Remark: Letting $U(g) = A - V(g)$, the identity $V(g_1) \cup \cdots \cup V(g_m) = V(g_1 \cdots g_m)$ translates to $U(g_1) \cap \cdots \cap U(g_m) = U(g_1 \cdots g_m)$. This suggests to define a topology on A whose basis of open sets consists of the sets $U(g)$. In this topology (called the Zariski topology), the sets of the form $V(g)$ are closed sets. Also, when $g_1, \ldots, g_m \in A[X_1, \ldots, X_n]$ and $n \geq 2$, understanding the structure of the closed sets of the form $V(g_1) \cap \cdots \cap V(g_m)$ is quite difficult, and it is the object of algebraic geometry (at least, its classical part).

When $f \in A[X_1, \ldots, X_n]$ and $n \geq 2$, one should not apply Proposition 30.27 abusively. For example, let

$$f(X, Y) = X^2 + Y^2 - 1,$$

considered as a polynomial in $\mathbb{R}[X, Y]$. Since $\mathbb{R}$ is an infinite field and since

$$f\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right) = \frac{(1-t^2)^2}{(1+t^2)^2} + \frac{(2t)^2}{(1+t^2)^2} - 1 = 0,$$

for every $t \in \mathbb{R}$, it would be tempting to say that $f = 0$. But what's wrong with the above reasoning is that there are no two infinite subsets R_1, R_2 of $\mathbb{R}$ such that $f(\alpha_1, \alpha_2) = 0$ for all $(\alpha_1, \alpha_2) \in \mathbb{R}^2$. For every $\alpha_1 \in \mathbb{R}$, there are at most two $\alpha_2 \in \mathbb{R}$ such that $f(\alpha_1, \alpha_2) = 0$. What the example shows though, is that a nonnull polynomial $f \in A[X_1, \ldots, X_n]$ where $n \geq 2$ can have an infinite number of zeros. This is in contrast with nonnull polynomials in one variables over an infinite field (which have a number of roots bounded by their degree).

We now look at polynomial interpolation.

30.7 Polynomial Interpolation (Lagrange, Newton, Hermite)

Let K be a field. First, we consider the following interpolation problem: Given a sequence $(\alpha_1, \ldots, \alpha_{m+1})$ of pairwise distinct scalars in K and any sequence $(\beta_1, \ldots, \beta_{m+1})$ of scalars in K, where the β_j are not necessarily distinct, find a polynomial $P(X)$ of degree $\leq m$ such that

$$P(\alpha_1) = \beta_1, \ldots, P(\alpha_{m+1}) = \beta_{m+1}.$$

First, observe that if such a polynomial exists, then it is unique. Indeed, this is a consequence of Proposition 30.24. Thus, we just have to find any polynomial of degree $\leq m$. Consider the following so-called *Lagrange polynomials*:

$$L_i(X) = \frac{(X-\alpha_1)\cdots(X-\alpha_{i-1})(X-\alpha_{i+1})\cdots(X-\alpha_{m+1})}{(\alpha_i-\alpha_1)\cdots(\alpha_i-\alpha_{i-1})(\alpha_i-\alpha_{i+1})\cdots(\alpha_i-\alpha_{m+1})}.$$

Note that $L(\alpha_i) = 1$ and that $L(\alpha_j) = 0$, for all $j \neq i$. But then,

$$P(X) = \beta_1 L_1 + \cdots + \beta_{m+1} L_{m+1}$$