Proof. Our proof is adapted from Vapnik [182] (Chapter 10, Theorem 10.1). For any separating hyperplane H, since

$$\begin{aligned} d(u_i, H) &= w^\top u_i - b & i &= 1, \ldots, p \\ d(v_j, H) &= -w^\top v_j + b & j &= 1, \ldots, q, \end{aligned}$$

and since the smallest distance to H is

$$\begin{aligned} \delta &= \min\{d(u_i, H),\ d(v_j, H) \mid 1 \leq i \leq p,\ 1 \leq j \leq q\} \\ &= \min\{w^\top u_i - b,\ -w^\top v_j + b \mid 1 \leq i \leq p,\ 1 \leq j \leq q\} \\ &= \min\{\min\{w^\top u_i - b \mid 1 \leq i \leq p\}, \min\{-w^\top v_j + b \mid 1 \leq j \leq q\}\} \\ &= \min\{\min\{w^\top u_i \mid 1 \leq i \leq p\} - b\}, \min\{-w^\top v_j \mid 1 \leq j \leq q\} + b\} \\ &= \min\{\min\{w^\top u_i \mid 1 \leq i \leq p\} - b\}, -\max\{w^\top v_j \mid 1 \leq j \leq q\} + b\} \\ &= \min\{c_1(w) - b, -c_2(w) + b\}, \end{aligned}$$

in order for δ to be maximal we must have

$$c_1(w) - b = -c_2(w) + b,$$

which yields

$$b = \frac{c_1(w) + c_2(w)}{2}.$$

In this case,

$$c_1(w) - b = \frac{c_1(w) - c_2(w)}{2} = -c_2(w) + b,$$

so the maximum margin δ is indeed obtained when $\rho(w) = (c_1(w) - c_2(w))/2$ is maximal over U. Conversely, it is easy to see that any hyperplane of equation $w^\top x - b = 0$ associated with a w maximizing ρ over U and $b = (c_1(w) + c_2(w))/2$ is an optimal solution.

It remains to show that an optimal separating hyperplane exists and is unique. Since the unit ball is compact, U (as defined in Theorem 50.13) is compact, and since the function $w \mapsto \rho(w)$ is continuous, it achieves its maximum for some w_0 such that $\|w_0\| \leq 1$. Actually, we must have $\|w_0\| = 1$, since otherwise, by the reasoning used in Proposition 50.12, $w_0/\|w_0\|$ would be an even better solution. Therefore, w_0 is on the boundary of U. But ρ is a concave function (as an infimum of affine functions), so if it had two distinct maxima w_0 and w_0' with $\|w_0\| = \|w_0'\| = 1$, these would be global maxima since U is also convex, so we would have $\rho(w_0) = \rho(w_0')$ and then ρ would also have the same value along the segment (w_0, w_0') and in particular at $(w_0 + w_0')/2$, an interior point of U, a contradiction. $\square$

We can proceed with the above formulation (SVM_{h1}) but there is a way to reformulate the problem so that the constraints are all *affine*, which might be preferable since they will be *automatically qualified*.