Multiply the differential equation by any arbitrary *test function* $v \in V$, obtaining

$$-u''(x)v(x) + c(x)u(x)v(x) = f(x)v(x), \tag{$*$}$$

and integrate this equation! We get

$$-\int_0^1 u''(x)v(x)dx + \int_0^1 c(x)u(x)v(x)dx = \int_0^1 f(x)v(x)dx. \tag{$\dagger$}$$

Now, the trick is to use integration by parts on the first term. Recall that

$$(u'v)' = u''v + u'v',$$

and to be careful about discontinuities, write

$$\int_0^1 u''(x)v(x)dx = \sum_{i=0}^{N} \int_{x_i}^{x_{i+1}} u''(x)v(x)dx.$$

Using integration by parts, we have

$$\begin{aligned}\int_{x_i}^{x_{i+1}} u''(x)v(x)dx &= \int_{x_i}^{x_{i+1}} (u'(x)v(x))'dx - \int_{x_i}^{x_{i+1}} u'(x)v'(x)dx \\ &= [u'(x)v(x)]_{x=x_i}^{x=x_{i+1}} - \int_{x_i}^{x_{i+1}} u'(x)v'(x)dx \\ &= u'(x_{i+1})v(x_{i+1}) - u'(x_i)v(x_i) - \int_{x_i}^{x_{i+1}} u'(x)v'(x)dx.\end{aligned}$$

It follows that

$$\begin{aligned}\int_0^1 u''(x)v(x)dx &= \sum_{i=0}^{N} \int_{x_i}^{x_{i+1}} u''(x)v(x)dx \\ &= \sum_{i=0}^{N} \left(u'(x_{i+1})v(x_{i+1}) - u'(x_i)v(x_i) - \int_{x_i}^{x_{i+1}} u'(x)v'(x)dx \right) \\ &= u'(1)v(1) - u'(0)v(0) - \int_0^1 u'(x)v'(x)dx.\end{aligned}$$

However, the test function v satisfies the boundary conditions $v(0) = v(1) = 0$ (recall that $v \in V$), so we get

$$\int_0^1 u''(x)v(x)dx = -\int_0^1 u'(x)v'(x)dx.$$

Consequently, the equation ($\dagger$) becomes

$$\int_0^1 u'(x)v'(x)dx + \int_0^1 c(x)u(x)v(x)dx = \int_0^1 f(x)v(x)dx,$$