while

$$\pi_2 = g_2(f)h_2(f) = \frac{1}{2}(X_f^2 + 2\mathrm{id}) = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}.$$

Although it is not entirely obvious, π_1 and π_2 are indeed projections since

$$\pi_1^2 = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} = \pi_1,$$

and

$$\pi_2^2 = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} = \pi_2.$$

Furthermore observe that $\pi_1 + \pi_2 = \mathrm{id}$. The primary decomposition theorem implies that $\mathbb{R}^3 = W_1 \oplus W_2$ where

$$\begin{aligned} W_1 &= \pi_1(\mathbb{R}^3) = \mathrm{Ker}\,(p_1(f)) = \mathrm{Ker}\,(X^2+2) = \mathrm{Ker}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \mathrm{span}\{(0,1,0),(1,0,-1)\}, \\ W_2 &= \pi_2(\mathbb{R}^3) = \mathrm{Ker}\,(p_2(f)) = \mathrm{Ker}\,(X) = \mathrm{span}\{(1,0,1)\}. \end{aligned}$$

See Figure 31.1.

If all the eigenvalues of f belong to the field K, we obtain the following result.

Theorem 31.11. *(Primary Decomposition Theorem, Version 2) Let $f\colon E \to E$ be a linear map on the finite-dimensional vector space E over the field K. If all the eigenvalues $\lambda_1, \ldots, \lambda_k$ of f belong to K, write*

$$m = (X-\lambda_1)^{r_1} \cdots (X-\lambda_k)^{r_k}$$

for the minimal polynomial of f,

$$\chi_f = (X-\lambda_1)^{n_1} \cdots (X-\lambda_k)^{n_k}$$

for the characteristic polynomial of f, with $1 \le r_i \le n_i$, and let

$$W_i = \mathrm{Ker}\,(\lambda_i \mathrm{id} - f)^{r_i}, \quad i = 1, \ldots, k.$$

Then

(a) $E = W_1 \oplus \cdots \oplus W_k$.