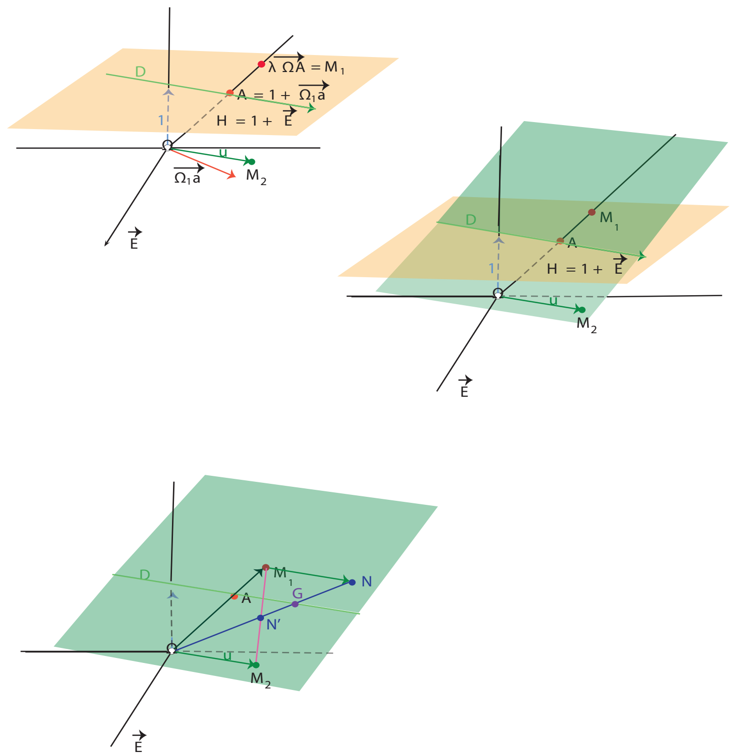

Figure 25.7: The geometric construction of $\langle a, \lambda\rangle + u$.

defined by a universal property are unique up to isomorphism. This property is left as an exercise.

Proposition 25.5. *Given any affine space* $\big(E, \overrightarrow{E}\big)$ *and any vector space* $\overrightarrow{F}$, *for any affine map* $f \colon E \to \overrightarrow{F}$, *there is a unique linear map* $\widehat{f} \colon \widehat{E} \to \overrightarrow{F}$ *extending* f *such that*

$$\widehat{f}(u \mathbin{\widehat{+}} \lambda a) = \lambda f(a) + \overrightarrow{f}(u)$$

for all $a \in E$, *all* $u \in \overrightarrow{E}$, *and all* $\lambda \in \mathbb{R}$, *where* $\overrightarrow{f}$ *is the linear map associated with* f. *In particular, when* $\lambda \neq 0$, *we have*

$$\widehat{f}(u \mathbin{\widehat{+}} \lambda a) = \lambda f(a + \lambda^{-1} u).$$

Proof. Assuming that $\widehat{f}$ exists, recall that from Proposition 25.1, for every $a \in E$, every element of $\widehat{E}$ can be written uniquely as $u \mathbin{\widehat{+}} \lambda a$. By linearity of $\widehat{f}$ and since $\widehat{f}$ extends f, we have

$$\widehat{f}(u \mathbin{\widehat{+}} \lambda a) = \widehat{f}(u) + \lambda \widehat{f}(a) = \widehat{f}(u) + \lambda f(a) = \lambda f(a) + \widehat{f}(u).$$

If $\lambda = 1$, since $a \mathbin{\widehat{+}} u$ and $a + u$ are identified, and since $\widehat{f}$ extends f, we must have

$$f(a) + \widehat{f}(u) = \widehat{f}(a) + \widehat{f}(u) = \widehat{f}(a \mathbin{\widehat{+}} u) = f(a + u) = f(a) + \overrightarrow{f}(u),$$