Proof. The addition operation $+\colon E \times E \to E$ is uniformly continuous because

$$\|(u' + v') - (u'' + v'')\| \leq \|u' - u''\| + \|v' - v''\|.$$

It is not hard to show that $\widehat{E} \times \widehat{E}$ is a complete metric space and that $E \times E$ is dense in $\widehat{E} \times \widehat{E}$. Then, by Theorem 37.52, the uniformly continuous function $+$ has a unique continuous extension $+\colon \widehat{E} \times \widehat{E} \to \widehat{E}$.

The map $\cdot\colon \mathbb{R} \times E \to E$ is not uniformly continuous, but for any fixed $\lambda \in \mathbb{R}$, the map $L_\lambda\colon E \to E$ given by $L_\lambda(u) = \lambda \cdot u$ is uniformly continuous, so by Theorem 37.52 the function L_λ has a unique continuous extension $L_\lambda\colon \widehat{E} \to \widehat{E}$, which we use to define the scalar multiplication $\cdot\colon \mathbb{R} \times \widehat{E} \to \widehat{E}$. It is easily checked that with the above addition and scalar multiplication, $\widehat{E}$ is a vector space.

Since the norm $\| \ \|$ on E is uniformly continuous, it has a unique continuous extension $\| \ \|_{\widehat{E}}\colon \widehat{E} \to \mathbb{R}_+$. The identities $\|u + v\| \leq \|u\| + \|v\|$ and $\|\lambda u\| \leq |\lambda| \, \|u\|$ extend to $\widehat{E}$ by continuity. The equation

$$d(u, v) = \|u - v\|$$

also extends to $\widehat{E}$ by continuity and yields

$$\widehat{d}(\alpha, \beta) = \|\alpha - \beta\|_{\widehat{E}},$$

which shows that $\| \ \|_{\widehat{E}}$ is indeed a norm, and that the metric $\widehat{d}$ is associated to it. Finally, it is easy to verify that the map φ is linear. The uniqueness of the structure of normed vector space follows from the uniqueness of continuous extensions in Theorem 37.52. $\square$

Theorem 37.63 and Theorem 37.52 will be used to show that every Hermitian space can be embedded in a Hilbert space.

The following version of Theorem 37.52 for normed vector spaces is needed in the theory of integration.

Theorem 37.64. *Let E and F be two normed vector spaces, let E_0 be a dense subspace of E, and let $f_0\colon E_0 \to F$ be a continuous function. If f_0 is uniformly continuous and if F is complete, then there is a unique uniformly continuous function $f\colon E \to F$ extending f_0. Furthermore, if f_0 is a continuous linear map, then f is also a linear continuous map, and $\|f\| = \|f_0\|$.*

Proof. We only need to prove the second statement. Given any two vectors $x, y \in E$, since E_0 is dense on E we can pick sequences (x_n) and (y_n) of vectors $x_n, y_n \in E_0$ such that $x = \lim_{n \mapsto \infty} x_n$ and $y = \lim_{n \mapsto \infty} y_n$. Since addition and scalar multiplication are continuous, we get

$$\begin{aligned} x + y &= \lim_{n \mapsto \infty} (x_n + y_n) \\ \lambda x &= \lim_{n \mapsto \infty} (\lambda x_n) \end{aligned}$$