Conversely, assume that $\dim(U \cap V) \geq 1$. Pick a basis $(w_1, \ldots, w_r)$ of $W = U \cap V$, and extend this basis to a basis $(w_1, \ldots, w_r, w_{r+1}, \ldots, w_p)$ of U and to a basis $(w_1, \ldots, w_r, w_{p+1}, \ldots, w_{p+q-r})$ of V. By Corollary 34.26, $(u_1, \ldots, u_p)$ is also basis of U, so

$$u_1 \wedge \cdots \wedge u_p = a\, w_1 \wedge \cdots \wedge w_r \wedge w_{r+1} \wedge \cdots \wedge w_p$$

for some $a \in K$, and $(v_1, \ldots, v_q)$ is also basis of V, so

$$v_1 \wedge \cdots \wedge v_q = b\, w_1 \cdots \wedge w_r \wedge w_{p+1} \wedge \cdots \wedge w_{p+q-r}$$

for some $b \in K$, and thus

$$u \wedge v = u_1 \wedge \cdots \wedge u_p \wedge v_1 \wedge \cdots \wedge v_q = 0$$

since it contains some repeated w_i, with $1 \leq i \leq r$. $\square$

As an application of Proposition 34.31, consider two projective lines D_1 and D_2 in $\mathbb{RP}^3$, which means that D_1 and D_2 correspond to two 2-planes in $\mathbb{R}^4$, and thus by Proposition 34.30, to two points in $\mathbb{RP}^{\binom{4}{2}-1} = \mathbb{RP}^5$. These two points correspond to the 2-vectors

$$z = a_{1,2}e_1 \wedge e_2 + a_{1,3}e_1 \wedge e_3 + a_{1,4}e_1 \wedge e_4 + a_{2,3}e_2 \wedge e_3 + a_{2,4}e_2 \wedge e_4 + a_{3,4}e_3 \wedge e_4$$

and

$$z' = a'_{1,2}e_1 \wedge e_2 + a'_{1,3}e_1 \wedge e_3 + a'_{1,4}e_1 \wedge e_4 + a'_{2,3}e_2 \wedge e_3 + a'_{2,4}e_2 \wedge e_4 + a'_{3,4}e_3 \wedge e_4$$

whose Plücker coordinates, (where $a_{i,j} = \lambda_{ij}$), satisfy the equation

$$\lambda_{12}\lambda_{34} - \lambda_{13}\lambda_{24} + \lambda_{14}\lambda_{23} = 0$$

of the Klein quadric, and D_1 and D_2 intersect iff $z \wedge z' = 0$ iff

$$a_{1,2}a'_{3,4} - a_{1,3}a'_{3,4} + a_{1,4}a'_{2,3} + a_{2,3}a'_{1,4} - a_{2,4}a'_{1,3} + a_{3,4}a'_{1,2} = 0.$$

Observe that for D_1 fixed, this is a linear condition. This fact is very helpful for solving problems involving intersections of lines. A famous problem is to find how many lines in $\mathbb{RP}^3$ meet four given lines in general position. The answer is at most 2.

34.10 Vector-Valued Alternating Forms

The purpose of this section is to present the technical background needed to understand vector-valued differential forms, in particular in the case of Lie groups where differential forms taking their values in a Lie algebra arise naturally.

In this section the vector space E is assumed to have *finite dimension*. We know that there is a canonical isomorphism $\bigwedge^n(E^*) \cong \mathrm{Alt}^n(E; K)$ between alternating n-forms and