The linear map f has an eigenspace $E(1, f) = \mathrm{Ker}\,(f - \mathrm{id})$ of dimension p for the eigenvalue 1, and an eigenspace $E(-1, f) = \mathrm{Ker}\,(f + \mathrm{id})$ of dimension q for the eigenvalue -1. If $\det(f) = +1$ (f is a rotation), the dimension q of $E(-1, f)$ must be even, and the entries in $-I_q$ can be paired to form two-dimensional blocks, if we wish. In this case, every rotation in $\mathbf{SO}(n)$ has a matrix of the form

$$\begin{pmatrix} A_1 & \ldots & & \\ \vdots & \ddots & \vdots & \\ & \ldots & A_m & \\ \ldots & & & I_{n-2m} \end{pmatrix}$$

where the first m blocks A_j are of the form

$$A_j = \begin{pmatrix} \cos\theta_j & -\sin\theta_j \\ \sin\theta_j & \cos\theta_j \end{pmatrix}$$

with $0 < \theta_j \leq \pi$.

Theorem 17.16 can be used to prove a version of the Cartan–Dieudonné theorem.

Theorem 17.17. *Let E be a Euclidean space of dimension $n \geq 2$. For every isometry $f \in \mathbf{O}(E)$, if $p = \dim(E(1, f)) = \dim(\mathrm{Ker}\,(f - \mathrm{id}))$, then f is the composition of $n - p$ reflections, and $n - p$ is minimal.*

Proof. From Theorem 17.16 there are r subspaces $F_1, \ldots, F_r$, each of dimension 2, such that

$$E = E(1, f) \oplus E(-1, f) \oplus F_1 \oplus \cdots \oplus F_r,$$

and all the summands are pairwise orthogonal. Furthermore, the restriction r_i of f to each F_i is a rotation $r_i \neq \pm\mathrm{id}$. Each 2D rotation r_i can be written as the composition $r_i = s_i' \circ s_i$ of two reflections s_i and s_i' about lines in F_i (forming an angle $\theta_i/2$). We can extend s_i and s_i' to hyperplane reflections in E by making them the identity on $F_i^{\perp}$. Then

$$s_r' \circ s_r \circ \cdots \circ s_1' \circ s_1$$

agrees with f on $F_1 \oplus \cdots \oplus F_r$ and is the identity on $E(1, f) \oplus E(-1, f)$. If $E(-1, f)$ has an orthonormal basis of eigenvectors $(v_1, \ldots, v_q)$, letting s_j'' be the reflection about the hyperplane $(v_j)^{\perp}$, it is clear that

$$s_q'' \circ \cdots \circ s_1''$$

agrees with f on $E(-1, f)$ and is the identity on $E(1, f) \oplus F_1 \oplus \cdots \oplus F_r$. But then

$$f = s_q'' \circ \cdots \circ s_1'' \circ s_r' \circ s_r \circ \cdots \circ s_1' \circ s_1,$$

the composition of $2r + q = n - p$ reflections.