Each $\mathfrak{b}_i$ is an ideal in $\mathbb{Z}/30\mathbb{Z}$. Furthermore

$$\mathbb{Z}/30\mathbb{Z} = (\mathbb{Z}/30\mathbb{Z})/(2\mathbb{Z}/30\mathbb{Z}) \times (\mathbb{Z}/30\mathbb{Z})/(3\mathbb{Z}/30\mathbb{Z}) \times (\mathbb{Z}/30\mathbb{Z})/(5\mathbb{Z}/30\mathbb{Z}),$$

where

$$e_1 = (1, 0, 0) \to \overline{15}, \qquad e_2 = (0, 1, 0) \to \overline{10}, \qquad e_3 = (0, 0, 1) \to \overline{6},$$

since

$$\begin{gathered} \overline{15}^2 = \overline{15}, \qquad \overline{10}^2 = \overline{10}, \qquad \overline{6}^2 = \overline{6} \\ \overline{15}\,\overline{10} = \overline{15}\,\overline{6} = \overline{10}\,\overline{6} = 0, \qquad \overline{15} + \overline{10} + \overline{6} = \overline{1}. \end{gathered}$$

Note that $\overline{15}$ corresponds to $\overline{1} \in (\mathbb{Z}/30\mathbb{Z})/(2\mathbb{Z}/30\mathbb{Z})$, $\overline{10}$ corresponds to $\overline{1} \in (\mathbb{Z}/30\mathbb{Z})/(3\mathbb{Z}/30\mathbb{Z})$, while $\overline{6}$ corresponds to $\overline{1} \in (\mathbb{Z}/30\mathbb{Z})/(5\mathbb{Z}/30\mathbb{Z})$.

32.3 Noetherian Rings and Hilbert's Basis Theorem

Given a (commutative) ring A (with unit element 1), an ideal $\mathfrak{A} \subseteq A$ is said to be *finitely generated* if there exists a finite set $\{a_1, \ldots, a_n\}$ of elements from $\mathfrak{A}$ so that

$$\mathfrak{A} = (a_1, \ldots, a_n) = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid \lambda_i \in A,\ 1 \leq i \leq n\}.$$

If K is a field, it turns out that every polynomial ideal $\mathfrak{A}$ in $K[X_1, \ldots, X_m]$ is finitely generated. This fact due to Hilbert and known as Hilbert's basis theorem, has very important consequences. For example, in algebraic geometry, one is interested in the zero locus of a set of polyomial equations, i.e., the set, $V(\mathcal{P})$, of n-tuples $(\lambda_1, \ldots, \lambda_n) \in K^n$ so that

$$P_i(\lambda_1, \ldots, \lambda_n) = 0$$

for all polynomials $P_i(X_1, \ldots, X_n)$ in some given family, $\mathcal{P} = (P_i)_{i \in I}$. However, it is clear that

$$V(\mathcal{P}) = V(\mathfrak{A}),$$

where $\mathfrak{A}$ is the ideal generated by $\mathcal{P}$. Then, Hilbert's basis theorem says that $V(\mathfrak{A})$ is actually defined by a *finite* number of polynomials (any set of generators of $\mathfrak{A}$), even if $\mathcal{P}$ is infinite.

The property that every ideal in a ring is finitely generated is equivalent to other natural properties, one of which is the so-called *ascending chain condition*, abbreviated *a.c.c.* Before proving Hilbert's basis theorem, we explore the equivalence of these conditions.

Definition 32.4. Let A be a commutative ring with unit 1. We say that A satisfies the *ascending chain condition*, for short, the *a.c.c*, if for every ascending chain of ideals

$$\mathfrak{A}_1 \subseteq \mathfrak{A}_2 \subseteq \cdots \subseteq \mathfrak{A}_i \subseteq \cdots,$$