The following proposition show that the Frobenius norm is a matrix norm satisfying other nice properties.

Proposition 9.7. *The Frobenius norm $\| \, \|_F$ on $\mathrm{M}_n(\mathbb{C})$ satisfies the following properties:*

(1) It is a matrix norm; that is, $\|AB\|_F \leq \|A\|_F \|B\|_F$, for all $A, B \in \mathrm{M}_n(\mathbb{C})$.

(2) It is unitarily invariant, which means that for all unitary matrices U, V, we have

$$\|A\|_F = \|UA\|_F = \|AV\|_F = \|UAV\|_F .$$

(3) $\sqrt{\rho(A^*A)} \leq \|A\|_F \leq \sqrt{n}\sqrt{\rho(A^*A)}$, for all $A \in \mathrm{M}_n(\mathbb{C})$.

Proof. (1) The only property that requires a proof is the fact $\|AB\|_F \leq \|A\|_F \|B\|_F$. This follows from the Cauchy–Schwarz inequality:

$$\begin{aligned}
\|AB\|_F^2 &= \sum_{i,j=1}^n \left| \sum_{k=1}^n a_{ik} b_{kj} \right|^2 \\
&\leq \sum_{i,j=1}^n \left(\sum_{h=1}^n |a_{ih}|^2 \right) \left(\sum_{k=1}^n |b_{kj}|^2 \right) \\
&= \left(\sum_{i,h=1}^n |a_{ih}|^2 \right) \left(\sum_{k,j=1}^n |b_{kj}|^2 \right) = \|A\|_F^2 \|B\|_F^2 .
\end{aligned}$$

(2) We have

$$\|A\|_F^2 = \mathrm{tr}(AA^*) = \mathrm{tr}(AVV^*A^*) = \mathrm{tr}(AV(AV)^*) = \|AV\|_F^2 ,$$

and

$$\|A\|_F^2 = \mathrm{tr}(A^*A) = \mathrm{tr}(A^*U^*UA) = \|UA\|_F^2 .$$

The identity

$$\|A\|_F = \|UAV\|_F$$

follows from the previous two.

(3) It is shown in Section 15.1 that the trace of a matrix is equal to the sum of its eigenvalues. Furthermore, A^*A is symmetric positive semidefinite (which means that its eigenvalues are nonnegative), so $\rho(A^*A)$ is the largest eigenvalue of A^*A and

$$\rho(A^*A) \leq \mathrm{tr}(A^*A) \leq n\rho(A^*A),$$

which yields (3) by taking square roots. $\square$

Remark: The Frobenius norm is also known as the *Hilbert-Schmidt norm* or the *Schur norm*. So many famous names associated with such a simple thing!