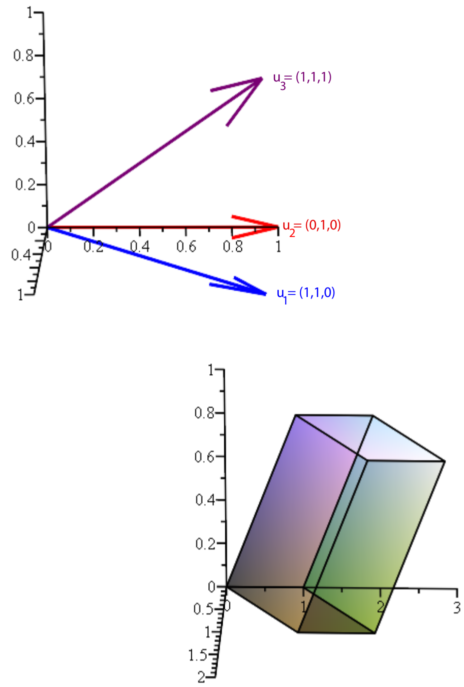

Figure 7.2: The parallelepiped in $\mathbb{R}^3$ spanned by the vectors $u_1 = (1, 1, 0)$, $u_2 = (0, 1, 0)$, and $u_3 = (0, 0, 1)$.

Example 7.2. Consider the so-called *Vandermonde determinant*

$$V(x_1, \ldots, x_n) = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_n \\ x_1^2 & x_2^2 & \ldots & x_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \ldots & x_n^{n-1} \end{vmatrix}.$$

We claim that

$$V(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n} (x_j - x_i),$$

with $V(x_1, \ldots, x_n) = 1$, when $n = 1$. We prove it by induction on $n \geq 1$. The case $n = 1$ is obvious. Assume $n \geq 2$. We proceed as follows: multiply Row $n - 1$ by x_1 and subtract it from Row n (the last row), then multiply Row $n - 2$ by x_1 and subtract it from Row $n - 1$, etc, multiply Row $i - 1$ by x_1 and subtract it from row i, until we reach Row 1. We obtain