matrix, by Corollary 4.6, the matrix of f with respect to the basis $(v_1, \ldots, v_n)$ is $P^{-1}M(f)P$. By Proposition 7.9, we have

$$\det(P^{-1}M(f)P) = \det(P^{-1})\det(M(f))\det(P) = \det(P^{-1})\det(P)\det(M(f)) = \det(M(f)).$$

Thus, $\det(f)$ is indeed independent of the basis of E.

Definition 7.8. Given a vector space E of finite dimension, for any linear map $f \colon E \to E$, we define the *determinant* $\det(f)$ *of* f as the determinant $\det(M(f))$ of the matrix of f in any basis (since, from the discussion just before this definition, this determinant does not depend on the basis).

Then we have the following proposition.

Proposition 7.13. *Given any vector space E of finite dimension n, a linear map $f \colon E \to E$ is invertible iff* $\det(f) \neq 0$.

Proof. The linear map $f \colon E \to E$ is invertible iff its matrix $M(f)$ in any basis is invertible (by Proposition 4.2), iff $\det(M(f)) \neq 0$, by Proposition 7.10. □

Given a vector space of finite dimension n, it is easily seen that the set of bijective linear maps $f \colon E \to E$ such that $\det(f) = 1$ is a group under composition. This group is a subgroup of the general linear group $\mathbf{GL}(E)$. It is called the *special linear group (of E)*, and it is denoted by $\mathbf{SL}(E)$, or when $E = K^n$, by $\mathbf{SL}(n, K)$, or even by $\mathbf{SL}(n)$.

7.7 The Cayley–Hamilton Theorem

We next discuss an interesting and important application of Proposition 7.10, the *Cayley–Hamilton theorem*. The results of this section apply to matrices over any commutative ring K. First we need the concept of the characteristic polynomial of a matrix.

Definition 7.9. If K is any commutative ring, for every $n \times n$ matrix $A \in \mathrm{M}_n(K)$, the *characteristic polynomial* $P_A(X)$ of A is the determinant

$$P_A(X) = \det(XI - A).$$

The characteristic polynomial $P_A(X)$ is a polynomial in $K[X]$, the ring of polynomials in the indeterminate X with coefficients in the ring K. For example, when $n = 2$, if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then

$$P_A(X) = \begin{vmatrix} X - a & -b \\ -c & X - d \end{vmatrix} = X^2 - (a + d)X + ad - bc.$$