3.8 Quotient Spaces

Let E be a vector space, and let M be any subspace of E. The subspace M induces a relation $\equiv_M$ on E, defined as follows: For all $u, v \in E$,

$u \equiv_M v$ iff $u - v \in M$.

We have the following simple proposition.

Proposition 3.22. *Given any vector space E and any subspace M of E, the relation $\equiv_M$ is an equivalence relation with the following two congruential properties:*

1. *If $u_1 \equiv_M v_1$ and $u_2 \equiv_M v_2$, then $u_1 + u_2 \equiv_M v_1 + v_2$, and*
2. *if $u \equiv_M v$, then $\lambda u \equiv_M \lambda v$.*

Proof. It is obvious that $\equiv_M$ is an equivalence relation. Note that $u_1 \equiv_M v_1$ and $u_2 \equiv_M v_2$ are equivalent to $u_1 - v_1 = w_1$ and $u_2 - v_2 = w_2$, with $w_1, w_2 \in M$, and thus,

$$(u_1 + u_2) - (v_1 + v_2) = w_1 + w_2,$$

and $w_1 + w_2 \in M$, since M is a subspace of E. Thus, we have $u_1 + u_2 \equiv_M v_1 + v_2$. If $u - v = w$, with $w \in M$, then

$$\lambda u - \lambda v = \lambda w,$$

and $\lambda w \in M$, since M is a subspace of E, and thus $\lambda u \equiv_M \lambda v$. $\square$

Proposition 3.22 shows that we can define addition and multiplication by a scalar on the set E/M of equivalence classes of the equivalence relation $\equiv_M$.

Definition 3.25. Given any vector space E and any subspace M of E, we define the following operations of addition and multiplication by a scalar on the set E/M of equivalence classes of the equivalence relation $\equiv_M$ as follows: for any two equivalence classes $[u], [v] \in E/M$, we have

$$[u] + [v] = [u + v],$$
$$\lambda[u] = [\lambda u].$$

By Proposition 3.22, the above operations do not depend on the specific choice of representatives in the equivalence classes $[u], [v] \in E/M$. It is also immediate to verify that E/M is a vector space. The function $\pi\colon E \to E/F$, defined such that $\pi(u) = [u]$ for every $u \in E$, is a surjective linear map called the *natural projection of E onto E/F*. The vector space E/M is called the *quotient space of E by the subspace M*.

Given any linear map $f\colon E \to F$, we know that $\operatorname{Ker} f$ is a subspace of E, and it is immediately verified that $\operatorname{Im} f$ is isomorphic to the quotient space $E/\operatorname{Ker} f$.