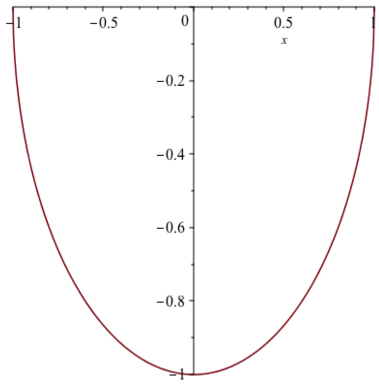

Figure 51.16: The graph of the function in Example 51.7.

assuming that the affine hull of $\mathbf{epi}(f)$ has dimension $m+1$. See Figure (1) of Figure 51.17. The inclusion $\subseteq$ is obvious, so we only need to prove the reverse inclusion. Then for any $z \in \mathrm{int}(\mathrm{dom}(f))$, we can find a convex polyhedral subset $P = \mathrm{conv}(a_1, \ldots, a_{m+1})$ with $a_1, \ldots, a_{m+1} \in \mathrm{dom}(f)$ such that $z \in \mathrm{int}(P)$. Let

$$\alpha = \max\{f(a_1), \ldots, f(a_{m+1})\}.$$

Since any $x \in P$ can be expressed as

$$x = \lambda_1 a_1 + \cdots + \lambda_{m+1} a_{m+1}, \quad \lambda_1 + \cdots + \lambda_{m+1} = 1,\ \lambda_i \geq 0,$$

and since f is convex we have

$$f(x) \leq \lambda_1 f(a_1) + \cdots + \lambda_{m+1} f(a_{m+1}) \leq (\lambda_1 + \cdots + \lambda_{m+1})\alpha = \alpha \quad \text{for all } x \in P.$$

The above shows that the open subset

$$\{(x, \mu) \in \mathbb{R}^{m+1} \mid x \in \mathrm{int}(P),\ \alpha < \mu\}$$

is contained in $\mathbf{epi}(f)$. See Figure (2) of Figure 51.17. In particular, for every $\mu > \alpha$, we have

$$(z, \mu) \in \mathrm{int}(\mathbf{epi}(f)).$$

Thus for any $\beta \in \mathbb{R}$ such that $\beta > f(z)$, we see that (z, β) belongs to the relative interior of the vertical line segment $\{(z, \mu) \in \mathbb{R}^{m+1} \mid f(z) \leq \mu \leq \alpha + \beta + 1\}$ which meets $\mathrm{int}(\mathbf{epi}(f))$. See Figure (3) of Figure 51.17. By Proposition 51.12, $(z, \beta) \in \mathrm{int}(\mathbf{epi}(f))$. ☐

We can now prove the following important theorem.

Theorem 51.14. *Let f be a proper convex function on $\mathbb{R}^n$. For any $x \in \mathbf{relint}(\mathrm{dom}(f))$, there is a nonvertical supporting hyperplane $\mathcal{H}$ to $\mathbf{epi}(f)$ at $(x, f(x))$. Consequently f is subdiffentiable for all $x \in \mathbf{relint}(\mathrm{dom}(f))$, and there is some affine form $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ such that $f(y) \geq \varphi(y)$ for all $y \in \mathbb{R}^n$.*