and since both $\rho_{v,-\theta}\circ\rho_{u,\theta}$ and $h_{v-u}\circ h_{v-e^{-i\theta}u}$ are the identity on the orthogonal complement of $\{u,v\}$, they are equal. Since we also have

$$\begin{aligned}
h_{u+v}(u) &= -v,\\
h_{u+v}(v) &= -u,\\
h_{u+e^{i\theta}v}(u) &= -e^{i\theta}v,\\
h_{u+e^{i\theta}v}(v) &= -e^{-i\theta}u,
\end{aligned}$$

it is immediately verified that

$$h_{v-u}\circ h_{v-e^{-i\theta}u} = h_{u+v}\circ h_{u+e^{i\theta}v}.$$

□

We will use Proposition 28.3 as follows.

Proposition 28.4. *Let E be a nontrivial Hermitian space, and let $(u_1,\ldots,u_n)$ be some orthonormal basis for E. For any $\theta_1,\ldots,\theta_n$ such that $\theta_1+\cdots+\theta_n=0$, if $f\in\mathbf{U}(n)$ is the isometry defined such that*

$$f(u_j) = e^{i\theta_j}u_j,$$

for all j, $1\leq j\leq n$, then f is a rotation ($f\in\mathbf{SU}(n)$), and

$$\begin{aligned}
f &= \rho_{u_n,\theta_n}\circ\cdots\circ\rho_{u_1,\theta_1}\\
&= \rho_{u_n,-(\theta_1+\cdots+\theta_{n-1})}\circ\rho_{u_{n-1},\theta_1+\cdots+\theta_{n-1}}\circ\cdots\circ\rho_{u_2,-\theta_1}\circ\rho_{u_1,\theta_1}\\
&= h_{u_n-u_{n-1}}\circ h_{u_n-e^{-i(\theta_1+\cdots+\theta_{n-1})}u_{n-1}}\circ\cdots\circ h_{u_2-u_1}\circ h_{u_2-e^{-i\theta_1}u_1}\\
&= h_{u_{n-1}+u_n}\circ h_{u_{n-1}+e^{i(\theta_1+\cdots+\theta_{n-1})}u_n}\circ\cdots\circ h_{u_1+u_2}\circ h_{u_1+e^{i\theta_1}u_2}.
\end{aligned}$$

Proof. It is obvious from the definitions that

$$f = \rho_{u_n,\theta_n}\circ\cdots\circ\rho_{u_1,\theta_1},$$

and since the determinant of f is

$$D(f) = e^{i\theta_1}\cdots e^{i\theta_n} = e^{i(\theta_1+\cdots+\theta_n)}$$

and $\theta_1+\cdots+\theta_n=0$, we have $D(f)=e^0=1$, and f is a rotation. Letting

$$f_k = \rho_{u_k,-(\theta_1+\cdots+\theta_{k-1})}\circ\rho_{u_{k-1},\theta_1+\cdots+\theta_{k-1}}\circ\cdots\circ\rho_{u_3,-(\theta_1+\theta_2)}\circ\rho_{u_2,\theta_1+\theta_2}\circ\rho_{u_2,-\theta_1}\circ\rho_{u_1,\theta_1},$$

we prove by induction on k, $2\leq k\leq n$, that

$$f_k(u_j) = \begin{cases} e^{i\theta_j}u_j & \text{if } 1\leq j\leq k-1,\\ e^{-i(\theta_1+\cdots+\theta_{k-1})}u_k & \text{if } j=k, \text{ and}\\ u_j & \text{if } k+1\leq j\leq n.\end{cases}$$