where $c_i \in \mathbb{Z}$ for $i = 1, \ldots, n$. The left hand side of the preceding line is equivalent to

$$\frac{c_1 p_1 q_2 \cdots q_n + \cdots + c_n p_n q_1 \cdots q_{n-1}}{q_1 q_2 \cdots q_n},$$

where the numerator is an element of the ideal in $\mathbb{Z}$ spanned by $(c_1, c_2, \cdots, c_n)$. Since $\mathbb{Z}$ is a PID, there exists $a \in \mathbb{Z}$ such that (a) is the ideal spanned by $(c_1, c_2, \cdots, c_n)$. Thus

$$c_1 \frac{p_1}{q_1} + \cdots + c_n \frac{p_n}{q_n} = \frac{ma}{q_1 q_2 \cdots q_n} = \frac{r}{s},$$

where $m \in \mathbb{Z}$. Set

$$\frac{a}{q_1 q_2 \cdots q_n} = \frac{a_1}{b}, \qquad (a_1, b) = 1.$$

Then if $\mathbb{Q}$ was a finitely generated $\mathbb{Z}$-module, we deduce that for all $x \in \mathbb{Q}$

$$x = \frac{r}{s} = m\frac{a_1}{b},$$

whenever a_1/b is a fixed rational number, clearly a contradiction. (In particular let $x = 1/p$ where p is a fixed prime $p > b$. If $ma_1/b = 1/p$, then $ma_1 \in \mathbb{Z}$ with $ma_1 = b_1/p$, an impossiblity since $(b_1, p) = 1$ and $p > b_1$.)

Definition 3.11 can be generalized to rings and yields free modules.

Definition 35.2. Given a commutative ring A and any (nonempty) set I, let $A^{(I)}$ be the subset of the cartesian product A^I consisting of all families $(\lambda_i)_{i\in I}$ with finite support of scalars in A.[2] We define addition and multiplication by a scalar as follows:

$$(\lambda_i)_{i\in I} + (\mu_i)_{i\in I} = (\lambda_i + \mu_i)_{i\in I},$$

and

$$\lambda \cdot (\mu_i)_{i\in I} = (\lambda\mu_i)_{i\in I}.$$

It is immediately verified that addition and multiplication by a scalar are well defined. Thus, $A^{(I)}$ is a module. Furthermore, because families with finite support are considered, the family $(e_i)_{i\in I}$ of vectors e_i, defined such that $(e_i)_j = 0$ if $j \neq i$ and $(e_i)_i = 1$, is clearly a basis of the module $A^{(I)}$. When $I = \{1, \ldots, n\}$, we denote $A^{(I)}$ by A^n. The function $\iota\colon I \to A^{(I)}$, such that $\iota(i) = e_i$ for every $i \in I$, is clearly an injection.

Definition 35.3. An A-module M is *free* iff it has a basis.

The module $A^{(I)}$ is a free module.

All definitions from Section 3.7 apply to modules, linear maps, kernel, image, except the definition of rank, which has to be defined differently. Propositions 3.17, 3.18, 3.19, and

[2] Where A^I denotes the set of all functions from I to A.