and $\mu_j \neq 0$ for some $j \in I$. But then,

$$v = \sum_{i \in I} \lambda_i u_i + 0 = \sum_{i \in I} \lambda_i u_i + \sum_{i \in I} \mu_i u_i = \sum_{i \in I} (\lambda_i + \mu_i) u_i,$$

with $\lambda_j \neq \lambda_j + \mu_j$ since $\mu_j \neq 0$, contradicting the assumption that $(\lambda_i)_{i \in I}$ is the unique family such that $v = \sum_{i \in I} \lambda_i u_i$. $\square$

Definition 3.10. If $(u_i)_{i \in I}$ is a basis of a vector space E, for any vector $v \in E$, if $(x_i)_{i \in I}$ is the unique family of scalars in K such that

$$v = \sum_{i \in I} x_i u_i,$$

each x_i is called the *component (or coordinate) of index i of v with respect to the basis* $(u_i)_{i \in I}$.

Given a field K and any (nonempty) set I, we can form a vector space $K^{(I)}$ which, in some sense, is the standard vector space of dimension $|I|$.

Definition 3.11. Given a field K and any (nonempty) set I, let $K^{(I)}$ be the subset of the cartesian product K^I consisting of all families $(\lambda_i)_{i \in I}$ with finite support of scalars in K.[3] We define addition and multiplication by a scalar as follows:

$$(\lambda_i)_{i \in I} + (\mu_i)_{i \in I} = (\lambda_i + \mu_i)_{i \in I},$$

and

$$\lambda \cdot (\mu_i)_{i \in I} = (\lambda \mu_i)_{i \in I}.$$

It is immediately verified that addition and multiplication by a scalar are well defined. Thus, $K^{(I)}$ is a vector space. Furthermore, because families with finite support are considered, the family $(e_i)_{i \in I}$ of vectors e_i, defined such that $(e_i)_j = 0$ if $j \neq i$ and $(e_i)_i = 1$, is clearly a basis of the vector space $K^{(I)}$. When $I = \{1, \ldots, n\}$, we denote $K^{(I)}$ by K^n. The function $\iota \colon I \to K^{(I)}$, such that $\iota(i) = e_i$ for every $i \in I$, is clearly an injection.

When I is a finite set, $K^{(I)} = K^I$, but this is false when I is infinite. In fact, $\dim(K^{(I)}) = |I|$, but $\dim(K^I)$ is strictly greater when I is infinite.

3.6 Matrices

In Section 2.1 we introduced informally the notion of a matrix. In this section we define matrices precisely, and also introduce some operations on matrices. It turns out that matrices form a vector space equipped with a multiplication operation which is associative, but noncommutative. We will explain in Section 4.1 how matrices can be used to represent linear maps, defined in the next section.

[3]Where K^I denotes the set of all functions from I to K.