and thus

$$\|u - w\| \leq d + \epsilon.$$

Since the above holds for every $\epsilon > 0$, we have $\|u - w\| = d$. Thus, $w \in X_n$ for all $n \geq 1$, which proves that $\bigcap_{n \geq 1} X_n = \{w\}$. Now any $z \in X$ such that $\|u - z\| = d(u, X) = d$ also belongs to every X_n, and thus $z = w$, proving the uniqueness of w, which we denote as $p_X(u)$. See Figure 48.4.

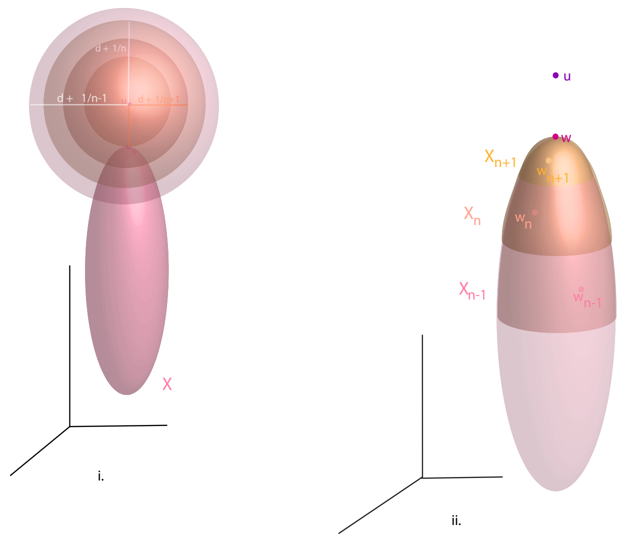

Figure 48.4: Let X be the solid pink ellipsoid with $p_X(u) = w$ at its apex. Each X_n is the intersection of X and a solid sphere centered at u with radius $d + 1/n$. These intersections are the colored "caps" of Figure ii. The Cauchy sequence $(w_n)_{n \geq 1}$ is obtained by selecting a point in each colored X_n.

(2) Let $z \in X$. Since X is convex, $v = (1 - \lambda)p_X(u) + \lambda z \in X$ for every λ, $0 \leq \lambda \leq 1$. Then by the definition of u, we have

$$\|u - v\| \geq \|u - p_X(u)\|$$

for all λ, $0 \leq \lambda \leq 1$, and since

$$\begin{aligned}\|u - v\|^2 &= \|u - p_X(u) - \lambda(z - p_X(u))\|^2 \\ &= \|u - p_X(u)\|^2 + \lambda^2\|z - p_X(u)\|^2 - 2\lambda\Re\,\langle u - p_X(u), z - p_X(u)\rangle\,,\end{aligned}$$

for all λ, $0 < \lambda \leq 1$, we get

$$\Re\,\langle u - p_X(u), z - p_X(u)\rangle = \frac{1}{2\lambda}\left(\|u - p_X(u)\|^2 - \|u - v\|^2\right) + \frac{\lambda}{2}\|z - p_X(u)\|^2. \qquad (\dagger)$$

Since

$$\|u - v\| \geq \|u - p_X(u)\|,$$