If

$$f = s_t \circ \cdots \circ s_1,$$

for t reflections s_i, it is clear that

$$F = \bigcap_{i=1}^{t} E(1, s_i) \subseteq E(1, f),$$

where $E(1, s_i)$ is the hyperplane defining the reflection s_i. By the Grassmann relation, if we intersect $t \leq n$ hyperplanes, the dimension of their intersection is at least $n - t$. Thus, $n - t \leq p$, that is, $t \geq n - p$, and $n - p$ is the smallest number of reflections composing f. $\square$

As a corollary of Theorem 17.17, we obtain the following fact: If the dimension n of the Euclidean space E is odd, then every rotation $f \in \mathbf{SO}(E)$ admits 1 as an eigenvalue.

Proof. The characteristic polynomial $\det(XI - f)$ of f has odd degree n and has real coefficients, so it must have some real root λ. Since f is an isometry, its n eigenvalues are of the form, $+1, -1$, and $e^{\pm i\theta}$, with $0 < \theta < \pi$, so $\lambda = \pm 1$. Now the eigenvalues $e^{\pm i\theta}$ appear in conjugate pairs, and since n is odd, the number of real eigenvalues of f is odd. This implies that $+1$ is an eigenvalue of f, since otherwise -1 would be the only real eigenvalue of f, and since its multiplicity is odd, we would have $\det(f) = -1$, contradicting the fact that f is a rotation. $\square$

When $n = 3$, we obtain the result due to Euler which says that every 3D rotation R has an invariant axis D, and that restricted to the plane orthogonal to D, it is a 2D rotation. Furthermore, if (a, b, c) is a unit vector defining the axis D of the rotation R and if the angle of the rotation is θ, if B is the skew-symmetric matrix

$$B = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$

then the Rodigues formula (Proposition 12.15) states that

$$R = I + \sin\theta B + (1 - \cos\theta) B^2.$$

The theorems of this section and of the previous section can be immediately translated in terms of matrices. The matrix versions of these theorems is often used in applications so we briefly present them in the section.