Problem 16.4. Let $\mathrm{Ad}\colon \mathbf{SU}(2) \to \mathbf{GL}(\mathfrak{su}(2))$ be the map defined such that for every $q \in \mathbf{SU}(2)$,

$$\mathrm{Ad}_q(A) = qAq^*, \quad A \in \mathfrak{su}(2),$$

where q^* is the inverse of q (since $\mathbf{SU}(2)$ is a unitary group) Prove that the map Ad_q is an invertible linear map from $\mathfrak{su}(2)$ to itself and that Ad is a group homomorphism.

Problem 16.5. Prove that every Hermitian matrix with zero trace is of the form $x\sigma_3 + y\sigma_2 + z\sigma_1$, with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Check that $\mathbf{i} = i\sigma_3$, $\mathbf{j} = i\sigma_2$, and that $\mathbf{k} = i\sigma_1$.

Problem 16.6. If

$$B = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix},$$

and if we write $\theta = \sqrt{u_1^2 + u_2^2 + u_3^2}$ (with $0 \leq \theta \leq \pi$), then the Rodrigues formula says that

$$e^B = I + \frac{\sin\theta}{\theta}B + \frac{(1-\cos\theta)}{\theta^2}B^2, \quad \theta \neq 0,$$

with $e^0 = I$. Check that $\mathrm{tr}(e^B) = 1 + 2\cos\theta$. Prove that the quaternion q corresponding to the rotation $R = e^B$ (with $B \neq 0$) is given by

$$q = \left[\cos\left(\frac{\theta}{2}\right), \sin\left(\frac{\theta}{2}\right)\left(\frac{u_1}{\theta}, \frac{u_2}{\theta}, \frac{u_3}{\theta}\right)\right].$$

Problem 16.7. For every matrix $A \in \mathfrak{su}(2)$, with

$$A = \begin{pmatrix} iu_1 & u_2 + iu_3 \\ -u_2 + iu_3 & -iu_1 \end{pmatrix},$$

prove that if we write $\theta = \sqrt{u_1^2 + u_2^2 + u_3^2}$, then

$$e^A = \cos\theta I + \frac{\sin\theta}{\theta}A, \quad \theta \neq 0,$$

and $e^0 = I$. Conclude that e^A is a unit quaternion representing the rotation of angle 2θ and axis (u_1, u_2, u_3) (or I when $\theta = k\pi$, $k \in \mathbb{Z}$).

Problem 16.8. Write a `Matlab` program implementing the method of Section 16.4 for finding a unit quaternion corresponding to a rotation matrix.

Problem 16.9. Show that there is a very simple method for producing an orthonormal frame in $\mathbb{R}^4$ whose first vector is any given nonnull vector (a, b, c, d).