22.6 Summary

The main concepts and results of this chapter are listed below:

- For any linear map $f\colon E \to E$ on a Euclidean space E, the maps $f^* \circ f$ and $f \circ f^*$ are self-adjoint and positive semidefinite.
- The *singular values* of a linear map.
- *Positive semidefinite* and *positive definite* self-adjoint maps.
- Relationships between $\mathrm{Im}\, f$, $\mathrm{Ker}\, f$, $\mathrm{Im}\, f^*$, and $\mathrm{Ker}\, f^*$.
- The *singular value decomposition theorem* for square matrices (Theorem 22.5).
- The *SVD* of matrix.
- The *polar decomposition* of a matrix.
- The *Weyl inequalities*.
- The *singular value decomposition theorem* for $m \times n$ matrices (Theorem 22.7).
- Ky Fan k-norms, Ky Fan p-k-norms, Schatten p-norms.

22.7 Problems

Problem 22.1. (1) Let A be a real $n \times n$ matrix and consider the $(2n) \times (2n)$ real symmetric matrix

$$S = \begin{pmatrix} 0 & A \\ A^\top & 0 \end{pmatrix}.$$

Suppose that A has rank r. If $A = V \Sigma U^\top$ is an SVD for A, with $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ and $\sigma_1 \geq \cdots \geq \sigma_r > 0$, denoting the columns of U by u_k and the columns of V by v_k, prove that σ_k is an eigenvalue of S with corresponding eigenvector $\begin{pmatrix} v_k \\ u_k \end{pmatrix}$ for $k = 1, \ldots, n$, and that $-\sigma_k$ is an eigenvalue of S with corresponding eigenvector $\begin{pmatrix} v_k \\ -u_k \end{pmatrix}$ for $k = 1, \ldots, n$.

Hint. We have $Au_k = \sigma_k v_k$ for $k = 1, \ldots, n$. Show that $A^\top v_k = \sigma_k u_k$ for $k = 1, \ldots, n$.

(2) Prove that the $2n$ eigenvectors of S in (1) are pairwise orthogonal. Check that if A has rank r, then S has rank $2r$.

(3) Now assume that A is a real $m \times n$ matrix and consider the $(m + n) \times (m + n)$ real symmetric matrix

$$S = \begin{pmatrix} 0 & A \\ A^\top & 0 \end{pmatrix}.$$