(1) Prove that if $x_0 > 0$, then $x_k > 0$ and

$$\begin{aligned} x_{k+1} - \sqrt{\alpha} &= \frac{1}{2x_k}(x_k - \sqrt{\alpha})^2 \\ x_{k+2} - x_{k+1} &= \frac{1}{2x_{k+1}}(\alpha - x_{k+1}^2) \end{aligned}$$

for all $k \geq 0$. Deduce that Newton's method converges to $\sqrt{\alpha}$ for any $x_0 > 0$.

(2) Prove that if $x_0 < 0$, then Newton's method converges to $-\sqrt{\alpha}$.

Problem 41.2. (1) If $\alpha > 0$ and $f(x) = x^2 - \alpha$, show that the conditions of Theorem 41.1 are satisfied by any $\beta \in (0, 1)$ and any x_0 such that

$$|x_0^2 - \alpha| \leq \frac{4\beta(1-\beta)}{(\beta+2)^2} x_0^2,$$

with

$$r = \frac{\beta}{\beta + 2} x_0, \qquad M = \frac{\beta + 2}{4x_0}.$$

(2) Prove that the maximum of the function defined on $[0, 1]$ by

$$\beta \mapsto \frac{4\beta(1-\beta)}{(\beta+2)^2}$$

has a maximum for $\beta = 2/5$. For this value of β, check that $r = (1/6)x_0$, $M = 3/(5x_0)$, and

$$\frac{6}{7}\alpha \leq x_0^2 \leq \frac{6}{5}\alpha.$$

Problem 41.3. Consider generalizing Problem 41.1 to the matrix function f given by $f(X) = X^2 - C$, where X and C are two real $n \times n$ matrices with C symmetric positive definite. The first step is to determine for which A does the inverse df_A^{-1} exist. Let g be the function given by $g(X) = X^2$. From Problem 39.1 we know that the derivative at A of the function g is $dg_A(X) = AX + XA$, and obviously $df_A = dg_A$. Thus we are led to figure out when the linear matrix map $X \mapsto AX + XA$ is invertible. This can be done using the Kronecker product.

Given an $m \times n$ matrix $A = (a_{ij})$ and a $p \times q$ matrix $B = (b_{ij})$, the *Kronecker product* (or *tensor product*) $A \otimes B$ of A and B is the $mp \times nq$ matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}.$$