Chapter 22

Singular Value Decomposition and Polar Form

22.1 Properties of $f^* \circ f$

In this section we assume that we are dealing with real Euclidean spaces. Let $f\colon E \to E$ be any linear map. In general, it may not be possible to diagonalize f. We show that every linear map can be diagonalized if we are willing to use *two* orthonormal bases. This is the celebrated *singular value decomposition (SVD)*. A close cousin of the SVD is the *polar form* of a linear map, which shows how a linear map can be decomposed into its purely rotational component (perhaps with a flip) and its purely stretching part.

The key observation is that $f^* \circ f$ is self-adjoint since

$$\langle (f^* \circ f)(u), v\rangle = \langle f(u), f(v)\rangle = \langle u, (f^* \circ f)(v)\rangle.$$

Similarly, $f \circ f^*$ is self-adjoint.

The fact that $f^* \circ f$ and $f \circ f^*$ are self-adjoint is very important, because by Theorem 17.8, it implies that *$f^* \circ f$ and $f \circ f^*$ can be diagonalized and that they have real eigenvalues.* In fact, these *eigenvalues are all nonnegative* as shown in the following proposition.

Proposition 22.1. *The eigenvalues of $f^* \circ f$ and $f \circ f^*$ are nonnegative.*

Proof. If u is an eigenvector of $f^* \circ f$ for the eigenvalue λ, then

$$\langle (f^* \circ f)(u), u\rangle = \langle f(u), f(u)\rangle$$

and

$$\langle (f^* \circ f)(u), u\rangle = \lambda\langle u, u\rangle,$$

and thus

$$\lambda\langle u, u\rangle = \langle f(u), f(u)\rangle,$$

which implies that $\lambda \geq 0$, since $\langle -, -\rangle$ is positive definite. A similar proof applies to $f \circ f^*$. $\square$