and since J is elliptic, for all $u, w \in V$ we can write

$$\begin{aligned}
\langle \nabla^2 J_u(w), w\rangle &= \lim_{\theta \mapsto 0} \frac{\langle \nabla J_{u+\theta w} - \nabla J_u, w\rangle}{\theta} \\
&= \lim_{\theta \mapsto 0} \frac{\langle \nabla J_{u+\theta w} - \nabla J_u, \theta w\rangle}{\theta^2} \\
&\geq \theta \left\| w \right\|^2.
\end{aligned}$$

Conversely, assume that the condition

$$\langle \nabla^2 J_u(w), w\rangle \geq \alpha \left\| w \right\|^2 \quad \text{for all } u, w \in V$$

holds. If we define the function $g \colon V \to \mathbb{R}$ by

$$g(w) = \langle \nabla J_w, v - u\rangle = dJ_w(v-u) = \mathrm{D}_{v-u} J(w),$$

where u and v are fixed vectors in V, then we have

$$dg_{u+\theta(v-u)}(v-u) = \mathrm{D}_{v-u} g(u+\theta(v-u)) = \mathrm{D}_{v-u}\mathrm{D}_{v-u} J(u+\theta(v-u)) = \mathrm{D}^2 J_{u+\theta(v-u)}(v-u, v-u)$$

and we can apply the Taylor–MacLaurin formula (Theorem 39.25 with $m = 0$) to g, and we get

$$\begin{aligned}
\langle \nabla J_v - \nabla J_u, v - u\rangle &= g(v) - g(u) \\
&= dg_{u+\theta(v-u)}(v-u) \qquad (0 < \theta < 1) \\
&= \mathrm{D}^2 J_{u+\theta(v-u)}(v-u, v-u) \\
&= \langle \nabla^2 J_{u+\theta(v-u)}(v-u), v-u\rangle \\
&\geq \alpha \left\| v - u \right\|^2,
\end{aligned}$$

which shows that J is elliptic. $\square$

Corollary 49.9. *If $J \colon \mathbb{R}^n \to \mathbb{R}$ is a quadratic function given by*

$$J(v) = \frac{1}{2}\langle Av, v\rangle - \langle b, v\rangle$$

(where A is a symmetric $n \times n$ matrix and $\langle -, -\rangle$ is the standard Eucidean inner product), then J is elliptic iff A is positive definite.

This a consequence of Theorem 49.8 because

$$\langle \nabla^2 J_u(w), w\rangle = \langle Aw, w\rangle \geq \lambda_1 \left\| w \right\|^2$$

where λ_1 is the smallest eigenvalue of A; see Proposition 17.24 (Rayleigh–Ritz). Note that by Proposition 17.24 (Rayleigh–Ritz), we also have the folllowing corollary.