Similarly, there is a basis of $2^n \times 2^n = 2^{2n}$ vectors h_{ij} ($2^n \times 2^n$ matrices) for the 2D Haar transform, in the sense that for any $2^n \times 2^n$ matrix A, its matrix C of Haar coefficients is given by

$$C = \sum_{i=1}^{2^n} \sum_{j=1}^{2^n} a_{ij} h_{ij}.$$

If the columns of W^{-1} are $w'_1, \ldots, w'_{2^n}$, then

$$h_{ij} = w'_i (w'_j)^\top.$$

We leave it as exercise to compute the bases (w_{ij}) and (h_{ij}) for $n = 2$, and to display the corresponding images using the command `imagesc`.

5.6 Hadamard Matrices

There is another famous family of matrices somewhat similar to Haar matrices, but these matrices have entries $+1$ and -1 (no zero entries).

Definition 5.6. A real $n \times n$ matrix H is a *Hadamard matrix* if $h_{ij} = \pm 1$ for all i, j such that $1 \leq i, j \leq n$ and if

$$H^\top H = nI_n.$$

Thus the columns of a Hadamard matrix are pairwise orthogonal. Because H is a square matrix, the equation $H^\top H = nI_n$ shows that H is invertible, so we also have $HH^\top = nI_n$. The following matrices are example of Hadamard matrices:

$$H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad H_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix},$$

and

$$H_8 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}.$$

A natural question is to determine the positive integers n for which a Hadamard matrix of dimension n exists, but surprisingly this is an *open problem*. The *Hadamard conjecture* is that for every positive integer of the form $n = 4k$, there is a Hadamard matrix of dimension n.

What is known is a necessary condition and various sufficient conditions.