

Going the other way, given a polar decomposition $A = R_1 S$, where R_1 is orthogonal and S is positive semidefinite symmetric, there is an orthogonal matrix R_2 and a positive semidefinite diagonal matrix D such that $S = R_2 D R_2^\top$, and thus

$$A = R_1 R_2 D R_2^\top = V D U^\top,$$

where $V = R_1 R_2$ and $U = R_2$ are orthogonal.

Example 22.6. Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $A = R_1 S$, where $R_1 = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$ and $S = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$. This is the polar decomposition of Example 22.5. Observe that

$$S = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = R_2 D R_2^\top.$$

Set $U = R_2$ and $V = R_1 R_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ to obtain the SVD decomposition of Example 22.4.

The eigenvalues and the singular values of a matrix are typically not related in any obvious way. For example, the $n \times n$ matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 2 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 & 2 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & 2 \\ 0 & 0 & \dots & 0 & 0 & 0 & 1 \end{pmatrix}$$

has the eigenvalue 1 with multiplicity n , but its singular values, $\sigma_1 \geq \dots \geq \sigma_n$, which are the positive square roots of the eigenvalues of the matrix $B = A^\top A$ with

$$B = \begin{pmatrix} 1 & 2 & 0 & 0 & \dots & 0 & 0 \\ 2 & 5 & 2 & 0 & \dots & 0 & 0 \\ 0 & 2 & 5 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 2 & 5 & 2 & 0 \\ 0 & 0 & \dots & 0 & 2 & 5 & 2 \\ 0 & 0 & \dots & 0 & 0 & 2 & 5 \end{pmatrix}$$

have a wide spread, since

$$\frac{\sigma_1}{\sigma_n} = \text{cond}_2(A) \geq 2^{n-1}.$$