Therefore, given $w_{12}, w_{13}, w_{23} \in F$, the function h given by

$$\begin{aligned} h(u_1e_1 + u_2e_2 + u_3e_3, v_1e_1 + v_2e_2 + v_3e_3) &= (u_1v_2 - u_2v_1)w_{12} + (u_1v_3 - u_3v_1)w_{13} \\ &\quad + (u_2v_3 - u_3v_2)w_{23} \end{aligned}$$

is clearly bilinear and alternating, and by construction $h(e_i, e_j) = w_{ij}$, with $1 \leq i < j \leq 3$ does the job.

We now prove the assertion that tensors u_I with $|I| = n$ generate $\bigwedge^n(E)$ for arbitrary n.

Proposition 34.7. *Given any vector space E, if E has finite dimension $d = \dim(E)$, then for all $n > d$, the exterior power $\bigwedge^n(E)$ is trivial; that is $\bigwedge^n(E) = (0)$. If $n \leq d$ or if E is infinite dimensional, then for every ordered basis $((u_i)_{i\in\Sigma}, \leq)$, the family (u_I) is basis of $\bigwedge^n(E)$, where I ranges over finite nonempty subsets of Σ of size $|I| = n$.*

Proof. First assume that E has finite dimension $d = \dim(E)$ and that $n > d$. We know that $\bigwedge^n(E)$ is generated by the tensors of the form $v_1 \wedge \cdots \wedge v_n$, with $v_i \in E$. If $u_1, \ldots, u_d$ is a basis of E, as every v_i is a linear combination of the u_j, when we expand $v_1 \wedge \cdots \wedge v_n$ using multilinearity, we get a linear combination of the form

$$v_1 \wedge \cdots \wedge v_n = \sum_{(j_1,\ldots,j_n)} \lambda_{(j_1,\ldots,j_n)}\, u_{j_1} \wedge \cdots \wedge u_{j_n},$$

where each $(j_1, \ldots, j_n)$ is some sequence of integers $j_k \in \{1, \ldots, d\}$. As $n > d$, each sequence $(j_1, \ldots, j_n)$ must contain two identical elements. By alternation, $u_{j_1} \wedge \cdots \wedge u_{j_n} = 0$, and so $v_1 \wedge \cdots \wedge v_n = 0$. It follows that $\bigwedge^n(E) = (0)$.

Now assume that either $\dim(E) = d$ and $n \leq d$, or that E is infinite dimensional. The argument below shows that the u_I are nonzero and linearly independent. As usual, let $u_i^* \in E^*$ be the linear form given by

$$u_i^*(u_j) = \delta_{ij}.$$

For any nonempty subset $I = \{i_1, \ldots, i_n\} \subseteq \Sigma$ with $i_1 < \cdots < i_n$, for any n vectors $v_1, \ldots, v_n \in E$, let

$$l_I(v_1, \ldots, v_n) = \det(u_{i_j}^*(v_k)) = \begin{vmatrix} u_{i_1}^*(v_1) & \cdots & u_{i_1}^*(v_n) \\ \vdots & \ddots & \vdots \\ u_{i_n}^*(v_1) & \cdots & u_{i_n}^*(v_n) \end{vmatrix}.$$

If we let the n-tuple $(v_1, \ldots, v_n)$ vary we obtain a map l_I from E^n to K, and it is easy to check that this map is alternating multilinear. Thus l_I induces a unique linear map $L_I \colon \bigwedge^n(E) \to K$ making the following diagram commute.

$$\begin{tikzcd} E^n \arrow[r, "\iota_\wedge"] \arrow[dr, "l_I"'] & \bigwedge^n(E) \arrow[d, "L_I"] \\ & K \end{tikzcd}$$