By a previous remark, $p_f \leq p_m$ and $q_f \leq q_m$, the first inequality being strict if there is some i such that $0 < \lambda_i < K$, and the second inequality being strict if there is some j such that $0 < \mu_j < K$. This will be the case under the **Standard Margin Hypothesis**.

Observe that a small value of ν keeps p_f and q_f small, which is achieved if the δ-slab is narrow (to avoid having points on the wrong sides of the margin hyperplanes). A large value of ν allows p_m and q_m to be fairly large, which is achieved if the δ-slab is wide. Thus the smaller ν is, the narrower the δ-slab is, and the larger ν is, the wider the δ-slab is. This is the opposite of the behavior that we witnessed in ν-regression (see Section 56.1).

54.7 Existence of Support Vectors for $(\mathrm{SVM}_{s2'})$

We now consider the issue of the existence of support vectors. We will show that in the "generic case" there is always some blue support vector and some red support vector. The term generic has to do with the choice of ν and will be explained below.

Given any real numbers u, v, x, y, if $\max\{u, v\} < \min\{x, y\}$, then $u < x$ and $v < y$. This is because $u, v \leq \max\{u, v\} < \min\{x, y\} \leq x, y$. Consequently, since by Proposition 54.1, $\max\{2p_f/(p+q), 2q_f/(p+q)\} \leq \nu$, if $\nu < \min\{2p/(p+q), 2q/(p+q)\}$, then $p_f < p$ and $q_f < q$, and since $p_{sf} \leq p_f$ and $q_{sf} \leq q_f$, we also have $p_{sf} < p$ and $q_{sf} < q$. This implies that there are constraints corresponding to some $i \notin E_\lambda$ (in which case $\epsilon_i = 0$) and to some $j \notin E_\mu$ (in which case $\xi_j = 0$), of the form

$$\begin{aligned} w^\top u_i - b &\geq \eta & \quad & i \notin E_\lambda \\ -w^\top v_j + b &\geq \eta & & j \notin E_\mu. \end{aligned}$$

If $w^\top u_i - b = \eta$ for some $i \notin E_\lambda$ and $-w^\top v_j + b = \eta$ for some $j \notin E_\mu$, then we have a blue support vector and a red support vector. Otherwise, we show how to modify b and η to obtain an optimal solution with a blue support vector and a red support vector.

Proposition 54.2. *For every optimal solution $(w, b, \eta, \epsilon, \xi)$ of Problem $(\mathrm{SVM}_{s2'})$ with $w \neq 0$ and $\eta > 0$, if*

$$\nu < \min\{2p/(p+q), 2q/(p+q)\}$$

and if either no u_i is a support vector or no v_j is a support vector, then there is another optimal solution (for the same w) with some i_0 such that $\epsilon_{i_0} = 0$ and $w^\top u_{i_0} - b = \eta$, and there is some j_0 such that $\xi_{j_0} = 0$ and $-w^\top v_{j_0} + b = \eta$; in other words, some u_{i_0} and some v_{j_0} is a support vector; in particular, $p_{sf} < p$ and $q_{sf} < q$.

Proof. We just explained that $p_{sf} < p$ and $q_{sf} < q$, so the following constraints hold:

$$\begin{aligned} w^\top u_i - b &= \eta - \epsilon_i & \quad \epsilon_i &> 0 & \quad & i \in E_\lambda \\ -w^\top v_j + b &= \eta - \xi_j & \xi_j &> 0 & & j \in E_\mu \\ w^\top u_i - b &\geq \eta & & & & i \notin E_\lambda \\ -w^\top v_j + b &\geq \eta & & & & j \notin E_\mu, \end{aligned}$$