for all $u \in \overrightarrow{E}$, f will be affine if we can show that g is linear, and f will be an affine isometry if we can show that g is a linear isometry.

Observe that

$$\begin{aligned} g(v) - g(u) &= \overrightarrow{f(\Omega)f(\Omega + v)} - \overrightarrow{f(\Omega)f(\Omega + u)} \\ &= \overrightarrow{f(\Omega + u)f(\Omega + v)}. \end{aligned}$$

Then, the hypothesis

$$\|\overrightarrow{f(a)f(b)}\| = \|\overrightarrow{ab}\|$$

for all $a, b \in E$, implies that

$$\|g(v) - g(u)\| = \|\overrightarrow{f(\Omega + u)f(\Omega + v)}\| = \|\overrightarrow{(\Omega + u)(\Omega + v)}\| = \|v - u\|.$$

Thus, g preserves the distance. Also, by definition, we have

$$g(0) = 0.$$

Thus, we can apply Lemma 12.12, which shows that g is indeed a linear isometry, and thus f is an affine isometry. $\square$

In order to understand the structure of affine isometries, it is important to investigate the fixed points of an affine map.

27.3 Fixed Points of Affine Maps

Recall that $E\big(1, \overrightarrow{f}\big)$ denotes the eigenspace of the linear map $\overrightarrow{f}$ associated with the scalar 1, that is, the subspace consisting of all vectors $u \in \overrightarrow{E}$ such that $\overrightarrow{f}(u) = u$. Clearly, $\mathrm{Ker}\,\big(\overrightarrow{f} - \mathrm{id}\big) = E\big(1, \overrightarrow{f}\big)$. Given some origin $\Omega \in E$, since

$$f(a) = f(\Omega + \overrightarrow{\Omega a}) = f(\Omega) + \overrightarrow{f}(\overrightarrow{\Omega a}),$$

we have $\overrightarrow{f(\Omega)f(a)} = \overrightarrow{f}(\overrightarrow{\Omega a})$, and thus

$$\overrightarrow{\Omega f(a)} = \overrightarrow{\Omega f(\Omega)} + \overrightarrow{f}(\overrightarrow{\Omega a}).$$

From the above, we get

$$\overrightarrow{\Omega f(a)} - \overrightarrow{\Omega a} = \overrightarrow{\Omega f(\Omega)} + \overrightarrow{f}(\overrightarrow{\Omega a}) - \overrightarrow{\Omega a}.$$

Using this, we show the following lemma, which holds for arbitrary affine spaces of finite dimension and for arbitrary affine maps.