It can be shown (and you may use these facts without proof) that $\otimes$ is associative and that

$$\begin{aligned}(A \otimes B)(C \otimes D) &= AC \otimes BD \\ (A \otimes B)^\top &= A^\top \otimes B^\top,\end{aligned}$$

whenever AC and BD are well defined.

Check that

$$W_{n,n} = \begin{pmatrix} I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} & I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix},$$

and that

$$W_n = \begin{pmatrix} W_{n-1} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} & I_{2^{n-1}} \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix}.$$

Use the above to reprove that

$$W_{n,n} W_{n,n}^\top = 2 I_{2^n}.$$

Let

$$B_1 = 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

and for $n \geq 1$,

$$B_{n+1} = 2 \begin{pmatrix} B_n & 0 \\ 0 & I_{2^n} \end{pmatrix}.$$

Prove that

$$W_n^\top W_n = B_n, \quad \text{for all } n \geq 1.$$

(3) The matrix $W_{n,i}$ is obtained from the matrix $W_{i,i}$ $(1 \leq i \leq n-1)$ as follows:

$$W_{n,i} = \begin{pmatrix} W_{i,i} & 0_{2^i, 2^n - 2^i} \\ 0_{2^n - 2^i, 2^i} & I_{2^n - 2^i} \end{pmatrix}.$$

It consists of four blocks, where $0_{2^i, 2^n - 2^i}$ and $0_{2^n - 2^i, 2^i}$ are matrices of zeros and $I_{2^n - 2^i}$ is the identity matrix of dimension $2^n - 2^i$.

Explain what $W_{n,i}$ does to c and prove that

$$W_{n,n} W_{n,n-1} \cdots W_{n,1} = W_n,$$

where W_n is the Haar matrix of dimension 2^n.

Hint. Use induction on k, with the induction hypothesis

$$W_{n,k} W_{n,k-1} \cdots W_{n,1} = \begin{pmatrix} W_k & 0_{2^k, 2^n - 2^k} \\ 0_{2^n - 2^k, 2^k} & I_{2^n - 2^k} \end{pmatrix}.$$