$(x_1, x_2)$- plane, so that $g = 0$. We let $V$ be the subspace of $C^2(\Omega) \cap C^1(\overline{\Omega})$ consisting of functions $v$ such that $v = 0$ on $\Gamma$.

As before, we multiply the PDE by a test function $v \in V$, getting

$$-\Delta u(x)v(x) = f(x)v(x),$$

and we "integrate by parts." In this case, this means that we use a version of Stokes formula known as *Green's first identity*, which says that

$$\int_\Omega -\Delta u \, v \, dx = \int_\Omega (\mathrm{grad}\, u) \cdot (\mathrm{grad}\, v) \, dx - \int_\Gamma (\mathrm{grad}\, u) \cdot n \, v d\sigma$$

(where $n$ denotes the outward pointing unit normal to the surface). Because $v = 0$ on $\Gamma$, the integral $\int_\Gamma$ drops out, and we get an equation of the form

$$a(u, v) = \widetilde{f}(v) \quad \text{for all } v \in V,$$

where $a$ is the bilinear form given by

$$a(u, v) = \int_\Omega \left( \frac{\partial u}{\partial x_1} \frac{\partial v}{\partial x_1} + \frac{\partial u}{\partial x_2} \frac{\partial v}{\partial x_2} \right) dx$$

and $\widetilde{f}$ is the linear form given by

$$\widetilde{f}(v) = \int_\Omega f v dx.$$

We get the same equation as in section 19.2, but over a set of functions defined on a two-dimensional domain. As before, we can choose a finite-dimensional subspace $V_a$ of $V$ and consider the discrete problem with respect to $V_a$. Again, if we pick a basis $(w_1, \ldots, w_n)$ of $V_a$, a vector $u = u_1 w_1 + \cdots + u_n w_n$ is a solution of the Weak Formulation of our problem iff $\mathbf{u} = (u_1, \ldots, u_n)$ is a solution of the linear system

$$A\mathbf{u} = b,$$

with $A = (a(w_i, w_j))$ and $b = (\widetilde{f}(w_j))$. However, the integrals that give the entries in $A$ and $b$ are much more complicated.

An approach to deal with this problem is the *method of finite elements*. The idea is to also discretize the boundary curve $\Gamma$. If we assume that $\Gamma$ is a *polygonal line*, then we can *triangulate* the domain $\Omega$, and then we consider spaces of functions which are piecewise defined on the triangles of the triangulation of $\Omega$. The simplest functions are piecewise affine and look like tents erected above groups of triangles. Again, we can define base functions with small support, so that the matrix $A$ is tridiagonal by blocks.

The finite element method is a vast subject and it is presented in many books of various degrees of difficulty and obscurity. Let us simply state three important requirements of the finite element method: