Example 50.6. Going back to Example 50.5 where we considered the linear program (P)

$$\begin{aligned}&\text{minimize} \quad c^\top v\\&\text{subject to} \quad Av \le b, \;\; v \ge 0,\end{aligned}$$

with A an $m \times n$ matrix, the Lagrangian $L(v, \mu, \nu)$ is given by

$$L(v, \mu, \nu) = -b^\top \mu + (c + A^\top \mu - \nu)^\top v,$$

and we found that the dual function $G(\mu, \nu) = \inf_{v \in \mathbb{R}^n} L(v, \mu, \nu)$ is given for all $\mu \ge 0$ and $\nu \ge 0$ by

$$G(\mu, \nu) = \begin{cases} -b^\top \mu & \text{if } A^\top \mu - \nu + c = 0, \\ -\infty & \text{otherwise.} \end{cases}$$

The hypotheses of Theorem 50.17(1) certainly fail since there are infinitely $u_{\mu,\nu} \in \mathbb{R}^n$ such that $G(\mu, \nu) = \inf_{v \in \mathbb{R}^n} L(v, \mu, \nu) = L(u_{\mu,\nu}, \mu, \nu)$. Therefore, the dual function G is no help in finding a solution of the Primal Problem (P). As we saw earlier, if we consider the modified dual Problem (D_1) then strong duality holds, but this *does not* follow from Theorem 50.17, and a different proof is required.

Thus, we have the somewhat counter-intuitive situation that the *general* theory of Lagrange duality does not apply, at least directly, to linear programming, a fact that is not sufficiently emphasized in many expositions. A separate treatment of duality is required.

Unlike the case of linear programming, which needs a separate treatment, Theorem 50.17 applies to the optimization problem involving a convex quadratic objective function and a set of affine inequality constraints. So in some sense, convex quadratic programming is simpler than linear programming!

Example 50.7. Consider the quadratic objective function

$$J(v) = \frac{1}{2} v^\top A v - v^\top b,$$

where A is an $n \times n$ matrix which is symmetric positive definite, $b \in \mathbb{R}^n$, and the constraints are affine inequality constraints of the form

$$Cv \le d,$$

where C is an $m \times n$ matrix and $d \in \mathbb{R}^m$. For the time being, we do not assume that C has rank m. Since A is symmetric positive definite, J is strictly convex, as implied by Proposition 40.11 (see Example 40.6). The Lagrangian of this quadratic optimization problem is given by

$$\begin{aligned}L(v, \mu) &= \frac{1}{2} v^\top A v - v^\top b + (Cv - d)^\top \mu \\ &= \frac{1}{2} v^\top A v - v^\top (b - C^\top \mu) - \mu^\top d.\end{aligned}$$