is in fact the value of the determinant of A (which, as we shall see shortly, is also equal to the determinant of $A^\top$). However, working directly with the above definition is quite awkward, and we will proceed via a slightly indirect route

Remark: The reader might have been puzzled by the fact that it is the transpose matrix $A^\top$ rather than A itself that appears in Lemma 7.4. The reason is that if we want the generic term in the determinant to be

$$\epsilon(\pi)a_{\pi(1)\,1}\cdots a_{\pi(n)\,n},$$

where the permutation applies to the first index, then we have to express the v_js in terms of the u_is in terms of $A^\top$ as we did. Furthermore, since

$$v_j = a_{1\,j}u_1 + \cdots + a_{i\,j}u_i + \cdots + a_{n\,j}u_n,$$

we see that v_j corresponds to the jth column of the matrix A, and so the determinant is viewed as a function of the *columns* of A.

The literature is split on this point. Some authors prefer to define a determinant as we did. Others use A itself, which amounts to viewing det as a function of the rows, in which case we get the expression

$$\sum_{\sigma\in\mathfrak{S}_n}\epsilon(\sigma)a_{1\,\sigma(1)}\cdots a_{n\,\sigma(n)}.$$

Corollary 7.7 show that these two expressions are equal, so it doesn't matter which is chosen. This is a matter of taste.

7.3 Definition of a Determinant

Recall that the set of all square $n\times n$-matrices with coefficients in a field K is denoted by $\mathrm{M}_n(K)$.

Definition 7.4. A *determinant* is defined as any map

$$D\colon \mathrm{M}_n(K)\to K,$$

which, when viewed as a map on $(K^n)^n$, i.e., a map of the n columns of a matrix, is n-linear alternating and such that $D(I_n)=1$ for the identity matrix I_n. Equivalently, we can consider a vector space E of dimension n, some fixed basis $(e_1,\ldots,e_n)$, and define

$$D\colon E^n\to K$$

as an n-linear alternating map such that $D(e_1,\ldots,e_n)=1$.