29.6 Totally Isotropic Subspaces

In this section, we deal with ϵ-Hermitian forms, $\varphi\colon E \times E \to K$. In general, E may have subspaces U such that $U \cap U^{\perp} \neq (0)$, or worse, such that $U \subseteq U^{\perp}$ (that is, φ is zero on U). We will see that such subspaces play a crucial in the decomposition of E into orthogonal subspaces.

Definition 29.17. Given an ϵ-Hermitian forms $\varphi\colon E \times E \to K$, a nonzero vector $u \in E$ is said to be *isotropic* if $\varphi(u, u) = 0$. It is convenient to consider 0 to be isotropic. Given any subspace U of E, the subspace $\mathrm{rad}(U) = U \cap U^{\perp}$ is called the *radical* of U. We say that

(i) U is *degenerate* if $\mathrm{rad}(U) \neq (0)$ (equivalently if there is some nonzero vector $u \in U$ such that $x \in U^{\perp}$). Otherwise, we say that U is *nondegenerate*.

(ii) U is *totally isotropic* if $U \subseteq U^{\perp}$ (equivalently if the restriction of φ to U is zero).

By definition, the trivial subspace $U = (0)$ $(= \{0\})$ is nondegenerate. Observe that a subspace U is nondegenerate iff the restriction of φ to U is nondegenerate. A degenerate subspace is sometimes called an *isotropic* subspace. Other authors say that a subspace U is *isotropic* if it contains some (nonzero) isotropic vector. A subspace which has no nonzero isotropic vector is often called *anisotropic*. The space of all isotropic vectors is a cone often called the *light cone* (a terminology coming from the theory of relativity). This is not to be confused with the cone of silence (from Get Smart)! It should also be noted that some authors (such as Serre) use the term *isotropic* instead of *totally isotropic*. The apparent lack of standard terminology is almost as bad as in graph theory!

It is clear that any direct sum of pairwise orthogonal totally isotropic subspaces is totally isotropic. Thus, every totally isotropic subspace is contained in some maximal totally isotropic subspace. Here is another fact that we will use all the time: if V is a totally isotropic subspace and if U is a subspace of V, then U is totally isotropic.

This is because by definition V is isotropic if $V \subseteq V^{\perp}$, and since $U \subseteq V$ we get $V^{\perp} \subseteq U^{\perp}$, so $U \subseteq V \subseteq V^{\perp} \subseteq U^{\perp}$, which shows that U is totally isotropic.

First, let us show that in order to sudy an ϵ-Hermitian form on a space E, it suffices to restrict our attention to nondegenerate forms.

Proposition 29.19. *Given an ϵ-Hermitian form $\varphi\colon E \times E \to K$ on E, we have:*

(a) If U and V are any two orthogonal subspaces of E, then

$$\mathrm{rad}(U + V) = \mathrm{rad}(U) + \mathrm{rad}(V).$$

(b) $\mathrm{rad}(\mathrm{rad}(E)) = \mathrm{rad}(E)$.