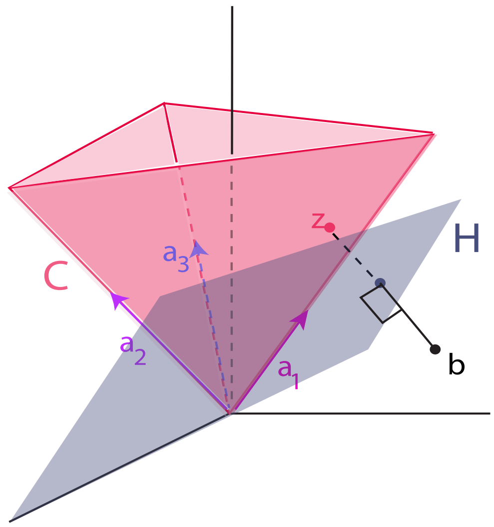

Figure 47.2: The hyperplane H, perpendicular to $z - b$, separates the point b from $C = \text{cone}(\{a_1, a_2, a_3\})$.

Proof of the Farkas–Minkowski proposition. Let $C = \text{cone}(\{a_1, \ldots, a_m\})$ be a polyhedral cone (nonempty) and assume that $b \notin C$. By Proposition 44.2, the polyhedral cone is closed, and by Proposition 47.5 there is some $z \in C$ such that $d(b, C) = \|b - z\|$; that is, z is a point of C closest to b. Since $b \notin C$ and $z \in C$ we have $u = z - b \neq 0$, and we claim that the linear hyperplane H orthogonal to u does the job, as illustrated in Figure 47.2.

First let us show that

$$\langle u, z\rangle = \langle z - b, z\rangle = 0. \tag{$*_1$}$$

This is trivial if $z = 0$, so assume $z \neq 0$. If $\langle u, z\rangle \neq 0$, then either $\langle u, z\rangle > 0$ or $\langle u, z\rangle < 0$. In either case we show that we can find some point $z' \in C$ closer to b than z is, a contradiction.

Case 1: $\langle u, z\rangle > 0$.

Let $z' = (1 - \alpha)z$ for any α such that $0 < \alpha < 1$. Then $z' \in C$ and since $u = z - b$,

$$z' - b = (1 - \alpha)z - (z - u) = u - \alpha z,$$

so

$$\|z' - b\|^2 = \|u - \alpha z\|^2 = \|u\|^2 - 2\alpha\langle u, z\rangle + \alpha^2 \|z\|^2 .$$

If we pick $\alpha > 0$ such that $\alpha < 2\langle u, z\rangle / \|z\|^2$, then $-2\alpha\langle u, z\rangle + \alpha^2 \|z\|^2 < 0$, so $\|z' - b\|^2 < \|u\|^2 = \|z - b\|^2$, contradicting the fact that z is a point of C closest to b.

Case 2: $\langle u, z\rangle < 0$.

Let $z' = (1 + \alpha)z$ for any α such that $\alpha \geq -1$. Then $z' \in C$ and since $u = z - b$, we have $z' - b = (1 + \alpha)z - (z - u) = u + \alpha z$ so

$$\|z' - b\|^2 = \|u + \alpha z\|^2 = \|u\|^2 + 2\alpha\langle u, z\rangle + \alpha^2 \|z\|^2 ,$$