Definition 30.12. Given a ring A and any nonnull polynomial $f \in A[X]$, given any $\alpha \in A$, the unique $h \geq 0$ such that f is divisible by $(X-\alpha)^h$ but not by $(X-\alpha)^{h+1}$ is called the *order, or multiplicity, of* α. We have $h = 0$ iff α is not a root of f, and when α is a root of f, if $h = 1$, we call α a *simple root*, if $h = 2$, a *double root*, and generally, a root of multiplicity $h \geq 2$ is called a *multiple root*.

Observe that Proposition 30.20 (2) implies that if $A \subseteq B$, where A and B are rings, for every nonnull polynomial $f \in A[X]$, if $\alpha \in A$ is a root of f, then the multiplicity of α with respect to $f \in A[X]$ and the multiplicity of α with respect to f considered as a polynomial in $B[X]$, is the same.

We now show that if the ring A is an integral domain, the number of roots of a nonzero polynomial is at most its degree.

Proposition 30.21. *Let $f, g \in A[X]$ be nonnull polynomials, let $\alpha \in A$, and let $h \geq 0$ and $k \geq 0$ be the multiplicities of α with respect to f and g. The following properties hold.*

(1) If l is the multiplicity of α with respect to $(f+g)$, then $l \geq \min(h,k)$. If $h \neq k$, then $l = \min(h,k)$.

(2) If m is the multiplicity of α with respect to fg, then $m \geq h+k$. If A is an integral domain, then $m = h+k$.

Proof. (1) We have $f(X) = (X-\alpha)^h f_1(X)$, $g(X) = (X-\alpha)^k g_1(X)$, with $f_1(\alpha) \neq 0$ and $g_1(\alpha) \neq 0$. Clearly, $l \geq \min(h,k)$. If $h \neq k$, assume $h < k$. Then, we have

$$f(X) + g(X) = (X-\alpha)^h f_1(X) + (X-\alpha)^k g_1(X) = (X-\alpha)^h (f_1(X) + (X-\alpha)^{k-h} g_1(X)),$$

and since $(f_1(X) + (X-\alpha)^{k-h} g_1(X))(\alpha) = f_1(\alpha) \neq 0$, we have $l = h = \min(h,k)$.

(2) We have

$$f(X)g(X) = (X-\alpha)^{h+k} f_1(X) g_1(X),$$

with $f_1(\alpha) \neq 0$ and $g_1(\alpha) \neq 0$. Clearly, $m \geq h+k$. If A is an integral domain, then $f_1(\alpha)g_1(\alpha) \neq 0$, and so $m = h+k$. ☐

Proposition 30.22. *Let A be an integral domain. Let f be any nonnull polynomial $f \in A[X]$ and let $\alpha_1, \ldots, \alpha_m \in A$ be $m \geq 1$ distinct roots of f of respective multiplicities $k_1, \ldots, k_m$. Then, we have*

$$f(X) = (X-\alpha_1)^{k_1} \cdots (X-\alpha_m)^{k_m} g(X),$$

where $g \in A[X]$ and $g(\alpha_i) \neq 0$ for all i, $1 \leq i \leq m$.

Proof. We proceed by induction on m. The case $m = 1$ is obvious in view of Definition 30.12 (which itself, is justified by Proposition 30.20). If $m \geq 2$, by the induction hypothesis, we have

$$f(X) = (X-\alpha_1)^{k_1} \cdots (X-\alpha_{m-1})^{k_{m-1}} g_1(X),$$