Definition 26.12. Let $h\colon \mathbb{P}(E) \to \mathbb{P}(E)$ be a homology of axis $\mathbb{P}(H)$ with $h \neq \mathrm{id}$, where $h = \mathbb{P}(f)$ for some linear isomorphism $f\colon E \to E$. The fixed point $O = [u]$ associated with the vector u involved in the definition of f, which is unique up to a scalar, is called the *center* of h. If $O \in \mathbb{P}(H)$, then h is called a *projective transvection* (or *elation*).

The same geometric construction that we used in the case of the projective plane shows that a homology is determined by its center O, its axis $\mathbb{P}(H)$, and a pair of points A and $A' = h(A)$, with $A \neq O$ and $A \notin \mathbb{P}(H)$. As a kind of converse, we have the following proposition which is easily shown; see Vienne [185] (Chapter IV, Proposition 8).

Proposition 26.25. *Let $\mathbb{P}(H)$ be a hyperplane of $\mathbb{P}(E)$ and let $O \in \mathbb{P}(E)$ be a point. For any pair of distinct points (A, A') such that O, A, A' are collinear and $A, A' \notin \mathbb{P}(H) \cup \{O\}$, there is a unique homology $h\colon \mathbb{P}(E) \to \mathbb{P}(E)$ of centrer O and axis $\mathbb{P}(H)$ such that $h(A) = A'$.*

Remark: From the proof of Proposition 8.23, since every dilatation can be represented by a matrix of the form

$$\begin{pmatrix} \alpha & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

we see that by choosing the hyperplane at infinity to be $x_1 = 0$, on the affine hyperplane $x_1 = 1$, a homology becomes a central magnification by α^{-1}. Similarly, since every transvection can be represented by a matrix of the form

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ \alpha & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

we see that by choosing the hyperplane at infinity to be $x_1 = 0$, on the affine hyperplane $x_1 = 1$, an elation becomes a translation.

Theorem 8.26 immediately yields the following result showing that the group of homographies $\mathbf{PGL}(E)$ is generated by the homologies.

Theorem 26.26. *Let E be any finite-dimensional vector space over a field K of characteristic not equal to 2. Then, the group of homographies $\mathbf{PGL}(E)$ is generated by the homologies.*

When $E = \mathbb{R}^3$, we saw earlier that the involutions of $\mathbb{RP}^2$ have a nice structure. In particular, if an involution has two fixed points, then it is a harmonic homology.

If $\dim(E) \geq 4$, it is harder to characterize the involutions of $\mathbb{P}(E)$, but it is possible. The case where the linear isomorphism $f\colon E \to E$ defining the involutive homography $h = \mathbb{P}(f)$