**Remark:** Many books on quantum mechanics use the so-called Dirac notation to denote objects in the Hilbert space $E$ and operators in its dual space $E'$. In the Dirac notation, an element of $E$ is denoted as $|x\rangle$, and an element of $E'$ is denoted as $\langle t|$. The scalar product is denoted as $\langle t| \cdot |x\rangle$. This uses the isomorphism between $E$ and $E'$, except that the inner product is assumed to be semi-linear on the left rather than on the right.

Proposition 48.9 allows us to define the adjoint of a linear map, as in the Hermitian case (see Proposition 14.8). Actually, we can prove a slightly more general result which is used in optimization theory.

If $\varphi\colon E\times E \to \mathbb{C}$ is a sesquilinear map on a normed vector space $(E, \| \|)$, then Proposition 37.59 is immediately adapted to prove that $\varphi$ is continuous iff there is some constant $k \geq 0$ such that

$$|\varphi(u, v)| \leq k \,\|u\| \,\|v\| \quad \text{for all } u, v \in E.$$

Thus we define $\|\varphi\|$ as in Definition 37.42 by

$$\|\varphi\| = \sup \{|\varphi(x, y)| \mid \|x\| \leq 1,\ \|y\| \leq 1,\ x, y \in E\}.$$

**Proposition 48.10.** *Given a Hilbert space $E$, for every continuous sesquilinear map $\varphi\colon E \times E \to \mathbb{C}$, there is a unique continuous linear map $f_\varphi\colon E \to E$, such that*

$$\varphi(u, v) = \langle u, f_\varphi(v)\rangle \quad \textit{for all } u, v \in E.$$

*We also have $\|f_\varphi\| = \|\varphi\|$. If $\varphi$ is Hermitian, then $f_\varphi$ is self-adjoint, that is*

$$\langle u, f_\varphi(v)\rangle = \langle f_\varphi(u), v\rangle \quad \textit{for all } u, v \in E.$$

*Proof.* The proof is adapted from Rudin [141] (Theorem 12.8). To define the function $f_\varphi$, we proceed as follows. For any fixed $v \in E$, define the linear map $\varphi_v$ by

$$\varphi_v(u) = \varphi(u, v) \quad \text{for all } u \in E.$$

Since $\varphi$ is continuous, $\varphi_v$ is continuous. So by Proposition 48.9, there is a unique vector in $E$ that we denote $f_\varphi(v)$ such that

$$\varphi_v(u) = \langle u, f_\varphi(v)\rangle \quad \text{for all } u \in E,$$

and $\|f_\varphi(v)\| = \|\varphi_v\|$. Let us check that the map $v \mapsto f_\varphi(v)$ is linear.

We have

$$\begin{aligned}
\varphi(u, v_1 + v_2) &= \varphi(u, v_1) + \varphi(u, v_2) && \varphi \text{ is additive}\\
&= \langle u, f_\varphi(v_1)\rangle + \langle u, f_\varphi(v_2)\rangle && \text{by definition of } f_\varphi\\
&= \langle u, f_\varphi(v_1) + f_\varphi(v_2)\rangle && \langle -, -\rangle \text{ is additive}
\end{aligned}$$