46.4 The Simplex Algorithm Using Tableaux

We now describe a formalism for presenting the simplex algorithm, namely *(full) tableaux*. This is the traditional formalism used in all books, modulo minor variations. A particularly nice feature of the tableau formalism is that the update of a tableau can be performed using elementary row operations *identical* to the operations used during the reduction of a matrix to row reduced echelon form (rref). What differs is the criterion for the choice of the pivot.

Since the quantities $c_j - c_K\gamma^j_K$ play a crucial role in determining which column A^j should come into the basis, the notation $\overline{c}_j$ is used to denote $c_j - c_K\gamma^j_K$, which is called the *reduced cost* of the variable x_j. The reduced costs actually depend on K so to be very precise we should denote them by $(\overline{c}_K)_j$, but to simplify notation we write $\overline{c}_j$ instead of $(\overline{c}_K)_j$. We will see shortly how $(\overline{c}_{K^+})_i$ is computed in terms of $(\overline{c}_K)_i$.

Observe that the data needed to execute the next step of the simplex algorithm are

(1) The current basic solution u_K and its basis $K = (k_1, \ldots, k_m)$.

(2) The reduced costs $\overline{c}_j = c_j - c_K A_K^{-1} A^j = c_j - c_K\gamma^j_K$, for all $j \notin K$.

(3) The vectors $\gamma^j_K = (\gamma^j_{k_i})_{i=1}^m$ for all $j \notin K$, that allow us to express each A^j as $A_K\gamma^j_K$.

All this information can be packed into a $(m+1)\times(n+1)$ matrix called a *(full) tableau* organized as follows:

$c_K u_K$	$\overline{c}_1$	$\cdots$	$\overline{c}_j$	$\cdots$	$\overline{c}_n$
u_{k_1}	γ^1_1	$\cdots$	γ^j_1	$\cdots$	γ^n_1
$\vdots$	$\vdots$		$\vdots$		$\vdots$
u_{k_m}	γ^1_m	$\cdots$	γ^j_m	$\cdots$	γ^n_m

It is convenient to think as the first row as Row 0, and of the first column as Column 0. Row 0 contains the current value of the objective function and the reduced costs. Column 0, except for its top entry, contains the components of the current basic solution u_K, and the remaining columns, except for their top entry, contain the vectors γ^j_K. Observe that the γ^j_K corresponding to indices j in K constitute a permutation of the identity matrix I_m. The entry $\gamma^{j^+}_{k^-}$ is called the *pivot* element. A tableau together with the new basis $K^+ = (K - \{k^-\}) \cup \{j^+\}$ contains all the data needed to compute the new u_{K^+}, the new $\gamma^j_{K^+}$, and the new reduced costs $(\overline{c}_{K^+})_j$.

If we define the $m \times n$ matrix Γ as the matrix $\Gamma = [\gamma^1_K \ \cdots \ \gamma^n_K]$ whose jth column is γ^j_K, and $\overline{c}$ as the row vector $\overline{c} = (\overline{c}_1 \ \cdots \ \overline{c}_n)$, then the above tableau is denoted concisely by

$c_K u_K$	$\overline{c}$
u_K	Γ