program simpler than the primal program. If $d^*$ is the optimal value of the dual program and if $p^*$ is the optimal value of the primal program, we always have

$$d^* \leq p^*,$$

which is known as *weak duality*. Under certain conditions, $d^* = p^*$, that is, the duality gap is zero, in which case we say that *strong duality* holds. Also, under certain conditions, a solution of the dual yields a solution of the primal, and if the primal has an optimal solution, then the dual has an optimal solution, but beware that the converse is generally false (see Theorem 50.17). We also show how to deal with equality constraints, and discuss the use of conjugate functions to find the dual function. Our coverage of Lagrangian duality is quite thorough, but we do not discuss more general orderings such as the semidefinite ordering. For these topics which belong to convex optimization, the reader is referred to Boyd and Vandenberghe [29].

Our approach in this chapter is very much inspired by Ciarlet [41] because we find it one of the more direct, and it is general enough to accomodate Hilbert spaces. The field of nonlinear optimization and convex optimization is vast and there are many books on the subject. Among those we recommend (in alphabetic order) Bertsekas [16, 17, 18], Bertsekas, Nedić, and Ozdaglar [19], Boyd and Vandenberghe [29], Luenberger [116], and Luenberger and Ye [117].

## 50.1 The Cone of Feasible Directions

Let $V$ be a normed vector space and let $U$ be a nonempty subset of $V$. For any point $u \in U$, consider any converging sequence $(u_k)_{k\geq 0}$ of vectors $u_k \in U$ having $u$ as their limit, with $u_k \neq u$ for all $k \geq 0$, and look at the sequence of "unit chords,"

$$\frac{u_k - u}{\|u_k - u\|}.$$

This sequence could oscillate forever, or it could have a limit, some unit vector $\widehat{w} \in V$. In the second case, all nonzero vectors $\lambda\widehat{w}$ for all $\lambda > 0$, belong to an object called the cone of feasible directions at $u$. First, we need to define the notion of cone.

**Definition 50.1.** Given a (real) vector space $V$, a nonempty subset $C \subseteq V$ is a *cone with apex* 0 (for short, a *cone*), if for any $v \in V$, if $v \in C$, then $\lambda v \in C$ for all $\lambda > 0$ ($\lambda \in \mathbb{R}$). For any $u \in V$, a *cone with apex* $u$ is any nonempty subset of the form $u + C = \{u + v \mid v \in C\}$, where $C$ is a cone with apex 0; see Figure 50.1.

Observe that a cone with apex 0 (or $u$) is not necessarily convex, and that 0 does not necessarily belong to $C$ (resp. $u$ does not necessarily belong to $u + C$) (although in the case of the cone of feasible directions $C(u)$ we have $0 \in C(u)$). The condition for being a cone only asserts that if a nonzero vector $v$ belongs to $C$, then the open ray $\{\lambda v \mid \lambda > 0\}$ (resp. the affine open ray $u + \{\lambda v \mid \lambda > 0\}$) also belongs to $C$.