29.10 Witt's Theorem

Witt's theorem was referred to as a "scandal" by Emil Artin. What he meant by this is that one had to wait until 1936 (Witt [190]) to formulate and prove a theorem at once so simple in its statement and underlying concepts, and so useful in various domains (geometry, arithmetic of quadratic forms).[1]

Besides Witt's original proof (Witt [190]), Chevalley's proof [37] seems to be the "best" proof that applies to the symmetric as well as the skew-symmetric case. The proof in Bourbaki [24] is based on Chevalley's proof, and so are a number of other proofs. This is the one we follow (slightly reorganized). In the symmetric case, Serre's exposition is hard to beat (see Serre [157], Chapter IV).

The following observation is one of the key ingredients in the proof of Theorem 29.45.

Proposition 29.44. *Given a finite-dimensional space E equipped with an ϵ-Hermitan form φ, if U_1 and U_2 are two subspaces of E such that $U_1 \cap U_2 = (0)$ and if we have metric linear maps $f_1\colon U_1 \to E$ and $f_2\colon U_2 \to E$ such that*

$$\varphi(f_1(u_1), f_2(u_2)) = \varphi(u_1, u_2) \quad \textit{for } u_i \in U_i \ (i = 1, 2), \tag{$*$}$$

then the linear map $f\colon U_1 \oplus U_2 \to E$ given by $f(u_1 + u_2) = f_1(u_1) + f_2(u_2)$ extends f_1 and f_2 and is metric. Furthermore, if f_1 and f_2 are injective, then so if f.

Proof. Indeed, since f_1 and f_2 are metric and using $(*)$, we have

$$\begin{aligned}
\varphi(f_1(u_1) + f_2(u_2), f_1(v_1) + f_2(v_2)) &= \varphi(f_1(u_1), f_1(v_1)) + \varphi(f_1(u_1), f_2(v_2)) \\
&\quad + \varphi(f_2(u_2), f_1(v_1)) + \varphi(f_2(u_2), f_2(v_2)) \\
&= \varphi(u_1, v_1) + \varphi(u_1, v_2) + \varphi(u_2, v_1) + \varphi(u_2, v_2) \\
&= \varphi(u_1 + u_2, v_2 + v_2).
\end{aligned}$$

Thus f is a metric map extending f_1 and f_2. ∎

Theorem 29.45. *(Witt, 1936) Let E and E' be two finite-dimensional spaces respectively equipped with two nondegenerate ϵ-Hermitan forms φ and φ' satisfying condition (T), and assume that there is an isometry between (E, φ) and (E', φ'). For any subspace U of E, every injective metric linear map f from U into E' extends to an isometry from E to E'.*

Proof. Since (E, φ) and (E', φ') are isometric, we may assume that $E' = E$ and $\varphi' = \varphi$ (if $h\colon E \to E'$ is an isometry, then $h^{-1} \circ f$ is an injective metric map from U into E. The details are left to the reader).

[1] Curiously, some references to Witt's paper claim its date of publication to be 1936, but others say 1937. The answer to this mystery is that Volume 176 of *Crelle Journal* was published in four issues. The cover page of volume 176 mentions the year 1937, but Witt's paper is dated May 1936. This is not the only paper of Witt appearing in this volume!