Now we have shown that $\lambda(u) = \lambda(v)$, for any two distinct basis vectors in B, which proves that $\lambda(u)$ is independent of $u \in G$, and proves that $g = \lambda f$. $\square$

Proposition 26.4 shows that the projective linear group $\mathbf{PGL}(E)$ is isomorphic to the quotient group of the linear group $\mathbf{GL}(E)$ modulo the subgroup $K^*\mathrm{id}_E$ (where $K^* = K - \{0\}$). Using projective frames, we prove the following useful result.

Proposition 26.5. *Given two nontrivial vector spaces E and F of the same dimension $n+1$, for any two projective frames $(a_i)_{1\leq i\leq n+2}$ for $\mathbf{P}(E)$ and $(b_i)_{1\leq i\leq n+2}$ for $\mathbf{P}(F)$, there is a unique projectivity $h\colon \mathbf{P}(E) \to \mathbf{P}(F)$ such that $h(a_i) = b_i$ for $1 \leq i \leq n+2$.*

Proof. Let $(u_1, \ldots, u_{n+1})$ be a basis of E associated with the projective frame $(a_i)_{1\leq i\leq n+2}$, and let $(v_1, \ldots, v_{n+1})$ be a basis of F associated with the projective frame $(b_i)_{1\leq i\leq n+2}$. Since $(u_1, \ldots, u_{n+1})$ is a basis, there is a unique linear bijection $g\colon E \to F$ such that $g(u_i) = v_i$, for $1 \leq i \leq n+1$. Clearly, $h = \mathbf{P}(g)$ is a projectivity such that $h(a_i) = b_i$, for $1 \leq i \leq n+2$. Let $h'\colon \mathbf{P}(E) \to \mathbf{P}(F)$ be any projectivity such that $h'(a_i) = b_i$, for $1 \leq i \leq n+2$. By definition, there is a linear isomorphism $f\colon E \to F$ such that $h' = \mathbf{P}(f)$. Since $h'(a_i) = b_i$, for $1 \leq i \leq n+2$, we must have $f(u_i) = \lambda_i v_i$, for some $\lambda_i \in K - \{0\}$, where $1 \leq i \leq n+1$, and

$$f(u_1 + \cdots + u_{n+1}) = \lambda(v_1 + \cdots + v_{n+1}),$$

for some $\lambda \in K - \{0\}$. By linearity of f, we have

$$\lambda_1 v_1 + \cdots + \lambda_{n+1} v_{n+1} = \lambda v_1 + \cdots + \lambda v_{n+1},$$

and since $(v_1, \ldots, v_{n+1})$ is a basis of F, we must have

$$\lambda_1 = \cdots = \lambda_{n+1} = \lambda.$$

This shows that $f = \lambda g$, and thus that

$$h' = \mathbf{P}(f) = \mathbf{P}(g) = h,$$

and h is uniquely determined. $\square$

The above proposition and Proposition 26.4 are false if K is a skew field. Also, Proposition 26.5 fails if $(b_i)_{1\leq i\leq n+2}$ is not a projective frame, or if a_{n+2} is dropped.

As a corollary of Proposition 26.5, given a projective space $\mathbf{P}(E)$, two distinct projective lines D and D' in $\mathbf{P}(E)$, three distinct points a, b, c on D, and any three distinct points a', b', c' on D', there is a unique projectivity from D to D', mapping a to a', b to b', and c to c'. This is because, as we mentioned earlier, any three distinct points on a line form a projective frame.

Remark: As in the affine case, there is "fundamental theorem of projective geometry." For simplicity, we state this theorem assuming that vector spaces are over the field $K = \mathbb{R}$. Given