Example 23.3. Consider the following real diagonal form of the normal matrix

$$A = \begin{pmatrix} -2.7500 & 2.1651 & -0.8660 & 0.5000 \\ 2.1651 & -0.2500 & -1.5000 & 0.8660 \\ 0.8660 & 1.5000 & 0.7500 & -0.4330 \\ -0.5000 & -0.8660 & -0.4330 & 0.2500 \end{pmatrix} = U\Lambda U^\top,$$

with

$$U = \begin{pmatrix} \cos(\pi/3) & 0 & \sin(\pi/3) & 0 \\ \sin(\pi/3) & 0 & -\cos(\pi/3) & 0 \\ 0 & \cos(\pi/6) & 0 & \sin(\pi/6) \\ 0 & -\cos(\pi/6) & 0 & \sin(\pi/6) \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 1 & -2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We obtain

$$\Lambda^+ = \begin{pmatrix} 1/5 & 2/5 & 0 & 0 \\ -2/5 & 1/5 & 0 & 0 \\ 0 & 0 & -1/4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and the pseudo-inverse of A is

$$A^+ = U\Lambda^+ U^\top = \begin{pmatrix} -0.1375 & 0.1949 & 0.1732 & -0.1000 \\ 0.1949 & 0.0875 & 0.3000 & -0.1732 \\ -0.1732 & -0.3000 & 0.1500 & -0.0866 \\ 0.1000 & 0.1732 & -0.0866 & 0.0500 \end{pmatrix},$$

which agrees with `pinv(A)`.

The following properties, due to Penrose, characterize the pseudo-inverse of a matrix. We have already proved that the pseudo-inverse satisfies these equations. For a proof of the converse, see Kincaid and Cheney [102].

Proposition 23.8. *Given any $m \times n$ matrix A (real or complex), the pseudo-inverse A^+ of A is the unique $n \times m$ matrix satisfying the following properties:*

$$\begin{aligned} AA^+A &= A, \\ A^+AA^+ &= A^+, \\ (AA^+)^\top &= AA^+, \\ (A^+A)^\top &= A^+A. \end{aligned}$$

23.3 Data Compression and SVD

Among the many applications of SVD, a very useful one is *data compression*, notably for images. In order to make precise the notion of closeness of matrices, we use the notion of