Fortunately, every $n \times m$-matrix A can be written as

$$A = VDU^\top$$

where U and V are orthogonal and D is a rectangular diagonal matrix with non-negative entries (*singular value decomposition, or SVD*); see Chapter 22.

The SVD can be used to solve an inconsistent system. It is shown in Chapter 23 that there is a vector x of smallest norm minimizing $\|Ax - b\|_2$. It is given by the (Penrose) *pseudo-inverse* of A (itself given by the SVD).

It has been observed that solving in the least-squares sense may give too much weight to "outliers," that is, points clearly outside the best-fit plane. In this case, it is preferable to minimize (the ℓ^1-norm)

$$\sum_{i=1}^{n} |ax_i + by_i + d - z_i|.$$

This does not appear to be a linear problem, but we can use a trick to convert this minimization problem into a linear program (which means a problem involving linear constraints).

Note that $|x| = \max\{x, -x\}$. So by introducing new variables $e_1, \ldots, e_n$, our minimization problem is equivalent to the linear program (LP):

$$\begin{aligned} \text{minimize} \qquad & e_1 + \cdots + e_n \\ \text{subject to} \qquad & ax_i + by_i + d - z_i \leq e_i \\ & -(ax_i + by_i + d - z_i) \leq e_i \\ & 1 \leq i \leq n. \end{aligned}$$

Observe that the constraints are equivalent to

$$e_i \geq |ax_i + by_i + d - z_i|, \qquad 1 \leq i \leq n.$$

For an optimal solution, we must have equality, since otherwise we could decrease some e_i and get an even better solution. Of course, we are no longer dealing with "pure" linear algebra, since our constraints are inequalities.

We prefer not getting into linear programming right now, but the above example provides a good reason to learn more about linear programming!

9.7 Limits of Sequences and Series

If $x \in \mathbb{R}$ or $x \in \mathbb{C}$ and if $|x| < 1$, it is well known that the sums $\sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$ converge to the limit $1/(1-x)$ when n goes to infinity, and we write

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}.$$