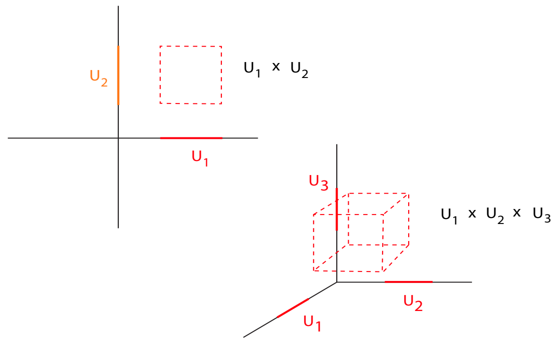

Figure 37.12: Examples of open sets in the product topology for $\mathbb{R}^2$ and $\mathbb{R}^3$ induced by the Euclidean metric.

It is easy to show that

$$\begin{aligned} d_\infty((x_1,\ldots,x_n),(y_1,\ldots,y_n)) &\leq d_2((x_1,\ldots,x_n),(y_1,\ldots,y_n)) \leq d_1((x_1,\ldots,x_n),(y_1,\ldots,y_n)) \\ &\leq nd_\infty((x_1,\ldots,x_n),(y_1,\ldots,y_n)), \end{aligned}$$

so these distances define the same topology, which is the product topology.

If each $(E_i, \|\ \|_{E_i})$ is a normed vector space, there are three natural norms that can be defined on $E_1 \times \cdots \times E_n$:

$$\begin{aligned} \|(x_1,\ldots,x_n)\|_1 &= \|x_1\|_{E_1} + \cdots + \|x_n\|_{E_n}, \\ \|(x_1,\ldots,x_n)\|_2 &= \left(\|x_1\|_{E_1}^2 + \cdots + \|x_n\|_{E_n}^2\right)^{\frac{1}{2}}, \\ \|(x_1,\ldots,x_n)\|_\infty &= \max\left\{\|x_1\|_{E_1},\ldots,\|x_n\|_{E_n}\right\}. \end{aligned}$$

It is easy to show that

$$\|(x_1,\ldots,x_n)\|_\infty \leq \|(x_1,\ldots,x_n)\|_2 \leq \|(x_1,\ldots,x_n)\|_1 \leq n\|(x_1,\ldots,x_n)\|_\infty,$$

so these norms define the same topology, which is the product topology. It can also be verified that when $E_i = \mathbb{R}$, with the standard topology induced by $|x-y|$, the topology product on $\mathbb{R}^n$ is the standard topology induced by the Euclidean norm.

Definition 37.13. Two metrics d and d' on a space E are *equivalent* if they induce the same topology $\mathcal{O}$ on E (i.e., they define the same family $\mathcal{O}$ of open sets). Similarly, two norms $\|\ \|$ and $\|\ \|'$ on a space E are *equivalent* if they induce the same topology $\mathcal{O}$ on E.

Given a topological space $(E,\mathcal{O})$, it is often useful, as in Proposition 37.7, to define the topology $\mathcal{O}$ in terms of a subfamily $\mathcal{B}$ of subsets of E.