Proof. Assume $x \in \overline{A}$. By Proposition 37.13, there is some sequence (a_n) of points $a_n \in A$ which converges to x. Consequently (a_n) is a Cauchy sequence in E, and thus a Cauchy sequence in A (since $a_n \in A$ for all n). Since A is complete, the sequence (a_n) has a limit $a \in A$, but since E is a metric space it is Hausdorff, so $a = x$, which shows that $x \in A$; that is, A is closed. $\square$

Proposition 37.51. *Let (E, d) be a metric space, and let A be a subset of E. If E is complete and if A is closed in E, then A is complete.*

Proof. Let (a_n) be a Cauchy sequence in A. The sequence (a_n) is also a Cauchy sequence in E, and since E is complete, it has a limit $x \in E$. But $a_n \in A$ for all n, so by Proposition 37.13 we must have $x \in \overline{A}$. Since A is closed, actually $x \in A$, which proves that A is complete. $\square$

An arbitrary metric space (E, d) is not necessarily complete, but there is a construction of a metric space $(\widehat{E}, \widehat{d})$ such that $\widehat{E}$ is complete, and there is a continuous (injective) distance-preserving map $\varphi\colon E \to \widehat{E}$ such that $\varphi(E)$ is dense in $\widehat{E}$. This is a generalization of the construction of the set $\mathbb{R}$ of real numbers from the set $\mathbb{Q}$ of rational numbers in terms of Cauchy sequences. This construction can be immediately adapted to a normed vector space $(E, \| \|)$ to embed $(E, \| \|)$ into a complete normed vector space $(\widehat{E}, \| \|_{\widehat{E}})$ (a Banach space). This construction is used heavily in integration theory, where E is a set of functions.

37.9 Completion of a Metric Space

In order to prove a kind of uniqueness result for the completion $(\widehat{E}, \widehat{d})$ of a metric space (E, d), we need the following result about extending a uniformly continuous function.

Recall that E_0 is dense in E iff $\overline{E_0} = E$. Since E is a metric space, by Proposition 37.13, this means that for every $x \in E$, there is some sequence (x_n) converging to x, with $x_n \in E_0$.

Theorem 37.52. *Let E and F be two metric spaces, let E_0 be a dense subspace of E, and let $f_0\colon E_0 \to F$ be a continuous function. If f_0 is uniformly continuous and if F is complete, then there is a unique uniformly continuous function $f\colon E \to F$ extending f_0.*

Proof. We follow Schwartz's proof; see Schwartz [149] (Chapter XI, Section 3, Theorem 1).

Step 1. We begin by constructing a function $f\colon E \to F$ extending f_0. Since E_0 is dense in E, for every $x \in E$, there is some sequence (x_n) converging to x, with $x_n \in E_0$. Then the sequence (x_n) is a Cauchy sequence in E. We claim that $(f_0(x_n))$ is a Cauchy sequence in F.

Proof of the claim. For every $\epsilon > 0$, since f_0 is uniformly continuous, there is some $\eta > 0$ such that for all $(y, z) \in E_0$, if $d(y, z) \leq \eta$, then $d(f_0(y), f_0(z)) \leq \epsilon$. Since (x_n) is a Cauchy sequence with $x_n \in E_0$, there is some integer $p > 0$ such that if $m, n \geq p$, then $d(x_m, x_n) \leq \eta$, thus $d(f_0(x_m), f_0(x_n)) \leq \epsilon$, which proves that $(f_0(x_n))$ is a Cauchy sequence in F. $\square$