The last part of Proposition 28.1 shows that the Cartan–Dieudonné is salvaged, since we can send u to v by a sequence of two Hermitian reflections when $u \neq v$ and $\|u\| = \|v\|$, and since the inverse of a Hermitian reflection is a Hermitian reflection. Actually, because we are over the complex field, a linear map always have (complex) eigenvalues, and we can get a slightly improved result.

Theorem 28.2. *Let E be a Hermitian space of dimension $n \geq 1$. Every isometry $f \in \mathbf{U}(E)$ is the composition $f = \rho_n \circ \rho_{n-1} \circ \cdots \circ \rho_1$ of n isometries ρ_j, where each ρ_j is either the identity or a Hermitian reflection (possibly a standard hyperplane reflection). When $n \geq 2$, the identity is the composition of any hyperplane reflection with itself.*

Proof. We prove by induction on n that there is an orthonormal basis of eigenvectors $(u_1, \ldots, u_n)$ of f such that

$$f(u_j) = e^{i\theta_j} u_j,$$

where $e^{i\theta_j}$ is an eigenvalue associated with u_j, for all j, $1 \leq j \leq n$.

When $n = 1$, every isometry $f \in \mathbf{U}(E)$ is either the identity or a Hermitian reflection ρ_θ, since for any nonnull vector u, we have $f(u) = e^{i\theta} u$ for some θ. We let u_1 be any nonnull unit vector.

Let us now consider the case where $n \geq 2$. Since $\mathbb{C}$ is algebraically closed, the characteristic polynomial $\det(f - \lambda \mathrm{id})$ of f has n complex roots which must be the form $e^{i\theta}$, since they have absolute value 1. Pick any such eigenvalue $e^{i\theta_1}$, and pick any eigenvector $u_1 \neq 0$ of f for $e^{i\theta_1}$ of unit length. If $F = \mathbb{C}u_1$ is the subspace spanned by u_1, we have $f(F) = F$, since $f(u_1) = e^{i\theta_1} u_1$. Since $f(F) = F$ and f is an isometry, it is easy to see that $f(F^\perp) \subseteq F^\perp$, and by Proposition 14.13, we have $E = F \oplus F^\perp$. Furthermore, it is obvious that the restriction of f to $F^\perp$ is unitary. Since $\dim(F^\perp) = n - 1$, we can apply the induction hypothesis to $F^\perp$, and we get an orthonormal basis of eigenvectors $(u_2, \ldots, u_n)$ for $F^\perp$ such that

$$f(u_j) = e^{i\theta_j} u_j,$$

where $e^{i\theta_j}$ is an eigenvalue associated with u_j, for all j, $2 \leq j \leq n$ Since $E = F \oplus F^\perp$ and $F = \mathbb{C}u_1$, the claim is proved. But then, if ρ_j is the Hermitian reflection about the hyperplane H_j orthogonal to u_j and of angle θ_j, it is obvious that

$$f = \rho_{\theta_n} \circ \cdots \circ \rho_{\theta_1}.$$

When $n \geq 2$, we have $\mathrm{id} = s \circ s$ for every reflection s. ∎

Remarks:

(1) Any isometry $f \in \mathbf{U}(n)$ can be express as $f = \rho_\theta \circ g$, where $g \in \mathbf{SU}(n)$ is a rotation, and ρ_θ is a Hermitian reflection. Indeed, by the above theorem, with respect to the basis $(u_1, \ldots, u_n)$, $\det(f) = e^{i(\theta_1 + \cdots + \theta_n)}$, and letting $\theta = \theta_1 + \cdots + \theta_n$ and ρ_θ be the Hermitian