and we have

$$QRQ^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Observe that $QRQ^{-1} \notin \mathbf{SO}(2)$, so $\mathbf{SO}(2)$ is not a normal subgroup of $\mathbf{SO}(3)$.

Let T and $A \in \mathbf{GL}(2, \mathbb{R})$ be the following matrices

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We have

$$A^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = A,$$

and

$$ATA^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

The matrix T is upper triangular, but ATA^{-1} is not, so the group of 2×2 upper triangular matrices is not a normal subgroup of $\mathbf{GL}(2, \mathbb{R})$.

Let $Q \in V$ and $A \in \mathbb{GL}(2, \mathbb{R})$ be the following matrices

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We have

$$A^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

and

$$AQA^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix}.$$

Clearly $AQA^{-1} \notin V$, which shows that the Klein four group is not a normal subgroup of $\mathbf{GL}(2, \mathbb{R})$.

The reader should check that the subgroups $n\mathbb{Z}$, $\mathbf{GL}^+(n, \mathbb{R})$, $\mathbf{SL}(n, \mathbb{R})$, and $\mathbf{SO}(n, \mathbb{R})$ as a subgroup of $\mathbf{O}(n, \mathbb{R})$, are normal subgroups.

If N is a normal subgroup of G, the equivalence relation $\sim$ induced by left cosets (see Definition 2.5) is the same as the equivalence induced by right cosets. Furthermore, this equivalence relation is a *congruence*, which means that: For all $g_1, g_2, g_1', g_2' \in G$,