(3) A sequence (x_k) of points of Ω defined by

$$x_{k+1} = x_k - (A_k(x_\ell))^{-1}(f(x_k)), \quad k \geq 0, \tag{$**$}$$

where for every integer $k \geq 0$, the integer ℓ satisfies the condition

$$0 \leq \ell \leq k.$$

With $\Delta x_k = x_{k+1} - x_k$, Equation $(**)$ is equivalent to solving the equation

$$A_k(x_\ell)(\Delta x_k) = -f(x_k)$$

and setting $x_{k+1} = x_k + \Delta x_k$. The function $A_k(x)$ usually depends on f'.

Definition 41.1 gives us enough flexibility to capture all the situations that we have previously discussed:

	Function	Index
Variant 1:	$A_k(x) = f'(x)$,	$\ell = k$
Variant 2:	$A_k(x) = f'(x)$,	$\ell = \min\{rp, k\}$, if $rp \leq k \leq (r+1)p - 1, r \geq 0$
Variant 3:	$A_k(x) = f'(x)$,	$\ell = 0$
Variant 4:	$A_k(x) = A_0$,	

where A_0 is a linear isomorphism from X to Y. The first case corresponds to Newton's original method and the others to the variants that we just discussed. We could also have $A_k(x) = A_k$, a fixed linear isomorphism independent of $x \in \Omega$.

Example 41.2. Consider the matrix function f given by

$$f(X) = A - X^{-1},$$

with A and X invertible $n \times n$ matrices. If we apply Variant 1 of Newton's method starting with any $n \times n$ matrix X_0, since the derivative of the function g given by $g(X) = X^{-1}$ is $dg_X(Y) = -X^{-1}YX^{-1}$, we have

$$f'_X(Y) = X^{-1}YX^{-1},$$

so

$$(f'_X)^{-1}(Y) = XYX$$

and the Newton step is

$$X_{k+1} = X_k - (f'_{X_k})^{-1}(f(X_k)) = X_k - X_k(A - X_k^{-1})X_k,$$

which yields the sequence (X_k) with

$$X_{k+1} = X_k(2I - AX_k), \quad k \geq 0.$$

In Problem 41.5, it is shown that Newton's method converges to A^{-1} iff the spectral radius of $I - X_0A$ is strictly smaller than 1, that is, $\rho(I - X_0A) < 1$.