Since the left-hand side is independent of w, it is a lower bound for the right-hand side for all w, so we obtain $(*_1)$:

$$\sup_{\mu\in M}\inf_{v\in\Omega} L(v,\mu) \leq \inf_{v\in\Omega}\sup_{\mu\in M} L(v,\mu).$$

To obtain the reverse inequality, we use the fact that (u,λ) is a saddle point, so

$$\inf_{v\in\Omega}\sup_{\mu\in M} L(v,\mu) \leq \sup_{\mu\in M} L(u,\mu) = L(u,\lambda)$$

and

$$L(u,\lambda) = \inf_{v\in\Omega} L(v,\lambda) \leq \sup_{\mu\in M}\inf_{v\in\Omega} L(v,\mu),$$

and these imply that

$$\inf_{v\in\Omega}\sup_{\mu\in M} L(v,\mu) \leq \sup_{\mu\in M}\inf_{v\in\Omega} L(v,\mu), \qquad (*_2)$$

as desired. $\square$

We now return to our main Minimization Problem (P):

$$\begin{aligned} &\text{minimize} && J(v)\\ &\text{subject to} && \varphi_i(v)\leq 0, \quad i=1,\ldots,m, \end{aligned}$$

where $J\colon \Omega\to\mathbb{R}$ and the constraints $\varphi_i\colon\Omega\to\mathbb{R}$ are some functions defined on some open subset Ω of some finite-dimensional Euclidean vector space V (more generally, a real Hilbert space V).

Definition 50.8. The *Lagrangian* of the Minimization Problem (P) defined above is the function $L\colon \Omega\times\mathbb{R}^m_+\to\mathbb{R}$ given by

$$L(v,\mu) = J(v) + \sum_{i=1}^m \mu_i\varphi_i(v),$$

with $\mu = (\mu_1,\ldots,\mu_m)$. The numbers μ_i are called *generalized Lagrange multipliers*.

The following theorem shows that under some suitable conditions, every solution u of the Problem (P) is the first argument of a saddle point (u,λ) of the Lagrangian L, and conversely, if (u,λ) is a saddle point of the Lagrangian L, then u is a solution of the Problem (P).

Theorem 50.15. *Consider Problem (P) defined above where $J\colon\Omega\to\mathbb{R}$ and the constraints $\varphi_i\colon\Omega\to\mathbb{R}$ are some functions defined on some open subset Ω of some finite-dimensional Euclidean vector space V (more generally, a real Hilbert space V). The following facts hold.*

(1) If $(u,\lambda)\in\Omega\times\mathbb{R}^m_+$ is a saddle point of the Lagrangian L associated with Problem (P), then $u\in U$, u is a solution of Problem (P), and $J(u)=L(u,\lambda)$.