Example 12.8. Consider the vector space $\mathrm{M}_n(\mathbb{R})$ of real $n \times n$ matrices with the inner product

$$\langle A, B \rangle = \mathrm{tr}(A^\top B).$$

Let $s\colon \mathrm{M}_n(\mathbb{R}) \to \mathbb{R}$ be the function given by

$$s(A) = \sum_{i,j=1}^{n} a_{ij},$$

where $A = (a_{ij})$. It is immediately verified that s is a linear form. It is easy to check that the unique matrix Z such that

$$\langle Z, A \rangle = s(A) \quad \text{for all } A \in \mathrm{M}_n(\mathbb{R})$$

is the matrix $Z = \mathbf{ones}(n, n)$ whose entries are all equal to 1.

12.3 Adjoint of a Linear Map

The existence of the isomorphism $\flat\colon E \to E^*$ is crucial to the existence of adjoint maps. The importance of adjoint maps stems from the fact that the linear maps arising in physical problems are often self-adjoint, which means that $f = f^*$. Moreover, self-adjoint maps can be diagonalized over orthonormal bases of eigenvectors. This is the key to the solution of many problems in mechanics and engineering in general (see Strang [169]).

Let E be a Euclidean space of finite dimension n, and let $f\colon E \to E$ be a linear map. For every $u \in E$, the map

$$v \mapsto u \cdot f(v)$$

is clearly a linear form in E^*, and by Theorem 12.6, there is a unique vector in E denoted by $f^*(u)$ such that

$$f^*(u) \cdot v = u \cdot f(v),$$

for every $v \in E$. The following simple proposition shows that the map f^* is linear.

Proposition 12.8. *Given a Euclidean space E of finite dimension, for every linear map $f\colon E \to E$, there is a unique linear map $f^*\colon E \to E$ such that*

$$f^*(u) \cdot v = u \cdot f(v), \quad \textit{for all } u, v \in E.$$

Proof. Given $u_1, u_2 \in E$, since the inner product is bilinear, we have

$$(u_1 + u_2) \cdot f(v) = u_1 \cdot f(v) + u_2 \cdot f(v),$$

for all $v \in E$, and

$$(f^*(u_1) + f^*(u_2)) \cdot v = f^*(u_1) \cdot v + f^*(u_2) \cdot v,$$