and

$$\det(qAq^*) = \det(q)\det(A)\det(q^*) = \det(A) = -(x^2 + y^2 + z^2).$$

We can embed $\mathbb{R}^3$ into the space of Hermitian matrices with zero trace by

$$\varphi(x, y, z) = x\sigma_3 + y\sigma_2 + z\sigma_1.$$

Note that

$$\varphi = -i\psi \quad \text{and} \quad \varphi^{-1} = i\psi^{-1}.$$

Definition 16.5. The unit quaternion $q \in \mathbf{SU}(2)$ induces a map r_q on $\mathbb{R}^3$ by

$$r_q(x, y, z) = \varphi^{-1}(q\varphi(x, y, z)q^*) = \varphi^{-1}(q(x\sigma_3 + y\sigma_2 + z\sigma_1)q^*).$$

The map r_q is clearly linear since φ is linear.

Proposition 16.1. *For every unit quaternion $q \in \mathbf{SU}(2)$, the linear map r_q is orthogonal, that is, $r_q \in \mathbf{O}(3)$.*

Proof. Since

$$-\left\|(x, y, z)\right\|^2 = -(x^2 + y^2 + z^2) = \det(x\sigma^3 + y\sigma^2 + z\sigma_1) = \det(\varphi(x, y, z)),$$

we have

$$\begin{aligned} -\left\|r_q(x, y, z)\right\|^2 &= \det(\varphi(r_q(x, y, z))) = \det(q(x\sigma_3 + y\sigma_2 + z\sigma_1)q^*) \\ &= \det(x\sigma_3 + y\sigma_2 + z\sigma_1) = -\left\|(x, y, z)^2\right\|, \end{aligned}$$

and we deduce that r_q is an isometry. Thus, $r_q \in \mathbf{O}(3)$. $\square$

In fact, r_q is a rotation, and we can show this by finding the fixed points of r_q. Let q be a unit quaternion of the form

$$q = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$$

with $\alpha = a + ib$, $\beta = c + id$, and $a^2 + b^2 + c^2 + d^2 = 1$ $(a, b, c, d \in \mathbb{R})$.

If $b = c = d = 0$, then $q = I$ and r_q is the identity so we may assume that $(b, c, d) \neq (0, 0, 0)$.

Proposition 16.2. *If $(b, c, d) \neq (0, 0, 0)$, then the fixed points of r_q are solutions (x, y, z) of the linear system*

$$\begin{aligned} -dy + cz &= 0 \\ cx - by &= 0 \\ dx - bz &= 0. \end{aligned}$$

This linear system has the nontrivial solution (b, c, d) and has rank 2. Therefore, r_q has the eigenvalue 1 with multiplicity 1, and r_q is a rotation whose axis is determined by (b, c, d).