Proof. As we suggested earlier, let us compute $Q(x) + G(\lambda)$, assuming that the constraint $B^\top x = f$ holds. Eliminating f, since $b^\top x = x^\top b$ and $\lambda^\top B^\top x = x^\top B\lambda$, we get

$$\begin{aligned} Q(x) + G(\lambda) &= \frac{1}{2} x^\top A^{-1} x - b^\top x + \frac{1}{2}(B\lambda - b)^\top A(B\lambda - b) + \lambda^\top f \\ &= \frac{1}{2}(A^{-1}x + B\lambda - b)^\top A(A^{-1}x + B\lambda - b). \end{aligned}$$

Since A is positive definite, the last expression is nonnegative. In fact, it is null iff

$$A^{-1}x + B\lambda - b = 0,$$

that is,

$$A^{-1}x + B\lambda = b.$$

But then the unique constrained minimum of $Q(x)$ subject to $B^\top x = f$ is equal to the unique maximum of $-G(\lambda)$ exactly when $B^\top x = f$ and $A^{-1}x + B\lambda = b$, which proves the proposition. □

We can confirm that the maximum of $-G(\lambda)$, or equivalently the minimum of

$$G(\lambda) = \frac{1}{2}(B\lambda - b)^\top A(B\lambda - b) + \lambda^\top f,$$

corresponds to value of λ obtained by solving the system

$$\begin{pmatrix} A^{-1} & B \\ B^\top & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} b \\ f \end{pmatrix}.$$

Indeed, since

$$G(\lambda) = \frac{1}{2}\lambda^\top B^\top AB\lambda - \lambda^\top B^\top Ab + \lambda^\top f + \frac{1}{2} b^\top b,$$

and $B^\top AB$ is symmetric positive definite, by Proposition 42.2, the global minimum of $G(\lambda)$ is obtained when

$$B^\top AB\lambda - B^\top Ab + f = 0,$$

that is, $\lambda = (B^\top AB)^{-1}(B^\top Ab - f)$, as we found earlier.

Remarks:

(1) There is a form of duality going on in this situation. The constrained minimization of $Q(x)$ subject to $B^\top x = f$ is called the *primal problem*, and the unconstrained maximization of $-G(\lambda)$ is called the *dual problem*. Duality is the fact stated slightly loosely as

$$\min_x Q(x) = \max_\lambda -G(\lambda).$$

A general treatment of duality in constrained minimization problems is given in Section 50.7.