Proposition 2.19. *Given a homomorphism of rings $h\colon A \to B$, the rings $A/\mathrm{Ker}\ h$ and $\mathrm{Im}\ h = h(A)$ are isomorphic.*

A field is a commutative ring K for which $K - \{0\}$ is a group under multiplication.

Definition 2.22. A set K is a *field* if it is a ring and the following properties hold:

(F1) $0 \neq 1$;

(F2) For every $a \in K$, if $a \neq 0$, then a has an inverse w.r.t. $*$;

(F3) $*$ is commutative.

Let $K^* = K - \{0\}$. Observe that (F1) and (F2) are equivalent to the fact that K^* is a group w.r.t. $*$ with identity element 1. If $*$ is not commutative but (F1) and (F2) hold, we say that we have a *skew field* (or *noncommutative field*).

Note that we are assuming that the operation $*$ of a field is commutative. This convention is not universally adopted, but since $*$ will be commutative for most fields we will encounter, we may as well include this condition in the definition.

Example 2.9.

1. The rings $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are fields.
2. The set of (formal) fractions $f(X)/g(X)$ of polynomials $f(X), g(X) \in \mathbb{R}[X]$, where $g(X)$ is not the null polynomial, is a field.
3. The ring $\mathcal{C}(a, b)$ of continuous functions $f\colon (a, b) \to \mathbb{R}$ such that $f(x) \neq 0$ for all $x \in (a, b)$ is a field.
4. Using Proposition 2.17, it is easy to see that the ring $\mathbb{Z}/p\mathbb{Z}$ is a field iff p is prime.
5. If d is a square-free positive integer and if $d \geq 2$, the set
$$\mathbb{Q}(\sqrt{d}) = \{a + b\sqrt{d} \in \mathbb{R} \mid a, b \in \mathbb{Q}\}$$
is a field. If $z = a + b\sqrt{d} \in \mathbb{Q}(\sqrt{d})$ and $\overline{z} = a - b\sqrt{d}$, then it is easy to check that if $z \neq 0$, then $z^{-1} = \overline{z}/(z\overline{z})$.
6. Similarly, If $d \geq 1$ is a square-free positive integer, the set of complex numbers
$$\mathbb{Q}(\sqrt{-d}) = \{a + ib\sqrt{d} \in \mathbb{C} \mid a, b \in \mathbb{Q}\}$$
is a field. If $z = a + ib\sqrt{d} \in \mathbb{Q}(\sqrt{-d})$ and $\overline{z} = a - ib\sqrt{d}$, then it is easy to check that if $z \neq 0$, then $z^{-1} = \overline{z}/(z\overline{z})$.