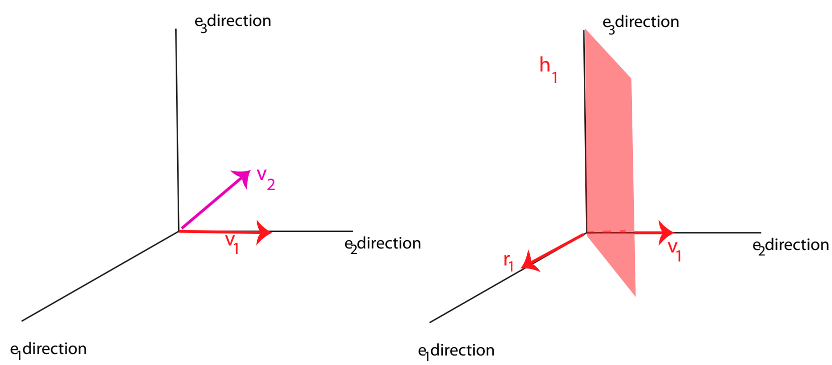

Figure 13.5: The construction of $r_1 = h_1(v_1)$ in Proposition 13.3.

where $u'_{k+1} \in U'_k$ and $u''_{k+1} \in U''_k$. See Figure 13.6. Let

$$r_{k+1,k+1} = \|u''_{k+1}\|.$$

If $u''_{k+1} = r_{k+1,k+1}\, e_{k+1}$, we let $h_{k+1} = \mathrm{id}$. Otherwise, there is a unique hyperplane reflection h_{k+1} such that

$$h_{k+1}(u''_{k+1}) = r_{k+1,k+1}\, e_{k+1},$$

defined such that

$$h_{k+1}(u) = u - 2\,\frac{(u \cdot w_{k+1})}{\|w_{k+1}\|^2}\, w_{k+1}$$

for all $u \in E$, where

$$w_{k+1} = r_{k+1,k+1}\, e_{k+1} - u''_{k+1}.$$

The map h_{k+1} is the reflection about the hyperplane H_{k+1} orthogonal to the vector $w_{k+1} = r_{k+1,k+1}\, e_{k+1} - u''_{k+1}$. However, since $u''_{k+1}, e_{k+1} \in U''_k$ and U'_k is orthogonal to U''_k, the subspace U'_k is contained in H_{k+1}, and thus, the vectors $(r_1, \ldots, r_k)$ and u'_{k+1}, which belong to U'_k, are invariant under h_{k+1}. This proves that

$$h_{k+1}(u_{k+1}) = h_{k+1}(u'_{k+1}) + h_{k+1}(u''_{k+1}) = u'_{k+1} + r_{k+1,k+1}\, e_{k+1}$$

is a linear combination of $(e_1, \ldots, e_{k+1})$. Letting

$$r_{k+1} = h_{k+1}(u_{k+1}) = u'_{k+1} + r_{k+1,k+1}\, e_{k+1},$$

since $u_{k+1} = h_k \circ \cdots \circ h_2 \circ h_1(v_{k+1})$, the vector

$$r_{k+1} = h_{k+1} \circ \cdots \circ h_2 \circ h_1(v_{k+1})$$

is a linear combination of $(e_1, \ldots, e_{k+1})$. See Figure 13.7. The coefficient of r_{k+1} over e_{k+1} is $r_{k+1,k+1} = \|u''_{k+1}\|$, which is nonnegative. This concludes the induction step, and thus the proof. $\square$