Let us now consider the case of an arbitrary symmetric matrix A.

Proposition 42.5. *If A is an $n \times n$ symmetric matrix, then the function*

$$f(x) = \frac{1}{2} x^\top A x - x^\top b$$

has a minimum value iff $A \succeq 0$ and $(I - AA^+)b = 0$, in which case this minimum value is

$$p^* = -\frac{1}{2} b^\top A^+ b.$$

Furthermore, if A is diagonalized as $A = U^\top \Sigma U$ (with U orthogonal), then the optimal value is achieved by all $x \in \mathbb{R}^n$ of the form

$$x = A^+ b + U^\top \begin{pmatrix} 0 \\ z \end{pmatrix},$$

for any $z \in \mathbb{R}^{n-r}$, where r is the rank of A.

Proof. The case that A is invertible is taken care of by Proposition 42.4, so we may assume that A is singular. If A has rank $r < n$, then we can diagonalize A as

$$A = U^\top \begin{pmatrix} \Sigma_r & 0 \\ 0 & 0 \end{pmatrix} U,$$

where U is an orthogonal matrix and where Σ_r is an $r \times r$ diagonal invertible matrix. Then we have

$$\begin{aligned} f(x) &= \frac{1}{2} x^\top U^\top \begin{pmatrix} \Sigma_r & 0 \\ 0 & 0 \end{pmatrix} U x - x^\top U^\top U b \\ &= \frac{1}{2} (Ux)^\top \begin{pmatrix} \Sigma_r & 0 \\ 0 & 0 \end{pmatrix} U x - (Ux)^\top U b. \end{aligned}$$

If we write

$$Ux = \begin{pmatrix} y \\ z \end{pmatrix} \quad \text{and} \quad Ub = \begin{pmatrix} c \\ d \end{pmatrix},$$

with $y, c \in \mathbb{R}^r$ and $z, d \in \mathbb{R}^{n-r}$, we get

$$\begin{aligned} f(x) &= \frac{1}{2} (Ux)^\top \begin{pmatrix} \Sigma_r & 0 \\ 0 & 0 \end{pmatrix} U x - (Ux)^\top U b \\ &= \frac{1}{2} (y^\top \; z^\top) \begin{pmatrix} \Sigma_r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} - (y^\top \; z^\top) \begin{pmatrix} c \\ d \end{pmatrix} \\ &= \frac{1}{2} y^\top \Sigma_r y - y^\top c - z^\top d. \end{aligned}$$