we also have
$$\rho(2\alpha - \rho \|C\|_2^2) \left\|u^k - u\right\|^2 \leq \left\|\lambda^k - \lambda\right\|^2 - \left\|\lambda^{k+1} - \lambda\right\|^2.$$
So if
$$0 < \rho < \frac{2\alpha}{\|C\|_2^2},$$
then $\rho(2\alpha - \rho \|C\|_2^2) > 0$, and we conclude that
$$\lim_{k \mapsto \infty} \left\|u^k - u\right\| = 0,$$
that is, the sequence $(u^k)_{k\geq 0}$ converges to u.

Step 5. Convergence of the sequence $(\lambda^k)_{k\geq 0}$ to λ if C has rank m.

Since the sequence $(\left\|\lambda^k - \lambda\right\|)_{k\geq 0}$ is nonincreasing, the sequence $(\lambda^k)_{k\geq 0}$ is bounded, and thus it has a convergent subsequence $(\lambda^{i(k)})_{i\geq 0}$ whose limit is some $\lambda' \in \mathbb{R}^m_+$. Since J' is continuous, by $(\dagger_2)$ we have
$$\nabla J_u + C^\top \lambda' = \lim_{i \mapsto \infty} (\nabla J_{u^{i(k)}} + C^\top \lambda^{i(k)}) = 0. \tag{$*_6$}$$

If C has rank m, then $\mathrm{Im}(C) = \mathbb{R}^m$, which is equivalent to $\mathrm{Ker}\,(C^\top) = (0)$, so $C^\top$ is injective and since by $(\dagger_1)$ we also have $\nabla J_u + C^\top \lambda = 0$, we conclude that $\lambda' = \lambda$. The above reasoning applies to any subsequence of $(\lambda^k)_{k\geq 0}$, so $(\lambda^k)_{k\geq 0}$ converges to λ. $\square$

In the special case where J is an elliptic quadratic functional
$$J(v) = \frac{1}{2}\langle Av, v\rangle - \langle b, v\rangle,$$
where A is symmetric positive definite, by $(\dagger_2)$ an iteration of Uzawa's method gives
$$\begin{aligned} &Au^k - b + C^\top \lambda^k = 0 \\ &\lambda_i^{k+1} = \max\{(\lambda^k + \rho(Cu^k - d))_i, 0\}, \quad 1 \leq i \leq m. \end{aligned}$$
Theorem 50.21 implies that Uzawa's method converges if
$$0 < \rho < \frac{2\lambda_1}{\|C\|_2^2},$$
where λ_1 is the smallest eigenvalue of A.

If we solve for u^k using the first equation, we get
$$\lambda^{k+1} = p_+(\lambda^k + \rho(-CA^{-1}C^\top \lambda^k + CA^{-1}b - d)). \tag{$*_7$}$$
In Example 50.7 we showed that the gradient of the dual function G is given by
$$\nabla G_\mu = Cu_\mu - d = -CA^{-1}C^\top \mu + CA^{-1}b - d,$$
so $(*_7)$ can be written as
$$\lambda^{k+1} = p_+(\lambda^k + \rho \nabla G_{\lambda^k});$$
this shows that Uzawa's method is indeed the gradient method with fixed stepsize applied to the dual program.