recall the definition of a Cauchy sequence and of a complete metric space.

Definition 37.38. Given a metric space, (E, d), a sequence, $(x_n)_{n\in\mathbb{N}}$, in E is a *Cauchy sequence* if the following condition holds: for every $\epsilon > 0$, there is some $p \geq 0$, such that, for all $m, n \geq p$, then $d(x_m, x_n) \leq \epsilon$.

If every Cauchy sequence in (E, d) converges we say that (E, d) is a *complete metric space*.

First let us show the following proposition:

Proposition 37.48. *Given a metric space, E, if a Cauchy sequence, (x_n), has some accumulation point, a, then a is the limit of the sequence, (x_n).*

Proof. Since (x_n) is a Cauchy sequence, for every $\epsilon > 0$, there is some $p \geq 0$, such that, for all $m, n \geq p$, then $d(x_m, x_n) \leq \epsilon/2$. Since a is an accumulation point for (x_n), for infinitely many n, we have $d(x_n, a) \leq \epsilon/2$, and thus, for at least some $n \geq p$, we have $d(x_n, a) \leq \epsilon/2$. Then, for all $m \geq p$,

$$d(x_m, a) \leq d(x_m, x_n) + d(x_n, a) \leq \epsilon,$$

which shows that a is the limit of the sequence (x_n). □

We can now prove the following theorem.

Theorem 37.49. *A metric space, E, is compact iff it is precompact and complete.*

Proof. Let E be compact. For every $\epsilon > 0$, the family of all open balls of radius ϵ is an open cover for E and since E is compact, there is a finite subcover, $B_0(a_0, \epsilon) \cup \cdots \cup B_0(a_n, \epsilon)$, of E by open balls of radius ϵ. Thus E is precompact. Since E is compact, by Theorem 37.47, every sequence, (x_n), has some accumulation point. Thus every Cauchy sequence, (x_n), has some accumulation point, a, and, by Proposition 37.48, a is the limit of (x_n). Thus, E is complete.

Now assume that E is precompact and complete. We prove that every sequence, (x_n), has an accumulation point. By the other direction of Theorem 37.47, this shows that E is compact. Given any sequence, (x_n), we construct a Cauchy subsequence, (y_n), of (x_n) as follows: Since E is precompact, letting $\epsilon = 1$, there exists a finite cover, $\mathcal{U}_1$, of E by open balls of radius 1. Thus some open ball, B_o^0, in the cover, $\mathcal{U}_1$, contains infinitely many elements from the sequence (x_n). Let y_0 be any element of (x_n) in B_o^0. By induction, assume that a sequence of open balls, $(B_o^i)_{1\leq i\leq m}$, has been defined, such that every ball, B_o^i, has radius $\frac{1}{2^i}$, contains infinitely many elements from the sequence (x_n) and contains some y_i from (x_n) such that

$$d(y_i, y_{i+1}) \leq \frac{1}{2^i},$$

for all i, $0 \leq i \leq m-1$. See Figure 37.45. Then letting $\epsilon = \frac{1}{2^{m+1}}$, because E is precompact, there is some finite cover, $\mathcal{U}_{m+1}$, of E by open balls of radius ϵ and thus, of the open ball B_o^m.