sequence

$$
\begin{aligned}
u^0 &= (10, 15, 5, -2, 1, 3, 1, 1)\\
u^1 &= (10 + 15, 10 - 15, 5, -2, 1, 3, 1, 1) = (25, -5, 5, -2, 1, 3, 1, 1)\\
u^2 &= (25 + 5, 25 - 5, -5 + (-2), -5 - (-2), 1, 3, 1, 1) = (30, 20, -7, -3, 1, 3, 1, 1)\\
u^3 &= (30 + 1, 30 - 1, 20 + 3, 20 - 3, -7 + 1, -7 - 1, -3 + 1, -3 - 1)\\
&= (31, 29, 23, 17, -6, -8, -2, -4),
\end{aligned}
$$

which gives back $u = (31, 29, 23, 17, -6, -8, -2, -4)$.

5.3 Kronecker Product Construction of Haar Matrices

There is another recursive method for constructing the Haar matrix W_n of dimension 2^n that makes it clearer why the columns of W_n are pairwise orthogonal, and why the above algorithms are indeed correct (which nobody seems to prove!). If we split W_n into two $2^n \times 2^{n-1}$ matrices, then the second matrix containing the last 2^{n-1} columns of W_n has a very simple structure: it consists of the vector

$$\underbrace{(1, -1, 0, \ldots, 0)}_{2^n}$$

and $2^{n-1} - 1$ shifted copies of it, as illustrated below for $n = 3$:

$$\begin{pmatrix}
1 & 0 & 0 & 0\\
-1 & 0 & 0 & 0\\
0 & 1 & 0 & 0\\
0 & -1 & 0 & 0\\
0 & 0 & 1 & 0\\
0 & 0 & -1 & 0\\
0 & 0 & 0 & 1\\
0 & 0 & 0 & -1
\end{pmatrix}.$$

Observe that this matrix can be obtained from the identity matrix $I_{2^{n-1}}$, in our example

$$I_4 = \begin{pmatrix}
1 & 0 & 0 & 0\\
0 & 1 & 0 & 0\\
0 & 0 & 1 & 0\\
0 & 0 & 0 & 1
\end{pmatrix},$$

by forming the $2^n \times 2^{n-1}$ matrix obtained by replacing each 1 by the column vector

$$\begin{pmatrix} 1\\ -1 \end{pmatrix}$$