- The (real) vector space $\mathfrak{su}(2)$ of 2×2 Hermitian matrices with zero trace.
- The group homomorphism $r\colon \mathbf{SU}(2) \to \mathbf{SO}(3)$; $\mathrm{Ker}\,(r) = \{+I, -I\}$.
- The matrix representation R_q of the rotation r_q induced by a unit quaternion q.
- Surjectivity of the homomorphism $r\colon \mathbf{SU}(2) \to \mathbf{SO}(3)$.
- The exponential map $\exp\colon \mathfrak{su}(2) \to \mathbf{SU}(2)$.
- Surjectivity of the exponential map $\exp\colon \mathfrak{su}(2) \to \mathbf{SU}(2)$.
- Finding a logarithm of a quaternion.
- Quaternion interpolation.
- Shoemake's slerp interpolation formula.
- Sections $s\colon \mathbf{SO}(3) \to \mathbf{SU}(2)$ of $r\colon \mathbf{SU}(2) \to \mathbf{SO}(3)$.

16.9 Problems

Problem 16.1. Verify the quaternion identities

$$\begin{aligned}
&\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -\mathbf{1},\\
&\mathbf{ij} = -\mathbf{ji} = \mathbf{k},\\
&\mathbf{jk} = -\mathbf{kj} = \mathbf{i},\\
&\mathbf{ki} = -\mathbf{ik} = \mathbf{j}.
\end{aligned}$$

Problem 16.2. Check that for every quaternion $X = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, we have

$$XX^* = X^*X = (a^2 + b^2 + c^2 + d^2)\mathbf{1}.$$

Conclude that if $X \neq 0$, then X is invertible and its inverse is given by

$$X^{-1} = (a^2 + b^2 + c^2 + d^2)^{-1} X^*.$$

Problem 16.3. Given any two quaternions $X = a\mathbf{1}+b\mathbf{i}+c\mathbf{j}+d\mathbf{k}$ and $Y = a'\mathbf{1}+b'\mathbf{i}+c'\mathbf{j}+d'\mathbf{k}$, prove that

$$\begin{aligned}
XY &= (aa' - bb' - cc' - dd')\mathbf{1} + (ab' + ba' + cd' - dc')\mathbf{i}\\
&\quad + (ac' + ca' + db' - bd')\mathbf{j} + (ad' + da' + bc' - cb')\mathbf{k}.
\end{aligned}$$

Also prove that if $X = [a, U]$ and $Y = [a', U']$, the quaternion product XY can be expressed as

$$XY = [aa' - U \cdot U',\ aU' + a'U + U \times U'].$$