is the unique desired polynomial, since clearly, $P(\alpha_i) = \beta_i$. Such a polynomial is called a *Lagrange interpolant*. Also note that the polynomials $(L_1, \ldots, L_{m+1})$ form a basis of the vector space of all polynomials of degree $\leq m$. Indeed, if we had

$$\lambda_1 L_1(X) + \cdots + \lambda_{m+1} L_{m+1}(X) = 0,$$

setting X to α_i, we would get $\lambda_i = 0$. Thus, the L_i are linearly independent, and by the previous argument, they are a set of generators. We we call $(L_1, \ldots, L_{m+1})$ the *Lagrange basis* (of order $m+1$).

It is known from numerical analysis that from a computational point of view, the Lagrange basis is not very good. Newton proposed another solution, the method of divided differences.

Consider the polynomial $P(X)$ of degree $\leq m$, called the *Newton interpolant*,

$$P(X) = \lambda_0 + \lambda_1(X - \alpha_1) + \lambda_2(X - \alpha_1)(X - \alpha_2) + \cdots + \lambda_m(X - \alpha_1)(X - \alpha_2)\cdots(X - \alpha_m).$$

Then, the λ_i can be determined by successively setting X to, $\alpha_1, \alpha_2, \ldots, \alpha_{m+1}$. More precisely, we define inductively the polynomials $Q(X)$ and $Q(\alpha_1, \ldots, \alpha_i, X)$, for $1 \leq i \leq m$, as follows:

$$\begin{aligned}
Q(X) &= P(X) \\
Q_1(\alpha_1, X) &= \frac{Q(X) - Q(\alpha_1)}{X - \alpha_1} \\
Q(\alpha_1, \alpha_2, X) &= \frac{Q(\alpha_1, X) - Q(\alpha_1, \alpha_2)}{X - \alpha_2} \\
&\cdots \\
Q(\alpha_1, \ldots, \alpha_i, X) &= \frac{Q(\alpha_1, \ldots, \alpha_{i-1}, X) - Q(\alpha_1, \ldots, \alpha_{i-1}, \alpha_i)}{X - \alpha_i}, \\
&\cdots \\
Q(\alpha_1, \ldots, \alpha_m, X) &= \frac{Q(\alpha_1, \ldots, \alpha_{m-1}, X) - Q(\alpha_1, \ldots, \alpha_{m-1}, \alpha_m)}{X - \alpha_m}.
\end{aligned}$$

By induction on i, $1 \leq i \leq m-1$, it is easily verified that

$$\begin{aligned}
Q(X) &= P(X), \\
Q(\alpha_1, \ldots, \alpha_i, X) &= \lambda_i + \lambda_{i+1}(X - \alpha_{i+1}) + \cdots + \lambda_m(X - \alpha_{i+1})\cdots(X - \alpha_m), \\
Q(\alpha_1, \ldots, \alpha_m, X) &= \lambda_m.
\end{aligned}$$

From the above expressions, it is clear that

$$\begin{aligned}
\lambda_0 &= Q(\alpha_1), \\
\lambda_i &= Q(\alpha_1, \ldots, \alpha_i, \alpha_{i+1}), \\
\lambda_m &= Q(\alpha_1, \ldots, \alpha_m, \alpha_{m+1}).
\end{aligned}$$