Proof. By Hölder's inequality (Corollary 9.2), for all $x, y \in \mathbb{C}^n$, we have

$$|\langle x, y\rangle| \leq \|x\|_p \|y\|_q,$$

so

$$\|y\|_p^D = \sup_{\substack{x \in \mathbb{C}^n \\ \|x\|_p = 1}} |\langle x, y\rangle| \leq \|y\|_q.$$

For the converse, we consider the cases $p = 1$, $1 < p < +\infty$, and $p = +\infty$. First assume $p = 1$. The result is obvious for $y = 0$, so assume $y \neq 0$. Given y, if we pick $x_j = 1$ for some index j such that $\|y\|_\infty = \max_{1 \leq i \leq n} |y_i| = |y_j|$, and $x_k = 0$ for $k \neq j$, then $|\langle x, y\rangle| = |y_j| = \|y\|_\infty$, so $\|y\|_1^D = \|y\|_\infty$.

Now we turn to the case $1 < p < +\infty$. Then we also have $1 < q < +\infty$, and the equation $1/p + 1/q = 1$ is equivalent to $pq = p + q$, that is, $p(q-1) = q$. Pick $z_j = y_j |y_j|^{q-2}$ for $j = 1, \ldots, n$, so that

$$\|z\|_p = \left(\sum_{j=1}^n |z_j|^p\right)^{1/p} = \left(\sum_{j=1}^n |y_j|^{(q-1)p}\right)^{1/p} = \left(\sum_{j=1}^n |y_j|^q\right)^{1/p}.$$

Then if $x = z / \|z\|_p$, we have

$$|\langle x, y\rangle| = \frac{\left|\sum_{j=1}^n z_j \overline{y_j}\right|}{\|z\|_p} = \frac{\left|\sum_{j=1}^n y_j \overline{y_j} |y_j|^{q-2}\right|}{\|z\|_p} = \frac{\sum_{j=1}^n |y_j|^q}{\left(\sum_{j=1}^n |y_j|^q\right)^{1/p}} = \left(\sum_{j=1}^n |y_j|^q\right)^{1/q} = \|y\|_q.$$

Thus $\|y\|_p^D = \|y\|_q$.

Finally, if $p = \infty$, then pick $x_j = y_j/|y_j|$ if $y_j \neq 0$, and $x_j = 0$ if $y_j = 0$. Then

$$|\langle x, y\rangle| = \left|\sum_{y_j \neq 0}^n y_j \overline{y_j}/|y_j|\right| = \sum_{y_j \neq 0} |y_j| = \|y\|_1.$$

Thus $\|y\|_\infty^D = \|y\|_1$. ∎

We can show that the dual of the spectral norm is the *trace norm* (or *nuclear norm*) also discussed in Section 22.5. Recall from Proposition 9.10 that the spectral norm $\|A\|_2$ of a matrix A is the square root of the largest eigenvalue of A^*A, that is, the largest singular value of A.

Proposition 14.31. *The dual of the spectral norm is given by*

$$\|A\|_2^D = \sigma_1 + \cdots + \sigma_r,$$

where $\sigma_1 > \cdots > \sigma_r > 0$ are the singular values of $A \in \mathrm{M}_n(\mathbb{C})$ (which has rank r).