Therefore, we are in the situation described in the previous sections with $B = M^{-1}N$ and $c = M^{-1}b$. In fact, since $A = M - N$, we have

$$B = M^{-1}N = M^{-1}(M - A) = I - M^{-1}A, \tag{$*$}$$

which shows that $I - B = M^{-1}A$ is invertible. The iterative method associated with the matrix $B = M^{-1}N$ is given by

$$u_{k+1} = M^{-1}Nu_k + M^{-1}b, \quad k \geq 0, \tag{$\dagger$}$$

starting from any arbitrary vector u_0. From a practical point of view, we do not invert M, and instead we solve iteratively the systems

$$Mu_{k+1} = Nu_k + b, \quad k \geq 0.$$

Various methods correspond to various ways of choosing M and N from A. The first two methods choose M and N as disjoint submatrices of A, but the relaxation method allows some overlapping of M and N.

To describe the various choices of M and N, it is convenient to write A in terms of three submatrices D, E, F, as

$$A = D - E - F,$$

where the only nonzero entries in D are the diagonal entries in A, the only nonzero entries in E are the negatives of nonzero entries in A below the the diagonal, and the only nonzero entries in F are the negatives of nonzero entries in A above the diagonal. More explicitly, if

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n-1} & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n-1} & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n-1} & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1\,1} & a_{n-1\,2} & a_{n-1\,3} & \cdots & a_{n-1\,n-1} & a_{n-1\,n} \\ a_{n\,1} & a_{n\,2} & a_{n\,3} & \cdots & a_{n\,n-1} & a_{n\,n} \end{pmatrix},$$

then

$$D = \begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_{22} & 0 & \cdots & 0 & 0 \\ 0 & 0 & a_{33} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1\,n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & a_{n\,n} \end{pmatrix},$$