Part (a) of Theorem 11.4 shows that

$$\dim(E) \leq \dim(E^*).$$

When E is of finite dimension n and $(u_1, \ldots, u_n)$ is a basis of E, by part (c), the family $(u_1^*, \ldots, u_n^*)$ is a basis of the dual space E^*, called the *dual basis* of $(u_1, \ldots, u_n)$. This fact was also proven directly in Theorem 3.23.

Define the function $\mathcal{E}$ ($\mathcal{E}$ for equations) from subspaces of E to subspaces of E^* and the function $\mathcal{Z}$ ($\mathcal{Z}$ for zeros) from subspaces of E^* to subspaces of E by

$$\begin{aligned} \mathcal{E}(V) &= V^0, \quad V \subseteq E \\ \mathcal{Z}(U) &= U^0, \quad U \subseteq E^*. \end{aligned}$$

By Parts (c) and (d) of Theorem 11.4,

$$\begin{aligned} (\mathcal{Z} \circ \mathcal{E})(V) &= V^{00} = V \\ (\mathcal{E} \circ \mathcal{Z})(U) &= U^{00} = U, \end{aligned}$$

so $\mathcal{Z} \circ \mathcal{E} = \mathrm{id}$ and $\mathcal{E} \circ \mathcal{Z} = \mathrm{id}$, and the maps $\mathcal{E}$ and $\mathcal{Z}$ are inverse bijections. These maps set up a *duality* between subspaces of E and subspaces of E^*. In particular, every subspace $V \subseteq E$ of dimension m is the set of common zeros of the space of linear forms (equations) V^0, which has dimension $n - m$. This confirms the claim we made about the dimension of the subpsace defined by a set of linear equations.

One should be careful that this bijection does not extend to subspaces of E^* of infinite dimension.

When E is of infinite dimension, for every basis $(u_i)_{i \in I}$ of E, the family $(u_i^*)_{i \in I}$ of coordinate forms is never a basis of E^*. It is linearly independent, but it is "too small" to generate E^*. For example, if $E = \mathbb{R}^{(\mathbb{N})}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$, the map $f \colon E \to \mathbb{R}$ that sums the nonzero coordinates of a vector in E is a linear form, but it is easy to see that it cannot be expressed as a linear combination of coordinate forms. As a consequence, when E is of infinite dimension, E and E^* are not isomorphic.

We now discuss some applications of the duality theorem.

Problem 1 . Suppose that V is a subspace of $\mathbb{R}^n$ of dimension m and that $(v_1, \ldots, v_m)$ is a basis of V. The problem is to find a basis of V^0.

We first extend $(v_1, \ldots, v_m)$ to a basis $(v_1, \ldots, v_n)$ of $\mathbb{R}^n$, and then by part (c) of Theorem 11.4, we know that $(v_{m+1}^*, \ldots, v_n^*)$ is a basis of V^0.

Example 11.6. For example, suppose that V is the subspace of $\mathbb{R}^4$ spanned by the two linearly independent vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix},$$