Problem 45.2. Convert the following program to standard form:

$$\begin{aligned}
\text{maximize} \quad & 3x_1 - 2x_2 \\
\text{subject to} \quad & \\
& 2x_1 - x_2 \leq 4 \\
& x_1 + 3x_2 \geq 5 \\
& x_2 \geq 0.
\end{aligned}$$

Problem 45.3. The notion of basic feasible solution for linear programs where the constraints are of the form $Ax \leq b$, $x \geq 0$ is defined as follows. A basic feasible solution of a (general) linear program with n variables is a feasible solution for which some n linearly independent constraints hold with equality.

Prove that the definition of a basic feasible solution for linear programs in standard form is a special case of the above definition.

Problem 45.4. Consider the linear program

$$\begin{aligned}
\text{maximize} \quad & x_1 + x_2 \\
\text{subject to} \quad & \\
& x_1 + x_2 \leq 1.
\end{aligned}$$

Show that none of the optimal solutions are basic.

Problem 45.5. The *standard n-simplex* is the subset Δ^n of $\mathbb{R}^{n+1}$ given by

$$\Delta^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1 + \cdots + x_{n+1} = 1,\ x_i \geq 0,\ 1 \leq i \leq n+1\}.$$

(1) Prove that Δ^n is convex and that it is the convex hull of the $n+1$ vectors $e_1, \ldots e_{n+1}$, where e_i is the ith canonical unit basis vector, $i = 1, \ldots, n+1$.

(2) Prove that Δ^n is the intersection of $n+1$ half spaces and determine the hyperplanes defining these half-spaces.

Remark: The volume under the standard simplex Δ^n is $1/(n+1)!$.

Problem 45.6. The n-dimensional *cross-polytope* is the subset XP_n of $\mathbb{R}^n$ given by

$$XP_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid |x_1| + \cdots + |x_n| \leq 1\}.$$

(1) Prove that XP_n is convex and that it is the convex hull of the $2n$ vectors $\pm e_i$, where e_i is the ith canonical unit basis vector, $i = 1, \ldots, n$.

(2) Prove that XP_n is the intersection of 2^n half spaces and determine the hyperplanes defining these half-spaces.

Remark: The volume of XP_n is $2^n/n!$.