(1) The eigenvalues of f are strictly positive iff

$$\langle f(u), u\rangle > 0 \quad \textit{for all } u \neq 0.$$

(2) The eigenvalues of f are nonnegative iff

$$\langle f(u), u\rangle \geq 0 \quad \textit{for all } u \neq 0.$$

Proof. Since f is self-adjoint, by the spectral theorem (Theorem 17.8), f has real eigenvalues $\lambda_1, \ldots, \lambda_n$, and there is some orthonormal basis $(e_1, \ldots, e_n)$, where e_i is an eigenvector for λ_i. With respect to this basis, every vector $u \in E$ can be written in a unique way as $u = \sum_{i=1}^n x_i u_i$ for some $x_i \in \mathbb{R}$. Since each e_i is eigenvector associated with $\lambda_i \in \mathbb{R}$, we have

$$f\left(\sum_{i=1}^n x_i e_i\right) = \sum_{i=1}^n x_i f(e_i) = \sum_{i=1}^n \lambda_i x_i e_i,$$

and using the bilinearity of the inner product, we have

$$\begin{aligned}\langle f(u), u\rangle &= \left\langle f\left(\sum_{i=1}^n x_i e_i\right), \sum_{j=1}^n x_j e_j \right\rangle \\ &= \left\langle \sum_{i=1}^n \lambda_i x_i e_i, \sum_{j=1}^n x_j e_j \right\rangle \\ &= \sum_{i=1}^n \sum_{j=1}^n \lambda_i x_i x_j \langle e_i, e_j\rangle,\end{aligned}$$

and since $(e_1, \ldots, e_n)$, is an orthonormal basis, we obtain

$$\langle f(u), u\rangle = \sum_{i=1}^n \lambda_i x_i^2. \tag{$\dagger$}$$

(1) If $\lambda_i > 0$ for $i = 1, \ldots, n$, for any $u \neq 0$, we have $x_i \neq 0$ for some i, so $\langle f(u), u\rangle = \sum_{i=1}^n \lambda_i x_i^2 > 0$.

Conversely, if $\langle f(u), u\rangle > 0$ for all $u \neq 0$, by picking $u = e_i$, we get

$$\langle f(e_i), e_i\rangle = \langle \lambda_i e_i, e_i\rangle = \lambda_i \langle e_i, e_i\rangle = \lambda_i,$$

so $\lambda_i > 0$ for $i = 1, \ldots, n$.

(2) If $\lambda_i \geq 0$ for $i = 1, \ldots, n$, for any $u \neq 0$, then $\langle f(u), u\rangle = \sum_{i=1}^n \lambda_i x_i^2 \geq 0$.

Conversely, if $\langle f(u), u\rangle \geq 0$ for all $u \neq 0$, by picking $u = e_i$, we get

$$\langle f(e_i), e_i\rangle = \langle \lambda_i e_i, e_i\rangle = \lambda_i \langle e_i, e_i\rangle = \lambda_i,$$

so $\lambda_i \geq 0$ for $i = 1, \ldots, n$. $\square$