Definition 37.28. Given a topological space E, for any subset A of E, an *open cover* $(U_i)_{i \in I}$ *of* A is a family of open subsets of E such that $A \subseteq \bigcup_{i \in I} U_i$. An *open subcover* of an open cover $(U_i)_{i \in I}$ of A is any subfamily $(U_j)_{j \in J}$ which is an open cover of A, with $J \subseteq I$. An open cover $(U_i)_{i \in I}$ of A is *finite* if I is finite. See Figure 37.28. The topological space E is *compact* if it is Hausdorff and for every open cover $(U_i)_{i \in I}$ of E, there is a finite open subcover $(U_j)_{j \in J}$ of E. Given any subset A of E, we say that A is *compact* if it is compact with respect to the subspace topology. We say that A is *relatively compact* if its closure $\overline{A}$ is compact.

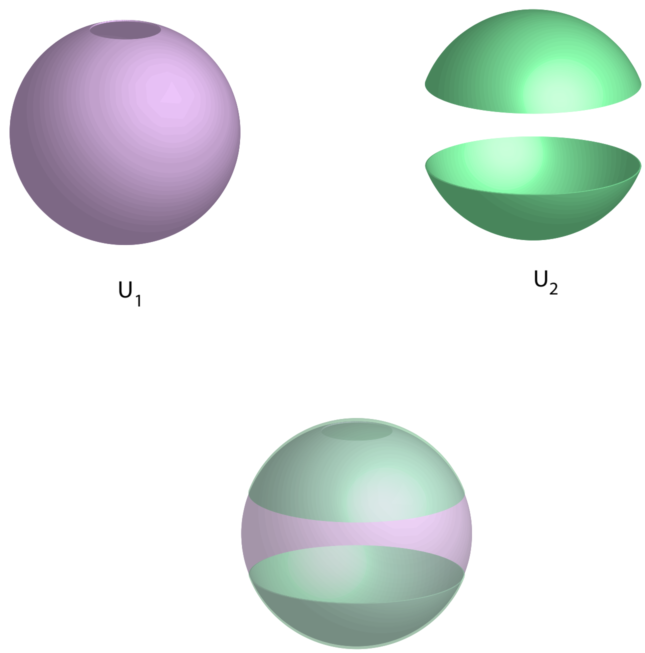

Figure 37.28: An open cover of S^2 using two open sets induced by the Euclidean topology of $\mathbb{R}^3$.

It is immediately verified that a subset A of E is compact in the subspace topology relative to A iff for every open cover $(U_i)_{i \in I}$ of A by open subsets of E, there is a finite open subcover $(U_j)_{j \in J}$ of A. The property that every open cover contains a finite open subcover is often called the *Heine-Borel-Lebesgue* property. By considering complements, a Hausdorff space is compact iff for every family $(F_i)_{i \in I}$ of closed sets, if $\bigcap_{i \in I} F_i = \emptyset$, then $\bigcap_{j \in J} F_j = \emptyset$ for some finite subset J of I.

Definition 37.28 requires that a compact space be Hausdorff. There are books in which a compact space is not necessarily required to be Hausdorff. Following Schwartz, we prefer calling such a space *quasi-compact*.

Another equivalent and useful characterization can be given in terms of families having the finite intersection property.