Suppose that A has rank r. If $A = V\Sigma U^\top$ is an SVD for A, prove that σ_k is an eigenvalue of S with corresponding eigenvector $\begin{pmatrix} v_k \\ u_k \end{pmatrix}$ for $k = 1, \ldots, r$, and that $-\sigma_k$ is an eigenvalue of S with corresponding eigenvector $\begin{pmatrix} v_k \\ -u_k \end{pmatrix}$ for $k = 1, \ldots, r$.

Find the remaining $m + n - 2r$ eigenvectors of S associated with the eigenvalue 0.

(4) Prove that these $m + n$ eigenvectors of S are pairwise orthogonal.

Problem 22.2. Let A be a real $m \times n$ matrix of rank r.

(1) Consider the $(m + n) \times (m + n)$ real symmetric matrix

$$S = \begin{pmatrix} 0 & A \\ A^\top & 0 \end{pmatrix}$$

and prove that

$$\begin{pmatrix} I_m & z^{-1}A \\ 0 & I_n \end{pmatrix} \begin{pmatrix} zI_m & -A \\ -A^\top & zI_n \end{pmatrix} = \begin{pmatrix} zI_m - z^{-1}AA^\top & 0 \\ -A^\top & zI_n \end{pmatrix}. \tag{$*$}$$

Use the Equation $(*)$ to prove that if if $n \geq m$, then

$$\det(zI_{m+n} - S) = z^{n-m} \det(z^2 I_m - AA^\top).$$

Permute the two matrices on the lefthand side of Equation $(*)$ to obtain another equation and use this equation to prove that if $m \geq n$, then

$$\det(zI_{m+n} - S) = z^{m-n} \det(z^2 I_n - A^\top A).$$

(2) Prove that the eigenvalues of S are $\pm\sigma_1, \ldots, \pm\sigma_r$, with $m + n - 2r$ additional zeros.

Problem 22.3. Let B be a real bidiagonal matrix of the form

$$B = \begin{pmatrix} a_1 & b_1 & 0 & \cdots & 0 \\ 0 & a_2 & b_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{n-1} & b_{n-1} \\ 0 & 0 & \cdots & 0 & a_n \end{pmatrix}.$$

Let A be the $(2n) \times (2n)$ symmetric matrix

$$A = \begin{pmatrix} 0 & B^\top \\ B & 0 \end{pmatrix},$$

and let P be the permutation matrix given by $P = [e_1, e_{n+1}, e_2, e_{n+2}, \cdots, e_n, e_{2n}]$.