We proceed by induction on the dimension r of U. Since the proof is quite intricate, we spell out the general plan of attack. For the induction step, we first show that we can reduce the situation to what we call *Case (H)*, namely that the subspace of U left fixed by f is a hyperplane H in U. Then, the set $D = \{f(u) - u \mid u \in U\}$ is a line in U and it turns out that $D^{\perp}$ is a hyperplane in E. We now introduce *Hypothesis (V)*, which says we can find a nontrivial subspace V of E orthogonal to D and such that $V \cap U = V \cap f(U) = (0)$. We show that if Hypothesis (V) holds, then f can be extended to an isometry of $U \oplus V$. It is then possible to further extend f to an isometry of E.

To prove that Hypothesis (V) holds we consider two cases. In Case (a), we obtain some V such that $E = U \oplus V$ and we are done. In Case (b), we obtain some V such that $D^{\perp} = U \oplus V$. We are then reduced to the situation where $U = D^{\perp}$ is a hyperplane in E and f is an isometry of U. To finish the proof we pick any $v \notin U$, so that $E = U \oplus Kv$, and we find some $v_1 \in E$ such that

$$\begin{aligned} \varphi(f(u), v_1) &= \varphi(u, v) \quad \text{for all } u \in U \\ \varphi(v_1, v_1) &= \varphi(v, v). \end{aligned}$$

Then, by Proposition 29.44, we can extend f to a metric map g of $U + Kv = E$ such that $g(v) = v_1$. The argument used to find v_1 makes use of (†) (see below) and is bit tricky. We also makes use of Property (T) in the form of Lemma 29.28.

We now go back to the proof. The case $r = 0$ is trivial. For the induction step, $r \geq 1$ so $U \neq (0)$, and let H be any hyperplane in U. Let $f \colon U \to E$ be an injective metric linear map. By the induction hypothesis, the restriction f_0 of f to H extends to an isometry g_0 of E. If g_0 extends f, we are done. Otherwise, H is the subspace of elements of U left fixed by $g_0^{-1} \circ f$. If the theorem holds in this situation, namely the subspace of U left fixed by $g_0^{-1} \circ f$ is a hyperplane H in U, then we have an isometry g_1 of E extending $g_0^{-1} \circ f$, and $g_0 \circ g_1$ is an isometry of E extending f. Therefore, we are reduced to the following situation:

Case (H). The subspace of U left fixed by f is a hyperplane H in U.

In this case, the set $D = \{f(u) - u \mid u \in U\}$ is a line in U (a one-dimensional subspace). For all $u, v \in U$, we have

$$\varphi(f(u), f(v) - v) = \varphi(f(u), f(v)) - \varphi(f(u), v) = \varphi(u, v) - \varphi(f(u), v) = \varphi(u - f(u), v),$$

that is

$$\varphi(f(u), f(v) - v) = \varphi(u - f(u), v) \quad \text{for all } u, v \in U, \tag{$**$}$$

and if $u \in H$, which means that $f(u) = u$, we get $u \in D^{\perp}$. Therefore, $H \subseteq D^{\perp}$. Since φ is nondegenerate, we have $\dim(D) + \dim(D^{\perp}) = \dim(E)$, and since $\dim(D) = 1$, the subspace $D^{\perp}$ is a hyperplane in E.

Hypothesis (V). We can find a nontrivial subspace V of E orthogonal to D and such that $V \cap U = V \cap f(U) = (0)$.

Claim. Hypothesis (V) implies that f can be extended to an isometry of $U \oplus V$.