condition on the gradient of J. This time the space V can be infinite dimensional.

Proposition 49.14. *Let $J\colon V \to \mathbb{R}$ be a continuously differentiable functional defined on a Hilbert space V. Suppose there exists two constants $\alpha > 0$ and $M > 0$ such that*

$$\langle \nabla J_v - \nabla J_u, v - u\rangle \geq \alpha \left\|v - u\right\|^2 \quad \text{for all } u, v \in V,$$

and the Lipschitz condition

$$\left\|\nabla J_v - \nabla J_u\right\| \leq M \left\|v - u\right\| \quad \text{for all } u, v \in V.$$

If there exists two real numbers $a, b \in \mathbb{R}$ such that

$$0 < a \leq \rho_k \leq b \leq \frac{2\alpha}{M^2} \quad \text{for all } k \geq 0,$$

then the gradient method with variable stepsize parameter converges. Furthermore, there is some constant $\beta > 0$ (depending on α, M, a, b) such that

$$\beta < 1 \quad \text{and} \quad \left\|u_k - u\right\| \leq \beta^k \left\|u_0 - u\right\|,$$

where $u \in V$ is the unique minimum of J.

Proof. By hypothesis the functional J is elliptic, so by Theorem 49.8(2) it has a unique minimum u characterized by the fact that $\nabla J_u = 0$. Then since $u_{k+1} = u_k - \rho_k \nabla J_{u_k}$, we can write

$$u_{k+1} - u = (u_k - u) - \rho_k(\nabla J_{u_k} - \nabla J_u). \tag{$*$}$$

Using the inequalities

$$\langle \nabla J_{u_k} - \nabla J_u, u_k - u\rangle \geq \alpha \left\|u_k - u\right\|^2$$

and

$$\left\|\nabla J_{u_k} - \nabla J_u\right\| \leq M \left\|u_k - u\right\|,$$

and assuming that $\rho_k > 0$, it follows that

$$\begin{aligned}
\left\|u_{k+1} - u\right\|^2 &= \left\|u_k - u\right\|^2 - 2\rho_k \langle \nabla J_{u_k} - \nabla J_u, u_k - u\rangle + \rho_k^2 \left\|\nabla J_{u_k} - \nabla J_u\right\|^2 \\
&\leq \left(1 - 2\alpha\rho_k + M^2\rho_k^2\right) \left\|u_k - u\right\|^2.
\end{aligned}$$

Consider the function

$$T(\rho) = M^2\rho^2 - 2\alpha\rho + 1.$$

Its graph is a parabola intersecting the y-axis at $y = 1$ for $\rho = 0$, it has a minimum for $\rho = \alpha/M^2$, and it also has the value $y = 1$ for $\rho = 2\alpha/M^2$; see Figure 49.7. Therefore if we pick a, b and ρ_k such that

$$0 < a \leq \rho_k \leq b < \frac{2\alpha}{M^2},$$