Chapter 36

The Rational Canonical Form and Other Normal Forms

36.1 The Torsion Module Associated With An Endomorphism

We saw in Section 7.7 that given a linear map $f\colon E \to E$ from a K-vector space E into itself, we can define a scalar multiplication $\cdot\colon K[X] \times E \to E$ that makes E into a $K]X]$-module. If E is finite-dimensional, this $K[X]$-module denoted by E_f is a torsion module, and the main results of this chapter yield important direct sum decompositions of E into subspaces invariant under f.

Recall that given any polynomial $p(X) = a_0X^n + a_1X^{n-1} + \cdots + a_n$ with coefficients in the field K, we define the *linear map* $p(f)\colon E \to E$ by

$$p(f) = a_0 f^n + a_1 f^{n-1} + \cdots + a_n \mathrm{id},$$

where $f^k = f \circ \cdots \circ f$, the k-fold composition of f with itself. Note that

$$p(f)(u) = a_0 f^n(u) + a_1 f^{n-1}(u) + \cdots + a_n u,$$

for every vector $u \in E$. Then, we define the scalar multiplication $\cdot\colon K[X] \times E \to E$ by polynomials as follows: for every polynomial $p(X) \in K[X]$, for every $u \in E$,

$$p(X) \cdot u = p(f)(u).^1$$

It is easy to verify that this scalar multiplication satisfies the axioms M1, M2, M3, M4:

$$\begin{aligned} p \cdot (u + v) &= p \cdot u + p \cdot v \\ (p + q) \cdot u &= p \cdot u + q \cdot u \\ (pq) \cdot u &= p \cdot (q \cdot u) \\ 1 \cdot u &= u, \end{aligned}$$

[1]If necessary to avoid confusion, we use the notion $p(X) \cdot_f u$ instead of $p(X) \cdot u$.