Thus, we see that the exponential of a $2 \times 2$ skew-symmetric matrix is a rotation matrix. This property generalizes to any dimension. An explicit formula when $n = 3$ (the Rodrigues' formula) is given in Section 12.7.

**Proposition 9.23.** *If $B$ is an $n \times n$ (real) skew symmetric matrix, that is, $B^\top = -B$, then $Q = e^B$ is an orthogonal matrix, that is*

$$Q^\top Q = QQ^\top = I.$$

*Proof.* Since $B^\top = -B$, we have

$$Q^\top = (e^B)^\top = e^{B^\top} = e^{-B}.$$

Since $B$ and $-B$ commute, we have

$$Q^\top Q = e^{-B} e^B = e^{-B+B} = e^0 = I.$$

Similarly,

$$QQ^\top = e^B e^{-B} = e^{B-B} = e^0 = I,$$

which concludes the proof. $\square$

It can also be shown that $\det(Q) = \det(e^B) = 1$, but this requires a better understanding of the eigenvalues of $e^B$ (see Section 15.5). Furthermore, for every $n \times n$ rotation matrix $Q$ (an orthogonal matrix $Q$ such that $\det(Q) = 1$), there is a skew symmetric matrix $B$ such that $Q = e^B$. This is a fundamental property which has applications in robotics for $n = 3$.

All familiar series have matrix analogs. For example, if $\|A\| < 1$ (where $\| \; \|$ is an operator norm), then the series $\sum_{k=0}^\infty A^k$ converges absolutely, and it can be shown that its limit is $(I - A)^{-1}$.

Another interesting series is the logarithm. For any $n \times n$ complex matrix $A$, if $\|A\| < 1$ (where $\| \; \|$ is an operator norm), then the series

$$\log(I + A) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{A^k}{k}$$

converges absolutely.

## 9.9 Summary

The main concepts and results of this chapter are listed below:

- *Norms* and *normed vector spaces*.
- The *triangle inequality*.