Lemma 37.44. *Given a metric space, E, if every sequence, (x_n), has an accumulation point, for every open cover, $(U_i)_{i \in I}$, of E, there is some $\delta > 0$ (a Lebesgue number for $(U_i)_{i \in I}$) such that, for every open ball, $B_0(a, \epsilon)$, of radius $\epsilon \leq \delta$, there is some open subset, U_i, such that $B_0(a, \epsilon) \subseteq U_i$. See Figure 37.41*

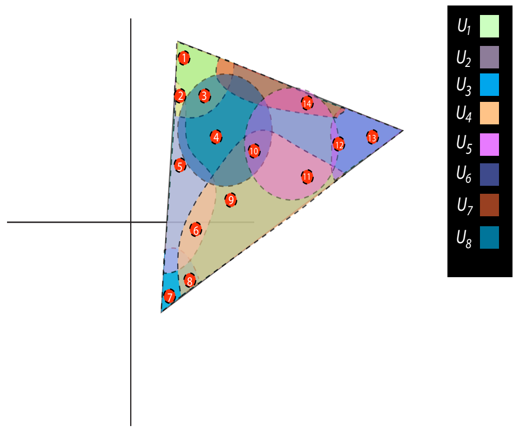

Figure 37.41: The space E the closed triangular region of $\mathbb{R}^2$. It's open cover is $(U_i)_{i=1}^{8}$. The Lebesgue number is the radius of the small orange balls labelled 1 through 14. Each open ball of this radius entirely contained within at least one U_i. For example, Ball 2 is contained in both U_1 and U_2.

Proof. If there was no δ with the above property, then, for every natural number, n, there would be some open ball, $B_0(a_n, 1/n)$, which is not contained in any open set, U_i, of the open cover, $(U_i)_{i \in I}$. However, the sequence, (a_n), has some accumulation point, a, and since $(U_i)_{i \in I}$ is an open cover of E, there is some U_i such that $a \in U_i$. Since U_i is open, there is some open ball of center a and radius ϵ contained in U_i. Now, since a is an accumulation point of the sequence, (a_n), every open set containing a contains a_n for infinitely many n and thus, there is some n large enough so that

$$1/n \leq \epsilon/2 \quad \text{and} \quad a_n \in B_0(a, \epsilon/2),$$

which implies that

$$B_0(a_n, 1/n) \subseteq B_0(a, \epsilon) \subseteq U_i,$$

a contradiction. $\square$

By a previous remark, since the proof of Proposition 37.43 implies that in a compact topological space, every sequence has some accumulation point, by Lemma 37.44, in a compact metric space, every open cover has a Lebesgue number. This fact can be used to prove another important property of compact metric spaces, the uniform continuity theorem.