(1) *The space* $(A, \mathcal{U})$ *is a topological space.*

(2) *If* E *is a metric space with metric* d*, then the restriction* $d_A \colon A \times A \to \mathbb{R}_+$ *of the metric* d *to* A *defines a metric space. Furthermore, the topology induced by the metric* d_A *agrees with the topology defined by* $\mathcal{U}$*, as above.*

Proof. Left as an exercise. □

Proposition 37.5 suggests the following definition.

Definition 37.11. Given a topological space $(E, \mathcal{O})$, given any subset A of E, the *subspace topology on* A *induced by* $\mathcal{O}$ is the family $\mathcal{U}$ of open sets defined such that

$$\mathcal{U} = \{U \cap A \mid U \in \mathcal{O}\}$$

is the family of all subsets of A obtained as the intersection of any open set in $\mathcal{O}$ with A. We say that $(A, \mathcal{U})$ has the *subspace topology*. If (E, d) is a metric space, the restriction $d_A \colon A \times A \to \mathbb{R}_+$ of the metric d to A is called the *subspace metric*.

For example, if $E = \mathbb{R}^n$ and d is the Euclidean metric, we obtain the subspace topology on the closed n-cube

$$\{(x_1, \ldots, x_n) \in E \mid a_i \leq x_i \leq b_i,\ 1 \leq i \leq n\}.$$

See Figure 37.11,

One should realize that every open set $U \in \mathcal{O}$ which is entirely contained in A is also in the family $\mathcal{U}$, but $\mathcal{U}$ may contain open sets that are not in $\mathcal{O}$. For example, if $E = \mathbb{R}$ with $|x - y|$, and $A = [a, b]$, then sets of the form $[a, c)$, with $a < c < b$ belong to $\mathcal{U}$, but they are not open sets for $\mathbb{R}$ under $|x - y|$. However, there is agreement in the following situation.

Proposition 37.6. *Given a topological space* $(E, \mathcal{O})$*, given any subset* A *of* E*, if* $\mathcal{U}$ *is the subspace topology, then the following properties hold.*

(1) *If* A *is an open set* $A \in \mathcal{O}$*, then every open set* $U \in \mathcal{U}$ *is an open set* $U \in \mathcal{O}$*.*

(2) *If* A *is a closed set in* E*, then every closed set w.r.t. the subspace topology is a closed set w.r.t.* $\mathcal{O}$*.*

Proof. Left as an exercise. □

The concept of product topology is also useful. We have the following proposition.

Proposition 37.7. *Given* n *topological spaces* $(E_i, \mathcal{O}_i)$*, let* $\mathcal{B}$ *be the family of subsets of* $E_1 \times \cdots \times E_n$ *defined as follows:*

$$\mathcal{B} = \{U_1 \times \cdots \times U_n \mid U_i \in \mathcal{O}_i,\ 1 \leq i \leq n\},$$

and let $\mathcal{P}$ *be the family consisting of arbitrary unions of sets in* $\mathcal{B}$*, including* $\emptyset$*. Then* $\mathcal{P}$ *is a topology on* $E_1 \times \cdots \times E_n$*.*