Interestingly, under certain hypotheses, it is possible to prove that the sequence of approximate solutions $(u^k)_{k\geq 0}$ converges to the minimizer $u$ of $J$ over $U$, even if the sequence $(\lambda^k)_{k\geq 0}$ does not converge. We prove such a result when the constraints $\varphi_i$ are *affine*.

**Theorem 50.21.** *Suppose $J\colon \mathbb{R}^n \to \mathbb{R}$ is an elliptic functional, which means that $J$ is continuously differentiable on $\mathbb{R}^n$, and there is some constant $\alpha > 0$ such that*

$$\langle \nabla J_v - \nabla J_u, v - u\rangle \geq \alpha \left\| v - u \right\|^2 \quad \text{for all } u, v \in \mathbb{R}^n,$$

*and that $U$ is a nonempty closed convex subset given by*

$$U = \{v \in \mathbb{R}^n \mid Cv \leq d\},$$

*where $C$ is a real $m \times n$ matrix and $d \in \mathbb{R}^m$. If the scalar $\rho$ satisfies the condition*

$$0 < \rho < \frac{2\alpha}{\left\|C\right\|_2^2},$$

*where $\left\|C\right\|_2$ is the spectral norm of $C$, then the sequence $(u^k)_{k\geq 0}$ computed by Uzawa's method converges to the unique minimizer $u \in U$ of $J$.*

*Furthermore, if $C$ has rank $m$, then the sequence $(\lambda^k)_{k\geq 0}$ converges to the unique maximizer of the Dual Problem $(D)$.*

*Proof.*

*Step 1*. We establish algebraic conditions relating the unique minimizer $u \in U$ of $J$ over $U$ and some $\lambda \in \mathbb{R}^m_+$ such that $(u, \lambda)$ is a saddle point.

Since $J$ is elliptic and $U$ is nonempty closed and convex, by Theorem 49.8, the functional $J$ is strictly convex, so it has a unique minimizer $u \in U$. Since $J$ is convex and the constraints are affine, by Theorem 50.17(2) the Dual Problem $(D)$ has at least one solution. By Theorem 50.15(2), there is some $\lambda \in \mathbb{R}^m_+$ such that $(u, \lambda)$ is a saddle point of the Lagrangian $L$.

If we define the affine function $\varphi$ by

$$\varphi(v) = (\varphi_1(v), \ldots, \varphi_m(v)) = Cv - d,$$

then the Lagrangian $L(v, \mu)$ can be written as

$$L(v, \mu) = J(v) + \sum_{i=1}^m \mu_i \varphi_i(v) = J(v) + \langle C^\top \mu, v\rangle - \langle \mu, d\rangle.$$

Since

$$L(u, \lambda) = \inf_{v\in\mathbb{R}^n} L(v, \lambda),$$

by Theorem 40.13(4) we must have

$$\nabla J_u + C^\top \lambda = 0, \tag{$*_1$}$$