and on the other hand the continuity of a implies that

$$a(v, v) \leq \|a\| \, \|v\|^2 ,$$

so we get

$$\sqrt{\alpha} \, \|v\| \leq (a(v, v))^{1/2} \leq \sqrt{\|a\|} \, \|v\| .$$

The above also shows that the norm $v \mapsto (a(v, v))^{1/2}$ induced by the inner product a is equivalent to the norm induced by the inner product $\langle -, - \rangle$ on V. Thus h is still continuous with respect to the norm $v \mapsto (a(v, v))^{1/2}$. Then by the Riesz representation theorem (Proposition 48.9), there is some unique $c \in V$ such that

$$h(v) = a(c, v) \quad \text{for all } v \in V.$$

Consequently, we can express $J(v)$ as

$$J(v) = \frac{1}{2}a(v, v) - a(c, v) = \frac{1}{2}a(v - c, v - c) - \frac{1}{2}a(c, c).$$

But then minimizing $J(v)$ over U is equivalent to minimizing $(a(v - c, v - c))^{1/2}$ over $v \in U$, and by the projection lemma (Proposition 48.5(1)) this is equivalent to finding the projection $p_U(c)$ of c on the closed convex set U with respect to the inner product a. Therefore, there is a unique $u = p_U(c) \in U$ such that

$$J(u) = \inf_{v \in U} J(v).$$

Also by Proposition 48.5(2), this unique element $u \in U$ is characterized by the condition

$$a(u - c, v - u) \geq 0 \quad \text{for all } v \in U.$$

Since

$$a(u - c, v - u) = a(u, v - u) - a(c, v - u) = a(u, v - u) - h(v - u),$$

the above inequality is equivalent to

$$a(u, v - u) \geq h(v - u) \quad \text{for all } v \in U. \tag{$*$}$$

If U is a subspace of V, then by Proposition 48.5(3) we have the condition

$$a(u - c, v) = 0 \quad \text{for all } v \in U,$$

which is equivalent to

$$a(u, v) = a(c, v) = h(v) \quad \text{for all } v \in U, \tag{$**$}$$

a claimed. $\square$