Problem 9.10. Prove that for any real or complex square matrix A, we have

$$\|A\|_2^2 \leq \|A\|_1 \, \|A\|_\infty \, ,$$

where the above norms are operator norms.

Hint. Use Proposition 9.10 (among other things, it shows that $\|A\|_1 = \left\|A^\top\right\|_\infty$).

Problem 9.11. Show that the map $A \mapsto \rho(A)$ (where $\rho(A)$ is the spectral radius of A) is neither a norm nor a matrix norm. In particular, find two 2×2 matrices A and B such that

$$\rho(A + B) > \rho(A) + \rho(B) = 0 \quad \text{and} \quad \rho(AB) > \rho(A)\rho(B) = 0.$$

Problem 9.12. Define the map $A \mapsto M(A)$ (defined on $n \times n$ real or complex $n \times n$ matrices) by

$$M(A) = \max\{|a_{ij}| \mid 1 \leq i, j \leq n\}.$$

(1) Prove that

$$M(AB) \leq nM(A)M(B)$$

for all $n \times n$ matrices A and B.

(2) Give a counter-example of the inequality

$$M(AB) \leq M(A)M(B).$$

(3) Prove that the map $A \mapsto \|A\|_M$ given by

$$\|A\|_M = nM(A) = n \max\{|a_{ij}| \mid 1 \leq i, j \leq n\}$$

is a matrix norm.

Problem 9.13. Let S be a real symmetric positive definite matrix.

(1) Use the Cholesky factorization to prove that there is some upper-triangular matrix C, unique if its diagonal elements are strictly positive, such that $S = C^\top C$.

(2) For any $x \in \mathbb{R}^n$, define

$$\|x\|_S = (x^\top S x)^{1/2}.$$

Prove that

$$\|x\|_S = \|Cx\|_2 \, ,$$

and that the map $x \mapsto \|x\|_S$ is a norm.

Problem 9.14. Let A be a real 2×2 matrix

$$A = \begin{pmatrix} a_{1\,1} & a_{1\,2} \\ a_{2\,1} & a_{2\,2} \end{pmatrix}.$$