The base case was treated in Proposition 28.3. Now, the proof of Proposition 28.3 also showed that

$$\begin{aligned}\rho_{u_{k+1},\,-(\theta_1+\cdots+\theta_k)} \circ \rho_{u_k,\,\theta_1+\cdots+\theta_k}(u_k) &= e^{i(\theta_1+\cdots+\theta_k)}u_k,\\ \rho_{u_{k+1},\,-(\theta_1+\cdots+\theta_k)} \circ \rho_{u_k,\,\theta_1+\cdots+\theta_k}(u_{k+1}) &= e^{-i(\theta_1+\cdots+\theta_k)}u_{k+1},\end{aligned}$$

and thus, using the induction hypothesis for k ($2 \le k \le n-1$), we have

$$\begin{aligned}f_{k+1}(u_j) &= \rho_{u_{k+1},\,-(\theta_1+\cdots+\theta_k)} \circ \rho_{u_k,\,\theta_1+\cdots+\theta_k} \circ f_k(u_j) = e^{i\theta_j}u_j, \quad 1 \le j \le k-1,\\ f_{k+1}(u_k) &= \rho_{u_{k+1},\,-(\theta_1+\cdots+\theta_k)} \circ \rho_{u_k,\,\theta_1+\cdots+\theta_k} \circ f_k(u_k) = e^{i(\theta_1+\cdots+\theta_k)}e^{-i(\theta_1+\cdots+\theta_{k-1})}u_k = e^{i\theta_k}u_k,\\ f_{k+1}(u_{k+1}) &= \rho_{u_{k+1},\,-(\theta_1+\cdots+\theta_k)} \circ \rho_{u_k,\,\theta_1+\cdots+\theta_k} \circ f_k(u_{k+1}) = e^{-i(\theta_1+\cdots+\theta_k)}u_{k+1},\\ f_{k+1}(u_j) &= \rho_{u_{k+1},\,-(\theta_1+\cdots+\theta_k)} \circ \rho_{u_k,\,\theta_1+\cdots+\theta_k} \circ f_k(u_j) = u_j, \quad k+1 \le j \le n,\end{aligned}$$

which proves the induction step.

As a summary, we proved that

$$f_n(u_j) = \begin{cases} e^{i\theta_j}u_j & \text{if } 1 \le j \le n-1,\\ e^{-i(\theta_1+\cdots+\theta_{n-1})}u_n & \text{when } j = n,\end{cases}$$

but since $\theta_1 + \cdots + \theta_n = 0$, we have $\theta_n = -(\theta_1 + \cdots + \theta_{n-1})$, and the last expression is in fact

$$f_n(u_n) = e^{i\theta_n}u_n.$$

Therefore, we proved that

$$f = \rho_{u_n,\theta_n} \circ \cdots \circ \rho_{u_1,\theta_1} = \rho_{u_n,\,-(\theta_1+\cdots+\theta_{n-1})} \circ \rho_{u_{n-1},\,\theta_1+\cdots+\theta_{n-1}} \circ \cdots \circ \rho_{u_2,\,-\theta_1} \circ \rho_{u_1,\theta_1},$$

and using Proposition 28.3, we also have

$$\begin{aligned}f &= \rho_{u_n,\,-(\theta_1+\cdots+\theta_{n-1})} \circ \rho_{u_{n-1},\,\theta_1+\cdots+\theta_{n-1}} \circ \cdots \circ \rho_{u_2,\,-\theta_1} \circ \rho_{u_1,\theta_1}\\ &= h_{u_n - u_{n-1}} \circ h_{u_n - e^{-i(\theta_1+\cdots+\theta_{n-1})}u_{n-1}} \circ \cdots \circ h_{u_2 - u_1} \circ h_{u_2 - e^{-i\theta_1}u_1}\\ &= h_{u_{n-1}+u_n} \circ h_{u_{n-1} + e^{i(\theta_1+\cdots+\theta_{n-1})}u_n} \circ \cdots \circ h_{u_1+u_2} \circ h_{u_1 + e^{i\theta_1}u_2},\end{aligned}$$

which completes the proof. $\square$

We finally get our improved version of the Cartan–Dieudonné theorem.

Theorem 28.5. *Let E be a Hermitian space of dimension $n \ge 1$. Every rotation $f \in \mathbf{SU}(E)$ different from the identity is the composition of at most $2n-2$ standard hyperplane reflections. Every isometry $f \in \mathbf{U}(E)$ different from the identity is the composition of at most $2n-1$ isometries, all standard hyperplane reflections, except for possibly one Hermitian reflection. When $n \ge 2$, the identity is the composition of any reflection with itself.*