We leave as an exercise the fact that every subspace $V \neq E$ of a vector space E is the intersection of all hyperplanes that contain V. We now consider the notion of transpose of a linear map and of a matrix.

11.6 Transpose of a Linear Map and of a Matrix

Given a linear map $f\colon E \to F$, it is possible to define a map $f^\top\colon F^* \to E^*$ which has some interesting properties.

Definition 11.5. Given a linear map $f\colon E \to F$, the *transpose* $f^\top\colon F^* \to E^*$ *of* f is the linear map defined such that

$$f^\top(v^*) = v^* \circ f, \quad \text{for every } v^* \in F^*,$$

as shown in the diagram below:

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[dr, "f^\top(v^*)"'] & F \arrow[d, "v^*"] \\ & K. \end{tikzcd}$$

Equivalently, the linear map $f^\top\colon F^* \to E^*$ is defined such that

$$\langle v^*, f(u)\rangle = \langle f^\top(v^*), u\rangle, \tag{$*$}$$

for all $u \in E$ and all $v^* \in F^*$.

It is easy to verify that the following properties hold:

$$\begin{aligned} (f+g)^\top &= f^\top + g^\top \\ (g \circ f)^\top &= f^\top \circ g^\top \\ \mathrm{id}_E^\top &= \mathrm{id}_{E^*}. \end{aligned}$$

Note the reversal of composition on the right-hand side of $(g \circ f)^\top = f^\top \circ g^\top$.

The equation $(g \circ f)^\top = f^\top \circ g^\top$ implies the following useful proposition.

Proposition 11.8. *If $f\colon E \to F$ is any linear map, then the following properties hold:*

(1) If f is injective, then $f^\top$ is surjective.

(2) If f is surjective, then $f^\top$ is injective.