15.7 Problems

Problem 15.1. Let A be the following 2×2 matrix

$$A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$

(1) Prove that A has the eigenvalue 0 with multiplicity 2 and that $A^2 = 0$.

(2) Let A be any real 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Prove that if $bc > 0$, then A has two distinct real eigenvalues. Prove that if $a, b, c, d > 0$, then there is a positive eigenvector u associated with the largest of the two eigenvalues of A, which means that if $u = (u_1, u_2)$, then $u_1 > 0$ and $u_2 > 0$.

(3) Suppose now that A is any complex 2×2 matrix as in (2). Prove that if A has the eigenvalue 0 with multiplicity 2, then $A^2 = 0$. Prove that if A is real symmetric, then $A = 0$.

Problem 15.2. Let A be any complex $n \times n$ matrix. Prove that if A has the eigenvalue 0 with multiplicity n, then $A^n = 0$. Give an example of a matrix A such that $A^n = 0$ but $A \neq 0$.

Problem 15.3. Let A be a complex 2×2 matrix, and let λ_1 and λ_2 be the eigenvalues of A. Prove that if $\lambda_1 \neq \lambda_2$, then

$$e^A = \frac{\lambda_1 e^{\lambda_2} - \lambda_2 e^{\lambda_1}}{\lambda_1 - \lambda_2} I + \frac{e^{\lambda_1} - e^{\lambda_2}}{\lambda_1 - \lambda_2} A.$$

Problem 15.4. Let A be the real symmetric 2×2 matrix

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

(1) Prove that the eigenvalues of A are real and given by

$$\lambda_1 = \frac{a + c + \sqrt{4b^2 + (a - c)^2}}{2}, \quad \lambda_2 = \frac{a + c - \sqrt{4b^2 + (a - c)^2}}{2}.$$

(2) Prove that A has a double eigenvalue ($\lambda_1 = \lambda_2 = a$) if and only if $b = 0$ and $a = c$; that is, A is a diagonal matrix.

(3) Prove that the eigenvalues of A are nonnegative iff $b^2 \leq ac$ and $a + c \geq 0$.

(4) Prove that the eigenvalues of A are positive iff $b^2 < ac$, $a > 0$ and $c > 0$.