The natural number r is called the *free rank* or *Betti number* of the module M. The generators $\alpha_1, \ldots, \alpha_m$ of the ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_m$ (defined up to a unit) are often called the *invariant factors* of M (in the notation of Theorem 35.25, the generators of the ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_m$ are denoted by $a_q, \ldots, a_{s+1}$, $s \leq q$).

As corollaries of Theorem 35.25, we obtain again the following facts established in Section 35.1:

1. A finitely generated module over a PID is the direct sum of its torsion module and a free module.

2. A finitely generated torsion-free module over a PID is free.

It turns out that the ideals $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots \subseteq \mathfrak{a}_m \neq A$ are uniquely determined by the module M. Uniqueness proofs found in most books tend to be intricate and not very intuitive. The shortest proof that we are aware of is from Bourbaki [26] (Chapter VII, Section 4), and uses wedge products.

The following preliminary results are needed.

Proposition 35.26. *If A is a commutative ring and if $\mathfrak{a}_1, \ldots, \mathfrak{a}_m$ are ideals of A, then there is an isomorphism*

$$A/\mathfrak{a}_1 \otimes \cdots \otimes A/\mathfrak{a}_m \approx A/(\mathfrak{a}_1 + \cdots + \mathfrak{a}_m).$$

Sketch of proof. We proceed by induction on m. For $m = 2$, we define the map $\varphi\colon A/\mathfrak{a}_1 \times A/\mathfrak{a}_2 \to A/(\mathfrak{a}_1 + \mathfrak{a}_2)$ by

$$\varphi(\overline{a}, \overline{b}) = ab \pmod{\mathfrak{a}_1 + \mathfrak{a}_2}.$$

It is well-defined because if $a' = a + a_1$ and $b' = b + a_2$ with $a_1 \in \mathfrak{a}_1$ and $a_2 \in \mathfrak{a}_2$, then

$$a'b' = (a + a_1)(b + a_2) = ab + ba_1 + aa_2 + a_1a_2,$$

and so

$$a'b' \equiv ab \pmod{\mathfrak{a}_1 + \mathfrak{a}_2}.$$

It is also clear that this map is bilinear, so it induces a linear map $\varphi\colon A/\mathfrak{a}_1 \otimes A/\mathfrak{a}_2 \to A/(\mathfrak{a}_1 + \mathfrak{a}_2)$ such that $\varphi(\overline{a} \otimes \overline{b}) = ab \pmod{\mathfrak{a}_1 + \mathfrak{a}_2}$.

Next, observe that any arbitrary tensor

$$\overline{a}_1 \otimes \overline{b}_1 + \cdots + \overline{a}_n \otimes \overline{b}_n$$

in $A/\mathfrak{a}_1 \otimes A/\mathfrak{a}_2$ can be rewritten as

$$\overline{1} \otimes (\overline{a_1b_1} + \cdots + \overline{a_nb_n}),$$

which is of the form $\overline{1} \otimes \overline{s}$, with $s \in A$. We can use this fact to show that φ is injective and surjective, and thus an isomorphism.