thus $W^\perp \cap U \subseteq W \cap U$, and since U is totally isotropic, $U \subseteq U^\perp$, which yields

$$W^\perp \cap U \subseteq W \cap U \subseteq W \cap U^\perp = (0),$$

contradicting the fact that $U \cap W^\perp \neq (0)$.

Therefore, there is some $u \in W^\perp$ such that $u \notin W + U^\perp$. Since $U \subseteq U^\perp$, we can add to u any vector $z \in W^\perp \cap U \subseteq U^\perp$ so that $u + z \in W^\perp$ and $u + z \notin W + U^\perp$ (if $u + z \in W + U^\perp$, since $z \in U^\perp$, then $u \in W + U^\perp$, a contradiction). Since $W^\perp \cap U \neq (0)$ is totally isotropic and $u \notin W + U^\perp = (W^\perp \cap U)^\perp$, we can invoke Lemma 29.28 to find a $z \in W^\perp \cap U$ such that $\varphi(u + z, u + z) = 0$. See Figure 29.1. If we write $x = u + z$, then $x \notin W + U^\perp$, so $W' = W + Kx$ is a totally isotropic subspace of dimension $s + 1$. Furthermore, we claim that $W' \cap U^\perp = 0$.

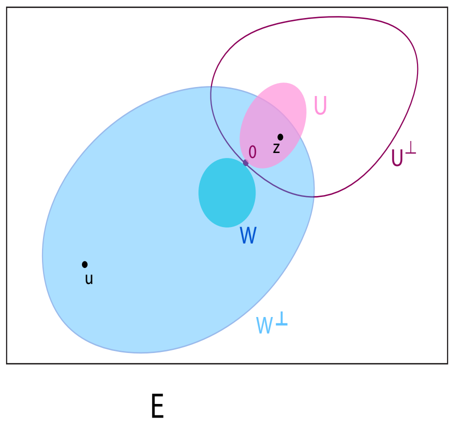

Figure 29.1: A schematic illustration of W and $x = u + z$

Otherwise, we would have $y = w + \lambda x \in U^\perp$, for some $w \in W$ and some $\lambda \in K$, and then we would have $\lambda x = -w + y \in W + U^\perp$. If $\lambda \neq 0$, then $x \in W + U^\perp$, a contradiction. Therefore, $\lambda = 0$, $y = w$, and since $y \in U^\perp$ and $w \in W$, we have $y \in W \cap U^\perp = (0)$, which means that $y = 0$. Therefore, W' is the required subspace and this completes the proof. $\square$

Here are some consequences of Proposition 29.29. If we set $W = (0)$ in Proposition 29.29(2), then we get the following theorem showing that if E is not anisotropic (there is some nonzero isotropic vector) then weak nontrivial Witt decompositions exist.

Theorem 29.30. *Let φ be an ϵ-Hermitian form on E which is nondegenerate and satisfies property (T). For any totally isotropic subspace U of E of finite dimension $r \geq 1$, there exists a totally isotropic subspace U' of dimension r such that $U \cap U' = (0)$ and $U \oplus U'$ is nondegenerate. As a consequence, if E is not anisotropic, then $(U, U', (U \oplus U')^\perp)$ is a weak nontrivial Witt decomposition for E. Furthermore, by Proposition 29.29(1), the block A in the matrix of φ is the identity matrix.*