Theorem 19.1 shows that every solution u of our boundary problem (BP) is a solution (in fact, unique) of the equation (WF).

The equation (WF) is called the *weak form* or *variational equation* associated with the boundary problem. This idea to derive these equations is due to *Ritz and Galerkin*.

Now, the natural question is whether the variational equation (WF) has a solution, and whether this solution, if it exists, is also a solution of the boundary problem (it must belong to $C^2([0,1])$, which is far from obvious). Then, (BP) and (WF) would be equivalent.

Some fancy tools of analysis can be used to prove these assertions. The first difficulty is that the vector space V is not the right space of solutions, because in order for the variational problem to have a solution, it must be complete. So, we must construct a completion of the vector space V. This can be done and we get the *Sobolev space* $H_0^1(0,1)$. Then, the question of the regularity of the "weak solution" can also be tackled.

We will not worry about all this. Instead, let us find *approximations* of the problem (WF). Instead of using the infinite-dimensional vector space V, we consider *finite-dimensional* subspaces V_a (with $\dim(V_a) = n$) of V, and we consider the *discrete problem*:

Find a function $u^{(a)} \in V_a$, such that

$$a(u^{(a)}, v) = \widetilde{f}(v), \quad \text{for all } v \in V_a. \tag{DWF}$$

Since V_a is finite dimensional (of dimension n), let us pick a basis of functions $(w_1, \ldots, w_n)$ in V_a, so that every function $u \in V_a$ can we written as

$$u = u_1 w_1 + \cdots + u_n w_n.$$

Then, the equation (DWF) holds iff

$$a(u, w_j) = \widetilde{f}(w_j), \quad j = 1, \ldots, n,$$

and by plugging $u_1 w_1 + \cdots + u_n w_n$ for u, we get a system of k linear equations

$$\sum_{i=1}^{n} a(w_i, w_j) u_i = \widetilde{f}(w_j), \quad 1 \leq j \leq n.$$

Because $a(v, v) \geq \frac{1}{2} \|v\|_{V_a}$, the bilinear form a is symmetric positive definite, and thus the matrix $(a(w_i, w_j))$ is symmetric positive definite, and thus invertible. Therefore, (DWF) has a solution given by a *linear system*!

From a practical point of view, we have to compute the integrals

$$a_{ij} = a(w_i, w_j) = \int_0^1 (w_i' w_j' + c w_i w_j) dx,$$

and

$$b_j = \widetilde{f}(w_j) = \int_0^1 f(x) w_j(x) dx.$$