is a symmetric tridiagonal matrix.

(1) Prove that for any isometry $f\colon E \to E$, we have $f = f^* = f^{-1}$ iff $f \circ f = \mathrm{id}$.

Prove that if f and h are self-adjoint linear maps ($f^* = f$ and $h^* = h$), then $h \circ f \circ h$ is a self-adjoint linear map.

(2) Let V_k be the subspace spanned by $(e_{k+1}, \ldots, e_n)$. Proceed by induction. For the base case, proceed as follows.

Let

$$f(e_1) = a_1^0 e_1 + \cdots + a_n^0 e_n,$$

and let

$$r_{1,2} = \left\| a_2^0 e_2 + \cdots + a_n^0 e_n \right\|.$$

Find an isometry h_1 (reflection or id) such that

$$h_1(f(e_1) - a_1^0 e_1) = r_{1,2}\, e_2.$$

Observe that

$$w_1 = r_{1,2}\, e_2 + a_1^0 e_1 - f(e_1) \in V_1,$$

and prove that $h_1(e_1) = e_1$, so that

$$h_1 \circ f \circ h_1(e_1) = a_1^0 e_1 + r_{1,2}\, e_2.$$

Let $f_1 = h_1 \circ f \circ h_1$.

Assuming by induction that

$$f_k = h_k \circ \cdots \circ h_1 \circ f \circ h_1 \circ \cdots \circ h_k$$

has a tridiagonal matrix up to the kth row and column, $1 \leq k \leq n-3$, let

$$f_k(e_{k+1}) = a_k^k e_k + a_{k+1}^k e_{k+1} + \cdots + a_n^k e_n,$$

and let

$$r_{k+1,\,k+2} = \left\| a_{k+2}^k e_{k+2} + \cdots + a_n^k e_n \right\|.$$

Find an isometry h_{k+1} (reflection or id) such that

$$h_{k+1}(f_k(e_{k+1}) - a_k^k e_k - a_{k+1}^k e_{k+1}) = r_{k+1,\,k+2}\, e_{k+2}.$$

Observe that

$$w_{k+1} = r_{k+1,\,k+2}\, e_{k+2} + a_k^k e_k + a_{k+1}^k e_{k+1} - f_k(e_{k+1}) \in V_{k+1},$$

and prove that $h_{k+1}(e_k) = e_k$ and $h_{k+1}(e_{k+1}) = e_{k+1}$, so that

$$h_{k+1} \circ f_k \circ h_{k+1}(e_{k+1}) = a_k^k e_k + a_{k+1}^k e_{k+1} + r_{k+1,\,k+2}\, e_{k+2}.$$