Proof. For each k, $1 \leq k \leq n$, every $v^k \in E_k$ can be written uniquely as

$$v^k = \sum_{j \in I_k} v_j^k u_j^k,$$

for some family of scalars $(v_j^k)_{j \in I_k}$. Let F be any nontrivial vector space. We show that for every family

$$(w_{i_1,\ldots,i_n})_{(i_1,\ldots,i_n) \in I_1 \times \ldots \times I_n},$$

of vectors in F, there is some linear map $h\colon E_1 \otimes \cdots \otimes E_n \to F$ such that

$$h(u_{i_1}^1 \otimes \cdots \otimes u_{i_n}^n) = w_{i_1,\ldots,i_n}.$$

Then by Proposition 33.4, it follows that

$$(u_{i_1}^1 \otimes \cdots \otimes u_{i_n}^n)_{(i_1,\ldots,i_n) \in I_1 \times \ldots \times I_n}$$

is linearly independent. However, since $(u_i^k)_{i \in I_k}$ is a basis for E_k, the $u_{i_1}^1 \otimes \cdots \otimes u_{i_n}^n$ also generate $E_1 \otimes \cdots \otimes E_n$, and thus, they form a basis of $E_1 \otimes \cdots \otimes E_n$.

We define the function $f\colon E_1 \times \cdots \times E_n \to F$ as follows: For any n nonempty finite subsets $J_1, \ldots, J_n$ such that $J_k \subseteq I_k$ for $k = 1, \ldots, n$,

$$f(\sum_{j_1 \in J_1} v_{j_1}^1 u_{j_1}^1, \ldots, \sum_{j_n \in J_n} v_{j_n}^n u_{j_n}^n) = \sum_{j_1 \in J_1, \ldots, j_n \in J_n} v_{j_1}^1 \cdots v_{j_n}^n \, w_{j_1,\ldots,j_n}.$$

It is immediately verified that f is multilinear. By the universal mapping property of the tensor product, the linear map $f_\otimes\colon E_1 \otimes \cdots \otimes E_n \to F$ such that $f = f_\otimes \circ \varphi$, is the desired map h. $\square$

In particular, when each I_k is finite and of size $m_k = \dim(E_k)$, we see that the dimension of the tensor product $E_1 \otimes \cdots \otimes E_n$ is $m_1 \cdots m_n$. As a corollary of Proposition 33.12, if $(u_i^k)_{i \in I_k}$ is a basis for E_k, $1 \leq k \leq n$, then every tensor $z \in E_1 \otimes \cdots \otimes E_n$ can be written in a unique way as

$$z = \sum_{(i_1,\ldots,i_n) \, \in \, I_1 \times \ldots \times I_n} \lambda_{i_1,\ldots,i_n} \, u_{i_1}^1 \otimes \cdots \otimes u_{i_n}^n,$$

for some unique family of scalars $\lambda_{i_1,\ldots,i_n} \in K$, all zero except for a finite number.

33.4 Some Useful Isomorphisms for Tensor Products

Proposition 33.13. *Given three vector spaces E, F, G, there exists unique canonical isomorphisms*

(1) $E \otimes F \cong F \otimes E$