Pick any nonzero $w \in C^*(u)$, which means that $(\varphi_i')_u(w) \leq 0$ for all $i \in I(u)$. For any sequence $(\epsilon_k)_{k\geq 0}$ of reals $\epsilon_k > 0$ such that $\lim_{k\mapsto\infty} \epsilon_k = 0$, let $(u_k)_{k\geq 0}$ be the sequence of vectors in V given by

$$u_k = u + \epsilon_k w.$$

We have $u_k - u = \epsilon_k w \neq 0$ for all $k \geq 0$ and $\lim_{k\mapsto\infty} u_k = u$. Furthermore, since the functions φ_i are continuous for all $i \notin I$, we have

$$0 > \varphi_i(u) = \lim_{k\mapsto\infty} \varphi_i(u_k),$$

and since φ_i is affine and $\varphi_i(u) = 0$ for all $i \in I$, we have $\varphi_i(u) = h_i(u) + c_i = 0$, so

$$\varphi_i(u_k) = h_i(u_k) + c_i = h_i(u_k) - h_i(u) = h_i(u_k - u) = (\varphi_i')_u(u_k - u) = \epsilon_k(\varphi_i')_u(w) \leq 0, \quad (*_0)$$

which implies that $u_k \in U$ for all k large enough. Since

$$\frac{u_k - u}{\|u_k - u\|} = \frac{w}{\|w\|} \quad \text{for all } k \geq 0,$$

we conclude that $w \in C(u)$. See Figure 50.9.

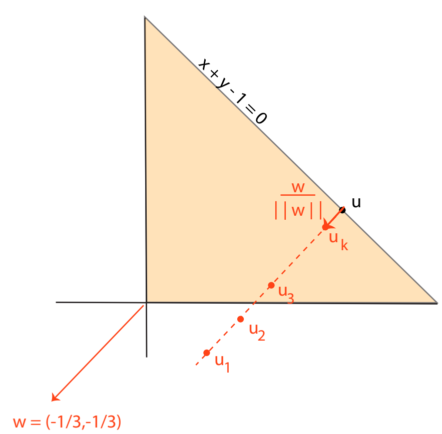

Figure 50.9: Let U be the peach triangle bounded by the lines $y = 0$, $x = 0$, and $y = -x + 1$. Let u satisfy the affine constraint $\varphi(x, y) = y + x - 1$. Since $\varphi'_{(x,y)} = (1\ 1)$, set $w = (-1, -1)$ and approach u along the line $u + tw$.

(2)(b) Let us now consider the case where some function φ_i is not affine for some $i \in I(u)$. Let $w \neq 0$ be some vector in V such that Condition (b) of Definition 50.5 holds, namely: for all $i \in I(u)$, we have

(i) $(\varphi_i')_u(w) \leq 0$.

(ii) If φ_i is not affine, then $(\varphi_i')_u(w) < 0$.