reflection about the hyperplane orthogonal to u_1 and of angle θ, since $\rho_\theta \circ \rho_{-\theta} = \mathrm{id}$, we have

$$f = (\rho_\theta \circ \rho_{-\theta}) \circ f = \rho_\theta \circ (\rho_{-\theta} \circ f).$$

Letting $g = \rho_{-\theta} \circ f$, it is obvious that $\det(g) = 1$. As a consequence, there is a bijection between $S^1 \times \mathbf{SU}(n)$ and $\mathbf{U}(n)$, where S^1 is the unit circle (which corresponds to the group of complex numbers $e^{i\theta}$ of unit length). In fact, it is a homeomorphism.

(2) We abandoned the style of proof used in theorem 27.1, because in the Hermitian case, eigenvalues and eigenvectors always exist, and the proof is simpler that way (in the real case, an isometry may not have any real eigenvalues!). The sacrifice is that the theorem yields no information on the number of (standard) hyperplane reflections. We shall rectify this situation shortly.

We will now reveal the beautiful trick (found in Mneimné and Testard [127]) that allows us to prove that every rotation in $\mathbf{SU}(n)$ is the composition of at most $2n - 2$ (standard) hyperplane reflections. For what follows, it is more convenient to denote a standard reflection about the hyperplane H as h_u (it is trivial that these do not depend on the choice of u in $H^\perp$). Then, given any two distinct orthogonal vectors u, v such that $\|u\| = \|v\|$, consider the composition $\rho_{v,-\theta} \circ \rho_{u,\theta}$. The trick is that this composition can be expressed as two standard hyperplane reflections! This wonderful fact is proved in the next Proposition.

Proposition 28.3. *Let E be a nontrivial Hermitian space. For any two distinct orthogonal vectors u, v such that $\|u\| = \|v\|$, we have*

$$\rho_{v,-\theta} \circ \rho_{u,\theta} = h_{v-u} \circ h_{v-e^{-i\theta}u} = h_{u+v} \circ h_{u+e^{i\theta}v}.$$

Proof. Since u and v are orthogonal, each one is in the hyperplane orthogonal to the other, and thus,

$$\begin{aligned}
\rho_{u,\theta}(u) &= e^{i\theta}u,\\
\rho_{u,\theta}(v) &= v,\\
\rho_{v,-\theta}(u) &= u,\\
\rho_{v,-\theta}(v) &= e^{-i\theta}v,\\
h_{v-u}(u) &= v,\\
h_{v-u}(v) &= u,\\
h_{v-e^{-i\theta}u}(u) &= e^{i\theta}v,\\
h_{v-e^{-i\theta}u}(v) &= e^{-i\theta}u.
\end{aligned}$$

Consequently, using linearity,

$$\begin{aligned}
\rho_{v,-\theta} \circ \rho_{u,\theta}(u) &= e^{i\theta}u,\\
\rho_{v,-\theta} \circ \rho_{u,\theta}(v) &= e^{-i\theta}v,\\
h_{v-u} \circ h_{v-e^{-i\theta}u}(u) &= e^{i\theta}u,\\
h_{v-u} \circ h_{v-e^{-i\theta}u}(v) &= e^{-i\theta}v,
\end{aligned}$$