Using Proposition 26.23 we obtain the following characterization of homologies. Write $\dim(E) = n + 1$.

Proposition 26.24. *If $h\colon \mathbb{P}(E) \to \mathbb{P}(E)$ is a homology of axis $\mathbb{P}(H)$ and if $h \neq \mathrm{id}$, then for any linear isomorphism $f\colon E \to E$ such that $h = \mathbb{P}(f)$, the following properties hold:*

(1) Either f is a dilatation of hyperplane H and of direction u for some nonzero $u \in E - H$ uniquely defined up to a scalar;

(2) Or f is a transvection of hyperplane H and direction u for some nonzero $u \in H$ uniquely defined up to a scalar.

In both cases, $O = [u] \in \mathbb{P}(E)$ is a fixed point of h, and h has no other fixed points besides O and points in $\mathbb{P}(H)$. In Case (1), $O \notin \mathbb{P}(H)$, whereas in Case (2), $O \in \mathbb{P}(H)$. Furthermore, for any point $M \in \mathbb{P}(E)$, if $M \neq O$ and if $M \notin \mathbb{P}(H)$, then the line $\langle M, h(M)\rangle$ passes through O. If $\dim(E) \geq 3$, the point O is the only point satisfying the above property.

Proof. Since the restriction of $h = \mathbb{P}(f)$ to $\mathbb{P}(H)$ is the identity, and since $\mathbb{P}(f) = \mathbb{P}(\mathrm{id}_H)$, by Proposition 26.4 we have $f = \lambda \mathrm{id}_H$ on H for some nonzero scalar $\lambda \in K$. Then $g = \lambda^{-1} f$ is the identity on H, so by Proposition 26.23 we obtain (1) and (2).

In Case (1), we have $g(u) = \alpha u$, so $\mathbb{P}(g)([u]) = \mathbb{P}(f)([u]) = [u]$. In Case (2), $g(u) = u$, so again $\mathbb{P}(g)([u]) = \mathbb{P}(f)([u]) = [u]$. Therefore, $O = [u]$ is a fixed point of $\mathbb{P}(f)$. In Case (1), the eigenvalues of f are 1 with multiplicity n and α with multiplicity 1. If $Q = [v] \neq O$ was a fixed point of h not in $\mathbb{P}(H)$, then v would be an eigenvector corresponding to a nonzero eigenvalue λ of f with $\lambda \neq 1, \alpha$, and then f would have $n + 2$ eigenvalues (counted with multiplicty), which is absurd. In Case (2), the only eigenvalue of f is 1, with multiplicity n, so f not diagonalizable, and as above, a vector v such that $Q = [v]$ is a fixed point of h not in $\mathbb{P}(H)$ would be an eigenvector corresponding to a nonzero eigenvalue $\lambda \neq 1$ of f, so f would be diagonalizable, a contradiction.

Since in Case (1), for any $x \neq u$ and $x \notin H$ we have $x = \lambda u + h$ for some unique $h \in H$ and some unique $\lambda \neq 0$, so

$$g(x) = g(\lambda u) + g(h) = \lambda\alpha u + h = \lambda u + h + (\lambda\alpha - \lambda)u = x + \lambda(\alpha - 1)u,$$

which shows that $O, [x]$ and $\mathbb{P}(g)([x]) = \mathbb{P}(f)([x])$ are collinear. In Case (2), for any $x \neq u$ and $x \notin H$ we have

$$g(x) = x + \varphi(x)u,$$

which also shows that $O, [x]$ and $\mathbb{P}(g)([x]) = \mathbb{P}(f)([x])$ are collinear. The last property is left as an exercise (see Vienne [185], Chapter 4, Proposition 7). ☐

Proposition 26.24 suggests the following definition.