53.6 Problems

Problem 53.1. Referring back to Example 53.3, prove that if $\varphi_1 \colon X \to \mathbb{R}^{n_1}$ and $\varphi_2 \colon X \to \mathbb{R}^{n_2}$ are two feature maps and if $\kappa_1(x,y) = \langle \varphi_1(x), \varphi_1(y) \rangle$ and $\kappa_2(x,y) = \langle \varphi_2(x), \varphi_2(y) \rangle$ are the corresponding kernel functions, then the map defined by

$$\kappa(x,y) = \kappa_1(x,y)\kappa_2(x,y)$$

is a kernel function, for the feature space $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and the feature map

$$\varphi(x)_{(i,j)} = (\varphi_1(x))_i (\varphi_2(x))_j, \qquad 1 \leq i \leq n_1,\ 1 \leq j \leq n_2.$$

Problem 53.2. Referring back to Example 53.3, prove that the feature embedding $\varphi \colon X \to \mathbb{R}^{\binom{n+m-1}{m}}$ given by

$$\varphi_{(i_1,\ldots,i_n)}(x) = \binom{m}{i_1 \cdots i_n}^{1/2} (\varphi_1(x))_1^{i_1} (\varphi_1(x))_1^{i_2} \cdots (\varphi_1(x))_1^{i_n}, \quad i_1 + i_2 + \cdots + i_n = m,\ i_j \in \mathbb{N},$$

where the n-tuples $(i_1, \ldots, i_n)$ are ordered lexicographically, defines the kernel function κ given by $\kappa(x,y) = (\kappa_1(x,y))^m$.

Problem 53.3. In Example 53.6, prove that for any two subsets A_1 and A_2 of D,

$$\langle \varphi(A_1), \varphi(A_2) \rangle = 2^{|A_1 \cap A_2|},$$

the number of common subsets of A_1 and A_2.

Problem 53.4. Prove that the pointwise limit of positive definite kernels is also a positive definite kernel.

Problem 53.5. Prove that if κ_1 is a positive definite kernel, then

$$\kappa(x,y) = e^{-\frac{\kappa_1(x,x) + \kappa_1(y,y) - 2\kappa_1(x,y)}{2\sigma^2}}$$

is a positive definite kernel.