Proposition 45.3 implies the following important result.

Theorem 45.4. *Let (P_2) be any standard linear program with objective function cx, where $Ax = b$ and A is an $m \times n$ matrix of rank m. If (P_2) has some feasible solution and if it is bounded above, then some basic feasible solution $\widetilde{x}$ is an optimal solution of (P_2).*

Proof. By Proposition 45.3, for any feasible solution x there is some basic feasible solution $\widetilde{x}$ such that $cx \leq c\widetilde{x}$. But there are only finitely many basic feasible solutions, so one of them has to yield the maximum of the objective function. $\square$

Geometrically, basic solutions are exactly the vertices of the polyhedron $\mathcal{P}(A, b)$, a notion that we now define.

Definition 45.8. Given an $\mathcal{H}$-polyhedron $\mathcal{P} \subseteq \mathbb{R}^n$, a *vertex* of $\mathcal{P}$ is a point $v \in \mathcal{P}$ with property that there is some nonzero linear form $c \in (\mathbb{R}^n)^*$ and some $\mu \in \mathbb{R}$, such that v is the unique point of $\mathcal{P}$ for which the map $x \mapsto cx$ has the maximum value μ ; that is, $cy < cv = \mu$ for all $y \in \mathcal{P} - \{v\}$. Geometrically, this means that the hyperplane of equation $cy = \mu$ touches $\mathcal{P}$ exactly at v. More generally, a convex subset F of $\mathcal{P}$ is a *k-dimensional face* of $\mathcal{P}$ if F has dimension k and if there is some affine form $\varphi(x) = cx - \mu$ such that $cy = \mu$ for all $y \in F$, and $cy < \mu$ for all $y \in \mathcal{P} - F$. A 1-dimensional face is called an *edge*.

The concept of a vertex is illustrated in Figure 45.4, while the concept of an edge is illustrated in Figure 45.5.

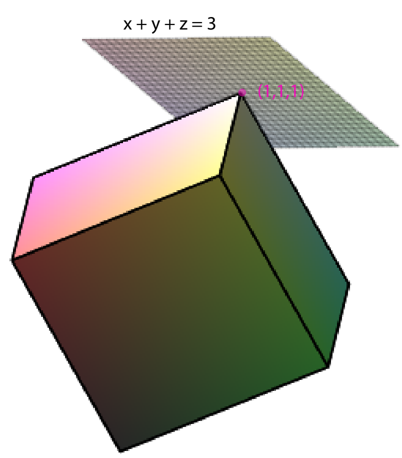

Figure 45.4: The cube centered at the origin with diagonal through $(-1, -1, -1)$ and $(1, 1, 1)$ has eight vertices. The vertex $(1, 1, 1)$ is associated with the linear form $x + y + z = 3$.

Since a k-dimensional face F of $\mathcal{P}$ is equal to the intersection of the hyperplane $H(\varphi)$ of equation $cx = \mu$ with $\mathcal{P}$, it is indeed convex and the notion of dimension makes sense.