all $\lambda, \mu \in K$, we have

$$\begin{aligned}
\varphi(u_1 + u_2, v) &= \varphi(u_1, v) + \varphi(u_2, v)\\
\varphi(u, v_1 + v_2) &= \varphi(u, v_1) + \varphi(u, v_2)\\
\varphi(\lambda u, v) &= \lambda\varphi(u, v)\\
\varphi(u, \mu v) &= \mu\varphi(u, v).
\end{aligned}$$

A bilinear form as in Definition 29.1 is sometimes called a *pairing*. The first two conditions imply that $\varphi(0, v) = \varphi(u, 0) = 0$ for all $u \in E$ and all $v \in F$.

If $E = F$, observe that

$$\begin{aligned}
\varphi(\lambda u + \mu v, \lambda u + \mu v) &= \lambda\varphi(u, \lambda u + \mu v) + \mu\varphi(v, \lambda u + \mu v)\\
&= \lambda^2\varphi(u, u) + \lambda\mu\varphi(u, v) + \lambda\mu\varphi(v, u) + \mu^2\varphi(v, v).
\end{aligned}$$

If we let $\lambda = \mu = 1$, we get

$$\varphi(u + v, u + v) = \varphi(u, u) + \varphi(u, v) + \varphi(v, u) + \varphi(v, v).$$

If φ is *symmetric*, which means that

$$\varphi(u, v) = \varphi(v, u) \quad \text{for all } u, v \in E,$$

then

$$2\varphi(u, v) = \varphi(u + v, u + v) - \varphi(u, u) - \varphi(v, v). \tag{$*$}$$

The function Φ defined such that

$$\Phi(u) = \varphi(u, u) \quad u \in E,$$

is called the *quadratic form* associated with φ. If the field K is not of characteristic 2, then φ is completely determined by its quadratic form Φ. The symmetric bilinear form φ is called the *polar form* of Φ. This suggests the following definition.

Definition 29.2. A function $\Phi\colon E \to K$ is a *quadratic form* on E if the following conditions hold:

(1) We have $\Phi(\lambda u) = \lambda^2\Phi(u)$, for all $u \in E$ and all $\lambda \in E$.

(2) The map φ' given by $\varphi'(u, v) = \Phi(u + v) - \Phi(u) - \Phi(v)$ is bilinear. Obviously, the map φ' is symmetric.

Since $\Phi(x + x) = \Phi(2x) = 4\Phi(x)$, we have

$$\varphi'(u, u) = 2\Phi(u) \quad u \in E.$$