(2) Let $\varphi_L\colon \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$ be the map defined by

$$\varphi_L((x_1, x_2, x_3, x_4), (y_1, y_2, y_3, , y_4)) = x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4.$$

Prove that φ is a bilinear nondegenerate pairing.

Show that there exist nonzero vectors $x \in \mathbb{R}^4$ such that $\varphi_L(x, x) = 0$.

Remark: The vector space $\mathbb{R}^4$ equipped with the above bilinear form called the *Lorentz form* is called *Minkowski space*.

Problem 11.4. Given any two subspaces V_1, V_2 of a finite-dimensional vector space E, prove that

$$\begin{aligned}(V_1 + V_2)^0 &= V_1^0 \cap V_2^0\\ (V_1 \cap V_2)^0 &= V_1^0 + V_2^0.\end{aligned}$$

Beware that in the second equation, V_1 and V_2 are subspaces of E, not E^*.

Hint. To prove the second equation, prove the inclusions $V_1^0 + V_2^0 \subseteq (V_1 \cap V_2)^0$ and $(V_1 \cap V_2)^0 \subseteq V_1^0 + V_2^0$. Proving the second inclusion is a little tricky. First, prove that we can pick a subspace W_1 of V_1 and a subspace W_2 of V_2 such that

1. V_1 is the direct sum $V_1 = (V_1 \cap V_2) \oplus W_1$.

2. V_2 is the direct sum $V_2 = (V_1 \cap V_2) \oplus W_2$.

3. $V_1 + V_2$ is the direct sum $V_1 + V_2 = (V_1 \cap V_2) \oplus W_1 \oplus W_2$.

Problem 11.5. (1) Let A be any $n \times n$ matrix such that the sum of the entries of every row of A is the same (say c_1), and the sum of entries of every column of A is the same (say c_2). Prove that $c_1 = c_2$.

(2) Prove that for any $n \geq 2$, the $2n - 2$ equations asserting that the sum of the entries of every row of A is the same, and the sum of entries of every column of A is the same are lineary independent. For example, when $n = 4$, we have the following 6 equations

$$\begin{aligned}
a_{11} + a_{12} + a_{13} + a_{14} - a_{21} - a_{22} - a_{23} - a_{24} &= 0\\
a_{21} + a_{22} + a_{23} + a_{24} - a_{31} - a_{32} - a_{33} - a_{34} &= 0\\
a_{31} + a_{32} + a_{33} + a_{34} - a_{41} - a_{42} - a_{43} - a_{44} &= 0\\
a_{11} + a_{21} + a_{31} + a_{41} - a_{12} - a_{22} - a_{32} - a_{42} &= 0\\
a_{12} + a_{22} + a_{32} + a_{42} - a_{13} - a_{23} - a_{33} - a_{43} &= 0\\
a_{13} + a_{23} + a_{33} + a_{43} - a_{14} - a_{24} - a_{34} - a_{44} &= 0.
\end{aligned}$$

Hint. Group the equations as above; that is, first list the $n - 1$ equations relating the rows, and then list the $n - 1$ equations relating the columns. Prove that the first $n - 1$ equations