Definition 51.18. Let f be a proper convex function on $\mathbb{R}^n$. We denote by $\inf f$ the quantity

$$\inf f = \inf_{x \in \mathrm{dom}(f)} f(x).$$

This is the minimum of the function f over $\mathbb{R}^n$ (it may be equal to $-\infty$).

For every $\alpha \in \mathbb{R}$, we have the sublevel set

$$\mathrm{sublev}_\alpha(f) = \{x \in \mathbb{R}^n \mid f(x) \leq \alpha\}.$$

By Proposition 51.2, we know that the sublevel sets $\mathrm{sublev}_\alpha(f)$ are convex and that

$$\mathrm{dom}(f) = \bigcup_{\alpha \in \mathbb{R}} \mathrm{sublev}_\alpha(f).$$

Observe that $\mathrm{sublev}_\alpha(f) = \emptyset$ if $\alpha < \inf f$. If $\inf f > -\infty$, then for $\alpha = \inf f$, the sublevel set $\mathrm{sublev}_\alpha(f)$ consists of the set of vectors where f achieves it minimum.

Definition 51.19. Let f be a proper convex function on $\mathbb{R}^n$. If $\inf f > -\infty$, then the sublevel set $\mathrm{sublev}_{\inf f}(f)$ is called the *minimum set* of f (this set may be empty). See Figure 51.24.

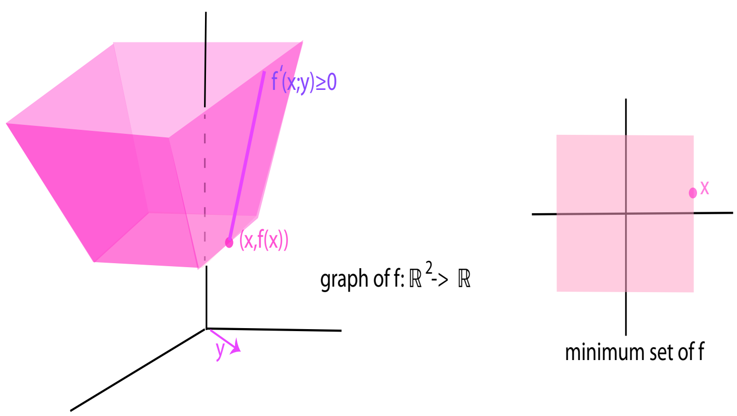

Figure 51.24: Let f be the proper convex function whose graph is the surface of the upward facing pink trough. The minimum set of f is the light pink square of $\mathbb{R}^2$ which maps to the bottom surface of the trough in $\mathbb{R}^3$. For any x in the minimum set, $f'(x; y) \geq 0$, a fact substantiated by Proposition 51.34.

It is important to determine whether the minimum set is empty or nonempty, or whether it contains a single point. As we noted in Theorem 40.13(2), if f is strictly convex then the minimum set contains at most one point.