($h_{i+1i} \neq 0$ for $i = 1, \ldots, n-1$), then the columns of index $2, \ldots, n$ of U are determined by the first column of U up to sign; see Demmel [48] (Theorem 4.9) and Golub and Van Loan [80] (Theorem 7.4.2) for the proof in the case of real matrices. Actually, the proof is not difficult and will be the object of a homework exercise. In the case of a single shift, an implicit shift generates $A_{k+1} = Q_k^* A_k Q_k$ without having to compute a QR-factorization of $A_k - \sigma_k I$. For real matrices, this is done by applying a sequence of Givens rotations which perform a bulge chasing process (a Givens rotation is an orthogonal block diagonal matrix consisting of a single block which is a 2D rotation, the other diagonal entries being equal to 1). Similarly, in the case of a double shift, $A_{k+2} = (Q_k Q_{k+1})^* A_k Q_k Q_{k+1}$ is generated without having to compute the QR-factorizations of $A_k - \sigma_k I$ and $A_{k+1} - \overline{\sigma}_k I$. Again, $(Q_k Q_{k+1})^* A_k Q_k Q_{k+1}$ is generated by applying some simple orthogonal matrices which perform a bulge chasing process. See Demmel [48] (Section 4.4.8) and Golub and Van Loan [80] (Section 7.5) for further explanations regarding implicit shifting involving bulge chasing in the case of real matrices. Watkins [187, 188] discusses bulge chasing in the more general case of complex matrices.

The `Matlab` function for finding the eigenvalues and the eigenvectors of a matrix A is `eig` and is called as `[U, D] = eig(A)`. It is implemented using an optimized version of the QR-algorithm with implicit shifts.

If the dimension of the matrix A is very large, we can find approximations of some of the eigenvalues of A by using a truncated version of the reduction to Hessenberg form due to Arnoldi in general and to Lanczos in the symmetric (or Hermitian) tridiagonal case.

18.4 Krylov Subspaces; Arnoldi Iteration

In this section, we denote the dimension of the square real or complex matrix A by m rather than n, to make it easier for the reader to follow Trefethen and Bau exposition [176], which is particularly lucid.

Suppose that the $m \times m$ matrix A has been reduced to the upper Hessenberg form H, as $A = UHU^*$. For any $n \leq m$ (typically much smaller than m), consider the $(n+1) \times n$ upper left block

$$\widetilde{H}_n = \begin{pmatrix} h_{11} & h_{12} & h_{13} & \cdots & h_{1n} \\ h_{21} & h_{22} & h_{23} & \cdots & h_{2n} \\ 0 & h_{32} & h_{33} & \cdots & h_{3n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & h_{nn-1} & h_{nn} \\ 0 & \cdots & 0 & 0 & h_{n+1n} \end{pmatrix}$$