(2) For all $h_1, h_2 \in H$, we have $h_1 h_2 \in H$;

(3) For all $h \in H$, we have $h^{-1} \in H$.

The proof of the following proposition is left as an exercise.

Proposition 2.5. *Given a group G, a subset $H \subseteq G$ is a subgroup of G iff H is nonempty and whenever $h_1, h_2 \in H$, then $h_1 h_2^{-1} \in H$.*

If the group G is finite, then the following criterion can be used.

Proposition 2.6. *Given a finite group G, a subset $H \subseteq G$ is a subgroup of G iff*

(1) $e \in H$;

(2) H is closed under multiplication.

Proof. We just have to prove that Condition (3) of Definition 2.4 holds. For any $a \in H$, since the left translation L_a is bijective, its restriction to H is injective, and since H is finite, it is also bijective. Since $e \in H$, there is a unique $b \in H$ such that $L_a(b) = ab = e$. However, if a^{-1} is the inverse of a in G, we also have $L_a(a^{-1}) = aa^{-1} = e$, and by injectivity of L_a, we have $a^{-1} = b \in H$. ☐

Example 2.2.

1. For any integer $n \in \mathbb{Z}$, the set
$$n\mathbb{Z} = \{nk \mid k \in \mathbb{Z}\}$$
is a subgroup of the group $\mathbb{Z}$.

2. The set of matrices
$$\mathbf{GL}^{+}(n, \mathbb{R}) = \{A \in \mathbf{GL}(n, \mathbb{R}) \mid \det(A) > 0\}$$
is a subgroup of the group $\mathbf{GL}(n, \mathbb{R})$.

3. The group $\mathbf{SL}(n, \mathbb{R})$ is a subgroup of the group $\mathbf{GL}(n, \mathbb{R})$.

4. The group $\mathbf{O}(n)$ is a subgroup of the group $\mathbf{GL}(n, \mathbb{R})$.

5. The group $\mathbf{SO}(n)$ is a subgroup of the group $\mathbf{O}(n)$, and a subgroup of the group $\mathbf{SL}(n, \mathbb{R})$.