We confess that the use of the same letter D for the member of $\mathcal{D}_n$ being defined, and for members of $\mathcal{D}_{n-1}$, may be slightly confusing. We considered using subscripts to distinguish, but this seems to complicate things unnecessarily. One should not worry too much anyway, since it will turn out that each $\mathcal{D}_n$ contains just one map.

Each $(-1)^{i+j} D(A_{i\,j})$ is called the *cofactor* of $a_{i\,j}$, and the inductive expression for $D(A)$ is called a *Laplace expansion of D according to the i-th Row*. Given a matrix $A \in \mathrm{M}_n(K)$, each $D(A)$ is called a *determinant* of A.

We can think of each member of $\mathcal{D}_n$ as an *algorithm* to evaluate "the" determinant of A. The main point is that these algorithms, which recursively evaluate a determinant using all possible Laplace row expansions, all yield the *same result*, $\det(A)$.

We will prove shortly that $D(A)$ is uniquely defined (at the moment, it is not clear that $\mathcal{D}_n$ consists of a single map). Assuming this fact, given a $n \times n$-matrix $A = (a_{i\,j})$,

$$A = \begin{pmatrix} a_{1\,1} & a_{1\,2} & \dots & a_{1\,n} \\ a_{2\,1} & a_{2\,2} & \dots & a_{2\,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n\,1} & a_{n\,2} & \dots & a_{n\,n} \end{pmatrix},$$

its determinant is denoted by $D(A)$ or $\det(A)$, or more explicitly by

$$\det(A) = \begin{vmatrix} a_{1\,1} & a_{1\,2} & \dots & a_{1\,n} \\ a_{2\,1} & a_{2\,2} & \dots & a_{2\,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n\,1} & a_{n\,2} & \dots & a_{n\,n} \end{vmatrix}.$$

Let us first consider some examples.

Example 7.1.

1. When $n = 2$, if
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$
then by expanding according to any row, we have
$$D(A) = ad - bc.$$

2. When $n = 3$, if
$$A = \begin{pmatrix} a_{1\,1} & a_{1\,2} & a_{1\,3} \\ a_{2\,1} & a_{2\,2} & a_{2\,3} \\ a_{3\,1} & a_{3\,2} & a_{3\,3} \end{pmatrix},$$