Problem 48.6. Let V be a real Hilbert space, $(a_1, \ldots, a_m)$ a sequence of m vectors in V, b some vector in V, $(\alpha_1, \ldots, \alpha_m)$ a sequence of m real numbers, and β some real number. Prove that the inclusion

$$\{w \in V \mid \langle a_i, w\rangle \geq \alpha_i,\ 1 \leq i \leq m\} \subseteq \{w \in V \mid \langle b, w\rangle \geq \beta\}$$

holds if and only if there exist $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ such that $\lambda_i \geq 0$ for $i = 1, \ldots, m$ and

$$b = \sum_{i=1}^{m} \lambda_i a_i$$
$$\beta \leq \sum_{i=1}^{m} \lambda_i \alpha_i.$$