Remark. Observe that $\mathrm{int}(C) \subseteq \mathbf{relint}(C)$. Rockafellar denotes the relative interior of a set C by $\mathbf{ri}(C)$.

The following result from Rockafellar [138] (Theorem 7.2) tells us that an improper convex function mostly takes infinite values, except perhaps at relative boundary points of its effective domain.

Proposition 51.4. *If f is an improper convex function, then $f(x) = -\infty$ for every $x \in \mathbf{relint}(\mathrm{dom}(f))$. Thus an improper convex function takes infinite values, except at relative boundary points of its effective domain.*

Example 51.2 illustrates Proposition 51.4.

The following result also holds; see Rockafellar [138] (Corollary 7.2.3).

Proposition 51.5. *If f is a convex function whose effective domain is relatively open, which means that $\mathbf{relint}(\mathrm{dom}(f)) = \mathrm{dom}(f)$, then either $f(x) > -\infty$ for all $x \in \mathbb{R}^n$, or $f(x) = \pm\infty$ for all $x \in \mathbb{R}^n$.*

We also have the following result showing that the closure of a proper convex function does not differ much from the original function; see Rockafellar [138] (Theorem 7.4).

Proposition 51.6. *Let $f \colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function. Then $\mathrm{cl}(f)$ is a closed proper convex function, and $\mathrm{cl}(f)$ agrees with f on $\mathrm{dom}(f)$ except possibly at relative boundary points.*

Example 51.3. For an example of Propositions 51.6 and 51.5, let $f \colon \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be the proper convex function

$$f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ +\infty & \text{if } |x| \geq 1. \end{cases}$$

Then $\mathrm{cl}(f)$ is

$$\mathrm{cl}f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ +\infty & \text{if } |x| > 1, \end{cases}$$

and $\mathrm{cl}f(x) = f(x)$ whenever $x \in (-\infty, 1) = \mathbf{relint}(\mathrm{dom}(f)) = \mathrm{dom}(f)$. Furthermore, since $\mathbf{relint}(\mathrm{dom}(f)) = \mathrm{dom}(f)$, $f(x) > -\infty$ for all $x \in \mathbb{R}$. See Figure 51.5.

Small miracle: *the indicator function I_C of any closed convex set is proper and closed.* Indeed, for any $\alpha \in \mathbb{R}$ the sublevel set $\{x \in \mathbb{R}^n \mid I_C(x) \leq \alpha\}$ is either empty if $\alpha < 0$, or equal to C if $\alpha \geq 0$, and C is closed.

We now discuss briefly continuity properties of convex functions. The fact that a convex function f can take the values $\pm\infty$ causes a difficulty, so we consider the restriction of f to its effective domain. There is still a problem because an improper function may take the value $-\infty$. However, if we consider any subset C of $\mathrm{dom}(f)$ which is relatively open, which means that $\mathbf{relint}(C) = C$, then $C \subseteq \mathbf{relint}(\mathrm{dom}(f))$, so by Proposition 51.4, the function