if we let $\theta = \sqrt{a^2 + b^2 + c^2}$ and

$$B = \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix},$$

then we have the following result known as *Rodrigues' formula* (1840). The (real) vector space of $n \times n$ skew symmetric matrices is denoted by $\mathfrak{so}(n)$.

Proposition 12.15. *The exponential map* $\exp\colon \mathfrak{so}(3) \to \mathbf{SO}(3)$ *is given by*

$$e^A = \cos\theta\, I_3 + \frac{\sin\theta}{\theta} A + \frac{(1-\cos\theta)}{\theta^2} B,$$

or, equivalently, by

$$e^A = I_3 + \frac{\sin\theta}{\theta} A + \frac{(1-\cos\theta)}{\theta^2} A^2$$

if $\theta \neq 0$, *with* $e^{0_3} = I_3$.

Proof sketch. First observe that

$$A^2 = -\theta^2 I_3 + B,$$

since

$$\begin{aligned} A^2 &= \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} = \begin{pmatrix} -c^2 - b^2 & ba & ca \\ ab & -c^2 - a^2 & cb \\ ac & cb & -b^2 - a^2 \end{pmatrix} \\ &= \begin{pmatrix} -a^2 - b^2 - c^2 & 0 & 0 \\ 0 & -a^2 - b^2 - c^2 & 0 \\ 0 & 0 & -a^2 - b^2 - c^2 \end{pmatrix} + \begin{pmatrix} a^2 & ba & ca \\ ab & b^2 & cb \\ ac & cb & c^2 \end{pmatrix} \\ &= -\theta^2 I_3 + B, \end{aligned}$$

and that

$$AB = BA = 0.$$

From the above, deduce that

$$A^3 = -\theta^2 A,$$

and for any $k \geq 0$,

$$\begin{aligned} A^{4k+1} &= \theta^{4k} A, \\ A^{4k+2} &= \theta^{4k} A^2, \\ A^{4k+3} &= -\theta^{4k+2} A, \\ A^{4k+4} &= -\theta^{4k+2} A^2. \end{aligned}$$