Else compute

$$\theta^+ = \min\left\{-\frac{yA^j - c_j}{z^*A^j} \;\middle|\; j \notin J,\ z^*A^j < 0\right\}, \qquad y^+ = y + \theta^+ z^*,$$

and

$$J^+ = \{j \in \{1, \ldots, n\} \mid y^+ A^j = c_j\}.$$

Go back to Step 3.

The following proposition shows that at each iteration we can start the Program (RP) with the optimal solution obtained at the previous iteration.

Proposition 47.14. *Every $j \in J$ such that A^j is in the basis of the optimal solution ξ^* belongs to the next index set J^+.*

Proof. Such an index $j \in J$ correspond to a variable ξ_j such that $\xi_j > 0$, so by complementary slackness, the constraint $z^*A^j \geq 0$ of the Dual Program (DRP) must be an equality, that is, $z^*A^j = 0$. But then we have

$$y^+A^j = yA^j + \theta^+ z^* A^j = c_j,$$

which shows that $j \in J^+$. $\square$

If (u^*, ξ^*) with the basis K^* is the optimal solution of the Program (RP), Proposition 47.14 together with the last property of Theorem 47.12 allows us to restart the (RP) in Step 3 with $(u^*, \xi^*)_{K^*}$ as initial solution (with basis K^*). For every $j \in J - J^+$, column j is deleted, and for every $j \in J^+ - J$, the new column A^j is computed by multiplying $\widehat{A}_{K^*}^{-1}$ and A^j, but $\widehat{A}_{K^*}^{-1}$ is the matrix $\Gamma^*[1{:}m; p+1{:}p+m]$ consisting of the last m columns of Γ^* in the final tableau, and the new reduced $\overline{c}_j$ is given by $c_j - z^*A^j$. Reusing the optimal solution of the previous (RP) may improve efficiency significantly.

Another crucial observation is that for any index $j_0 \in N$ such that $\theta^+ = (yA^{j_0} - c_{j_0})/(-z^*A^{j_0})$, we have

$$y^+A_{j_0} = yA_{j_0} + \theta^+ z^* A^{j_0} = c_{j_0},$$

and so $j_0 \in J^+$. This fact that be used to ensure that the primal-dual algorithm terminates in a finite number of steps (using a pivot rule that prevents cycling); see Papadimitriou and Steiglitz [134] (Theorem 5.4).

It remains to discuss how to pick some initial feasible solution y of the Dual Program (D). If $c_j \leq 0$ for $j = 1, \ldots, n$, then we can pick $y = 0$. If we are dealing with a minimization problem, the weight c_j are often nonnegative, so from the point of view of maximization we will have $-c_j \leq 0$ for all j, and we will be able to use $y = 0$ as a starting point.

Going back to our primal problem in maximization form and its dual in minimization form, we still need to deal with the situation where $c_j > 0$ for some j, in which case there