if we let

$$\Lambda(u) = -\frac{\partial J}{\partial x_2}(u) \circ \left(\frac{\partial \varphi}{\partial x_2}(u)\right)^{-1},$$

then we get

$$\begin{aligned} dJ(u) &= \frac{\partial J}{\partial x_1}(u) + \frac{\partial J}{\partial x_2}(u) \\ &= \frac{\partial J}{\partial x_2}(u) \circ \left(\frac{\partial \varphi}{\partial x_2}(u)\right)^{-1} \circ \left(\frac{\partial \varphi}{\partial x_1}(u) + \frac{\partial \varphi}{\partial x_2}(u)\right) \\ &= -\Lambda(u) \circ d\varphi(u), \end{aligned}$$

which yields $dJ(u) + \Lambda(u) \circ d\varphi(u) = 0$, as claimed. □

Finally, we prove Theorem 40.2.

Proof of Theorem 40.2. The linear independence of the m linear forms $d\varphi_i(u)$ is equivalent to the fact that the $m \times n$ matrix $A = \big((\partial\varphi_i/\partial x_j)(u)\big)$ has rank m. By reordering the columns, we may assume that the first m columns are linearly independent. To conform to the set-up of Theorem 40.4 we define E_1 and E_2 as

$$E_1 = \left\{ \sum_{i=m+1}^{n} v_i e_i \mid (v_{m+1}, \ldots, v_n) \in \mathbb{R}^{n-m} \right\}, \quad E_2 = \left\{ \sum_{i=1}^{m} v_i e_i \mid (v_1, \ldots, v_m) \in \mathbb{R}^{m} \right\}.$$

If we let $\psi \colon \Omega \to \mathbb{R}^m$ be the function defined by

$$\psi(v_{m+1}, \ldots, v_n, v_1, \ldots, v_m) = (\varphi_1(v), \ldots, \varphi_m(v))$$

for all $(v_{m+1}, \ldots, v_n, v_1, \ldots, v_m) \in \Omega$, with $v = (v_1, \ldots, v_n)$, then we see that $\partial\psi/\partial x_2(u)$ is invertible and both $\partial\psi/\partial x_2(u)$ and its inverse are continuous, so that Theorem 40.4 applies, and there is some (continuous) linear form $\Lambda(u) \in \mathcal{L}(\mathbb{R}^m; \mathbb{R})$ such that

$$dJ(u) + \Lambda(u) \circ d\psi(u_{m+1}, \ldots, u_n, u_1, \ldots, u_m) = 0,$$

namely

$$dJ(u) + \Lambda(u) \circ d\varphi(u) = 0.$$

However, $\Lambda(u)$ is defined by some m-tuple $(\lambda_1(u), \ldots, \lambda_m(u)) \in \mathbb{R}^m$, and in view of the definition of φ, the above equation is equivalent to

$$dJ(u) + \lambda_1(u) d\varphi_1(u) + \cdots + \lambda_m(u) d\varphi_m(u) = 0.$$

The uniqueness of the $\lambda_i(u)$ is a consequence of the linear independence of the $d\varphi_i(u)$. □

We now investigate conditions for the existence of extrema involving the second derivative of J.