As we will see later, most bilinear forms that we will encounter are equivalent to one whose matrix is of the following form:

1. I_n, $-I_n$.
2. If $p+q=n$, with $p,q\geq 1$,
$$I_{p,q}=\begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$$
3. If $n=2m$,
$$J_{m.m}=\begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$$
4. If $n=2m$,
$$A_{m,m}=I_{m.m}J_{m.m}=\begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}.$$

If we make changes of bases given by matrices P and Q, so that $x=Px'$ and $y=Qy'$, then the new matrix expressing φ is $P^\top MQ$. In particular, if $E=F$ and the same basis is used, then the new matrix is $P^\top MP$. This shows that if φ is nondegenerate, then the determinant of φ is determined up to a square element.

Observe that if φ is a symmetric bilinear form ($E=F$) and if K does not have characteristic 2, then by Theorem 29.4, there is a basis of E with respect to which the matrix M representing φ is a diagonal matrix. If $K=\mathbb{R}$ or $K=\mathbb{C}$, this allows us to classify completely the symmetric bilinear forms. Recall that $\Phi(u)=\varphi(u,u)$ for all $u\in E$.

Proposition 29.6. *Given any bilinear form $\varphi\colon E\times E\to K$ with $\dim(E)=n$, if φ is symmetric and K does not have characteristic 2, then there is a basis $(e_1,\ldots,e_n)$ of E such that*
$$\Phi\left(\sum_{i=1}^n x_ie_i\right)=\sum_{i=1}^r \lambda_i x_i^2,$$
for some $\lambda_i\in K-\{0\}$ and with $r\leq n$. Furthermore, if $K=\mathbb{C}$, then there is a basis $(e_1,\ldots,e_n)$ of E such that
$$\Phi\left(\sum_{i=1}^n x_ie_i\right)=\sum_{i=1}^r x_i^2,$$
and if $K=\mathbb{R}$, then there is a basis $(e_1,\ldots,e_n)$ of E such that
$$\Phi\left(\sum_{i=1}^n x_ie_i\right)=\sum_{i=1}^p x_i^2-\sum_{i=p+1}^{p+q} x_i^2,$$
with $0\leq p,q$ and $p+q\leq n$.