37.13 Normed Affine Spaces

For geometric applications, we will need to consider affine spaces $(E, \overrightarrow{E})$ where the associated space of translations $\overrightarrow{E}$ is a vector space equipped with a norm.

Definition 37.44. Given an affine space $(E, \overrightarrow{E})$, where the space of translations $\overrightarrow{E}$ is a vector space over $\mathbb{R}$ or $\mathbb{C}$, we say that $(E, \overrightarrow{E})$ is a *normed affine space* if $\overrightarrow{E}$ is a normed vector space with norm $\| \, \|$.

Given a normed affine space, there is a natural metric on E itself, defined such that

$$d(a, b) = \|\overrightarrow{ab}\|.$$

Observe that this metric is invariant under translation, that is,

$$d(a + u, b + u) = d(a, b).$$

Also, for every fixed $a \in E$ and $\lambda > 0$, if we consider the map $h\colon E \to E$, defined such that,

$$h(x) = a + \lambda\overrightarrow{ax},$$

then $d(h(x), h(y)) = \lambda d(x, y)$.

Note that the map $(a, b) \mapsto \overrightarrow{ab}$ from $E \times E$ to $\overrightarrow{E}$ is continuous, and similarly for the map $a \mapsto a + u$ from $E \times \overrightarrow{E}$ to E. In fact, the map $u \mapsto a + u$ is a homeomorphism from $\overrightarrow{E}$ to E_a.

Of course, $\mathbb{R}^n$ is a normed affine space under the Euclidean metric, and it is also complete.

If an affine space E is a finite direct sum $(E_1, a_1) \oplus \cdots \oplus (E_m, a_m)$, and each E_i is also a normed affine space with norm $\| \, \|_i$, we make $(E_1, a_1) \oplus \cdots \oplus (E_m, a_m)$ into a normed affine space, by giving it the norm

$$\|(x_1, \ldots, x_n)\| = \max(\|x_1\|_1, \ldots, \|x_n\|_n).$$

Similarly, the finite product $E_1 \times \cdots \times E_m$ is made into a normed affine space, under the same norm.

We are now ready to define the derivative (or differential) of a map between two normed affine spaces. This will lead to tangent spaces to curves and surfaces (in normed affine spaces).

37.14 Futher Readings

A thorough treatment of general topology can be found in Munkres [131, 130], Dixmier [51], Lang [111, 112], Schwartz [150, 149], Bredon [30], and the classic, Seifert and Threlfall [155].