where $E_1 = Z(u_1, f) \cong \mathbb{R}[X]/(X)$, $E_2 = Z(u_2, f) \cong \mathbb{R}[X]/(X)$, $E_3 = Z(u_3, f) \cong \mathbb{R}[X]/(X-2)$, and $E_4 = Z(u_4, f) \cong \mathbb{R}[X]/(X-2)$. The subspaces E_1 and E_2 correspond to one-dimensional spaces spanned by eigenvectors associated with eigenvalue 0, while E_3 and E_4 correspond to one-dimensional spaces spanned by eigenvectors associated with eigenvalue 2. If we let $u_1 = (-1, 0, 0, 1)$, $u_2 = (0, -1, 1, 0)$, $u_3 = (1, 0, 0, 1)$ and $u_4 = (0, 1, 1, 0)$, Theorem 36.15 gives

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$

as the rational canonical form associated with the cyclic decomposition $\mathbb{R}^4 = E_1 \oplus E_2 \oplus E_3 \oplus E_4$.

As we pointed earlier, unlike the similarity invariants, the elementary divisors may change when we pass to a field extension.

We will now consider the special case where all the irreducible polynomials p_i are of the form $X - \lambda_i$; that is, when are the eigenvalues of f belong to K. In this case, we find again the Jordan form.

36.4 The Jordan Form Revisited

In this section, we assume that all the roots of the minimal polynomial of f belong to K. This will be the case if K is algebraically closed. The irreducible polynomials p_i of Theorem 36.14 are the polynomials $X - \lambda_i$, for the distinct eigenvalues λ_i of f. Then, each cyclic subspace $Z(u_j; f)$ has a minimal polynomial of the form $(X - \lambda)^m$, for some eigenvalue λ of f and some $m \geq 1$. It turns out that by choosing a suitable basis for the cyclic subspace $Z(u_j; f)$, the matrix of the restriction of f to $Z(u_j; f)$ is a Jordan block.

Proposition 36.16. *Let E be a finite-dimensional K-vector space and let $f \colon E \to E$ be a linear map. If E is a cyclic $K[X]$-module and if $(X - \lambda)^n$ is the minimal polynomial of f, then there is a basis of E of the form*

$$((f - \lambda \mathrm{id})^{n-1}(u), (f - \lambda \mathrm{id})^{n-2}(u), \ldots, (f - \lambda \mathrm{id})(u), u),$$

for some $u \in E$. With respect to this basis, the matrix of f is the Jordan block

$$J_n(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}.$$