17.3 Spectral Theorem for Normal Linear Maps

Given a Euclidean space E, our next step is to show that for every linear map $f\colon E \to E$ there is some subspace W of dimension 1 or 2 such that $f(W) \subseteq W$. When $\dim(W) = 1$, the subspace W is actually an eigenspace for some real eigenvalue of f. Furthermore, when f is normal, there is a subspace W of dimension 1 or 2 such that $f(W) \subseteq W$ **and** $f^*(W) \subseteq W$. The difficulty is that the eigenvalues of f are not necessarily real. One way to get around this problem is to complexify both the vector space E and the inner product $\langle -, - \rangle$ as we did in Section 17.2.

Given any subspace W of a Euclidean space E, recall that the *orthogonal complement* $W^\perp$ of W is the subspace defined such that

$$W^\perp = \{u \in E \mid \langle u, w \rangle = 0, \text{ for all } w \in W\}.$$

Recall from Proposition 12.11 that $E = W \oplus W^\perp$ (this can be easily shown, for example, by constructing an orthonormal basis of E using the Gram–Schmidt orthonormalization procedure). The same result also holds for Hermitian spaces; see Proposition 14.13.

As a warm up for the proof of Theorem 17.12, let us prove that every self-adjoint map on a Euclidean space can be diagonalized with respect to an orthonormal basis of eigenvectors.

Theorem 17.8. *(Spectral theorem for self-adjoint linear maps on a Euclidean space) Given a Euclidean space E of dimension n, for every self-adjoint linear map $f\colon E \to E$, there is an orthonormal basis $(e_1, \ldots, e_n)$ of eigenvectors of f such that the matrix of f w.r.t. this basis is a diagonal matrix*

$$\begin{pmatrix} \lambda_1 & & \ldots & \\ & \lambda_2 & \ldots & \\ \vdots & \vdots & \ddots & \vdots \\ & & \ldots & \lambda_n \end{pmatrix},$$

with $\lambda_i \in \mathbb{R}$.

Proof. We proceed by induction on the dimension n of E as follows. If $n = 1$, the result is trivial. Assume now that $n \geq 2$. From Proposition 17.6, all the eigenvalues of f are real, so pick some eigenvalue $\lambda \in \mathbb{R}$, and let w be some eigenvector for λ. By dividing w by its norm, we may assume that w is a unit vector. Let W be the subspace of dimension 1 spanned by w. Clearly, $f(W) \subseteq W$. We claim that $f(W^\perp) \subseteq W^\perp$, where $W^\perp$ is the orthogonal complement of W.

Indeed, for any $v \in W^\perp$, that is, if $\langle v, w \rangle = 0$, because f is self-adjoint and $f(w) = \lambda w$, we have

$$\begin{aligned} \langle f(v), w \rangle &= \langle v, f(w) \rangle \\ &= \langle v, \lambda w \rangle \\ &= \lambda \langle v, w \rangle = 0 \end{aligned}$$