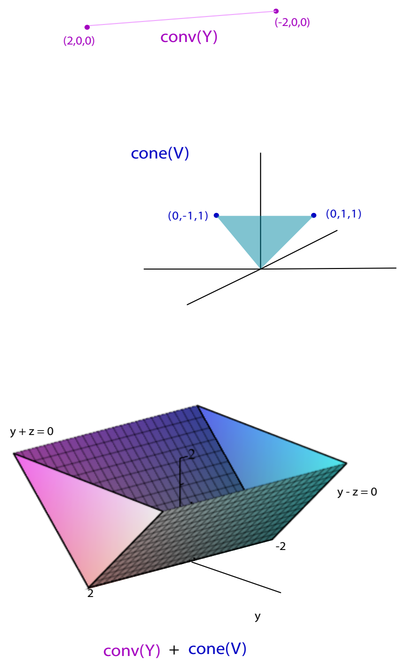

Figure 44.6: The "triangular trough" determined by the inequalities $y - z \leq 0$, $y + z \geq 0$, and $-2 \leq x \leq 2$ is an $\mathcal{H}$-polyhedron and an $\mathcal{V}$-polyhedron, where $Y = \{(2, 0, 0), (-2, 0, 0)\}$ and $V = \{(0, 1, 1), (0, -1, 1)\}$.

Since $\lambda_i^{(k)} \geq 0$ for $i = 1, \ldots, m$ and all $k \geq 0$, we have $x_i \geq 0$ for $i = 1, \ldots, m$, so $x \in \mathrm{cone}(\{a_1, \ldots, a_m\})$.

Next, assume that x belongs to the polyhedral cone C. Consider a positive combination

$$x = \lambda_1 a_1 + \cdots + \lambda_k a_k, \qquad (*_1)$$

for some nonzero $a_1, \ldots, a_k \in C$, with $\lambda_i \geq 0$ and with k *minimal*. Since k is minimal, we must have $\lambda_i > 0$ for $i = 1, \ldots, k$. We claim that $(a_1, \ldots, a_k)$ are linearly independent.

If not, there is some nontrivial linear combination

$$\mu_1 a_1 + \cdots + \mu_k a_k = 0, \qquad (*_2)$$

and since the a_i are nonzero, $\mu_j \neq 0$ for some at least some j. We may assume that $\mu_j < 0$ for some j (otherwise, we consider the family $(-\mu_i)_{1 \leq i \leq k}$), so let

$$J = \{j \in \{1, \ldots, k\} \mid \mu_j < 0\}.$$

For any $t \in \mathbb{R}$, since $x = \lambda_1 a_1 + \cdots + \lambda_k a_k$, using $(*_2)$ we get

$$x = (\lambda_1 + t\mu_1)a_1 + \cdots + (\lambda_k + t\mu_k)a_k, \qquad (*_3)$$