We can substitute the matrix A for the variable X in the polynomial $P_A(X)$, obtaining a *matrix* P_A. If we write

$$P_A(X) = X^n + c_1 X^{n-1} + \cdots + c_n,$$

then

$$P_A = A^n + c_1 A^{n-1} + \cdots + c_n I.$$

We have the following remarkable theorem.

Theorem 7.14. *(Cayley–Hamilton) If K is any commutative ring, for every $n \times n$ matrix $A \in \mathrm{M}_n(K)$, if we let*

$$P_A(X) = X^n + c_1 X^{n-1} + \cdots + c_n$$

be the characteristic polynomial of A, then

$$P_A = A^n + c_1 A^{n-1} + \cdots + c_n I = 0.$$

Proof. We can view the matrix $B = XI - A$ as a matrix with coefficients in the polynomial ring $K[X]$, and then we can form the matrix $\widetilde{B}$ which is the transpose of the matrix of cofactors of elements of B. Each entry in $\widetilde{B}$ is an $(n-1) \times (n-1)$ determinant, and thus a polynomial of degree a most $n-1$, so we can write $\widetilde{B}$ as

$$\widetilde{B} = X^{n-1} B_0 + X^{n-2} B_1 + \cdots + B_{n-1},$$

for some $n \times n$ matrices $B_0, \ldots, B_{n-1}$ with coefficients in K. For example, when $n = 2$, we have

$$B = \begin{pmatrix} X-a & -b \\ -c & X-d \end{pmatrix}, \quad \widetilde{B} = \begin{pmatrix} X-d & b \\ c & X-a \end{pmatrix} = X \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -d & b \\ c & -a \end{pmatrix}.$$

By Proposition 7.10, we have

$$B\widetilde{B} = \det(B) I = P_A(X) I.$$

On the other hand, we have

$$B\widetilde{B} = (XI - A)(X^{n-1} B_0 + X^{n-2} B_1 + \cdots + X^{n-j-1} B_j + \cdots + B_{n-1}),$$

and by multiplying out the right-hand side, we get

$$B\widetilde{B} = X^n D_0 + X^{n-1} D_1 + \cdots + X^{n-j} D_j + \cdots + D_n,$$

with

$$\begin{aligned} D_0 &= B_0 \\ D_1 &= B_1 - AB_0 \\ &\vdots \\ D_j &= B_j - AB_{j-1} \\ &\vdots \\ D_{n-1} &= B_{n-1} - AB_{n-2} \\ D_n &= -AB_{n-1}. \end{aligned}$$