We leave it as an exercise for the reader to verify Equation $(*)$ for arbitrary nonnegative integers m and n.

Another useful canonical isomorphism (of K-algebras) is given below.

Proposition 33.32. *For any two vector spaces E and F, there is a canonical isomorphism (of K-algebras)*

$$\mathrm{S}(E \oplus F) \cong \mathrm{S}(E) \otimes \mathrm{S}(F).$$

33.12 Problems

Problem 33.1. Prove Proposition 33.4.

Problem 33.2. Given two linear maps $f\colon E \to E'$ and $g\colon F \to F'$, we defined the unique linear map

$$f \otimes g\colon E \otimes F \to E' \otimes F'$$

by

$$(f \otimes g)(u \otimes v) = f(u) \otimes g(v),$$

for all $u \in E$ and all $v \in F$. See Proposition 33.9. Thus $f \otimes g \in \mathrm{Hom}(E \otimes F, E' \otimes F')$. If we denote the tensor product $E \otimes F$ by $T(E, F)$, and we assume that E, E' and F, F' are finite dimensional, pick bases and show that the map induced by $f \otimes g \mapsto T(f, g)$ is an isomorphism

$$\mathrm{Hom}(E, F) \otimes \mathrm{Hom}(E', F') \cong \mathrm{Hom}(E \otimes F, E' \otimes F').$$

Problem 33.3. Adjust the proof of Proposition 33.13 (2) to show that

$$E \otimes (F \otimes G) \cong E \otimes F \otimes G,$$

whenever E, F, and G are arbitrary vector spaces.

Problem 33.4. Given a fixed vector space G, for any two vector spaces M and N and every linear map $f\colon M \to N$, we defined $\tau_G(f) = f \otimes \mathrm{id}_G$ to be the unique linear map making the following diagram commute.

$$\begin{tikzcd}
M \times G \arrow[r, "\iota_{M\otimes}"] \arrow[d, "f \times \mathrm{id}_G"'] & M \otimes G \arrow[d, "f \otimes \mathrm{id}_G"] \\
N \times G \arrow[r, "\iota_{N\otimes}"'] & N \otimes G
\end{tikzcd}$$

See the proof of Proposition 33.13 (3). Show that

(1) $\tau_G(0) = 0$,

(2) $\tau_G(\mathrm{id}_M) = (\mathrm{id}_M \otimes \mathrm{id}_G) = \mathrm{id}_{M \otimes G}$,

(3) If $f'\colon M \to N$ is another linear map, then $\tau_G(f + f') = \tau_G(f) + \tau_G(f')$.