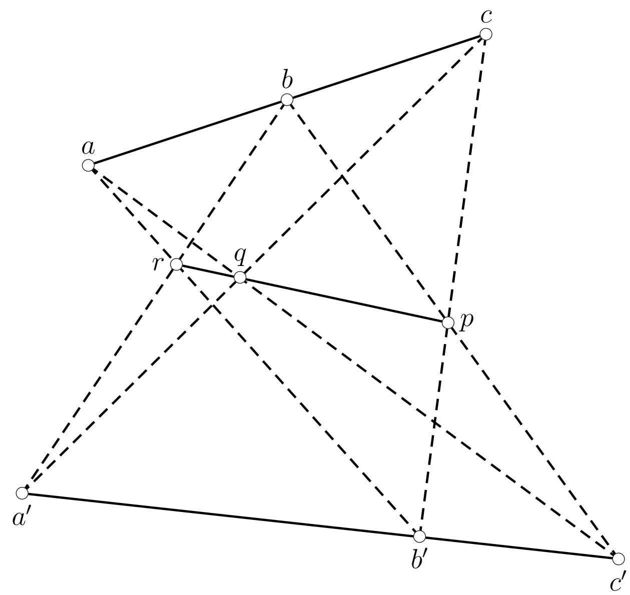

Figure 26.20: Pappus's theorem (projective version).

a, b, c, a', b', c' are distinct from the intersection of D and D', then the intersection points $p = \langle b, c'\rangle \cap \langle b', c\rangle$, $q = \langle a, c'\rangle \cap \langle a', c\rangle$, and $r = \langle a, b'\rangle \cap \langle a', b\rangle$ are collinear.

Proof. First, since any two lines in a projective plane intersect in a single point, the points p, q, r are well defined. Choose $\Delta = \langle p, r\rangle$ as the line at infinity, and consider the affine plane $X = \mathbf{P}(E) - \Delta$. Since $\langle a, b'\rangle$ and $\langle a', b\rangle$ intersect at a point at infinity r on Δ, $\langle a, b'\rangle$ and $\langle a', b\rangle$ are parallel, and similarly $\langle b, c'\rangle$ and $\langle b', c\rangle$ are parallel. Thus, by the affine version of Pappus's theorem (Proposition 24.12), the lines $\langle a, c'\rangle$ and $\langle a', c\rangle$ are parallel, which means that their intersection q is on the line at infinity $\Delta = \langle p, r\rangle$, which means that p, q, r are collinear. $\square$

By working in the projective completion of an affine plane, we can obtain an improved version of Pappus's theorem for affine planes. The reader will have to figure out how to deal with the special cases where some of p, q, r go to infinity.

Now, we prove a projective version of Desargues's theorem slightly more general than that given in Proposition 26.7. It is interesting that the proof is radically different, depending on the dimension of the projective space $\mathbf{P}(E)$. This is not surprising. In axiomatic presentations of projective plane geometry, Desargues's theorem is independent of the other axioms. Desargues's theorem is illustrated in Figure 26.21.

Proposition 26.19. *(Desargues) Let $\mathbf{P}(E)$ be a projective space. Given two triangles (a, b, c) and (a', b', c'), where the points a, b, c, a', b', c' are pairwise distinct and the lines $A = \langle b, c\rangle$, $B = \langle a, c\rangle$, $C = \langle a, b\rangle$, $A' = \langle b', c'\rangle$, $B' = \langle a', c'\rangle$, $C' = \langle a', b'\rangle$ are pairwise distinct, if the*