Proposition 6.20. *Let E be a vector space. If $E = U \oplus V$ and $E = U \oplus W$, then there is an isomorphism $f : V \to W$ between V and W.*

Proof. Let R be the relation between V and W, defined such that

$$\langle v, w\rangle \in R \quad \text{iff} \quad w - v \in U.$$

We claim that R is a functional relation that defines a linear isomorphism $f : V \to W$ between V and W, where $f(v) = w$ iff $\langle v, w\rangle \in R$ (R is the graph of f). If $w - v \in U$ and $w' - v \in U$, then $w' - w \in U$, and since $U \oplus W$ is a direct sum, $U \cap W = (0)$, and thus $w' - w = 0$, that is $w' = w$. Thus, R is functional. Similarly, if $w - v \in U$ and $w - v' \in U$, then $v' - v \in U$, and since $U \oplus V$ is a direct sum, $U \cap V = (0)$, and $v' = v$. Thus, f is injective. Since $E = U \oplus V$, for every $w \in W$, there exists a unique pair $\langle u, v\rangle \in U \times V$, such that $w = u + v$. Then, $w - v \in U$, and f is surjective. We also need to verify that f is linear. If

$$w - v = u$$

and

$$w' - v' = u',$$

where $u, u' \in U$, then, we have

$$(w + w') - (v + v') = (u + u'),$$

where $u + u' \in U$. Similarly, if

$$w - v = u$$

where $u \in U$, then we have

$$\lambda w - \lambda v = \lambda u,$$

where $\lambda u \in U$. Thus, f is linear. □

Given a vector space E and any subspace U of E, Proposition 6.20 shows that the dimension of any subspace V such that $E = U \oplus V$ depends only on U. We call $\dim(V)$ the *codimension* of U, and we denote it by $\mathrm{codim}(U)$. A subspace U of codimension 1 is called a *hyperplane*.

The notion of rank of a linear map or of a matrix is an important one, both theoretically and practically, since it is the key to the solvability of linear equations. Recall from Definition 3.19 that the *rank* $\mathrm{rk}(f)$ of a linear map $f : E \to F$ is the dimension $\dim(\mathrm{Im}\, f)$ of the image subspace $\mathrm{Im}\, f$ of F.

We have the following simple proposition.

Proposition 6.21. *Given a linear map $f : E \to F$, the following properties hold:*

(i) $\mathrm{rk}(f) = \mathrm{codim}(\mathrm{Ker}\, f)$.