We define

$$\epsilon(u,v) = \frac{f((a,b)+(u,v)) - f(a,b) - (f(u,b)+f(a,v))}{\|(u,v)\|_1},$$

and observe that the continuity of f implies

$$\begin{aligned}\|f((a,b)+(u,v)) - f(a,b) - (f(u,b)+f(a,v))\| &= \|f(u,v)\| \\ &\leq C\,\|u\|_1\,\|v\|_2 \leq C\,(\|u\|_1 + \|v\|_2)^2 .\end{aligned}$$

Hence

$$\|\epsilon(u,v)\| = \left\|\frac{f(u,v)}{\|(u,v)\|_1}\right\| = \frac{\|f(u,v)\|}{\|(u,v)\|_1} \leq \frac{C\,(\|u\|_1 + \|v\|_2)^2}{\|u\|_1 + \|v\|_2} = C\,(\|u\|_1 + \|v\|_2) = C\,\|(u,v)\|_1\,,$$

which in turn implies

$$\lim_{(u,v)\mapsto(0,0)} \epsilon(u,v) = 0.$$

□

We now state the very useful *chain rule*.

Theorem 39.6. *Given three normed affine spaces E, F, and G, let A be an open set in E, and let B an open set in F. For any functions $f\colon A \to F$ and $g\colon B \to G$, such that $f(A) \subseteq B$, for any $a \in A$, if $\mathrm{D}f(a)$ exists and $\mathrm{D}g(f(a))$ exists, then $\mathrm{D}(g\circ f)(a)$ exists, and*

$$\mathrm{D}(g\circ f)(a) = \mathrm{D}g(f(a)) \circ \mathrm{D}f(a).$$

Proof. Since f is differentiable at a and g is differentiable at $b = f(a)$ for every η such that $0 < \eta < 1$ there is some $\rho > 0$ such that for all s, t, if $\|s\| \leq \rho$ and $\|t\| \leq \rho$ then

$$\begin{aligned} f(a+s) &= f(a) + \mathrm{D}f_a(s) + \epsilon_1(s) \\ g(b+t) &= g(b) + \mathrm{D}g_b(t) + \epsilon_2(t), \end{aligned}$$

with $\|\epsilon_1(s)\| \leq \eta\,\|s\|$ and $\|\epsilon_2(t)\| \leq \eta\,\|t\|$. Since $\mathrm{D}f_a$ and $\mathrm{D}g_b$ are continuous, we have

$$\|\mathrm{D}f_a(s)\| \leq \|\mathrm{D}f_a\|\,\|s\| \quad \text{and} \quad \|\mathrm{D}g_b(t)\| \leq \|\mathrm{D}g_b\|\,\|t\|\,,$$

which, since $\|\epsilon_1(s)\| \leq \eta\,\|s\|$ and $\eta < 1$, implies that

$$\|\mathrm{D}f_a(s) + \epsilon_1(s)\| \leq \|\mathrm{D}f_a\|\,\|s\| + \|\epsilon_1(s)\| \leq \|\mathrm{D}f_a\|\,\|s\| + \eta\,\|s\| \leq (\|\mathrm{D}f_a\| + 1)\,\|s\|\,.$$

Consequently, if $\|s\| < \rho/(\|\mathrm{D}f_a\| + 1)$, we have

$$\|\epsilon_2(\mathrm{D}f_a(s) + \epsilon_1(s))\| \leq \eta(\|\mathrm{D}f_a\| + 1)\,\|s\| \tag{$*_1$}$$

and

$$\|\mathrm{D}g_b(\epsilon_1(s))\| \leq \|\mathrm{D}g_b\|\,\|\epsilon_1(s)\| \leq \eta\,\|\mathrm{D}g_b\|\,\|s\|\,. \tag{$*_2$}$$