Proposition 37.1. *Given a metric space E with metric d, the family $\mathcal{O}$ of all open sets defined in Definition 37.4 satisfies the following properties:*

(O1) For every finite family $(U_i)_{1\leq i\leq n}$ of sets $U_i \in \mathcal{O}$, we have $U_1 \cap \cdots \cap U_n \in \mathcal{O}$, i.e., $\mathcal{O}$ is closed under finite intersections.

(O2) For every arbitrary family $(U_i)_{i\in I}$ of sets $U_i \in \mathcal{O}$, we have $\bigcup_{i\in I} U_i \in \mathcal{O}$, i.e., $\mathcal{O}$ is closed under arbitrary unions.

(O3) $\emptyset \in \mathcal{O}$, and $E \in \mathcal{O}$, i.e., $\emptyset$ and E belong to $\mathcal{O}$.

Furthermore, for any two distinct points $a \neq b$ in E, there exist two open sets U_a and U_b such that, $a \in U_a$, $b \in U_b$, and $U_a \cap U_b = \emptyset$.

Proof. It is straightforward. For the last point, letting $\rho = d(a, b)/3$ (in fact $\rho = d(a, b)/2$ works too), we can pick $U_a = B_0(a, \rho)$ and $U_b = B_0(b, \rho)$. By the triangle inequality, we must have $U_a \cap U_b = \emptyset$. □

The above proposition leads to the very general concept of a topological space.

One should be careful that, in general, the family of open sets is not closed under infinite intersections. For example, in $\mathbb{R}$ under the metric $|x - y|$, letting $U_n = (-1/n, +1/n)$, each U_n is open, but $\bigcap_n U_n = \{0\}$, which is not open.

Later on, given any nonempty subset A of a metric space (E, d), we will need to know that certain special sets containing A are open.

Definition 37.5. Let (E, d) be a metric space. For any nonempty subset A of E and any $x \in E$, let
$$d(x, A) = \inf_{a\in A} d(x, a).$$

Proposition 37.2. *Let (E, d) be a metric space. For any nonempty subset A of E and for any two points $x, y \in E$, we have*
$$|d(x, A) - d(y, A)| \leq d(x, y).$$

Proof. For all $a \in A$ we have
$$d(x, a) \leq d(x, y) + d(y, a),$$
which implies
$$\begin{aligned} d(x, A) &= \inf_{a\in A} d(x, a) \\ &\leq \inf_{a\in A}(d(x, y) + d(y, a)) \\ &= d(x, y) + \inf_{a\in A} d(y, a) \\ &= d(x, y) + d(y, A). \end{aligned}$$