provided that $\beta \neq 1$, i.e., $b \neq c$. When $b = c$, we agree that $\mathrm{ratio}(a, b, c) = \infty$. We warn our readers that other authors define the ratio of a, b, c as $-\mathrm{ratio}(a, b, c) = \frac{\overrightarrow{ba}}{\overrightarrow{bc}}$. Since affine maps preserve barycenters, it is clear that affine maps preserve the ratio of three points.

24.8 Affine Groups

We now take a quick look at the bijective affine maps. Given an affine space E, the set of affine bijections $f\colon E \to E$ is clearly a group, called the *affine group of* E, and denoted by $\mathbf{GA}(E)$. Recall that the group of bijective linear maps of the vector space $\overrightarrow{E}$ is denoted by $\mathbf{GL}(\overrightarrow{E})$. Then, the map $f \mapsto \overrightarrow{f}$ defines a group homomorphism $L\colon \mathbf{GA}(E) \to \mathbf{GL}(\overrightarrow{E})$. The kernel of this map is the set of translations on E.

The subset of all linear maps of the form $\lambda\,\mathrm{id}_{\overrightarrow{E}}$, where $\lambda \in \mathbb{R} - \{0\}$, is a subgroup of $\mathbf{GL}(\overrightarrow{E})$, and is denoted by $\mathbb{R}^*\mathrm{id}_{\overrightarrow{E}}$ (where $\lambda\,\mathrm{id}_{\overrightarrow{E}}(u) = \lambda u$, and $\mathbb{R}^* = \mathbb{R} - \{0\}$). The subgroup $\mathbf{DIL}(E) = L^{-1}(\mathbb{R}^*\mathrm{id}_{\overrightarrow{E}})$ of $\mathbf{GA}(E)$ is particularly interesting. It turns out that it is the disjoint union of the translations and of the dilatations of ratio $\lambda \neq 1$. The elements of $\mathbf{DIL}(E)$ are called *affine dilatations*.

Given any point $a \in E$, and any scalar $\lambda \in \mathbb{R}$, a *dilatation or central dilatation (or homothety) of center* a *and ratio* λ is a map $H_{a,\lambda}$ defined such that

$$H_{a,\lambda}(x) = a + \lambda\overrightarrow{ax},$$

for every $x \in E$.

Remark: The terminology does not seem to be universally agreed upon. The terms *affine dilatation* and *central dilatation* are used by Pedoe [136]. Snapper and Troyer use the term *dilation* for an affine dilatation and *magnification* for a central dilatation [162]. Samuel uses *homothety* for a central dilatation, a direct translation of the French "homothétie" [142]. Since dilation is shorter than dilatation and somewhat easier to pronounce, perhaps we should use that!

Observe that $H_{a,\lambda}(a) = a$, and when $\lambda \neq 0$ and $x \neq a$, $H_{a,\lambda}(x)$ is on the line defined by a and x, and is obtained by "scaling" $\overrightarrow{ax}$ by λ.

Figure 24.20 shows the effect of a central dilatation of center d. The triangle (a, b, c) is magnified to the triangle (a', b', c'). Note how every line is mapped to a parallel line.

When $\lambda = 1$, $H_{a,1}$ is the identity. Note that $\overrightarrow{H_{a,\lambda}} = \lambda\,\mathrm{id}_{\overrightarrow{E}}$. When $\lambda \neq 0$, it is clear that $H_{a,\lambda}$ is an affine bijection. It is immediately verified that

$$H_{a,\lambda} \circ H_{a,\mu} = H_{a,\lambda\mu}.$$

We have the following useful result.