If the field K is not of characteristic 2, then $\varphi = \frac{1}{2}\varphi'$ is the unique symmetric bilinear form such that that $\varphi(u,u) = \Phi(u)$ for all $u \in E$. The bilinear form $\varphi = \frac{1}{2}\varphi'$ is called the *polar form* of Φ. In this case, there is a bijection between the set of bilinear forms on E and the set of quadratic forms on E.

If K is a field of characteristic 2, then φ' is *alternating*, which means that

$$\varphi'(u,u) = 0 \quad \text{for all } u \in E.$$

Thus if K is a field of characteristic 2, then Φ cannot be recovered from the symmetric bilinear form φ'.

If $(e_1, \ldots, e_n)$ is a basis of E, it is easy to show that

$$\Phi\Big(\sum_{i=1}^{n} \lambda_i e_i\Big) = \sum_{i=1}^{n} \lambda_i^2 \Phi(e_i) + \sum_{i \neq j} \lambda_i \lambda_j \varphi'(e_i, e_j).$$

This shows that the quadratic form Φ is completely determined by the scalars $\Phi(e_i)$ and $\varphi'(e_i, e_j)$ $(i \neq j)$. Furthermore, given any bilinear form $\psi\colon E \times E \to K$ (not necessarily symmetric) we can define a quadratic form Φ by setting $\Phi(x) = \psi(x,x)$, and we immediately check that the symmetric bilinear form φ' associated with Φ is given by $\varphi'(u,v) = \psi(u,v) + \psi(v,u)$. Using the above facts, it is not hard to prove that given any quadratic form Φ, there is some (nonsymmetric) bilinear form ψ such that $\Phi(u) = \psi(u,u)$ for all $u \in E$ (see Bourbaki [24], Section §3.4, Proposition 2). Thus, quadratic forms are more general than symmetric bilinear forms (except in characteristic $\neq 2$).

Definition 29.3. Given any bilinear form $\varphi\colon E \times E \to K$ where K is a field of any characteristic, we say that φ is *alternating* if

$$\varphi(u,u) = 0 \quad \text{for all } u \in E,$$

and *skew-symmetric* if

$$\varphi(v,u) = -\varphi(u,v) \quad \text{for all } u, v \in E.$$

If K is a field of any characteristic, the identity

$$\varphi(u+v, u+v) = \varphi(u,u) + \varphi(u,v) + \varphi(v,u) + \varphi(v,v)$$

shows that if φ is alternating, then

$$\varphi(v,u) = -\varphi(u,v) \quad \text{for all } u, v \in E,$$

that is, φ is skew-symmetric. Conversely, if the field K is not of characteristic 2, then a skew-symmetric bilinear map is alternating, since $\varphi(u,u) = -\varphi(u,u)$ implies $\varphi(u,u) = 0$.

An important consequence of bilinearity is that a pairing yields a linear map from E into F^* and a linear map from F into E^* (where $E^* = \mathrm{Hom}_K(E,K)$, the *dual* of E, is the set of linear maps from E to K, called *linear forms*).