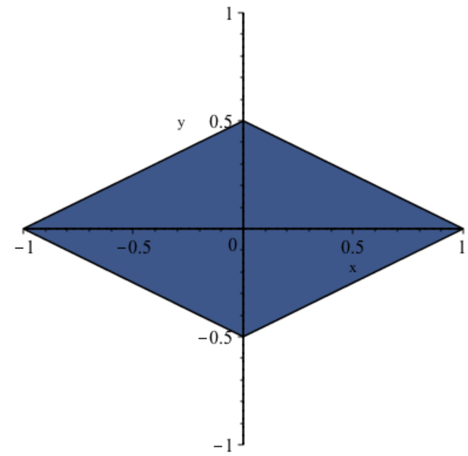

Figure 9.5: The unit closed unit ball $\{(u_1, u_2) \in \mathbb{R}^2 \mid \|(u_1, u_2)\| \leq 1\}$, where $\|(u_1, u_2)\| = |u_1| + 2|u_2|$.

so that by summing up these equations we get

$$\sum_{i=1}^{n} (|u_i| + |v_i|)^p = \sum_{i=1}^{n} |u_i|(|u_i| + |v_i|)^{p-1} + \sum_{i=1}^{n} |v_i|(|u_i| + |v_i|)^{p-1},$$

and using Inequality $(**)$, with $V \in E$ where $V_i = (|u_i| + |v_i|)^{p-1}$, we get

$$\sum_{i=1}^{n} (|u_i| + |v_i|)^p \leq \|u\|_p \|V\|_q + \|v\|_p \|V\|_q = (\|u\|_p + \|v\|_p)\bigg(\sum_{i=1}^{n} (|u_i| + |v_i|)^{(p-1)q}\bigg)^{1/q}.$$

However, $1/p + 1/q = 1$ implies $pq = p + q$, that is, $(p-1)q = p$, so we have

$$\sum_{i=1}^{n} (|u_i| + |v_i|)^p \leq (\|u\|_p + \|v\|_p)\bigg(\sum_{i=1}^{n} (|u_i| + |v_i|)^{p}\bigg)^{1/q},$$

which yields

$$\bigg(\sum_{i=1}^{n} (|u_i| + |v_i|)^p\bigg)^{1-1/q} = \bigg(\sum_{i=1}^{n} (|u_i| + |v_i|)^p\bigg)^{1/p} \leq \|u\|_p + \|v\|_p.$$

Since $|u_i + v_i| \leq |u_i| + |v_i|$, the above implies the triangle inequality $\|u + v\|_p \leq \|u\|_p + \|v\|_p$, as claimed. $\square$

For $p > 1$ and $1/p + 1/q = 1$, the inequality

$$\sum_{i=1}^{n} |u_i v_i| \leq \bigg(\sum_{i=1}^{n} |u_i|^p\bigg)^{1/p} \bigg(\sum_{i=1}^{n} |v_i|^q\bigg)^{1/q}$$