Thus, some open ball, B_o^{m+1}, in the cover, $\mathcal{U}_{m+1}$, contains infinitely many elements from the sequence, (x_n), and we let y_{m+1} be any element of (x_n) in B_o^{m+1}. Thus, we have defined by induction a sequence, (y_n), which is a subsequence of, (x_n), and such that

$$d(y_i, y_{i+1}) \leq \frac{1}{2^i},$$

for all i. However, for all $m, n \geq 1$, we have

$$d(y_m, y_n) \leq d(y_m, y_{m+1}) + \cdots + d(y_{n-1}, y_n) \leq \sum_{i=m}^{n} \frac{1}{2^i} \leq \frac{1}{2^{m-1}},$$

and thus, (y_n) is a Cauchy sequence Since E is complete, the sequence, (y_n), has a limit, and since it is a subsequence of (x_n), the sequence, (x_n), has some accumulation point. $\square$

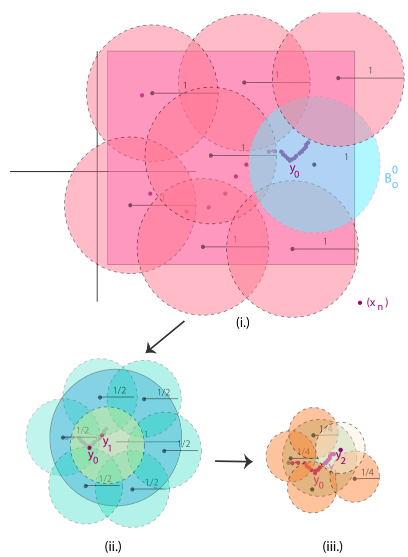

Figure 37.45: The first three stages of the construction of the Cauchy sequence (y_n), where E is the pink square region of $\mathbb{R}^2$. The original sequence (x_n) is illustrated with plum colored dots. Figure $(i.)$ covers E with ball of radius 1 and shows the selection of B_o^0 and y_0. Figure $(ii.)$ covers B_o^0 with balls of radius $1/2$ and selects the yellow ball as B_o^1 with point y_1. Figure $(iii.)$ covers B_o^1 with balls of radius $1/4$ and selects the pale peach ball as B_o^2 with point y_2.

Another useful property of a complete metric space is that a subset is closed iff it is complete. This is shown in the following two propositions.

Proposition 37.50. *Let (E, d) be a metric space, and let A be a subset of E. If A is complete (which means that every Cauchy sequence of elements in A converges to some point of A), then A is closed in E.*