8.9 SPD Matrices and the Cholesky Decomposition

Definition 8.4. A real $n \times n$ matrix A is *symmetric positive definite*, for short *SPD*, iff it is symmetric and if

$$x^\top Ax > 0 \quad \text{for all } x \in \mathbb{R}^n \text{ with } x \neq 0.$$

The following facts about a symmetric positive definite matrix A are easily established (some left as an exercise):

(1) The matrix A is invertible. (Indeed, if $Ax = 0$, then $x^\top Ax = 0$, which implies $x = 0$.)

(2) We have $a_{i\,i} > 0$ for $i = 1, \ldots, n$. (Just observe that for $x = e_i$, the ith canonical basis vector of $\mathbb{R}^n$, we have $e_i^\top Ae_i = a_{i\,i} > 0$.)

(3) For every $n \times n$ real invertible matrix Z, the matrix $Z^\top AZ$ is real symmetric positive definite iff A is real symmetric positive definite.

(4) The set of $n \times n$ real symmetric positive definite matrices is convex. This means that if A and B are two $n \times n$ symmetric positive definite matrices, then for any $\lambda \in \mathbb{R}$ such that $0 \leq \lambda \leq 1$, the matrix $(1 - \lambda)A + \lambda B$ is also symmetric positive definite. Clearly since A and B are symmetric, $(1 - \lambda)A + \lambda B$ is also symmetric. For any nonzero $x \in \mathbb{R}^n$, we have $x^\top Ax > 0$ and $x^\top Bx > 0$, so

$$x^\top((1 - \lambda)A + \lambda B)x = (1 - \lambda)x^\top Ax + \lambda x^\top Bx > 0,$$

because $0 \leq \lambda \leq 1$, so $1-\lambda \geq 0$ and $\lambda \geq 0$, and $1-\lambda$ and λ can't be zero simultaneously.

(5) The set of $n \times n$ real symmetric positive definite matrices is a cone. This means that if A is symmetric positive definite and if $\lambda > 0$ is any real, then λA is symmetric positive definite. Clearly λA is symmetric, and for nonzero $x \in \mathbb{R}^n$, we have $x^\top Ax > 0$, and since $\lambda > 0$, we have $x^\top \lambda Ax = \lambda x^\top Ax > 0$.

Remark: Given a complex $m \times n$ matrix A, we define the matrix $\overline{A}$ as the $m \times n$ matrix $\overline{A} = (\overline{a_{ij}})$. Then we define A^* as the $n \times m$ matrix $A^* = (\overline{A})^\top = \overline{(A^\top)}$. The $n \times n$ complex matrix A is *Hermitian* if $A^* = A$. This is the complex analog of the notion of a real symmetric matrix.

Definition 8.5. A complex $n \times n$ matrix A is *Hermitian positive definite*, for short *HPD*, if it is Hermitian and if

$$z^* Az > 0 \quad \text{for all } z \in \mathbb{C}^n \text{ with } z \neq 0.$$

It is easily verified that Properties (1)-(5) hold for Hermitian positive definite matrices; replace $\top$ by $*$.