24.6 Affine Independence and Affine Frames

Corresponding to the notion of linear independence in vector spaces, we have the notion of affine independence. Given a family $(a_i)_{i\in I}$ of points in an affine space E, we will reduce the notion of (affine) independence of these points to the (linear) independence of the families $(\overrightarrow{a_ia_j})_{j\in(I-\{i\})}$ of vectors obtained by choosing any a_i as an origin. First, the following proposition shows that it is sufficient to consider only one of these families.

Proposition 24.4. *Given an affine space $\langle E, \overrightarrow{E}, +\rangle$, let $(a_i)_{i\in I}$ be a family of points in E. If the family $(\overrightarrow{a_ia_j})_{j\in(I-\{i\})}$ is linearly independent for some $i \in I$, then $(\overrightarrow{a_ia_j})_{j\in(I-\{i\})}$ is linearly independent for every $i \in I$.*

Proof. Assume that the family $(\overrightarrow{a_ia_j})_{j\in(I-\{i\})}$ is linearly independent for some specific $i \in I$. Let $k \in I$ with $k \neq i$, and assume that there are some scalars $(\lambda_j)_{j\in(I-\{k\})}$ such that

$$\sum_{j\in(I-\{k\})} \lambda_j\overrightarrow{a_ka_j} = 0.$$

Since

$$\overrightarrow{a_ka_j} = \overrightarrow{a_ka_i} + \overrightarrow{a_ia_j},$$

we have

$$\begin{aligned}
\sum_{j\in(I-\{k\})} \lambda_j\overrightarrow{a_ka_j} &= \sum_{j\in(I-\{k\})} \lambda_j\overrightarrow{a_ka_i} + \sum_{j\in(I-\{k\})} \lambda_j\overrightarrow{a_ia_j}, \\
&= \sum_{j\in(I-\{k\})} \lambda_j\overrightarrow{a_ka_i} + \sum_{j\in(I-\{i,k\})} \lambda_j\overrightarrow{a_ia_j}, \\
&= \sum_{j\in(I-\{i,k\})} \lambda_j\overrightarrow{a_ia_j} - \Bigg(\sum_{j\in(I-\{k\})} \lambda_j\Bigg)\overrightarrow{a_ia_k},
\end{aligned}$$

and thus

$$\sum_{j\in(I-\{i,k\})} \lambda_j\overrightarrow{a_ia_j} - \Bigg(\sum_{j\in(I-\{k\})} \lambda_j\Bigg)\overrightarrow{a_ia_k} = 0.$$

Since the family $(\overrightarrow{a_ia_j})_{j\in(I-\{i\})}$ is linearly independent, we must have $\lambda_j = 0$ for all $j \in (I - \{i, k\})$ and $\sum_{j\in(I-\{k\})} \lambda_j = 0$, which implies that $\lambda_j = 0$ for all $j \in (I - \{k\})$. ☐

We define affine independence as follows.

Definition 24.4. Given an affine space $\langle E, \overrightarrow{E}, +\rangle$, a family $(a_i)_{i\in I}$ of points in E is *affinely independent* if the family $(\overrightarrow{a_ia_j})_{j\in(I-\{i\})}$ is linearly independent for some $i \in I$.