Chapter 33

Tensor Algebras and Symmetric Algebras

Tensors are creatures that we would prefer did not exist but keep showing up whenever multilinearity manifests itself.

One of the goals of differential geometry is to be able to generalize "calculus on $\mathbb{R}^n$" to spaces more general than $\mathbb{R}^n$, namely manifolds. We would like to differentiate functions $f\colon M \to \mathbb{R}$ defined on a manifold, optimize functions (find their minima or maxima), but also to integrate such functions, as well as compute areas and volumes of subspaces of our manifold.

The suitable notion of differentiation is the notion of tangent map, a linear notion. One of the main discoveries made at the beginning of the twentieth century by Poincaré and Élie Cartan, is that the "right" approach to integration is to integrate *differential forms*, and not functions. To integrate a function f, we integrate the form $f\omega$, where ω is a *volume form* on the manifold M. The formalism of differential forms takes care of the process of the change of variables quite automatically, and allows for a very clean statement of *Stokes' formula*.

Differential forms can be combined using a notion of product called the wedge product, but what really gives power to the formalism of differential forms is the magical operation d of *exterior differentiation*. Given a form ω, we obtain another form $d\omega$, and remarkably, the following equation holds

$$dd\omega = 0.$$

As silly as it looks, the above equation lies at the core of the notion of cohomology, a powerful algebraic tool to understand the topology of manifolds, and more generally of topological spaces.

Élie Cartan had many of the intuitions that lead to the cohomology of differential forms, but it was George de Rham who defined it rigorously and proved some important theorems about it. It turns out that the notion of Laplacian can also be defined on differential forms using a device due to Hodge, and some important theorems can be obtained: the Hodge