see Jacobson [98], Section 3.10, just after Formula (33).

If all the roots, $\lambda_1, \ldots, \lambda_n$, of the polynomial $\det(XI - A)$ belong to the field K, then we can write

$$\chi_A(X) = \det(XI - A) = (X - \lambda_1) \cdots (X - \lambda_n),$$

where some of the λ_i's may appear more than once. Consequently,

$$\chi_A(X) = \det(XI - A) = X^n - \sigma_1(\lambda)X^{n-1} + \cdots + (-1)^k \sigma_k(\lambda)X^{n-k} + \cdots + (-1)^n \sigma_n(\lambda),$$

where

$$\sigma_k(\lambda) = \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ |I|=k}} \prod_{i \in I} \lambda_i,$$

the *kth elementary symmetric polynomial (or function)* of the λ_i's, where $\lambda = (\lambda_1, \ldots, \lambda_n)$. The elementary symmetric polynomial $\sigma_k(\lambda)$ is often denoted $E_k(\lambda)$, but this notation may be confusing in the context of linear algebra. For $n = 5$, the elementary symmetric polynomials are listed below:

$$\begin{aligned}
\sigma_0(\lambda) &= 1 \\
\sigma_1(\lambda) &= \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 \\
\sigma_2(\lambda) &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_1\lambda_5 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_2\lambda_5 \\
&\quad + \lambda_3\lambda_4 + \lambda_3\lambda_5 + \lambda_4\lambda_5 \\
\sigma_3(\lambda) &= \lambda_3\lambda_4\lambda_5 + \lambda_2\lambda_4\lambda_5 + \lambda_2\lambda_3\lambda_5 + \lambda_2\lambda_3\lambda_4 + \lambda_1\lambda_4\lambda_5 \\
&\quad + \lambda_1\lambda_3\lambda_5 + \lambda_1\lambda_3\lambda_4 + \lambda_1\lambda_2\lambda_5 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_2\lambda_3 \\
\sigma_4(\lambda) &= \lambda_1\lambda_2\lambda_3\lambda_4 + \lambda_1\lambda_2\lambda_3\lambda_5 + \lambda_1\lambda_2\lambda_4\lambda_5 + \lambda_1\lambda_3\lambda_4\lambda_5 + \lambda_2\lambda_3\lambda_4\lambda_5 \\
\sigma_5(\lambda) &= \lambda_1\lambda_2\lambda_3\lambda_4\lambda_5.
\end{aligned}$$

Since

$$\begin{aligned}
\chi_A(X) &= X^n - \tau_1(A)X^{n-1} + \cdots + (-1)^k \tau_k(A)X^{n-k} + \cdots + (-1)^n \tau_n(A) \\
&= X^n - \sigma_1(\lambda)X^{n-1} + \cdots + (-1)^k \sigma_k(\lambda)X^{n-k} + \cdots + (-1)^n \sigma_n(\lambda),
\end{aligned}$$

we have

$$\sigma_k(\lambda) = \tau_k(A), \quad k = 1, \ldots, n,$$

and in particular, the product of the eigenvalues of f is equal to $\det(A) = \det(f)$, and the sum of the eigenvalues of f is equal to the trace $\mathrm{tr}(A) = \mathrm{tr}(f)$, of f; for the record,

$$\begin{aligned}
\mathrm{tr}(f) &= \lambda_1 + \cdots + \lambda_n \\
\det(f) &= \lambda_1 \cdots \lambda_n,
\end{aligned}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of f (and A), where some of the λ_i's may appear more than once. In particular, f is not invertible iff it admits 0 has an eigenvalue (since f is singular iff $\lambda_1 \cdots \lambda_n = \det(f) = 0$).