Because h yields a unique linear map $h_\odot \colon \mathrm{S}^2(E) \to F$ such that

$$h_\odot(e_i \odot e_j) = w_{ij}, \quad 1 \leq i \leq j \leq 3,$$

by Proposition 33.4, the vectors

$$e_1 \odot e_1, \quad e_1 \odot e_2, \quad e_1 \odot e_3, \quad e_2 \odot e_2, \quad e_2 \odot e_3, \quad e_3 \odot e_3$$

are linearly independent. This suggests understanding how a symmetric bilinear function $f \colon E^2 \to F$ is expressed in terms of its values $f(e_i, e_j)$ on the basis vectors (e_1, e_2, e_3), and this can be done easily. Using bilinearity and symmetry, we obtain

$$\begin{aligned} f(u_1e_1 + u_2e_2 + u_3e_3, v_1e_1 + v_2e_2 + v_3e_3) &= u_1v_1f(e_1, e_1) + (u_1v_2 + u_2v_1)f(e_1, e_2) \\ &\quad + (u_1v_3 + u_3v_1)f(e_1, e_3) + u_2v_2f(e_2, e_2) \\ &\quad + (u_2v_3 + u_3v_2)f(e_2, e_3) + u_3v_3f(e_3, e_3). \end{aligned}$$

Therefore, given $w_{11}, w_{12}, w_{13}, w_{22}, w_{23}, w_{33} \in F$, the function h given by

$$\begin{aligned} h(u_1e_1 + u_2e_2 + u_3e_3, v_1e_1 + v_2e_2 + v_3e_3) &= u_1v_1w_{11} + (u_1v_2 + u_2v_1)w_{12} \\ &\quad + (u_1v_3 + u_3v_1)w_{13} + u_2v_2w_{22} \\ &\quad + (u_2v_3 + u_3v_2)w_{23} + u_3v_3w_{33} \end{aligned}$$

is clearly bilinear symmetric, and by construction $h(e_i, e_j) = w_{ij}$, so it does the job.

The generalization of this argument to any $m \geq 2$ and to a space E of any dimension (even infinite) is conceptually clear, but notationally messy. If $\dim(E) = n$ and if $(e_1, \ldots, e_n)$ is a basis of E, for any m vectors $v_j = \sum_{i=1}^n u_{i,j}e_i$ in E, for any symmetric multilinear map $f \colon E^m \to F$, we have

$$\begin{aligned} &f(v_1, \ldots, v_m) \\ &= \sum_{k_1+\cdots+k_n=m} \left(\sum_{\substack{I_1\cup\cdots\cup I_n=\{1,\ldots,m\} \\ I_i\cap I_j=\emptyset,\, i\neq j,\, |I_j|=k_j}} \left(\prod_{i_1\in I_1} u_{1,i_1} \right) \cdots \left(\prod_{i_n\in I_n} u_{n,i_n} \right) \right) f(\underbrace{e_1, \ldots, e_1}_{k_1}, \ldots, \underbrace{e_n, \ldots, e_n}_{k_n}). \end{aligned}$$

Definition 33.19. Given any set J of $n \geq 1$ elements, say $J = \{j_1, \ldots, j_n\}$, and given any $m \geq 2$, for any sequence $(k_1 \ldots, k_n)$ of natural numbers $k_i \in \mathbb{N}$ such that $k_1 + \cdots + k_n = m$, the multiset M of size m

$$M = \{\underbrace{j_1, \ldots, j_1}_{k_1}, \underbrace{j_2, \ldots, j_2}_{k_2}, \ldots, \underbrace{j_n, \ldots, j_n}_{k_n}\}$$

is denoted by $M(m, J, k_1, \ldots, k_n)$. Note that $M(j_i) = k_i$, for $i = 1, \ldots, n$. Given any $k \geq 1$, and any $u \in E$, we denote $\underbrace{u \odot \cdots \odot u}_{k}$ as $u^{\odot k}$.