It follows that the equivalence class of an element $g \in G$ is the coset gH (resp. Hg). Since L_g is a bijection between H and gH, the cosets gH all have the same cardinality. The map $L_{g^{-1}} \circ R_g$ is a bijection between the left coset gH and the right coset Hg, so they also have the same cardinality. Since the distinct cosets gH form a partition of G, we obtain the following fact:

Proposition 2.8. *(Lagrange) For any finite group G and any subgroup H of G, the order h of H divides the order n of G.*

Definition 2.6. Given a finite group G and a subgroup H of G, if $n = |G|$ and $h = |H|$, then the ratio n/h is denoted by $(G : H)$ and is called the *index of H in G*.

The index $(G : H)$ is the number of left (and right) cosets of H in G. Proposition 2.8 can be stated as

$$|G| = (G : H)|H|.$$

The set of left cosets of H in G (which, in general, is **not** a group) is denoted G/H. The "points" of G/H are obtained by "collapsing" all the elements in a coset into a single element.

Example 2.3.

1. Let n be any positive integer, and consider the subgroup $n\mathbb{Z}$ of $\mathbb{Z}$ (under addition). The coset of 0 is the set $\{0\}$, and the coset of any nonzero integer $m \in \mathbb{Z}$ is

   $$m + n\mathbb{Z} = \{m + nk \mid k \in \mathbb{Z}\}.$$

   By dividing m by n, we have $m = nq + r$ for some unique r such that $0 \leq r \leq n - 1$. But then we see that r is the smallest positive element of the coset $m + n\mathbb{Z}$. This implies that there is a bijection betwen the cosets of the subgroup $n\mathbb{Z}$ of $\mathbb{Z}$ and the set of residues $\{0, 1, \ldots, n - 1\}$ modulo n, or equivalently a bijection with $\mathbb{Z}/n\mathbb{Z}$.

2. The cosets of $\mathbf{SL}(n, \mathbb{R})$ in $\mathbf{GL}(n, \mathbb{R})$ are the sets of matrices

   $$A\,\mathbf{SL}(n, \mathbb{R}) = \{AB \mid B \in \mathbf{SL}(n, \mathbb{R})\},\ A \in \mathbf{GL}(n, \mathbb{R}).$$

   Since A is invertible, $\det(A) \neq 0$, and we can write $A = (\det(A))^{1/n}((\det(A))^{-1/n}A)$ if $\det(A) > 0$ and $A = (-\det(A))^{1/n}((-\det(A))^{-1/n}A)$ if $\det(A) < 0$. But we have $(\det(A))^{-1/n}A \in \mathbf{SL}(n, \mathbb{R})$ if $\det(A) > 0$ and $-(-\det(A))^{-1/n}A \in \mathbf{SL}(n, \mathbb{R})$ if $\det(A) < 0$, so the coset $A\,\mathbf{SL}(n, \mathbb{R})$ contains the matrix

   $$(\det(A))^{1/n}I_n \quad \text{if} \quad \det(A) > 0, \quad -(-\det(A))^{1/n}I_n \quad \text{if} \quad \det(A) < 0.$$

   It follows that there is a bijection between the cosets of $\mathbf{SL}(n, \mathbb{R})$ in $\mathbf{GL}(n, \mathbb{R})$ and $\mathbb{R}$.