3.5 Bases of a Vector Space

Given a vector space E, given a family $(v_i)_{i\in I}$, the subset V of E consisting of the null vector 0 and of all linear combinations of $(v_i)_{i\in I}$ is easily seen to be a subspace of E. The family $(v_i)_{i\in I}$ is an economical way of representing the entire subspace V, but such a family would be even nicer if it was not redundant. Subspaces having such an "efficient" generating family (called a basis) play an important role and motivate the following definition.

Definition 3.6. Given a vector space E and a subspace V of E, a family $(v_i)_{i\in I}$ of vectors $v_i \in V$ *spans* V or *generates* V iff for every $v \in V$, there is some family $(\lambda_i)_{i\in I}$ of scalars in K such that

$$v = \sum_{i\in I} \lambda_i v_i.$$

We also say that the elements of $(v_i)_{i\in I}$ are *generators* of V and that V is *spanned by* $(v_i)_{i\in I}$, or *generated by* $(v_i)_{i\in I}$. If a subspace V of E is generated by a finite family $(v_i)_{i\in I}$, we say that V is *finitely generated*. A family $(u_i)_{i\in I}$ that spans V and is linearly independent is called a *basis* of V.

Example 3.4.

1. In $\mathbb{R}^3$, the vectors $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, illustrated in Figure 3.9, form a basis.

2. The vectors $(1,1,1,1),(1,1,-1,-1),(1,-1,0,0),(0,0,1,-1)$ form a basis of $\mathbb{R}^4$ known as the *Haar basis*. This basis and its generalization to dimension 2^n are crucial in wavelet theory.

3. In the subspace of polynomials in $\mathbb{R}[X]$ of degree at most n, the polynomials $1, X, X^2, \ldots, X^n$ form a basis.

4. The *Bernstein polynomials* $\binom{n}{k}(1-X)^{n-k}X^k$ for $k = 0,\ldots,n$, also form a basis of that space. These polynomials play a major role in the theory of *spline curves*.

The first key result of linear algebra is that *every vector space E has a basis.* We begin with a crucial lemma which formalizes the mechanism for building a basis incrementally.

Lemma 3.6. *Given a linearly independent family $(u_i)_{i\in I}$ of elements of a vector space E, if $v \in E$ is not a linear combination of $(u_i)_{i\in I}$, then the family $(u_i)_{i\in I} \cup_k (v)$ obtained by adding v to the family $(u_i)_{i\in I}$ is linearly independent (where $k \notin I$).*

Proof. Assume that $\mu v + \sum_{i\in I} \lambda_i u_i = 0$, for any family $(\lambda_i)_{i\in I}$ of scalars in K. If $\mu \neq 0$, then μ has an inverse (because K is a field), and thus we have $v = -\sum_{i\in I}(\mu^{-1}\lambda_i)u_i$, showing that v is a linear combination of $(u_i)_{i\in I}$ and contradicting the hypothesis. Thus, $\mu = 0$. But then, we have $\sum_{i\in I} \lambda_i u_i = 0$, and since the family $(u_i)_{i\in I}$ is linearly independent, we have $\lambda_i = 0$ for all $i \in I$. $\square$