for some $\lambda_1, \ldots, \lambda_n \in K$, by multiplying both sides of the equation $A\lambda = 0$ by B we get

$$BA\lambda = I_n\lambda = \lambda = B0 = 0,$$

so $\lambda = 0$. Then since $(A^1, \ldots, A^n)$ are n linearly independent vectors in K^n, they form a basis of K^n. Consequently, for every vector $b \in K^n$, there is a unique column vector $(x_1, \ldots, x_n) \in K^n$ such that

$$Ax = x_1A^1 + \cdots + x_nA^n = b,$$

where x is the column vector

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Thus we can solve the n equations

$$Ax^j = e_j, \quad 1 \leq j \leq n,$$

where $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the jth canonical basis vector in K^n. These equations yield the matrix equation

$$AX = I_n,$$

where $X = (x^1 \cdots x^n)$ is the $n \times n$ matrix whose jth column is x^j. Consequently, X is a right inverse of A. Now A has a left inverse B and a right inverse X, so by Proposition 2.2, we have $X = B$, so A is invertible and its inverse is equal to B.

Let us now assume that there is a matrix C such that $AC = I_n$. We can repeat the previous proof with C playing the role of A and A playing the role of B to conclude that C is invertible and that $C^{-1} = A$. But then C^{-1} is invertible with inverse C, and since $C = (C^{-1})^{-1} = A^{-1}$, we conclude that A is invertible and that its inverse is equal to C. $\square$

Using Proposition 2.3 (or mimicking the computations in its proof), we note that if A and B are two $n \times n$ invertible matrices, then AB is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

An important criterion for a square matrix to be invertible is stated next. Another proof is provided in Proposition 4.4 .

Proposition 3.14. *A square matrix $A \in \mathrm{M}_n(K)$ is invertible iff its columns $(A^1, \ldots, A^n)$ are linearly independent.*

Proof. If A is invertible, then in particular it has a left inverse A^{-1}, so the first part of the proof of Proposition 3.13 with $B = A^{-1}$ proves that the columns $(A^1, \ldots, A^n)$ of A are linearly independent. This fact is also proven as part of the proof of Proposition 4.4. Conversely, assume that the columns $(A^1, \ldots, A^n)$ of A are linearly independent. The second part of the proof of Proposition 3.13 shows that A is invertible. $\square$