is given by

$$a(u_1, \ldots, u_p) = u_1 + \cdots + u_p,$$

with $u_i \in U_i$ for $i = 1, \ldots, p$.

(1) If we let $Z_i \subseteq U_1 \times \cdots \times U_p$ be given by

$$Z_i = \left\{ \left(u_1, \ldots, u_{i-1}, -\sum_{j=1, j\neq i}^{p} u_j, u_{i+1}, \ldots, u_p \right) \;\middle|\; \sum_{j=1, j\neq i}^{p} u_j \in U_i \cap \left(\sum_{j=1, j\neq i}^{p} U_j \right) \right\},$$

for $i = 1, \ldots, p$, then prove that

$$\operatorname{Ker} a = Z_1 = \cdots = Z_p.$$

In general, for any given i, the condition $U_i \cap \left(\sum_{j=1, j\neq i}^{p} U_j \right) = (0)$ does not necessarily imply that $Z_i = (0)$. Thus, let

$$Z = \left\{ \left(u_1, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_p \right) \;\middle|\; u_i = -\sum_{j=1, j\neq i}^{p} u_j,\ u_i \in U_i \cap \left(\sum_{j=1, j\neq i}^{p} U_j \right),\ 1 \leq i \leq p \right\}.$$

Since $\operatorname{Ker} a = Z_1 = \cdots = Z_p$, we have $Z = \operatorname{Ker} a$. Prove that if

$$U_i \cap \left(\sum_{j=1, j\neq i}^{p} U_j \right) = (0) \quad 1 \leq i \leq p,$$

then $Z = \operatorname{Ker} a = (0)$.

(2) Prove that $U_1 + \cdots + U_p$ is a direct sum iff

$$U_i \cap \left(\sum_{j=1, j\neq i}^{p} U_j \right) = (0) \quad 1 \leq i \leq p.$$

Problem 6.4. Assume that E is finite-dimensional, and let $f_i \colon E \to E$ be any $p \geq 2$ linear maps such that

$$f_1 + \cdots + f_p = \mathrm{id}_E.$$

Prove that the following properties are equivalent:

(1) $f_i^2 = f_i$, $1 \leq i \leq p$.

(2) $f_j \circ f_i = 0$, for all $i \neq j$, $1 \leq i, j \leq p$.

Hint. Use Problem 6.2.