for all $a, b \in X$. Let $\lambda = \max\{\lambda_1, \ldots, \lambda_n\}$. We claim that

$$D(F(A), F(B)) \leq \lambda D(A, B).$$

For any $x \in F(A) = f_1(A) \cup \cdots \cup f_n(A)$, there is some $a_i \in A_i$ such that $x = f_i(a_i)$ and since $\eta \geq D(A, B)$, there is some $b_i \in B$ such that

$$d(a_i, b_i) \leq \eta,$$

and thus,

$$d(x, f_i(b_i)) = d(f_i(a_i), f_i(b_i)) \leq \lambda_i d(a_i, b_i) \leq \lambda\eta.$$

This show that

$$F(A) \subseteq V_{\lambda\eta}(F(B)).$$

Similarly, we can prove that

$$F(B) \subseteq V_{\lambda\eta}(F(A)),$$

and since this holds for all $\eta \geq D(A, B)$, we proved that

$$D(F(A), F(B)) \leq \lambda D(A, B)$$

where $\lambda = \max\{\lambda_1, \ldots, \lambda_n\}$. Since $0 \leq \lambda_i < 1$, we have $0 \leq \lambda < 1$ and F is indeed a contracting mapping. $\square$

Theorem 38.1 justifies the existence of many familiar "self-similar" fractals. One of the best known fractals is the *Sierpinski gasket*.

Example 38.1. Consider an equilateral triangle with vertices a, b, c, and let f_1, f_2, f_3 be the dilatations of centers a, b, c and ratio $1/2$. The Sierpinski gasket is the invariant set of the ifs (f_1, f_2, f_3). The dilations f_1, f_2, f_3 can be defined explicitly as follows, assuming that $a = (-1/2, 0)$, $b = (1/2, 0)$, and $c = (0, \sqrt{3}/2)$. The contractions f_1, f_2, f_3 are specified by

$$\begin{aligned} x' &= \frac{1}{2}x - \frac{1}{4}, \\ y' &= \frac{1}{2}y, \end{aligned}$$

$$\begin{aligned} x' &= \frac{1}{2}x + \frac{1}{4}, \\ y' &= \frac{1}{2}y, \end{aligned}$$

and

$$\begin{aligned} x' &= \frac{1}{2}x, \\ y' &= \frac{1}{2}y + \frac{\sqrt{3}}{4}. \end{aligned}$$