$(b) \Rightarrow (c)$: If $v = \sum_{k\in K} c_k u_k$, then for every $\epsilon > 0$, there some finite subset I of K, such that

$$\Big\| v - \sum_{j\in J} c_j u_j \Big\| < \sqrt{\epsilon},$$

for every finite subset J of K such that $I \subseteq J$, and since we proved in (1) that

$$\|v\|^2 = \Big\| v - \sum_{j\in J} c_j u_j \Big\|^2 + \sum_{j\in J} |c_j|^2,$$

we get

$$\|v\|^2 - \sum_{j\in J} |c_j|^2 < \epsilon,$$

which proves that $(|c_k|^2)_{k\in K}$ is summable with sum $\|v\|^2$.

$(c) \Rightarrow (a)$: Finally, if $(|c_k|^2)_{k\in K}$ is summable with sum $\|v\|^2$, for every $\epsilon > 0$, there is some finite subset I of K such that

$$\|v\|^2 - \sum_{j\in J} |c_j|^2 < \epsilon^2$$

for every finite subset J of K such that $I \subseteq J$, and again, using the fact that

$$\|v\|^2 = \Big\| v - \sum_{j\in J} c_j u_j \Big\|^2 + \sum_{j\in J} |c_j|^2,$$

we get

$$\Big\| v - \sum_{j\in J} c_j u_j \Big\| < \epsilon,$$

which proves that $(c_k u_k)_{k\in K}$ is summable with sum $\sum_{k\in K} c_k u_k = v$, and $v \in V$.

(3) Since $\sum_{i\in I} |c_i|^2 \leq \|v\|^2$ for every finite subset I of K, by Proposition A.1(2), the family $(|c_k|^2)_{k\in K}$ is summable. The Bessel inequality

$$\sum_{k\in K} |c_k|^2 \leq \|v\|^2$$

is an obvious consequence of the inequality $\sum_{i\in I} |c_i|^2 \leq \|v\|^2$ (for every finite $I \subseteq K$). Now for every natural number $n \geq 1$, if K_n is the subset of K consisting of all c_k such that $|c_k| \geq 1/n$, the number of elements in K_n is at most

$$\sum_{k\in K_n} |nc_k|^2 \leq n^2 \sum_{k\in K} |c_k|^2 \leq n^2 \|v\|^2,$$

which is finite, and thus, at most a countable number of the c_k may be nonzero.