We still need to pick $y \in D$ so that $v_1 = v' + y$ satisfies $\varphi(v_1, v_1) = \varphi(v, v)$. However, since $v \notin U = D^{\perp}$, the vector v is not orthogonal D, and by Lemma 29.28, there is some $y_0 \in D$ such that

$$\varphi(v' + y_0, v' + y_0) = \varphi(v, v).$$

Then, if we let $v_1 = v' + y_0$, by Proposition 29.44, we can extend f to a metric map g of $U + Kv = E$ by setting $g(v) = v_1$. Since φ is nondegenerate, g is an isometry. $\square$

The first corollary of Witt's theorem is sometimes called the Witt's cancellation theorem.

Theorem 29.46. *(Witt Cancellation Theorem) Let (E_1, φ_1) and (E_2, φ_2) be two pairs of finite-dimensional spaces and nondegenerate ϵ-Hermitian forms satisfying condition (T), and assume that (E_1, φ_1) and (E_2, φ_2) are isometric. For any subspace U of E_1 and any subspace V of E_2, if there is an isometry $f \colon U \to V$, then there is an isometry $g \colon U^{\perp} \to V^{\perp}$.*

Proof. If $f \colon U \to V$ is an isometry between U and V, by Witt's theorem (Theorem 29.46), the linear map f extends to an isometry g between E_1 and E_2. We claim that g maps $U^{\perp}$ into $V^{\perp}$. This is because if $v \in U^{\perp}$, we have $\varphi_1(u, v) = 0$ for all $u \in U$, so

$$\varphi_2(g(u), g(v)) = \varphi_1(u, v) = 0 \quad \text{for all } u \in U,$$

and since g is a bijection between U and V, we have $g(U) = V$, so we see that $g(v)$ is orthogonal to V for every $v \in U^{\perp}$; that is, $g(U^{\perp}) \subseteq V^{\perp}$. Since g is a metric map and since φ_1 is nondegenerate, the restriction of g to $U^{\perp}$ is an isometry from $U^{\perp}$ to $V^{\perp}$. $\square$

A pair (E, φ) where E is finite-dimensional and φ is a nondegenerate ϵ-Hermitian form is often called an *ϵ-Hermitian space*. When $\epsilon = 1$ and φ is symmetric, we use the term *Euclidean* space or *quadratic* space. When $\epsilon = -1$ and φ is alternating, we use the term *symplectic* space. When $\epsilon = 1$ and the automorphism $\lambda \mapsto \overline{\lambda}$ is not the identity we use the term *Hermitian* space, and when $\epsilon = -1$, we use the term *skew-Hermitian* space.

We also have the following result showing that the group of isometries of an ϵ-Hermitian space is transitive on totally isotropic subspaces of the same dimension.

Theorem 29.47. *Let E be a finite-dimensional vector space and let φ be a nondegenerate ϵ-Hermitian form on E satisfying condition (T). Then for any two totally isotropic subspaces U and V of the same dimension, there is an isometry $f \in \mathbf{Isom}(\varphi)$ such that $f(U) = V$. Furthermore, every linear automorphism of U is induced by an isometry of E.*

Remark: Witt's cancelation theorem can be used to define an equivalence relation on ϵ-Hermitian spaces and to define a group structure on these equivalence classes. This way, we obtain the *Witt group*, but we will not discuss it here.

Witt's Theorem can be sharpened to isometries in $\mathbf{SO}(\varphi)$, but some condition on U is needed.