Proposition 10.7. *Given any matrix $A = D - E - F$, with A and D invertible, for any $\omega \neq 0$, we have*

$$\rho(\mathcal{L}_\omega) \geq |\omega - 1|,$$

where $\mathcal{L}_\omega = \left(\frac{D}{\omega} - E\right)^{-1}\left(\frac{1-\omega}{\omega}D + F\right)$. Therefore, the relaxation method (possibly by blocks) does not converge unless $\omega \in (0, 2)$. If we allow ω to be complex, then we must have

$$|\omega - 1| < 1$$

for the relaxation method to converge.

Proof. Observe that the product $\lambda_1 \cdots \lambda_n$ of the eigenvalues of $\mathcal{L}_\omega$, which is equal to $\det(\mathcal{L}_\omega)$, is given by

$$\lambda_1 \cdots \lambda_n = \det(\mathcal{L}_\omega) = \frac{\det\left(\dfrac{1-\omega}{\omega}D + F\right)}{\det\left(\dfrac{D}{\omega} - E\right)} = (1 - \omega)^n.$$

It follows that

$$\rho(\mathcal{L}_\omega) \geq |\lambda_1 \cdots \lambda_n|^{1/n} = |\omega - 1|.$$

The proof is the same if $\omega \in \mathbb{C}$. $\square$

10.5 Convergence of the Methods of Jacobi, Gauss–Seidel, and Relaxation for Tridiagonal Matrices

We now consider the case where A is a *tridiagonal matrix*, possibly by blocks. In this case, we obtain precise results about the spectral radius of J and $\mathcal{L}_\omega$, and as a consequence, about the convergence of these methods. We also obtain some information about the rate of convergence of these methods. We begin with the case $\omega = 1$, which is technically easier to deal with. The following proposition gives us the precise relationship between the spectral radii $\rho(J)$ and $\rho(\mathcal{L}_1)$ of the Jacobi matrix and the Gauss–Seidel matrix.

Proposition 10.8. *Let A be a tridiagonal matrix (possibly by blocks). If $\rho(J)$ is the spectral radius of the Jacobi matrix and $\rho(\mathcal{L}_1)$ is the spectral radius of the Gauss–Seidel matrix, then we have*

$$\rho(\mathcal{L}_1) = (\rho(J))^2.$$

Consequently, the method of Jacobi and the method of Gauss–Seidel both converge or both diverge simultaneously (even when A is tridiagonal by blocks); when they converge, the method of Gauss–Seidel converges faster than Jacobi's method.