Then, $\mathrm{S}^\bullet(V)$ automatically inherits a multiplication operation which is commutative, and since $T(V)$ is graded, that is

$$T(V) = \bigoplus_{m\geq 0} V^{\otimes m},$$

we have

$$\mathrm{S}^\bullet(V) = \bigoplus_{m\geq 0} V^{\otimes m}/(\mathfrak{I} \cap V^{\otimes m}).$$

However, it is easy to check that

$$\mathrm{S}^m(V) \cong V^{\otimes m}/(\mathfrak{I} \cap V^{\otimes m}),$$

so

$$\mathrm{S}^\bullet(V) \cong \mathrm{S}(V).$$

When V is of finite dimension n, $\mathrm{S}(V)$ corresponds to *the algebra of polynomials with coefficients in K in n variables* (this can be seen from Proposition 33.28). When V is of infinite dimension and $(u_i)_{i\in I}$ is a basis of V, the algebra $\mathrm{S}(V)$ corresponds to the algebra of polynomials in infinitely many variables in I. What's nice about the symmetric tensor algebra $\mathrm{S}(V)$ is that it provides an intrinsic definition of a polynomial algebra in any set of I variables.

It is also easy to see that $\mathrm{S}(V)$ satisfies the following universal mapping property.

Proposition 33.31. *Given any commutative K-algebra A, for any linear map $f\colon V \to A$, there is a unique K-algebra homomorphism $\overline{f}\colon \mathrm{S}(V) \to A$ so that*

$$f = \overline{f} \circ i,$$

as in the diagram below.

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[dr, "f"'] & \mathrm{S}(V) \arrow[d, "\overline{f}"] \\ & A \end{tikzcd}$$

Remark: If E is finite-dimensional, recall the isomorphism $\mu\colon \mathrm{S}^n(E^*) \longrightarrow \mathrm{Sym}^n(E; K)$ defined as the linear extension of the map given by

$$\mu(v_1^* \odot \cdots \odot v_n^*)(u_1, \ldots, u_n) = \sum_{\sigma\in\mathfrak{S}_n} v_{\sigma(1)}^*(u_1)\cdots v_{\sigma(n)}^*(u_n).$$

Now we have also a multiplication operation $\mathrm{S}^m(E^*) \times \mathrm{S}^n(E^*) \longrightarrow \mathrm{S}^{m+n}(E^*)$. The following question then arises: