Theorem 37.63 yields a quick proof of the fact that any Hermitian space E (with Hermitian product $\langle -, - \rangle$) can be embedded in a Hilbert space E_h.

Theorem 48.1. *Given a Hermitian space $(E, \langle -, - \rangle)$ (resp. Euclidean space), there is a Hilbert space $(E_h, \langle -, - \rangle_h)$ and a linear map $\varphi \colon E \to E_h$, such that*

$$\langle u, v \rangle = \langle \varphi(u), \varphi(v) \rangle_h$$

for all $u, v \in E$, and $\varphi(E)$ is dense in E_h. Furthermore, E_h is unique up to isomorphism.

Proof. Let $(\widehat{E}, \| \, \|_{\widehat{E}})$ be the Banach space, and let $\varphi \colon E \to \widehat{E}$ be the linear isometry, given by Theorem 37.63. Let $\|u\| = \sqrt{\langle u, u \rangle}$ (with $u \in E$) and $E_h = \widehat{E}$. If E is a real vector space, we know from Section refsec5bis that the inner product $\langle -, - \rangle$ can be expressed in terms of the norm $\|u\|$ by the polarity equation

$$\langle u, v \rangle = \frac{1}{2}(\|u + v\|^2 - \|u\|^2 - \|v\|^2),$$

and if E is a complex vector space, we know from Section 14.1 that we have the polarity equation

$$\langle u, v \rangle = \frac{1}{4}(\|u + v\|^2 - \|u - v\|^2 + i\|u + iv\|^2 - i\|u - iv\|^2).$$

By the Cauchy-Schwarz inequality, $|\langle u, v \rangle| \leq \|u\| \|v\|$, the map $\langle -, - \rangle \colon E \times E \to \mathbb{C}$ (resp. $\langle -, - \rangle \colon E \times E \to \mathbb{R}$) is continuous. However, it is not uniformly continuous, but we can get around this problem by using the polarity equations to extend it to a continuous map. By continuity, the polarity equations also hold in E_h, which shows that $\langle -, - \rangle$ extends to a positive definite Hermitian inner product (resp. Euclidean inner product) $\langle -, - \rangle_h$ on E_h induced by $\| \, \|_{\widehat{E}}$ extending $\langle -, - \rangle$. ☐

Remark: We followed the approach in Schwartz [149] (Chapter XXIII, Section 42. Theorem 2). For other approaches, see Munkres [131] (Chapter 7, Section 43), and Bourbaki [27].

One of the most important facts about finite-dimensional Hermitian (and Euclidean) spaces is that they have orthonormal bases. This implies that, up to isomorphism, *every finite-dimensional Hermitian space is isomorphic to* $\mathbb{C}^n$ (for some $n \in \mathbb{N}$) and that the inner product is given by

$$\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n) \rangle = \sum_{i=1}^{n} x_i \overline{y_i}.$$

Furthermore, every subspace W has an orthogonal complement $W^\perp$, and the inner product induces a natural duality between E and E^* (actually, between $\overline{E}$ and E^*) where E^* is the space of linear forms on E.

When E is a Hilbert space, E may be infinite dimensional, often of uncountable dimension. Thus, we can't expect that E always have an orthonormal basis. However, if we modify