and similarly,

$$\max_{x\neq 0, x\in\{u_1,\ldots,u_k\}^\perp} \frac{x^\top A x}{x^\top x} = \max_x \left\{ x^\top A x \mid (x \in \{u_1,\ldots,u_k\}^\perp) \wedge (x^\top x = 1) \right\}.$$

Since A is a symmetric matrix, its eigenvalues are real and it can be diagonalized with respect to an orthonormal basis of eigenvectors, so let $(u_1,\ldots,u_d)$ be such a basis. If we write

$$x = \sum_{i=1}^d x_i u_i,$$

a simple computation shows that

$$x^\top A x = \sum_{i=1}^d \lambda_i x_i^2.$$

If $x^\top x = 1$, then $\sum_{i=1}^d x_i^2 = 1$, and since we assumed that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$, we get

$$x^\top A x = \sum_{i=1}^d \lambda_i x_i^2 \leq \lambda_1 \left(\sum_{i=1}^d x_i^2 \right) = \lambda_1.$$

Thus,

$$\max_x \left\{ x^\top A x \mid x^\top x = 1 \right\} \leq \lambda_1,$$

and since this maximum is achieved for $e_1 = (1, 0, \ldots, 0)$, we conclude that

$$\max_x \left\{ x^\top A x \mid x^\top x = 1 \right\} = \lambda_1.$$

Next observe that $x \in \{u_1,\ldots,u_k\}^\perp$ and $x^\top x = 1$ iff $x_1 = \cdots = x_k = 0$ and $\sum_{i=1}^d x_i = 1$. Consequently, for such an x, we have

$$x^\top A x = \sum_{i=k+1}^d \lambda_i x_i^2 \leq \lambda_{k+1} \left(\sum_{i=k+1}^d x_i^2 \right) = \lambda_{k+1}.$$

Thus,

$$\max_x \left\{ x^\top A x \mid (x \in \{u_1,\ldots,u_k\}^\perp) \wedge (x^\top x = 1) \right\} \leq \lambda_{k+1},$$

and since this maximum is achieved for $e_{k+1} = (0, \ldots, 0, 1, 0, \ldots, 0)$ with a 1 in position $k+1$, we conclude that

$$\max_x \left\{ x^\top A x \mid (x \in \{u_1,\ldots,u_k\}^\perp) \wedge (x^\top x = 1) \right\} = \lambda_{k+1},$$

as claimed. $\square$