then the vector N^k is given by

$$
\begin{array}{c}
1 \\ \vdots \\ i_1 - 1 \\ i_1 \\ i_1 + 1 \\ \vdots \\ i_r - 1 \\ i_r \\ i_r + 1 \\ \vdots \\ j_k - 1 \\ j_k \\ j_k + 1 \\ \vdots \\ n
\end{array}
\begin{pmatrix}
0 \\ \vdots \\ 0 \\ -\alpha_1 \\ 0 \\ \vdots \\ 0 \\ -\alpha_r \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0
\end{pmatrix}.
$$

The presence of the 1 in position j_k guarantees that $N^1, \ldots, N^{n-r}$ are linearly independent.

As an illustration of the above method, consider the problem of finding a basis of the subspace V of $n \times n$ matrices $A \in \mathrm{M}_n(\mathbb{R})$ satisfying the following properties:

1. The sum of the entries in every row has the same value (say c_1);

2. The sum of the entries in every column has the same value (say c_2).

It turns out that $c_1 = c_2$ and that the $2n-2$ equations corresponding to the above conditions are linearly independent. We leave the proof of these facts as an interesting exercise. It can be shown using the duality theorem (Theorem 11.4) that the dimension of the space V of matrices satisying the above equations is $n^2 - (2n - 2)$. Let us consider the case $n = 4$. There are 6 equations, and the space V has dimension 10. The equations are

$$
\begin{aligned}
a_{11} + a_{12} + a_{13} + a_{14} - a_{21} - a_{22} - a_{23} - a_{24} &= 0 \\
a_{21} + a_{22} + a_{23} + a_{24} - a_{31} - a_{32} - a_{33} - a_{34} &= 0 \\
a_{31} + a_{32} + a_{33} + a_{34} - a_{41} - a_{42} - a_{43} - a_{44} &= 0 \\
a_{11} + a_{21} + a_{31} + a_{41} - a_{12} - a_{22} - a_{32} - a_{42} &= 0 \\
a_{12} + a_{22} + a_{32} + a_{42} - a_{13} - a_{23} - a_{33} - a_{43} &= 0 \\
a_{13} + a_{23} + a_{33} + a_{43} - a_{14} - a_{24} - a_{34} - a_{44} &= 0,
\end{aligned}
$$