ring, the structure A^n is a module such that the vectors e_i, with $(e_i)_i = 1$ and $(e_i)_j = 0$ for $j \neq i$, form a basis of A^n. Many properties of vector spaces still hold for A^n. Thus, A^n is a free module. As another example, when A is a commutative ring, $\mathrm{M}_{m,n}(A)$ is a free module with basis $(E_{i,j})_{1 \leq i \leq m, 1 \leq j \leq n}$. Polynomials over a commutative ring also form a free module of infinite dimension.

The properties listed in Proposition 3.16 are easily verified, although some of the computations are a bit tedious. A more conceptual proof is given in Proposition 4.1.

Proposition 3.16. *(1) Given any matrices $A \in \mathrm{M}_{m,n}(K)$, $B \in \mathrm{M}_{n,p}(K)$, and $C \in \mathrm{M}_{p,q}(K)$, we have*

$$(AB)C = A(BC);$$

that is, matrix multiplication is associative.

(2) Given any matrices $A, B \in \mathrm{M}_{m,n}(K)$, and $C, D \in \mathrm{M}_{n,p}(K)$, for all $\lambda \in K$, we have

$$\begin{aligned}
(A + B)C &= AC + BC \\
A(C + D) &= AC + AD \\
(\lambda A)C &= \lambda(AC) \\
A(\lambda C) &= \lambda(AC),
\end{aligned}$$

so that matrix multiplication $\cdot : \mathrm{M}_{m,n}(K) \times \mathrm{M}_{n,p}(K) \to \mathrm{M}_{m,p}(K)$ is bilinear.

The properties of Proposition 3.16 together with the fact that $AI_n = I_n A = A$ for all square $n \times n$ matrices show that $\mathrm{M}_n(K)$ is a ring with unit I_n (in fact, an associative algebra). This is a noncommutative ring with zero divisors, as shown by the following example.

Example 3.5. For example, letting A, B be the 2×2-matrices

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

then

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$BA = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Thus $AB \neq BA$, and $AB = 0$, even though both $A, B \neq 0$.