When the ideals $\mathfrak{a}_i$ form a chain of inclusions $\mathfrak{a}_1 \subseteq \cdots \subseteq \mathfrak{a}_n$, we get the following remarkable result.

Proposition 35.29. *Let A be a commutative ring and let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ be n ideals of A such that $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots \subseteq \mathfrak{a}_n$. If the module M is the direct sum of n cyclic modules*

$$M = A/\mathfrak{a}_1 \oplus \cdots \oplus A/\mathfrak{a}_n,$$

then for every p with $1 \leq p \leq n$, the ideal $\mathfrak{a}_p$ is the annihilator of the exterior power $\bigwedge^p M$. If $\mathfrak{a}_n \neq A$, then $\bigwedge^p M \neq (0)$ for $p = 1, \ldots, n$, and $\bigwedge^p M = (0)$ for $p > n$.

Proof. With the notation of Proposition 35.28, we have $\mathfrak{a}_H = \mathfrak{a}_{\max(H)}$, where $\max(H)$ is the greatest element in the set H. Since $\max(H) \geq p$ for any subset with p elements and since $\max(H) = p$ when $H = \{1, \ldots, p\}$, we see that

$$\mathfrak{a}_p = \bigcap_{\substack{H \subseteq \{1,\ldots,n\} \\ |H| = p}} \mathfrak{a}_H.$$

By Proposition 35.28, we have

$$\bigwedge^p M \approx \bigoplus_{\substack{H \subseteq \{1,\ldots,n\} \\ |H| = p}} A/\mathfrak{a}_H$$

which proves that $\mathfrak{a}_p$ is indeed the annihilator of $\bigwedge^p M$. The rest is clear. $\square$

Example 35.1 continued: Recall that M is the $\mathbb{Z}$-module generated by $\{e_1, e_2, e_3, e_4\}$ subject to $6e_3 = 0$, $2e_2 = 0$. Then

$$\begin{aligned}
\bigwedge^1 M &= \mathrm{span}\{e_1, e_2, e_3, e_4\} \\
\bigwedge^2 M &= \mathrm{span}\{e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4\} \\
\bigwedge^3 M &= \mathrm{span}\{e_1 \wedge e_2 \wedge e_3, e_1 \wedge e_2 \wedge e_4, e_1 \wedge e_3 \wedge e_4, e_2 \wedge e_3 \wedge e_4\} \\
\bigwedge^3 M &= \mathrm{span}\{e_1 \wedge e_2 \wedge e_3 \wedge e_4\}.
\end{aligned}$$

Since e_1 and e_2 are free, $e_1 \wedge e_2$ is also free. Since $6e_3 = 0$, each element of $\{e_1 \wedge e_3, e_2 \wedge e_3, e_1 \wedge e_2 \wedge e_3\}$ is annihilated by $6\mathbb{Z} = (6)$. Since $2e_4 = 0$, each element of $\{e_1 \wedge e_4, e_2 \wedge e_4, e_3 \wedge e_4, e_1 \wedge e_2 \wedge e_4, e_1 \wedge e_3 \wedge e_4, e_2 \wedge e_3 \wedge e_4, e_1 \wedge e_2 \wedge e_3 \wedge e_4\}$ is annihilated by $2\mathbb{Z} = (2)$.