(1) *The family $(u_k)_{k\in K}$ is a total orthogonal family.*

(2) *For every vector $v \in E$, if $(c_k)_{k\in K}$ are the Fourier coefficients of v, then the family $(c_k u_k)_{k\in K}$ is summable and $v = \sum_{k\in K} c_k u_k$.*

(3) *For every vector $v \in E$, we have the Parseval identity:*

$$\|v\|^2 = \sum_{k\in K} |c_k|^2.$$

(4) *For every vector $u \in E$, if $\langle u, u_k\rangle = 0$ for all $k \in K$, then $u = 0$.*

See Figure A.2.

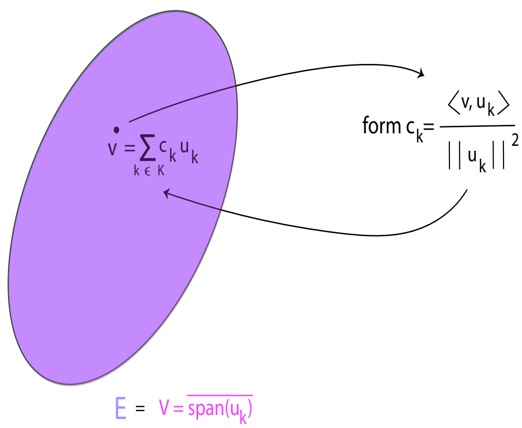

Figure A.2: A schematic illustration of Proposition A.5. Since $(u_k)_{k\in K}$ is a Hilbert basis, $V = E$. Then given a vector of E, if we form the Fourier coefficients c_k, then form the Fourier series $\sum_{k\in K} c_k u_k$, we are ensured that v is equal to its Fourier series.

Proof. The equivalence of (1), (2), and (3) is an immediate consequence of Proposition A.2 and Proposition A.4.

(4) If $(u_k)_{k\in K}$ is a total orthogonal family and $\langle u, u_k\rangle = 0$ for all $k \in K$, since $u = \sum_{k\in K} c_k u_k$ where $c_k = \langle u, u_k\rangle / \|u_k\|^2$, we have $c_k = 0$ for all $k \in K$, and $u = 0$.

Conversely, assume that the closure V of $(u_k)_{k\in K}$ is different from E. Then by Proposition 48.7, we have $E = V \oplus V^\perp$, where $V^\perp$ is the orthogonal complement of V, and $V^\perp$ is nontrivial since $V \neq E$. As a consequence, there is some nonnull vector $u \in V^\perp$. But then u is orthogonal to every vector in V, and in particular,

$$\langle u, u_k\rangle = 0$$

for all $k \in K$, which, by assumption, implies that $u = 0$, contradicting the fact that $u \neq 0$. $\square$