Thus, there is an isomorphism between the two Hilbert spaces $\mathrm{L}^2(T)$ and $\ell^2(\mathbb{Z})$, which is the deep reason why the Fourier coefficients "work." Theorem A.8 implies that the Fourier series $\sum_{k\in\mathbb{Z}} c_k e^{ikx}$ of a function $f \in \mathrm{L}^2(T)$ converges to f in the L^2-sense, i.e., in the mean-square sense. This does not necessarily imply that the Fourier series converges to f pointwise! This is a subtle issue, and for more on this subject, the reader is referred to Lang [111, 112] or Schwartz [152, 153].

We can also consider the set $\mathcal{C}([-1,1])$ of continuous functions $f\colon [-1,1] \to \mathbb{C}$. There is a Hilbert space $\mathrm{L}^2([-1,1])$ containing $\mathcal{C}([-1,1])$ and such that $\mathcal{C}([-1,1])$ is dense in $\mathrm{L}^2([-1,1])$, whose inner product is given by

$$\langle f, g\rangle = \int_{-1}^{1} f(x)\overline{g(x)}dx.$$

The Hilbert space $\mathrm{L}^2([-1,1])$ is the space of *Lebesgue square-integrable functions* over $[-1,1]$. The Legendre polynomials $P_n(x)$ defined in Example 5 of Section 12.2 (Chapter 12). form a Hilbert basis of $\mathrm{L}^2([-1,1])$. Recall that if we let f_n be the function

$$f_n(x) = (x^2-1)^n,$$

$P_n(x)$ is defined as follows:

$$P_0(x) = 1, \quad \text{and} \quad P_n(x) = \frac{1}{2^n n!}\, f_n^{(n)}(x),$$

where $f_n^{(n)}$ is the nth derivative of f_n. The reason for the leading coefficient is to get $P_n(1) = 1$. It can be shown with much efforts that

$$P_n(x) = \sum_{0\leq k\leq n/2} (-1)^k \, \frac{(2(n-k))!}{2^n(n-k)!k!(n-2k)!}\, x^{n-2k}.$$

A.3 Summary

The main concepts and results of this chapter are listed below:

- Hilbert space
- Orthogonal family, total orthogonal family.
- Hilbert basis.
- Fourier coefficients.
- Hamel bases, Schauder bases.
- Fourier series.