For the sake of brevity, we say that J has a *constrained local extremum* at u instead of saying that J has a *local extremum* at the point $u \in U$ *with respect to* U.

In most applications, we have $E_1 = \mathbb{R}^{n-m}$ and $E_2 = \mathbb{R}^m$ for some integers m, n such that $1 \leq m < n$, Ω is an open subset of $\mathbb{R}^n$, $J \colon \Omega \to \mathbb{R}$, and we have m functions $\varphi_i \colon \Omega \to \mathbb{R}$ defining the subset

$$U = \{v \in \Omega \mid \varphi_i(v) = 0,\ 1 \leq i \leq m\}.$$

Fortunately, there is a necessary condition for constrained local extrema in terms of *Lagrange multipliers*.

Theorem 40.2. *(Necessary condition for a constrained extremum in terms of Lagrange multipliers) Let Ω be an open subset of $\mathbb{R}^n$, consider m C^1-functions $\varphi_i \colon \Omega \to \mathbb{R}$ (with $1 \leq m < n$), let*

$$U = \{v \in \Omega \mid \varphi_i(v) = 0,\ 1 \leq i \leq m\},$$

and let $u \in U$ be a point such that the derivatives $d\varphi_i(u) \in \mathcal{L}(\mathbb{R}^n; \mathbb{R})$ are linearly independent; equivalently, assume that the $m \times n$ matrix $\big((\partial\varphi_i/\partial x_j)(u)\big)$ has rank m. If $J \colon \Omega \to \mathbb{R}$ is a function which is differentiable at $u \in U$ and if J has a local constrained extremum at u, then there exist m numbers $\lambda_i(u) \in \mathbb{R}$, uniquely defined, such that

$$dJ(u) + \lambda_1(u) d\varphi_1(u) + \cdots + \lambda_m(u) d\varphi_m(u) = 0;$$

equivalently,

$$\nabla J(u) + \lambda_1(u) \nabla\varphi_1(u) + \cdots + \lambda_m(u) \nabla\varphi_m(u) = 0.$$

Theorem 40.2 will be proven as a corollary of Theorem 40.4, which gives a more general formulation that applies to the situation where E_1 is an infinite-dimensional Banach space. To simplify the exposition we postpone a discussion of this theorem until we have presented several examples illustrating the method of Lagrange multipliers.

Definition 40.4. The numbers $\lambda_i(u)$ involved in Theorem 40.2 are called the *Lagrange multipliers* associated with the constrained extremum u (again, with some minor abuse of language).

The linear independence of the linear forms $d\varphi_i(u)$ is equivalent to the fact that the Jacobian matrix $\big((\partial\varphi_i/\partial x_j)(u)\big)$ of $\varphi = (\varphi_1, \ldots, \varphi_m)$ at u has rank m. If $m = 1$, the linear independence of the $d\varphi_i(u)$ reduces to the condition $\nabla\varphi_1(u) \neq 0$.

A fruitful way to reformulate the use of Lagrange multipliers is to introduce the notion of the Lagrangian associated with our constrained extremum problem.

Definition 40.5. The *Lagrangian* associated with our constrained extremum problem is the function $L \colon \Omega \times \mathbb{R}^m \to \mathbb{R}$ given by

$$L(v, \lambda) = J(v) + \lambda_1 \varphi_1(v) + \cdots + \lambda_m \varphi_m(v),$$

with $\lambda = (\lambda_1, \ldots, \lambda_m)$.