Definition 14.12. Let E be a Hermitian space of finite dimension. For any hyperplane H, for any nonnull vector w orthogonal to H, so that $E = H \oplus G$, where $G = \mathbb{C}w$, a *Hermitian reflection about H of angle θ* is a linear map of the form $\rho_{H,\theta} \colon E \to E$, defined such that

$$\rho_{H,\theta}(u) = p_H(u) + e^{i\theta} p_G(u),$$

for any unit complex number $e^{i\theta} \neq 1$ (i.e. $\theta \neq k2\pi$). For any nonzero vector $w \in E$, we denote by $\rho_{w,\theta}$ the Hermitian reflection given by $\rho_{H,\theta}$, where H is the hyperplane orthogonal to w.

Since $u = p_H(u) + p_G(u)$, the Hermitian reflection $\rho_{w,\theta}$ is also expressed as

$$\rho_{w,\theta}(u) = u + (e^{i\theta} - 1) p_G(u),$$

or as

$$\rho_{w,\theta}(u) = u + (e^{i\theta} - 1)\, \frac{(u \cdot w)}{\|w\|^2}\, w.$$

Note that the case of a standard hyperplane reflection is obtained when $e^{i\theta} = -1$, i.e., $\theta = \pi$. In this case,

$$\rho_{w,\pi}(u) = u - 2\, \frac{(u \cdot w)}{\|w\|^2}\, w,$$

and the matrix of such a reflection is a Householder matrix, as in Section 13.1, except that w may be a complex vector.

We leave as an easy exercise to check that $\rho_{w,\theta}$ is indeed an isometry, and that the inverse of $\rho_{w,\theta}$ is $\rho_{w,-\theta}$. If we pick an orthonormal basis $(e_1, \ldots, e_n)$ such that $(e_1, \ldots, e_{n-1})$ is an orthonormal basis of H, the matrix of $\rho_{w,\theta}$ is

$$\begin{pmatrix} I_{n-1} & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

We now come to the main surprise. Given any two distinct vectors u and v such that $\|u\| = \|v\|$, there isn't always a hyperplane reflection mapping u to v, but this can be done using two Hermitian reflections!

Proposition 14.19. *Let E be any nontrivial Hermitian space.*

(1) For any two vectors $u, v \in E$ such that $u \neq v$ and $\|u\| = \|v\|$, if $u \cdot v = e^{i\theta}|u \cdot v|$, then the (usual) reflection s about the hyperplane orthogonal to the vector $v - e^{-i\theta}u$ is such that $s(u) = e^{i\theta}v$.

(2) For any nonnull vector $v \in E$, for any unit complex number $e^{i\theta} \neq 1$, there is a Hermitian reflection $\rho_{v,\theta}$ such that

$$\rho_{v,\theta}(v) = e^{i\theta} v.$$

As a consequence, for u and v as in (1), we have $\rho_{v,-\theta} \circ s(u) = v$.