Proof. Since a is symmetric bilinear and h is linear, we have

$$\begin{aligned} J(u+\rho v) &= \frac{1}{2}a(u+\rho v, u+\rho v) - h(u+\rho v) \\ &= \frac{\rho^2}{2}a(v,v) + \rho a(u,v) + \frac{1}{2}a(u,u) - h(u) - \rho h(v) \\ &= \frac{\rho^2}{2}a(v,v) + \rho(a(u,v) - h(v)) + J(u). \end{aligned}$$

Since $dJ_u(v) = a(u,v) - h(v) = \langle Au - b, v\rangle$ and $\nabla J_u = Au - b$, we can also write

$$J(u+\rho v) = \frac{\rho^2}{2}a(v,v) + \rho\langle \nabla J_u, v\rangle + J(u),$$

as claimed. ∎

We have the following theorem about the existence and uniqueness of minima of quadratic functionals.

Theorem 49.4. *Given any real Hilbert space V, let $J\colon V \to \mathbb{R}$ be a quadratic functional of the form*

$$J(v) = \frac{1}{2}a(v,v) - h(v).$$

Assume that there is some real number $\alpha > 0$ such that

$$a(v,v) \geq \alpha \left\|v\right\|^2 \quad \text{for all } v \in V. \tag{$*_\alpha$}$$

If U is any nonempty, closed, convex subset of V, then there is a unique $u \in U$ such that

$$J(u) = \inf_{v\in U} J(v).$$

The element $u \in U$ satisfies the condition

$$a(u, v-u) \geq h(v-u) \quad \text{for all } v \in U. \tag{$*$}$$

Conversely (with the same assumptions on U as above), if an element $u \in U$ satisfies $(*)$, then

$$J(u) = \inf_{v\in U} J(v).$$

If U is a subspace of V, then the above inequalities are replaced by the equations

$$a(u,v) = h(v) \quad \text{for all } v \in U. \tag{$**$}$$

Proof. The key point is that the bilinear form a is actually an inner product in V. This is because it is positive definite, since $(*_\alpha)$ implies that

$$\sqrt{\alpha}\left\|v\right\| \leq (a(v,v))^{1/2},$$