because multiplication on the right by E_k^{-1} adds ℓ_i times column i to column k (of the matrix L_{k-1}) with $i > k$, and column i of L_{k-1} has only the nonzero entry 1 as its ith element. Since

$$L_k = E_1^{-1} \cdots E_k^{-1}, \quad 1 \leq k \leq n-1,$$

we conclude that $L = L_{n-1}$, proving our claim about the shape of L.

(3)

Step 1. Prove ($\dagger_1$).

First we prove by induction on k that

$$A_{k+1} = E_k^k \cdots E_1^k P_k \cdots P_1 A, \quad k = 1, \ldots, n-2.$$

For $k = 1$, we have $A_2 = E_1 P_1 A = E_1^1 P_1 A$, since $E_1^1 = E_1$, so our assertion holds trivially.

Now if $k \geq 2$,

$$A_{k+1} = E_k P_k A_k,$$

and by the induction hypothesis,

$$A_k = E_{k-1}^{k-1} \cdots E_2^{k-1} E_1^{k-1} P_{k-1} \cdots P_1 A.$$

Because P_k is either the identity or a transposition, $P_k^2 = I$, so by inserting occurrences of $P_k P_k$ as indicated below we can write

$$\begin{aligned}
A_{k+1} &= E_k P_k A_k \\
&= E_k P_k E_{k-1}^{k-1} \cdots E_2^{k-1} E_1^{k-1} P_{k-1} \cdots P_1 A \\
&= E_k P_k E_{k-1}^{k-1} (P_k P_k) \cdots (P_k P_k) E_2^{k-1} (P_k P_k) E_1^{k-1} (P_k P_k) P_{k-1} \cdots P_1 A \\
&= E_k (P_k E_{k-1}^{k-1} P_k) \cdots (P_k E_2^{k-1} P_k)(P_k E_1^{k-1} P_k) P_k P_{k-1} \cdots P_1 A.
\end{aligned}$$

Observe that P_k has been "moved" to the right of the elimination steps. However, by definition,

$$\begin{aligned}
E_j^k &= P_k E_j^{k-1} P_k, \quad j = 1, \ldots, k-1 \\
E_k^k &= E_k,
\end{aligned}$$

so we get

$$A_{k+1} = E_k^k E_{k-1}^k \cdots E_2^k E_1^k P_k \cdots P_1 A,$$

establishing the induction hypothesis. For $k = n-2$, we get

$$U = A_{n-1} = E_{n-1}^{n-1} \cdots E_1^{n-1} P_{n-1} \cdots P_1 A,$$

as claimed, and the factorization $PA = LU$ with

$$\begin{aligned}
P &= P_{n-1} \cdots P_1 \\
L &= (E_1^{n-1})^{-1} \cdots (E_{n-1}^{n-1})^{-1}
\end{aligned}$$