reasoning shows that f^{-1} restricts to a linear map $f^{-1} \mid N_p$ from N_p to M_p. But, $f \mid M_p$ and $f^{-1} \mid N_p$ are mutual inverses, so M_p and N_p are isomorphic.

Conversely, if $M_p \approx N_p$ for all $p \in P$, by Theorem 35.17, we get an isomorphism between $M = \bigoplus_{p\in P} M_p$ and $N = \bigoplus_{p\in P} N_p$. ☐

In view of Proposition 35.18, the direct sum of Theorem 35.17 in terms of its p-primary components is called the *canonical primary decomposition* of M.

If M is a finitely generated torsion-module, then Theorem 35.17 takes the following form.

Theorem 35.19. *(Primary Decomposition Theorem for finitely generated torsion modules) Let M be a finitely generated torsion-module over a PID A. If $\mathrm{Ann}(M) = (a)$ and if $a = up_1^{n_1} \cdots p_r^{n_r}$ is a factorization of a into prime factors, then M is the finite direct sum*

$$M = \bigoplus_{i=1}^{r} M(p_i^{n_i}).$$

Furthermore, the projection of M over $M(p_i^{n_i})$ is of the form $x \mapsto \gamma_i x$, for some $\gamma_i \in A$.

Proof. This is an immediate consequence of Proposition 35.16. ☐

Theorem 35.19 applies when $A = \mathbb{Z}$. In this case, M is a finitely generated torsion abelian group, and the theorem says that such a group is the direct sum of a finite number of groups whose elements have order some power of a prime number p. In particular, consider the $\mathbb{Z}$-module $\mathbb{Z}/10\mathbb{Z}$ where

$$\mathbb{Z}/10\mathbb{Z} = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}, \overline{7}, \overline{8}, \overline{9}\}.$$

Clearly $\mathbb{Z}/10\mathbb{Z}$ is generated by $\overline{1}$ and $\mathrm{Ann}\,(\mathbb{Z}/10\mathbb{Z}) = 10$. Theorem 35.19 implies that

$$\mathbb{Z}/10\mathbb{Z} = M(2) \oplus M(5),$$

where

$$\begin{aligned} M(2) &= \{\overline{x} \in M \mid 2\overline{x} = \overline{0}\} = \{\overline{0}, \overline{5}\} \\ M(5) &= \{\overline{x} \in M \mid 5\overline{x} = \overline{0}\} = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}, \overline{8}\}. \end{aligned}$$

Theorem 35.17 has several useful corollaries.

Proposition 35.20. *If M is a torsion module over a PID, for every submodule N of M, we have a direct sum*

$$N = \bigoplus_{p\in P} N \cap M_p.$$

Proof. It is easily verified that $N \cap M_p$ is the p-primary component of N. ☐