If φ_1 and φ_2 are symmetric bilinear forms, then $f^{*l} = f^{*r}$. This also holds if φ is ϵ-Hermitian. Indeed, since

$$\varphi_2(u, f(x)) = \varphi_1(f^{*r}(u), x),$$

we get

$$\epsilon\overline{\varphi_2(f(x), u)} = \epsilon\overline{\varphi_1(x, f^{*r}(u))},$$

and since $\lambda \mapsto \overline{\lambda}$ is an involution, we get

$$\varphi_2(f(x), u) = \varphi_1(x, f^{*r}(u)).$$

Since we also have

$$\varphi_2(f(x), u) = \varphi_1(x, f^{*l}(u)),$$

we obtain

$$\varphi_1(x, f^{*r}(u)) = \varphi_1(x, f^{*l}(u)) \quad \text{for all } x \in E_1, \text{ and all } u \in E_2,$$

and since φ_1 is nondegenerate, we conclude that $f^{*l} = f^{*r}$. Whenever $f^{*l} = f^{*r}$, we use the simpler notation f^*.

If $f\colon E_1 \to E_2$ and $g\colon E_1 \to E_2$ are two linear maps, we have the following properties:

$$\begin{aligned}(f+g)^{*l} &= f^{*l} + g^{*l}\\ \mathrm{id}^{*l} &= \mathrm{id}\\ (\lambda f)^{*l} &= \overline{\lambda} f^{*l},\end{aligned}$$

and similarly for right adjoints. If E_3 is another space, φ_3 is a sesquilinear form on E_3, and if l_{φ_2} and r_{φ_2} are bijective, then for any linear maps $f\colon E_1 \to E_2$ and $g\colon E_2 \to E_3$, we have

$$(g \circ f)^{*l} = f^{*l} \circ g^{*l},$$

and similarly for right adjoints. Furthermore, if $E_1 = E_2 = E$ and $\varphi\colon E \times E \to K$ is ϵ-Hermitian, for any linear map $f\colon E \to E$ (recall that in this case $f^{*l} = f^{*r} = f^*$), we have

$$f^{**} = \epsilon\overline{\epsilon} f.$$

29.5 Isometries Associated with Sesquilinear Forms

The notion of adjoint is a good tool to investigate the notion of isometry between spaces equipped with sesquilinear forms. First, we define metric maps and isometries.

Definition 29.15. If (E_1, φ_1) and (E_2, φ_2) are two pairs of spaces and sesquilinear maps $\varphi_1\colon E_1 \times E_1 \to K$ and $\varphi_2\colon E_2 \times E_2 \to K$, a *metric map* from (E_1, φ_1) to (E_2, φ_2) is a linear map $f\colon E_1 \to E_2$ such that

$$\varphi_1(u, v) = \varphi_2(f(u), f(v)) \quad \text{for all } u, v \in E_1.$$

We say that φ_1 and φ_2 are *equivalent* iff there is a metric map $f\colon E_1 \to E_2$ which is bijective. Such a metric map is called an *isometry*.