,$$

and by multiplying these equations, we get

$$a + b_2 \cdots b_n = 1,$$

where a is a sum of terms each containing some a_j as a factor, so $a \in \mathfrak{a}_1$ and $b_2 \cdots b_n \in \mathfrak{a}_2 \cdots \mathfrak{a}_n$, which shows that

$$\mathfrak{a}_1 + \mathfrak{a}_2 \cdots \mathfrak{a}_n = A,$$

as claimed. It follows that

$$\mathfrak{a}_1 \cap \mathfrak{a}_2 \cap \cdots \cap \mathfrak{a}_n = \mathfrak{a}_1 \cap (\mathfrak{a}_2 \cdots \mathfrak{a}_n) = \mathfrak{a}_1 \mathfrak{a}_2 \cdots \mathfrak{a}_n.$$

Let us now prove that θ is surjective by induction. The case $n = 2$ is Theorem 32.14. Let $x_1, \ldots, x_n$ be any $n \geq 3$ elements of A. First, applying Theorem 32.14 to $\mathfrak{a}_1$ and $\mathfrak{a}_2 \cdots \mathfrak{a}_n$, we can find $y_1 \in A$ such that

$$\begin{aligned} y_1 &\equiv 1 \pmod{\mathfrak{a}_1} \\ y_1 &\equiv 0 \pmod{\mathfrak{a}_2 \cdots \mathfrak{a}_n}. \end{aligned}$$

By the induction hypothesis, we can find $y_2, \ldots, y_n \in A$ such that for all i, j with $2 \leq i, j \leq n$,

$$\begin{aligned} y_i &\equiv 1 \pmod{\mathfrak{a}_i} \\ y_i &\equiv 0 \pmod{\mathfrak{a}_j}, \quad j \neq i. \end{aligned}$$