If Y is a constant vector field, it is immediately verified that the map

$$X \mapsto \mathrm{D}_Y X(a)$$

is a linear map called the *derivative* of the vector field X, and denoted by $\mathrm{D}X(a)$. If $f\colon E \to \mathbb{R}$ is a function, we define $\mathrm{D}_Y f(a)$ as the limit (if it exists)

$$\lim_{t\to 0,\, t\in U} \frac{f(a+tY(a)) - f(a)}{t},$$

where $U = \{t \in \mathbb{R} \mid a + tY(a) \in \Omega,\, t \neq 0\}$. It is the *directional derivative of f w.r.t. the vector field Y at a*, and it is also often denoted by $Y(f)(a)$, or $Y(f)_a$.

From now on, we assume that all the vector fields and all the functions under consideration are smooth (C^∞). The set $C^\infty(\Omega)$ of smooth C^∞-functions $f\colon \Omega \to \mathbb{R}$ is a ring. Given a smooth vector field X and a smooth function f (both over Ω), the vector field fX is defined such that $(fX)(a) = f(a)X(a)$, and it is immediately verified that it is smooth. Thus, the set $\mathcal{X}(\Omega)$ of smooth vector fields over Ω is a $C^\infty(\Omega)$-module.

The following proposition is left as an exercise. It shows that $\mathrm{D}_Y X(a)$ is a $\mathbb{R}$-bilinear map on $\mathcal{X}(\Omega)$, is $C^\infty(\Omega)$-linear in Y, and satisfies the Leibniz derivation rules with respect to X.

Proposition 39.28. *The covariant derivative* $\mathrm{D}_Y X(a)$ *satisfies the following properties:*

$$\begin{aligned}
\mathrm{D}_{(Y_1+Y_2)} X(a) &= \mathrm{D}_{Y_1} X(a) + \mathrm{D}_{Y_2} X(a),\\
\mathrm{D}_{fY} X(a) &= f(a)\mathrm{D}_Y X(a),\\
\mathrm{D}_Y (X_1 + X_2)(a) &= \mathrm{D}_Y X_1(a) + \mathrm{D}_Y X_2(a),\\
\mathrm{D}_Y fX(a) &= \mathrm{D}_Y f(a) X(a) + f(a)\mathrm{D}_Y X(a),
\end{aligned}$$

where X, Y, X_1, X_2, Y_1, Y_2 are smooth vector fields over Ω, and $f\colon E \to \mathbb{R}$ is a smooth function.

In differential geometry, the above properties are taken as the axioms of *affine connections*, in order to define covariant derivatives of vector fields over manifolds. In many cases, the vector field Y is the tangent field of some smooth curve $\gamma\colon]-\eta, \eta[\to E$. If so, the following proposition holds.

Proposition 39.29. *Given a smooth curve $\gamma\colon]-\eta, \eta[\to E$, letting Y be the vector field defined on $\gamma(]-\eta, \eta[)$ such that*

$$Y(\gamma(u)) = \frac{d\gamma}{dt}(u),$$

for any vector field X defined on $\gamma(]-\eta, \eta[)$, we have

$$\mathrm{D}_Y X(a) = \frac{d}{dt}\Big[X(\gamma(t))\Big](0),$$

where $a = \gamma(0)$.