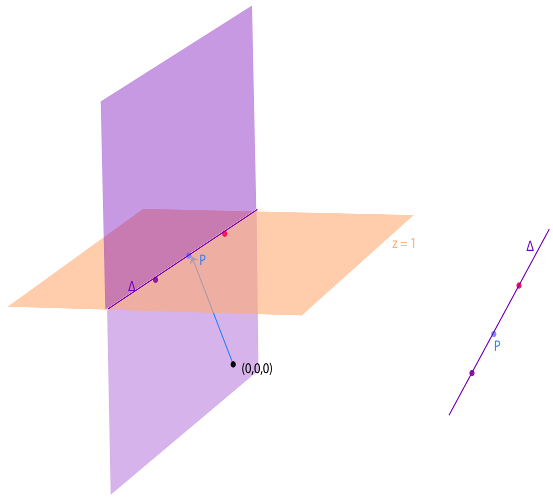

Figure 26.29: Case (VI): The left figure is the hyperplane representation of $\mathbb{RP}^2$ and a homography with fixed point P and invariant line Δ. The purple (linear) hyperplane maps to itself in a manner which is not the identity.

Observe that in Cases (III) and (IV), the homography h has a line Δ of fixed points, as well as a fixed point P. In Case (III), $P \notin \Delta$, and in Case (IV), $P \in \Delta$. This kind of homography is called a *homology*. The point P is called the *center* and the line Δ is called the *axis* (or *base*). Some authors only use the term homology when $P \notin \Delta$, and when $P \in \Delta$, they use the term *elation*. When $P \in \Delta$, other authors use the term *projective transvection*, which we prefer. The center is usually denoted by O (instead of P).

One of the nice features of homologies (and projective transvections) is that there is a nice geometric construction of the image $h(M)$ of a point M in terms of the center O, the axis Δ, and any pair (A, A') where $A' = h(A)$, $A \neq O$, and $A \notin \Delta$.

This construction is possible because for any point $M \neq O$, the line $\langle M, h(M)\rangle$ passes through O. This can be proved using Desargues' Theorem; for example, see Silder [161] (Chapter 4, Section 4.2). We will prove this property for a generalization of homologies to any projective space $\mathbb{P}(E)$, where E is a vector space of any finite dimension.

For the construction, first assume that $M \neq O$ is not on the line $\langle A, A'\rangle$. In this case, the line $\langle A, M\rangle$ intersects Δ in some point I. Since $I \in \Delta$, it is fixed by h, so the image of the line $\langle A, I\rangle$ is the line $\langle A', I\rangle$, and since M is on the line $\langle A, I\rangle$, its image $M' = h(M)$ is on the line $\langle A', I\rangle$. But $M' = h(M)$ is also on the line $\langle O, M\rangle$, which implies that $M' = h(M)$ is the intersection point of the lines $\langle A', I\rangle$ and $\langle O, M\rangle$; see Figure 26.30.

If $M \neq O$ is on the line $\langle A, A'\rangle$, then we use the construction of the image B' of some point $B \neq O$ and not on $\langle A, A'\rangle$ as before, and then repeat the construction by finding the intersection J of $\langle M, B\rangle$ and Δ, and then $M' = h(M)$ is the intersection point of $\langle B', J\rangle$ and $\langle A, A'\rangle$; see Figure 26.31.