Proof. For every $\rho_k \geq 0$, define the function $g_k \colon V \to U$ by

$$g_k(v) = p_U(v - \rho_k \nabla J_v).$$

By Proposition 48.6, the projection map p_U has Lipschitz constant 1, so using the inequalities assumed to hold in the proposition, we have

$$\begin{aligned}
\|g_k(v_1) - g_k(v_2)\|^2 &= \|p_U(v_1 - \rho_k \nabla J_{v_1}) - p_U(v_2 - \rho_k \nabla J_{v_2})\|^2 \\
&\leq \|(v_1 - v_2) - \rho_k(\nabla J_{v_1} - \nabla J_{v_2})\|^2 \\
&= \|v_1 - v_2\|^2 - 2\rho_k \langle \nabla J_{v_1} - \nabla J_{v_2}, v_1 - v_2\rangle + \rho_k^2 \|\nabla J_{v_1} - \nabla J_{v_2}\|^2 \\
&\leq \left(1 - 2\alpha\rho_k + M^2\rho_k^2\right) \|v_1 - v_2\|^2 .
\end{aligned}$$

As in the proof of Proposition 49.14, we know that if a and b satisfy the conditions $0 < a \leq \rho_k \leq b \leq \frac{2\alpha}{M^2}$, then there is some β such that

$$\left(1 - 2\alpha\rho_k + M^2\rho_k^2\right)^{1/2} \leq \beta < 1 \quad \text{for all } k \geq 0.$$

Since the minimizing point $u \in U$ is a fixed point of g_k for all k, by letting $v_1 = u_k$ and $v_2 = u$, we get

$$\|u_{k+1} - u\| = \|g_k(u_k) - g_k(u)\| \leq \beta \|u_k - u\| ,$$

which proves the convergence of the sequence $(u_k)_{k \geq 0}$. ☐

In the case of an elliptic quadratic functional

$$J(v) = \frac{1}{2}\langle Av, a\rangle - \langle b, v\rangle$$

defined on $\mathbb{R}^n$, the reasoning just after the proof of Proposition 49.14 can be immediately adapted to show that convergence takes place as long as a, b and ρ_k are chosen such that

$$0 < a \leq \rho_k \leq b \leq \frac{2}{\lambda_n}.$$

In theory, Proposition 49.18 gives a guarantee of the convergence of the projected-gradient method. Unfortunately, because computing the projection $p_U(v)$ effectively is generally impossible, the range of practical applications of Proposition 49.18 is rather limited. One exception is the case where U is a product $\prod_{i=1}^m [a_i, b_i]$ of closed intervals (where $a_i = -\infty$ or $b_i = +\infty$ is possible). In this case, it is not hard to show that

$$p_U(w)_i = \begin{cases} a_i & \text{if } w_i < a_i \\ w_i & \text{if } a_i \leq w_i \leq b_i \\ b_i & \text{if } b_i < w_i. \end{cases}$$