where $\langle u, v\rangle$ is the inner product in $L^2([0,L])$. The fact that it is legitimate to move $\partial^2/\partial t^2$ outside of the integral needs to be justified rigorously, but we won't do it here.

For the second term, we get

$$-\int_0^L \frac{\partial^2 u}{\partial x^2}(x,t)v(x)dx = -\left[\frac{\partial u}{\partial x}(x,t)v(x)\right]_{x=0}^{x=L} + \int_0^L \frac{\partial u}{\partial x}(x,t)\frac{dv}{dx}(x)dx,$$

and because $v \in V$, we have $v(0) = v(L) = 0$, so we obtain

$$-\int_0^L \frac{\partial^2 u}{\partial x^2}(x,t)v(x)dx = \int_0^L \frac{\partial u}{\partial x}(x,t)\frac{dv}{dx}(x)dx.$$

Our integrated equation becomes

$$\frac{d^2}{dt^2}\langle u, v\rangle + c^2 \int_0^L \frac{\partial u}{\partial x}(x,t)\frac{dv}{dx}(x)dx = 0, \quad \text{for all } v \in V \quad \text{and all } t \geq 0.$$

It is natural to introduce the bilinear form $a\colon V \times V \to \mathbb{R}$ given by

$$a(u,v) = \int_0^L \frac{\partial u}{\partial x}(x,t)\frac{\partial v}{\partial x}(x,t)dx,$$

where, for every $t \in \mathbb{R}_+$, the functions $u(x,t)$ and (v,t) belong to V. Actually, we have to replace V by the subspace of the Sobolev space $H_0^1(0,L)$ consisting of the functions such that $v(0) = v(L) = 0$. Then, the weak formulation (variational formulation) of our problem is this:

Find a function $u \in V$ such that

$$\begin{aligned}
\frac{d^2}{dt^2}\langle u, v\rangle + a(u,v) = 0, &\qquad \text{for all } v \in V \quad \text{and all } t \geq 0 \\
u(x,0) = u_{i,0}(x), &\quad 0 \leq x \leq L \quad \text{(intitial condition)}, \\
\frac{\partial u}{\partial t}(x,0) = u_{i,1}(x), &\quad 0 \leq x \leq L \quad \text{(intitial condition)}.
\end{aligned}$$

It can be shown that there is a positive constant $\alpha > 0$ such that

$$a(u,u) \geq \alpha \left\|u\right\|_{H_0^1}^2 \quad \text{for all } v \in V$$

(Poincaré's inequality), which shows that a is positive definite on V. The above method is known as the method of *Rayleigh-Ritz*.

A study of the above equation requires some sophisticated tools of analysis which go far beyond the scope of these notes. Let us just say that there is a countable sequence of solutions with separated variables of the form

$$u_k^{(1)} = \sin\left(\frac{k\pi x}{L}\right)\cos\left(\frac{k\pi ct}{L}\right), \quad u_k^{(2)} = \sin\left(\frac{k\pi x}{L}\right)\sin\left(\frac{k\pi ct}{L}\right), \quad k \in \mathbb{N}_+,$$