**Theorem 39.15.** *(Inverse Function Theorem) Let $E$ and $F$ be complete normed affine spaces, let $A$ be an open subset of $E$, and let $f\colon A \to F$ be a $C^1$-function on $A$. The following properties hold:*

*(1) For every $a \in A$, if $\mathrm{D}f(a)$ is a linear isomorphism (which means that both $\mathrm{D}f(a)$ and $(\mathrm{D}f(a))^{-1}$ are linear and continuous),[2] then there exist some open subset $U \subseteq A$ containing $a$, and some open subset $V$ of $F$ containing $f(a)$, such that $f$ is a diffeomorphism from $U$ to $V = f(U)$. Furthermore,*

$$\mathrm{D}f^{-1}(f(a)) = (\mathrm{D}f(a))^{-1}.$$

*For every neighborhood $N$ of $a$, its image $f(N)$ is a neighborhood of $f(a)$, and for every open ball $U \subseteq A$ of center $a$, its image $f(U)$ contains some open ball of center $f(a)$.*

*(2) If $\mathrm{D}f(a)$ is invertible for every $a \in A$, then $B = f(A)$ is an open subset of $F$, and $f$ is a local diffeomorphism from $A$ to $B$. Furthermore, if $f$ is injective, then $f$ is a diffeomorphism from $A$ to $B$.*

Proofs of the inverse function theorem can be found in Schwartz [151], Lang [111], Abraham and Marsden [1], and Cartan [34].

The idea of Schwartz's proof is that if we define the function $f_1\colon F \times \Omega \to F$ by

$$f_1(y, z) = f(z) - y,$$

then an inverse $g = f^{-1}$ of $f$ is an implicit solution of the equation $f_1(y, z) = 0$, since $f_1(y, g(y)) = f(g(y)) - y = 0$. Observe that the roles of $E$ and $F$ are switched, but this is not a problem since $F$ is complete. The proof consists in checking that the conditions of Theorem 39.14 apply.

Part (1) of Theorem 39.15 is often referred to as the "(local) inverse function theorem." It plays an important role in the study of manifolds and (ordinary) differential equations.

If $E$ and $F$ are both of finite dimension, and some frames have been chosen, the invertibility of $\mathrm{D}f(a)$ is equivalent to the fact that the Jacobian determinant $\det(J(f)(a))$ is nonnull. The case where $\mathrm{D}f(a)$ is just injective or just surjective is also important for defining manifolds, using implicit definitions.

**Definition 39.9.** Let $E$ and $F$ be normed affine spaces, where $E$ and $F$ are of finite dimension (or both $E$ and $F$ are complete), and let $A$ be an open subset of $E$. For any $a \in A$, a $C^1$-function $f\colon A \to F$ is an *immersion at $a$* if $\mathrm{D}f(a)$ is injective. A $C^1$-function $f\colon A \to F$ is a *submersion at $a$* if $\mathrm{D}f(a)$ is surjective. A $C^1$-function $f\colon A \to F$ is an *immersion on $A$* (resp. *a submersion on $A$*) if $\mathrm{D}f(a)$ is injective (resp. surjective) for every $a \in A$.

[2] Actually, since $E$ and $F$ are Banach spaces, by the Open Mapping Theorem, it is sufficient to assume that $\mathrm{D}f(a)$ is continuous and bijective; see Lang [111].