If $I = (1, \ldots, n)$, we also write $\sum_{i=1}^{n} a_i$ instead of $\sum_{i \in I} a_i$. Since $+$ is associative, Proposition 3.2 shows that the sum $\sum_{i=1}^{n} a_i$ is independent of the grouping of its elements, which justifies the use the notation $a_1 + \cdots + a_n$ (without any parentheses).

If we also assume that our associative binary operation on A is commutative, then we can show that the sum $\sum_{i \in I} a_i$ does not depend on the ordering of the index set I.

Proposition 3.3. *Given any nonempty set A equipped with an associative and commutative binary operation $+\colon A \times A \to A$, for any two nonempty finite sequences I and J of distinct natural numbers such that J is a permutation of I (in other words, the underlying sets of I and J are identical), for every sequence $(a_i)_{i \in I}$ of elements in A, we have*

$$\sum_{\alpha \in I} a_\alpha = \sum_{\alpha \in J} a_\alpha.$$

Proof. We proceed by induction on the number p of elements in I. If $p = 1$, we have $I = J$ and the proposition holds trivially.

If $p > 1$, to simplify notation, assume that $I = (1, \ldots, p)$ and that J is a permutation $(i_1, \ldots, i_p)$ of I. First, assume that $2 \leq i_1 \leq p-1$, let J' be the sequence obtained from J by deleting i_1, I' be the sequence obtained from I by deleting i_1, and let $P = (1, 2, \ldots, i_1 - 1)$ and $Q = (i_1 + 1, \ldots, p-1, p)$. Observe that the sequence I' is the concatenation of the sequences P and Q. By the induction hypothesis applied to J' and I', and then by Proposition 3.2 applied to I' and its partition (P, Q), we have

$$\sum_{\alpha \in J'} a_\alpha = \sum_{\alpha \in I'} a_\alpha = \left(\sum_{i=1}^{i_1 - 1} a_i\right) + \left(\sum_{i=i_1+1}^{p} a_i\right).$$

If we add the lefthand side to a_{i_1}, by definition we get

$$\sum_{\alpha \in J} a_\alpha.$$

If we add the righthand side to a_{i_1}, we get

$$a_{i_1} + \left(\left(\sum_{i=1}^{i_1 - 1} a_i\right) + \left(\sum_{i=i_1+1}^{p} a_i\right)\right).$$

Using associativity, we get

$$a_{i_1} + \left(\left(\sum_{i=1}^{i_1 - 1} a_i\right) + \left(\sum_{i=i_1+1}^{p} a_i\right)\right) = \left(a_{i_1} + \left(\sum_{i=1}^{i_1 - 1} a_i\right)\right) + \left(\sum_{i=i_1+1}^{p} a_i\right),$$