First we will show that such maps D exist, using an inductive definition that also gives a recursive method for computing determinants. Actually, we will define a family $(\mathcal{D}_n)_{n\geq 1}$ of (finite) sets of maps $D\colon \mathrm{M}_n(K) \to K$. Second we will show that determinants are in fact uniquely defined, that is, we will show that each $\mathcal{D}_n$ consists of a *single map*. This will show the equivalence of the direct definition $\det(A)$ of Lemma 7.4 with the inductive definition $D(A)$. Finally, we will prove some basic properties of determinants, using the uniqueness theorem.

Given a matrix $A \in \mathrm{M}_n(K)$, we denote its n columns by $A^1, \ldots, A^n$. In order to describe the recursive process to define a determinant we need the notion of a minor.

Definition 7.5. Given any $n \times n$ matrix with $n \geq 2$, for any two indices i, j with $1 \leq i, j \leq n$, let A_{ij} be the $(n-1) \times (n-1)$ matrix obtained by deleting Row i and Column j from A and called a *minor*:

$$A_{ij} = \begin{pmatrix} & & & & \times & & \\ & & & & \times & & \\ \times & \times & \times & \times & \times & \times & \times \\ & & & & \times & & \\ & & & & \times & & \\ & & & & \times & & \\ & & & & \times & & \end{pmatrix}.$$

For example, if

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

then

$$A_{2\,3} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

Definition 7.6. For every $n \geq 1$, we define a finite set $\mathcal{D}_n$ of maps $D\colon \mathrm{M}_n(K) \to K$ inductively as follows:

When $n = 1$, $\mathcal{D}_1$ consists of the single map D such that, $D(A) = a$, where $A = (a)$, with $a \in K$.

Assume that $\mathcal{D}_{n-1}$ has been defined, where $n \geq 2$. Then $\mathcal{D}_n$ consists of all the maps D such that, for some i, $1 \leq i \leq n$,

$$D(A) = (-1)^{i+1} a_{i\,1} D(A_{i\,1}) + \cdots + (-1)^{i+n} a_{i\,n} D(A_{i\,n}),$$

where for every j, $1 \leq j \leq n$, $D(A_{i\,j})$ is the result of applying any D in $\mathcal{D}_{n-1}$ to the minor $A_{i\,j}$.