namely,

$$E_{2,1;1/2}, E_{3,2;2/3}, E_{4,3;3/4}, \cdots, E_{n,n-1;(n-1)/n}.$$

It is also natural to conjecture that the lower triangular matrix L_n such that

$$L_n U_n = A_n$$

is given by

$$L_n = E_{2,1;-1/2} E_{3,2;-2/3} E_{4,3;-3/4} \cdots E_{n,n-1;-(n-1)/n},$$

that is,

$$L_n = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2/3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3/4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4/5 & 1 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \cdots & -(n-1)/n & 1 \end{pmatrix}.$$

Prove the above conjectures.

(6) Prove that the last column of A_n^{-1} is

$$\begin{pmatrix} 1/(n+1) \\ 2/(n+1) \\ \vdots \\ n/(n+1) \end{pmatrix}.$$

Problem 8.10. (1) Let A be any invertible 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Prove that there is an invertible matrix S such that

$$SA = \begin{pmatrix} 1 & 0 \\ 0 & ad - bc \end{pmatrix},$$

where S is the product of at most four elementary matrices of the form $E_{i,j;\beta}$.

Conclude that every matrix A in $\mathbf{SL}(2)$ (the group of invertible 2×2 matrices A with $\det(A) = +1$) is the product of at most four elementary matrices of the form $E_{i,j;\beta}$.

For any $a \neq 0, 1$, give an explicit factorization as above for

$$A = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}.$$