Linear maps such that $f^{-1} = f^*$, or equivalently

$$f^* \circ f = f \circ f^* = \mathrm{id},$$

also play an important role. They are *linear isometries*, or *isometries*. Rotations are special kinds of isometries. Another important class of linear maps are the linear maps satisfying the property

$$f^* \circ f = f \circ f^*,$$

called *normal linear maps*. We will see later on that normal maps can always be diagonalized over orthonormal bases of eigenvectors, but this will require using a Hermitian inner product (over $\mathbb{C}$).

Given two Euclidean spaces E and F, where the inner product on E is denoted by $\langle -, - \rangle_1$ and the inner product on F is denoted by $\langle -, - \rangle_2$, given any linear map $f\colon E \to F$, it is immediately verified that the proof of Proposition 12.8 can be adapted to show that there is a unique linear map $f^*\colon F \to E$ such that

$$\langle f(u), v \rangle_2 = \langle u, f^*(v) \rangle_1$$

for all $u \in E$ and all $v \in F$. The linear map f^* is also called the *adjoint of f*.

The following properties immediately follow from the definition of the adjoint map:

(1) For any linear map $f\colon E \to F$, we have

$$f^{**} = f.$$

(2) For any two linear maps $f, g\colon E \to F$ and any scalar $\lambda \in \mathbb{R}$:

$$\begin{aligned}(f + g)^* &= f^* + g^* \\ (\lambda f)^* &= \lambda f^*.\end{aligned}$$

(3) If E, F, G are Euclidean spaces with respective inner products $\langle -, - \rangle_1, \langle -, - \rangle_2$, and $\langle -, - \rangle_3$, and if $f\colon E \to F$ and $g\colon F \to G$ are two linear maps, then

$$(g \circ f)^* = f^* \circ g^*.$$

Remark: Given any basis for E and any basis for F, it is possible to characterize the matrix of the adjoint f^* of f in terms of the matrix of f and the Gram matrices defining the inner products; see Problem 12.5. We will do so with respect to orthonormal bases in Proposition 12.14(2). Also, since inner products are symmetric, the adjoint f^* of f is also characterized by

$$f(u) \cdot v = u \cdot f^*(v),$$

for all $u, v \in E$.