(1) The matrix L_{sym} is symmetric and positive semidefinite. In fact,

$$x^\top L_{\mathrm{sym}} x = \frac{1}{2}\sum_{i,j=1}^{m} w_{i\,j}\left(\frac{x_i}{\sqrt{d_i}} - \frac{x_j}{\sqrt{d_j}}\right)^2 \quad \text{for all } x \in \mathbb{R}^m.$$

(2) The normalized graph Laplacians L_{sym} and L_{rw} have the same spectrum $(0 = \nu_1 \leq \nu_2 \leq \ldots \leq \nu_m)$, and a vector $u \neq 0$ is an eigenvector of L_{rw} for λ iff $D^{1/2}u$ is an eigenvector of L_{sym} for λ.

(3) The graph Laplacians L and L_{sym} are symmetric and positive semidefinite.

(4) A vector $u \neq 0$ is a solution of the generalized eigenvalue problem $Lu = \lambda Du$ iff $D^{1/2}u$ is an eigenvector of L_{sym} for the eigenvalue λ iff u is an eigenvector of L_{rw} for the eigenvalue λ.

(5) The graph Laplacians, L and L_{rw} have the same nullspace. For any vector u, we have $u \in \mathrm{Ker}\,(L)$ iff $D^{1/2}u \in \mathrm{Ker}\,(L_{\mathrm{sym}})$.

(6) The vector $\mathbf{1}$ is in the nullspace of L_{rw}, and $D^{1/2}\mathbf{1}$ is in the nullspace of L_{sym}.

(7) For every eigenvalue ν_i of the normalized graph Laplacian L_{sym}, we have $0 \leq \nu_i \leq 2$. Furthermore, $\nu_m = 2$ iff the underlying graph of G contains a nontrivial connected bipartite component.

(8) If $m \geq 2$ and if the underlying graph of G is not a complete graph,[1] then $\nu_2 \leq 1$. Furthermore the underlying graph of G is a complete graph iff $\nu_2 = \frac{m}{m-1}$.

(9) If $m \geq 2$ and if the underlying graph of G is connected, then $\nu_2 > 0$.

(10) If $m \geq 2$ and if the underlying graph of G has no isolated vertices, then $\nu_m \geq \frac{m}{m-1}$.

Proof. (1) We have $L_{\mathrm{sym}} = D^{-1/2}LD^{-1/2}$, and $D^{-1/2}$ is a symmetric invertible matrix (since it is an invertible diagonal matrix). It is a well-known fact of linear algebra that if B is an invertible matrix, then a matrix S is symmetric, positive semidefinite iff $BSB^\top$ is symmetric, positive semidefinite. Since L is symmetric, positive semidefinite, so is $L_{\mathrm{sym}} = D^{-1/2}LD^{-1/2}$. The formula

$$x^\top L_{\mathrm{sym}} x = \frac{1}{2}\sum_{i,j=1}^{m} w_{i\,j}\left(\frac{x_i}{\sqrt{d_i}} - \frac{x_j}{\sqrt{d_j}}\right)^2 \quad \text{for all } x \in \mathbb{R}^m$$

follows immediately from Proposition 20.4 by replacing x by $D^{-1/2}x$, and also shows that L_{sym} is positive semidefinite.

(2) Since

$$L_{\mathrm{rw}} = D^{-1/2}L_{\mathrm{sym}}D^{1/2},$$

[1] Recall that an undirected graph is complete if for any two distinct nodes u, v, there is an edge $\{u, v\}$.