(6) Solve the system

$$\begin{pmatrix} 100 & 99 \\ 99 & 98 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 199 \\ 197 \end{pmatrix}.$$

Perturb the right-hand side b by

$$\Delta b = \begin{pmatrix} -0.0097 \\ 0.0106 \end{pmatrix}$$

and solve the new system

$$A_m y = b + \Delta b$$

where $y = (y_1, y_2)$. Check that

$$\Delta x = y - x = \begin{pmatrix} 2 \\ -2.0203 \end{pmatrix}.$$

Compute $\|x\|_2$, $\|\Delta x\|_2$, $\|b\|_2$, $\|\Delta b\|_2$, and estimate

$$c = \frac{\|\Delta x\|_2}{\|x\|_2} \left(\frac{\|\Delta b\|_2}{\|b\|_2} \right)^{-1}.$$

Check that

$$c \approx \mathrm{cond}_2(A_m) \approx 39,206.$$

Problem 9.15. Consider a real 2×2 matrix with zero trace of the form

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}.$$

(1) Prove that

$$A^2 = (a^2 + bc) I_2 = -\det(A) I_2.$$

If $a^2 + bc = 0$, prove that

$$e^A = I_2 + A.$$

(2) If $a^2 + bc < 0$, let $\omega > 0$ be such that $\omega^2 = -(a^2 + bc)$. Prove that

$$e^A = \cos\omega\, I_2 + \frac{\sin\omega}{\omega} A.$$

(3) If $a^2 + bc > 0$, let $\omega > 0$ be such that $\omega^2 = a^2 + bc$. Prove that

$$e^A = \cosh\omega\, I_2 + \frac{\sinh\omega}{\omega} A.$$

(3) Prove that in all cases

$$\det\left(e^A\right) = 1 \quad \text{and} \quad \mathrm{tr}(A) \geq -2.$$

(4) Prove that there exist some real 2×2 matrix B with $\det(B) = 1$ such that there is no real 2×2 matrix A with zero trace such that $e^A = B$.