Proposition 50.11. *The invertibility of the KKT-matrix*

$$\begin{pmatrix} P & A^\top \\ A & 0 \end{pmatrix}$$

is equivalent to the following conditions:

(1) For all $x \in \mathbb{R}^n$, if $Ax = 0$ with $x \neq 0$, then $x^\top Px > 0$; that is, P is positive definite on the kernel of A.

(2) The kernels of A and P only have 0 in common ($(\mathrm{Ker}\, A) \cap (\mathrm{Ker}\, P) = \{0\}$).

(3) There is some $n \times (n-m)$ matrix F such that $\mathrm{Im}(F) = \mathrm{Ker}\,(A)$ and $F^\top PF$ is symmetric positive definite.

(4) There is some symmetric positive semidefinite matrix Q such that $P + A^\top QA$ is symmetric positive definite. In fact, $Q = I$ works.

Proof sketch. Recall from Proposition 6.19 that a square matrix B is invertible iff its kernel is reduced to $\{0\}$; equivalently, for all x, if $Bx = 0$, then $x = 0$. Assume that Condition (1) holds. We have

$$\begin{pmatrix} P & A^\top \\ A & 0 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

iff

$$Pv + A^\top w = 0, \quad Av = 0. \tag{$*$}$$

We deduce that

$$v^\top Pv + v^\top A^\top w = 0,$$

and since

$$v^\top A^\top w = (Av)^\top w = 0w = 0,$$

we obtain $v^\top Pv = 0$. Since Condition (1) holds, because $v \in \mathrm{Ker}\, A$, we deduce that $v = 0$. Then $A^\top w = 0$, but since the $m \times n$ matrix A has rank m, the $n \times m$ matrix $A^\top$ also has rank m, so its columns are linearly independent, and so $w = 0$. Therefore the KKT-matrix is invertible.

Conversely, assume that the KKT-matrix is invertible, yet the assumptions of Condition (1) fail. This means there is some $v \neq 0$ such that $Av = 0$ and $v^\top Pv = 0$. We claim that $Pv = 0$. This is because if P is a symmetric positive semidefinite matrix, then for any v, we have $v^\top Pv = 0$ iff $Pv = 0$.

If $Pv = 0$, then obviously $v^\top Pv = 0$, so assume the converse, namely $v^\top Pv = 0$. Since P is a symmetric positive semidefinite matrix, it can be diagonalized as

$$P = R^\top \Sigma R,$$