Since $(f_j)_{j\in J}$ is a basis of F, for every $j \in J'$, we must have

$$\sum_{i\in I'} \lambda_{ij} e_i^*(x) = 0, \quad \text{for all } x \in E.$$

But then $(e_i^*)_{i\in I'}$ would be linearly dependent, contradicting the fact that $(e_i^*)_{i\in I}$ is a basis of E^*, so we must have

$$\lambda_{ij} = 0, \quad \text{for all } i \in I' \text{ and all } j \in J',$$

which shows that $\omega = 0$. Therefore, $\alpha_\otimes$ is injective.

(2) Let $(e_j)_{1\leq j\leq n}$ be a finite basis of E, and as usual, let $e_j^* \in E^*$ be the linear form defined by

$$e_j^*(e_k) = \delta_{j,k},$$

where $\delta_{j,k} = 1$ iff $j = k$ and 0 otherwise. We know that $(e_j^*)_{1\leq j\leq n}$ is a basis of E^* (this is where we use the finite dimension of E). For any linear map $f \in \mathrm{Hom}(E, F)$, for every $x = x_1e_1 + \cdots + x_ne_n \in E$, we have

$$f(x) = f(x_1e_1 + \cdots + x_ne_n) = x_1f(e_1) + \cdots + x_nf(e_n) = e_1^*(x)f(e_1) + \cdots + e_n^*(x)f(e_n).$$

Consequently, every linear map $f \in \mathrm{Hom}(E, F)$ can be expressed as

$$f(x) = e_1^*(x)f_1 + \cdots + e_n^*(x)f_n,$$

for some $f_i \in F$. Furthermore, if we apply f to e_i, we get $f(e_i) = f_i$, so the f_i are unique. Observe that

$$(\alpha_\otimes(e_1^* \otimes f_1 + \cdots + e_n^* \otimes f_n))(x) = \sum_{i=1}^n (\alpha_\otimes(e_i^* \otimes f_i))(x) = \sum_{i=1}^n e_i^*(x)f_i.$$

Thus, $\alpha_\otimes$ is surjective, so $\alpha_\otimes$ is a bijection.

(3) Let $(f_1, \ldots, f_m)$ be a finite basis of F, and let $(f_1^*, \ldots, f_m^*)$ be its dual basis. Given any linear map $h\colon E \to F$, for all $u \in E$, since $f_i^*(f_j) = \delta_{ij}$, we have

$$h(u) = \sum_{i=1}^m f_i^*(h(u))f_i.$$

If

$$h(u) = \sum_{j=1}^m v_j^*(u)f_j \quad \text{for all } u \in E \tag{$*$}$$

for some linear forms $(v_1^*, \ldots, v_m^*) \in (E^*)^m$, then

$$f_i^*(h(u)) = \sum_{j=1}^m v_j^*(u)f_i^*(f_j) = v_i^*(u) \quad \text{for all } u \in E,$$