and we will be done.

Since $\dim(E) \geq 2$, pick v to be any nonzero vector in $\overrightarrow{E}$ such that u and v are linearly independent, and let us evaluate $\langle \overrightarrow{f}(u+v), \overrightarrow{f}(w)\rangle$ for any $w \in \overrightarrow{E}$. We have

$$\begin{aligned}\langle \overrightarrow{f}(u+v), \overrightarrow{f}(w)\rangle &= \varphi_{u+v}(w)\\ &= \rho(u+v)\langle u+v, w\rangle\\ &= \rho(u+v)\langle u, w\rangle + \rho(u+v)\langle v, w\rangle\end{aligned}$$

and

$$\begin{aligned}\langle \overrightarrow{f}(u+v), \overrightarrow{f}(w)\rangle &= \langle \overrightarrow{f}(u) + \overrightarrow{f}(v), \overrightarrow{f}(w)\rangle\\ &= \langle \overrightarrow{f}(u), \overrightarrow{f}(w)\rangle + \langle \overrightarrow{f}(v), \overrightarrow{f}(w)\rangle\\ &= \rho(u)\langle u, w\rangle + \rho(v)\langle v, w\rangle,\end{aligned}$$

so we get

$$\langle (\rho(u+v) - \rho(u))u + (\rho(u+v) - \rho(v))v, w\rangle = 0 \quad \text{for all } w \in \overrightarrow{E},$$

which implies that

$$(\rho(u+v) - \rho(u))u + (\rho(u+v) - \rho(v))v = 0.$$

Since u and v are linearly independent, we must have

$$\rho(u+v) = \rho(u) = \rho(v).$$

This proves that $\rho(u)$ is a constant ρ independent of u, as claimed.

The converse is trivial. $\square$

Remark: Let $f \in \mathbf{GA}(E)$ be an affine similarity of ratio ρ. If either $\rho \neq 1$ or $\rho = 1$ and $\overrightarrow{f} \in \mathbf{O}(E)$ does not admit the eigenvalue 1, then f has a unique fixed point.

Indeed, we have $\overrightarrow{f} = \rho\overrightarrow{g}$ for some $\rho > 0$ and some linear isometry $\overrightarrow{g} \in \mathbf{O}(E)$, so for any origin $a \in E$, the point $a + u$ is a fixed point of f iff

$$f(a+u) = a + u$$

iff

$$f(a) + \overrightarrow{f}(u) = a + u$$

iff

$$\rho\overrightarrow{g}(u) = \overrightarrow{f(a)a} + u$$

iff

$$(\overrightarrow{g} - \rho^{-1}\mathrm{id})(u) = \rho^{-1}\overrightarrow{f(a)a}.$$