(7) There is a polynomial P (with complex coefficients) such that $f^* = P(f)$.

Problem 14.9. Recall from Problem 13.7 that a complex $n \times n$ matrix H is *upper Hessenberg* if $h_{jk} = 0$ for all (j, k) such that $j - k \geq 0$. Adapt the proof of Problem 13.7 to prove that given any complex $n \times n$-matrix A, there are $n - 2 \geq 1$ complex matrices $H_1, \ldots, H_{n-2}$, Householder matrices or the identity, such that

$$B = H_{n-2} \cdots H_1 A H_1 \cdots H_{n-2}$$

is upper Hessenberg.

Problem 14.10. Prove that all $y \in \mathbb{C}^n$,

$$\begin{aligned} \|y\|_1^D &= \|y\|_\infty \\ \|y\|_\infty^D &= \|y\|_1 \\ \|y\|_2^D &= \|y\|_2 \,. \end{aligned}$$

Problem 14.11. The purpose of this problem is to complete each of the matrices A_0, B_0, C_0 of Section 14.7 to a matrix A in such way that the nuclear norm $\|A\|_N$ is minimized.

(1) Prove that the squares σ_1^2 and σ_2^2 of the singular values of

$$A = \begin{pmatrix} 1 & 2 \\ c & d \end{pmatrix}$$

are the zeros of the equation

$$\lambda^2 - (5 + c^2 + d^2)\lambda + (2c - d)^2 = 0.$$

(2) Using the fact that

$$\|A\|_N = \sigma_1 + \sigma_2 = \sqrt{\sigma_1^2 + \sigma_2^2 + 2\sigma_1\sigma_2},$$

prove that

$$\|A\|_N^2 = 5 + c^2 + d^2 + 2|2c - d|.$$

Consider the cases where $2c - d \geq 0$ and $2c - d \leq 0$, and show that in both cases we must have $c = -2d$, and that the minimum of $f(c, d) = 5 + c^2 + d^2 + 2|2c - d|$ is achieved by $c = d = 0$. Conclude that the matrix A completing A_0 that minimizes $\|A\|_N$ is

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}.$$

(3) Prove that the squares σ_1^2 and σ_2^2 of the singular values of

$$A = \begin{pmatrix} 1 & b \\ c & 4 \end{pmatrix}$$