and by adding up these inequalities, we obtain

$$V^{k+1} \leq V^0 - \rho \sum_{j=0}^{k} \left(\left\| r^{j+1} \right\|_2^2 + \left\| B(z^{j+1} - z^j) \right\|_2^2 \right),$$

which implies that

$$\rho \sum_{j=0}^{k} \left(\left\| r^{j+1} \right\|_2^2 + \left\| B(z^{j+1} - z^j) \right\|_2^2 \right) \leq V_0 - V^{k+1} \leq V^0, \tag{B}$$

since $V^{k+1} \leq V^0$.

Step 2. Prove that the sequence (r^k) converges to 0, and that the sequences (Ax^{k+1}) and (Bz^{k+1}) also converge.

Inequality (B) implies that the series $\sum_{k=1}^{\infty} r^k$ and $\sum_{k=0}^{\infty} B(z^{k+1} - z^k)$ converge absolutely. In particular, the sequence (r^k) converges to 0.

The nth partial sum of the series $\sum_{k=0}^{\infty} B(z^{k+1} - z^k)$ is

$$\sum_{k=0}^{n} B(z^{k+1} - z^k) = B(z^{n+1} - z^0),$$

and since the series $\sum_{k=0}^{\infty} B(z^{k+1} - z^k)$ converges, we deduce that the sequence (Bz^{k+1}) converges. Since $Ax^{k+1} + Bz^{k+1} - c = r^{k+1}$, the convergence of (r^{k+1}) and (Bz^{k+1}) implies that the sequence (Ax^{k+1}) also converges.

Step 3. Prove that the sequences (x^{k+1}) and (z^{k+1}) converge. By Assumption (2), the matrices $A^\top A$ and $B^\top B$ are invertible, so multiplying each vector Ax^{k+1} by $(A^\top A)^{-1}A^\top$, if the sequence (Ax^{k+1}) converges to u, then the sequence (x^{k+1}) converges to $(A^\top A)^{-1}A^\top u$. Siimilarly, if the sequence (Bz^{k+1}) converges to v, then the sequence (z^{k+1}) converges to $(B^\top B)^{-1}B^\top v$.

Step 4. Prove that the sequence (λ^k) converges.

Recall that

$$\lambda^{k+1} = \lambda^k + \rho r^{k+1}.$$

It follows by induction that

$$\lambda^{k+p} = \lambda^k + \rho(r^{k+1} + \cdots + \rho^{k+p}), \quad p \geq 2.$$

As a consequence, we get

$$\left\| \lambda^{k+p} - \lambda^k \right\| \leq \rho(\left\| r^{k+1} \right\| + \cdots + \left\| r^{k+p} \right\|).$$