Theorem 39.13. *Given two normed affine spaces E and F, where E is of finite dimension n, and where $(a_0, (u_1, \ldots, u_n))$ is a frame of E, given any open subset A of E, given any function $f\colon A \to F$, the derivative $\mathrm{D}f\colon A \to \mathcal{L}(\overrightarrow{E}; \overrightarrow{F})$ is defined and continuous on A iff every partial derivative $\partial_j f$ (or $\dfrac{\partial f}{\partial x_j}$) is defined and continuous on A, for all j, $1 \leq j \leq n$. As a corollary, if F is of finite dimension m, and $(b_0, (v_1, \ldots, v_m))$ is a frame of F, the derivative $\mathrm{D}f\colon A \to \mathcal{L}(\overrightarrow{E}; \overrightarrow{F})$ is defined and continuous on A iff every partial derivative $\partial_j f_i$ (or $\dfrac{\partial f_i}{\partial x_j}$) is defined and continuous on A, for all i, j, $1 \leq i \leq m$, $1 \leq j \leq n$.*

Theorem 39.13 gives a necessary and sufficient condition for the existence and continuity of the derivative of a function on an open set. It should be noted that a more general version of Theorem 39.13 holds, assuming that $E = (E_1, a_1) \oplus \cdots \oplus (E_n, a_n)$, or $E = E_1 \times \cdots \times E_n$, and using the more general partial derivatives $\mathrm{D}_j f$ introduced before Proposition 39.11.

Definition 39.7. Given two normed affine spaces E and F, and an open subset A of E, we say that a function $f\colon A \to F$ is *of class C^0 on A or a C^0-function on A* if f is continuous on A. We say that $f\colon A \to F$ is *of class C^1 on A or a C^1-function on A* if $\mathrm{D}f$ exists and is continuous on A.

Since the existence of the derivative on an open set implies continuity, a C^1-function is of course a C^0-function. Theorem 39.13 gives a necessary and sufficient condition for a function f to be a C^1-function (when E is of finite dimension). It is easy to show that the composition of C^1-functions (on appropriate open sets) is a C^1-function.

39.4 The Implicit and The Inverse Function Theorems

Given three normed affine spaces E, F, and G, given a function $f\colon E \times F \to G$, given any $c \in G$, it may happen that the equation

$$f(x, y) = c$$

has the property that, for some open sets $A \subseteq E$, and $B \subseteq F$, there is a function $g\colon A \to B$, such that

$$f(x, g(x)) = c,$$

for all $x \in A$. Such a situation is usually very rare, but if some solution $(a, b) \in E \times F$ such that $f(a, b) = c$ is known, under certain conditions, for some small open sets $A \subseteq E$ containing a and $B \subseteq F$ containing b, the existence of a unique $g\colon A \to B$, such that

$$f(x, g(x)) = c,$$

for all $x \in A$, can be shown. Under certain conditions, it can also be shown that g is continuous, and differentiable. Such a theorem, known as the *implicit function theorem*, can be proven.