## 17.7 The Courant–Fischer Theorem; Perturbation Results

Another useful tool to prove eigenvalue equalities is the Courant–Fischer characterization of the eigenvalues of a symmetric matrix, also known as the Min-max (and Max-min) theorem.

**Theorem 17.27.** *(Courant–Fischer) Let $A$ be a symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. If $\mathcal{V}_k$ denotes the set of subspaces of $\mathbb{R}^n$ of dimension $k$, then*

$$\lambda_k = \max_{W \in \mathcal{V}_{n-k+1}} \min_{x \in W, x \neq 0} \frac{x^\top A x}{x^\top x}$$

$$\lambda_k = \min_{W \in \mathcal{V}_k} \max_{x \in W, x \neq 0} \frac{x^\top A x}{x^\top x}.$$

*Proof.* Let us consider the second equality, the proof of the first equality being similar. Let $(u_1, \ldots, u_n)$ be any orthonormal basis of eigenvectors of $A$, where $u_i$ is a unit eigenvector associated with $\lambda_i$. Observe that the space $V_k$ spanned by $(u_1, \ldots, u_k)$ has dimension $k$, and by Proposition 17.23, we have

$$\lambda_k = \max_{x \neq 0, x \in V_k} \frac{x^\top A x}{x^\top x} \geq \inf_{W \in \mathcal{V}_k} \max_{x \in W, x \neq 0} \frac{x^\top A x}{x^\top x}.$$

Therefore, we need to prove the reverse inequality; that is, we have to show that

$$\lambda_k \leq \max_{x \neq 0, x \in W} \frac{x^\top A x}{x^\top x}, \quad \text{for all} \quad W \in \mathcal{V}_k.$$

Now for any $W \in \mathcal{V}_k$, if we can prove that $W \cap V_{k-1}^\perp \neq (0)$, then for any nonzero $v \in W \cap V_{k-1}^\perp$, by Proposition 17.24 , we have

$$\lambda_k = \min_{x \neq 0, x \in V_{k-1}^\perp} \frac{x^\top A x}{x^\top x} \leq \frac{v^\top A v}{v^\top v} \leq \max_{x \in W, x \neq 0} \frac{x^\top A x}{x^\top x}.$$

It remains to prove that $\dim(W \cap V_{k-1}^\perp) \geq 1$. However, $\dim(V_{k-1}) = k - 1$, so $\dim(V_{k-1}^\perp) = n - k + 1$, and by hypothesis $\dim(W) = k$. By the Grassmann relation,

$$\dim(W) + \dim(V_{k-1}^\perp) = \dim(W \cap V_{k-1}^\perp) + \dim(W + V_{k-1}^\perp),$$

and since $\dim(W + V_{k-1}^\perp) \leq \dim(\mathbb{R}^n) = n$, we get

$$k + n - k + 1 \leq \dim(W \cap V_{k-1}^\perp) + n;$$

that is, $1 \leq \dim(W \cap V_{k-1}^\perp)$, as claimed. Thus we proved that

$$\lambda_k = \inf_{W \in \mathcal{V}_k} \max_{x \in W, x \neq 0} \frac{x^\top A x}{x^\top x},$$

but since the inf is achieved for the subspace $V_k$, the equation also holds with inf replaced by min. ∎