such that a does not divide g_j. Pick i and j minimal such that a does not divide f_i and a does not divide g_j. The coefficient c_{i+j} of X^{i+j} in $f(X)g(X)$ is

$$c_{i+j} = f_0 g_{i+j} + f_1 g_{i+j-1} + \cdots + f_i g_j + \cdots + f_{i+j} g_0$$

(letting $f_h = 0$ if $h > m$ and $g_k = 0$ if $k > n$). From the choice of i and j, a cannot divide $f_i g_j$, since a being irreducible, by (2′) of Proposition 32.2, a would divide f_i or g_j. However, by the choice of i and j, a divides every other nonnull term in the sum for c_{i+j}, and since a is irreducible and divides $f(X)g(X)$, by Proposition 32.4, a divides c_{i+j}, which implies that a divides $f_i g_j$, a contradiction. Thus, either a divides $f(X)$ or a divides $g(X)$. $\square$

As a corollary, we get the following proposition.

Proposition 32.6. *Let A be a UFD. For any $a \in A$, $a \neq 0$, if a divides the product $f(X)g(X)$ of two polynomials $f(X), g(X) \in A[X]$ and $f(X)$ is irreducible and of degree at least 1, then a divides $g(X)$.*

Proof. The Proposition is trivial is a is a unit. Otherwise, $a = a_1 \cdots a_m$ where $a_i \in A$ is irreducible. Using induction and applying Lemma 32.5, we conclude that a divides $g(X)$. $\square$

We now show that Lemma 32.5 also applies to the case where a is an irreducible polynomial. This requires a little excursion involving the fraction field F of A.

Remark: If A is a UFD, it is possible to prove the uniqueness condition (2) for $A[X]$ directly without using the fraction field of A, see Malliavin [119], Chapter 3.

Given an integral domain A, we can construct a field F such that every element of F is of the form a/b, where $a, b \in A$, $b \neq 0$, using essentially the method for constructing the field $\mathbb{Q}$ of rational numbers from the ring $\mathbb{Z}$ of integers.

Proposition 32.7. *Let A be an integral domain.*

(1) There is a field F and an injective ring homomorphism $i\colon A \to F$ such that every element of F is of the form $i(a)i(b)^{-1}$, where $a, b \in A$, $b \neq 0$.

(2) For every field K and every injective ring homomorphism $h\colon A \to K$, there is a (unique) field homomorphism $\widehat{h}\colon F \to K$ such that

$$\widehat{h}(i(a)i(b)^{-1}) = h(a)h(b)^{-1}$$

for all $a, b \in A$, $b \neq 0$.

(3) The field F in (1) is unique up to isomorphism.