It is immediately verified that if H is a hyperplane in E defined by a nonzero linear form φ so that $H = \operatorname{Ker}\varphi$, then for any nonzero $\alpha \in K$, the linear form $\alpha\varphi$ is a nonzero linear form that also defines H, that is, $H = \operatorname{Ker}\alpha\varphi$. This fact with the second part of Proposition 6.22 shows that a hyperplane H in E is defined by the one-dimensional subspace of the dual E^* of E consisting of all the linear forms that vanish on H (including the zero linear form). This is an instance of duality.

6.4 Summary

The main concepts and results of this chapter are listed below:

- *Direct products, sums, direct sums.*
- *Projections.*
- The fundamental equation

  $$\dim(E) = \dim(\operatorname{Ker} f) + \dim(\operatorname{Im} f) = \dim(\operatorname{Ker} f) + \operatorname{rk}(f)$$

  (Proposition 6.16).
- *Grassmann's relation*

  $$\dim(U) + \dim(V) = \dim(U + V) + \dim(U \cap V).$$

- Characterizations of a bijective linear map $f\colon E \to F$.
- *Rank* of a matrix.

6.5 Problems

Problem 6.1. Let V and W be two subspaces of a vector space E. Prove that if $V \cup W$ is a subspace of E, then either $V \subseteq W$ or $W \subseteq V$.

Problem 6.2. Prove that for every vector space E, if $f\colon E \to E$ is an idempotent linear map, i.e., $f \circ f = f$, then we have a direct sum

$$E = \operatorname{Ker} f \oplus \operatorname{Im} f,$$

so that f is the projection onto its image $\operatorname{Im} f$.

Problem 6.3. Let $U_1, \ldots, U_p$ be any $p \geq 2$ subspaces of some vector space E and recall that the linear map

$$a\colon U_1 \times \cdots \times U_p \to E$$