

# Chapter 45

## Linear Programs

In this chapter we introduce linear programs and the basic notions relating to this concept. We define the  $\mathcal{H}$ -polyhedron  $\mathcal{P}(A, b)$  of feasible solutions. Then we define bounded and unbounded linear programs and the notion of optimal solution. We define slack variables and the important notion of *linear program in standard form*.

We show that if a linear program in standard form has a feasible solution and is bounded above, then it has an optimal solution. This is not an obvious result and the proof relies on the fact that a polyhedral cone is closed (this result was shown in the previous chapter).

Next we show that in order to find optimal solutions it suffices to consider solutions of a special form called *basic feasible solutions*. We prove that if a linear program in standard form has a feasible solution and is bounded above, then some basic feasible solution is an optimal solution (Theorem 45.4).

Geometrically, a basic feasible solution corresponds to a *vertex*. In Theorem 45.6 we prove that a basic feasible solution of a linear program in standard form is a vertex of the polyhedron  $\mathcal{P}(A, b)$ . Finally, we prove that if a linear program in standard form has some feasible solution, then it has a basic feasible solution (see Theorem 45.7). This fact allows the simplex algorithm described in the next chapter to get started.

### 45.1 Linear Programs, Feasible Solutions, Optimal Solutions

The purpose of linear programming is to solve the following type of optimization problem.