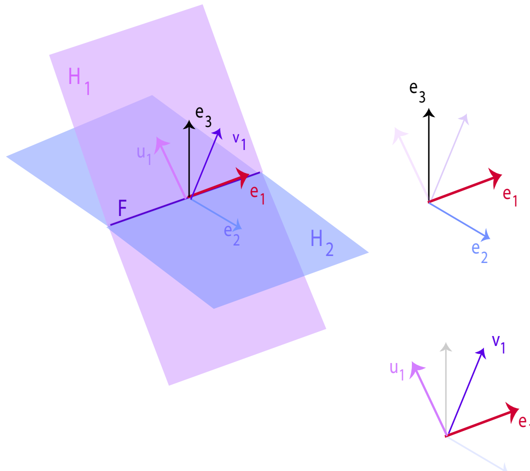

Figure 27.6: An illustration of the hyperplanes H_1, H_2, their intersection F, and the two orthonormal basis utilized in the proof of Proposition 27.4.

As a consequence, the matrix A_1 of h_1 over the basis $(e_1, \ldots, e_n)$ is of the form

$$A_1 = \begin{pmatrix} I_{n-2} & 0 & 0 \\ 0 & \cos 2\theta_1 & \sin 2\theta_1 \\ 0 & \sin 2\theta_1 & -\cos 2\theta_1 \end{pmatrix}.$$

Similarly, the matrix A_2 of h_2 over the basis $(e_1, \ldots, e_n)$ is of the form

$$A_2 = \begin{pmatrix} I_{n-2} & 0 & 0 \\ 0 & \cos 2\theta_2 & \sin 2\theta_2 \\ 0 & \sin 2\theta_2 & -\cos 2\theta_2 \end{pmatrix}.$$

Observe that both A_1 and A_2 have the eigenvalues -1 and $+1$ with multiplicity $n-1$. The trick is to observe that if we change the last entry in I_{n-2} from $+1$ to -1 (which is possible since $n \geq 3$), we have the following product A_2A_1:

$$\begin{pmatrix} I_{n-3} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \cos 2\theta_2 & \sin 2\theta_2 \\ 0 & 0 & \sin 2\theta_2 & -\cos 2\theta_2 \end{pmatrix} \begin{pmatrix} I_{n-3} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \cos 2\theta_1 & \sin 2\theta_1 \\ 0 & 0 & \sin 2\theta_1 & -\cos 2\theta_1 \end{pmatrix}.$$

Now, the two matrices above are clearly orthogonal, and they have the eigenvalues $-1, -1$, and $+1$ with multiplicity $n-2$, which implies that the corresponding isometries leave invariant a subspace of dimension $n-2$ and act as $-\mathrm{id}$ on its orthogonal complement (which has dimension 2). This means that the above two matrices represent two flips f_1 and f_2 such that $h_2 \circ h_1 = f_2 \circ f_1$. See Figure 27.7. $\square$