Also run the `Matlab` `rref` function and compare results.

Your program probably disagrees with `rref` even for small values of n. The problem is that some pivots are very small and the normalization step (to make the pivot 1) causes roundoff errors. Use a tolerance parameter to fix this problem.

What can you conjecture about the rank of A?

(4) Prove that the matrix A has the following row reduced form:

$$R = \begin{pmatrix} 1 & 0 & -1 & -2 & \cdots & -(n-2) \\ 0 & 1 & 2 & 3 & \cdots & n-1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Deduce from the above that A has rank 2.

Hint. Some well chosen sequence of row operations.

(5) Use your program to show that if you add any number greater than or equal to $(2/25)n^2$ to every diagonal entry of A you get an invertible matrix! In fact, running the `Matlab` function `chol` should tell you that these matrices are SPD (symmetric, positive definite).

Problem 8.15. Let A be an $n \times n$ complex Hermitian positive definite matrix. Prove that the lower-triangular matrix B with positive diagonal entries such that $A = BB^*$ is given by the following formulae: For $j = 1, \ldots, n$,

$$b_{j\,j} = \left(a_{j\,j} - \sum_{k=1}^{j-1} |b_{j\,k}|^2 \right)^{1/2},$$

and for $i = j+1, \ldots, n$ (and $j = 1, \ldots, n-1$)

$$\overline{b}_{i\,j} = \left(a_{i\,j} - \sum_{k=1}^{j-1} b_{i\,k} \overline{b}_{j\,k} \right) / b_{j\,j}.$$

Problem 8.16. (Permutations and permutation matrices) A permutation can be viewed as an operation permuting the rows of a matrix. For example, the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$$

corresponds to the matrix

$$P_\pi = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$