Another way (due to Jean Dieudonné) to prove that a section s of ρ is not a homomorphism is to prove that any unit quaternion is the product of two unit pure quaternions. Indeed, if $q = [a, u]$ is a unit quaternion, if we let $q_1 = [0, u_1]$, where u_1 is any unit vector orthogonal to u, then

$$q_1 q = [-u_1 \cdot u, a u_1 + u_1 \times u] = [0, a u_1 + u_1 \times u] = q_2$$

is a nonzero unit pure quaternion. This is because if $a \neq 0$ then $a u_1 + u_1 \times u \neq 0$ (since $u_1 \times u$ is orthogonal to $a u_1 \neq 0$), and if $a = 0$ then $u \neq 0$, so $u_1 \times u \neq 0$ (since u_1 is orthogonal to u). But then, $q_1^{-1} = [0, -u_1]$ is a unit pure quaternion and we have

$$q = q_1^{-1} q_2,$$

a product of two pure unit quaternions.

We also observe that for any two pure quaternions q_1, q_2, there is some unit quaternion q such that

$$q_2 = q q_1 q^{-1}.$$

This is just a restatement of the fact that the group $\mathbf{SO}(3)$ is transitive. Since the kernel of $\rho\colon \mathbf{SU}(2) \to \mathbf{SO}(3)$ is $\{I, -I\}$, the subgroup $s(\mathbf{SO}(3))$ would be a normal subgroup of index 2 in $\mathbf{SU}(2)$. Then we would have a surjective homomorphism η from $\mathbf{SU}(2)$ onto the quotient group $\mathbf{SU}(2)/s(\mathbf{SO}(3))$, which is isomorphic to $\{1, -1\}$. Now, since any two pure quaternions are conjugate of each other, η would have a constant value on the unit pure quaternions. Since $\mathbf{k} = \mathbf{ij}$, we would have

$$\eta(\mathbf{k}) = \eta(\mathbf{ij}) = (\eta(\mathbf{i}))^2 = 1.$$

Consequently, η would map all pure unit quaternions to 1. But since every unit quaternion is the product of two pure quaternions, η would map every unit quaternion to 1, contradicting the fact that it is surjective onto $\{-1, 1\}$.

16.8 Summary

The main concepts and results of this chapter are listed below:

- The group $\mathbf{SU}(2)$ of unit quaternions.
- The skew field $\mathbb{H}$ of quaternions.
- Hamilton's identities.
- The (real) vector space $\mathfrak{su}(2)$ of 2×2 skew Hermitian matrices with zero trace.
- The adjoint representation of $\mathbf{SU}(2)$.