Definition 2.23. A homomorphism $h\colon K_1 \to K_2$ between two fields K_1 and K_2 is just a homomorphism between the rings K_1 and K_2.

However, because K_1^* and K_2^* are groups under multiplication, a homomorphism of fields must be injective.

Proof. First, observe that for any $x \neq 0$,

$$1 = h(1) = h(xx^{-1}) = h(x)h(x^{-1})$$

and

$$1 = h(1) = h(x^{-1}x) = h(x^{-1})h(x),$$

so $h(x) \neq 0$ and

$$h(x^{-1}) = h(x)^{-1}.$$

But then, if $h(x) = 0$, we must have $x = 0$. Consequently, h is injective. □

Definition 2.24. A field homomorphism $h\colon K_1 \to K_2$ is an *isomorphism* iff there is a homomorphism $g\colon K_2 \to K_1$ such that $g \circ f = \mathrm{id}_{K_1}$ and $f \circ g = \mathrm{id}_{K_2}$. An isomorphism from a field to itself is called an *automorphism*.

Then, just as in the case of rings, g is unique and denoted by h^{-1}, and a bijective field homomorphism $h\colon K_1 \to K_2$ is an isomorphism.

Definition 2.25. Since every homomorphism $h\colon K_1 \to K_2$ between two fields is injective, the image $f(K_1)$ of K_1 is a subfield of K_2. We say that K_2 is an *extension* of K_1.

For example, $\mathbb{R}$ is an extension of $\mathbb{Q}$ and $\mathbb{C}$ is an extension of $\mathbb{R}$. The fields $\mathbb{Q}(\sqrt{d})$ and $\mathbb{Q}(\sqrt{-d})$ are extensions of $\mathbb{Q}$, the field $\mathbb{R}$ is an extension of $\mathbb{Q}(\sqrt{d})$ and the field $\mathbb{C}$ is an extension of $\mathbb{Q}(\sqrt{-d})$.

Definition 2.26. A field K is said to be *algebraically closed* if every polynomial $p(X)$ with coefficients in K has some root in K; that is, there is some $a \in K$ such that $p(a) = 0$.

It can be shown that every field K has some minimal extension Ω which is algebraically closed, called an *algebraic closure* of K. For example, $\mathbb{C}$ is the algebraic closure of $\mathbb{R}$. The algebraic closure of $\mathbb{Q}$ is called the *field of algebraic numbers*. This field consists of all complex numbers that are zeros of a polynomial with coefficients in $\mathbb{Q}$.

Definition 2.27. Given a field K and an automorphism $h\colon K \to K$ of K, it is easy to check that the set

$$\mathrm{Fix}(h) = \{a \in K \mid h(a) = a\}$$

of elements of K fixed by h is a subfield of K called the *field fixed by h*.