there is some $\lambda \neq 0$ such that

$$\lambda(u_1 + \cdots + u_{n+1}) = v_1 + \cdots + v_{n+1} = \lambda_1 u_1 + \cdots + \lambda_{n+1} u_{n+1},$$

and thus we have

$$(\lambda - \lambda_1)u_1 + \cdots + (\lambda - \lambda_{n+1})u_{n+1} = 0,$$

and since $(u_1, \ldots, u_{n+1})$ is a basis, we have $\lambda_i = \lambda$ for all i, $1 \leq i \leq n+1$, which implies $\lambda_1 = \cdots = \lambda_{n+1} = \lambda$. $\square$

Proposition 26.2 shows that a projective frame determines a unique basis of E, up to a (nonzero) scalar. This would not necessarily be the case if we did not have a point a_{n+2} such that $a_{n+2} = p(u_1 + \cdots + u_{n+1})$.

When $n = 0$, the projective space consists of a single point a, and there is only one projective frame, the pair (a, a). When $n = 1$, the projective space is a line, and a projective frame consists of any three pairwise distinct points a, b, c on this line. When $n = 2$, the projective space is a plane, and a projective frame consists of any four distinct points a, b, c, d such that a, b, c are the vertices of a nondegenerate triangle and d is not on any of the lines determined by the sides of this triangle. These examples of projective frames are illustrated in Figure 26.7. The reader can easily generalize to higher dimensions.

Given a projective frame $(a_i)_{1 \leq i \leq n+2}$ of $\mathbf{P}(E)$, let $(u_1, \ldots, u_{n+1})$ be a basis of E associated with $(a_i)_{1 \leq i \leq n+2}$. For every $a \in \mathbf{P}(E)$, there is some $u \in E - \{0\}$ such that

$$a = [u]_\sim = \{\lambda u \mid \lambda \in K - \{0\}\},$$

the equivalence class of u, and the set

$$\{(x_1, \ldots, x_{n+1}) \in K^{n+1} \mid v = x_1 u_1 + \cdots + x_{n+1} u_{n+1},\ \ v \in [u]_\sim = a\}$$

of coordinates of all the vectors in the equivalence class $[u]_\sim$ is called the *set of homogeneous coordinates of a over the basis* $(u_1, \ldots, u_{n+1})$.

Note that for each homogeneous coordinate $(x_1, \ldots, x_{n+1})$ we must have $x_i \neq 0$ for some i, $1 \leq i \leq n+1$, and any two homogeneous coordinates $(x_1, \ldots, x_{n+1})$ and $(y_1, \ldots, y_{n+1})$ for a differ by a nonzero scalar, i.e., there is some $\lambda \neq 0$ such that $y_i = \lambda x_i$, $1 \leq i \leq n+1$. Homogeneous coordinates $(x_1, \ldots, x_{n+1})$ are sometimes denoted by $(x_1 \colon \cdots : x_{n+1})$, for instance in algebraic geometry.

By Proposition 26.2, any other basis $(v_1, \ldots, v_{n+1})$ associated with the projective frame $(a_i)_{1 \leq i \leq n+2}$ differs from $(u_1, \ldots, u_{n+1})$ by a nonzero scalar, which implies that the set of homogeneous coordinates of $a \in \mathbf{P}(E)$ over the basis $(v_1, \ldots, v_{n+1})$ is identical to the set of homogeneous coordinates of $a \in \mathbf{P}(E)$ over the basis $(u_1, \ldots, u_{n+1})$. Consequently, we can associate a unique set of homogeneous coordinates to every point $a \in \mathbf{P}(E)$ with respect to the projective frame $(a_i)_{1 \leq i \leq n+2}$. With respect to this projective frame, note that a_{n+2} has homogeneous coordinates $(1, \ldots, 1)$, and that a_i has homogeneous coordinates $(0, \ldots, 1, \ldots, 0)$, where the 1 is in the ith position, where $1 \leq i \leq n+1$. We summarize the above discussion in the following definition.