and thus

$$v_j^* = \sum_{i=1}^n b_{j\,i} u_i^*.$$

Similar calculations show that

$$u_i^* = \sum_{j=1}^n a_{i\,j} v_j^*.$$

This means that the change of basis from the dual basis $(u_1^*, \ldots, u_n^*)$ to the dual basis $(v_1^*, \ldots, v_n^*)$ is $(P^{-1})^\top$. Since

$$\varphi^* = \sum_{i=1}^n \varphi_i u_i^* = \sum_{i=1}^n \varphi_i \sum_{j=1}^n a_{ij} v_j^* = \sum_{j=1}^n \left(\sum_{i=1}^n a_{ij}\varphi_i \right) v_j = \sum_{i=1}^n \varphi_i' v_i^*,$$

we get

$$\varphi_j' = \sum_{i=1}^n a_{i\,j} \varphi_i,$$

so the new coordinates φ_j' are expressed in terms of the old coordinates φ_i using the matrix $P^\top$. If we use the row vectors $(\varphi_1, \ldots, \varphi_n)$ and $(\varphi_1', \ldots, \varphi_n')$, we have

$$(\varphi_1', \ldots, \varphi_n') = (\varphi_1, \ldots, \varphi_n)P.$$

These facts are summarized in the following proposition.

Proposition 11.1. *Let $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ be two bases of E, and let $P = (a_{i\,j})$ be the change of basis matrix from $(u_1, \ldots, u_n)$ to $(v_1, \ldots, v_n)$, so that*

$$v_j = \sum_{i=1}^n a_{i\,j} u_i.$$

Then the change of basis from the dual basis $(u_1^*, \ldots, u_n^*)$ to the dual basis $(v_1^*, \ldots, v_n^*)$ is $(P^{-1})^\top$, and for any linear form φ, the new coordinates φ_j' of φ are expressed in terms of the old coordinates φ_i of φ using the matrix $P^\top$; that is,

$$(\varphi_1', \ldots, \varphi_n') = (\varphi_1, \ldots, \varphi_n)P.$$

To best understand the preceding paragraph, recall Example 3.1, in which $E = \mathbb{R}^2$, $u_1 = (1, 0)$, $u_2 = (0, 1)$, and $v_1 = (1, 1)$, $v_2 = (-1, 1)$. Then P, the change of basis matrix from (u_1, u_2) to (v_1, v_2), is given by

$$P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

with $(v_1, v_2) = (u_1, u_2)P$, and $(u_1, u_2) = (v_1, v_2)P^{-1}$, where

$$P^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}.$$