Proof. First observe that since $M(\alpha)$ is annihilated by α, we can view $M(\alpha)$ as a $A/(\alpha)$-module. By the Chinese remainder theorem (Theorem 32.15) applied to the ideals $(up_1^{n_1}) = (p_1^{n_1}), (p_2^{n_2}), \ldots, (p_r^{n_r})$, we have an isomorphism

$$A/(\alpha) \approx A/(p_1^{n_1}) \times \cdots \times A/(p_r^{n_r}).$$

Since we also have isomorphisms

$$A/(p_i^{n_i}) \approx (A/(\alpha))/((p_i^{n_i})/(\alpha)),$$

we can apply Proposition 35.15, and we get a direct sum

$$M(\alpha) = N_1 \oplus \cdots \oplus N_r,$$

where N_i is the $A/(\alpha)$-submodule of $M(\alpha)$ annihilated by $(p_i^{n_i})/(\alpha)$, and the projections onto the N_i are of the form stated in the proposition. However, N_i is just the A-module $M(p_i^{n_i})$ annihilated by $p_i^{n_i}$, because every nonzero element of $(p_i^{n_i})/(\alpha)$ is an equivalence class modulo (α) of the form $\overline{ap_i^{n_i}}$ for some nonzero $a \in A$, and by definition, $x \in N_i$ iff

$$0 = \overline{ap_i^{n_i}}\, x = ap_i^{n_i} x, \quad \text{for all } a \in A - \{0\},$$

in particular for $a = 1$, which implies that $x \in M(p_i^{n_i})$.

The inclusion $M(p_i^{n_i}) \subseteq M(\alpha) \cap M_{p_i}$ is clear. Conversely, pick $x \in M(\alpha) \cap M_{p_i}$, which means that $\alpha x = 0$ and $p_i^s x = 0$ for some $s \geq 1$. If $s < n_i$, we are done, so assume $s \geq n_i$. Since $p_i^{n_i}$ is a gcd of α and p_i^s, by Bezout, we can write

$$p_i^{n_i} = \lambda p_i^s + \mu\alpha$$

for some $\lambda, \mu \in A$, and then $p_i^{n_i} x = \lambda p_i^s x + \mu\alpha x = 0$, which shows that $x \in M(p_i^{n_i})$, as desired. $\square$

Here is an example of Proposition 35.16. Let $M = \mathbb{Z}/60\mathbb{Z}$, where M is considered as a $\mathbb{Z}$-module. A element in M is denoted by $\overline{x}$, where x is an integer with $0 \leq x \leq 59$. Let $\alpha = 6$ and define

$$M(6) = \{\overline{x} \in M \mid 6\overline{x} = \overline{0}\} = \{\overline{0}, \overline{10}, \overline{20}, \overline{30}, \overline{40}, \overline{50}\}.$$

Since $6 = 2 \cdot 3$, Proposition 35.16 implies that $M(6) = M(2) \oplus M(3)$, where

$$M(2) = \{\overline{x} \in M \mid 2\overline{x} = \overline{0}\} = \{\overline{0}, \overline{30}\}$$
$$M(3) = \{\overline{x} \in M \mid 3\overline{x} = \overline{0}\} = \{\overline{0}, \overline{20}, \overline{40}\}.$$

Recall that if M is a torsion module over a ring A which is an integral domain, then every finite set of elements $x_1, \ldots, x_n$ in M is annihilated by $a = a_1 \cdots a_n$, where each a_i annihilates x_i.

Since A is a PID, we can pick a set P of irreducible elements of A such that every nonzero nonunit of A has a unique factorization up to a unit. Then, we have the following structure theorem for torsion modules which holds even for modules that are not finitely generated.