Thus, there is a bijection between K and the set of equivalence classes containing some representative of the form $(x,1)$, and we denote the class $[x,1]$ by x. The equivalence class $[1,0]$ is denoted by ∞ and it is called the point at infinity. Thus, the projective line $\mathbb{P}^1_K$ is in bijection with $K \cup \{\infty\}$. The three points $\infty = [1,0]$, $0 = [0,1]$, and $1 = [1,1]$, form a projective frame for $\mathbb{P}^1_K$. The projective frame $(\infty, 0, 1)$ is often called the *canonical frame of* $\mathbb{P}^1_K$.

Homogeneous coordinates are also very useful to handle hyperplanes in terms of equations. If $(a_i)_{1\leq i\leq n+2}$ is a projective frame for $\mathbf{P}(E)$ associated with a basis $(u_1,\ldots,u_{n+1})$ for E, a nonnull linear form f is determined by $n+1$ scalars $\alpha_1,\ldots,\alpha_{n+1}$ (not all null), and a point $x \in \mathbf{P}(E)$ of homogeneous coordinates $(x_1,\ldots,x_{n+1})$ belongs to the projective hyperplane $\mathbf{P}(H)$ of equation f iff

$$\alpha_1 x_1 + \cdots + \alpha_{n+1}x_{n+1} = 0.$$

In particular, if $\mathbf{P}(E)$ is a projective plane, a line is defined by an equation of the form $\alpha x + \beta y + \gamma z = 0$. If $\mathbf{P}(E)$ is a projective space, a plane is defined by an equation of the form $\alpha x + \beta y + \gamma z + \delta w = 0$.

As an application, let us find the coordinates of the intersection point of two distinct lines in a projective plane $\mathbf{P}(E)$ (with respect to some projective frame (a_1, a_2, a_3, a_4)). If D and D' are two lines of equations

$$\alpha x + \beta y + \gamma z = 0 \quad \text{and} \quad \alpha' x + \beta' y + \gamma' z = 0, \tag{$*$}$$

then D and D' are distinct lines iff the matrix

$$\begin{pmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{pmatrix}$$

has rank 2. We claim that the intersection Q of the lines D and D' has homogeneous coordinates

$$(\beta\gamma' - \beta'\gamma \colon \gamma\alpha' - \gamma'\alpha \colon \alpha\beta' - \alpha'\beta); \tag{$\dagger$}$$

in other words, it is the projective point corresponding to the cross-product

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \times \begin{pmatrix} \alpha' \\ \beta' \\ \gamma' \end{pmatrix},$$

as illustrated in Figure 26.8.

Indeed, the homogeneous coordinates of the intersection Q of D and D' must satisfy simultaneously the two equations $(*)$, and since the two determinants

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} \alpha' & \beta' & \gamma' \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix}$$