any $a \in \mathfrak{A}$, the ideal $\mathfrak{B} + (a)$ (where $\mathfrak{B} + (a) = \{b + \lambda a \mid b \in \mathfrak{B},\ \lambda \in A\}$) is also finitely generated (since $\mathfrak{B}$ is finitely generated), and by maximality, we have

$$\mathfrak{B} = \mathfrak{B} + (a).$$

So, we get $a \in \mathfrak{B}$ for all $a \in \mathfrak{A}$, and thus, $\mathfrak{A} = \mathfrak{B}$, and $\mathfrak{A}$ is finitely generated. $\square$

Definition 32.5. A commutative ring A (with unit 1) is called *noetherian* if it satisfies the a.c.c. condition. A *noetherian domain* is a noetherian ring that is also a domain.

By Proposition 32.17 and Proposition 32.18, a noetherian ring can also be defined as a ring that either satisfies the maximal property or such that every ideal is finitely generated. The proof of Hilbert's basis theorem will make use the following lemma:

Lemma 32.19. *Let A be a (commutative) ring. For every ideal $\mathfrak{A}$ in $A[X]$, for every $i \geq 0$, let $L_i(\mathfrak{A})$ denote the set of elements of A consisting of 0 and of the coefficients of X^i in all the polynomials $f(X) \in \mathfrak{A}$ which are of degree i. Then, the $L_i(\mathfrak{A})$'s form an ascending chain of ideals in A. Furthermore, if $\mathfrak{B}$ is any ideal of $A[X]$ so that $\mathfrak{A} \subseteq \mathfrak{B}$ and if $L_i(\mathfrak{A}) = L_i(\mathfrak{B})$ for all $i \geq 0$, then $\mathfrak{A} = \mathfrak{B}$.*

Proof. That $L_i(\mathfrak{A})$ is an ideal and that $L_i(\mathfrak{A}) \subseteq L_{i+1}(\mathfrak{A})$ follows from the fact that if $f(X) \in \mathfrak{A}$ and $g(X) \in \mathfrak{A}$, then $f(X) + g(X)$, $\lambda f(X)$, and $Xf(X)$ all belong to $\mathfrak{A}$. Now, let $g(X)$ be any polynomial in $\mathfrak{B}$, and assume that $g(X)$ has degree n. Since $L_n(\mathfrak{A}) = L_n(\mathfrak{B})$, there is some polynomial $f_n(X)$ in $\mathfrak{A}$, of degree n, so that $g(X) - f_n(X)$ is of degree at most $n-1$. Now, since $\mathfrak{A} \subseteq \mathfrak{B}$, the polynomial $g(X) - f_n(X)$ belongs to $\mathfrak{B}$. Using this process, we can define by induction a sequence of polynomials $f_{n+i}(X) \in \mathfrak{A}$, so that each $f_{n+i}(X)$ is either zero or has degree $n - i$, and

$$g(X) - (f_n(X) + f_{n+1}(X) + \cdots + f_{n+i}(X))$$

is of degree at most $n - i - 1$. Note that this last polynomial must be zero when $i = n$, and thus, $g(X) \in \mathfrak{A}$. $\square$

We now prove Hilbert's basis theorem. The proof is substantially Hilbert's original proof. A slightly shorter proof can be given but it is not as transparent as Hilbert's proof (see the remark just after the proof of Theorem 32.20, and Zariski and Samuel [194], Chapter IV, Section 1, Theorem 1).

Theorem 32.20. *(Hilbert's basis theorem) If A is a noetherian ring, then $A[X]$ is also a noetherian ring.*

Proof. Let $\mathfrak{A}$ be any ideal in $A[X]$, and denote by $\mathcal{L}$ the set of elements of A consisting of 0 and of all the coefficients of the highest degree terms of all the polynomials in $\mathfrak{A}$. Observe that

$$\mathcal{L} = \bigcup_i L_i(\mathfrak{A}).$$