and since

$$G(\lambda) = L(u, \lambda) = \sup_{\mu \in \mathbb{R}^m_+} L(u, \mu),$$

by Theorem 40.13(3) (and since maximing a function g is equivalent to minimizing $-g$), we must have

$$G'_\lambda(\mu - \lambda) \leq 0 \quad \text{for all } \mu \in \mathbb{R}^m_+,$$

and since as noted earlier $\nabla G_\lambda = \varphi(u)$, we get

$$\langle \varphi(u), \mu - \lambda \rangle \leq 0 \quad \text{for all } \mu \in \mathbb{R}^m_+. \tag{$*_2$}$$

As in the proof of Proposition 49.18, $(*_2)$ can be expressed as follows for every $\rho > 0$:

$$\langle \lambda - (\lambda + \rho\varphi(u)), \mu - \lambda \rangle \geq 0 \quad \text{for all } \mu \in \mathbb{R}^m_+, \tag{$**_2$}$$

which shows that λ can be viewed as the projection onto $\mathbb{R}^m_+$ of the vector $\lambda + \rho\varphi(u)$. In summary we obtain the equations

$$(\dagger_1) \qquad \begin{cases} \nabla J_u + C^\top \lambda = 0 \\ \lambda = p_+(\lambda + \rho\varphi(u)). \end{cases}$$

Step 2. We establish algebraic conditions relating the unique solution u_k of the minimization problem arising during an iteration of Uzawa's method in (UZ) and λ^k.

Observe that the Lagrangian $L(v, \mu)$ is strictly convex as a function of v (as the sum of a strictly convex function and an affine function). As in the proof of Theorem 49.8(1) and using Cauchy–Schwarz, we have

$$\begin{aligned} J(v) + \langle C^\top \mu, v \rangle &\geq J(0) + \langle \nabla J_0, v \rangle + \frac{\alpha}{2} \|v\|^2 + \langle C^\top \mu, v \rangle \\ &\geq J(0) - \|\nabla J_0\| \|v\| - \left\|C^\top \mu\right\| \|v\| + \frac{\alpha}{2} \|v\|^2, \end{aligned}$$

and the term $(-\|\nabla J_0\| - \left\|C^\top \mu\right\| \|v\| + \frac{\alpha}{2} \|v\|) \|v\|$ goes to $+\infty$ when $\|v\|$ tends to $+\infty$, so $L(v, \mu)$ is coercive as a function of v. Therefore, the minimization problem find u^k such that

$$J(u^k) + \sum_{i=1}^m \lambda^k_i \varphi_i(u^k) = \inf_{v \in \mathbb{R}^n} \left(J(v) + \sum_{i=1}^m \lambda^k_i \varphi_i(v) \right)$$

has a unique solution $u^k \in \mathbb{R}^n$. It follows from Theorem 40.13(4) that the vector u^k must satisfy the equation

$$\nabla J_{u^k} + C^\top \lambda^k = 0, \tag{$*_3$}$$

and since by definition of Uzawa's method

$$\lambda^{k+1} = p_+(\lambda^k + \rho\varphi(u^k)), \tag{$*_4$}$$