**Proposition 15.13.** *Given any complex $n \times n$ matrix $A$, if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A$, then $e^{\lambda_1}, \ldots, e^{\lambda_n}$ are the eigenvalues of $e^A$. Furthermore, if $u$ is an eigenvector of $A$ for $\lambda_i$, then $u$ is an eigenvector of $e^A$ for $e^{\lambda_i}$.*

*Proof.* By Theorem 15.5, there is an invertible matrix $P$ and an upper triangular matrix $T$ such that

$$A = PTP^{-1}.$$

By Proposition 15.12,

$$e^{PTP^{-1}} = Pe^T P^{-1}.$$

Note that $e^T = \sum_{p\geq 0} \frac{T^p}{p!}$ is upper triangular since $T^p$ is upper triangular for all $p \geq 0$. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the diagonal entries of $T$, the properties of matrix multiplication, when combined with an induction on $p$, imply that the diagonal entries of $T^p$ are $\lambda_1^p, \lambda_2^p, \ldots, \lambda_n^p$. This in turn implies that the diagonal entries of $e^T$ are $\sum_{p\geq 0} \frac{\lambda_i^p}{p!} = e^{\lambda_i}$ for $1 \leq i \leq n$. Since $A$ and $T$ are similar matrices, we know that they have the same eigenvalues, namely the diagonal entries $\lambda_1, \ldots, \lambda_n$ of $T$. Since $e^A = e^{PTP^{-1}} = Pe^T P^{-1}$, and $e^T$ is upper triangular, we use the same argument to conclude that both $e^A$ and $e^T$ have the same eigenvalues, which are the diagonal entries of $e^T$, where the diagonal entries of $e^T$ are of the form $e^{\lambda_1}, \ldots, e^{\lambda_n}$. Now, if $u$ is an eigenvector of $A$ for the eigenvalue $\lambda$, a simple induction shows that $u$ is an eigenvector of $A^n$ for the eigenvalue $\lambda^n$, from which is follows that

$$\begin{aligned} e^A u &= \left[I + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + \ldots\right] u = u + Au + \frac{A^2}{2!}u + \frac{A^3}{3!}u + \ldots \\ &= u + \lambda u + \frac{\lambda^2}{2!}u + \frac{\lambda^3}{3!}u + \cdots = \left[1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \ldots\right] u = e^\lambda u, \end{aligned}$$

which shows that $u$ is an eigenvector of $e^A$ for $e^\lambda$. $\square$

As a consequence, we obtain the following result.

**Proposition 15.14.** *For every complex (or real) square matrix $A$, we have*

$$\det(e^A) = e^{\mathrm{tr}(A)},$$

*where $\mathrm{tr}(A)$ is the trace of $A$, i.e., the sum $a_{1\,1} + \cdots + a_{n\,n}$ of its diagonal entries.*

*Proof.* The trace of a matrix $A$ is equal to the sum of the eigenvalues of $A$. The determinant of a matrix is equal to the product of its eigenvalues, and if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A$, then by Proposition 15.13, $e^{\lambda_1}, \ldots, e^{\lambda_n}$ are the eigenvalues of $e^A$, and thus

$$\det\left(e^A\right) = e^{\lambda_1} \cdots e^{\lambda_n} = e^{\lambda_1 + \cdots + \lambda_n} = e^{\mathrm{tr}(A)},$$

as desired. $\square$