Property ($2'$) follows by Lemma 32.9. By Proposition 32.2, $A[X]$ is a UFD. $\square$

As a corollary of Theorem 32.10 and using induction, we note that for any field K, the polynomial ring $K[X_1, \ldots, X_n]$ is a UFD.

For the sake of completeness, we shall prove that every PID is a UFD. First, we review the notion of gcd and the characterization of gcd's in a PID.

Given an integral domain A, for any two elements $a, b \in A$, $a, b \neq 0$, we say that $d \in A$ ($d \neq 0$) is a *greatest common divisor (gcd)* of a and b if

(1) d divides both a and b.

(2) For any $h \in A$ ($h \neq 0$), if h divides both a and b, then h divides d.

We also say that a and b are *relatively prime* if 1 is a gcd of a and b.

Note that a and b are relatively prime iff every gcd of a and b is a unit. If A is a PID, then gcd's are characterized as follows.

Proposition 32.11. *Let A be a PID.*

(1) For any $a, b, d \in A$ ($a, b, d \neq 0$), d is a gcd of a and b iff

$$(d) = (a, b) = (a) + (b),$$

i.e., d generates the principal ideal generated by a and b.

(2) (Bezout identity) Two nonnull elements $a, b \in A$ are relatively prime iff there are some $x, y \in A$ such that

$$ax + by = 1.$$

Proof. (1) Recall that the ideal generated by a and b is the set

$$(a) + (b) = aA + bA = \{ax + by \mid x, y \in A\}.$$

First, assume that d is a gcd of a and b. If so, $a \in Ad$, $b \in Ad$, and thus, $(a) \subseteq (d)$ and $(b) \subseteq (d)$, so that

$$(a) + (b) \subseteq (d).$$

Since A is a PID, there is some $t \in A$, $t \neq 0$, such that

$$(a) + (b) = (t),$$

and thus, $(a) \subseteq (t)$ and $(b) \subseteq (t)$, which means that t divides both a and b. Since d is a gcd of a and b, t must divide d. But then,

$$(d) \subseteq (t) = (a) + (b),$$