1. For any $\beta \neq 0$, if β is an eigenvalue of $\mathcal{L}_1$, then $\beta^{1/2}$ and $-\beta^{1/2}$ are both eigenvalues of J, where $\beta^{1/2}$ is one of the complex square roots of β.

2. For any $\alpha \neq 0$, α is an eigenvalues of J iff $-\alpha$ is an eigenvalues of J, and if α is an eigenvalues of J, then α^2 is an eigenvalue of $\mathcal{L}_1$.

The above immediately implies that $\rho(\mathcal{L}_1) = (\rho(J))^2$. $\square$

We now consider the more general situation where ω is any real in $(0, 2)$.

Proposition 10.9. *Let A be a tridiagonal matrix (possibly by blocks), and assume that the eigenvalues of the Jacobi matrix are all real. If $\omega \in (0, 2)$, then the method of Jacobi and the method of relaxation both converge or both diverge simultaneously (even when A is tridiagonal by blocks). When they converge, the function $\omega \mapsto \rho(\mathcal{L}_\omega)$ (for $\omega \in (0, 2)$) has a unique minimum equal to $\omega_0 - 1$ for*

$$\omega_0 = \frac{2}{1 + \sqrt{1 - (\rho(J))^2}},$$

where $1 < \omega_0 < 2$ if $\rho(J) > 0$.

Proof. The proof is very technical and can be found in Serre [156] and Ciarlet [41]. As in the proof of the previous proposition, we begin by showing that the eigenvalues of the matrix $\mathcal{L}_\omega$ are the zeros of the polynomial

$$q_{\mathcal{L}_\omega}(\lambda) = \det\left(\frac{\lambda + \omega - 1}{\omega} D - \lambda E - F\right) = \det\left(\frac{D}{\omega} - E\right) p_{\mathcal{L}_\omega}(\lambda),$$

where $p_{\mathcal{L}_\omega}(\lambda)$ is the characteristic polynomial of $\mathcal{L}_\omega$. Then using the preliminary fact from Proposition 10.8, it is easy to show that

$$q_{\mathcal{L}_\omega}(\lambda^2) = \lambda^n q_J\left(\frac{\lambda^2 + \omega - 1}{\lambda\omega}\right),$$

for all $\lambda \in \mathbb{C}$, with $\lambda \neq 0$. This time we cannot extend the above equation to $\lambda = 0$. This leads us to consider the equation

$$\frac{\lambda^2 + \omega - 1}{\lambda\omega} = \alpha,$$

which is equivalent to

$$\lambda^2 - \alpha\omega\lambda + \omega - 1 = 0,$$

for all $\lambda \neq 0$. Since $\lambda \neq 0$, the above equivalence does not hold for $\omega = 1$, but this is not a problem since the case $\omega = 1$ has already been considered in the previous proposition. Then we can show the following: