What we just sketched is a general method to deal with rational curves. We can use our "hat construction" to embed an affine space $\mathcal{E}$ into a vector space $\widehat{\mathcal{E}}$ having one more dimension, then construct the projective space $\mathbf{P}(\widehat{\mathcal{E}})$. This turns out to be the "projective completion" of the affine space $\mathcal{E}$. Then we can define a rational curve in $\mathbf{P}(\widehat{\mathcal{E}})$, basically as the central projection of a polynomial curve in $\widehat{\mathcal{E}}$ back onto $\mathbf{P}(\widehat{\mathcal{E}})$. The same approach can be used to deal with rational surfaces. Due to the lack of space, such a presentation is omitted. However, it can be found on the web; see http://www.cis.upenn.edu/~jean/gbooks/geom2.html.

More generally, the projective completion of an affine space is a very convenient tool to handle "points at infinity" in a clean fashion.

This chapter contains a brief presentation of concepts of projective geometry. The following concepts are presented: projective spaces, projective frames, homogeneous coordinates, projective maps, projective hyperplanes, multiprojective maps, affine patches. The projective completion of an affine space is presented using the "hat construction." The theorems of Pappus and Desargues are proved, using the method in which points are "sent to infinity." We also discuss the cross-ratio and duality. The chapter ends with a very brief explanation of the use of the complexification of a projective space in order to define the notion of angle and orthogonality in a projective setting. We also include a short section on applications of projective geometry, notably to computer vision (camera calibration), efficient communication, and error-correcting codes.

26.2 Projective Spaces

As in the case of affine geometry, our presentation of projective geometry is rather sketchy. For a systematic treatment of projective geometry, we recommend Berger [11, 12], Samuel [142], Pedoe [136], Coxeter [45, 46, 43, 44], Beutelspacher and Rosenbaum [22], Fresnel [65], Sidler [161], Tisseron [175], Lehmann and Bkouche [115], Vienne [185], and the classical treatise by Veblen and Young [183, 184], which, although slightly old-fashioned, is definitely worth reading. Emil Artin's famous book [6] contains, among other things, an axiomatic presentation of projective geometry, and a wealth of geometric material presented from an algebraic point of view. Other "oldies but goodies" include the beautiful books by Darboux [47] and Klein [103]. For a development of projective geometry addressing the delicate problem of orientation, see Stolfi [167], and for an approach geared towards computer graphics, see Penna and Patterson [137].

First, we define projective spaces, allowing the field K to be arbitrary (which does no harm, and is needed to allow finite and complex projective spaces). Roughly speaking, every projective concept is a linear–algebraic concept "up to a scalar." For spaces, this is made precise as follows.

Definition 26.1. Given a vector space E over a field K, the *projective space* $\mathbf{P}(E)$ *induced by* E is the set $(E - \{0\})/\sim$ of equivalence classes of nonzero vectors in E under the