(R3) $*$ is distributive w.r.t. $+$.

The identity element for addition is denoted 0, and the additive inverse of $a \in A$ is denoted by $-a$. More explicitly, the axioms of a ring are the following equations which hold for all $a, b, c \in A$:

$$
\begin{aligned}
a + (b + c) &= (a + b) + c && \text{(associativity of } +\text{)} && (2.1)\\
a + b &= b + a && \text{(commutativity of } +\text{)} && (2.2)\\
a + 0 &= 0 + a = a && \text{(zero)} && (2.3)\\
a + (-a) &= (-a) + a = 0 && \text{(additive inverse)} && (2.4)\\
a * (b * c) &= (a * b) * c && \text{(associativity of } *\text{)} && (2.5)\\
a * 1 &= 1 * a = a && \text{(identity for } *\text{)} && (2.6)\\
(a + b) * c &= (a * c) + (b * c) && \text{(distributivity)} && (2.7)\\
a * (b + c) &= (a * b) + (a * c) && \text{(distributivity)} && (2.8)
\end{aligned}
$$

The ring A is *commutative* if

$$a * b = b * a \quad \text{for all } a, b \in A.$$

From (2.7) and (2.8), we easily obtain

$$
\begin{aligned}
a * 0 &= 0 * a = 0 && (2.9)\\
a * (-b) &= (-a) * b = -(a * b). && (2.10)
\end{aligned}
$$

Note that (2.9) implies that if $1 = 0$, then $a = 0$ for all $a \in A$, and thus, $A = \{0\}$. The ring $A = \{0\}$ is called the *trivial ring*. A ring for which $1 \neq 0$ is called *nontrivial*. The multiplication $a * b$ of two elements $a, b \in A$ is often denoted by ab.

Example 2.6.

1. The additive groups $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$, are commutative rings.

2. For any positive integer $n \in \mathbb{N}$, the group $\mathbb{Z}/n\mathbb{Z}$ is a group under addition. We can also define a multiplication operation by

$$\overline{a} \cdot \overline{b} = \overline{ab} = \overline{ab \bmod n},$$

for all $a, b \in \mathbb{Z}$. The reader will easily check that the ring axioms are satisfied, with $\overline{0}$ as zero and $\overline{1}$ as multiplicative unit. The resulting ring is denoted by $\mathbb{Z}/n\mathbb{Z}$.[2]

3. The group $\mathbb{R}[X]$ of polynomials in one variable with real coefficients is a ring under multiplication of polynomials. It is a commutative ring.

[2]The notation $\mathbb{Z}_n$ is sometimes used instead of $\mathbb{Z}/n\mathbb{Z}$ but it clashes with the notation for the *n-adic integers* so we prefer not to use it.