36.2 The Rational Canonical Form

Let E be a finite-dimensional vector space over a field K, and let $f\colon E \to E$ be an endomorphism of E. We know from Section 36.1 that there is a $K[X]$-module E_f associated with f, and that E_f is a finitely generated torsion module over the PID $K[X]$. In this chapter, we show how Theorems from Sections 35.4 and 35.5 yield important results about the structure of the linear map f.

Recall that the annihilator of a subspace V is an ideal (p) uniquely defined by a monic polynomial p called the *minimal polynomial* of V.

Our first result is obtained by translating the primary decomposition theorem, Theorem 35.19. It is not too surprising that we obtain again Theorem 31.10!

Theorem 36.4. *(Primary Decomposition Theorem) Let $f\colon E \to E$ be a linear map on the finite-dimensional vector space E over the field K. Write the minimal polynomial m of f as*

$$m = p_1^{r_1} \cdots p_k^{r_k},$$

where the p_i are distinct irreducible monic polynomials over K, and the r_i are positive integers. Let

$$W_i = \mathrm{Ker}\,(p_i(f)^{r_i}), \quad i = 1, \ldots, k.$$

Then

(a) $E = W_1 \oplus \cdots \oplus W_k$.

(b) Each W_i is invariant under f and the projection from W onto W_i is given by a polynomial in f.

(c) The minimal polynomial of the restriction $f \mid W_i$ of f to W_i is $p_i^{r_i}$.

Example 36.1. Let $f\colon \mathbb{R}^4 \to \mathbb{R}^4$ be defined as $f(x, y, z, w) = (x + w, y + z, y + z, x + w)$. In terms of the standard basis, f has the matrix representation

$$M = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

A basic calculation shows that $\chi_f(X) = X^2(X-2)^2$ and that $m_f(X) = X(X-2)$. The primary decomposition theorem implies that

$$\mathbb{R}^4 = W_1 \oplus W_2, \qquad W_1 = \mathrm{Ker}\,(M), \qquad W_2 = \mathrm{Ker}\,(M - 2I).$$

Note that $\mathrm{Ker}\,(M)$ corresponds to the eigenspace associated with eigenvalue 0 and has basis $([-1, 0, 0, 1], [0, -1, 1, 0])$, while $\mathrm{Ker}\,(M - 2I)$ corresponds to the eigenspace associated with eigenvalue 2 and has basis $([1, 0, 0, 1], [0, 1, 1, 0])$.