A. In other words, transposition exchanges the rows and the columns of a matrix. Here is an example. If A is the 5×6 matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 1 & 2 & 3 & 4 & 5 \\ 8 & 7 & 1 & 2 & 3 & 4 \\ 9 & 8 & 7 & 1 & 2 & 3 \\ 10 & 9 & 8 & 7 & 1 & 2 \end{pmatrix},$$

then $A^\top$ is the 6×5 matrix

$$A^\top = \begin{pmatrix} 1 & 7 & 8 & 9 & 10 \\ 2 & 1 & 7 & 8 & 9 \\ 3 & 2 & 1 & 7 & 8 \\ 4 & 3 & 2 & 1 & 7 \\ 5 & 4 & 3 & 2 & 1 \\ 6 & 5 & 4 & 3 & 2 \end{pmatrix}.$$

The following observation will be useful later on when we discuss the SVD. Given any $m \times n$ matrix A and any $n \times p$ matrix B, if we denote the columns of A by $A^1, \ldots, A^n$ and the rows of B by $B_1, \ldots, B_n$, then we have

$$AB = A^1 B_1 + \cdots + A^n B_n.$$

For every square matrix A of dimension n, it is immediately verified that $AI_n = I_n A = A$.

Definition 3.16. For any square matrix A of dimension n, if a matrix B such that $AB = BA = I_n$ exists, then it is unique, and it is called the *inverse* of A. The matrix B is also denoted by A^{-1}. An invertible matrix is also called a *nonsingular* matrix, and a matrix that is not invertible is called a *singular* matrix.

The following result is a matrix analog of Proposition 3.21.

Proposition 3.13. *If a square matrix $A \in \mathrm{M}_n(K)$ has a left inverse, that is a matrix B such that $BA = I_n$, or a right inverse, that is a matrix C such that $AC = I_n$, then A is actually invertible. Furthermore, $B = A^{-1}$ and $C = A^{-1}$.*

Proof. Proposition 3.13 follows from Proposition 3.21 and the fact that matrices represent linear maps. We can also give a direct proof as follows. Suppose that there is a matrix B such that $BA = I_n$. This implies that the columns $A^1, \ldots, A^n$ of A are linearly independent, because if

$$A\lambda = \lambda_1 A^1 + \cdots + \lambda_n A^n = 0,$$

where $\lambda \in K^n$ is the column vector

$$\lambda = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix},$$