If f is strictly convex, the above reasoning does not work, because a strict inequality is not necessarily preserved by "passing to the limit." We have recourse to the following trick: for any ω such that $0 < \omega < 1$, observe that

$$(1-\lambda)u + \lambda v = u + \lambda(v-u) = \frac{\omega - \lambda}{\omega} u + \frac{\lambda}{\omega}(u + \omega(v-u)).$$

If we assume that $0 < \lambda \leq \omega$, the convexity of f yields

$$f(u + \lambda(v-u)) = f\left(\left(1 - \frac{\lambda}{\omega}\right) u + \frac{\lambda}{\omega}(u + \omega(v-u))\right) \leq \frac{\omega - \lambda}{\omega} f(u) + \frac{\lambda}{\omega} f(u + \omega(v-u)).$$

If we subtract $f(u)$ to both sides, we get

$$\frac{f(u + \lambda(v-u)) - f(u)}{\lambda} \leq \frac{f(u + \omega(v-u)) - f(u)}{\omega}.$$

Now since $0 < \omega < 1$ and f is strictly convex,

$$f(u + \omega(v-u)) = f((1-\omega)u + \omega v) < (1-\omega) f(u) + \omega f(v),$$

which implies that

$$\frac{f(u + \omega(v-u)) - f(u)}{\omega} < f(v) - f(u),$$

and thus we get

$$\frac{f(u + \lambda(v-u)) - f(u)}{\lambda} \leq \frac{f(u + \omega(v-u)) - f(u)}{\omega} < f(v) - f(u).$$

If we let λ go to 0, by passing to the limit we get

$$df(u)(v-u) \leq \frac{f(u + \omega(v-u)) - f(u)}{\omega} < f(v) - f(u),$$

which yields the desired strict inequality.

Let us now consider the converse of (1); that is, assume that

$$f(v) \geq f(u) + df(u)(v-u) \quad \text{for all } u, v \in U.$$

For any two distinct points $u, v \in U$ and for any λ with $0 < \lambda < 1$, we get

$$\begin{aligned} f(v) &\geq f(v + \lambda(u-v)) - \lambda df(v + \lambda(u-v))(u-v) \\ f(u) &\geq f(v + \lambda(u-v)) + (1-\lambda) df(v + \lambda(u-v))(u-v), \end{aligned}$$

and if we multiply the first inequality by $1-\lambda$ and the second inequality by λ and them add up the resulting inequalities, we get

$$(1-\lambda) f(v) + \lambda f(u) \geq f(v + \lambda(u-v)) = f((1-\lambda)v + \lambda u),$$

which proves that f is convex.

The proof of the converse of (2) is similar, except that the inequalities are replaced by strict inequalities. $\square$