are clearly linear. Similarly, the maps $\mathrm{in}_i\colon E_i \to E_1 \times \cdots \times E_p$ given by

$$\mathrm{in}_i(u_i) = (0, \ldots, 0, u_i, 0, \ldots, 0)$$

are injective and linear. If $\dim(E_i) = n_i$ and if $(e^i_1, \ldots, e^i_{n_i})$ is a basis of $E_i$ for $i = 1, \ldots, p$, then it is easy to see that the $n_1 + \cdots + n_p$ vectors

$$\begin{array}{ccc}
(e^1_1, 0, \ldots, 0), & \ldots, & (e^1_{n_1}, 0, \ldots, 0), \\
\vdots & \vdots & \vdots \\
(0, \ldots, 0, e^i_1, 0, \ldots, 0), & \ldots, & (0, \ldots, 0, e^i_{n_i}, 0, \ldots, 0), \\
\vdots & \vdots & \vdots \\
(0, \ldots, 0, e^p_1), & \ldots, & (0, \ldots, 0, e^p_{n_p})
\end{array}$$

form a basis of $E_1 \times \cdots \times E_p$, and so

$$\dim(E_1 \times \cdots \times E_p) = \dim(E_1) + \cdots + \dim(E_p).$$

Let us now consider a vector space $E$ and $p$ subspaces $U_1, \ldots, U_p$ of $E$. We have a map

$$a\colon U_1 \times \cdots \times U_p \to E$$

given by

$$a(u_1, \ldots, u_p) = u_1 + \cdots + u_p,$$

with $u_i \in U_i$ for $i = 1, \ldots, p$. It is clear that this map is linear, and so its image is a subspace of $E$ denoted by

$$U_1 + \cdots + U_p$$

and called the *sum* of the subspaces $U_1, \ldots, U_p$. By definition,

$$U_1 + \cdots + U_p = \{u_1 + \cdots + u_p \mid u_i \in U_i,\ 1 \leq i \leq p\},$$

and it is immediately verified that $U_1 + \cdots + U_p$ is the smallest subspace of $E$ containing $U_1, \ldots, U_p$. This also implies that $U_1 + \cdots + U_p$ does not depend on the order of the factors $U_i$; in particular,

$$U_1 + U_2 = U_2 + U_1.$$

**Definition 6.3.** For any vector space $E$ and any $p \geq 2$ subspaces $U_1, \ldots, U_p$ of $E$, if the map $a$ defined above is injective, then the sum $U_1 + \cdots + U_p$ is called a *direct sum* and it is denoted by

$$U_1 \oplus \cdots \oplus U_p.$$

The space $E$ is the *direct sum* of the subspaces $U_i$ if

$$E = U_1 \oplus \cdots \oplus U_p.$$