Proof. Since E is Hausdorff, for every $a \in A$, there are some disjoint open sets, U_a and V_a, containing a and b respectively. Thus, the family, $(U_a)_{a\in A}$, forms an open cover of A. Since A is compact there is a finite open subcover, $(U_j)_{j\in J}$, of A, where $J \subseteq A$, and then $\bigcup_{j\in J} U_j$ is an open set containing A disjoint from the open set $\bigcap_{j\in J} V_j$ containing b. This shows that every point, b, in the complement of A belongs to some open set in this complement and thus, that the complement is open, i.e., that A is closed. See Figure 37.31. ☐

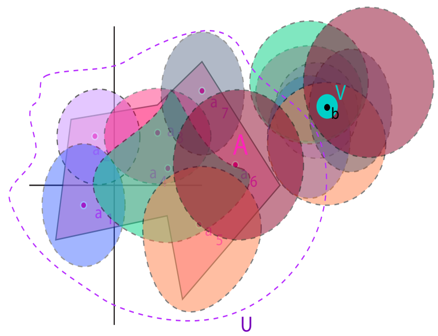

Figure 37.31: For the pink compact set A, U is the union of the seven disks which cover A, while V is the intersection of the seven open sets containing b.

Actually, the proof of Proposition 37.26 can be used to show the following useful property:

Proposition 37.27. *Given a topological Hausdorff space E, for every pair of compact disjoint subsets A and B, there exist disjoint open sets U and V, such that $A \subseteq U$ and $B \subseteq V$.*

Proof. We repeat the argument of Proposition 37.26 with B playing the role of b and use Proposition 37.26 to find disjoint open sets U_a containing $a \in A$, and V_a containing B. ☐

The following proposition shows that in a compact topological space, every closed set is compact:

Proposition 37.28. *Given a compact topological space, E, every closed set is compact.*

Proof. Since A is closed, $E - A$ is open and from any open cover, $(U_i)_{i\in I}$, of A, we can form an open cover of E by adding $E - A$ to $(U_i)_{i\in I}$ and, since E is compact, a finite subcover, $(U_j)_{j\in J} \cup \{E - A\}$, of E can be extracted such that $(U_j)_{j\in J}$ is a finite subcover of A. See Figure 37.32. ☐

Remark: Proposition 37.28 also holds for quasi-compact spaces, i.e., the Hausdorff separation property is not needed.