Proposition 51.2. *If f is any convex function on $\mathbb{R}^n$, then for every $\alpha \in \mathbb{R} \cup \{-\infty, +\infty\}$, the sublevel sets* $\mathrm{sublev}_\alpha(f)$ *and* $\mathrm{sublev}_{<\alpha}(f)$ *are convex.*

Definition 51.7. A function $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$ is *lower semi-continuous* if the sublevel sets $\mathrm{sublev}_\alpha(f) = \{x \in \mathbb{R}^n \mid f(x) \leq \alpha\}$ are closed for all $\alpha \in \mathbb{R}$.

Observe that the improper convex function of Example 51.2 is not lower semi-continuous since $\mathrm{sublev}_\alpha(f) = (-1, 1)$ whenever $-\infty < \alpha < 0$. This result reflects the fact that the epigraph is not closed as shown in the following proposition; see Rockafellar [138] (Theorem 7.1).

Proposition 51.3. *Let $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$ be any function. The following properties are equivalent:*

(1) The function f is lower semi-continuous.

(2) The epigraph of f is a closed set in $\mathbb{R}^{n+1}$.

The notion of the closure of convex function plays an important role. It is a bit subtle because a convex function may be improper.

Definition 51.8. Let $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$ be any function. The function whose epigraph is the closure of the epigraph $\mathbf{epi}(f)$ of f (in $\mathbb{R}^{n+1}$) is called the *lower semi-continuous hull* of f. If f is a convex function and if $f(x) > -\infty$ for all $x \in \mathbb{R}^n$, then the *closure* $\mathrm{cl}(f)$ *of* f is equal to its lower semi-continuous hull, else if $f(x) = -\infty$ for some $x \in \mathbb{R}^n$, then the *closure* $\mathrm{cl}(f)$ *of* f is the constant function with value $-\infty$. A convex function f is *closed* if $f = \mathrm{cl}(f)$.

Definition 51.8 implies that there are *only two closed improper convex functions*: the constant function with value $-\infty$ and the constant function with value $+\infty$. Also, by Proposition 51.3, *a proper convex function is closed iff it is equal to its lower semi-continuous hull iff its epigraph is nonempty and closed*.

Given a convex set C in $\mathbb{R}^n$, the interior $\mathrm{int}(C)$ of C (the largest open subset of $\mathbb{R}^n$ contained in C) is often not interesting because C may have dimension smaller than n. For example, a (closed) triangle in $\mathbb{R}^3$ has empty interior.

The remedy is to consider the affine hull $\mathrm{aff}(C)$ of C, which is the smallest affine set containing C; see Section 44.2. The dimension of C is the dimension of $\mathrm{aff}(C)$. Then the relative interior of C is the interior of C in $\mathrm{aff}(C)$ endowed with the subspace topology induced on $\mathrm{aff}(C)$. More explicitly, we can make the following definition.

Definition 51.9. Let C be a subset of $\mathbb{R}^n$. The *relative interior* of C is the set

$$\mathbf{relint}(C) = \{x \in C \mid B_\epsilon(x) \cap \mathrm{aff}(C) \subseteq C \quad \text{for some } \epsilon > 0\},$$

where $B_\epsilon(x) = \{y \in \mathbb{R}^n \mid \|x - y\|_2 < \epsilon\}$, the open ball of center x and radius ϵ. The *relative boundary* of C is defined as $\overline{C} - \mathbf{relint}(C)$, where $\overline{C}$ is the closure of C in $\mathbb{R}^n$ (the smallest closed subset of $\mathbb{R}^n$ containing C).