Definition 50.7. Let $L\colon \Omega \times M \to \mathbb{R}$ be a function defined on a set of the form $\Omega \times M$, where Ω and M are open subsets of two normed vector spaces. A point $(u, \lambda) \in \Omega \times M$ is a *saddle point* of L if u is a minimum of the function $L(-, \lambda)\colon \Omega \to \mathbb{R}$ given by $v \mapsto L(v, \lambda)$ for all $v \in \Omega$ and λ fixed, and λ is a maximum of the function $L(u, -)\colon M \to \mathbb{R}$ given by $\mu \mapsto L(u, \mu)$ for all $\mu \in M$ and u fixed; equivalently,

$$\sup_{\mu \in M} L(u, \mu) = L(u, \lambda) = \inf_{v \in \Omega} L(v, \lambda).$$

Note that the order of the arguments u and λ is important. The second set M will be the set of generalized multipliers, and this is why we use the symbol M. Typically, $M = \mathbb{R}^m_+$.

A saddle point is often depicted as a mountain pass, which explains the terminology; see Figure 50.17. However, this is a bit misleading since other situations are possible; see Figure 50.18.

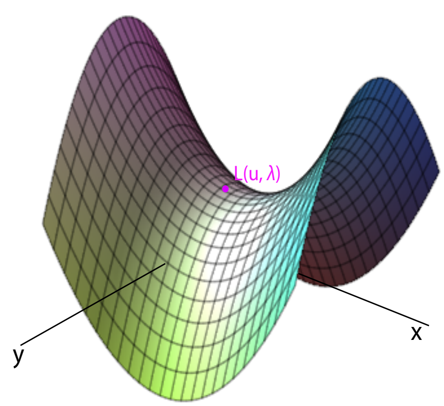

Figure 50.17: A three-dimensional rendition of a saddle point $L(u, \lambda)$ for the function $L(u, \lambda) = u^2 - \lambda^2$. The plane $x = u$ provides a maximum as the apex of a downward opening parabola, while the plane $y = \lambda$ provides a minimum as the apex of an upward opening parabola.

Proposition 50.14. *If (u, λ) is a saddle point of a function $L\colon \Omega \times M \to \mathbb{R}$, then*

$$\sup_{\mu \in M} \inf_{v \in \Omega} L(v, \mu) = L(u, \lambda) = \inf_{v \in \Omega} \sup_{\mu \in M} L(v, \mu).$$

Proof. First we prove that the following inequality always holds:

$$\sup_{\mu \in M} \inf_{v \in \Omega} L(v, \mu) \leq \inf_{v \in \Omega} \sup_{\mu \in M} L(v, \mu). \qquad (*_1)$$