As an exercise, the reader should verify that dual ascent (with $\alpha^k = \rho$) yields the equations

$$\begin{aligned}
x^{k+1} &= \frac{\lambda^k}{2\beta} \\
y^{k+1} &= -\frac{\lambda^k}{\beta} \\
\lambda^{k+1} &= \left(1 + \frac{2\rho}{\beta}\right)\lambda^k,
\end{aligned}$$

and so the method diverges, except for $\lambda^0 = 0$, which is the optimal solution.

The method of multipliers converges under conditions that are far more general than the dual ascent. However, the addition of the penalty term has the negative effect that even if J is separable, then the Lagrangian L_ρ is not separable. Thus the basic method of multipliers cannot be used for decomposition and is not parallelizable. The next method deals with the problem of separability.

52.3 ADMM: Alternating Direction Method of Multipliers

The alternating direction method of multipliers, for short ADMM, combines the decomposability of dual ascent with the superior convergence properties of the method of multipliers. It can be viewed as an approximation of the method of multipliers, but it is generally superior.

The idea is to split the function J into two independent parts, as $J(x, z) = f(x) + g(z)$, and to consider the Minimization Problem (P_{admm}),

$$\begin{aligned}
&\text{minimize} && f(x) + g(z) \\
&\text{subject to} && Ax + Bz = c,
\end{aligned}$$

for some $p \times n$ matrix A, some $p \times m$ matrix B, and with $x \in \mathbb{R}^n$, $z \in \mathbb{R}^m$, and $c \in \mathbb{R}^p$. We also assume that f and g are convex. Further conditions will be added later.

As in the method of multipliers, we form the *augmented Lagrangian*

$$L_\rho(x, z, \lambda) = f(x) + g(z) + \lambda^\top(Ax + Bz - c) + (\rho/2)\left\|Ax + Bz - c\right\|_2^2,$$

with $\lambda \in \mathbb{R}^p$ and for some $\rho > 0$.

Given some initial values (z^0, λ^0), the *ADMM method* consists of the following iterative steps:

$$\begin{aligned}
x^{k+1} &= \arg\min_x L_\rho(x, z^k, \lambda^k) \\
z^{k+1} &= \arg\min_z L_\rho(x^{k+1}, z, \lambda^k) \\
\lambda^{k+1} &= \lambda^k + \rho(Ax^{k+1} + Bz^{k+1} - c).
\end{aligned}$$