We proceed by induction. The base case $k = 1$ is trivial. For the induction step, from $(*_2)$, we have

$$P_k A_{k+1} = A P_k.$$

Since $A_{k+1} = R_k Q_k = Q_{k+1} R_{k+1}$, we have

$$P_{k+1}\mathcal{R}_{k+1} = P_k Q_{k+1} R_{k+1} \mathcal{R}_k = P_k A_{k+1} \mathcal{R}_k = A P_k \mathcal{R}_k = A A^k = A^{k+1}$$

establishing the induction step.

Step 2. We will express the matrix P_k as $P_k = Q\widetilde{Q}_k D_k$, in terms of a diagonal matrix D_k with unit entries, with Q and $\widetilde{Q}_k$ unitary.

Let $P = QR$, a QR-factorization of P (with R an upper triangular matrix with positive diagonal entries), and $P^{-1} = LU$, an LU-factorization of P^{-1}. Since $A = P\Lambda P^{-1}$, we have

$$A^k = P\Lambda^k P^{-1} = QR\Lambda^k LU = QR(\Lambda^k L \Lambda^{-k})\Lambda^k U. \tag{$*_4$}$$

Here, Λ^{-k} is the diagonal matrix with entries λ_i^{-k}. The reason for introducing the matrix $\Lambda^k L \Lambda^{-k}$ is that its asymptotic behavior is easy to determine. Indeed, we have

$$(\Lambda^k L \Lambda^{-k})_{ij} = \begin{cases} 0 & \text{if } i < j \\ 1 & \text{if } i = j \\ \left(\dfrac{\lambda_i}{\lambda_j}\right)^k L_{ij} & \text{if } i > j. \end{cases}$$

The hypothesis that $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n| > 0$ implies that

$$\lim_{k \mapsto \infty} \Lambda^k L \Lambda^{-k} = I. \tag{$\dagger$}$$

Note that it is to obtain this limit that we made the hypothesis on the moduli of the eigenvalues. We can write

$$\Lambda^k L \Lambda^{-k} = I + F_k, \quad \text{with} \quad \lim_{k \mapsto \infty} F_k = 0,$$

and consequently, since $R(\Lambda^k L \Lambda^{-k}) = R(I + F_k) = R + RF_k R^{-1} R = (I + RF_k R^{-1})R$, we have

$$R(\Lambda^k L \Lambda^{-k}) = (I + RF_k R^{-1})R. \tag{$*_5$}$$

By Proposition 9.11(1), since $\lim_{k\mapsto\infty} F_k = 0$, and thus $\lim_{k\mapsto\infty} RF_k R^{-1} = 0$, the matrices $I + RF_k R^{-1}$ are invertible for k large enough. Consequently for k large enough, we have a QR-factorization

$$I + RF_k R^{-1} = \widetilde{Q}_k \widetilde{R}_k, \tag{$*_6$}$$

with $(\widetilde{R}_k)_{ii} > 0$ for $i = 1, \ldots, n$. Since the matrices $\widetilde{Q}_k$ are unitary, we have $\left\| \widetilde{Q}_k \right\|_2 = 1$, so the sequence $(\widetilde{Q}_k)$ is bounded. It follows that it has a convergent subsequence $(\widetilde{Q}_\ell)$ that converges to some matrix $\widetilde{Q}$, which is also unitary. Since

$$\widetilde{R}_\ell = (\widetilde{Q}_\ell)^*(I + RF_\ell R^{-1}),$$