This is because if

$$z = x_1 u + x_2 v + x_3 w = y_1 u + y_2 v + y_3 w,$$

then by using our (linear!) operations on vectors, we get

$$(y_1 - x_1)u + (y_2 - x_2)v + (y_3 - x_3)w = 0,$$

which implies that

$$y_1 - x_1 = y_2 - x_2 = y_3 - x_3 = 0,$$

by linear independence. Thus,

$$y_1 = x_1, \quad y_2 = x_2, \quad y_3 = x_3,$$

which shows that $z$ has a unique expression as a linear combination, as claimed. Then our equation

$$x_1 u + x_2 v + x_3 w = b$$

has a *unique solution*, and indeed, we can check that

$$\begin{aligned} x_1 &= 1.4 \\ x_2 &= -0.4 \\ x_3 &= -0.4 \end{aligned}$$

is the solution.

But then, *how do we determine that some vectors are linearly independent?*

One answer is to compute a numerical quantity $\det(u, v, w)$, called the *determinant* of $(u, v, w)$, and to check that it is nonzero. In our case, it turns out that

$$\det(u, v, w) = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & -2 & -2 \end{vmatrix} = 15,$$

which confirms that $u, v, w$ are linearly independent.

Other methods, which are much better for systems with a large number of variables, consist of computing an LU-decomposition or a QR-decomposition, or an SVD of the *matrix* consisting of the three columns $u, v, w$,

$$A = \begin{pmatrix} u & v & w \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 1 & -2 & -2 \end{pmatrix}.$$

If we form the vector of unknowns

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$