so that if $x_j \in E_j$, then

$$f(x_j) = f_j(x_j) = \sum_{i=1}^{m} f_{ij}(x_j), \quad \text{with } f_{ij}(x_j) \in F_i. \tag{$\dagger_1$}$$

Observe that we are summing over the index i, which eventually corresponds to summing the entries in the jth column of the matrix representing f; see Definition 6.7.

We see that for every vector $x = x_1 + \cdots + x_n \in E$, with $x_j \in E_j$, we have

$$f(x) = \sum_{j=1}^{n} f_j(x_j) = \sum_{j=1}^{n}\sum_{i=1}^{m} f_{ij}(x_j) = \sum_{i=1}^{m}\sum_{j=1}^{n} f_{ij}(x_j).$$

Observe that conversely, given a family $(f_{ij})_{1\leq i\leq m, 1\leq j\leq n}$ of linear maps $f_{ij}\colon E_j \to F_i$, we obtain the linear map $f\colon E \to F$ defined such that for every $x = x_1 + \cdots + x_n \in E$, with $x_j \in E_j$, we have

$$f(x) = \sum_{i=1}^{m}\sum_{j=1}^{n} f_{ij}(x_j).$$

As a consequence, it is easy to check that there is an isomorphism between the vector spaces

$$\mathrm{Hom}(E, F) \quad \text{and} \quad \prod_{\substack{1\leq i\leq m\\ 1\leq j\leq n}} \mathrm{Hom}(E_j, F_i).$$

Example 6.1. Suppose that $E = E_1 \oplus E_2$ and $F = F_1 \oplus F_2 \oplus F_3$, and that we have six maps $f_{ij}\colon E_j \to F_i$ for $i = 1, 2, 3$ and $j = 1, 2$. For any $x = x_1 + x_2$, with $x_1 \in E_1$ and $x_2 \in E_2$, we have

$$\begin{aligned}
y_1 &= f_{11}(x_1) + f_{12}(x_2) \in F_1\\
y_2 &= f_{21}(x_1) + f_{22}(x_2) \in F_2\\
y_3 &= f_{31}(x_1) + f_{32}(x_2) \in F_3,
\end{aligned}$$

$$\begin{aligned}
f_1(x_1) &= f_{11}(x_1) + f_{21}(x_1) + f_{31}(x_1)\\
f_2(x_2) &= f_{12}(x_2) + f_{22}(x_2) + f_{32}(x_2),
\end{aligned}$$

and

$$\begin{aligned}
f(x) &= f_1(x_1) + f_2(x_2) = y_1 + y_2 + y_3\\
&= f_{11}(x_1) + f_{12}(x_2) + f_{21}(x_1) + f_{22}(x_2) + f_{31}(x_1) + f_{32}(x_2).
\end{aligned}$$

These equations suggest the matrix notation

$$\begin{pmatrix} y_1\\ y_2\\ y_3 \end{pmatrix} = \begin{pmatrix} f_{11} & f_{12}\\ f_{21} & f_{22}\\ f_{31} & f_{32} \end{pmatrix} \begin{pmatrix} x_1\\ x_2 \end{pmatrix}.$$