Householder matrices are symmetric and orthogonal. It is easily checked that over an orthonormal basis $(e_1, \ldots, e_n)$, a hyperplane reflection about a hyperplane H orthogonal to a nonzero vector w is represented by the matrix

$$H = I_n - 2\,\frac{WW^\top}{\|W\|^2},$$

where W is the column vector of the coordinates of w over the basis $(e_1, \ldots, e_n)$. Since

$$p_G(u) = \frac{(u \cdot w)}{\|w\|^2}\, w,$$

the matrix representing p_G is

$$\frac{WW^\top}{W^\top W},$$

and since $p_H + p_G = \mathrm{id}$, the matrix representing p_H is

$$I_n - \frac{WW^\top}{W^\top W}.$$

These formulae can be used to derive a formula for a rotation of $\mathbb{R}^3$, given the direction w of its axis of rotation and given the angle θ of rotation.

The following fact is the key to the proof that every isometry can be decomposed as a product of reflections.

Proposition 13.2. *Let E be any nontrivial Euclidean space. For any two vectors $u, v \in E$, if $\|u\| = \|v\|$, then there is a hyperplane H such that the reflection s about H maps u to v, and if $u \neq v$, then this reflection is unique. See Figure 13.3.*

Proof. If $u = v$, then any hyperplane containing u does the job. Otherwise, we must have $H = \{v - u\}^\perp$, and by the above formula,

$$s(u) = u - 2\,\frac{(u \cdot (v-u))}{\|(v-u)\|^2}\,(v-u) = u + \frac{2\|u\|^2 - 2u \cdot v}{\|(v-u)\|^2}\,(v-u),$$

and since

$$\|(v-u)\|^2 = \|u\|^2 + \|v\|^2 - 2u \cdot v$$

and $\|u\| = \|v\|$, we have

$$\|(v-u)\|^2 = 2\|u\|^2 - 2u \cdot v,$$

and thus, $s(u) = v$. $\square$

If E is a complex vector space and the inner product is Hermitian, Proposition 13.2 is false. The problem is that the vector $v - u$ does not work unless the inner product $u \cdot v$ is real! The proposition can be salvaged enough to yield the QR-decomposition in terms of Householder transformations; see Section 14.5.

We now show that hyperplane reflections can be used to obtain another proof of the QR-decomposition.