Because the trace function is invariant under permutation of its arguments ($\mathrm{tr}(XY) = \mathrm{tr}(YX)$), we see that the m-th derivatives in Example 39.12 are indeed symmetric multilinear maps.

If E is of finite dimension n, and $(a_0, (e_1, \ldots, e_n))$ is a frame for E, $\mathrm{D}^m f(a)$ is a symmetric m-multilinear map, and we have

$$\mathrm{D}^m f(a)(u_1, \ldots, u_m) = \sum_j u_{1,j_1} \cdots u_{m,j_m} \frac{\partial^m f}{\partial x_{j_1} \ldots \partial x_{j_m}}(a),$$

where j ranges over all functions $j\colon \{1, \ldots, m\} \to \{1, \ldots, n\}$, for any m vectors

$$u_j = u_{j,1}e_1 + \cdots + u_{j,n}e_n.$$

The concept of C^1-function is generalized to the concept of C^m-function, and Theorem 39.13 can also be generalized.

Definition 39.18. Given two normed affine spaces E and F, and an open subset A of E, for any $m \geq 1$, we say that a function $f\colon A \to F$ is *of class C^m on A or a C^m-function on A* if $\mathrm{D}^k f$ exists and is continuous on A for every k, $1 \leq k \leq m$. We say that $f\colon A \to F$ is *of class C^∞ on A or a C^∞-function on A* if $\mathrm{D}^k f$ exists and is continuous on A for every $k \geq 1$. A C^∞-function (on A) is also called a *smooth function* (on A). A *C^m-diffeomorphism $f\colon A \to B$ between A and B* (where A is an open subset of E and B is an open subset of B) is a bijection between A and $B = f(A)$, such that both $f\colon A \to B$ and its inverse $f^{-1}\colon B \to A$ are C^m-functions.

Equivalently, f is a C^m-function on A if f is a C^1-function on A and $\mathrm{D}f$ is a C^{m-1}-function on A.

We have the following theorem giving a necessary and sufficient condition for f to a C^m-function on A. A generalization to the case where $E = (E_1, a_1) \oplus \cdots \oplus (E_n, a_n)$ also holds.

Theorem 39.22. *Given two normed affine spaces E and F, where E is of finite dimension n, and where $(a_0, (u_1, \ldots, u_n))$ is a frame of E, given any open subset A of E, given any function $f\colon A \to F$, for any $m \geq 1$, the derivative $\mathrm{D}^m f$ is a C^m-function on A iff every partial derivative $\mathrm{D}_{u_{j_k}} \ldots \mathrm{D}_{u_{j_1}} f$ (or $\frac{\partial^k f}{\partial x_{j_1} \ldots \partial x_{j_k}}(a)$) is defined and continuous on A, for all k, $1 \leq k \leq m$, and all $j_1, \ldots, j_k \in \{1, \ldots, n\}$. As a corollary, if F is of finite dimension p, and $(b_0, (v_1, \ldots, v_p))$ is a frame of F, the derivative $\mathrm{D}^m f$ is defined and continuous on A iff every partial derivative $\mathrm{D}_{u_{j_k}} \ldots \mathrm{D}_{u_{j_1}} f_i$ (or $\frac{\partial^k f_i}{\partial x_{j_1} \ldots \partial x_{j_k}}(a)$) is defined and continuous on A, for all k, $1 \leq k \leq m$, for all i, $1 \leq i \leq p$, and all $j_1, \ldots, j_k \in \{1, \ldots, n\}$.*

Definition 39.19. When $E = \mathbb{R}$ (or $E = \mathbb{C}$), for any $a \in E$, $\mathrm{D}^m f(a)(1, \ldots, 1)$ is a vector in $\overrightarrow{F}$, called the *mth-order vector derivative*. As in the case $m = 1$, we will usually identify the multilinear map $\mathrm{D}^m f(a)$ with the vector $\mathrm{D}^m f(a)(1, \ldots, 1)$.