Unravelling Definition 3.3, a family $(u_i)_{i\in I}$ is linearly dependent iff either I consists of a single element, say i, and $u_i = 0$, or $|I| \geq 2$ and some u_j in the family can be expressed as a linear combination of the other vectors in the family. Indeed, in the second case, there is some family $(\lambda_i)_{i\in I}$ of scalars in K such that

$$\sum_{i\in I} \lambda_i u_i = 0 \quad \text{and} \quad \lambda_j \neq 0 \text{ for some } j \in I,$$

and since $|I| \geq 2$, the set $I - \{j\}$ is nonempty and we get

$$u_j = \sum_{i\in(I-\{j\})} -\lambda_j^{-1}\lambda_i u_i.$$

Observe that one of the reasons for defining linear dependence for families of vectors rather than for sets of vectors is that our definition allows multiple occurrences of a vector. This is important because a matrix may contain identical columns, and we would like to say that these columns are linearly dependent. The definition of linear dependence for sets does not allow us to do that.

The above also shows that a family $(u_i)_{i\in I}$ is linearly independent iff either $I = \emptyset$, or I consists of a single element i and $u_i \neq 0$, or $|I| \geq 2$ and no vector u_j in the family can be expressed as a linear combination of the other vectors in the family.

When I is nonempty, if the family $(u_i)_{i\in I}$ is linearly independent, note that $u_i \neq 0$ for all $i \in I$. Otherwise, if $u_i = 0$ for some $i \in I$, then we get a nontrivial linear dependence $\sum_{i\in I} \lambda_i u_i = 0$ by picking any nonzero λ_i and letting $\lambda_k = 0$ for all $k \in I$ with $k \neq i$, since $\lambda_i 0 = 0$. If $|I| \geq 2$, we must also have $u_i \neq u_j$ for all $i, j \in I$ with $i \neq j$, since otherwise we get a nontrivial linear dependence by picking $\lambda_i = \lambda$ and $\lambda_j = -\lambda$ for any nonzero λ, and letting $\lambda_k = 0$ for all $k \in I$ with $k \neq i, j$.

Thus, the definition of linear independence implies that *a nontrivial linearly independent family is actually a set*. This explains why certain authors choose to define linear independence for sets of vectors. The problem with this approach is that linear dependence, which is the logical negation of linear independence, is then only defined for sets of vectors. However, as we pointed out earlier, it is really desirable to define linear dependence for families allowing multiple occurrences of the same vector.

Example 3.2.

1. Any two distinct scalars $\lambda, \mu \neq 0$ in K are linearly dependent.

2. In $\mathbb{R}^3$, the vectors $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ are linearly independent. See Figure 3.7.

3. In $\mathbb{R}^4$, the vectors $(1, 1, 1, 1)$, $(0, 1, 1, 1)$, $(0, 0, 1, 1)$, and $(0, 0, 0, 1)$ are linearly independent.