Observe that Property (3) of Proposition 9.7 says that

$$\|A\|_2 \leq \|A\|_F \leq \sqrt{n}\,\|A\|_2\,,$$

which shows that the Frobenius norm is an upper bound on the spectral norm. The Frobenius norm is much easier to compute than the spectral norm.

The reader will check that the above proof still holds if the matrix A is real (change unitary to orthogonal), confirming the fact that $\|A\|_{\mathbb{R}} = \|A\|$ for the vector norms $\|\ \|_1$, $\|\ \|_2$, and $\|\ \|_\infty$. It is also easy to verify that the proof goes through for *rectangular* $m \times n$ matrices, with the same formulae. Similarly, the Frobenius norm given by

$$\|A\|_F = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2\right)^{1/2} = \sqrt{\mathrm{tr}(A^*A)} = \sqrt{\mathrm{tr}(AA^*)}$$

is also a norm on rectangular matrices. For these norms, whenever AB makes sense, we have

$$\|AB\| \leq \|A\|\,\|B\|\,.$$

Remark: It can be shown that for any two real numbers $p, q \geq 1$ such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$, we have

$$\|A^*\|_q = \|A\|_p = \sup\{\Re(y^*Ax) \mid \|x\|_p = 1, \|y\|_q = 1\} = \sup\{|\langle Ax, y\rangle| \mid \|x\|_p = 1, \|y\|_q = 1\},$$

where $\|A^*\|_q$ and $\|A\|_p$ are the operator norms.

Remark: Let $(E, \|\ \|)$ and $(F, \|\ \|)$ be two normed vector spaces (for simplicity of notation, we use the same symbol $\|\ \|$ for the norms on E and F; this should not cause any confusion). Recall that a function $f\colon E \to F$ is *continuous* if for every $a \in E$, for every $\epsilon > 0$, there is some $\eta > 0$ such that for all $x \in E$,

$$\text{if} \quad \|x - a\| \leq \eta \quad \text{then} \quad \|f(x) - f(a)\| \leq \epsilon.$$

It is not hard to show that a *linear map* $f\colon E \to F$ is continuous iff there is some constant $C \geq 0$ such that

$$\|f(x)\| \leq C\,\|x\| \text{ for all } x \in E.$$

If so, we say that f is *bounded* (or a *linear bounded operator*). We let $\mathcal{L}(E; F)$ denote the set of all continuous (equivalently, bounded) linear maps from E to F. Then we can define the *operator norm* (or *subordinate norm*) $\|\ \|$ on $\mathcal{L}(E; F)$ as follows: for every $f \in \mathcal{L}(E; F)$,

$$\|f\| = \sup_{\substack{x \in E \\ x \neq 0}} \frac{\|f(x)\|}{\|x\|} = \sup_{\substack{x \in E \\ \|x\| = 1}} \|f(x)\|\,,$$