The nuclear norm can be generalized to $m \times n$ matrices (see Section 22.5). The nuclear norm $\sigma_1 + \cdots + \sigma_r$ of an $m \times n$ matrix A (where r is the rank of A) is denoted by $\|A\|_N$. The nuclear norm plays an important role in *matrix completion*. The problem is this. Given a matrix A_0 with missing entries (missing data), one would like to fill in the missing entries in A_0 to obtain a matrix A of minimal rank. For example, consider the matrices

$$A_0 = \begin{pmatrix} 1 & 2 \\ * & * \end{pmatrix}, \qquad B_0 = \begin{pmatrix} 1 & * \\ * & 4 \end{pmatrix}, \qquad C_0 = \begin{pmatrix} 1 & 2 \\ 3 & * \end{pmatrix}.$$

All can be completed with rank 1. For A_0, use any multiple of $(1, 2)$ for the second row. For B_0, use any numbers b and c such that $bc = 4$. For C_0, the only possibility is $d = 6$.

A famous example of this problem is the *Netflix competition*. The ratings of m films by n viewers goes into A_0. But the customers didn't see all the movies. Many ratings were missing. Those had to be predicted by a recommender system. The nuclear norm gave a good solution that needed to be adjusted for human psychology.

Since the rank of a matrix is not a norm, in order to solve the matrix completion problem we can use the following "convex relaxation." Let A_0 be an incomplete $m \times n$ matrix:

Minimize $\|A\|_N$ subject to $A = A_0$ in the known entries.

The above problem has been extensively studied, in particular by Candès and Recht. Roughly, they showed that if A is an $n \times n$ matrix of rank r and K entries are known in A, then if K is large enough ($K > Cn^{5/4} r \log n$), with high probability, the recovery of A is perfect. See Strang [171] for details (Section III.5).

We close this section by stating the following duality theorem.

Theorem 14.32. *If E is a finite-dimensional Hermitian space, then for any norm $\| \; \|$ on E, we have*

$$\|y\|^{DD} = \|y\|$$

for all $y \in E$.

Proof. By Proposition 14.29, we have

$$|\langle x, y\rangle| \leq \|x\|^D \|y\|,$$

so we get

$$\|y\|^{DD} = \sup_{\|x\|^D = 1} |\langle x, y\rangle| \leq \|y\|, \quad \text{for all } y \in E.$$

It remains to prove that

$$\|y\| \leq \|y\|^{DD}, \quad \text{for all } y \in E.$$

Proofs of this fact can be found in Horn and Johnson [95] (Section 5.5), and in Serre [156] (Chapter 7). The proof makes use of the fact that a nonempty, closed, convex set has a supporting hyperplane through each of its boundary points, a result known as *Minkowski's*