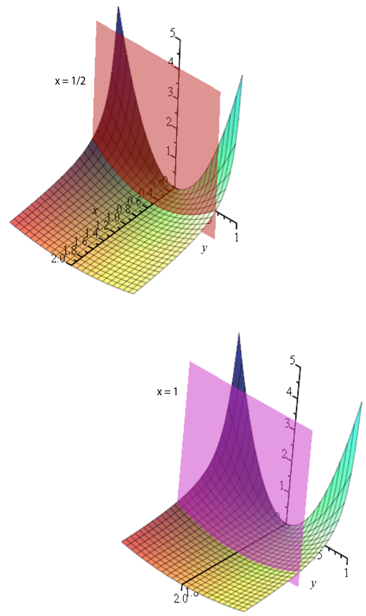

Figure 51.6: The proper convex function of Example 51.4. When intersected by vertical planes of the form $x = \alpha$, for $\alpha > 0$, the trace is an upward parabola. When α is close to zero, this parabola approximates the positive z axis.

$x = y^2/(2\alpha)$ is α for any $\alpha > 0$. See Figure 51.7 However, it is easy to see that the limit along any line segment from $(0,0)$ to a point in the open right half-plane is 0.

We conclude this quick tour of the basic properties of convex functions with a result involving the Lipschitz condition.

Definition 51.11. Let $f\colon E \to F$ be a function between normed vector spaces E and F, and let U be a nonempty subset of E. We say that f *Lipschitzian on* U (or *has the Lipschitz condition on* U) if there is some $c \geq 0$ such that

$$\|f(x) - f(y)\|_F \leq c\,\|x - y\|_E \quad \text{for all } x, y \in U.$$

Obviously, if f is Lipschitzian on U it is uniformly continuous on U. The following result is proven in Rockafellar [138] (Theorem 10.4).

Proposition 51.8. *Let f be a proper convex function, and let S be any (nonempty) closed bounded subset of* $\textbf{relint}(\mathrm{dom}(f))$. *Then f is Lipschitzian on S.*

In particular, a finite convex function on $\mathbb{R}^n$ is Lipschitzian on every compact subset of $\mathbb{R}^n$. However, such a function may not be Lipschitzian on $\mathbb{R}^n$ as a whole.