where $U = \{h \in \mathbb{R} \mid a + h \in A,\ h < 0\}$.

If a function f as in Definition 39.1 has a derivative $f'(a)$ at a, then it is continuous at a. If f is differentiable on A, then f is continuous on A. The composition of differentiable functions is differentiable.

Remark: A function f has a derivative $f'(a)$ at a iff the derivative of f on the left at a and the derivative of f on the right at a exist, and if they are equal. Also, if the derivative of f on the left at a exists, then f is continuous on the left at a (and similarly on the right).

We would like to extend the notion of derivative to functions $f \colon A \to F$, where E and F are normed affine spaces, and A is some nonempty open subset of E. The first difficulty is to make sense of the quotient

$$\frac{f(a+h) - f(a)}{h}.$$

If E and F are normed affine spaces, it will be notationally convenient to assume that the vector space associated with E is denoted by $\overrightarrow{E}$, and that the vector space associated with F is denoted as $\overrightarrow{F}$.

Since F is a normed affine space, making sense of $f(a+h)-f(a)$ is easy: we can define this as $\overrightarrow{f(a)f(a+h)}$, the unique vector translating $f(a)$ to $f(a+h)$. We should note however, that this quantity is a vector and not a point. Nevertheless, in defining derivatives, it is notationally more pleasant to denote $\overrightarrow{f(a)f(a+h)}$ by $f(a+h) - f(a)$. Thus, in the rest of this chapter, the vector $\overrightarrow{ab}$ will be denoted by $b - a$. But now, how do we define the quotient by a vector? Well, we don't!

A first possibility is to consider the *directional derivative* with respect to a vector $u \neq 0$ in $\overrightarrow{E}$. We can consider the vector $f(a + tu) - f(a)$, where $t \in \mathbb{R}$ (or $t \in \mathbb{C}$). Now,

$$\frac{f(a+tu) - f(a)}{t}$$

makes sense. The idea is that in E, the points of the form $a + tu$ for t in some small interval $[-\epsilon, +\epsilon]$ in $\mathbb{R}$ (or $\mathbb{C}$) form a line segment $[r, s]$ in A containing a, and that the image of this line segment defines a small curve segment on $f(A)$. This curve segment is defined by the map $t \mapsto f(a + tu)$, from $[r, s]$ to F, and the directional derivative $\mathrm{D}_u f(a)$ defines the direction of the tangent line at a to this curve; see Figure 39.1. This leads us to the following definition.

Definition 39.2. Let E and F be two normed affine spaces, let A be a nonempty open subset of E, and let $f \colon A \to F$ be any function. For any $a \in A$, for any $u \neq 0$ in $\overrightarrow{E}$, the *directional derivative of f at a w.r.t. the vector u*, denoted by $\mathrm{D}_u f(a)$, is the limit (if it exists)

$$\lim_{t \to 0,\, t \in U} \frac{f(a+tu) - f(a)}{t},$$

where $U = \{t \in \mathbb{R} \mid a + tu \in A,\ t \neq 0\}$ (or $U = \{t \in \mathbb{C} \mid a + tu \in A,\ t \neq 0\}$).