45.2 Basic Feasible Solutions and Vertices

If the system $Ax = b$ has a solution and if some row of A is a linear combination of other rows, then the corresponding equation is redundant, so we may assume that the rows of A are linearly independent; that is, we may assume that A has rank m, so $m \leq n$.

Definition 45.6. If A is an $m \times n$ matrix, for any nonempty subset K of $\{1, \ldots, n\}$, let A_K be the submatrix of A consisting of the columns of A whose indices belong to K. We denote the jth column of the matrix A by A^j.

Definition 45.7. Given a Linear Program (P_2)

$$\begin{aligned} &\text{maximize} \quad cx \\ &\text{subject to} \quad Ax = b \text{ and } x \geq 0, \end{aligned}$$

where A has rank m, a vector $x \in \mathbb{R}^n$ is a *basic feasible solution* of (P) if $x \in \mathcal{P}(A, b) \neq \emptyset$, and if there is some subset K of $\{1, \ldots, n\}$ of size m such that

(1) The matrix A_K is invertible (that is, the columns of A_K are linearly independent).

(2) $x_j = 0$ for all $j \notin K$.

The subset K is called a *basis* of x. Every index $k \in K$ is called *basic*, and every index $j \notin K$ is called *nonbasic*. Similarly, the columns A^k corresponding to indices $k \in K$ are called *basic*, and the columns A^j corresponding to indices $j \notin K$ are called *nonbasic*. The variables corresponding to basic indices $k \in K$ are called *basic variables*, and the variables corresponding to indices $j \notin K$ are called *nonbasic*.

For example, the linear program

$$\begin{aligned} &\text{maximize} \quad x_1 + x_2 \\ &\text{subject to} \quad x_1 + x_2 + x_3 = 1 \text{ and } x_1 \geq 0, \; x_2 \geq 0, \; x_3 \geq 0, \end{aligned} \qquad (*)$$

has three basic feasible solutions; the basic feasible solution $K = \{1\}$ corresponds to the point $(1, 0, 0)$; the basic feasible solution $K = \{2\}$ corresponds to the point $(0, 1, 0)$; the basic feasible solution $K = \{3\}$ corresponds to the point $(0, 0, 1)$. Each of these points corresponds to the vertices of the slanted purple triangle illustrated in Figure 45.3. The vertices $(1, 0, 0)$ and $(0, 1, 0)$ optimize the objective function with a value of 1.

We now show that if the Standard Linear Program (P_2) as in Definition 45.7 has some feasible solution and is bounded above, then some basic feasible solution is an optimal solution. We follow Matousek and Gardner [123] (Chapter 4, Section 2, Theorem 4.2.3).

First we obtain a more convenient characterization of a basic feasible solution.

Proposition 45.2. *Given any Standard Linear Program (P_2) where $Ax = b$ and A is an $m \times n$ matrix of rank m, for any feasible solution x, if $J_> = \{j \in \{1, \ldots, n\} \mid x_j > 0\}$, then x is a basic feasible solution iff the columns of the matrix $A_{J_>}$ are linearly independent.*