quadric Ω in two conjugate points I_Δ and J_Δ (also called circular points). It can be shown that the angle θ between D_1 and D_2 is defined by Laguerre's formula:

$$[(D_1)_\infty, (D_2)_\infty, I_\Delta, J_\Delta] = [D_1, D_2, D_{I_\Delta}, D_{J_\Delta}] = e^{i2\theta},$$

where D_{I_Δ} and D_{J_Δ} are the lines joining the intersection $D_1 \cap D_2$ of D_1 and D_2 to the circular points I_Δ and J_Δ.

As in the case of a plane, the above considerations show that it is not necessary to assume that $\big(E, \overrightarrow{E}\big)$ is a real Euclidean space to define the angle of two lines and orthogonality. Instead, it is enough to assume that a nondegenerate real quadric Ω in the hyperplane at infinity H and without real points is given. In particular, when $n = 3$, the absolute quadric Ω is a nondegenerate real conic consisting of complex points at infinity. We say that Ω provides a *similarity structure* on $\widetilde{E}_{\mathbb{C}}$.

It is also possible to show that the real projectivities of $\widetilde{E}_{\mathbb{C}}$ that leave both the hyperplane H at infinity and the absolute quadric Ω (globally) invariant form a group which is none other than the group of affine similarities; see Lehmann and Bkouche [115] (Chapter 10, page 321), and Berger [11] (Chapter 8, Proposition 8.8.6.4).

Definition 26.14. Let $(E, \overrightarrow{E}, \langle -, - \rangle)$ be a Euclidean affine space of finite dimension. An *affine similarity* of $(E, \overrightarrow{E})$ is an invertible affine map $f \in \mathbf{GA}(E)$ such that if $\overrightarrow{f}$ is the linear map associated with f, then there is some positive real $\rho > 0$ satisfying the condition $\left\| \overrightarrow{f}(u) \right\| = \rho \left\| u \right\|$ for all $u \in \overrightarrow{E}$. The number ρ is called the *ratio* of the affine similarity f.

If $f \in \mathbf{GA}(E)$ is an affine similarity of ratio ρ, let $\overrightarrow{g} = \rho^{-1}\overrightarrow{f}$. Since $\rho > 0$, we have

$$\left\| \overrightarrow{g}(u) \right\| = \left\| \rho^{-1}\overrightarrow{f}(u) \right\| = \rho^{-1} \left\| \overrightarrow{f}(u) \right\| = \rho^{-1}\rho \left\| u \right\| = \left\| u \right\|$$

for all $u \in \overrightarrow{E}$, and by Proposition 12.12, the map $\overrightarrow{g} = \rho^{-1}\overrightarrow{f}$ is an isometry; that is, $\overrightarrow{g} \in \mathbf{O}(E)$.

Consequently, every affine similarity f of E can be written as the composition of an isometry (a member of $\mathbf{O}(E)$), a central dilatation, and a translation. For example, when $n = 2$, a similarity is a transformation of the form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & -\epsilon b \\ b & \epsilon a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c \\ c' \end{pmatrix},$$

with $\epsilon = \pm 1$ and $a, b, c, c' \in \mathbb{R}$. We have the following result showing that the affine similarities of the plane can be viewed as special kinds of projectivities of $\mathbb{CP}^2$.

Proposition 26.29. *If a projectivity h of $\mathbb{CP}^2$ leaves the set of circular points $\{I, J\}$ fixed and maps the affine space $\mathbb{R}^2$ into itself (where $\mathbb{R}^2$ is viewed as the subspace of all points $(x, y, 1)$ with $x, y \in \mathbb{R}$), then h is an affine similarity.*