where A is the matrix expressing the v_j in terms of the e_i, we have

$$\langle v_1 \wedge \cdots \wedge v_n, v_1 \wedge \cdots \wedge v_n \rangle_\wedge = \det(A)^2 \langle e_1 \wedge \cdots \wedge e_n, e_1 \wedge \cdots \wedge e_n \rangle = \det(A)^2.$$

As a consequence, $\det(A) = \sqrt{\det(\langle v_i, v_j \rangle)}$, and

$$v_1 \wedge \cdots \wedge v_n = \sqrt{\det(\langle v_i, v_j \rangle)}\, e_1 \wedge \cdots \wedge e_n,$$

from which it follows that

$$*(1) = \frac{1}{\sqrt{\det(\langle v_i, v_j \rangle)}}\, v_1 \wedge \cdots \wedge v_n$$

(see Jost [101], Chapter 2, Lemma 2.1.3). □

34.7 Left and Right Hooks ⊛

In this section *all vector spaces are assumed to have finite dimension*. Say $\dim(E) = n$. Using our nonsingular pairing

$$\langle -, - \rangle\colon \bigwedge^p E^* \times \bigwedge^p E \longrightarrow K \qquad (1 \leq p \leq n)$$

defined on generators by

$$\langle u_1^* \wedge \cdots \wedge u_p^*, v_1 \wedge \cdots \wedge u_p \rangle = \det(u_i^*(v_j)),$$

we define various contraction operations (partial evaluation operators)

$$\lrcorner \colon \bigwedge^p E \times \bigwedge^{p+q} E^* \longrightarrow \bigwedge^q E^* \qquad \text{(left hook)}$$

and

$$\llcorner \colon \bigwedge^{p+q} E^* \times \bigwedge^p E \longrightarrow \bigwedge^q E^* \qquad \text{(right hook)},$$

as well as the versions obtained by replacing E by E^* and E^{**} by E. We begin with the *left interior product or left hook*, $\lrcorner$.

Let $u \in \bigwedge^p E$. For any q such that $p + q \leq n$, multiplication on the right by u is a linear map

$$\wedge_R(u)\colon \bigwedge^q E \longrightarrow \bigwedge^{p+q} E$$

given by

$$v \mapsto v \wedge u$$