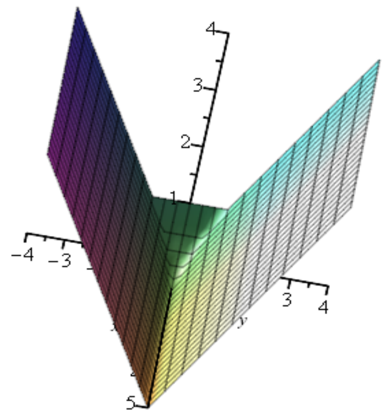

Figure 51.21: The graph of the function from Example 51.11 with a view along the positive x axis.

Corollary 51.20. *Let f be a convex function on $\mathbb{R}^n$, and let $x \in \mathbb{R}^n$ such that $f(x)$ is finite. If f is differentiable at x, then f is proper and $x \in \mathrm{int}(\mathrm{dom}(f))$.*

The following theorem shows that proper convex functions are differentiable almost everywhere.

Theorem 51.21. *Let f be a proper convex function on $\mathbb{R}^n$, and let D be the set of vectors where f is differentiable. Then D is a dense subset of $\mathrm{int}(\mathrm{dom}(f))$, and its complement in $\mathrm{int}(\mathrm{dom}(f))$ has measure zero. Furthermore, the gradient map $x \mapsto \nabla f_x$ is continuous on D.*

Theorem 51.21 is proven in Rockafellar [138] (Theorem 25.5).

Remark: If $f\colon (a,b) \to \mathbb{R}$ is a finite convex function on an open interval of $\mathbb{R}$, then the set D where f is differentiable is dense in (a,b), and $(a,b) - D$ is at most countable. The map f' is continuous and nondecreasing on D. See Rockafellar [138] (Theorem 25.3).

We also have the following result showing that in "most cases" the subdifferential $\partial f(x)$ can be constructed from the gradient map; see Rockafellar [138] (Theorem 25.6).

Theorem 51.22. *Let f be a closed proper convex function on $\mathbb{R}^n$. If $\mathrm{int}(\mathrm{dom}(f)) \neq \emptyset$, then for every $x \in \mathrm{dom}(f)$, we have*

$$\partial f(x) = \overline{\mathrm{conv}(S(x))} + N_{\mathrm{dom}(f)}(x)$$

where $N_{\mathrm{dom}(f)}(x)$ is the normal cone to $\mathrm{dom}(f)$ at x, and $S(x)$ is the set of all limits of sequences of the form $\nabla f_{x_1}, \nabla f_{x_2}, \ldots, \nabla f_{x_p}, \ldots$, where $x_1, x_2, \ldots, x_p, \ldots$ is a sequence in $\mathrm{dom}(f)$ converging to x such that each ∇f_{x_p} is defined.