(b) *The function* $g\colon A \to B$ *is continuous.*

If we also assume that

(5) *The derivative* $\mathrm{D}f(a,b)$ *exists;*

then

(c) *The derivative* $\mathrm{D}g(a)$ *exists, and*

$$\mathrm{D}g(a) = -\Big(\frac{\partial f}{\partial y}(a,b)\Big)^{-1} \circ \frac{\partial f}{\partial x}(a,b);$$

and if in addition

(6) $\dfrac{\partial f}{\partial x}\colon \Omega \to \mathcal{L}(\overrightarrow{E};\overrightarrow{G})$ *is also continuous (and thus, in view of (3), f is C^1 on Ω);*

then

(d) *The derivative* $\mathrm{D}g\colon A \to \mathcal{L}(\overrightarrow{E};\overrightarrow{F})$ *is continuous, and*

$$\mathrm{D}g(x) = -\Big(\frac{\partial f}{\partial y}(x,g(x))\Big)^{-1} \circ \frac{\partial f}{\partial x}(x,g(x)),$$

for all $x \in A$.

Example 39.7. Going back to Example 39.6, write $x = (x_1, x_2)$ and $y = x_3$, so that the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ are given in terms of their Jacobian matrices by

$$\frac{\partial f}{\partial x}(x,y) = \begin{pmatrix} 2x_1 & 2x_2 \end{pmatrix}$$
$$\frac{\partial f}{\partial y}(x,y) = 2x_3.$$

If $0 < |b| \leq 1$ and $\|a\|_2^2 + b^2 - 1 = 0$, then Conditions (3) and (4) are satisfied. Conditions (1) and (2) obviously hold. Since $df_{(a,b)}$ is given by its Jacobian matrix as

$$df_{(a,b)} = \begin{pmatrix} 2a_1 & 2a_2 & 2b \end{pmatrix},$$

Condition (5) holds, and clearly, Condition (6) also holds.

Theorem 39.14 implies that there is some open subset A of $\mathbb{R}^2$ containing a, some open subset B of $\mathbb{R}$ containing b, and a unique function $g\colon A \to B$ such that

$$f(x, g(x)) = 0$$

for all $x \in A$. In fact, we can pick A to be the open unit disk in $\mathbb{R}$, $B = (0, 2)$, and if $0 < b \leq 1$, then

$$g(x_1, x_2) = \sqrt{1 - x_1^2 - x_2^2},$$