Chapter 38

A Detour On Fractals

38.1 Iterated Function Systems and Fractals

A pleasant application of the Hausdorff distance and of the fixed point theorem for contracting mappings is a method for defining a class of "self-similar" fractals. For this, we can use iterated function systems.

Definition 38.1. Given a metric space, (X, d), an *iterated function system*, for short, an *ifs*, is a finite sequence of functions, $(f_1, \ldots, f_n)$, where each $f_i \colon X \to X$ is a contracting mapping. A nonempty compact subset, K, of X is an *invariant set (or attractor)* for the ifs, $(f_1, \ldots, f_n)$, if

$$K = f_1(K) \cup \cdots \cup f_n(K).$$

The major result about ifs's is the following:

Theorem 38.1. *If (X, d) is a nonempty complete metric space, then every iterated function system, $(f_1, \ldots, f_n)$, has a unique invariant set, A, which is a nonempty compact subset of X. Furthermore, for every nonempty compact subset, A_0, of X, this invariant set, A, if the limit of the sequence, (A_m), where $A_{m+1} = f_1(A_m) \cup \cdots \cup f_n(A_m)$.*

Proof. Since X is complete, by Theorem 37.55, the space $(\mathcal{K}(X), D)$ is a complete metric space. The theorem will follow from Theorem 37.54 if we can show that the map, $F \colon \mathcal{K}(X) \to \mathcal{K}(X)$, defined such that

$$F(K) = f_1(K) \cup \cdots \cup f_n(K),$$

for every nonempty compact set, K, is a contracting mapping. Let A, B be any two nonempty compact subsets of X and consider any $\eta \geq D(A, B)$. Since each $f_i \colon X \to X$ is a contracting mapping, there is some λ_i, with $0 \leq \lambda_i < 1$, such that

$$d(f_i(a), f_i(b)) \leq \lambda_i d(a, b),$$