Since

$$P_A(X)I = (X^n + c_1X^{n-1} + \cdots + c_n)I,$$

the equality

$$X^nD_0 + X^{n-1}D_1 + \cdots + D_n = (X^n + c_1X^{n-1} + \cdots + c_n)I$$

is an equality between two matrices, so it requires that all corresponding entries are equal, and since these are polynomials, the coefficients of these polynomials must be identical, which is equivalent to the set of equations

$$\begin{aligned} I &= B_0 \\ c_1I &= B_1 - AB_0 \\ &\vdots \\ c_jI &= B_j - AB_{j-1} \\ &\vdots \\ c_{n-1}I &= B_{n-1} - AB_{n-2} \\ c_nI &= -AB_{n-1}, \end{aligned}$$

for all $j$, with $1 \leq j \leq n-1$. If, as in the table below,

$$\begin{aligned} A^n &= A^nB_0 \\ c_1A^{n-1} &= A^{n-1}(B_1 - AB_0) \\ &\vdots \\ c_jA^{n-j} &= A^{n-j}(B_j - AB_{j-1}) \\ &\vdots \\ c_{n-1}A &= A(B_{n-1} - AB_{n-2}) \\ c_nI &= -AB_{n-1}, \end{aligned}$$

we multiply the first equation by $A^n$, the last by $I$, and generally the $(j+1)$th by $A^{n-j}$, when we add up all these new equations, we see that the right-hand side adds up to 0, and we get our desired equation

$$A^n + c_1A^{n-1} + \cdots + c_nI = 0,$$

as claimed. □

As a concrete example, when $n = 2$, the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

satisfies the equation

$$A^2 - (a+d)A + (ad - bc)I = 0.$$