Problem 7.4. Prove that if $n \geq 3$, then

$$\det \begin{pmatrix} 1+x_1y_1 & 1+x_1y_2 & \cdots & 1+x_1y_n \\ 1+x_2y_1 & 1+x_2y_2 & \cdots & 1+x_2y_n \\ \vdots & \vdots & \vdots & \vdots \\ 1+x_ny_1 & 1+x_ny_2 & \cdots & 1+x_ny_n \end{pmatrix} = 0.$$

Problem 7.5. Prove that

$$\begin{vmatrix} 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 25 \\ 9 & 16 & 25 & 36 \\ 16 & 25 & 36 & 49 \end{vmatrix} = 0.$$

Problem 7.6. Consider the $n \times n$ symmetric matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 & \ldots & 0 & 0 \\ 2 & 5 & 2 & 0 & \ldots & 0 & 0 \\ 0 & 2 & 5 & 2 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 2 & 5 & 2 & 0 \\ 0 & 0 & \ldots & 0 & 2 & 5 & 2 \\ 0 & 0 & \ldots & 0 & 0 & 2 & 5 \end{pmatrix}.$$

(1) Find an upper-triangular matrix R such that $A = R^\top R$.

(2) Prove that $\det(A) = 1$.

(3) Consider the sequence

$$\begin{aligned} p_0(\lambda) &= 1 \\ p_1(\lambda) &= 1 - \lambda \\ p_k(\lambda) &= (5 - \lambda)p_{k-1}(\lambda) - 4p_{k-2}(\lambda) \quad 2 \leq k \leq n. \end{aligned}$$

Prove that

$$\det(A - \lambda I) = p_n(\lambda).$$

Remark: It can be shown that $p_n(\lambda)$ has n distinct (real) roots and that the roots of $p_k(\lambda)$ separate the roots of $p_{k+1}(\lambda)$.

Problem 7.7. Let B be the $n \times n$ matrix ($n \geq 3$) given by

$$B = \begin{pmatrix} 1 & -1 & -1 & -1 & \cdots & -1 & -1 \\ 1 & -1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & -1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & -1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & -1 & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 & -1 \end{pmatrix}.$$