Example 50.9. Consider the following problem:

$$\begin{aligned} &\text{minimize} \quad \|v\| \\ &\text{subject to} \quad Av = b, \end{aligned}$$

where $\| \; \|$ is any norm on $\mathbb{R}^n$. Using the result of Example 50.8(6), we obtain

$$G(\nu) = -b^\top \nu - \left\|-A^\top \nu\right\|^*,$$

that is,

$$G(\nu) = \begin{cases} -b^\top \nu & \text{if } \left\|A^\top \nu\right\|^D \leq 1 \\ -\infty & \text{otherwise.} \end{cases}$$

In the special case where $\| \; \| = \| \; \|_2$, we also have $\| \; \|^D = \| \; \|_2$.

Another interesting application is to the entropy minimization problem.

Example 50.10. Consider the following problem known as *entropy minimization*:

$$\begin{aligned} &\text{minimize} \quad f(x) = \sum_{i=1}^n x_i \log x_i \\ &\text{subject to} \quad Ax \leq b \\ &\qquad\qquad\quad \mathbf{1}^\top x = 1, \end{aligned}$$

where $\mathrm{dom}(f) = \{x \in \mathbb{R}^n \mid x \geq 0\}$. By Example 50.8(3), the conjugate of the negative entropy function $u \log u$ is e^{v-1}, so we easily see that

$$f^*(y) = \sum_{i=1}^n e^{y_i - 1},$$

which is defined on $\mathbb{R}^n$. Proposition 50.20 implies that the dual function $G(\lambda, \mu)$ of the entropy minimization problem is given by

$$G(\lambda, \mu) = -b^\top \lambda - \mu - e^{-\mu-1} \sum_{i=1}^n e^{-(A^i)^\top \lambda},$$

for all $\lambda \in \mathbb{R}^n_+$ and all $\mu \in \mathbb{R}$, where A^i is the ith column of A. It follows that the dual program is:

$$\begin{aligned} &\text{maximize} \quad -b^\top \lambda - \mu - e^{-\mu-1} \sum_{i=1}^n e^{-(A^i)^\top \lambda} \\ &\text{subject to} \quad \lambda \geq 0. \end{aligned}$$