Proposition 34.14. *When E is finite dimensional, the maps $\mu\colon \bigwedge^n(E^*) \longrightarrow \mathrm{Alt}^n(E;K)$ induced by the linear extensions of the maps given by*

$$\mu(v_1^* \wedge \cdots \wedge v_n^*)(u_1, \ldots, u_n) = \det(v_j^*(u_i))$$

yield a canonical isomorphism of algebras $\mu\colon \bigwedge(E^*) \longrightarrow \mathrm{Alt}(E)$, where the multiplication in $\mathrm{Alt}(E)$ is defined by the maps $\wedge\colon \mathrm{Alt}^m(E;K) \times \mathrm{Alt}^n(E;K) \longrightarrow \mathrm{Alt}^{m+n}(E;K)$, with

$$(f \wedge g)(u_1, \ldots, u_{m+n}) = \sum_{\sigma \in \mathrm{shuffle}(m,n)} \mathrm{sgn}(\sigma)\, f(u_{\sigma(1)}, \ldots, u_{\sigma(m)}) g(u_{\sigma(m+1)}, \ldots, u_{\sigma(m+n)}),$$

where $\mathrm{shuffle}(m,n)$ *consists of all (m,n)-"shuffles," that is, permutations σ of $\{1, \ldots m+n\}$ such that $\sigma(1) < \cdots < \sigma(m)$ and $\sigma(m+1) < \cdots < \sigma(m+n)$.*

Remark: The algebra $\bigwedge(E)$ is a graded algebra. Given two graded algebras E and F, we can make a new tensor product $E \mathbin{\widehat{\otimes}} F$, where $E \mathbin{\widehat{\otimes}} F$ is equal to $E \otimes F$ as a vector space, but with a skew-commutative multiplication given by

$$(a \otimes b) \wedge (c \otimes d) = (-1)^{\deg(b)\deg(c)} (ac) \otimes (bd),$$

where $a \in E^m, b \in F^p$, $c \in E^n, d \in F^q$. Then, it can be shown that

$$\bigwedge(E \oplus F) \cong \bigwedge(E) \mathbin{\widehat{\otimes}} \bigwedge(F).$$

34.6 The Hodge ∗-Operator

In order to define a generalization of the Laplacian that applies to differential forms on a Riemannian manifold, we need to define isomorphisms

$$\bigwedge^k V \longrightarrow \bigwedge^{n-k} V,$$

for any Euclidean vector space V of dimension n and any k, with $0 \leq k \leq n$. If $\langle -, - \rangle$ denotes the inner product on V, we define an inner product on $\bigwedge^k V$, denoted $\langle -, - \rangle_\wedge$, by setting

$$\langle u_1 \wedge \cdots \wedge u_k, v_1 \wedge \cdots \wedge v_k \rangle_\wedge = \det(\langle u_i, v_j \rangle),$$

for all $u_i, v_i \in V$, and extending $\langle -, - \rangle_\wedge$ by bilinearity.

It is easy to show that if $(e_1, \ldots, e_n)$ is an orthonormal basis of V, then the basis of $\bigwedge^k V$ consisting of the e_I (where $I = \{i_1, \ldots, i_k\}$, with $1 \leq i_1 < \cdots < i_k \leq n$) is an orthonormal basis of $\bigwedge^k V$. Since the inner product on V induces an inner product on V^* (recall that $\langle \omega_1, \omega_2 \rangle = \langle \omega_1^\sharp, \omega_2^\sharp \rangle$, for all $\omega_1, \omega_2 \in V^*$), we also get an inner product on $\bigwedge^k V^*$.