for every $B \in \mathrm{M}_n(\mathbb{C})$. Then it is easy to verify that the above function is the matrix norm subordinate to the vector norm

$$v \mapsto \left\|(UD_\delta)^{-1}v\right\|_\infty.$$

Furthermore, for every $\epsilon > 0$, we can pick δ so that

$$\sum_{j=i+1}^{n} |\delta^{j-i}t_{ij}| \leq \epsilon, \quad 1 \leq i \leq n-1,$$

and by definition of the norm $\| \, \|_\infty$, we get

$$\|A\| \leq \rho(A) + \epsilon,$$

which shows that the norm that we have constructed satisfies the required properties. ☐

Note that equality is generally not possible; consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

for which $\rho(A) = 0 < \|A\|$, since $A \neq 0$.

9.5 Condition Numbers of Matrices

Unfortunately, there exist linear systems $Ax = b$ whose solutions are not stable under small perturbations of either b or A. For example, consider the system

$$\begin{pmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 32 \\ 23 \\ 33 \\ 31 \end{pmatrix}.$$

The reader should check that it has the solution $x = (1, 1, 1, 1)$. If we perturb slightly the right-hand side as $b + \Delta b$, where

$$\Delta b = \begin{pmatrix} 0.1 \\ -0.1 \\ 0.1 \\ -0.1 \end{pmatrix},$$

we obtain the new system

$$\begin{pmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{pmatrix} \begin{pmatrix} x_1 + \Delta x_1 \\ x_2 + \Delta x_2 \\ x_3 + \Delta x_3 \\ x_4 + \Delta x_4 \end{pmatrix} = \begin{pmatrix} 32.1 \\ 22.9 \\ 33.1 \\ 30.9 \end{pmatrix}.$$