we have

$$\overset{\bullet}{\bigwedge}(V) = \bigoplus_{m\geq 0} V^{\otimes m}/(\mathfrak{I}_a \cap V^{\otimes m}).$$

However, it is easy to check that

$$\bigwedge^m(V) \cong V^{\otimes m}/(\mathfrak{I}_a \cap V^{\otimes m}),$$

so

$$\overset{\bullet}{\bigwedge}(V) \cong \bigwedge(V).$$

When V has finite dimension d, we actually have a finite direct sum (coproduct)

$$\bigwedge(V) = \bigoplus_{m=0}^{d} \bigwedge^m(V),$$

and since each $\bigwedge^m(V)$ has dimension $\binom{d}{m}$, we deduce that

$$\dim(\bigwedge(V)) = 2^d = 2^{\dim(V)}.$$

The multiplication, $\wedge\colon \bigwedge^m(V) \times \bigwedge^n(V) \to \bigwedge^{m+n}(V)$, is skew-symmetric in the following precise sense:

Proposition 34.12. *For all $\alpha \in \bigwedge^m(V)$ and all $\beta \in \bigwedge^n(V)$, we have*

$$\beta \wedge \alpha = (-1)^{mn} \alpha \wedge \beta.$$

Proof. Since $v \wedge u = -u \wedge v$ for all $u, v \in V$, Proposition 34.12 follows by induction. □

Since $\alpha \wedge \alpha = 0$ for every *simple* (also called *decomposable*) tensor $\alpha = u_1 \wedge \cdots \wedge u_n$, it seems natural to infer that $\alpha \wedge \alpha = 0$ for *every* tensor $\alpha \in \bigwedge(V)$. If we consider the case where $\dim(V) \leq 3$, we can indeed prove the above assertion. However, if $\dim(V) \geq 4$, the above fact is generally false! For example, when $\dim(V) = 4$, if (u_1, u_2, u_3, u_4) is a basis for V, for $\alpha = u_1 \wedge u_2 + u_3 \wedge u_4$, we check that

$$\alpha \wedge \alpha = 2u_1 \wedge u_2 \wedge u_3 \wedge u_4,$$

which is nonzero. However, if $\alpha \in \bigwedge^m E$ with m odd, since m^2 is also odd, we have

$$\alpha \wedge \alpha = (-1)^{m^2} \alpha \wedge \alpha = -\alpha \wedge \alpha,$$

so indeed $\alpha \wedge \alpha = 0$ (if K is not a field of characteristic 2).