where $\lambda_1, \mu_1 \in \mathbb{R}$, with $\mu_1 > 0$. However, $W^\perp$ has dimension $n-2$, and by Proposition 17.9, $f\big(W^\perp\big) \subseteq W^\perp$. Since the restriction of f to $W^\perp$ is also normal, we conclude by applying the induction hypothesis to $W^\perp$. $\square$

After this relatively hard work, we can easily obtain some nice normal forms for the matrices of self-adjoint, skew-self-adjoint, and orthogonal linear maps. However, for the sake of completeness (and since we have all the tools to so do), we go back to the case of a Hermitian space and show that normal linear maps can be diagonalized with respect to an orthonormal basis. The proof is a slight generalization of the proof of Theorem 17.6.

Theorem 17.13. *(Spectral theorem for normal linear maps on a Hermitian space) Given a Hermitian space E of dimension n, for every normal linear map $f\colon E \to E$ there is an orthonormal basis $(e_1, \ldots, e_n)$ of eigenvectors of f such that the matrix of f w.r.t. this basis is a diagonal matrix*

$$\begin{pmatrix} \lambda_1 & & \ldots & \\ & \lambda_2 & \ldots & \\ \vdots & \vdots & \ddots & \vdots \\ & & \ldots & \lambda_n \end{pmatrix},$$

where $\lambda_j \in \mathbb{C}$.

Proof. We proceed by induction on the dimension n of E as follows. If $n = 1$, the result is trivial. Assume now that $n \geq 2$. Since $\mathbb{C}$ is algebraically closed (i.e., every polynomial has a root in $\mathbb{C}$), the linear map $f\colon E \to E$ has some eigenvalue $\lambda \in \mathbb{C}$, and let w be some unit eigenvector for λ. Let W be the subspace of dimension 1 spanned by w. Clearly, $f(W) \subseteq W$. By Proposition 17.3, w is an eigenvector of f^* for $\overline{\lambda}$, and thus $f^*(W) \subseteq W$. By Proposition 17.9, we also have $f(W^\perp) \subseteq W^\perp$. The restriction of f to $W^\perp$ is still normal, and we conclude by applying the induction hypothesis to $W^\perp$ (whose dimension is $n-1$). $\square$

Theorem 17.13 implies that (complex) self-adjoint, skew-self-adjoint, and orthogonal linear maps can be diagonalized with respect to an orthonormal basis of eigenvectors. In this latter case, though, an orthogonal map is called a *unitary* map. Proposition 17.5 also shows that the eigenvalues of a self-adjoint linear map are real, and Proposition 17.7 shows that the eigenvalues of a skew self-adjoint map are pure imaginary or zero, and that the eigenvalues of a unitary map have absolute value 1.

Remark: There is a converse to Theorem 17.13, namely, if there is an orthonormal basis $(e_1, \ldots, e_n)$ of eigenvectors of f, then f is normal. We leave the easy proof as an exercise.

In the next section we specialize Theorem 17.12 to self-adjoint, skew-self-adjoint, and orthogonal linear maps. Due to the additional structure, we obtain more precise normal forms.