We conclude our quick study of affine isometries by proving a result that plays a major role in characterizing the affine isometries. This result may be viewed as a generalization of Chasles's theorem about the direct rigid motions in $\mathbb{E}^3$.

Theorem 27.10. *Let E be a Euclidean affine space of finite dimension n. For every affine isometry $f\colon E \to E$, there is a unique affine isometry $g\colon E \to E$ and a unique translation $t = t_\tau$, with $\overrightarrow{f}(\tau) = \tau$ (i.e., $\tau \in \mathrm{Ker}\,(\overrightarrow{f} - \mathrm{id})$), such that the set $\mathrm{Fix}(g) = \{a \in E \mid g(a) = a\}$ of fixed points of g is a nonempty affine subspace of E of direction*

$$\overrightarrow{G} = \mathrm{Ker}\,\big(\overrightarrow{f} - \mathrm{id}\big) = E\big(1, \overrightarrow{f}\big),$$

and such that

$$f = t \circ g \quad \text{and} \quad t \circ g = g \circ t.$$

Furthermore, we have the following additional properties:

(a) $f = g$ and $\tau = 0$ iff f has some fixed point, i.e., iff $\mathrm{Fix}(f) \neq \emptyset$.

(b) If f has no fixed points, i.e., $\mathrm{Fix}(f) = \emptyset$, then $\dim\big(\mathrm{Ker}\,\big(\overrightarrow{f} - \mathrm{id}\big)\big) \geq 1$.

Proof. The proof rests on the following two key facts:

(1) If we can find some $x \in E$ such that $\overrightarrow{xf(x)} = \tau$ belongs to $\mathrm{Ker}\,\big(\overrightarrow{f} - \mathrm{id}\big)$, we get the existence of g and τ.

(2) $\overrightarrow{E} = \mathrm{Ker}\,\big(\overrightarrow{f} - \mathrm{id}\big) \oplus \mathrm{Im}\,\big(\overrightarrow{f} - \mathrm{id}\big)$, and the spaces $\mathrm{Ker}\,\big(\overrightarrow{f} - \mathrm{id}\big)$ and $\mathrm{Im}\,\big(\overrightarrow{f} - \mathrm{id}\big)$ are orthogonal. This implies the uniqueness of g and τ.

First, we prove that for every isometry $h\colon \overrightarrow{E} \to \overrightarrow{E}$, $\mathrm{Ker}\,(h - \mathrm{id})$ and $\mathrm{Im}\,(h - \mathrm{id})$ are orthogonal and that

$$\overrightarrow{E} = \mathrm{Ker}\,(h - \mathrm{id}) \oplus \mathrm{Im}\,(h - \mathrm{id}).$$

Recall that

$$\dim\big(\overrightarrow{E}\big) = \dim(\mathrm{Ker}\,\varphi) + \dim(\mathrm{Im}\,\varphi),$$

for any linear map $\varphi\colon \overrightarrow{E} \to \overrightarrow{E}$; see Theorem 6.16. To show that we have a direct sum, we prove orthogonality. Let $u \in \mathrm{Ker}\,(h - \mathrm{id})$, so that $h(u) = u$, let $v \in \overrightarrow{E}$, and compute

$$u \cdot (h(v) - v) = u \cdot h(v) - u \cdot v = h(u) \cdot h(v) - u \cdot v = 0,$$

since $h(u) = u$ and h is an isometry.

Next, assume that there is some $x \in E$ such that $\overrightarrow{xf(x)} = \tau$ belongs to the space $\mathrm{Ker}\,\big(\overrightarrow{f} - \mathrm{id}\big)$. If we define $g\colon E \to E$ such that

$$g = t_{(-\tau)} \circ f,$$