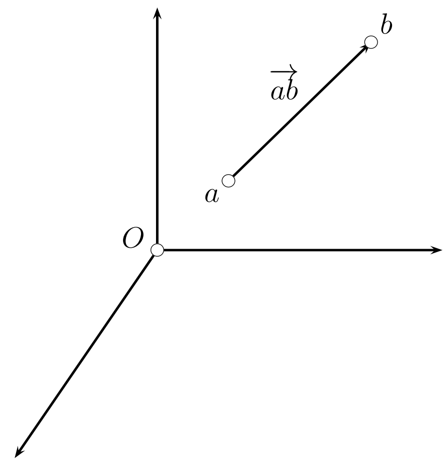

Figure 24.1: Points and free vectors.

addition being understood as addition in $\mathbb{R}^3$. Then, in the standard frame, given a point $x = (x_1, x_2, x_3)$, the position of x is the vector $\overrightarrow{Ox} = (x_1, x_2, x_3)$, which coincides with the point itself. In the standard frame, points and vectors are identified. Points and free vectors are illustrated in Figure 24.1.

What if we pick a frame with a different origin, say $\Omega = (\omega_1, \omega_2, \omega_3)$, but the same basis vectors (e_1, e_2, e_3)? This time, the point $x = (x_1, x_2, x_3)$ is defined by two position vectors:

$$\overrightarrow{Ox} = (x_1, x_2, x_3)$$

in the frame $(O, (e_1, e_2, e_3))$ and

$$\overrightarrow{\Omega x} = (x_1 - \omega_1, x_2 - \omega_2, x_3 - \omega_3)$$

in the frame $(\Omega, (e_1, e_2, e_3))$. See Figure 24.2.

This is because

$$\overrightarrow{Ox} = \overrightarrow{O\Omega} + \overrightarrow{\Omega x} \quad \text{and} \quad \overrightarrow{O\Omega} = (\omega_1, \omega_2, \omega_3).$$

We note that in the second frame $(\Omega, (e_1, e_2, e_3))$, points and position vectors are no longer identified. This gives us evidence that points are not vectors. It may be computationally convenient to deal with points using position vectors, but such a treatment is not frame invariant, which has undesirable effects.

Inspired by physics, we deem it important to define points and properties of points that are frame invariant. An undesirable side effect of the present approach shows up if we attempt to define linear combinations of points. First, let us review the notion of linear combination of vectors. Given two vectors u and v of coordinates (u_1, u_2, u_3) and (v_1, v_2, v_3) with respect