Definition 3.22. The set of all linear maps between two vector spaces E and F is denoted by $\mathrm{Hom}(E, F)$ or by $\mathcal{L}(E; F)$ (the notation $\mathcal{L}(E; F)$ is usually reserved to the set of continuous linear maps, where E and F are normed vector spaces). When we wish to be more precise and specify the field K over which the vector spaces E and F are defined we write $\mathrm{Hom}_K(E, F)$.

The set $\mathrm{Hom}(E, F)$ is a vector space under the operations defined in Example 3.1, namely

$$(f + g)(x) = f(x) + g(x)$$

for all $x \in E$, and

$$(\lambda f)(x) = \lambda f(x)$$

for all $x \in E$. The point worth checking carefully is that λf is indeed a linear map, which uses the commutativity of $*$ in the field K (typically, $K = \mathbb{R}$ or $K = \mathbb{C}$). Indeed, we have

$$(\lambda f)(\mu x) = \lambda f(\mu x) = \lambda\mu f(x) = \mu\lambda f(x) = \mu(\lambda f)(x).$$

When E and F have finite dimensions, the vector space $\mathrm{Hom}(E, F)$ also has finite dimension, as we shall see shortly.

Definition 3.23. When $E = F$, a linear map $f : E \to E$ is also called an *endomorphism*. The space $\mathrm{Hom}(E, E)$ is also denoted by $\mathrm{End}(E)$.

It is also important to note that composition confers to $\mathrm{Hom}(E, E)$ a ring structure. Indeed, composition is an operation $\circ : \mathrm{Hom}(E, E) \times \mathrm{Hom}(E, E) \to \mathrm{Hom}(E, E)$, which is associative and has an identity id_E, and the distributivity properties hold:

$$\begin{aligned}(g_1 + g_2) \circ f &= g_1 \circ f + g_2 \circ f;\\ g \circ (f_1 + f_2) &= g \circ f_1 + g \circ f_2.\end{aligned}$$

The ring $\mathrm{Hom}(E, E)$ is an example of a noncommutative ring.

It is easily seen that the set of bijective linear maps $f : E \to E$ is a group under composition.

Definition 3.24. Bijective linear maps $f : E \to E$ are also called *automorphisms*. The group of automorphisms of E is called the *general linear group (of E)*, and it is denoted by $\mathbf{GL}(E)$, or by $\mathrm{Aut}(E)$, or when $E = \mathbb{R}^n$, by $\mathbf{GL}(n, \mathbb{R})$, or even by $\mathbf{GL}(n)$.

Although in this book, we will not have many occasions to use quotient spaces, they are fundamental in algebra. The next section may be omitted until needed.