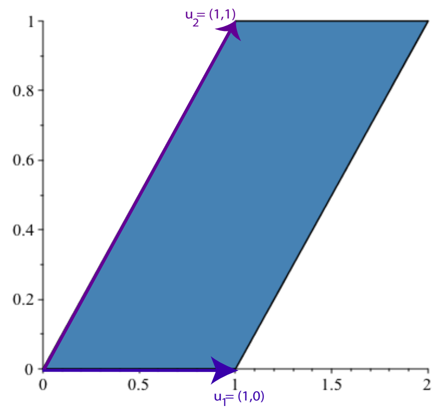

Figure 7.1: The parallelogram in $\mathbb{R}^w$ spanned by the vectors $u_1 = (1, 0)$ and $u_2 = (1, 1)$.

Corollary 7.7. *For every matrix $A \in \mathrm{M}_n(K)$, we have* $\det(A) = \det(A^\top)$.

Proof. By Theorem 7.6, we have

$$\det(A) = \sum_{\pi \in \mathfrak{S}_n} \epsilon(\pi) a_{\pi(1)\,1} \cdots a_{\pi(n)\,n},$$

where the sum ranges over all permutations π on $\{1, \ldots, n\}$. Since a permutation is invertible, every product

$$a_{\pi(1)\,1} \cdots a_{\pi(n)\,n}$$

can be rewritten as

$$a_{1\,\pi^{-1}(1)} \cdots a_{n\,\pi^{-1}(n)},$$

and since $\epsilon(\pi^{-1}) = \epsilon(\pi)$ and the sum is taken over all permutations on $\{1, \ldots, n\}$, we have

$$\sum_{\pi \in \mathfrak{S}_n} \epsilon(\pi) a_{\pi(1)\,1} \cdots a_{\pi(n)\,n} = \sum_{\sigma \in \mathfrak{S}_n} \epsilon(\sigma) a_{1\,\sigma(1)} \cdots a_{n\,\sigma(n)},$$

where π and σ range over all permutations. But it is immediately verified that

$$\det(A^\top) = \sum_{\sigma \in \mathfrak{S}_n} \epsilon(\sigma) a_{1\,\sigma(1)} \cdots a_{n\,\sigma(n)}. \qquad \square$$

A useful consequence of Corollary 7.7 is that the determinant of a matrix is also a multilinear alternating map of its *rows*. This fact, combined with the fact that the determinant of a matrix is a multilinear alternating map of its columns, is often useful for finding short-cuts in computing determinants. We illustrate this point on the following example which shows up in polynomial interpolation.