Remark: When considering a family $(a_i)_{i\in I}$, there is no reason to assume that I is ordered. The crucial point is that every element of the family is uniquely indexed by an element of I. Thus, unless specified otherwise, we do not assume that the elements of an index set are ordered.

If A is an abelian group with identity 0, we say that a family $(a_i)_{i\in I}$ has *finite support* if $a_i = 0$ for all $i \in I - J$, where J is a finite subset of I (the support of the family).

Given two disjoint sets I and J, the union of two families $(u_i)_{i\in I}$ and $(v_j)_{j\in J}$, denoted as $(u_i)_{i\in I} \cup (v_j)_{j\in J}$, is the family $(w_k)_{k\in(I\cup J)}$ defined such that $w_k = u_k$ if $k \in I$, and $w_k = v_k$ if $k \in J$. Given a family $(u_i)_{i\in I}$ and any element v, we denote by $(u_i)_{i\in I} \cup_k (v)$ the family $(w_i)_{i\in I\cup\{k\}}$ defined such that, $w_i = u_i$ if $i \in I$, and $w_k = v$, where k is any index such that $k \notin I$. Given a family $(u_i)_{i\in I}$, a *subfamily* of $(u_i)_{i\in I}$ is a family $(u_j)_{j\in J}$ where J is any subset of I.

In this chapter, unless specified otherwise, *is assumed that all families of scalars have finite support*.

Definition 3.3. Let E be a vector space. A vector $v \in E$ is a *linear combination of a family* $(u_i)_{i\in I}$ *of elements of* E iff there is a family $(\lambda_i)_{i\in I}$ of scalars in K such that

$$v = \sum_{i\in I} \lambda_i u_i.$$

When $I = \emptyset$, we stipulate that $v = 0$. (By Proposition 3.3, sums of the form $\sum_{i\in I} \lambda_i u_i$ are well defined.) We say that a family $(u_i)_{i\in I}$ is *linearly independent* iff for every family $(\lambda_i)_{i\in I}$ of scalars in K,

$$\sum_{i\in I} \lambda_i u_i = 0 \quad \text{implies that} \quad \lambda_i = 0 \ \text{for all } i \in I.$$

Equivalently, a family $(u_i)_{i\in I}$ is *linearly dependent* iff there is some family $(\lambda_i)_{i\in I}$ of scalars in K such that

$$\sum_{i\in I} \lambda_i u_i = 0 \quad \text{and} \quad \lambda_j \neq 0 \ \text{for some } j \in I.$$

We agree that when $I = \emptyset$, the family $\emptyset$ is linearly independent.

Observe that defining linear combinations for families of vectors rather than for sets of vectors has the advantage that *the vectors being combined need not be distinct*. For example, for $I = \{1, 2, 3\}$ and the families (u, v, u) and $(\lambda_1, \lambda_2, \lambda_1)$, the linear combination

$$\sum_{i\in I} \lambda_i u_i = \lambda_1 u + \lambda_2 v + \lambda_1 u$$

makes sense. Using sets of vectors in the definition of a linear combination does not allow such linear combinations; this is too restrictive.