if both (P) and (D) are feasible, $u \in U$ is an optimal solution of (P), $\lambda \in \mathbb{R}^m_+$ is an optimal solution of (D), and $J(u) = G(\lambda)$, then

$$\sum_{i=1}^{m} \lambda_i \varphi_i(u) = 0.$$

In other words, if the constraint φ_i is inactive at u, then $\lambda_i = 0$.

Proof. Since $J(u) = G(\lambda)$ we have

$$\begin{aligned} J(u) &= G(\lambda) \\ &= \inf_{v \in \Omega} \left(J(v) + \sum_{i=1}^{m} \lambda_i \varphi_i(v) \right) && \text{by definition of } G \\ &\leq J(u) + \sum_{i=1}^{m} \lambda_i \varphi_i(u) && \text{the greatest lower bound is a lower bound} \\ &\leq J(u) && \text{since } \lambda_i \geq 0, \varphi_i(u) \leq 0. \end{aligned}$$

which implies that $\sum_{i=1}^{m} \lambda_i \varphi_i(u) = 0$. ☐

Going back to Example 50.5, we see that weak duality says that for any feasible solution u of the Primal Problem (P), that is, some $u \in \mathbb{R}^n$ such that

$$Au \leq b, \;\; u \geq 0,$$

and for any feasible solution $\mu \in \mathbb{R}^m$ of the Dual Problem (D_1), that is,

$$A^\top \mu \geq -c, \;\; \mu \geq 0,$$

we have

$$-b^\top \mu \leq c^\top u.$$

Actually, if u and λ are optimal, then we know from Theorem 47.7 that strong duality holds, namely $-b^\top \mu = c^\top u$, but the proof of this fact is nontrivial.

The following theorem establishes a link between the solutions of the Primal Problem (P) and those of the Dual Problem (D). It also gives sufficient conditions for the duality gap to be zero.

Theorem 50.17. *Consider the Minimization Problem (P):*

$$\begin{aligned} &\text{minimize} && J(v) \\ &\text{subject to} && \varphi_i(v) \leq 0, \quad i = 1, \ldots, m, \end{aligned}$$

where the functions J and φ_i are defined on some open subset Ω of a finite-dimensional Euclidean vector space V (more generally, a real Hilbert space V).