(b) *We have a direct sum* $E = S \overset{\perp}{\oplus} (U \oplus W) \overset{\perp}{\oplus} D$.

(c) *The subspace* D *contains no nonzero isotropic vector* (D *is anisotropic*).

(d) *The subspace* W *is totally isotropic.*

Furthermore, U_1 *and* U_2 *are both finite dimensional, and we have* $\dim(U_1) = \dim(U_2)$, $\dim(W) = \dim(U)$, $\dim(S_1) = \dim(S_2)$, *and* $\operatorname{codim}(D) = 2\dim(F_1)$.

Proof. First observe that if X is a totally isotropic maximal subspace of E, then any isotropic vector $x \in E$ orthogonal to X must belong to X, since otherwise, $X + Kx$ would be a totally isotropic subspace strictly containing X, contradicting the maximality of X. As a consequence, if x_i is any isotropic vector such that $x_i \in U_i^{\perp}$ (for $i = 1, 2$), then $x_i \in U_i$.

We claim that

$$S_1 \cap S_2^{\perp} = (0) \quad \text{and} \quad S_2 \cap S_1^{\perp} = (0).$$

Assume that $y \in S_1$ is orthogonal to S_2. Since $U_1 = U \oplus S_1$ and U_1 is totally isotropic, y is orthogonal to U_1, and thus orthogonal to U, so that y is orthogonal to $U_2 = U \oplus S_2$. Since $S_1 \subseteq U_1$ and U_1 is totally isotropic, y is an isotropic vector orthogonal to U_2, which by a previous remark implies that $y \in U_2$. Then, since $S_1 \subseteq U_1$ and $U \oplus S_1$ is a direct sum, we have

$$y \in S_1 \cap U_2 = S_1 \cap U_1 \cap U_2 = S_1 \cap U = (0).$$

Therefore $S_1 \cap S_2^{\perp} = (0)$. A similar proof show that $S_2 \cap S_1^{\perp} = (0)$. If U_1 is finite-dimensional (the case where U_2 is finite-dimensional is similar), then S_1 is finite-dimensional, so by Proposition 29.13, $S_1^{\perp}$ has finite codimension. Since $S_2 \cap S_1^{\perp} = (0)$, and since any supplement of $S_1^{\perp}$ has finite dimension, we must have

$$\dim(S_2) \leq \operatorname{codim}(S_1^{\perp}) = \dim(S_1).$$

By a similar argument, $\dim(S_1) \leq \dim(S_2)$, so we have

$$\dim(S_1) = \dim(S_2).$$

By Proposition 29.29(1), we conclude that $S = S_1 + S_2$ is nondegenerate.

By Proposition 29.21, the subspace $N = S^{\perp} = (S_1 + S_2)^{\perp}$ is nondegenerate. Since $U_1 = U \oplus S_1$, $U_2 = U \oplus S_2$, and U_1, U_2 are totally isotropic, U is orthogonal to S_1 and to S_2, so $U \subseteq N$. Since U is totally isotropic, by Proposition 29.30 applied to N, there is a totally isotropic subspace W of N such that $\dim(W) = \dim(U)$, $U \cap W = (0)$, and $U + W$ is nondegenerate. Consequently, (d) is satisfied by W.

To satisfy (a) and (b), we pick D to be the orthogonal of $U \oplus W$ in N. Then, $N = (U \oplus W) \overset{\perp}{\oplus} D$ and $E = S \overset{\perp}{\oplus} N$, so $E = S \overset{\perp}{\oplus} (U \oplus W) \overset{\perp}{\oplus} D$.

As to (c), since D is orthogonal $U \oplus W$, D is orthogonal to U, and since $D \subseteq N$ and N is orthogonal to $S_1 + S_2$, D is orthogonal to S_1, so D is orthogonal to $U_1 = U \oplus S_1$. If $y \in D$