Type 3

$$S_c^{i,j} = \begin{pmatrix} 1 & & & & & & & & & & \\ & \ddots & & & & & & & & & \\ & & 1 & & & & & & & & \\ & & & 0 & 0 & \cdots & 0 & -i & & & \\ & & & 0 & -1 & \cdots & 0 & 0 & & & \\ & & & \vdots & \vdots & \ddots & \vdots & \vdots & & & \\ & & & 0 & 0 & \cdots & 1 & 0 & & & \\ & & & i & 0 & \cdots & 0 & 0 & & & \\ & & & & & & & & 1 & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & 1 \end{pmatrix}.$$

Prove that $S^{i,j}, S_c^{i,j} \in \mathbf{SU}(n)$, and using diagonal matrices as in Problem 12.12, prove that the matrices $S^{i,j}$ can be used to form the real part of a Hermitian matrix and the matrices $S_c^{i,j}$ can be used to form the imaginary part of a Hermitian matrix.

(3) Use (1) and (2) to prove that the matrices in $\mathbf{SU}(n)$ span all Hermitian matrices. It follows that $\mathbf{SU}(n)$ spans $\mathrm{M}_n(\mathbb{C})$ for $n \geq 3$.

Problem 14.7. Consider the complex matrix

$$A = \begin{pmatrix} i & 1 \\ 1 & -i \end{pmatrix}.$$

Check that this matrix is symmetric but not Hermitian. Prove that

$$\det(\lambda I - A) = \lambda^2,$$

and so the eigenvalues of A are $0, 0$.

Problem 14.8. Let $(E, \langle -, - \rangle)$ be a Hermitian space of finite dimension and let $f\colon E \to E$ be a linear map. Prove that the following conditions are equivalent.

(1) $f \circ f^* = f^* \circ f$ (f is normal).

(2) $\langle f(x), f(y) \rangle = \langle f^*(x), f^*(y) \rangle$ for all $x, y \in E$.

(3) $\|f(x)\| = \|f^*(x)\|$ for all $x \in E$.

(4) The map f can be diagonalized with respect to an orthonormal basis of eigenvectors.

(5) There exist some linear maps $g, h\colon E \to E$ such that, $g = g^*$, $\langle x, g(x) \rangle \geq 0$ for all $x \in E$, $h^{-1} = h^*$, and $f = g \circ h = h \circ g$.

(6) There exist some linear map $h\colon E \to E$ such that $h^{-1} = h^*$ and $f^* = h \circ f$.