Since the matrix $R^\top LR$ is symmetric, it has real eigenvalues. Actually, since L is positive semidefinite, so is $R^\top LR$. Then the trace of $R^\top LR$ is equal to the sum of its positive eigenvalues, and this is the energy $\mathcal{E}(R)$ of the graph drawing.

If R is the matrix of a graph drawing in $\mathbb{R}^n$, then for any $n \times n$ invertible matrix M, the map that assigns $\rho(v_i)M$ to v_i is another graph drawing of G, and these two drawings convey the same amount of information. From this point of view, *a graph drawing is determined by the column space of* R. Therefore, it is reasonable to assume that the columns of R are pairwise orthogonal and that they have unit length. Such a matrix satisfies the equation $R^\top R = I$.

Definition 21.3. If the matrix R of a graph drawing satisfies the equation $R^\top R = I$, then the corresponding drawing is called an *orthogonal graph drawing*.

This above condition also rules out trivial drawings. The following result tells us how to find minimum energy orthogonal balanced graph drawings, provided the graph is connected. Recall that

$$L\mathbf{1} = 0,$$

as we already observed.

Theorem 21.2. *Let $G = (V, W)$ be a weighted graph with $|V| = m$. If $L = D - W$ is the (unnormalized) Laplacian of G, and if the eigenvalues of L are $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_m$, then the minimal energy of any balanced orthogonal graph drawing of G in $\mathbb{R}^n$ is equal to $\lambda_2 + \cdots + \lambda_{n+1}$ (in particular, this implies that $n < m$). The $m \times n$ matrix R consisting of any unit eigenvectors $u_2, \ldots, u_{n+1}$ associated with $\lambda_2 \leq \ldots \leq \lambda_{n+1}$ yields a balanced orthogonal graph drawing of minimal energy; it satisfies the condition $R^\top R = I$.*

Proof. We present the proof given in Godsil and Royle [77] (Section 13.4, Theorem 13.4.1). The key point is that the sum of the n smallest eigenvalues of L is a lower bound for $\mathrm{tr}(R^\top LR)$. This can be shown using a Rayleigh ratio argument; see Proposition 17.25 (the Poincaré separation theorem). Then any n eigenvectors $(u_1, \ldots, u_n)$ associated with $\lambda_1, \ldots, \lambda_n$ achieve this bound. Because the first eigenvalue of L is $\lambda_1 = 0$ and because we are assuming that $\lambda_2 > 0$, we have $u_1 = \mathbf{1}/\sqrt{m}$. Since the u_j are pairwise orthogonal for $i = 2, \ldots, n$ and since u_i is orthogonal to $u_1 = \mathbf{1}/\sqrt{m}$, the entries in u_i add up to 0. Consequently, for any ℓ with $2 \leq \ell \leq n$, by deleting u_1 and using $(u_2, \ldots, u_\ell)$, we obtain a balanced orthogonal graph drawing in $\mathbb{R}^{\ell-1}$ with the same energy as the orthogonal graph drawing in $\mathbb{R}^\ell$ using $(u_1, u_2, \ldots, u_\ell)$. Conversely, from any balanced orthogonal drawing in $\mathbb{R}^{\ell-1}$ using $(u_2, \ldots, u_\ell)$, we obtain an orthogonal graph drawing in $\mathbb{R}^\ell$ using $(u_1, u_2, \ldots, u_\ell)$ with the same energy. Therefore, the minimum energy of a balanced orthogonal graph drawing in $\mathbb{R}^n$ is equal to the minimum energy of an orthogonal graph drawing in $\mathbb{R}^{n+1}$, and this minimum is $\lambda_2 + \cdots + \lambda_{n+1}$. ☐

Since $\mathbf{1}$ spans the nullspace of L, using u_1 (which belongs to $\mathrm{Ker}\, L$) as one of the vectors in R would have the effect that all points representing vertices of G would have the same