Tensor products can be defined in various ways, some more abstract than others. We try to stay down to earth, without excess.

Before proceeding any further, we review some facts about dual spaces and pairings. Pairings will be used to deal with dual spaces of tensors.

33.1 Linear Algebra Preliminaries: Dual Spaces and Pairings

We assume that we are dealing with vector spaces over a field K. As usual the *dual space* E^* of a vector space E is defined by $E^* = \mathrm{Hom}(E, K)$. The dual space E^* is the vector space consisting of all linear maps $\omega\colon E \to K$ with values in the field K.

A problem that comes up often is to decide when a space E is isomorphic to the dual F^* of some other space F (possibly equal to E). The notion of pairing due to Pontrjagin provides a very clean criterion.

Definition 33.1. Given two vector spaces E and F over a field K, a map $\langle -, -\rangle\colon E \times F \to K$ is a *nondegenerate pairing* iff it is bilinear and iff $\langle u, v\rangle = 0$ for all $v \in F$ implies $u = 0$, and $\langle u, v\rangle = 0$ for all $u \in E$ implies $v = 0$. A nondegenerate pairing induces two linear maps $\varphi\colon E \to F^*$ and $\psi\colon F \to E^*$ defined such that for all for all $u \in E$ and all $v \in F$, $\varphi(u)$ is the linear form in F^* and $\psi(v)$ is the linear form in E^* given by

$$\begin{aligned} \varphi(u)(y) &= \langle u, y\rangle \quad \text{for all } y \in F \\ \psi(v)(x) &= \langle x, v\rangle \quad \text{for all } x \in E. \end{aligned}$$

Schematically, $\varphi(u) = \langle u, -\rangle$ and $\psi(v) = \langle -, v\rangle$.

Proposition 33.1. *For every nondegenerate pairing $\langle -, -\rangle\colon E \times F \to K$, the induced maps $\varphi\colon E \to F^*$ and $\psi\colon F \to E^*$ are linear and injective. Furthermore, if E and F are finite dimensional, then $\varphi\colon E \to F^*$ and $\psi\colon F \to E^*$ are bijective.*

Proof. The maps $\varphi\colon E \to F^*$ and $\psi\colon F \to E^*$ are linear because $u, v \mapsto \langle u, v\rangle$ is bilinear. Assume that $\varphi(u) = 0$. This means that $\varphi(u)(y) = \langle u, y\rangle = 0$ for all $y \in F$, and as our pairing is nondegenerate, we must have $u = 0$. Similarly, ψ is injective. If E and F are finite dimensional, then $\dim(E) = \dim(E^*)$ and $\dim(F) = \dim(F^*)$. However, the injectivity of φ and ψ implies that that $\dim(E) \leq \dim(F^*)$ and $\dim(F) \leq \dim(E^*)$. Consequently $\dim(E) \leq \dim(F)$ and $\dim(F) \leq \dim(E)$, so $\dim(E) = \dim(F)$. Therefore, $\dim(E) = \dim(F^*)$ and φ is bijective (and similarly $\dim(F) = \dim(E^*)$ and ψ is bijective). $\square$

Proposition 33.1 shows that when E and F are finite dimensional, a nondegenerate pairing induces *canonical isomorphims* $\varphi\colon E \to F^*$ and $\psi\colon F \to E^*$; that is, isomorphisms that do not depend on the choice of bases. An important special case is the case where $E = F$ and we have an inner product (a symmetric, positive definite bilinear form) on E.