which proves that $\delta(X_n) \leq 1/n$. Now if we consider the sequence of closed sets $(\overline{X_n})$, we still have $\overline{X_{n+1}} \subseteq \overline{X_n}$, and by Proposition 48.4, $\delta(\overline{X_n}) = \delta(X_n) \leq 1/n$, which means that $\lim_{n\to\infty} \delta(\overline{X_n}) = 0$, and by Proposition 48.4, $\bigcap_{n=1}^{\infty} \overline{X_n}$ consists of a single element u. We claim that u is the sum of the family $(u_k)_{k\in K}$.

For every $\epsilon > 0$, there is some $n \geq 1$ such that $n > 2/\epsilon$, and since $u \in \overline{X_m}$ for all $m \geq 1$, there is some finite subset J_0 of K such that $I_n \subseteq J_0$ and

$$\|u - u_{J_0}\| < \epsilon/2,$$

where I_n is the finite subset of K involved in the definition of X_n. However, since $\delta(X_n) \leq 1/n$, for every finite subset J of K such that $I_n \subseteq J$, we have

$$\|u_J - u_{J_0}\| \leq 1/n < \epsilon/2,$$

and since

$$\|u - u_J\| \leq \|u - u_{J_0}\| + \|u_{J_0} - u_J\|,$$

we get

$$\|u - u_J\| < \epsilon$$

for every finite subset J of K with $I_n \subseteq J$, which proves that u is the sum of the family $(u_k)_{k\in K}$.

(2) Since every finite sum $\sum_{i\in I} r_i$ is bounded by the uniform bound B, the set of these finite sums has a least upper bound $r \leq B$. For every $\epsilon > 0$, since r is the least upper bound of the finite sums $\sum_{i\in I} r_i$ (where I finite, $I \subseteq K$), there is some finite $I \subseteq K$ such that

$$\left| r - \sum_{i\in I} r_i \right| < \epsilon,$$

and since $r_k \geq 0$ for all $k \in K$, we have

$$\sum_{i\in I} r_i \leq \sum_{j\in J} r_j$$

whenever $I \subseteq J$, which shows that

$$\left| r - \sum_{j\in J} r_j \right| \leq \left| r - \sum_{i\in I} r_i \right| < \epsilon$$

for every finite subset J such that $I \subseteq J \subseteq K$, proving that $(r_k)_{k\in K}$ is summable with sum $\sum_{k\in K} r_k = r$. $\square$