Proposition 17.28 also holds for Hermitian matrices.

A pretty result of Wielandt and Hoffman asserts that

$$\sum_{k=1}^{n} (\alpha_k - \beta_k)^2 \leq \|\Delta A\|_F^2 ,$$

where $\| \ \|_F$ is the Frobenius norm. However, the proof is significantly harder than the above proof; see Lax [113].

The Courant–Fischer theorem can also be used to prove some famous inequalities due to Hermann Weyl. These can also be viewed as perturbation results. Given two symmetric (or Hermitian) matrices A and B, let $\lambda_i(A), \lambda_i(B)$, and $\lambda_i(A+B)$ denote the ith eigenvalue of A, B, and $A+B$, respectively, arranged in nondecreasing order.

Proposition 17.29. *(Weyl) Given two symmetric (or Hermitian) $n \times n$ matrices A and B, the following inequalities hold: For all i, j, k with $1 \leq i, j, k \leq n$:*

1. *If $i + j = k + 1$, then*
$$\lambda_i(A) + \lambda_j(B) \leq \lambda_k(A+B).$$
2. *If $i + j = k + n$, then*
$$\lambda_k(A+B) \leq \lambda_i(A) + \lambda_j(B).$$

Proof. Observe that the first set of inequalities is obtained from the second set by replacing A by $-A$ and B by $-B$, so it is enough to prove the second set of inequalities. By the Courant–Fischer theorem, there is a subspace H of dimension $n - k + 1$ such that

$$\lambda_k(A+B) = \min_{x \in H, x \neq 0} \frac{x^\top (A+B) x}{x^\top x}.$$

Similarly, there exists a subspace F of dimension i and a subspace G of dimension j such that

$$\lambda_i(A) = \max_{x \in F, x \neq 0} \frac{x^\top A x}{x^\top x}, \quad \lambda_j(B) = \max_{x \in G, x \neq 0} \frac{x^\top B x}{x^\top x}.$$

We claim that $F \cap G \cap H \neq (0)$. To prove this, we use the Grassmann relation twice. First,

$$\dim(F \cap G \cap H) = \dim(F) + \dim(G \cap H) - \dim(F + (G \cap H)) \geq \dim(F) + \dim(G \cap H) - n,$$

and second,

$$\dim(G \cap H) = \dim(G) + \dim(H) - \dim(G + H) \geq \dim(G) + \dim(H) - n,$$

so

$$\dim(F \cap G \cap H) \geq \dim(F) + \dim(G) + \dim(H) - 2n.$$