When $\|r_n\|_2 = \|\widetilde{H}_n y - \|b\|_2 e_1\|_2$ is considered small enough, we stop and the approximate solution of $Ax = b$ is then

$$x_n = U_n y.$$

There are ways of improving efficiency of the "naive" version of GMRES that we just presented; see Trefethen and Bau [176] (Lecture 35). We now consider the case where A is a Hermitian (or symmetric) matrix.

18.6 The Hermitian Case; Lanczos Iteration

If A is an $m \times m$ symmetric or Hermitian matrix, then Arnoldi's method is simpler and much more efficient. Indeed, in this case, it is easy to see that the upper Hessenberg matrices H_n are also symmetric (Hermitian respectively), and thus tridiagonal. Also, the eigenvalues of A and H_n are real. It is convenient to write

$$H_n = \begin{pmatrix} \alpha_1 & \beta_1 & & & \\ \beta_1 & \alpha_2 & \beta_2 & & \\ & \beta_2 & \alpha_3 & \ddots & \\ & & \ddots & \ddots & \beta_{n-1} \\ & & & \beta_{n-1} & \alpha_n \end{pmatrix}.$$

The recurrence $(*_2)$ of Section 18.4 becomes the three-term recurrence

$$Au_n = \beta_{n-1}u_{n-1} + \alpha_n u_n + \beta_n u_{n+1}. \qquad (*_6)$$

We also have $\alpha_n = u_n^* A u_n$, so Arnoldi's algorithm becomes the following algorithm known as *Lanczos' algorithm* (or *Lanczos iteration*). The inner loop on j from 1 to n has been eliminated and replaced by a single assignment.

Given an arbitrary nonzero vector $b \in \mathbb{C}^m$, let $u_1 = b/\|b\|$;

for $n = 1, 2, 3, \ldots$ **do**

  $z := Au_n$;

    $\alpha_n := u_n^* z$;

    $z := z - \beta_{n-1}u_{n-1} - \alpha_n u_n$

  $\beta_n := \|z\|$;

  if $\beta_n = 0$ **quit**

  $u_{n+1} = z/\beta_n$

When $\beta_n = 0$, we say that we have a *breakdown* of the Lanczos iteration.

Versions of Proposition 18.5 and Theorem 18.6 apply to Lanczos iteration.

Besides being much more efficient than Arnoldi iteration, Lanczos iteration has the advantage that the *Rayleigh–Ritz method* for finding some of the eigenvalues of A as the eigenvalues