If $\langle x_0, y\rangle = \rho e^{i\theta}$, with $\rho \geq 0$, then

$$|\langle e^{-i\theta} x_0, y\rangle| = |e^{-i\theta}\langle x_0, y\rangle| = |e^{-i\theta}\rho e^{i\theta}| = \rho,$$

so

$$\|y\|^D = \rho = \langle e^{-i\theta} x_0, y\rangle, \tag{$*$}$$

with $\left\|e^{-i\theta} x_0\right\| = \|x_0\| = 1$. On the other hand,

$$\Re\langle x, y\rangle \leq |\langle x, y\rangle|,$$

so by $(*)$ we get

$$\|y\|^D = \sup_{\substack{x\in E\\ \|x\|=1}} |\langle x, y\rangle| = \sup_{\substack{x\in E\\ \|x\|=1}} \Re\langle x, y\rangle,$$

as claimed. □

Proposition 14.29. *For all $x, y \in E$, we have*

$$\begin{aligned}|\langle x, y\rangle| &\leq \|x\| \, \|y\|^D\\ |\langle x, y\rangle| &\leq \|x\|^D \, \|y\| .\end{aligned}$$

Proof. If $x = 0$, then $\langle x, y\rangle = 0$ and these inequalities are trivial. If $x \neq 0$, since $\|x/\|x\|\| = 1$, by definition of $\|y\|^D$, we have

$$|\langle x/\|x\| , y\rangle| \leq \sup_{\|z\|=1} |\langle z, y\rangle| = \|y\|^D ,$$

which yields

$$|\langle x, y\rangle| \leq \|x\| \, \|y\|^D .$$

The second inequality holds because $|\langle x, y\rangle| = |\langle y, x\rangle|$. □

It is not hard to show that for all $y \in \mathbb{C}^n$,

$$\begin{aligned}\|y\|_1^D &= \|y\|_\infty\\ \|y\|_\infty^D &= \|y\|_1\\ \|y\|_2^D &= \|y\|_2 .\end{aligned}$$

Thus, the Euclidean norm is autodual. More generally, the following proposition holds.

Proposition 14.30. *If $p, q \geq 1$ and $1/p + 1/q = 1$, or $p = 1$ and $q = \infty$, or $p = \infty$ and $q = 1$, then for all $y \in \mathbb{C}^n$, we have*

$$\|y\|_p^D = \|y\|_q .$$