where $C$ is a $(p+q) \times (n+1)$ matrix $C$ and $d \in \mathbb{R}^{p+q}$ is the vector given by

$$C = \begin{pmatrix} -u_1^\top & 1 \\ \vdots & \vdots \\ -u_p^\top & 1 \\ v_1^\top & -1 \\ \vdots & \vdots \\ v_q^\top & -1 \end{pmatrix}, \quad d = \begin{pmatrix} -1 \\ \vdots \\ -1 \end{pmatrix} = -\mathbf{1}_{p+q}.$$

If we let $X$ be the $n \times (p+q)$ matrix given by

$$X = \begin{pmatrix} -u_1 & \cdots & -u_p & v_1 & \cdots & v_q \end{pmatrix},$$

then

$$C = \begin{pmatrix} X^\top & \begin{matrix} \mathbf{1}_p \\ -\mathbf{1}_q \end{matrix} \end{pmatrix}$$

and so

$$C^\top = \begin{pmatrix} & X & \\ \mathbf{1}_p^\top & & -\mathbf{1}_q^\top \end{pmatrix}.$$

**Step 2:** Write the objective function in matrix form.

The objective function is given by

$$J(w, b) = \frac{1}{2} \begin{pmatrix} w^\top & b \end{pmatrix} \begin{pmatrix} I_n & 0_n \\ 0_n^\top & 0 \end{pmatrix} \begin{pmatrix} w \\ b \end{pmatrix}.$$

Note that the corresponding matrix is symmetric positive semidefinite, but it is *not* invertible. Thus the function $J$ is convex but not strictly convex.

**Step 3:** Write the Lagrangian in matrix form.

As in Example 50.7, we obtain the Lagrangian

$$L(w, b, \lambda, \mu) = \frac{1}{2} \begin{pmatrix} w^\top & b \end{pmatrix} \begin{pmatrix} I_n & 0_n \\ 0_n^\top & 0 \end{pmatrix} \begin{pmatrix} w \\ b \end{pmatrix} - \begin{pmatrix} w^\top & b \end{pmatrix} \left( 0_{n+1} - C^\top \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \right) + \begin{pmatrix} \lambda^\top & \mu^\top \end{pmatrix} \mathbf{1}_{p+q},$$

that is,

$$L(w, b, \lambda, \mu) = \frac{1}{2} \begin{pmatrix} w^\top & b \end{pmatrix} \begin{pmatrix} I_n & 0_n \\ 0_n^\top & 0 \end{pmatrix} \begin{pmatrix} w \\ b \end{pmatrix} + \begin{pmatrix} w^\top & b \end{pmatrix} \begin{pmatrix} X \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \\ \mathbf{1}_p^\top \lambda - \mathbf{1}_q^\top \mu \end{pmatrix} + \begin{pmatrix} \lambda^\top & \mu^\top \end{pmatrix} \mathbf{1}_{p+q}.$$

**Step 4:** Find the dual function $G(\lambda, \mu)$.

In order to find the dual function $G(\lambda, \mu)$, we need to minimize $L(w, b, \lambda, \mu)$ with respect to $w$ and $b$ and for this, since the objective function $J$ is convex and since $\mathbb{R}^{n+1}$ is convex