and the function $(\mu, \nu) \mapsto u_{\mu,\nu}$ is continuous (on $\mathbb{R}^m_+ \times \mathbb{R}^p$). Then the function G is differentiable for all $\mu \in \mathbb{R}^m_+$ and all $\nu \in \mathbb{R}^p$, and

$$G'_{\mu,\nu}(\xi, \zeta) = \sum_{i=1}^{m} \xi_i \varphi_i(u_{\mu,\nu}) + \sum_{j=1}^{p} \zeta_j \psi_j(u_{\mu,\nu}) \quad \text{for all } \xi \in \mathbb{R}^m \text{ and all } \zeta \in \mathbb{R}^p \ .$$

If (λ, η) is any solution of Problem (D):

$$\begin{aligned} &\text{maximize} \quad G(\mu, \nu) \\ &\text{subject to} \quad \mu \in \mathbb{R}^m_+,\ \nu \in \mathbb{R}^p, \end{aligned}$$

then the solution $u_{\lambda,\eta}$ of the corresponding Problem $(P_{\lambda,\eta})$ is a solution of Problem (P').

(2) Assume Problem (P') has some solution $u \in U$, and that Ω is convex (open), the functions φ_i $(1 \leq i \leq m)$ and J are convex, differentiable at u, and that the constraints are qualified. Then Problem (D') has a solution $(\lambda, \eta) \in \mathbb{R}^m_+ \times \mathbb{R}^p$, and $J(u) = G(\lambda, \eta)$; that is, the duality gap is zero.

In the next section we derive the dual function and the dual program of the optimization problem of Section 50.6 (Hard margin SVM), which involves both inequality and equality constraints. We also derive the KKT conditions associated with the dual program.

50.10 Dual of the Hard Margin Support Vector Machine

Recall the **Hard margin SVM** problem (SVM_{h2}):

$$\begin{aligned} &\text{minimize} \quad \frac{1}{2}\|w\|^2, \qquad w \in \mathbb{R}^n \\ &\text{subject to} \\ &\qquad w^\top u_i - b \geq 1 \qquad i = 1, \ldots, p \\ &\qquad - w^\top v_j + b \geq 1 \qquad j = 1, \ldots, q. \end{aligned}$$

We proceed in six steps.

Step 1: Write the constraints in matrix form.

The inequality constraints are written as

$$C \begin{pmatrix} w \\ b \end{pmatrix} \leq d,$$