The proof of Proposition 56.4 reveals that there are three critical values for ν:

$$\frac{2p_{sf}}{m}, \quad \frac{2q_{sf}}{m}, \quad \frac{p_{sf} + q_{sf}}{m}.$$

These values can be avoided by requiring the strict inequality

$$\max\left\{\frac{2p_{sf}}{m}, \frac{2q_{sf}}{m}\right\} < \nu.$$

Then the following corollary holds.

Theorem 56.5. *For every optimal solution* $(w, b, \epsilon, \xi, \xi')$ *with* $w \neq 0$ *and* $\epsilon > 0$*, if*

$$\max\left\{\frac{2p_{sf}}{m}, \frac{2q_{sf}}{m}\right\} < \nu < (m-1)/m,$$

then some x_{i_0} *is a blue support vector and some* x_{j_0} *is a red support vector (with* $i_0 \neq j_0$*).*

Proof. We proceed by contradiction. Suppose that for every optimal solution with $w \neq 0$ and $\epsilon > 0$ no x_i is a blue support vector or no x_i is a red support vector. Since $\nu < (m-1)/m$, Proposition 56.4 holds, so there is another optimal solution. But since the critical values of ν are avoided, the proof of Proposition 56.4 shows that the value of the objective function for this new optimal solution is strictly smaller than the original optimal value, a contradiction. $\square$

Remark: If an optimal solution has $\epsilon = 0$, then depending on the value of C there may not be any support vectors, or many.

If the primal has an optimal solution with $w \neq 0$ and $\epsilon > 0$, then by $(*_w)$ and since

$$\sum_{i=1}^m \lambda_i - \sum_{i=1}^m \mu_i = 0 \quad \text{and} \quad \lambda_i \mu_i = 0,$$

there is i_0 such that $\lambda_{i_0} > 0$ and some $j_0 \neq i_0$ such that $\mu_{j_0} > 0$.

Under the mild hypothesis called the **Standard Margin Hypothesis** that there is some i_0 such that $0 < \alpha_{i_0} < \frac{C}{m}$ *and* there is some $j_0 \neq i_0$ such that $0 < \mu_{j_0} < \frac{C}{m}$, in other words there is a blue support vector of type 1 and there is a red support vector of type 1, then by $(*)$ we have $\xi_{i_0} = 0, \xi'_{j_0} = 0$, and we have the two equations

$$\begin{aligned} w^\top x_{i_0} + b - y_{i_0} &= \epsilon \\ -w^\top x_{j_0} - b + y_{j_0} &= \epsilon, \end{aligned}$$