Proposition 8.27. *Let E be any finite-dimensional vector space. For every transvection $\tau_{\varphi,u}$ ($u \neq 0$) and every linear map $g \in \mathbf{GL}(E)$, the map $g \circ \tau_{\varphi,u} \circ g^{-1}$ is the transvection of hyperplane $g(H)$ and direction $g(u)$ (that is, $g \circ \tau_{\varphi,u} \circ g^{-1} = \tau_{\varphi \circ g^{-1}, g(u)}$). For every other transvection $\tau_{\psi,u'}$ ($u' \neq 0$), there is some $g \in \mathbf{GL}(E)$ such $\tau_{\psi,u'} = g \circ \tau_{\varphi,u} \circ g^{-1}$; in other words any two transvections ($\neq \mathrm{id}$) are conjugate in $\mathbf{GL}(E)$. Moreover, if $n \geq 3$, then the linear isomorphism g as above can be chosen so that $g \in \mathbf{SL}(E)$.*

Proof. We just need to prove that if $n \geq 3$, then for any two transvections $\tau_{\varphi,u}$ and $\tau_{\psi,u'}$ ($u, u' \neq 0$), there is some $g \in \mathbf{SL}(E)$ such that $\tau_{\psi,u'} = g \circ \tau_{\varphi,u} \circ g^{-1}$. As before, we pick a basis $(v, u, e_2, \ldots, e_{n-1})$ where $(u, e_2, \ldots, e_{n-1})$ is a basis of H, we pick a basis $(v', u', e_2', \ldots, e_{n-1}')$ where $(u', e_2', \ldots, e_{n-1}')$ is a basis of H', and we define g as the unique linear map such that $g(v) = v'$, $g(u) = u'$, and $g(e_i) = e_i'$, for $i = 1, \ldots, n-1$. But in this case, both H and $H' = g(H)$ have dimension at least 2, so in any basis of H' including u', there is some basis vector e_2' independent of u', and we can rescale e_2' in such a way that the matrix of g over the two bases has determinant $+1$. ☐

8.16 Summary

The main concepts and results of this chapter are listed below:

- One does not solve (large) linear systems by computing determinants.
- *Upper-triangular* (*lower-triangular*) matrices.
- Solving by *back-substitution* (*forward-substitution*).
- *Gaussian elimination.*
- Permuting rows.
- The *pivot* of an elimination step; *pivoting*.
- *Transposition matrix*; *elementary matrix*.
- The *Gaussian elimination theorem* (Theorem 8.1).
- *Gauss-Jordan factorization.*
- *LU-factorization*; Necessary and sufficient condition for the existence of an LU-factorization (Proposition 8.2).
- *LDU-factorization.*
- "$PA = LU$ theorem" (Theorem 8.5).
- *$LDL^\top$-factorization* of a symmetric matrix.