Definition 39.3. Let E and F be two normed affine spaces, let A be a nonempty open subset of E, and let $f\colon A \to F$ be any function. For any $a \in A$, we say that f is *differentiable at* $a \in A$ if there is a linear continuous map $L\colon \overrightarrow{E} \to \overrightarrow{F}$ and a function ϵ, such that

$$f(a+h) = f(a) + L(h) + \epsilon(h)\|h\|$$

for every $a + h \in A$, where $\epsilon(h)$ is defined for every h such that $a + h \in A$ and

$$\lim_{h\to 0,\, h\in U} \epsilon(h) = 0,$$

where $U = \{h \in \overrightarrow{E} \mid a + h \in A,\ h \neq 0\}$. The linear map L is denoted by $\mathrm{D}f(a)$, or $\mathrm{D}f_a$, or $df(a)$, or df_a, or $f'(a)$, and it is called the *Fréchet derivative*, or *derivative*, or *total derivative*, or *total differential*, or *differential*, of f at a; see Figure 39.3.

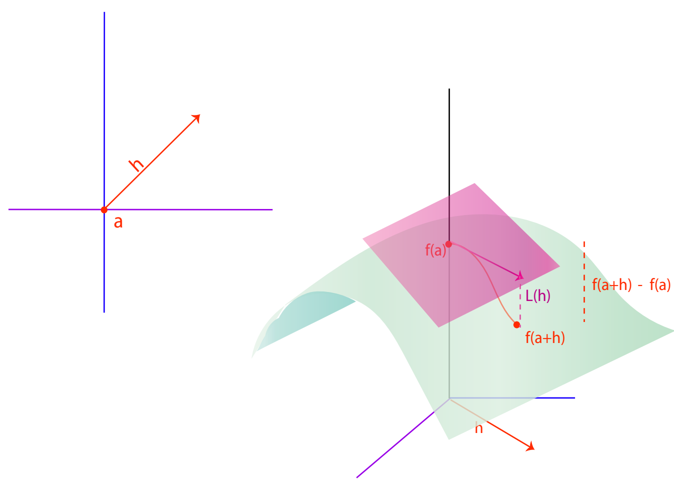

Figure 39.3: Let $f\colon \mathbb{R}^2 \to \mathbb{R}$. The graph of f is the green surface in $\mathbb{R}^3$. The linear map $L = \mathrm{D}f(a)$ is the pink tangent plane. For any vector $h \in \mathbb{R}^2$, $L(h)$ is approximately equal to $f(a+h) - f(a)$. Note that $L(h)$ is also the direction tangent to the curve $t \mapsto f(a + tu)$.

Since the map $h \mapsto a + h$ from $\overrightarrow{E}$ to E is continuous, and since A is open in E, the inverse image U of $A - \{a\}$ under the above map is open in $\overrightarrow{E}$, and it makes sense to say that

$$\lim_{h\to 0,\, h\in U} \epsilon(h) = 0.$$

Note that for every $h \in U$, since $h \neq 0$, $\epsilon(h)$ is uniquely determined since

$$\epsilon(h) = \frac{f(a+h) - f(a) - L(h)}{\|h\|},$$