Proof. For $i = 1, \ldots, n$, let $p_i \colon M \to M$ be the map given by

$$p_i(x) = e_i x, \quad x \in M.$$

The map p_i is clearly linear, and because of the properties satisfied by the e_is, we have

$$\begin{aligned} p_i^2 &= p_i \\ p_i p_j &= 0, \quad i \neq j \\ p_1 + \cdots + p_n &= \mathrm{id}. \end{aligned}$$

This shows that the p_i are projections, and by Proposition 6.8 (which also holds for modules), we have a direct sum

$$M = p_1(M) \oplus \cdots \oplus p_n(M) = e_1 M \oplus \cdots \oplus e_n M.$$

It remains to show that $M_i = e_i M$. Since $(1 - e_i)e_i = e_i - e_i^2 = e_i - e_i = 0$, we see that $e_i M$ is annihilated by $\mathfrak{b}_i = (1 - e_i)A$. Furthermore, for $i \neq j$, for any $x \in M$, we have $(1 - e_i)e_j x = (e_j - e_i e_j)x = e_j x$, so no nonzero element of $e_j M$ is annihilated by $1 - e_i$, and thus not annihilated by $\mathfrak{b}_i$. It follows that $e_i M = M_i$, as claimed. $\square$

Definition 35.8. Given an A-module M, for any nonzero $\alpha \in A$, let

$$M(\alpha) = \{x \in M \mid \alpha x = 0\},$$

the submodule of M annihilated by α. If α divides β, then $M(\alpha) \subseteq M(\beta)$, so we can define

$$M_\alpha = \bigcup_{n \geq 1} M(\alpha^n) = \{x \in M \mid (\exists n \geq 1)(\alpha^n x = 0)\},$$

the submodule of M consisting of all elements of M annihilated by some power of α.

If N is any submodule of M, it is clear that

$$N_\alpha = M \cap M_\alpha.$$

Recall that in a PID, an irreducible element is also called a *prime element*.

Definition 35.9. If A is a PID and p is a prime element in A, we say that a module M is *p-primary* if $M = M_p$.

Proposition 35.16. *Let M be module over a PID A. For every nonzero $\alpha \in A$, if*

$$\alpha = u p_1^{n_1} \cdots p_r^{n_r}$$

is a factorization of α into prime factors (where u is a unit), then the module $M(\alpha)$ annihilated by α is the direct sum

$$M(\alpha) = M(p_1^{n_1}) \oplus \cdots \oplus M(p_r^{n_r}).$$

Furthermore, the projection from $M(\alpha)$ onto $M(p_i^{n_i})$ is of the form $x \mapsto \gamma_i x$, for some $\gamma_i \in A$, and

$$M(p_i^{n_i}) = M(\alpha) \cap M_{p_i}.$$