*Proof.* First we need to prove that $\ell^2(K)$ is a vector space. Assume that $(x_k)_{k\in K}$ and $(y_k)_{k\in K}$ are in $\ell^2(K)$. This means that $(|x_k|^2)_{k\in K}$ and $(|y_k|^2)_{k\in K}$ are summable, which, in view of Proposition A.1(2), is equivalent to the existence of some positive bounds $A$ and $B$ such that $\sum_{i\in I}|x_i|^2 < A$ and $\sum_{i\in I}|y_i|^2 < B$, for every finite subset $I$ of $K$. To prove that $(|x_k+y_k|^2)_{k\in K}$ is summable, it is sufficient to prove that there is some $C>0$ such that $\sum_{i\in I}|x_i+y_i|^2 < C$ for every finite subset $I$ of $K$. However, the parallelogram inequality implies that

$$\sum_{i\in I}|x_i+y_i|^2 \leq \sum_{i\in I}2(|x_i|^2+|y_i|^2) \leq 2(A+B),$$

for every finite subset $I$ of $K$, and we conclude by Proposition A.1(2). Similarly, for every $\lambda\in\mathbb{C}$,

$$\sum_{i\in I}|\lambda x_i|^2 \leq \sum_{i\in I}|\lambda|^2|x_i|^2 \leq |\lambda|^2 A,$$

and $(\lambda_k x_k)_{k\in K}$ is summable. Therefore, $\ell^2(K)$ is a vector space.

By the Cauchy-Schwarz inequality,

$$\sum_{i\in I}|x_i\overline{y_i}| \leq \sum_{i\in I}|x_i||y_i| \leq \Big(\sum_{i\in I}|x_i|^2\Big)^{1/2}\Big(\sum_{i\in I}|xy_i|^2\Big)^{1/2} \leq \sum_{i\in I}(|x_i|^2+|y_i|^2)/2 \leq (A+B)/2,$$

for every finite subset $I$ of $K$. For the third inequality we used the fact that

$$4CD \leq (C+D)^2,$$

(with $C=\sum_{i\in I}|x_i|^2$ and $D=\sum_{i\in I}|y_i|^2$) which is equivalent to

$$(C-D)^2 \geq 0.$$

By Proposition A.1(2), $(|x_k\overline{y_k}|)_{k\in K}$ is summable. The customary language is that $(x_k\overline{y_k})_{k\in K}$ is absolutely summable. However, it is a standard fact that this implies that $(x_k\overline{y_k})_{k\in K}$ is summable (For every $\epsilon>0$, there is some finite subset $I$ of $K$ such that

$$\sum_{j\in J}|x_j\overline{y_j}| < \epsilon$$

for every finite subset $J$ of $K$ such that $I\cap J=\emptyset$, and thus

$$|\sum_{j\in J}x_j\overline{y_j}| \leq \sum_{i\in J}|x_j\overline{y_j}| < \epsilon,$$

proving that $(x_k\overline{y_k})_{k\in K}$ is a Cauchy family, and thus summable). We still have to prove that $\ell^2(K)$ is complete.

Consider a sequence $((\lambda_k^n)_{k\in K})_{n\geq 1}$ of sequences $(\lambda_k^n)_{k\in K}\in\ell^2(K)$, and assume that it is a Cauchy sequence. This means that for every $\epsilon>0$, there is some $N\geq 1$ such that

$$\sum_{k\in K}|\lambda_k^m-\lambda_k^n|^2 < \epsilon^2$$