equations

$$
\begin{aligned}
x_2 - x_1 &= 1 \\
x_1 + 6x_2 &= 15 \\
4x_1 - x_2 &= 10 \\
x_1 &= 0 \\
x_2 &= 0.
\end{aligned}
$$

In general, each constraint $a_i x \leq b_i$ corresponds to the affine form φ_i given by $\varphi_i(x) = a_i x - b_i$ and defines the half-space $H_-(\varphi_i)$, and each inequality $x_j \geq 0$ defines the half-space $H_+(x_j)$. The intersection of these half-spaces is the set of solutions of all these constraints. It is a (possibly empty) $\mathcal{H}$-polyhedron denoted $\mathcal{P}(A, b)$.

Definition 45.2. If $\mathcal{P}(A, b) = \emptyset$, we say that the Linear Program (P) has *no feasible solution*, and otherwise any $x \in \mathcal{P}(A, b)$ is called a *feasible solution* of (P).

The linear program shown in Example 45.2 obtained by reversing the direction of the inequalities $x_2 - x_1 \leq 1$ and $4x_1 - x_2 \leq 10$ in the linear program of Example 45.1 has no feasible solution; see Figure 45.2.

Example 45.2.

$$
\begin{aligned}
\text{maximize} \quad & x_1 + x_2 \\
\text{subject to} \quad & \\
& x_1 - x_2 \leq -1 \\
& x_1 + 6x_2 \leq 15 \\
& x_2 - 4x_1 \leq -10 \\
& x_1 \geq 0, \; x_2 \geq 0.
\end{aligned}
$$

Assume $\mathcal{P}(A, b) \neq \emptyset$, so that the Linear Program (P) has a feasible solution. In this case, consider the image $\{cx \in \mathbb{R} \mid x \in \mathcal{P}(A, b)\}$ of $\mathcal{P}(A, b)$ under the objective function $x \mapsto cx$.

Definition 45.3. If the set $\{cx \in \mathbb{R} \mid x \in \mathcal{P}(A, b)\}$ is unbounded above, then we say that the Linear Program (P) is *unbounded*.

The linear program shown in Example 45.3 obtained from the linear program of Example 45.1 by deleting the constraints $4x_1 - x_2 \leq 10$ and $x_1 + 6x_2 \leq 15$ is unbounded.

Example 45.3.

$$
\begin{aligned}
\text{maximize} \quad & x_1 + x_2 \\
\text{subject to} \quad & \\
& x_2 - x_1 \leq 1 \\
& x_1 \geq 0, \; x_2 \geq 0.
\end{aligned}
$$