The converse of Proposition 40.12 (2) is false as we see by considering the strictly convex function f given by $f(x) = x^4$ and its second derivative at $x = 0$.

Example 40.6. On the other hand, if f is a quadratic function of the form

$$f(u) = \frac{1}{2} u^\top A u - u^\top b$$

where A is a symmetric matrix, we know that

$$df(u)(v) = v^\top (Au - b),$$

so

$$\begin{aligned}
f(v) - f(u) - df(u)(v-u) &= \frac{1}{2} v^\top A v - v^\top b - \frac{1}{2} u^\top A u + u^\top b - (v-u)^\top (Au - b) \\
&= \frac{1}{2} v^\top A v - \frac{1}{2} u^\top A u - (v-u)^\top A u \\
&= \frac{1}{2} v^\top A v + \frac{1}{2} u^\top A u - v^\top A u \\
&= \frac{1}{2} (v-u)^\top A (v-u).
\end{aligned}$$

Therefore, Proposition 40.11 implies that if A is positive semidefinite, then f is convex and if A is positive definite, then f is strictly convex. The converse follows by Proposition 40.12.

We conclude this section by applying our previous theorems to convex functions defined on convex subsets. In this case local minima (resp. local maxima) are global minima (resp. global maxima). The next definition is the special case of Definition 40.1 in which $W = E$ but it does not hurt to state it explicitly.

Definition 40.9. Let $f\colon E \to \mathbb{R}$ be any function defined on some normed vector space (or more generally, any set). For any $u \in E$, we say that f has a *minimum* in u (resp. *maximum* in u) if

$$f(u) \leq f(v) \;\; (\text{resp. } f(u) \geq f(v)) \quad \text{for all } v \in E.$$

We say that f has a *strict minimum* in u (resp. *strict maximum* in u) if

$$f(u) < f(v) \;\; (\text{resp. } f(u) > f(v)) \quad \text{for all } v \in E - \{u\}.$$

If $U \subseteq E$ is a subset of E and $u \in U$, we say that f has a *minimum* in u (resp. *strict minimum* in u) *with respect to U* if

$$f(u) \leq f(v) \quad \text{for all } v \in U \quad (\text{resp. } f(u) < f(v) \quad \text{for all } v \in U - \{u\}),$$

and similarly for a *maximum* in u (resp. *strict maximum* in u) *with respect to U* with $\leq$ changed to $\geq$ and $<$ to $>$.