49.4 Elliptic Functionals

We begin by defining the notion of an elliptic functional which generalizes the notion of a quadratic function defined by a symmetric positive definite matrix. Elliptic functionals are well adapted to the types of iterative methods described in this section and lend themselves well to an analysis of the convergence of these methods.

Definition 49.7. Given a Hilbert space V, a functional $J\colon V \to \mathbb{R}$ is said to be *elliptic* if it is continuously differentiable on V, and if there is some constant $\alpha > 0$ such that

$$\langle \nabla J_v - \nabla J_u, v - u \rangle \geq \alpha \left\| v - u \right\|^2 \quad \text{for all } u, v \in V.$$

The following proposition gathers properties of elliptic functionals that will be used later to analyze the convergence of various gradient descent methods.

Theorem 49.8. *Let V be a Hilbert space.*

(1) *An elliptic functional $J\colon V \to \mathbb{R}$ is strictly convex and coercive. Furthermore, it satisfies the identity*

$$J(v) - J(u) \geq \langle \nabla J_u, v - u \rangle + \frac{\alpha}{2} \left\| v - u \right\|^2 \quad \textit{for all } u, v \in V.$$

(2) *If U is a nonempty, convex, closed subset of the Hilbert space V and if J is an elliptic functional, then Problem (P),*

$$\begin{aligned} &\text{find} \quad u \\ &\text{such that} \;\; u \in U \;\; \textit{and} \;\; J(u) = \inf_{v \in U} J(v) \end{aligned}$$

has a unique solution.

(3) *Suppose the set U is convex and that the functional J is elliptic. Then an element $u \in U$ is a solution of Problem (P) if and only if it satisfies the condition*

$$\langle \nabla J_u, v - u \rangle \geq 0 \quad \textit{for every } v \in U$$

in the general case, or

$$\nabla J_u = 0 \;\; \textit{if } U = V.$$

(4) *A functional J which is twice differentiable in V is elliptic if and only if*

$$\langle \nabla^2 J_u(w), w \rangle \geq \alpha \left\| w \right\|^2 \quad \textit{for all } u, w \in V.$$