Proposition 12.1. *We have*

$$\varphi(u,\, v) = \frac{1}{2}[\Phi(u+v) - \Phi(u) - \Phi(v)]$$

for all $u, v \in E$. We say that φ is the **polar form of** Φ.

Proof. By bilinearity and symmetry, we have

$$\begin{aligned}
\Phi(u+v) &= \varphi(u+v,\, u+v) \\
&= \varphi(u,\, u+v) + \varphi(v,\, u+v) \\
&= \varphi(u,\, u) + 2\varphi(u,\, v) + \varphi(v,\, v) \\
&= \Phi(u) + 2\varphi(u,\, v) + \Phi(v).
\end{aligned}$$

□

If E is finite-dimensional and if $\varphi\colon E \times E \to \mathbb{R}$ is a bilinear form on E, given any basis $(e_1, \ldots, e_n)$ of E, we can write $x = \sum_{i=1}^n x_i e_i$ and $y = \sum_{j=1}^n y_j e_j$, and we have

$$\varphi(x, y) = \varphi\Bigg(\sum_{i=1}^n x_i e_i, \sum_{j=1}^n y_j e_j\Bigg) = \sum_{i,j=1}^n x_i y_j \varphi(e_i, e_j).$$

If we let G be the matrix $G = (\varphi(e_i, e_j))$, and if x and y are the column vectors associated with $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$, then we can write

$$\varphi(x, y) = x^\top G y = y^\top G^\top x.$$

Note that we are committing an abuse of notation since $x = \sum_{i=1}^n x_i e_i$ is a vector in E, but the column vector associated with $(x_1, \ldots, x_n)$ belongs to $\mathbb{R}^n$. To avoid this minor abuse, we could denote the column vector associated with $(x_1, \ldots, x_n)$ by $\mathbf{x}$ (and similarly $\mathbf{y}$ for the column vector associated with $(y_1, \ldots, y_n)$), in wich case the "correct" expression for $\varphi(x, y)$ is

$$\varphi(x, y) = \mathbf{x}^\top G \mathbf{y}.$$

However, in view of the isomorphism between E and $\mathbb{R}^n$, to keep notation as simple as possible, we will use x and y instead of $\mathbf{x}$ and $\mathbf{y}$.

Also observe that φ is symmetric iff $G = G^\top$, and φ is positive definite iff the matrix G is positive definite, that is,

$$x^\top G x > 0 \quad \text{for all } x \in \mathbb{R}^n,\ x \neq 0.$$

The matrix G associated with an inner product is called the *Gram matrix* of the inner product with respect to the basis $(e_1, \ldots, e_n)$.

Conversely, if A is a symmetric positive definite $n \times n$ matrix, it is easy to check that the bilinear form

$$\langle x, y \rangle = x^\top A y$$