and

$$\eta_{4,k+1} \leq \frac{3}{4}\eta_{4,k}^2, \quad \text{for all} \quad k \geq 1.$$

Deduce from the above that the rate of convergence of $\eta_{i,k}$ is very fast, for $i = 3, 4$ (and $k \geq 1$).

Remark: If we let $\mu_2(a) = a$ for all a and $\eta_{2,k} = \epsilon_k$, we then proved that

$$\eta_{2,k+1} \leq \frac{1}{2}\eta_{2,k}^2, \quad \text{for all} \quad k \geq 1.$$

Problem 41.8. This is a continuation of Problem 41.7.

(1) Prove that for all $n \geq 2$, we have

$$\epsilon_{k+1} = \left(\frac{n-1}{n}\right)\frac{\epsilon_k^2\mu_n(\epsilon_k)}{(\epsilon_k+1)^{n-1}}, \quad \text{for all} \quad k \geq 0,$$

where μ_n is given by

$$\mu_n(a) = \frac{1}{2}n + \sum_{j=1}^{n-4}\frac{1}{n-1}\left((n-1)\binom{n-2}{j} + (n-2)\binom{n-2}{j+1} - \binom{n-2}{j+2}\right)a^j + \frac{n(n-2)}{n-1}a^{n-3} + a^{n-2}.$$

Furthermore, prove that μ_n can be expressed as

$$\mu_n(a) = \frac{1}{2}n + \frac{n(n-2)}{3}a + \sum_{j=2}^{n-4}\frac{(j+1)n}{(j+2)(n-1)}\binom{n-1}{j+1}a^j + \frac{n(n-2)}{n-1}a^{n-3} + a^{n-2}.$$

(2) Prove that for every j, with $1 \leq j \leq n-1$, the coefficient of a^j in $a\mu_n(a)$ is less than or equal to the coefficient of a^j in $(a+1)^{n-1} - 1$, and thus

$$a\mu_n(a) \leq (a+1)^{n-1} - 1, \quad \text{for all} \quad a \geq 0,$$

with strict inequality if $n \geq 3$. In fact, prove that if $n \geq 3$, then for every j, with $3 \leq j \leq n-2$, the coefficient of a^j in $a\mu_n(a)$ is strictly less than the coefficient of a^j in $(a+1)^{n-1} - 1$, and if $n \geq 4$, this also holds for $j = 2$.

Let $\eta_{n,k} = \mu_n(\epsilon_1)\epsilon_k$ $(n \geq 2)$. Prove that

$$\eta_{n,k+1} \leq \left(\frac{n-1}{n}\right)\eta_{n,k}^2, \quad \text{for all} \quad k \geq 1.$$