Proof. Given any two nonzero polynomials $u, v \in K[X]$, observe that u divides v iff $(v) \subseteq (u)$. Now, (2) can be restated as $(f) \subseteq (d)$, $(g) \subseteq (d)$, and $d \in (f) + (g)$, which is equivalent to $(d) = (f) + (g)$, namely (3).

If (2) holds, since $d = uf + vg$, whenever $h \in K[X]$ divides f and g, then h divides d, and d is a gcd of f and g.

Assume that d is a gcd of f and g. Then, since d divides f and d divides g, we have $(f) \subseteq (d)$ and $(g) \subseteq (d)$, and thus $(f) + (g) \subseteq (d)$, and $(f) + (g)$ is nonempty since f and g are nonzero. By Proposition 30.10, there exists a monic polynomial $d_1 \in K[X]$ such that $(d_1) = (f) + (g)$. Then, d_1 divides both f and g, and since d is a gcd of f and g, then d_1 divides d, which shows that $(d) \subseteq (d_1) = (f) + (g)$. Consequently, $(f) + (g) = (d)$, and (3) holds.

Since $(d) = (f) + (g)$ and f and g are nonzero, the last part of the proposition is obvious. $\square$

As a consequence of Proposition 30.11, two nonzero polynomials $f, g \in K[X]$ are relatively prime iff there exist $u, v \in K[X]$ such that

$$uf + vg = 1.$$

The identity

$$d = uf + vg$$

of part (2) of Proposition 30.11 is often called the *Bezout identity*.

We derive more useful consequences of Proposition 30.11.

Proposition 30.12. *Let K be a field and let $f, g \in K[X]$ be any two nonzero polynomials. For every gcd $d \in K[X]$ of f and g, the following properties hold:*

(1) For every nonzero polynomial $q \in K[X]$, the polynomial dq is a gcd of fq and gq.

(2) For every nonzero polynomial $q \in K[X]$, if q divides f and g, then d/q is a gcd of f/q and g/q.

Proof. (1) By Proposition 30.11 (2), d divides f and g, and there exist $u, v \in K[X]$, such that

$$d = uf + vg.$$

Then, dq divides fq and gq, and

$$dq = ufq + vgq.$$

By Proposition 30.11 (2), dq is a gcd of fq and gq. The proof of (2) is similar. $\square$

The following proposition is used often.