1. $u_r \notin W$.

2. $(g_r - \alpha_r \mathrm{id})(u_r) \in W$ for some scalars $\alpha_r \in K$.

Now since $u_r \in V_{r-1}$, we have $(f_i - \alpha_i \mathrm{id})(u_r) \in W$ for $i = 1, \ldots, r-1$, so $(f_i - \alpha_i \mathrm{id})(u_r) \in W$ for $i = 1, \ldots, r$ (since g_r is the restriction of f_r), which concludes the proof of the induction step. Finally, since every $f \in \mathcal{F}$ is the linear combination of $(f_1, \ldots, f_r)$, Condition (2) of the inductive claim implies Condition (2) of the proposition. $\square$

We can now prove the following result.

Proposition 31.9. *Let $\mathcal{F}$ be a nonempty finite commuting family of triangulable linear maps on a finite-dimensional vector space E. There exists a basis of E such that every linear map in $\mathcal{F}$ is represented in that basis by an upper triangular matrix.*

Proof. Let $n = \dim(E)$. We construct inductively a basis $(u_1, \ldots, u_n)$ of E such that if W_i is the subspace spanned by $(u_1 \ldots, u_i)$, then for every $f \in \mathcal{F}$,

$$f(u_i) = a^f_{1i} u_1 + \cdots + a^f_{ii} u_i,$$

for some $a^f_{ij} \in K$; that is, $f(u_i)$ belongs to the subspace W_i.

We begin by applying Proposition 31.8 to the subspace $W_0 = (0)$ to get u_1 so that for all $f \in \mathcal{F}$,

$$f(u_1) = \alpha^f_1 u_1.$$

For the induction step, since W_i invariant under $\mathcal{F}$, we apply Proposition 31.8 to the subspace W_i, to get $u_{i+1} \in E$ such that

1. $u_{i+1} \notin W_i$.

2. For every $f \in \mathcal{F}$, the vector $f(u_{i+1})$ belong to the subspace spanned by W_i and u_{i+1}.

Condition (1) implies that $(u_1, \ldots, u_i, u_{i+1})$ is linearly independent, and Condition (2) means that for every $f \in \mathcal{F}$,

$$f(u_{i+1}) = a^f_{1i+1} u_1 + \cdots + a^f_{i+1i+1} u_{i+1},$$

for some $a^f_{i+1j} \in K$, establishing the induction step. After n steps, each $f \in \mathcal{F}$ is represented by an upper triangular matrix. $\square$

Observe that if $\mathcal{F}$ consists of a single linear map f and if the minimal polynomial of f is of the form

$$m = (X - \lambda_1)^{r_1} \cdots (X - \lambda_k)^{r_k},$$

with all $\lambda_i \in K$, using Proposition 31.5 instead of Proposition 31.8, the proof of Proposition 31.9 yields another proof of Theorem 15.5.