obtained by settting the entries of the part of A_{k+1} below the diagonal to 0. Then we can find the corresponding eigenvectors by solving the linear system

$$Tv = t_{ii}v,$$

and since T is upper triangular, this can be done by bottom-up elimination. We leave it as an exercise to show that the following vectors $v^i = (v_1^i, \ldots, v_n^i)$ are eigenvectors:

$$v^1 = e_1,$$

and if $i = 2, \ldots, n$, then

$$v_j^i = \begin{cases} 0 & \text{if } i + 1 \leq j \leq n \\ 1 & \text{if } j = i \\ -\dfrac{t_{jj+1}v_{j+1}^i + \cdots + t_{ji}v_i^i}{t_{jj} - t_{ii}} & \text{if } i - 1 \geq j \geq 1. \end{cases}$$

Then the vectors $(P_k v^1, \ldots, P_k v^n)$ are a basis of (approximate) eigenvectors for A. In the special case where T is a diagonal matrix, then $v^i = e_i$ for $i = 1, \ldots, n$ and the columns of P_k are an orthonormal basis of (approximate) eigenvectors for A.

If A is a real matrix whose eigenvalues are not all real, then there is some complex pair of eigenvalues $\lambda + i\mu$ (with $\mu \neq 0$), and the QR-algorithm cannot converge to a matrix whose strictly lower-triangular part is zero. There is a way to deal with this situation using upper Hessenberg matrices which will be discussed in the next section.

Since the convergence of the QR method depends crucially only on the fact that the part of A_k below the diagonal goes to zero, it would be highly desirable if we could replace A by a similar matrix U^*AU easily computable from A having lots of zero strictly below the diagonal. We can't expect U^*AU to be a diagonal matrix (since this would mean that A was easily diagonalized), but it turns out that there is a way to construct a matrix $H = U^*AU$ which is almost triangular, except that it may have an extra nonzero diagonal below the main diagonal. Such matrices called Hessenberg matrices are discussed in the next section.

18.2 Hessenberg Matrices

Definition 18.1. An $n \times n$ matrix (real or complex) H is an *(upper) Hessenberg matrix* if it is almost triangular, except that it may have an extra nonzero diagonal below the main diagonal. Technically, $h_{jk} = 0$ for all (j, k) such that $j - k \geq 2$.

The 5×5 matrix below is an example of a Hessenberg matrix.

$$H = \begin{pmatrix} * & * & * & * & * \\ h_{21} & * & * & * & * \\ 0 & h_{32} & * & * & * \\ 0 & 0 & h_{43} & * & * \\ 0 & 0 & 0 & h_{54} & * \end{pmatrix}.$$