Artin's famous book [6], which contains an in-depth study of the orthogonal group, as well as other groups arising in geometry. It is still worth consulting some of the older classics, such as Hadamard [84, 85] and Rouché and de Comberousse [139]. The first edition of [84] was published in 1898 and finally reached its thirteenth edition in 1947! In this chapter it is assumed that all vector spaces are defined over the field $\mathbb{R}$ of real numbers unless specified otherwise (in a few cases, over the complex numbers $\mathbb{C}$).

First we define a Euclidean structure on a vector space. Technically, a Euclidean structure over a vector space E is provided by a symmetric bilinear form on the vector space satisfying some extra properties. Recall that a bilinear form $\varphi\colon E \times E \to \mathbb{R}$ is *definite* if for every $u \in E$, $u \neq 0$ implies that $\varphi(u, u) \neq 0$, and *positive* if for every $u \in E$, $\varphi(u, u) \geq 0$.

Definition 12.1. A *Euclidean space* is a real vector space E equipped with a symmetric bilinear form $\varphi\colon E\times E \to \mathbb{R}$ that is *positive definite*. More explicitly, $\varphi\colon E\times E \to \mathbb{R}$ satisfies the following axioms:

$$\begin{aligned}
\varphi(u_1 + u_2, v) &= \varphi(u_1, v) + \varphi(u_2, v),\\
\varphi(u, v_1 + v_2) &= \varphi(u, v_1) + \varphi(u, v_2),\\
\varphi(\lambda u, v) &= \lambda\varphi(u, v),\\
\varphi(u, \lambda v) &= \lambda\varphi(u, v),\\
\varphi(u, v) &= \varphi(v, u),\\
u &\neq 0 \text{ implies that } \varphi(u, u) > 0.
\end{aligned}$$

The real number $\varphi(u, v)$ is also called the *inner product (or scalar product) of u and v*. We also define the *quadratic form associated with φ* as the function $\Phi\colon E \to \mathbb{R}_+$ such that

$$\Phi(u) = \varphi(u, u),$$

for all $u \in E$.

Since φ is bilinear, we have $\varphi(0, 0) = 0$, and since it is positive definite, we have the stronger fact that

$$\varphi(u, u) = 0 \quad \text{iff} \quad u = 0,$$

that is, $\Phi(u) = 0$ iff $u = 0$.

Given an inner product $\varphi\colon E \times E \to \mathbb{R}$ on a vector space E, we also denote $\varphi(u, v)$ by

$$u \cdot v \quad \text{or} \quad \langle u, v\rangle \quad \text{or} \quad (u|v),$$

and $\sqrt{\Phi(u)}$ by $\|u\|$.

Example 12.1. The standard example of a Euclidean space is $\mathbb{R}^n$, under the inner product $\cdot$ defined such that

$$(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

This Euclidean space is denoted by $\mathbb{E}^n$.