Using the formula for the derivative of the inversion map and the chain rule we can show that

$$\mathrm{D}^2 f(A)(X_1, X_2) = -\mathrm{tr}(A^{-1}X_1A^{-1}X_2),$$

and so

$$Hf(A)(X_1, X_2) = -\mathrm{tr}(A^{-1}X_1A^{-1}X_2),$$

a formula which is far from obvious.

The function f can be generalized to matrices $A \in \mathbf{GL}^+(n, \mathbb{R})$, that is, $n \times n$ real invertible matrices of positive determinants, as

$$f(A) = \log\det(A).$$

It can be shown that the formulae

$$\begin{aligned} df_A(X) &= \mathrm{tr}(A^{-1}X) \\ \mathrm{D}^2 f(A)(X_1, X_2) &= -\mathrm{tr}(A^{-1}X_1A^{-1}X_2) \end{aligned}$$

also hold.

Example 39.11. If we restrict the function of Example 39.10 to symmetric positive definite matrices we obtain the function g defined by

$$g(a, b, c) = \log(ac - b^2).$$

We immediately verify that the Jacobian matrix of g is given by

$$dg_{a,b,c} = \frac{1}{ac - b^2}\begin{pmatrix} c & -2b & a \end{pmatrix}.$$

Computing second-order derivatives, we find that the Hessian matrix of g is given by

$$Hg(a, b, c) = \frac{1}{(ac - b^2)^2}\begin{pmatrix} -c^2 & 2bc & -b^2 \\ 2bc & -2(b^2 + ac) & 2ab \\ -b^2 & 2ab & -a^2 \end{pmatrix}.$$

Although this is not obvious, it can be shown that if $ac - b^2 > 0$ and $a, c > 0$, then the matrix $-Hg(a, b, c)$ is symmetric positive definite.

If F itself is of finite dimension, and $(b_0, (v_1, \ldots, v_m))$ is a frame for F, then $f = (f_1, \ldots, f_m)$, and each component $\mathrm{D}^2 f(a)_i(u, v)$ of $\mathrm{D}^2 f(a)(u, v)$ $(1 \leq i \leq m)$, can be written as

$$\mathrm{D}^2 f(a)_i(u, v) = U^\top \begin{pmatrix} \dfrac{\partial^2 f_i}{\partial x_1^2}(a) & \dfrac{\partial^2 f_i}{\partial x_1 \partial x_2}(a) & \cdots & \dfrac{\partial^2 f_i}{\partial x_1 \partial x_n}(a) \\ \dfrac{\partial^2 f_i}{\partial x_1 \partial x_2}(a) & \dfrac{\partial^2 f_i}{\partial x_2^2}(a) & \cdots & \dfrac{\partial^2 f_i}{\partial x_2 \partial x_n}(a) \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f_i}{\partial x_1 \partial x_n}(a) & \dfrac{\partial^2 f_i}{\partial x_2 \partial x_n}(a) & \cdots & \dfrac{\partial^2 f_i}{\partial x_n^2}(a) \end{pmatrix} V$$