Proof. Since M is finitely generated and nontrivial, there is a surjective homomorphism $\varphi\colon A^n \to M$ for some $n \geq 1$, and M is isomorphic to $A^n/\mathrm{Ker}\,(\varphi)$. Since $\mathrm{Ker}\,(\varphi)$ is a submodule of the free module A^n, by Proposition 35.23, $\mathrm{Ker}\,(\varphi)$ is a free module and there is a basis $(e_1, \ldots, e_n)$ of A^n and some nonzero elements $a_1, \ldots, a_q$ $(q \leq n)$ such that $(a_1e_1, \ldots, a_qe_q)$ is a basis of $\mathrm{Ker}\,(\varphi)$ and $a_1 \mid a_2 \mid \cdots \mid a_q$. Let $a_{q+1} = \ldots = a_n = 0$.

By Proposition 35.24, we have an isomorphism

$$A^n/\mathrm{Ker}\,(\varphi) \approx A/a_1A \oplus \cdots \oplus A/a_nA.$$

Whenever a_i is unit, the factor $A/a_iA = (0)$, so we can weed out the units. Let $r = n - q$, and let $s \in \mathbb{N}$ be the smallest index such that a_{s+1} is not a unit. Note that $s = 0$ means that there are no units. Also, as $M \neq (0)$, $s < n$. Then,

$$M \approx A^n/\mathrm{Ker}\,(\varphi) \approx A/a_{s+1}A \oplus \cdots \oplus A/a_nA.$$

Let $m = r + q - s = n - s$. Then, we have the sequence

$$\underbrace{a_{s+1}, \ldots, a_q}_{q-s}, \underbrace{a_{q+1}, \ldots, a_n}_{r=n-q},$$

where $a_{s+1} \mid a_{s+2} \mid \cdots \mid a_q$ are nonzero and nonunits and $a_{q+1} = \cdots = a_n = 0$, so we define the m ideals $\mathfrak{a}_i$ as follows:

$$\mathfrak{a}_i = \begin{cases} (0) & \text{if } 1 \leq i \leq r \\ a_{r+q+1-i}A & \text{if } r+1 \leq i \leq m. \end{cases}$$

With these definitions, the ideals $\mathfrak{a}_i$ are proper ideals and we have

$$\mathfrak{a}_i \subseteq \mathfrak{a}_{i+1}, \quad i = 1, \ldots, m-1.$$

When $r = 0$, since $a_{s+1} \mid a_{s+2} \mid \cdots \mid a_n$, it is clear that $\mathfrak{a}_1 = a_nA$ is the annihilator of M. The other statements of the theorem are clear. $\square$

Example 35.1. Here is an example of Theorem 35.25. Let M be a $\mathbb{Z}$-module with generators $\{e_1, e_2, e_3, e_4\}$ subject to the relations $6e_3 = 0$, $2e_4 = 0$. Then

$$M \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z},$$

where

$$\mathfrak{a}_1 = (0), \qquad \mathfrak{a}_2 = (0), \qquad \mathfrak{a}_3 = (6), \qquad \mathfrak{a}_4 = (2).$$