20.2 Laplacian Matrices of Graphs

Let us begin with directed graphs, although as we will see, graph Laplacians are fundamentally associated with undirected graph. The key proposition below shows how given an undirected graph G, for any orientation σ of G, $B^\sigma (B^\sigma)^\top$ relates to the adjacency matrix A (where B^σ is the incidence matrix of the directed graph G^σ). We reproduce the proof in Gallier [71] (see also Godsil and Royle [77]).

Proposition 20.2. *Given any undirected graph G, for any orientation σ of G, if B^σ is the incidence matrix of the directed graph G^σ, A is the adjacency matrix of G^σ, and D is the degree matrix such that $D_{i\,i} = d(v_i)$, then*

$$B^\sigma (B^\sigma)^\top = D - A.$$

Consequently, $L = B^\sigma (B^\sigma)^\top$ is independent of the orientation σ of G, and $D - A$ is symmetric and positive semidefinite; that is, the eigenvalues of $D - A$ are real and nonnegative.

Proof. The entry $B^\sigma (B^\sigma)^\top_{i\,j}$ is the inner product of the ith row b^σ_i, and the jth row b^σ_j of B^σ. If $i = j$, then as

$$b^\sigma_{i\,k} = \begin{cases} +1 & \text{if } s(e_k) = v_i \\ -1 & \text{if } t(e_k) = v_i \\ 0 & \text{otherwise} \end{cases}$$

we see that $b^\sigma_i \cdot b^\sigma_i = d(v_i)$. If $i \neq j$, then $b^\sigma_i \cdot b^\sigma_j \neq 0$ iff there is some edge e_k with $s(e_k) = v_i$ and $t(e_k) = v_j$ or vice-versa (which are mutually exclusive cases, since G^σ arises by orienting an undirected graph), in which case, $b^\sigma_i \cdot b^\sigma_j = -1$. Therefore,

$$B^\sigma (B^\sigma)^\top = D - A,$$

as claimed.

For every $x \in \mathbb{R}^m$, we have

$$x^\top L x = x^\top B^\sigma (B^\sigma)^\top x = ((B^\sigma)^\top x)^\top (B^\sigma)^\top x = \left\|(B^\sigma)^\top x\right\|_2^2 \geq 0,$$

since the Euclidean norm $\| \, \|_2$ is positive (definite). Therefore, $L = B^\sigma (B^\sigma)^\top$ is positive semidefinite. It is well-known that a real symmetric matrix is positive semidefinite iff its eigenvalues are nonnegative. ☐

Definition 20.15. The matrix $L = B^\sigma (B^\sigma)^\top = D - A$ is called the *(unnormalized) graph Laplacian* of the graph G^σ. The *(unnormalized) graph Laplacian* of an undirected graph $G = (V, E)$ is defined by

$$L = D - A.$$