

Problem 31.7. Let $f: E \rightarrow E$ be a linear map on a finite-dimensional vector space. Prove that if f has rank 1, then either f is diagonalizable or f is nilpotent but not both.

Problem 31.8. Find the Jordan form of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Problem 31.9. Let N be a 3×3 nilpotent matrix over \mathbb{C} . Prove that the matrix $A = I + (1/2)N - (1/8)N^2$ satisfies the equation

$$A^2 = I + N.$$

In other words, A is a square root of $I + N$.

Generalize the above fact to any $n \times n$ nilpotent matrix N over \mathbb{C} using the binomial series for $(1 + t)^{1/2}$.

Problem 31.10. Let K be an algebraically closed field (for example, $K = \mathbb{C}$). Prove that every 4×4 matrix is similar to a Jordan matrix of the following form:

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}, \quad \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}, \quad \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix},$$

$$\begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \mu & 1 \\ 0 & 0 & 0 & \mu \end{pmatrix}, \quad \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}.$$

Problem 31.11. In this problem the field K is of characteristic 0. Consider an $(r \times r)$ Jordan block

$$J_r(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}.$$

Prove that for any polynomial $f(X)$, we have

$$f(J_r(\lambda)) = \begin{pmatrix} f(\lambda) & f_1(\lambda) & f_2(\lambda) & \cdots & f_{r-1}(\lambda) \\ 0 & f(\lambda) & f_1(\lambda) & \cdots & f_{r-2}(\lambda) \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & f_1(\lambda) \\ 0 & 0 & 0 & \cdots & f(\lambda) \end{pmatrix},$$