Problem 50.5. (From Boyd and Vandenberghe [29], Problem 5.12) Given an $m \times n$ matrix A and any vector $b \in \mathbb{R}^n$, consider the problem

$$\begin{aligned} \text{minimize} \quad & -\sum_{i=1}^{m} \log(b_i - a_i x) \\ \text{subject to} \quad & Ax < b, \end{aligned}$$

where a_i is the ith row of A. This is called the *analytic centering problem*. It can be shown that the problem has a unique solution iff the open polyhedron $\{x \in \mathbb{R}^n \mid Ax < b\}$ is nonempty and bounded.

(1) Prove that necessary and sufficient conditions for the problem to have an optimal solution are

$$Ax < b, \quad \sum_{i=1}^{m} \frac{a_i^\top}{b_i - a_x x} = 0.$$

(2) Derive a dual program for the above program.
Hint. First introduce new variables y_i and equations $y_i = b_i - a_i x$.

Problem 50.6. (From Boyd and Vandenberghe [29], Problem 5.13) A *Boolean linear program* is the following optimization problem:

$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{subject to} \quad & Ax \leq b \\ & x_i \in \{0, 1\},\ i = 1, \ldots, n, \end{aligned}$$

where A is an $m \times n$ matrix, $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$. The fact that the solutions $x \in \mathbb{R}^n$ are constrained to have coordinates x_i taking the values 0 or 1 makes it a hard problem. The above problem can be stated as a program with quadratic constraints:

$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{subject to} \quad & Ax \leq b \\ & x_i(1 - x_i) = 0,\ i = 1, \ldots, n. \end{aligned}$$

(1) Derive the Lagrangian dual of the above program.

(2) A way to approximate a solution of the Boolean linear program is to consider its linear relaxation where the constraints $x_i \in \{0, 1\}$ are replaced by the *linear constraints* $0 \leq x_i \leq 1$:

$$\begin{aligned} \text{minimize} \quad & c^\top x \\ \text{subject to} \quad & Ax \leq b \\ & 0 \leq x_i \leq 1,\ i = 1, \ldots, n. \end{aligned}$$

Find the dual linear program of the above linear program. Show that the maxima of the dual programs in (1) and (2) are the same.