In general, there is no closed-form formula for the exponential e^A of a matrix A, but for skew symmetric matrices of dimension 2 and 3, there are explicit formulae. Everyone should enjoy computing the exponential e^A where

$$A = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}.$$

If we write

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

then

$$A = \theta J$$

The key property is that

$$J^2 = -I.$$

Proposition 9.22. *If $A = \theta J$, then*

$$e^A = \cos\theta I + \sin\theta J = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Proof. We have

$$\begin{aligned} A^{4n} &= \theta^{4n} I_2, \\ A^{4n+1} &= \theta^{4n+1} J, \\ A^{4n+2} &= -\theta^{4n+2} I_2, \\ A^{4n+3} &= -\theta^{4n+3} J, \end{aligned}$$

and so

$$e^A = I_2 + \frac{\theta}{1!}J - \frac{\theta^2}{2!}I_2 - \frac{\theta^3}{3!}J + \frac{\theta^4}{4!}I_2 + \frac{\theta^5}{5!}J - \frac{\theta^6}{6!}I_2 - \frac{\theta^7}{7!}J + \cdots.$$

Rearranging the order of the terms, we have

$$e^A = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) I_2 + \left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right) J.$$

We recognize the power series for $\cos\theta$ and $\sin\theta$, and thus

$$e^A = \cos\theta I_2 + \sin\theta J,$$

that is

$$e^A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

as claimed. $\square$