Recall that the augmented Lagrangian is given by

$$L_\rho(x, z, \lambda) = f(x) + g(z) + \lambda^\top (Ax + Bz - c) + (\rho/2) \left\| Ax + Bz - c \right\|_2^2 .$$

For z (and λ) fixed, we have

$$\begin{aligned} L_\rho(x, z, \lambda) &= f(x) + g(z) + \lambda^\top (Ax + Bz - c) + (\rho/2)(Ax + Bz - c)^\top (Ax + Bz - c) \\ &= f(x) + (\rho/2) x^\top A^\top Ax + (\lambda^\top + \rho(Bz - c)^\top) Ax \\ &\quad + g(z) + \lambda^\top (Bz - c) + (\rho/2)(Bz - c)^\top (Bz - c). \end{aligned}$$

Assume that (1) and (2) hold. Since $A^\top A$ is invertible, then it is symmetric positive definite, and by Proposition 51.37 the x-minimization step has a unique solution (the minimization problem succeeds with a unique minimizer).

Similarly, for x (and λ) fixed, we have

$$\begin{aligned} L_\rho(x, z, \lambda) &= f(x) + g(z) + \lambda^\top (Ax + Bz - c) + (\rho/2)(Ax + Bz - c)^\top (Ax + Bz - c) \\ &= g(z) + (\rho/2) z^\top B^\top Bz + (\lambda^\top + \rho(Ax - c)^\top) Bz \\ &\quad + f(x) + \lambda^\top (Ax - c) + (\rho/2)(Ax - c)^\top (Ax - c). \end{aligned}$$

Since $B^\top B$ is invertible, then it is symmetric positive definite, and by Proposition 51.37 the z-minimization step has a unique solution (the minimization problem succeeds with a unique minimizer).

By Theorem 51.41, Assumption (3) is equivalent to the fact that the KKT equations are satisfied by some triple (x^*, z^*, λ^*), namely

$$Ax^* + Bz^* - c = 0 \tag{$*$}$$

and

$$0 \in \partial f(x^*) + \partial g(z^*) + A^\top \lambda^* + B^\top \lambda^*, \tag{$\dagger$}$$

Assumption (3) is also equivalent to Conditions (a) and (b) of Theorem 51.41. In particular, our program has an optimal solution (x^*, z^*). By Theorem 51.43, λ^* is maximizer of the dual function $G(\lambda) = \inf_{x,z} L_0(x, z, \lambda)$ and strong duality holds, that is, $G(\lambda^*) = f(x^*) + g(z^*)$ (the duality gap is zero).

We will see after the proof of Theorem 52.1 that Assumption (2) is actually implied by Assumption (3). This allows us to prove a convergence result stronger than the convergence result proven in Boyd et al. [28] under the exact same assumptions (1) and (3).

Let p^* be the minimum value of $f + g$ over the convex set $\{(x, z) \in \mathbb{R}^{m+p} \mid Ax + Bz - c = 0\}$, and let (p^k) be the sequence given by $p^k = f(x^k) + g(z^k)$, and recall that $r^k = Ax^k + Bz^k - c$.

Our main goal is to prove the following result.

Theorem 52.1. *Suppose the following assumptions hold:*