Problem 40.5. Prove that the function f with domain $\mathrm{dom}(f) = \mathbb{R} - \{0\}$ given by $f(x) = 1/x^2$ has the property that $f''(x) > 0$ for all $x \in \mathrm{dom}(f)$, but it is not convex. Why isn't Proposition 40.12 applicable?

Problem 40.6. (1) Prove that the function $x \mapsto e^{ax}$ (on $\mathbb{R}$) is convex for any $a \in \mathbb{R}$.

(2) Prove that the function $x \mapsto x^a$ is convex on $\{x \in \mathbb{R} \mid x > 0\}$, for all $a \in \mathbb{R}$ such that $a \leq 0$ or $a \geq 1$.

Problem 40.7. (1) Prove that the function $x \mapsto |x|^p$ is convex on $\mathbb{R}$ for all $p \geq 1$.

(2) Prove that the function $x \mapsto \log x$ is concave on $\{x \in \mathbb{R} \mid x > 0\}$.

(3) Prove that the function $x \mapsto x \log x$ is convex on $\{x \in \mathbb{R} \mid x > 0\}$.

Problem 40.8. (1) Prove that the function f given by $f(x_1, \ldots, x_n) = \max\{x_1, \ldots, x_n\}$ is convex on $\mathbb{R}^n$.

(2) Prove that the function g given by $g(x_1, \ldots, x_n) = \log(e^{x_1} + \cdots + e^{x_n})$ is convex on $\mathbb{R}^n$.

Prove that

$$\max\{x_1, \ldots, x_n\} \leq g(x_1, \ldots, x_n) \leq \max\{x_1, \ldots, x_n\} + \log n.$$

Problem 40.9. In Problem 39.6, it was shown that

$$\begin{aligned} df_A(X) &= \mathrm{tr}(A^{-1}X) \\ \mathrm{D}^2 f(A)(X_1, X_2) &= -\mathrm{tr}(A^{-1}X_1A^{-1}X_2), \end{aligned}$$

for all $n \times n$ real matrices X, X_1, X_2, where f is the function defined on $\mathbf{GL}^+(n, \mathbb{R})$ (the $n \times n$ real invertible matrices of positive determinants), given by

$$f(A) = \log\det(A).$$

Assume that A is symmetric positive definite and that X is symmetric.

(1) Prove that the eigenvalues of $A^{-1}X$ are real (even though $A^{-1}X$ may **not** be symmetric).

Hint. Since A is symmetric positive definite, then so is A^{-1}, so we can write $A^{-1} = S^2$ for some symmetric positive definite matrix S, and then

$$A^{-1}X = S^2X = S(SXS)S^{-1}.$$

(2) Prove that the eigenvalues of $(A^{-1}X)^2$ are nonnegative. Deduce that

$$\mathrm{D}^2 f(A)(X, X) = -\mathrm{tr}((A^{-1}X)^2) < 0$$

for all nonzero symmetric matrices X and SPD matrices A. Conclude that the function $X \mapsto \log\det X$ is strictly concave on the set of symmetric positive definite matrices.