and

$$\begin{pmatrix} q_1^x & q_2^x & q_3^x \\ q_1^y & q_2^y & q_3^y \\ q_1^z & q_2^z & q_3^z \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} q_4^x \\ q_4^y \\ q_4^z \end{pmatrix},$$

and the matrix $A_{\mathcal{E}}$ of our linear map f with respect to the canonical basis is determined as follows.

Proposition 26.9. *With respect to the canonical basis $\mathcal{E} = (e_1, e_2, e_3)$, the matrix $A_{\mathcal{E}}$ of the unique homography h of $\mathbb{RP}^2$ mapping the projective frame $([p_1], [p_2], [p_3], [p_4])$ to the projective frame $[(q_1], [q_2], [q_3], [q_4])$ is given by*

$$A_{\mathcal{E}} = \begin{pmatrix} q_1^x & q_2^x & q_3^x \\ q_1^y & q_2^y & q_3^y \\ q_1^z & q_2^z & q_3^z \end{pmatrix} \begin{pmatrix} \frac{\lambda_1}{\alpha_1} & 0 & 0 \\ 0 & \frac{\lambda_2}{\alpha_2} & 0 \\ 0 & 0 & \frac{\lambda_3}{\alpha_3} \end{pmatrix} \begin{pmatrix} p_1^x & p_2^x & p_3^x \\ p_1^y & p_2^y & p_3^y \\ p_1^z & p_2^z & p_3^z \end{pmatrix}^{-1}.$$

Proof. Since $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ is the unique linear map given by

$$f(u_i) = v_i, \quad i = 1, \ldots, 3,$$

the map $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ is equal to the composition

$$f = f_{\mathcal{Q}} \circ g,$$

where $g\colon \mathbb{R}^3 \to \mathbb{R}^3$ is the unique linear map given by

$$g(u_i) = e_i, \quad i = 1, \ldots, 3,$$

and $f_{\mathcal{Q}}\colon \mathbb{R}^3 \to \mathbb{R}^3$ is the unique linear map given by

$$f_{\mathcal{Q}}(e_i) = v_i, \quad i = 1, \ldots, 3.$$

However, $g\colon \mathbb{R}^3 \to \mathbb{R}^3$ is the inverse of the unique linear map $f_{\mathcal{P}}\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$f_{\mathcal{P}}(e_i) = u_i, \quad i = 1, \ldots, 3,$$

so

$$f = f_{\mathcal{Q}} \circ f_{\mathcal{P}}^{-1}.$$

The matrix $B_{\mathcal{P}}$ representing $f_{\mathcal{P}}$ over the canonical basis $\mathcal{E}$ is

$$B_{\mathcal{P}} = \begin{pmatrix} \alpha_1 p_1^x & \alpha_2 p_2^x & \alpha_3 p_3^x \\ \alpha_1 p_1^y & \alpha_2 p_2^y & \alpha_3 p_3^y \\ \alpha_1 p_1^z & \alpha_2 p_2^z & \alpha_3 p_3^z \end{pmatrix} = \begin{pmatrix} p_1^x & p_2^x & p_3^x \\ p_1^y & p_2^y & p_3^y \\ p_1^z & p_2^z & p_3^z \end{pmatrix} \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix},$$