$M$ such that $M(i) = k_i$ for every $i$, $1 \leq i \leq n$, is denoted by $k_1 \cdot 1 + \cdots + k_n \cdot n$, or more simply, by $(k_1, \ldots, k_n)$, and $\deg(k_1 \cdot 1 + \cdots + k_n \cdot n) = k_1 + \cdots + k_n$ is the *size* or *degree* of $M$. The set of all multisets over $I$ is denoted by $\mathbb{N}^{(I)}$, and when $I = \{1, \ldots, n\}$, by $\mathbb{N}^{(n)}$.

Intuitively, the order of the elements of a multiset is irrelevant, but the multiplicity of each element is relevant, contrary to sets. Every $i \in I$ is identified with the multiset $M_i$ such that $M_i(i) = 1$ and $M_i(j) = 0$ for $j \neq i$. When $I = \{1\}$, the set $\mathbb{N}^{(1)}$ of multisets $k \cdot 1$ can be identified with $\mathbb{N}$ and $\{1\}^*$. We will denote $k \cdot 1$ simply by $k$.

However, beware that when $n \geq 2$, the set $\mathbb{N}^{(n)}$ of multisets cannot be identified with the set of strings in $\{1, \ldots, n\}^*$, because multiset union is commutative, but concatenation of strings in $\{1, \ldots, n\}^*$ is not commutative when $n \geq 2$. This is because in a multiset $k_1 \cdot 1 + \cdots + k_n \cdot n$, the order is irrelevant, whereas in a string, the order is relevant. For example, $2 \cdot 1 + 3 \cdot 2 = 3 \cdot 2 + 2 \cdot 1$, but $11222 \neq 22211$, as strings over $\{1, 2\}$.

Nevertheless, $\mathbb{N}^{(n)}$ and the set $\mathbb{N}^n$ of ordered $n$-tuples under component-wise addition are isomorphic under the map

$$k_1 \cdot 1 + \cdots + k_n \cdot n \mapsto (k_1, \ldots, k_n).$$

Thus, since the notation $(k_1, \ldots, k_n)$ is less cumbersome that $k_1 \cdot 1 + \cdots + k_n \cdot n$, it will be preferred. We just have to remember that the order of the $k_i$ is really irrelevant.

But when $I$ is infinite, beware that $\mathbb{N}^{(I)}$ and the set $\mathbb{N}^I$ of ordered $I$-tuples are not isomorphic.

We are now ready to define polynomials.

## 30.2 Polynomials

We begin with polynomials in one variable.

**Definition 30.2.** Given a ring $A$, we define the set $\mathcal{P}_A(1)$ of *polynomials over $A$ in one variable* as the set of functions $P \colon \mathbb{N} \to A$ such that $P(k) \neq 0$ for finitely many $k \in \mathbb{N}$. The polynomial such that $P(k) = 0$ for all $k \in \mathbb{N}$ is the *null (or zero) polynomial* and it is denoted by 0. We define addition of polynomials, multiplication by a scalar, and multiplication of polynomials, as follows: Given any three polynomials $P, Q, R \in \mathcal{P}_A(1)$, letting $a_k = P(k)$, $b_k = Q(k)$, and $c_k = R(k)$, for every $k \in \mathbb{N}$, we define $R = P + Q$ such that

$$c_k = a_k + b_k,$$

$R = \lambda P$ such that

$$c_k = \lambda a_k,$$

where $\lambda \in A$,