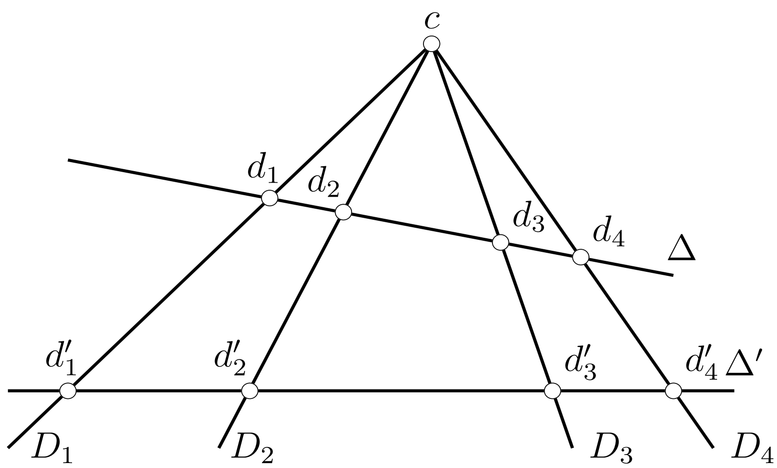

Figure 26.39: A pencil of lines and its cross-ratio with intersecting lines.

$G = \psi^{-1}(0)$. Let $a = \mathbf{P}(F) \cap \Delta$ and $b = \mathbf{P}(G) \cap \Delta$. There are some vectors $u, v \in D$ such that $a = p(u)$ and $b = p(v)$, and since φ and ψ are linearly independent, we have $a \neq b$, and we can choose φ and ψ such that $\varphi(v) = -1$ and $\psi(u) = 1$. Also, (u, v) is a basis of D. Then a point $p(\alpha u + \beta v)$ on Δ belongs to the hyperplane $H = p(\gamma\varphi + \delta\psi)$ of the pencil P iff

$$(\gamma\varphi + \delta\psi)(\alpha u + \beta v) = 0,$$

which, since $\varphi(u) = 0$, $\psi(v) = 0$, $\varphi(v) = -1$, and $\psi(u) = 1$, yields $\gamma\beta = \delta\alpha$, which is equivalent to $[\alpha, \beta] = [\gamma, \delta]$ in $\mathbf{P}(K^2)$. But then the map $h\colon P \to \Delta$ is a projectivity. Letting $d_i = \Delta \cap H_i$, since by Proposition 26.20 a projectivity of lines preserves the cross-ratio, we get $[d_1, d_2, d_3, d_4] = [H_1, H_2, H_3, H_4]$. $\square$

26.14 Complexification of a Real Projective Space

Notions such as orthogonality, angles, and distance between points are not projective concepts. In order to define such notions, one needs an inner product on the underlying vector space. We say that such notions belong to *Euclidean geometry*. At first glance, the fact that some important Euclidean concepts are not covered by projective geometry seems a major drawback of projective geometry. Fortunately, geometers of the nineteenth century (including Laguerre, Monge, Poncelet, Chasles, von Staudt, Cayley, and Klein) found an astute way of recovering certain Euclidean notions such as angles and orthogonality (also circles) by embedding real projective spaces into complex projective spaces. In the next two sections we will give a brief account of this method. More details can be found in Berger [11, 12], Pedoe [136], Samuel [142], Coxeter [43, 44], Sidler [161], Tisseron [175], Lehmann and Bkouche [115], and, of course, Volume II of Veblen and Young [184].