and since $\epsilon = \pm 1$, we have $\epsilon^2 = 1$, so we get

$$a' = -\epsilon b.$$

By adding and subtracting the equations on the third row we obtain

$$c = c' = 0.$$

Since A is invertible, $c'' \neq 0$, and since A is determined up to a nonzero scalar we may assume that $c'' = 1$, and we conclude that

$$A = \begin{pmatrix} a & -\epsilon b & a'' \\ b & \epsilon a & b'' \\ 0 & 0 & 1 \end{pmatrix}.$$

If h maps $\mathbb{R}^2$ into itself, then

$$\begin{pmatrix} a & -\epsilon b & a'' \\ b & \epsilon a & b'' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

must be real for all $x, y \in \mathbb{R}$, which implies that $a, b, a'', b'' \in \mathbb{R}$. $\square$

The following proposition from Berger [11] (Chapter 8, Proposition 8.8.5.1) gives a convenient characterization of the affine similarities.

Proposition 26.30. *Let $(E, \overrightarrow{E}, \langle -, - \rangle)$ be a Euclidean affine space of finite dimension $n \geq 2$. An affine map $f \in \mathbf{GA}(E)$ is an affine similarity iff $\overrightarrow{f}$ preserves orthogonality; that is, for any two vectors $u, v \in \overrightarrow{E}$, if $\langle u, v \rangle = 0$, then $\langle \overrightarrow{f}(u), \overrightarrow{f}(v) \rangle = 0$.*

Proof. Assume that $f \in \mathbf{GA}(E)$ is an affine map such that for any two vectors $u, v \in \overrightarrow{E}$, if $\langle u, v \rangle = 0$, then $\langle \overrightarrow{f}(u), \overrightarrow{f}(v) \rangle = 0$. Fix any nonzero $u \in \overrightarrow{E}$ and consider the linear form φ_u given by

$$\varphi_u(v) = \langle \overrightarrow{f}(u), \overrightarrow{f}(v) \rangle, \quad v \in \overrightarrow{E}.$$

Since $\overrightarrow{f}$ is invertible, $\varphi_u(u) \neq 0$. For any $v \in \overrightarrow{E}$ such that $\langle u, v \rangle = 0$, we have

$$\varphi_u(v) = \langle \overrightarrow{f}(u), \overrightarrow{f}(v) \rangle = 0,$$

thus φ_u is a nonzero linear form vanishing on the hyperplane H orthogonal to u, which is the kernel of the linear form $v \mapsto \langle u, v \rangle$. Therefore, there is some nonzero scalar $\rho(u) \in \mathbb{R}$ such that

$$\varphi_u(v) = \rho(u) \langle u, v \rangle \quad \text{for all } v \in \overrightarrow{E}.$$

Evaluating φ_u at u, we see that $\rho(u) > 0$. If we can show that $\rho(u)$ is a constant $\rho > 0$ independent of u, we will have shown that

$$\langle \overrightarrow{f}(u), \overrightarrow{f}(v) \rangle = \rho \langle u, v \rangle \quad \text{for all } u, v \in \overrightarrow{E},$$