is clear.

Step 2. Prove that the matrices $(E_j^k)^{-1}$ are lower-triangular. To achieve this, we prove that the matrices $\mathcal{E}_j^k$ are strictly lower triangular matrices of a very special form.

Since for $j = 1, \dots, n-2$, we have $E_j^j = E_j$,

$$E_j^k = P_k E_j^{k-1} P_k, \quad k = j+1, \dots, n-1,$$

since $E_{n-1}^{n-1} = E_{n-1}$ and $P_k^{-1} = P_k$, we get $(E_j^j)^{-1} = E_j^{-1}$ for $j = 1, \dots, n-1$, and for $j = 1, \dots, n-2$, we have

$$(E_j^k)^{-1} = P_k (E_j^{k-1})^{-1} P_k, \quad k = j+1, \dots, n-1.$$

Since

$$(E_j^{k-1})^{-1} = I + \mathcal{E}_j^{k-1}$$

and $P_k = P(k,i)$ is a transposition or $P_k = I$, so $P_k^2 = I$, and we get

$$(E_j^k)^{-1} = P_k (E_j^{k-1})^{-1} P_k = P_k (I + \mathcal{E}_j^{k-1}) P_k = P_k^2 + P_k\, \mathcal{E}_j^{k-1}\, P_k = I + P_k\, \mathcal{E}_j^{k-1}\, P_k.$$

Therefore, we have

$$(E_j^k)^{-1} = I + P_k\, \mathcal{E}_j^{k-1}\, P_k, \quad 1 \le j \le n-2,\ j+1 \le k \le n-1.$$

We prove for $j = 1, \dots, n-1$, that for $k = j, \dots, n-1$, each $\mathcal{E}_j^k$ is a lower triangular matrix of the form

$$\mathcal{E}_j^k = \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & \ell_{j+1j}^{(k)} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \ell_{nj}^{(k)} & 0 & \cdots & 0 \end{pmatrix},$$

and that

$$\mathcal{E}_j^k = P_k\, \mathcal{E}_j^{k-1}, \quad 1 \le j \le n-2,\ j+1 \le k \le n-1,$$

with $P_k = I$ or $P_k = P(k,i)$ for some i such that $k+1 \le i \le n$.

For each j $(1 \le j \le n-1)$ we proceed by induction on $k = j, \dots, n-1$. Since $(E_j^j)^{-1} = E_j^{-1}$ and since E_j^{-1} is of the above form, the base case holds.

For the induction step, we only need to consider the case where $P_k = P(k,i)$ is a transposition, since the case where $P_k = I$ is trivial. We have to figure out what $P_k\, \mathcal{E}_j^{k-1}\, P_k =$