8.8 Gaussian Elimination of Tridiagonal Matrices

Consider the tridiagonal matrix

$$A = \begin{pmatrix} b_1 & c_1 & & & & & \\ a_2 & b_2 & c_2 & & & & \\ & a_3 & b_3 & c_3 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & a_{n-2} & b_{n-2} & c_{n-2} & \\ & & & & a_{n-1} & b_{n-1} & c_{n-1} \\ & & & & & a_n & b_n \end{pmatrix}.$$

Define the sequence

$$\delta_0 = 1, \quad \delta_1 = b_1, \quad \delta_k = b_k\delta_{k-1} - a_kc_{k-1}\delta_{k-2}, \quad 2 \leq k \leq n.$$

Proposition 8.7. *If A is the tridiagonal matrix above, then $\delta_k = \det(A(1:k,1:k))$ for $k = 1, \ldots, n$.*

Proof. By expanding $\det(A(1:k,1:k))$ with respect to its last row, the proposition follows by induction on k. ☐

Theorem 8.8. *If A is the tridiagonal matrix above and $\delta_k \neq 0$ for $k = 1, \ldots, n$, then A has the following LU-factorization:*

$$A = \begin{pmatrix} 1 & & & & & \\ a_2\frac{\delta_0}{\delta_1} & 1 & & & & \\ & a_3\frac{\delta_1}{\delta_2} & 1 & & & \\ & & \ddots & \ddots & & \\ & & & a_{n-1}\frac{\delta_{n-3}}{\delta_{n-2}} & 1 & \\ & & & & a_n\frac{\delta_{n-2}}{\delta_{n-1}} & 1 \end{pmatrix} \begin{pmatrix} \frac{\delta_1}{\delta_0} & c_1 & & & & \\ & \frac{\delta_2}{\delta_1} & c_2 & & & \\ & & \frac{\delta_3}{\delta_2} & c_3 & & \\ & & & \ddots & \ddots & \\ & & & & \frac{\delta_{n-1}}{\delta_{n-2}} & c_{n-1} \\ & & & & & \frac{\delta_n}{\delta_{n-1}} \end{pmatrix}.$$

Proof. Since $\delta_k = \det(A(1:k,1:k)) \neq 0$ for $k = 1, \ldots, n$, by Theorem 8.5 (and Proposition 8.2), we know that A has a unique LU-factorization. Therefore, it suffices to check that the proposed factorization works. We easily check that

$$\begin{aligned} (LU)_{k\,k+1} &= c_k, \quad 1 \leq k \leq n-1 \\ (LU)_{k\,k-1} &= a_k, \quad 2 \leq k \leq n \\ (LU)_{k\,l} &= 0, \quad |k-l| \geq 2 \\ (LU)_{1\,1} &= \frac{\delta_1}{\delta_0} = b_1 \\ (LU)_{k\,k} &= \frac{a_kc_{k-1}\delta_{k-2} + \delta_k}{\delta_{k-1}} = b_k, \quad 2 \leq k \leq n, \end{aligned}$$