and if we pick

$$t = \min_{j \in J} \left(-\frac{\lambda_j}{\mu_j} \right) \geq 0,$$

we have $(\lambda_i + t\mu_i) \geq 0$ for $i = 1, \ldots, k$, but $\lambda_j + t\mu_j = 0$ for some $j \in J$, so $(*_3)$ is an expression of x with less that k nonzero coefficients, contradicting the minimality of k in $(*_1)$. Therefore, $(a_1, \ldots, a_k)$ are linearly independent.

Since a polyhedral cone C is spanned by finitely many vectors, there are finitely many primitive cones (corresponding to linearly independent subfamilies), and since every $x \in C$, belongs to some primitive cone, C is the union of a finite number of primitive cones. Since every primitive cone is closed, as a union of finitely many closed sets, C itself is closed.

The above facts are also proven in Matousek and Gardner [123] (Chapter 6, Section 5, Lemma 6.5.3, 6.5.4, and 6.5.5). $\square$

Another way to prove that a polyhedral cone C is closed is to show that C is also a $\mathcal{H}$-polyhedron. This takes even more work; see Gallier [73] (Chapter 4, Section 4, Proposition 4.16). Yet another proof is given in Lax [113] (Chapter 13, Theorem 1).

44.4 Summary

The main concepts and results of this chapter are listed below:

- Affine combination.
- Affine hull.
- Affine subspace; direction of an affine subspace, dimension of an affine subspace.
- Convex combination.
- Convex set, dimension of a convex set.
- Convex hull.
- Affine form.
- Affine hyperplane, half-spaces.
- Cone, polyhedral cone.
- $\mathcal{H}$-polyhedron, $\mathcal{H}$-polytope.
- $\mathcal{V}$-polyhedron, polytope.
- Primitive cone.