In order to prove that our constrained minimization problem has a unique solution, we proceed to prove that the constrained minimization of $Q(x)$ subject to $B^\top x = f$ is equivalent to the unconstrained maximization of another function $-G(\lambda)$. We get $G(\lambda)$ by minimizing the Lagrangian $L(x, \lambda)$ treated as a function of x alone. The function $-G(\lambda)$ is the *dual function* of the Lagrangian $L(x, \lambda)$. Here we are encountering a special case of the notion of dual function defined in Section 50.7.

Since A^{-1} is symmetric positive definite and

$$L(x, \lambda) = \frac{1}{2} x^\top A^{-1} x - (b - B\lambda)^\top x - \lambda^\top f,$$

by Proposition 42.2 the global minimum (with respect to x) of $L(x, \lambda)$ is obtained for the solution x of

$$A^{-1} x = b - B\lambda,$$

that is, when

$$x = A(b - B\lambda),$$

and the minimum of $L(x, \lambda)$ is

$$\min_x L(x, \lambda) = -\frac{1}{2} (B\lambda - b)^\top A (B\lambda - b) - \lambda^\top f.$$

Letting

$$G(\lambda) = \frac{1}{2} (B\lambda - b)^\top A (B\lambda - b) + \lambda^\top f,$$

we will show in Proposition 42.3 that the solution of the constrained minimization of $Q(x)$ subject to $B^\top x = f$ is equivalent to the unconstrained maximization of $-G(\lambda)$. This is a special case of the duality discussed in Section 50.7.

Of course, since we minimized $L(x, \lambda)$ with respect to x, we have

$$L(x, \lambda) \geq -G(\lambda)$$

for all x and all λ. However, when the constraint $B^\top x = f$ holds, $L(x, \lambda) = Q(x)$, and thus for any admissible x, which means that $B^\top x = f$, we have

$$\min_x Q(x) \geq \max_\lambda -G(\lambda).$$

In order to prove that the unique minimum of the constrained problem $Q(x)$ subject to $B^\top x = f$ is the unique maximum of $-G(\lambda)$, we compute $Q(x) + G(\lambda)$.

Proposition 42.3. *The quadratic constrained minimization problem of Definition 42.3 has a unique solution (x, λ) given by the system*

$$\begin{pmatrix} A^{-1} & B \\ B^\top & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} b \\ f \end{pmatrix}.$$

Furthermore, the component λ of the above solution is the unique value for which $-G(\lambda)$ is maximum.