43.2 Symmetric Positive Definite Matrices and Schur Complements

If we assume that our block matrix M is symmetric, so that A, D are symmetric and $C = B^\top$, then we see by Proposition 43.1 that M is expressed as

$$M = \begin{pmatrix} A & B \\ B^\top & D \end{pmatrix} = \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A - BD^{-1}B^\top & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix}^\top,$$

which shows that M is similar to a block diagonal matrix (obviously, the Schur complement, $A - BD^{-1}B^\top$, is symmetric). As a consequence, we have the following version of "Schur's trick" to check whether $M \succ 0$ for a symmetric matrix.

Proposition 43.3. *For any symmetric matrix M of the form*

$$M = \begin{pmatrix} A & B \\ B^\top & C \end{pmatrix},$$

if C is invertible, then the following properties hold:

(1) $M \succ 0$ iff $C \succ 0$ and $A - BC^{-1}B^\top \succ 0$.

(2) If $C \succ 0$, then $M \succeq 0$ iff $A - BC^{-1}B^\top \succeq 0$.

Proof. (1) Since C is invertible, we have

$$M = \begin{pmatrix} A & B \\ B^\top & C \end{pmatrix} = \begin{pmatrix} I & BC^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A - BC^{-1}B^\top & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} I & BC^{-1} \\ 0 & I \end{pmatrix}^\top. \qquad (*)$$

Observe that

$$\begin{pmatrix} I & BC^{-1} \\ 0 & I \end{pmatrix}^{-1} = \begin{pmatrix} I & -BC^{-1} \\ 0 & I \end{pmatrix},$$

so $(*)$ yields

$$\begin{pmatrix} I & -BC^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B \\ B^\top & C \end{pmatrix} \begin{pmatrix} I & -BC^{-1} \\ 0 & I \end{pmatrix}^\top = \begin{pmatrix} A - BC^{-1}B^\top & 0 \\ 0 & C \end{pmatrix},$$

and we know that for any symmetric matrix T, here $T = M$, and any invertible matrix N, here

$$N = \begin{pmatrix} I & -BC^{-1} \\ 0 & I \end{pmatrix},$$

the matrix T is positive definite ($T \succ 0$) iff $NTN^\top$ (which is obviously symmetric) is positive definite ($NTN^\top \succ 0$). But a block diagonal matrix is positive definite iff each diagonal block is positive definite, which concludes the proof.

(2) This is because for any symmetric matrix T and any invertible matrix N, we have $T \succeq 0$ iff $NTN^\top \succeq 0$. $\square$