Remark: Proposition 37.42 also holds for metric spaces.

As an illustration of Proposition 37.42 let (x_n) be the sequence $(1, -1, 1, -1, \dots)$. This sequence has two accumulation points, namely 1 and -1 since $(x_{2n+1}) = (1)$ and $(x_{2n}) = (-1)$.

In second-countable Hausdorff spaces, compactness can be characterized in terms of accumulation points (this is also true for metric spaces).

Proposition 37.43. *A second-countable topological Hausdorff space, E, is compact iff every sequence, (x_n), of E has some accumulation point in E.*

Proof. Assume that every sequence, (x_n), has some accumulation point. Let $(U_i)_{i \in I}$ be some open cover of E. By Proposition 37.41, there is a countable open subcover, $(O_n)_{n \geq 0}$, for E. Now, if E is not covered by any finite subcover of $(O_n)_{n \geq 0}$, we can define a sequence, (x_m), by induction as follows:

Let x_0 be arbitrary and for every $m \geq 1$, let x_m be some point in E not in $O_1 \cup \cdots \cup O_m$, which exists, since $O_1 \cup \cdots \cup O_m$ is not an open cover of E. We claim that the sequence, (x_m), does not have any accumulation point. Indeed, for every $l \in E$, since $(O_n)_{n \geq 0}$ is an open cover of E, there is some O_m such that $l \in O_m$, and by construction, every x_n with $n \geq m + 1$ does not belong to O_m, which means that $x_n \in O_m$ for only finitely many n and l is not an accumulation point. See Figure 37.39.

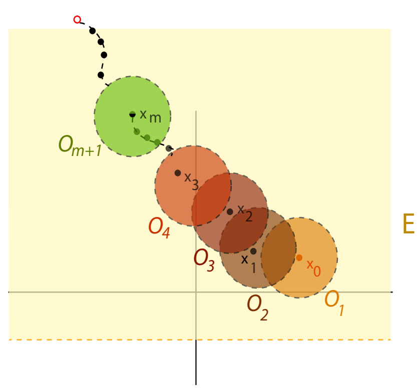

Figure 37.39: The space E is the open half plane above the line $y = -1$. S ince E is not compact, we inductively build a sequence, (x_n) that will have no accumulation point in E. Note the y coordinate of x_n approaches infinity.

Conversely, assume that E is compact, and let (x_n) be any sequence . If $l \in E$ is not an accumulation point of the sequence, then there is some open set, U_l, such that $l \in U_l$