are linearly independent, and that the last $n-1$ equations are also linearly independent. Then, find a relationship between the two groups of equations that will allow you to prove that they span subspace V^r and V^c such that $V^r \cap V^c = (0)$.

(3) Now consider *magic squares*. Such matrices satisfy the two conditions about the sum of the entries in each row and in each column to be the same number, and also the additional two constraints that the main descending and the main ascending diagonals add up to this common number. Traditionally, it is also required that the entries in a magic square are positive integers, but we will consider generalized magic square with arbitrary real entries. For example, in the case $n = 4$, we have the following system of 8 equations:

$$\begin{aligned}
a_{11} + a_{12} + a_{13} + a_{14} - a_{21} - a_{22} - a_{23} - a_{24} &= 0\\
a_{21} + a_{22} + a_{23} + a_{24} - a_{31} - a_{32} - a_{33} - a_{34} &= 0\\
a_{31} + a_{32} + a_{33} + a_{34} - a_{41} - a_{42} - a_{43} - a_{44} &= 0\\
a_{11} + a_{21} + a_{31} + a_{41} - a_{12} - a_{22} - a_{32} - a_{42} &= 0\\
a_{12} + a_{22} + a_{32} + a_{42} - a_{13} - a_{23} - a_{33} - a_{43} &= 0\\
a_{13} + a_{23} + a_{33} + a_{43} - a_{14} - a_{24} - a_{34} - a_{44} &= 0\\
a_{22} + a_{33} + a_{44} - a_{12} - a_{13} - a_{14} &= 0\\
a_{41} + a_{32} + a_{23} - a_{11} - a_{12} - a_{13} &= 0.
\end{aligned}$$

In general, the equation involving the descending diagonal is

$$a_{22} + a_{33} + \cdots + a_{nn} - a_{12} - a_{13} - \cdots - a_{1n} = 0 \tag{r}$$

and the equation involving the ascending diagonal is

$$a_{n1} + a_{n-12} + \cdots + a_{2n-1} - a_{11} - a_{12} - \cdots - a_{1n-1} = 0. \tag{c}$$

Prove that if $n \geq 3$, then the $2n$ equations asserting that a matrix is a generalized magic square are linearly independent.

Hint. Equations are really linear forms, so find some matrix annihilated by all equations except equation r, and some matrix annihilated by all equations except equation c.

Problem 11.6. Let $U_1, \ldots, U_p$ be some subspaces of a vector space E, and assume that they form a direct sum $U = U_1 \oplus \cdots \oplus U_p$. Let $j_i \colon U_i \to U_1 \oplus \cdots \oplus U_p$ be the canonical injections, and let $\pi_i \colon U_1^* \times \cdots \times U_p^* \to U_i^*$ be the canonical projections. Prove that there is an isomorphism f from $(U_1 \oplus \cdots \oplus U_p)^*$ to $U_1^* \times \cdots \times U_p^*$ such that

$$\pi_i \circ f = j_i^\top, \quad 1 \leq i \leq p.$$

Problem 11.7. Let U and V be two subspaces of a vector space E such that $E = U \oplus V$. Prove that

$$E^* = U^0 \oplus V^0.$$