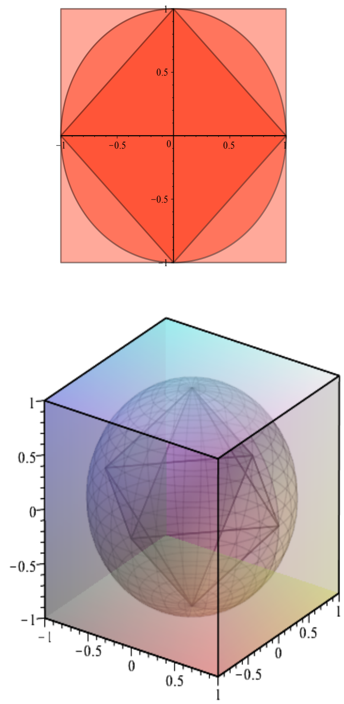

Figure 9.4: The relationships between the closed unit balls from the ℓ^1-norm, the Euclidean norm, and the sup-norm.

Since the above is trivial if $u = 0$ or $v = 0$, let us assume that $u \neq 0$ and $v \neq 0$. Then Inequality $(*)$ with $\alpha = |u_i| / \|u\|_p$ and $\beta = |v_i| / \|v\|_q$ yields

$$\frac{|u_i v_i|}{\|u\|_p \|v\|_q} \leq \frac{|u_i|^p}{p \|u\|_p^p} + \frac{|v_i|^q}{q \|v\|_q^q},$$

for $i = 1, \ldots, n$, and by summing up these inequalities, we get

$$\sum_{i=1}^{n} |u_i v_i| \leq \|u\|_p \|v\|_q,$$

as claimed. To finish the proof, we simply have to prove that property (N3) holds, since (N1) and (N2) are clear. For $i = 1, \ldots, n$, we can write

$$(|u_i| + |v_i|)^p = |u_i|(|u_i| + |v_i|)^{p-1} + |v_i|(|u_i| + |v_i|)^{p-1},$$