In addition, $d \neq 0$, and d is unique up to multiplication by a nonzero scalar in K.

As a consequence of Proposition 30.15, some polynomials $f_1, \ldots, f_n \in K[X]$ are relatively prime iff there exist $u_1, \ldots, u_n \in K[X]$ such that

$$u_1 f_1 + \cdots + u_n f_n = 1.$$

The identity

$$u_1 f_1 + \cdots + u_n f_n = 1$$

of part (2) of Proposition 30.15 is also called the *Bezout identity*.

We now consider the factorization of polynomials of a single variable into irreducible factors.

30.5 Factorization and Irreducible Factors in $K[X]$

Definition 30.9. Given a field K, a polynomial $p \in K[X]$ is *irreducible or indecomposable or prime* if $\deg(p) \geq 1$ and if p is not divisible by any polynomial $q \in K[X]$ such that $1 \leq \deg(q) < \deg(p)$. Equivalently, p is irreducible if $\deg(p) \geq 1$ and if $p = q_1 q_2$, then either $q_1 \in K$ or $q_2 \in K$ (and of course, $q_1 \neq 0$, $q_2 \neq 0$).

Example 30.2. Every polynomial $aX + b$ of degree 1 is irreducible. Over the field $\mathbb{R}$, the polynomial $X^2 + 1$ is irreducible (why?), but $X^3 + 1$ is not irreducible, since

$$X^3 + 1 = (X + 1)(X^2 - X + 1).$$

The polynomial $X^2 - X + 1$ is irreducible over $\mathbb{R}$ (why?). It would seem that $X^4 + 1$ is irreducible over $\mathbb{R}$, but in fact,

$$X^4 + 1 = (X^2 - \sqrt{2}X + 1)(X^2 + \sqrt{2}X + 1).$$

However, in view of the above factorization, $X^4 + 1$ is irreducible over $\mathbb{Q}$.

It can be shown that the irreducible polynomials over $\mathbb{R}$ are the polynomials of degree 1, or the polynomials of degree 2 of the form $aX^2 + bX + c$, for which $b^2 - 4ac < 0$ (i.e., those having no real roots). This is not easy to prove! Over the complex numbers $\mathbb{C}$, the only irreducible polynomials are those of degree 1. This is a version of a fact often referred to as the "Fundamental theorem of Algebra", or, as the French sometimes say, as "d'Alembert's theorem"!

We already observed that for any two nonzero polynomials $f, g \in K[X]$, f divides g iff $(g) \subseteq (f)$. In view of the definition of a maximal ideal given in Definition 30.4, we now prove that a polynomial $p \in K[X]$ is irreducible iff (p) is a maximal ideal in $K[X]$.

Proposition 30.16. *A polynomial $p \in K[X]$ is irreducible iff (p) is a maximal ideal in $K[X]$.*