The linear map $\overrightarrow{g} - \rho^{-1}\mathrm{id}$ is singular iff ρ^{-1} is an eigenvalue or $\overrightarrow{g}$, and since $\overrightarrow{g} \in \mathbf{O}(E)$ its eigenvalues have modulus 1, so if $\rho \neq 1$ or if $\rho = 1$ is not an eigenvalue of $\overrightarrow{g}$, then $\overrightarrow{g} - \rho^{-1}\mathrm{id}$ is invertible, and then there is a unique $u \in \overrightarrow{E}$ such that

$$(\overrightarrow{g} - \rho^{-1}\mathrm{id})(u) = \rho^{-1}\overrightarrow{f(a)a}.$$

For more details on the use of absolute quadrics to obtain some very sophisticated results, the reader should consult Berger [11, 12], Pedoe [136], Samuel [142], Coxeter [43], Sidler [161], Tisseron [175], Lehmann and Bkouche [115], and, of course, Volume II of Veblen and Young [184], which also explains how some non-Euclidean geometries are obtained by chosing the absolute quadric in an appropriate fashion (after Cayley and Klein).

26.16 Some Applications of Projective Geometry

Projective geometry is definitely a jewel of pure mathematics and one of the major mathematical achievements of the nineteenth century. It turns out to be a prerequisite for algebraic geometry, but to our surprise (and pleasure), it also turns out to have applications in engineering. In this short section we summarize some of these applications.

We first discuss applications of projective geometry to camera calibration, a crucial problem in computer vision. Our brief presentation follows quite closely Trucco and Verri [178] (Chapter 2 and Chapter 6). One should also consult Faugeras [59], or Jain, Katsuri, and Schunck [100].

The *pinhole* (or *perspective*) model of a camera is a typical example from computer vision that can be explained very simply in terms of projective transformations. A pinhole camera consists of a point $\mathbf{O}$ called the *center* or *focus of projection*, and a plane π (not containing $\mathbf{O}$) called the *image plane*. The distance f from the image plane π to the center $\mathbf{O}$ is called the *focal length*. The line through $\mathbf{O}$ and perpendicular to π is called the *optical axis*, and the point $\mathbf{o}$, intersection of the optical axis with the image plane is called the *principal point* or *image center*. The way the camera works is that a point P in 3D space is projected onto the image plane (the film) to a point p via the central projection of center $\mathbf{O}$.

It is assumed that an orthonormal frame $\mathcal{F}_c$ is attached to the camera, with its origin at $\mathbf{O}$ and its z-axis parallel to the optical axis. Such a frame is called the *camera reference frame*. With respect to the camera reference frame, it is very easy to write the equations relating the coordinates (x, y) (omitting $z = f$) of the image p (in the image plane π) of a point P of coordinates (X, Y, Z):

$$x = f\,\frac{X}{Z}, \quad y = f\,\frac{Y}{Z}.$$

Typically, points in 3D space are defined by their coordinates not with respect to the camera reference frame, but with respect to another frame $\mathcal{F}_w$, called the *world reference frame*.