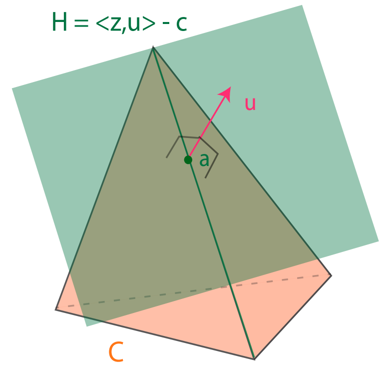

Figure 51.12: Let C be the solid peach tetrahedron in $\mathbb{R}^3$. The green plane H defined by $\varphi(z) = \langle z, u\rangle - c$ is a supporting hyperplane to C at a. The pink normal to H, namely the vector u, is also normal to C at a.

$(x, f(x))$. *In this case there is an affine form* φ *(over* $\mathbb{R}^n$*) such that* $f(y) \geq \varphi(y)$ *for all* $y \in \mathbb{R}^n$. *We can pick* φ *given by* $\varphi(y) = \langle y - x, u\rangle + f(x)$ *for all* $y \in \mathbb{R}^n$.

It is easy to see that $\partial f(x)$ is closed and convex. The set $\partial f(x)$ may be empty, or reduced to a single element. In $\partial f(x)$ consists of a single element it can be shown that f is finite near x, differentiable at x, and that $\partial f(x) = \{\nabla f_x\}$, the gradient of f at x.

Example 51.5. The ℓ^2 norm $f(x) = \|x\|_2$ is subdifferentiable for all $x \in \mathbb{R}^n$, in fact differentiable for all $x \neq 0$. For $x = 0$, the set $\partial f(0)$ consists of all $u \in \mathbb{R}^n$ such that

$$\|z\|_2 \geq \langle z, u\rangle \quad \text{for all } z \in \mathbb{R}^n,$$

namely (by Cauchy–Schwarz), the Euclidean unit ball $\{u \in \mathbb{R}^n \mid \|u\|_2 \leq 1\}$. See Figure 51.14.

Example 51.6. For the ℓ^∞ norm if $f(x) = \|x\|_\infty$, we leave it as an exercise to show that $\partial f(0)$ is the polyhedron

$$\partial f(0) = \mathrm{conv}\{\pm e_1, \ldots, \pm e_n\}.$$

See Figure 51.15. One can also work out what is $\partial f(x)$ if $x \neq 0$, but this is more complicated; see Rockafellar [138], page 215.

Example 51.7. The following function is an example of a proper convex function which is not subdifferentiable everywhere:

$$f(x) = \begin{cases} -(1 - |x|^2)^{1/2} & \text{if } |x| \leq 1 \\ +\infty & \text{otherwise.} \end{cases}$$