Example 50.11. Consider the Hard Margin Problem (SVM$_{h1}$):

$$
\begin{aligned}
&\text{maximize} \quad \delta \\
&\text{subject to} \\
&\qquad w^\top u_i - b \geq \delta \qquad i = 1, \ldots, p \\
&\qquad - w^\top v_j + b \geq \delta \qquad j = 1, \ldots, q \\
&\qquad \|w\|_2 \leq 1,
\end{aligned}
$$

which is converted to the following minimization problem:

$$
\begin{aligned}
&\text{minimize} \quad -2\delta \\
&\text{subject to} \\
&\qquad w^\top u_i - b \geq \delta \qquad i = 1, \ldots, p \\
&\qquad - w^\top v_j + b \geq \delta \qquad j = 1, \ldots, q \\
&\qquad \|w\|_2 \leq 1.
\end{aligned}
$$

We replaced δ by 2δ because this will make it easier to find a nice geometric interpretation. Recall from Section 50.5 that Problem (SVM$_{h1}$) has a an optimal solution iff $\delta > 0$, in which case $\|w\| = 1$.

The corresponding Lagrangian with $\lambda \in \mathbb{R}^p_+, \mu \in \mathbb{R}^q_+, \gamma \in \mathbb{R}^+$, is

$$
\begin{aligned}
L(w, b, \delta, \lambda, \mu, \gamma) &= -2\delta + \sum_{i=1}^p \lambda_i(\delta + b - w^\top u_i) + \sum_{j=1}^q \mu_j(\delta - b + w^\top v_j) + \gamma(\|w\|_2 - 1) \\
&= w^\top \left(- \sum_{i=1}^p \lambda_i u_i + \sum_{j=1}^q \mu_j v_j \right) + \gamma \|w\|_2 + \left(\sum_{i=1}^p \lambda_i - \sum_{j=1}^q \mu_j \right) b \\
&\quad + \left(-2 + \sum_{i=1}^p \lambda_i + \sum_{j=1}^q \mu_j \right) \delta - \gamma.
\end{aligned}
$$

Next to find the dual function $G(\lambda, \mu, \gamma)$ we need to minimize $L(w, b, \delta, \lambda, \mu, \gamma)$ with respect to w, b and δ, so its gradient with respect to w, b and δ must be zero. This implies that

$$
\begin{aligned}
\sum_{i=1}^p \lambda_i - \sum_{j=1}^q \mu_j &= 0 \\
-2 + \sum_{i=1}^p \lambda_i + \sum_{j=1}^q \mu_j &= 0,
\end{aligned}
$$

which yields

$$
\sum_{i=1}^p \lambda_i = \sum_{j=1}^q \mu_j = 1.
$$