12.4 Existence and Construction of Orthonormal Bases

We can also use Theorem 12.6 to show that any Euclidean space of finite dimension has an orthonormal basis.

Proposition 12.9. *Given any nontrivial Euclidean space E of finite dimension $n \geq 1$, there is an orthonormal basis $(u_1, \ldots, u_n)$ for E.*

Proof. We proceed by induction on n. When $n = 1$, take any nonnull vector $v \in E$, which exists since we assumed E nontrivial, and let

$$u = \frac{v}{\|v\|}.$$

If $n \geq 2$, again take any nonnull vector $v \in E$, and let

$$u_1 = \frac{v}{\|v\|}.$$

Consider the linear form φ_{u_1} associated with u_1. Since $u_1 \neq 0$, by Theorem 12.6, the linear form φ_{u_1} is nonnull, and its kernel is a hyperplane H. Since $\varphi_{u_1}(w) = 0$ iff $u_1 \cdot w = 0$, the hyperplane H is the orthogonal complement of $\{u_1\}$. Furthermore, since $u_1 \neq 0$ and the inner product is positive definite, $u_1 \cdot u_1 \neq 0$, and thus, $u_1 \notin H$, which implies that $E = H \oplus \mathbb{R}u_1$. However, since E is of finite dimension n, the hyperplane H has dimension $n-1$, and by the induction hypothesis, we can find an orthonormal basis $(u_2, \ldots, u_n)$ for H. Now because H and the one dimensional space $\mathbb{R}u_1$ are orthogonal and $E = H \oplus \mathbb{R}u_1$, it is clear that $(u_1, \ldots, u_n)$ is an orthonormal basis for E. $\square$

As a consequence of Proposition 12.9, given any Euclidean space of finite dimension n, if $(e_1, \ldots, e_n)$ is an orthonormal basis for E, then for any two vectors $u = u_1e_1 + \cdots + u_ne_n$ and $v = v_1e_1 + \cdots + v_ne_n$, the inner product $u \cdot v$ is expressed as

$$u \cdot v = (u_1e_1 + \cdots + u_ne_n) \cdot (v_1e_1 + \cdots + v_ne_n) = \sum_{i=1}^{n} u_iv_i,$$

and the norm $\|u\|$ as

$$\|u\| = \|u_1e_1 + \cdots + u_ne_n\| = \left(\sum_{i=1}^{n} u_i^2\right)^{1/2}.$$

The fact that a Euclidean space always has an orthonormal basis implies that any Gram matrix G can be written as

$$G = Q^\top Q,$$