as illustrated by the following diagram

$$\begin{tikzcd}
\mathcal{U}, E \arrow[r, "f", "M_{\mathcal{U},}(f)"'] & \mathcal{U}, E \arrow[d, "P^{-1}_{\mathcal{U}',\mathcal{U}}"', "\mathrm{id}_E"] \\
\mathcal{U}', E \arrow[u, "P_{\mathcal{U}',\mathcal{U}}", "\mathrm{id}_E"'] \arrow[r, "M_{\mathcal{U}'}(f)", "f"'] & \mathcal{U}', E.
\end{tikzcd}$$

Example 4.3. Let $E = \mathbb{R}^2$, $\mathcal{U} = (e_1, e_2)$ where $e_1 = (1, 0)$ and $e_2 = (0, 1)$ are the canonical basis vectors, let $\mathcal{V} = (v_1, v_2) = (e_1, e_1 - e_2)$, and let

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}.$$

The change of basis matrix $P = P_{\mathcal{V},\mathcal{U}}$ from $\mathcal{U}$ to $\mathcal{V}$ is

$$P = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix},$$

and we check that

$$P^{-1} = P.$$

Therefore, in the basis $\mathcal{V}$, the matrix representing the linear map f defined by A is

$$A' = P^{-1}AP = PAP = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = D,$$

a diagonal matrix. In the basis $\mathcal{V}$, it is clear what the action of f is: it is a stretch by a factor of 2 in the v_1 direction and it is the identity in the v_2 direction. Observe that v_1 and v_2 are not orthogonal.

What happened is that we *diagonalized* the matrix A. The diagonal entries 2 and 1 are the *eigenvalues* of A (and f), and v_1 and v_2 are corresponding *eigenvectors*. We will come back to eigenvalues and eigenvectors later on.

The above example showed that the same linear map can be represented by different matrices. This suggests making the following definition:

Definition 4.5. Two $n \times n$ matrices A and B are said to be *similar* iff there is some invertible matrix P such that

$$B = P^{-1}AP.$$

It is easily checked that similarity is an equivalence relation. From our previous considerations, *two $n \times n$ matrices A and B are similar iff they represent the same linear map with respect to two different bases.* The following surprising fact can be shown: Every square