Similarly

$$\sup_{\substack{x\in\mathbb{R}^n \\ x\neq 0}} \frac{\|Ax\|}{\|x\|} = \sup_{\substack{x\in\mathbb{R}^n \\ \|x\|=1}} \|Ax\| .$$

The above considerations justify the following definition.

Definition 9.7. If $\| \, \|$ is any norm on $\mathbb{C}^n$, we define the function $\| \, \|_{\mathrm{op}}$ on $\mathrm{M}_n(\mathbb{C})$ by

$$\|A\|_{\mathrm{op}} = \sup_{\substack{x\in\mathbb{C}^n \\ x\neq 0}} \frac{\|Ax\|}{\|x\|} = \sup_{\substack{x\in\mathbb{C}^n \\ \|x\|=1}} \|Ax\| .$$

The function $A \mapsto \|A\|_{\mathrm{op}}$ is called the *subordinate matrix norm* or *operator norm* induced by the norm $\| \, \|$.

Another notation for the operator norm of a matrix A (in particular, used by Horn and Johnson [95]), is $|||A|||$.

It is easy to check that the function $A \mapsto \|A\|_{\mathrm{op}}$ is indeed a norm, and by definition, it satisfies the property

$$\|Ax\| \leq \|A\|_{\mathrm{op}} \|x\| , \quad \text{for all } x \in \mathbb{C}^n.$$

A norm $\| \, \|_{\mathrm{op}}$ on $\mathrm{M}_n(\mathbb{C})$ satisfying the above property is said to be *subordinate* to the vector norm $\| \, \|$ on $\mathbb{C}^n$. As a consequence of the above inequality, we have

$$\|ABx\| \leq \|A\|_{\mathrm{op}} \|Bx\| \leq \|A\|_{\mathrm{op}} \|B\|_{\mathrm{op}} \|x\| ,$$

for all $x \in \mathbb{C}^n$, which implies that

$$\|AB\|_{\mathrm{op}} \leq \|A\|_{\mathrm{op}} \|B\|_{\mathrm{op}} \quad \text{for all } A, B \in \mathrm{M}_n(\mathbb{C}),$$

showing that $A \mapsto \|A\|_{\mathrm{op}}$ is a matrix norm (it is submultiplicative).

Observe that the operator norm is also defined by

$$\|A\|_{\mathrm{op}} = \inf\{\lambda \in \mathbb{R} \mid \|Ax\| \leq \lambda \|x\| , \text{ for all } x \in \mathbb{C}^n\}.$$

Since the function $x \mapsto \|Ax\|$ is continuous (because $| \, \|Ay\| - \|Ax\| \, | \leq \|Ay - Ax\| \leq C_A \|x - y\|$) and the unit sphere $S^{n-1} = \{x \in \mathbb{C}^n \mid \|x\| = 1\}$ is compact, there is some $x \in \mathbb{C}^n$ such that $\|x\| = 1$ and

$$\|Ax\| = \|A\|_{\mathrm{op}} .$$

Equivalently, there is some $x \in \mathbb{C}^n$ such that $x \neq 0$ and

$$\|Ax\| = \|A\|_{\mathrm{op}} \|x\| .$$

Consequently we can replace sup by max in the definition of $\|A\|_{\mathrm{op}}$ (and inf by min), namely

$$\|A\|_{\mathrm{op}} = \max_{\substack{x\in\mathbb{C}^n \\ \|x\|=1}} \|Ax\| .$$