By definition of g, the vector $u = q(f)(v)$ cannot be in W, since otherwise g would not be of minimal degree. However, $(*)$ implies that

$$\begin{aligned}(f - \lambda_j \mathrm{id})(u) &= (f - \lambda_j \mathrm{id})(q(f)(v)) \\ &= g(f)(v)\end{aligned}$$

is in W, which concludes the proof. $\square$

We can now prove the main result of this section.

Theorem 31.6. *Let $f\colon E \to E$ be a linear map on a finite-dimensional space E. Then f is diagonalizable iff its minimal polynomial m is of the form*

$$m = (X - \lambda_1) \cdots (X - \lambda_k),$$

where $\lambda_1, \ldots, \lambda_k$ are distinct elements of K.

Proof. We already showed in Proposition 31.2 that if f is diagonalizable, then its minimal polynomial is of the above form (where $\lambda_1, \ldots, \lambda_k$ are the distinct eigenvalues of f).

For the converse, let W be the subspace spanned by all the eigenvectors of f. If $W \neq E$, since W is invariant under f, by Proposition 31.5, there is some vector $u \notin W$ such that for some λ_j, we have

$$(f - \lambda_j \mathrm{id})(u) \in W.$$

Let $v = (f - \lambda_j \mathrm{id})(u) \in W$. Since $v \in W$, we can write

$$v = w_1 + \cdots + w_k$$

where $f(w_i) = \lambda_i w_i$ (either $w_i = 0$ or w_i is an eigenvector for λ_i), and so for every polynomial h, we have

$$h(f)(v) = h(\lambda_1) w_1 + \cdots + h(\lambda_k) w_k,$$

which shows that $h(f)(v) \in W$ for every polynomial h. We can write

$$m = (X - \lambda_j) q$$

for some polynomial q, and also

$$q - q(\lambda_j) = p(X - \lambda_j)$$

for some polynomial p. We know that $p(f)(v) \in W$, and since m is the minimal polynomial of f, we have

$$0 = m(f)(u) = (f - \lambda_j \mathrm{id})(q(f)(u)),$$

which implies that $q(f)(u) \in W$ (either $q(f)(u) = 0$, or it is an eigenvector associated with λ_j). However,

$$q(f)(u) - q(\lambda_j) u = p(f)((f - \lambda_j \mathrm{id})(u)) = p(f)(v),$$

and since $p(f)(v) \in W$ and $q(f)(u) \in W$, we conclude that $q(\lambda_j) u \in W$. But, $u \notin W$, which implies that $q(\lambda_j) = 0$, so λ_j is a double root of m, a contradiction. Therefore, we must have $W = E$. $\square$