Finally, we can prove that (2′) holds.

Lemma 32.9. *Let A be a UFD. Given any three nonnull polynomials $f(X), g(X), h(X) \in A[X]$, if $f(X)$ is irreducible and $f(X)$ divides the product $g(X)h(X)$, then either $f(X)$ divides $g(X)$ or $f(X)$ divides $h(X)$.*

Proof. If $f(X)$ has degree 0, then the result follows from Lemma 32.5. Thus, we may assume that the degree of $f(X)$ is $m \geq 1$. Let F be the fraction field of A. By Lemma 32.8, $f(X)$ is also irreducible in $F[X]$. Since $F[X]$ is a UFD (by Theorem 30.17), either $f(X)$ divides $g(X)$ or $f(X)$ divides $h(X)$, in $F[X]$. Assume that $f(X)$ divides $g(X)$, the other case being similar. Then,

$$g(X) = f(X)G(X),$$

for some $G(X) \in F[X]$. If a is the product the denominators of the coefficients of G, we have

$$ag(X) = q_1(X)f(X),$$

where $q_1(X) = aG(X) \in A[X]$. If a is a unit, we see that $f(X)$ divides $g(X)$. Otherwise, $a = a_1 \cdots a_n$, where $a_i \in A$ is irreducible. We prove by induction on n that

$$g(X) = q(X)f(X)$$

for some $q(X) \in A[X]$.

If $n = 1$, since $f(X)$ is irreducible and of degree $m \geq 1$ and

$$a_1 g(X) = q_1(X)f(X),$$

by Lemma 32.5, a_1 divides $q_1(X)$. Thus, $q_1(X) = a_1 q(X)$ where $q(X) \in A[X]$. Since $A[X]$ is an integral domain, we get

$$g(X) = q(X)f(X),$$

and $f(X)$ divides $g(X)$. If $n > 1$, from

$$a_1 \cdots a_n g(X) = q_1(X)f(X),$$

we note that a_1 divides $q_1(X)f(X)$, and as in the previous case, a_1 divides $q_1(X)$. Thus, $q_1(X) = a_1 q_2(X)$ where $q_2(X) \in A[X]$, and we get

$$a_2 \cdots a_n g(X) = q_2(X)f(X).$$

By the induction hypothesis, we get

$$g(X) = q(X)f(X)$$

for some $q(X) \in A[X]$, and $f(X)$ divides $g(X)$. $\square$

We finally obtain the fact that $A[X]$ is a UFD when A is.