Proof. Replace tensor product by n-th symmetric tensor power in the proof of Proposition 33.5. $\square$

We now give a construction that produces a symmetric n-th tensor power of a vector space E.

Theorem 33.24. *Given a vector space E, a symmetric n-th tensor power $(\mathrm{S}^n(E), \varphi)$ for E can be constructed ($n \geq 1$). Furthermore, denoting $\varphi(u_1, \ldots, u_n)$ as $u_1 \odot \cdots \odot u_n$, the symmetric tensor power $\mathrm{S}^n(E)$ is generated by the vectors $u_1 \odot \cdots \odot u_n$, where $u_1, \ldots, u_n \in E$, and for every symmetric multilinear map $f\colon E^n \to F$, the unique linear map $f_\odot\colon \mathrm{S}^n(E) \to F$ such that $f = f_\odot \circ \varphi$ is defined by*

$$f_\odot(u_1 \odot \cdots \odot u_n) = f(u_1, \ldots, u_n)$$

on the generators $u_1 \odot \cdots \odot u_n$ of $\mathrm{S}^n(E)$.

Proof. The tensor power $E^{\otimes n}$ is too big, and thus we define an appropriate quotient. Let C be the subspace of $E^{\otimes n}$ generated by the vectors of the form

$$u_1 \otimes \cdots \otimes u_n - u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)},$$

for all $u_i \in E$, and all permutations $\sigma\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$. We claim that the quotient space $(E^{\otimes n})/C$ does the job.

Let $p\colon E^{\otimes n} \to (E^{\otimes n})/C$ be the quotient map, and let $\varphi\colon E^n \to (E^{\otimes n})/C$ be the map given by

$$\varphi = p \circ \varphi_0,$$

where $\varphi_0\colon E^n \to E^{\otimes n}$ is the injection given by $\varphi_0(u_1, \ldots, u_n) = u_1 \otimes \cdots \otimes u_n$.

Let us denote $\varphi(u_1, \ldots, u_n)$ as $u_1 \odot \cdots \odot u_n$. It is clear that φ is symmetric. Since the vectors $u_1 \otimes \cdots \otimes u_n$ generate $E^{\otimes n}$, and p is surjective, the vectors $u_1 \odot \cdots \odot u_n$ generate $(E^{\otimes n})/C$.

It remains to show that $((E^{\otimes n})/C, \varphi)$ satisfies the universal mapping property. To this end we begin by proving that there is a map h such that $f = h \circ \varphi$. Given any symmetric multilinear map $f\colon E^n \to F$, by Theorem 33.6 there is a linear map $f_\otimes\colon E^{\otimes n} \to F$ such that $f = f_\otimes \circ \varphi_0$, as in the diagram below.

$$\begin{tikzcd} E^n \arrow[r, "\varphi_0"] \arrow[dr, "f"'] & E^{\otimes n} \arrow[d, "f_\otimes"] \\ & F \end{tikzcd}$$