and let $\| \ \|$ be a matrix norm such that

$$\|\mathrm{diag}(\alpha_1, \ldots, \alpha_n)\| = \max_{1 \le i \le n} |\alpha_i|,$$

for every diagonal matrix. Then for every perturbation matrix ΔA, if we write

$$B_i = \{z \in \mathbb{C} \mid |z - \lambda_i| \le \mathrm{cond}(P) \, \|\Delta A\|\},$$

for every eigenvalue λ of $A + \Delta A$, we have

$$\lambda \in \bigcup_{k=1}^{n} B_k.$$

Proof. Let λ be any eigenvalue of the matrix $A + \Delta A$. If $\lambda = \lambda_j$ for some j, then the result is trivial. Thus assume that $\lambda \neq \lambda_j$ for $j = 1, \ldots, n$. In this case the matrix $D - \lambda I$ is invertible (since its eigenvalues are $\lambda - \lambda_j$ for $j = 1, \ldots, n$), and we have

$$\begin{aligned} P^{-1}(A + \Delta A - \lambda I)P &= D - \lambda I + P^{-1}(\Delta A)P \\ &= (D - \lambda I)(I + (D - \lambda I)^{-1} P^{-1}(\Delta A)P). \end{aligned}$$

Since λ is an eigenvalue of $A + \Delta A$, the matrix $A + \Delta A - \lambda I$ is singular, so the matrix

$$I + (D - \lambda I)^{-1} P^{-1}(\Delta A)P$$

must also be singular. By Proposition 9.11(2), we have

$$1 \le \left\|(D - \lambda I)^{-1} P^{-1}(\Delta A)P\right\|,$$

and since $\| \ \|$ is a matrix norm,

$$\left\|(D - \lambda I)^{-1} P^{-1}(\Delta A)P\right\| \le \left\|(D - \lambda I)^{-1}\right\| \left\|P^{-1}\right\| \|\Delta A\| \|P\|,$$

so we have

$$1 \le \left\|(D - \lambda I)^{-1}\right\| \left\|P^{-1}\right\| \|\Delta A\| \|P\|.$$

Now $(D - \lambda I)^{-1}$ is a diagonal matrix with entries $1/(\lambda_i - \lambda)$, so by our assumption on the norm,

$$\left\|(D - \lambda I)^{-1}\right\| = \frac{1}{\min_i(|\lambda_i - \lambda|)}.$$

As a consequence, since there is some index k for which $\min_i(|\lambda_i - \lambda|) = |\lambda_k - \lambda|$, we have

$$\left\|(D - \lambda I)^{-1}\right\| = \frac{1}{|\lambda_k - \lambda|},$$

and we obtain

$$|\lambda - \lambda_k| \le \left\|P^{-1}\right\| \|\Delta A\| \|P\| = \mathrm{cond}(P) \, \|\Delta A\|,$$

which proves our result. ∎