For example, if $d \geq 2$ is square-free, then the map $c\colon \mathbb{Q}(\sqrt{d}) \to \mathbb{Q}(\sqrt{d})$ given by

$$c(a + b\sqrt{d}) = a - b\sqrt{d}$$

is an automorphism of $\mathbb{Q}(\sqrt{d})$, and $\mathrm{Fix}(c) = \mathbb{Q}$.

If K is a field, we have the ring homomorphism $h\colon \mathbb{Z} \to K$ given by $h(n) = n \cdot 1$. If h is injective, then K contains a copy of $\mathbb{Z}$, and since it is a field, it contains a copy of $\mathbb{Q}$. In this case, we say that K has *characteristic* 0. If h is not injective, then $h(\mathbb{Z})$ is a subring of K, and thus an integral domain, the kernel of h is a subgroup of $\mathbb{Z}$, which by Proposition 2.15 must be of the form $p\mathbb{Z}$ for some $p \geq 1$. By the first isomorphism theorem, $h(\mathbb{Z})$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ for some $p \geq 1$. But then, p must be prime since $\mathbb{Z}/p\mathbb{Z}$ is an integral domain iff it is a field iff p is prime. The prime p is called the *characteristic* of K, and we also says that K is of *finite characteristic*.

Definition 2.28. If K is a field, then either

(1) $n \cdot 1 \neq 0$ for all integer $n \geq 1$, in which case we say that K has *characteristic* 0, or

(2) There is some smallest prime number p such that $p \cdot 1 = 0$ called the *characteristic* of K, and we say K is of *finite characteristic*.

A field K of characteristic 0 contains a copy of $\mathbb{Q}$, thus is infinite. As we will see in Section 8.10, a finite field has nonzero characteristic p. However, there are infinite fields of nonzero characteristic.