Problem 7.11. Let A be an $n \times n$ matrix with integer entries. Prove that A^{-1} exists and has integer entries if and only if $\det(A) = \pm 1$.

Problem 7.12. Let A be an $n \times n$ real or complex matrix.

(1) Prove that if $A^\top = -A$ (A is *skew-symmetric*) and if n is odd, then $\det(A) = 0$.

(2) Prove that

$$\begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix} = (af - be + dc)^2.$$

Problem 7.13. A *Cauchy matrix* is a matrix of the form

$$\begin{pmatrix} \dfrac{1}{\lambda_1 - \sigma_1} & \dfrac{1}{\lambda_1 - \sigma_2} & \cdots & \dfrac{1}{\lambda_1 - \sigma_n} \\ \dfrac{1}{\lambda_2 - \sigma_1} & \dfrac{1}{\lambda_2 - \sigma_2} & \cdots & \dfrac{1}{\lambda_2 - \sigma_n} \\ \vdots & \vdots & \vdots & \vdots \\ \dfrac{1}{\lambda_n - \sigma_1} & \dfrac{1}{\lambda_n - \sigma_2} & \cdots & \dfrac{1}{\lambda_n - \sigma_n} \end{pmatrix}$$

where $\lambda_i \neq \sigma_j$, for all i, j, with $1 \leq i, j \leq n$. Prove that the determinant C_n of a Cauchy matrix as above is given by

$$C_n = \frac{\prod_{i=2}^n \prod_{j=1}^{i-1} (\lambda_i - \lambda_j)(\sigma_j - \sigma_i)}{\prod_{i=1}^n \prod_{j=1}^n (\lambda_i - \sigma_j)}.$$

Problem 7.14. Let $(\alpha_1, \ldots, \alpha_{m+1})$ be a sequence of pairwise distinct scalars in $\mathbb{R}$ and let $(\beta_1, \ldots, \beta_{m+1})$ be any sequence of scalars in $\mathbb{R}$, not necessarily distinct.

(1) Prove that there is a unique polynomial P of degree at most m such that

$$P(\alpha_i) = \beta_i, \quad 1 \leq i \leq m + 1.$$

Hint. Remember Vandermonde!

(2) Let $L_i(X)$ be the polynomial of degree m given by

$$L_i(X) = \frac{(X - \alpha_1) \cdots (X - \alpha_{i-1})(X - \alpha_{i+1}) \cdots (X - \alpha_{m+1})}{(\alpha_i - \alpha_1) \cdots (\alpha_i - \alpha_{i-1})(\alpha_i - \alpha_{i+1}) \cdots (\alpha_i - \alpha_{m+1})}, \quad 1 \leq i \leq m + 1.$$

The polynomials $L_i(X)$ are known as *Lagrange polynomial interpolants*. Prove that

$$L_i(\alpha_j) = \delta_{i\,j} \quad 1 \leq i, j \leq m + 1.$$