Theorem 37.55. *If (X, d) is a metric space, then the Hausdorff distance, D, on the set, $\mathcal{K}(X)$, of nonempty compact subsets of X is a distance. If (X, d) is complete, then $(\mathcal{K}(X), D)$ is complete and if (X, d) is compact, then $(\mathcal{K}(X), D)$ is compact.*

Proof. Since (nonempty) compact sets are bounded, $D(A, B)$ is well defined. Clearly D is symmetric. Assume that $D(A, B) = 0$. Then for every $\epsilon > 0$, $A \subseteq V_\epsilon(B)$, which means that for every $a \in A$, there is some $b \in B$ such that $d(a, b) \leq \epsilon$, and thus, that $A \subseteq \overline{B}$. Since Proposition 37.26 implies that B is closed, $\overline{B} = B$, and we have $A \subseteq B$. Similarly, $B \subseteq A$, and thus, $A = B$. Clearly, if $A = B$, we have $D(A, B) = 0$. It remains to prove the triangle inequality. Assume that $D(A, B) \leq \epsilon_1$ and that $D(B, C) \leq \epsilon_2$. We must show that $D(A, C) \leq \epsilon_1 + \epsilon_2$. This will be accomplished if we can show that $C \subseteq V_{\epsilon_1+\epsilon_2}(A)$ and $A \subseteq V_{\epsilon_1+\epsilon_2}(C)$. By assumption and definition of D, $B \subseteq V_{\epsilon_1}(A)$ and $C \subseteq V_{\epsilon_2}(B)$. Then

$$V_{\epsilon_2}(B) \subseteq V_{\epsilon_2}(V_{\epsilon_1}(A)),$$

and since a basic application of the triangle inequality implies that

$$V_{\epsilon_2}(V_{\epsilon_1}(A)) \subseteq V_{\epsilon_1+\epsilon_2}(A),$$

we get

$$C \subseteq V_{\epsilon_2}(B) \subseteq V_{\epsilon_1+\epsilon_2}(A).$$

See Figure 37.47.

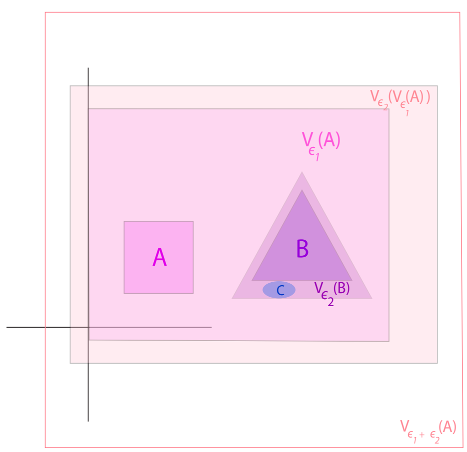

Figure 37.47: Let A be the small pink square and B be the small purple triangle in $\mathbb{R}^2$. The periwinkle oval C is contained in $V_{\epsilon_1+\epsilon_2}(A)$.

Similarly, the conditions $(A, B) \leq \epsilon_1$ and $D(B, C) \leq \epsilon_2$ imply that

$$A \subseteq V_{\epsilon_1}(B), \qquad B \subseteq V_{\epsilon_2}(C).$$