More generally, given any two vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, their *inner product* denoted $x \cdot y$, or $\langle x, y \rangle$, is the number

$$x \cdot y = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \sum_{i=1}^{n} x_i y_i.$$

Inner products play a very important role. First, we quantity

$$\|x\|_2 = \sqrt{x \cdot x} = (x_1^2 + \cdots + x_n^2)^{1/2}$$

is a generalization of the length of a vector, called the *Euclidean norm*, or *ℓ^2-norm*. Second, it can be shown that we have the inequality

$$|x \cdot y| \leq \|x\| \, \|y\| \,,$$

so if $x, y \neq 0$, the ratio $(x \cdot y)/(\|x\| \, \|y\|)$ can be viewed as the cosine of an angle, the angle between x and y. In particular, if $x \cdot y = 0$ then the vectors x and y make the angle $\pi/2$, that is, they are *orthogonal*. The (square) matrices Q that preserve the inner product, in the sense that $\langle Qx, Qy \rangle = \langle x, y \rangle$ for all $x, y \in \mathbb{R}^n$, also play a very important role. They can be thought of as generalized rotations.

Returning to matrices, if A is an $m \times n$ matrix consisting of n columns $A^1, \ldots, A^n$ (in $\mathbb{R}^m$), and B is a $n \times p$ matrix consisting of p columns $B^1, \ldots, B^p$ (in $\mathbb{R}^n$) we can form the p vectors (in $\mathbb{R}^m$)

$$AB^1, \ldots, AB^p.$$

These p vectors constitute the $m \times p$ matrix denoted AB, whose jth column is AB^j. But we know that the ith coordinate of AB^j is the inner product of the ith row of A by the jth column of B,

$$\begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{pmatrix} \cdot \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Thus we have defined a multiplication operation on matrices, namely if $A = (a_{ik})$ is a $m \times n$ matrix and if $B = (b_{jk})$ if $n \times p$ matrix, then their product AB is the $m \times n$ matrix whose entry on the ith row and the jth column is given by the inner product of the ith row of A by the jth column of B,

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Beware that unlike the multiplication of real (or complex) numbers, if A and B are two $n \times n$ matrices, in general, $AB \neq BA$.