Then

$$D^+ = \begin{pmatrix} 0.0583 & 0 & 0 & 0 \\ 0 & 0.8583 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$A^+ = UD^+V^\top = \begin{pmatrix} -0.5100 & -0.2200 & 0.0700 & 0.3600 \\ -0.2200 & -0.0900 & 0.0400 & 0.1700 \\ 0.0700 & 0.0400 & 0.0100 & -0.0200 \\ 0.3600 & 0.1700 & -0.0200 & -0.2100 \end{pmatrix},$$

which is also the result obtained by calling `pinv(A)`.

If A is an $m \times n$ matrix of rank n (and so $m \geq n$), it is immediately shown that the QR-decomposition in terms of Householder transformations applies as follows:

There are n $m \times m$ matrices $H_1, \ldots, H_n$, Householder matrices or the identity, and an upper triangular $m \times n$ matrix R of rank n such that

$$A = H_1 \cdots H_n R.$$

Then because each H_i is an isometry,

$$\|Ax - b\|_2 = \|Rx - H_n \cdots H_1 b\|_2,$$

and the least squares problem $Ax = b$ is equivalent to the system

$$Rx = H_n \cdots H_1 b.$$

Now the system

$$Rx = H_n \cdots H_1 b$$

is of the form

$$\begin{pmatrix} R_1 \\ 0_{m-n} \end{pmatrix} x = \begin{pmatrix} c \\ d \end{pmatrix},$$

where R_1 is an invertible $n \times n$ matrix (since A has rank n), $c \in \mathbb{R}^n$, and $d \in \mathbb{R}^{m-n}$, and the least squares solution of smallest norm is

$$x^+ = R_1^{-1} c.$$

Since R_1 is a triangular matrix, it is very easy to invert R_1.

The method of least squares is one of the most effective tools of the mathematical sciences. There are entire books devoted to it. Readers are advised to consult Strang [170], Golub and Van Loan [80], Demmel [48], and Trefethen and Bau [176], where extensions and applications of least squares (such as weighted least squares and recursive least squares) are described. Golub and Van Loan [80] also contains a very extensive bibliography, including a list of books on least squares.