This suggests the plan of attack for our second proof of the Cayley–Hamilton theorem. For simplicity, we prove the theorem for vector spaces over a field. The proof goes through for a free module over a commutative ring.

Theorem 7.15. *(Cayley–Hamilton) For every finite-dimensional vector space over a field K, for every linear map $f\colon E \to E$, for every basis $(e_1, \ldots, e_n)$, if A is the matrix over f over the basis $(e_1, \ldots, e_n)$ and if*

$$P_A(X) = X^n + c_1 X^{n-1} + \cdots + c_n$$

is the characteristic polynomial of A, then

$$P_A(f) = f^n + c_1 f^{n-1} + \cdots + c_n \mathrm{id} = 0.$$

Proof. Since the columns of A consist of the vector $f(e_j)$ expressed over the basis $(e_1, \ldots, e_n)$, we have

$$f(e_j) = \sum_{i=1}^{n} a_{i\,j} e_i, \quad 1 \leq j \leq n.$$

Using our action of $K[X]$ on E, the above equations can be expressed as

$$X \cdot e_j = \sum_{i=1}^{n} a_{i\,j} \cdot e_i, \quad 1 \leq j \leq n,$$

which yields

$$\sum_{i=1}^{j-1} -a_{i\,j} \cdot e_i + (X - a_{j\,j}) \cdot e_j + \sum_{i=j+1}^{n} -a_{i\,j} \cdot e_i = 0, \qquad 1 \leq j \leq n.$$

Observe that the transpose of the characteristic polynomial shows up, so the above system can be written as

$$\begin{pmatrix} X - a_{1\,1} & -a_{2\,1} & \cdots & -a_{n\,1} \\ -a_{1\,2} & X - a_{2\,2} & \cdots & -a_{n\,2} \\ \vdots & \vdots & \vdots & \vdots \\ -a_{1\,n} & -a_{2\,n} & \cdots & X - a_{n\,n} \end{pmatrix} \cdot \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

If we let $B = XI - A^{\top}$, then as in the previous proof, if $\widetilde{B}$ is the transpose of the matrix of cofactors of B, we have

$$\widetilde{B}B = \det(B)I = \det(XI - A^{\top})I = \det(XI - A)I = P_A I.$$

But since

$$B \cdot \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$