We now show that any two n-th exterior tensor powers (A_1, φ_1) and (A_2, φ_2) for E are isomorphic.

Proposition 34.3. *Given any two n-th exterior tensor powers (A_1, φ_1) and (A_2, φ_2) for E, there is an isomorphism $h\colon A_1 \to A_2$ such that*

$$\varphi_2 = h \circ \varphi_1.$$

Proof. Replace tensor product by n-th exterior tensor power in the proof of Proposition 33.5. $\square$

We next give a construction that produces an n-th exterior tensor power of a vector space E.

Theorem 34.4. *Given a vector space E, an n-th exterior tensor power $(\bigwedge^n(E), \varphi)$ for E can be constructed ($n \geq 1$). Furthermore, denoting $\varphi(u_1, \ldots, u_n)$ as $u_1 \wedge \cdots \wedge u_n$, the exterior tensor power $\bigwedge^n(E)$ is generated by the vectors $u_1 \wedge \cdots \wedge u_n$, where $u_1, \ldots, u_n \in E$, and for every alternating multilinear map $f\colon E^n \to F$, the unique linear map $f_\wedge\colon \bigwedge^n(E) \to F$ such that $f = f_\wedge \circ \varphi$ is defined by*

$$f_\wedge(u_1 \wedge \cdots \wedge u_n) = f(u_1, \ldots, u_n)$$

on the generators $u_1 \wedge \cdots \wedge u_n$ of $\bigwedge^n(E)$.

Proof sketch. We can give a quick proof using the tensor algebra $T(E)$. Let $\mathfrak{I}_a$ be the two-sided ideal of $T(E)$ generated by all tensors of the form $u \otimes u \in E^{\otimes 2}$. Then let

$$\bigwedge^n(E) = E^{\otimes n}/(\mathfrak{I}_a \cap E^{\otimes n})$$

and let π be the projection $\pi\colon E^{\otimes n} \to \bigwedge^n(E)$. If we let $u_1 \wedge \cdots \wedge u_n = \pi(u_1 \otimes \cdots \otimes u_n)$, it is easy to check that $(\bigwedge^n(E), \wedge)$ satisfies the conditions of Theorem 34.4. $\square$

Remark: We can also define

$$\bigwedge(E) = T(E)/\mathfrak{I}_a = \bigoplus_{n \geq 0} \bigwedge^n(E),$$

the *exterior algebra* of E. This is the skew-symmetric counterpart of $\mathrm{S}(E)$, and we will study it a little later.

For simplicity of notation, we may write $\bigwedge^n E$ for $\bigwedge^n(E)$. We also abbreviate "exterior tensor power" as "exterior power." Clearly, $\bigwedge^1(E) \cong E$, and it is convenient to set $\bigwedge^0(E) = K$.