we have

$$\left\| f\left(\frac{u}{\|u\|}\right)\right\| \leq k,$$

which implies that

$$\|f(u)\| \leq k\|u\|.$$

Thus, (3) holds.

If (3) holds, then for all $u, v \in E$, we have

$$\|f(v) - f(u)\| = \|f(v-u)\| \leq k\|v-u\|.$$

If $k = 0$, then f is the zero function, and continuity is obvious. Otherwise, if $k > 0$, for every $\epsilon > 0$, if $\|v-u\| \leq \frac{\epsilon}{k}$, then $\|f(v-u)\| \leq \epsilon$, which shows continuity at every $u \in E$. Finally, it is obvious that (4) implies (1). $\square$

Among other things, Proposition 37.56 shows that a linear map is continuous iff the image of the unit (closed) ball is bounded. Since a continuous linear map satisfies the condition $\|f(u)\| \leq k\|u\|$ (for some $k \geq 0$), it is also uniformly continuous.

If E and F are normed vector spaces, the set of all continuous linear maps $f\colon E \to F$ is denoted by $\mathcal{L}(E; F)$.

Using Proposition 37.56, we can define a norm on $\mathcal{L}(E; F)$ which makes it into a normed vector space. This definition has already been given in Chapter 9 (Definition 9.7) but for the reader's convenience, we repeat it here.

Definition 37.41. Given two normed vector spaces E and F, for every continuous linear map $f\colon E \to F$, we define the *operator norm* $\|f\|$ *of* f as

$$\|f\| = \inf\{k \geq 0 \mid \|f(x)\| \leq k\|x\|, \text{ for all } x \in E\} = \sup\{\|f(x)\| \mid \|x\| \leq 1\}.$$

From Definition 37.41, for every continuous linear map $f \in \mathcal{L}(E; F)$, we have

$$\|f(x)\| \leq \|f\|\|x\|,$$

for every $x \in E$. It is easy to verify that $\mathcal{L}(E; F)$ is a normed vector space under the norm of Definition 37.41. Furthermore, if E, F, G, are normed vector spaces, and $f\colon E \to F$ and $g\colon F \to G$ are continuous linear maps, we have

$$\|g \circ f\| \leq \|g\|\|f\|.$$

We can now show that when $E = \mathbb{R}^n$ or $E = \mathbb{C}^n$, with any of the norms $\| \ \|_1$, $\| \ \|_2$, or $\| \ \|_\infty$, then every linear map $f\colon E \to F$ is continuous.

Proposition 37.57. *If $E = \mathbb{R}^n$ or $E = \mathbb{C}^n$, with any of the norms $\| \ \|_1$, $\| \ \|_2$, or $\| \ \|_\infty$, and F is any normed vector space, then every linear map $f\colon E \to F$ is continuous.*