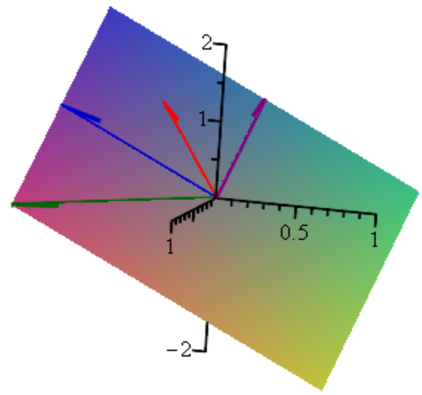

Figure 3.10: The subspace $x+y+z=0$ is the plane through the origin with normal $(1,1,1)$.

3. For any $n \geq 0$, the set of polynomials $f(X) \in \mathbb{R}[X]$ of degree at most n is a subspace of $\mathbb{R}[X]$.

4. The set of upper triangular $n \times n$ matrices is a subspace of the space of $n \times n$ matrices.

Proposition 3.5. *Given any vector space E, if S is any nonempty subset of E, then the smallest subspace $\langle S \rangle$ (or* Span(S)*) of E containing S is the set of all (finite) linear combinations of elements from S.*

Proof. We prove that the set $\mathrm{Span}(S)$ of all linear combinations of elements of S is a subspace of E, leaving as an exercise the verification that every subspace containing S also contains $\mathrm{Span}(S)$.

First, $\mathrm{Span}(S)$ is nonempty since it contains S (which is nonempty). If $u = \sum_{i\in I} \lambda_i u_i$ and $v = \sum_{j\in J} \mu_j v_j$ are any two linear combinations in $\mathrm{Span}(S)$, for any two scalars $\lambda, \mu \in K$,

$$\begin{aligned}
\lambda u + \mu v &= \lambda \sum_{i\in I} \lambda_i u_i + \mu \sum_{j\in J} \mu_j v_j \\
&= \sum_{i\in I} \lambda\lambda_i u_i + \sum_{j\in J} \mu\mu_j v_j \\
&= \sum_{i\in I-J} \lambda\lambda_i u_i + \sum_{i\in I\cap J} (\lambda\lambda_i + \mu\mu_i) u_i + \sum_{j\in J-I} \mu\mu_j v_j,
\end{aligned}$$

which is a linear combination with index set $I \cup J$, and thus $\lambda u + \mu v \in \mathrm{Span}(S)$, which proves that $\mathrm{Span}(S)$ is a subspace. $\square$

One might wonder what happens if we add extra conditions to the coefficients involved in forming linear combinations. Here are three natural restrictions which turn out to be important (as usual, we assume that our index sets are finite):