$t_{ii} = \overline{t_{ii}}$ for $i = 1, \ldots, n$, which means that the t_{ii} are real, so T is indeed a real diagonal matrix, say D.

If we apply this result to a (real) symmetric matrix A, we obtain the fact that all the eigenvalues of a symmetric matrix are real, and by applying Theorem 15.6 again, we conclude that $A = QDQ^{\top}$, where Q is orthogonal and D is a real diagonal matrix. $\square$

More general versions of Proposition 15.8 are proven in Chapter 17.

When a real matrix A has complex eigenvalues, there is a version of Theorem 15.6 involving only real matrices provided that we allow T to be block upper-triangular (the diagonal entries may be 2×2 matrices or real entries).

Theorem 15.6 is not a very practical result but it is a useful theoretical result to cope with matrices that cannot be diagonalized. For example, it can be used to prove that *every* complex matrix is the limit of a sequence of diagonalizable matrices that have distinct eigenvalues!

15.3 Location of Eigenvalues

If A is an $n \times n$ complex (or real) matrix A, it would be useful to know, even roughly, where the eigenvalues of A are located in the complex plane $\mathbb{C}$. The Gershgorin discs provide some precise information about this.

Definition 15.5. For any complex $n \times n$ matrix A, for $i = 1, \ldots, n$, let

$$R'_i(A) = \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{i\,j}|$$

and let

$$G(A) = \bigcup_{i=1}^{n} \{z \in \mathbb{C} \mid |z - a_{i\,i}| \leq R'_i(A)\}.$$

Each disc $\{z \in \mathbb{C} \mid |z - a_{i\,i}| \leq R'_i(A)\}$ is called a *Gershgorin disc* and their union $G(A)$ is called the *Gershgorin domain*. An example of Gershgorin domain for $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & i & 6 \\ 7 & 8 & 1+i \end{pmatrix}$ is illustrated in Figure 15.1.

Although easy to prove, the following theorem is very useful:

Theorem 15.9. *(Gershgorin's disc theorem) For any complex $n \times n$ matrix A, all the eigenvalues of A belong to the Gershgorin domain $G(A)$. Furthermore the following properties hold:*