shows that $\varphi\colon E \times E \to \mathbb{C}$ is continuous, and thus, that $\| \, \|$ is continuous.

If $\langle E, \varphi \rangle$ is only pre-Hilbertian, $\|u\|$ is called a *seminorm*. In this case, the condition

$$\|u\| = 0 \quad \text{implies} \quad u = 0$$

is not necessarily true. However, the Cauchy–Schwarz inequality shows that if $\|u\| = 0$, then $u \cdot v = 0$ for all $v \in E$.

Remark: As in the case of real vector spaces, a norm on a complex vector space is induced by some positive definite Hermitian product $\langle -, - \rangle$ iff it satisfies the *parallelogram law*:

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2).$$

This time the Hermitian product is recovered using the polarization identity from Proposition 14.1:

$$4\langle u, v \rangle = \|u + v\|^2 - \|u - v\|^2 + i \|u + iv\|^2 - i \|u - iv\|^2 .$$

It is easy to check that $\langle u, u \rangle = \|u\|^2$, and

$$\begin{aligned} \langle v, u \rangle &= \overline{\langle u, v \rangle} \\ \langle iu, v \rangle &= i \langle u, v \rangle, \end{aligned}$$

so it is enough to check linearity in the variable u, and only for real scalars. This is easily done by applying the proof from Section 12.1 to the real and imaginary part of $\langle u, v \rangle$; the details are left as an exercise.

We will now basically mirror the presentation of Euclidean geometry given in Chapter 12 rather quickly, leaving out most proofs, except when they need to be seriously amended.

14.2 Orthogonality, Duality, Adjoint of a Linear Map

In this section we assume that we are dealing with Hermitian spaces. We denote the Hermitian inner product by $u \cdot v$ or $\langle u, v \rangle$. The concepts of orthogonality, orthogonal family of vectors, orthonormal family of vectors, and orthogonal complement of a set of vectors are unchanged from the Euclidean case (Definition 12.2).

For example, the set $\mathcal{C}[-\pi, \pi]$ of continuous functions $f\colon [-\pi, \pi] \to \mathbb{C}$ is a Hermitian space under the product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) \overline{g(x)} dx,$$

and the family $(e^{ikx})_{k \in \mathbb{Z}}$ is orthogonal.

Propositions 12.4 and 12.5 hold without any changes. It is easy to show that

$$\left\| \sum_{i=1}^{n} u_i \right\|^2 = \sum_{i=1}^{n} \|u_i\|^2 + \sum_{1 \leq i < j \leq n} 2\Re(u_i \cdot u_j).$$