for some nonzero $c \in \mathbb{R}$. The above implies that the smallest subspaces W and W' of $\mathbb{R}^n$ such that $u_1 \wedge \cdots \wedge u_p \in \bigwedge^p W$ and $v_1 \wedge \cdots \wedge v_p \in \bigwedge^p W'$ are identical, so $W = W'$. By Corollary 34.26, this smallest subspace W has both $(u_1, \ldots, u_p)$ and $(v_1, \ldots, v_p)$ as bases, so the v_j are linear combinations of the u_i (and vice-versa), and $U = V$. ☐

Since any nonzero $z \in \bigwedge^p \mathbb{R}^n$ can be uniquely written as

$$z = \sum_I \lambda_I e_I$$

in terms of its Plücker coordinates (λ_I), every point of $\mathbb{RP}^{\binom{n}{p}-1}$ is defined by the Plücker coordinates (λ_I) viewed as homogeneous coordinates. The points of $\mathbb{RP}^{\binom{n}{p}-1}$ corresponding to one-dimensional spaces associated with decomposable alternating p-tensors are the points whose coordinates satisfy the Grassmann-Plücker's equations of Proposition 34.29. Therefore, the map i_G embeds the Grassmannian $G(p, n)$ as an algebraic variety in $\mathbb{RP}^{\binom{n}{p}-1}$ defined by equations of degree 2.

We can replace the field $\mathbb{R}$ by $\mathbb{C}$ in the above reasoning and we obtain an embedding of the complex Grassmannian $G_{\mathbb{C}}(p, n)$ as an algebraic variety in $\mathbb{CP}^{\binom{n}{p}-1}$ defined by equations of degree 2.

In particular, if $n = 4$ and $p = 2$, the equation

$$\lambda_{12}\lambda_{34} - \lambda_{13}\lambda_{24} + \lambda_{14}\lambda_{23} = 0$$

is the homogeneous equation of a quadric in $\mathbb{CP}^5$ known as the *Klein quadric*. The points on this quadric are in one-to-one correspondence with the lines in $\mathbb{CP}^3$.

There is also a simple algebraic criterion to decide whether the smallest subspaces U and V associated with two nonzero decomposable vectors $u_1 \wedge \cdots \wedge u_p$ and $v_1 \wedge \cdots \wedge v_q$ have a nontrivial intersection.

Proposition 34.31. *Let E be any n-dimensional vector space over a field K, and let U and V be the smallest subspaces of E associated with two nonzero decomposable vectors $u = u_1 \wedge \cdots \wedge u_p \in \bigwedge^p U$ and $v = v_1 \wedge \cdots \wedge v_q \in \bigwedge^q V$. The following properties hold:*

(1) We have $U \cap V = (0)$ iff $u \wedge v \neq 0$.

(2) If $U \cap V = (0)$, then $U + V$ is the least subspace associated with $u \wedge v$.

Proof. Assume $U \cap V = (0)$. We know by Corollary 34.26 that $(u_1, \ldots, u_p)$ is a basis of U and $(v_1, \ldots, v_q)$ is a basis of V. Since $U \cap V = (0)$, $(u_1, \ldots, u_p, v_1, \ldots, v_q)$ is a basis of $U + V$, and by Proposition 34.8, we have

$$u \wedge v = u_1 \wedge \cdots \wedge u_p \wedge v_1 \wedge \cdots \wedge v_q \neq 0.$$

This also proves (2).