Remark: The subspace consisting of all sequences $(z_k)_{k\in K}$ such that $z_k = 0$, except perhaps for finitely many k, provides an example of a subspace which is not closed in $\ell^2(K)$. Indeed, this space is strictly contained in $\ell^2(K)$, since there are countable sequences of nonnull elements in $\ell^2(K)$ (why?).

We just need two more propositions before being able to prove that every Hilbert space is isomorphic to some $\ell^2(K)$.

Proposition A.4. *Let E be a Hilbert space, and $(u_k)_{k\in K}$ an orthogonal family in E. The following properties hold:*

(1) For every family $(\lambda_k)_{k\in K} \in \ell^2(K)$, the family $(\lambda_k u_k)_{k\in K}$ is summable. Furthermore, $v = \sum_{k\in K} \lambda_k u_k$ is the only vector such that $c_k = \lambda_k$ for all $k \in K$, where the c_k are the Fourier coefficients of v.

(2) For any two families $(\lambda_k)_{k\in K} \in \ell^2(K)$ and $(\mu_k)_{k\in K} \in \ell^2(K)$, if $v = \sum_{k\in K} \lambda_k u_k$ and $w = \sum_{k\in K} \mu_k u_k$, we have the following equation, also called Parseval identity:

$$\langle v, w\rangle = \sum_{k\in K} \lambda_k \overline{\mu_k}.$$

Proof. (1) The fact that $(\lambda_k)_{k\in K} \in \ell^2(K)$ means that $(|\lambda_k|^2)_{k\in K}$ is summable. The proof given in Proposition A.2 (3) applies to the family $(|\lambda_k|^2)_{k\in K}$ (instead of $(|c_k|^2)_{k\in K}$), and yields the fact that $(\lambda_k u_k)_{k\in K}$ is summable. Letting $v = \sum_{k\in K} \lambda_k u_k$, recall that $c_k = \langle v, u_k\rangle / \|u_k\|^2$. Pick some $k \in K$. Since $\langle -, -\rangle$ is continuous, for every $\epsilon > 0$, there is some $\eta > 0$ such that

$$|\langle v, u_k\rangle - \langle w, u_k\rangle| < \epsilon \|u_k\|^2$$

whenever

$$\|v - w\| < \eta.$$

However, since for every $\eta > 0$, there is some finite subset I of K such that

$$\left\| v - \sum_{j\in J} \lambda_j u_j \right\| < \eta$$

for every finite subset J of K such that $I \subseteq J$, we can pick $J = I \cup \{k\}$ and letting $w = \sum_{j\in J} \lambda_j u_j$ we get

$$\left| \langle v, u_k\rangle - \left\langle \sum_{j\in J} \lambda_j u_j, u_k \right\rangle \right| < \epsilon \|u_k\|^2.$$

However,

$$\langle v, u_k\rangle = c_k \|u_k\|^2 \quad \text{and} \quad \left\langle \sum_{j\in J} \lambda_j u_j, u_k \right\rangle = \lambda_k \|u_k\|^2,$$

and thus, the above proves that $|c_k - \lambda_k| < \epsilon$ for every $\epsilon > 0$, and thus, that $c_k = \lambda_k$.