Proof. As discussed just after Definition 49.4, by Proposition 48.10, there is a unique continuous linear map $A\colon V \to V$ such that

$$a(u, v) = \langle Au, v\rangle \quad \text{for all } u, v \in V,$$

with $\|A\| = \|a\| = C$, and by the Riesz representation theorem (Proposition 48.9), there is a unique $b \in V$ such that

$$h(v) = \langle b, v\rangle \quad \text{for all } v \in V.$$

Consequently, J can be written as

$$J(v) = \frac{1}{2}\langle Av, v\rangle - \langle b, v\rangle \quad \text{for all } v \in V. \tag{$*_1$}$$

Since $\|A\| = \|a\| = C$, we have $\|Av\| \leq \|A\| \, \|v\| = C \, \|v\|$ for all $v \in V$. Using $(*_1)$, the inequality $(*)$ is equivalent to finding u such that

$$\langle Au, v - u\rangle \geq \langle b, v - u\rangle \quad \text{for all } v \in U. \tag{$*_2$}$$

Let $\rho > 0$ be a constant to be determined later. Then $(*_2)$ is equivalent to

$$\langle \rho b - \rho Au + u - u, v - u\rangle \leq 0 \quad \text{for all } v \in U. \tag{$*_3$}$$

By the projection lemma (Proposition 48.5 (1) and (2)), $(*_3)$ is equivalent to finding $u \in U$ such that

$$u = p_U(\rho b - \rho Au + u). \tag{$*_4$}$$

We are led to finding a fixed point of the function $F\colon U \to U$ given by

$$F(v) = p_U(\rho b - \rho Av + v).$$

By Proposition 48.6, the projection map p_U does not increase distance, so

$$\|F(v_1) - F(v_2)\| \leq \|v_1 - v_2 - \rho(Av_1 - Av_2)\| .$$

Since a is coercive we have

$$a(v, v) \geq \alpha \, \|v\|^2 ,$$

since $a(v, v) = \langle Av, v\rangle$ we have

$$\langle Av, v\rangle \geq \alpha \, \|v\|^2 \quad \text{for all } v \in V, \tag{$*_5$}$$

and since

$$\|Av\| \leq C \, \|v\| \quad \text{for all } v \in V, \tag{$*_6$}$$

we get

$$\begin{aligned}\|F(v_1) - F(v_2)\|^2 &\leq \|v_1 - v_2\|^2 - 2\rho\langle Av_1 - Av_2, v_1 - v_2\rangle + \rho^2 \, \|Av_1 - Av_2\|^2 \\ &\leq \left(1 - 2\rho\alpha + \rho^2 C^2\right) \|v_1 - v_2\|^2 .\end{aligned}$$