For $y = 0$, we get
$$f(x) = -z^\top d,$$
so if $d \neq 0$, the function f has no minimum. Therefore, if f has a minimum, then $d = 0$. However, $d = 0$ means that
$$Ub = \begin{pmatrix} c \\ 0 \end{pmatrix},$$
and we know from Proposition 23.5 that b is in the range of A (here, U is $V^\top$), which is equivalent to $(I - AA^+)b = 0$. If $d = 0$, then
$$f(x) = \frac{1}{2} y^\top \Sigma_r y - y^\top c.$$
Consider the function $g\colon \mathbb{R}^r \to \mathbb{R}$ given by
$$g(y) = \frac{1}{2} y^\top \Sigma_r y - y^\top c, \quad y \in \mathbb{R}^r.$$
Since
$$\begin{pmatrix} y \\ z \end{pmatrix} = U^\top x$$
and $U^\top$ is invertible (with inverse U), when x ranges over $\mathbb{R}^n$, y ranges over the whole of $\mathbb{R}^r$, and since $f(x) = g(y)$, the function f has a minimum iff g has a minimum. Since Σ_r is invertible, by Proposition 42.4, the function g has a minimum iff $\Sigma_r \succeq 0$, which is equivalent to $A \succeq 0$.

Therefore, we have proven that if f has a minimum, then $(I - AA^+)b = 0$ and $A \succeq 0$. Conversely, if $(I - AA^+)b = 0$, then
$$\begin{aligned} \left(\begin{pmatrix} I_r & 0 \\ 0 & I_{n-r} \end{pmatrix} - U^\top \begin{pmatrix} \Sigma_r & 0 \\ 0 & 0 \end{pmatrix} U U^\top \begin{pmatrix} \Sigma_r^{-1} & 0 \\ 0 & 0 \end{pmatrix} U \right) b &= \left(\begin{pmatrix} I_r & 0 \\ 0 & I_{n-r} \end{pmatrix} - U^\top \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} U \right) b \\ &= U^\top \begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix} Ub = 0, \end{aligned}$$
which implies that if
$$Ub = \begin{pmatrix} c \\ d \end{pmatrix},$$
then $d = 0$, so as above
$$f(x) = g(y) = \frac{1}{2} y^\top \Sigma_r y - y^\top c,$$
and because $A \succeq 0$, we also have $\Sigma_r \succeq 0$, so g and f have a minimum.

When the above conditions hold, since
$$A = U^\top \begin{pmatrix} \Sigma_r & 0 \\ 0 & 0 \end{pmatrix} U$$