3. Hermitian matrices ($\epsilon = 1$)

4. Skew-Hermitian matrices ($\epsilon = -1$).

Going back to a sesquilinear form $\varphi\colon E \times F \to K$, for any subspace U of E, it is easy to check that

$$U \subseteq (U^{\perp})^{\perp},$$

and that for any subspace V of F, we have

$$V \subseteq (V^{\perp})^{\perp}.$$

For simplicity of notation, we write $U^{\perp\perp}$ instead of $(U^{\perp})^{\perp}$ (and $V^{\perp\perp}$ instead of $(V^{\perp})^{\perp}$).

Given any two subspaces U_1 and U_2 of E, if $U_1 \subseteq U_2$, then $U_2^{\perp} \subseteq U_1^{\perp}$. Indeed, if $v \in U_2^{\perp}$ then $\varphi(u_2, v) = 0$ for all $u_2 \in U_2$, and since $U_1 \subseteq U_2$ this implies that $\varphi(u_1, v) = 0$ for all $u_1 \in U_1$, which shows that $v \in U_1^{\perp}$. Similarly for any two subspaces V_1, V_2 of F, if $V_1 \subseteq V_2$, then $V_2^{\perp} \subseteq V_1^{\perp}$. As a consequence,

$$U^{\perp} = U^{\perp\perp\perp}, \quad V^{\perp} = V^{\perp\perp\perp}.$$

First, we have $U^{\perp} \subseteq U^{\perp\perp\perp}$. Second, from $U \subseteq U^{\perp\perp}$, we get $U^{\perp\perp\perp} \subseteq U^{\perp}$, so $U^{\perp} = U^{\perp\perp\perp}$. The other equation is proved is a similar way.

Observe that φ is nondegenerate iff $E^{\perp} = \{0\}$ and $F^{\perp} = \{0\}$. Furthermore, since

$$\begin{aligned}\varphi(u + x, v) &= \varphi(u, v)\\ \varphi(u, v + y) &= \varphi(u, v)\end{aligned}$$

for any $x \in F^{\perp}$ and any $y \in E^{\perp}$, we see that we obtain by passing to the quotient a sesquilinear form

$$[\varphi]\colon (E/F^{\perp}) \times (F/E^{\perp}) \to K$$

which is nondegenerate.

Proposition 29.12. *For any sesquilinear form $\varphi\colon E \times F \to K$, the space $E/F^{\perp}$ is finite-dimensional iff the space $F/E^{\perp}$ is finite-dimensional; if so,* $\dim(E/F^{\perp}) = \dim(F/E^{\perp})$.

Proof. Since the sesquilinear form $[\varphi]\colon (E/F^{\perp}) \times (F/E^{\perp}) \to K$ is nondegenerate, the maps $l_{[\varphi]}\colon \overline{(E/F^{\perp})} \to (F/E^{\perp})^*$ and $r_{[\varphi]}\colon \overline{(F/E^{\perp})} \to (E/F^{\perp})^*$ are injective. If $\dim(E/F^{\perp}) = m$, then $\dim(E/F^{\perp}) = \dim((E/F^{\perp})^*)$, so by injectivity of $r_{[\varphi]}$, we have $\dim(F/E^{\perp}) = \dim(\overline{(F/E^{\perp})}) \leq m$. A similar reasoning using the injectivity of $l_{[\varphi]}$ applies if $\dim(F/E^{\perp}) = n$, and we get $\dim(E/F^{\perp}) = \dim(\overline{(E/F^{\perp})}) \leq n$. Therefore, $\dim(E/F^{\perp}) = m$ is finite iff $\dim(F/E^{\perp}) = n$ is finite, in which case $m = n$ by Proposition 29.1(d). $\square$