The space $\mathcal{K}_n(A, b) = \mathrm{Span}(b, Ab, \ldots, A^{n-1}b)$ is called a *Krylov subspace*. We can view Arnoldi's algorithm as the construction of an orthonormal basis for $\mathcal{K}_n(A, b)$. It is a sort of Gram–Schmidt procedure.

Equation $(*_2)$ shows that if K_n is the $m \times n$ matrix whose columns are the vectors $(b, Ab, \ldots, A^{n-1}b)$, then there is a $n \times n$ upper triangular matrix R_n such that

$$K_n = U_n R_n. \tag{$*_4$}$$

The above is called a *reduced QR factorization* of K_n.

Since $(u_1, \ldots, u_n)$ is an orthonormal system, the matrix $U_n^* U_{n+1}$ is the $n \times (n+1)$ matrix consisting of the identity matrix I_n plus an extra column of 0's, so $U_n^* U_{n+1} \widetilde{H}_n = U_n^* A U_n$ is obtained by deleting the last row of $\widetilde{H}_n$, namely H_n, and so

$$U_n^* A U_n = H_n. \tag{$*_5$}$$

We summarize the above facts in the following proposition.

Proposition 18.5. *If Arnoldi iteration run on an $m \times m$ matrix A starting with a nonzero vector $b \in \mathbb{C}^m$ does not have a breakdown at stage $n \leq m$, then the following properties hold:*

(1) If K_n is the $m \times n$ Krylov matrix associated with the vectors $(b, Ab, \ldots, A^{n-1}b)$ and if U_n is the $m \times n$ matrix of orthogonal vectors produced by Arnoldi iteration, then there is a QR-factorization

$$K_n = U_n R_n,$$

for some $n \times n$ upper triangular matrix R_n.

(2) The $m \times n$ upper Hessenberg matrices H_n produced by Arnoldi iteration are the projection of A onto the Krylov space $\mathcal{K}_n(A, b)$, that is,

$$H_n = U_n^* A U_n.$$

(3) The successive iterates are related by the formula

$$AU_n = U_{n+1} \widetilde{H}_n.$$

Remark: If Arnoldi iteration has a breakdown at stage n, that is, $h_{n+1} = 0$, then we found the first unreduced block of the Hessenberg matrix H. It can be shown that the eigenvalues of H_n are eigenvalues of A. So a breakdown is actually a good thing. In this case, we can pick some new nonzero vector u_{n+1} orthogonal to the vectors $(u_1, \ldots, u_n)$ as a new starting vector and run Arnoldi iteration again. Such a vector exists since the $(n+1)$th column of U works. So repeated application of Arnoldi yields a full Hessenberg reduction of A. However,