where $\theta \in [0, 2\pi[$, and that every improper orthogonal transformation is represented by a matrix of the form

$$S = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

In the first case, we call $\theta \in [0, 2\pi[$ the *measure* of the angle of rotation of R w.r.t. the orthonormal basis (e_1, e_2). In the second case, we have a reflection about a line, and it is easy to determine what this line is. It is also easy to see that S is the composition of a reflection about the x-axis with a rotation (of matrix R).

We refrained from calling θ "the angle of rotation," because there are some subtleties involved in defining rigorously the notion of angle of two vectors (or two lines). For example, note that with respect to the "opposite basis" (e_2, e_1), the measure θ must be changed to $2\pi - \theta$ (or $-\theta$ if we consider the quotient set $\mathbb{R}/2\pi$ of the real numbers modulo 2π).

It is easily shown that the group $\mathbf{SO}(2)$ of rotations in the plane is abelian. First, recall that every plane rotation is the product of two reflections (about lines), and that every isometry in $\mathbf{O}(2)$ is either a reflection or a rotation. To alleviate the notation, we will omit the composition operator $\circ$, and write rs instead of $r \circ s$. Now, if r is a rotation and s is a reflection, rs being in $\mathbf{O}(2)$ must be a reflection (since $\det(rs) = \det(r)\det(s) = -1$), and thus $(rs)^2 = \mathrm{id}$, since a reflection is an involution, which implies that

$$srs = r^{-1}.$$

Then, given two rotations r_1 and r_2, writing r_1 as $r_1 = s_2 s_1$ for two reflections s_1, s_2, we have

$$r_1 r_2 r_1^{-1} = s_2 s_1 r_2 (s_2 s_1)^{-1} = s_2 s_1 r_2 s_1^{-1} s_2^{-1} = s_2 s_1 r_2 s_1 s_2 = s_2 r_2^{-1} s_2 = r_2,$$

since $srs = r^{-1}$ for all reflections s and rotations r, and thus $r_1 r_2 = r_2 r_1$.

We can also perform the following calculation, using some elementary trigonometry:

$$\begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix} \begin{pmatrix} \cos\psi & \sin\psi \\ \sin\psi & -\cos\psi \end{pmatrix} = \begin{pmatrix} \cos(\varphi+\psi) & \sin(\varphi+\psi) \\ \sin(\varphi+\psi) & -\cos(\varphi+\psi) \end{pmatrix}.$$

The above also shows that the inverse of a rotation matrix

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is obtained by changing θ to $-\theta$ (or $2\pi - \theta$). Incidentally, note that in writing a rotation r as the product of two reflections $r = s_2 s_1$, the first reflection s_1 can be chosen arbitrarily, since $s_1^2 = \mathrm{id}$, $r = (rs_1)s_1$, and rs_1 is a reflection.

For $n = 3$, the only two choices for p are $p = 1$, which corresponds to the identity, or $p = 0$, in which case f is a rotation leaving a line invariant. This line D is called the *axis of*