Remark: Proposition 31.5 can be used to give a quick proof of Theorem 15.5.

31.3 Commuting Families of Diagonalizable and Triangulable Maps

Using Theorem 31.6, we can give a short proof about commuting diagonalizable linear maps.

Definition 31.4. If $\mathcal{F}$ is a family of linear maps on a vector space E, we say that $\mathcal{F}$ is a *commuting family* iff $f \circ g = g \circ f$ for all $f, g \in \mathcal{F}$.

Proposition 31.7. *Let $\mathcal{F}$ be a finite commuting family of diagonalizable linear maps on a vector space E. There exists a basis of E such that every linear map in $\mathcal{F}$ is represented in that basis by a diagonal matrix.*

Proof. We proceed by induction on $n = \dim(E)$. If $n = 1$, there is nothing to prove. If $n > 1$, there are two cases. If all linear maps in $\mathcal{F}$ are of the form $\lambda\mathrm{id}$ for some $\lambda \in K$, then the proposition holds trivially. In the second case, let $f \in \mathcal{F}$ be some linear map in $\mathcal{F}$ which is not a scalar multiple of the identity. In this case, f has at least two distinct eigenvalues $\lambda_1, \ldots, \lambda_k$, and because f is diagonalizable, E is the direct sum of the corresponding eigenspaces $E_{\lambda_1}, \ldots, E_{\lambda_k}$. For every index i, the eigenspace E_{λ_i} is invariant under f and under every other linear map g in $\mathcal{F}$, since for any $g \in \mathcal{F}$ and any $u \in E_{\lambda_i}$, because f and g commute, we have

$$f(g(u)) = g(f(u)) = g(\lambda_i u) = \lambda_i g(u)$$

so $g(u) \in E_{\lambda_i}$. Let $\mathcal{F}_i$ be the family obtained by restricting each $f \in \mathcal{F}$ to E_{λ_i}. By Proposition 31.3, the minimal polynomial of every linear map $f \mid E_{\lambda_i}$ in $\mathcal{F}_i$ divides the minimal polynomial m_f of f, and since f is diagonalizable, m_f is a product of distinct linear factors, so the minimal polynomial of $f \mid E_{\lambda_i}$ is also a product of distinct linear factors. By Theorem 31.6, the linear map $f \mid E_{\lambda_i}$ is diagonalizable. Since $k > 1$, we have $\dim(E_{\lambda_i}) < \dim(E)$ for $i = 1, \ldots, k$, and by the induction hypothesis, for each i there is a basis of E_{λ_i} over which $f \mid E_{\lambda_i}$ is represented by a diagonal matrix. Since the above argument holds for all i, by combining the bases of the E_{λ_i}, we obtain a basis of E such that the matrix of every linear map $f \in \mathcal{F}$ is represented by a diagonal matrix. ☐

Remark: Proposition 31.7 also holds for infinite commuting families $\mathcal{F}$ of diagonalizable linear maps, because E being finite dimensional, there is a finite subfamily of linearly independent linear maps in $\mathcal{F}$ spanning $\mathcal{F}$.

There is also an analogous result for commuting families of linear maps represented by upper triangular matrices. To prove this we need the following proposition.

Proposition 31.8. *Let $\mathcal{F}$ be a nonempty finite commuting family of triangulable linear maps on a finite-dimensional vector space E. Let W be a proper subspace of E which is invariant under $\mathcal{F}$. Then there exists a vector $u \in E$ such that:*