Proposition 29.5 also holds for sesquilinear forms and their matrix representations.

Observe that if φ is a Hermitian form ($E = F$) and if K does not have characteristic 2, then by Theorem 29.10, there is a basis of E with respect to which the matrix M representing φ is a diagonal matrix. If $K = \mathbb{C}$, then these entries are real, and this allows us to classify completely the Hermitian forms.

Proposition 29.11. *Given any Hermitian form $\varphi\colon E \times E \to \mathbb{C}$ with $\dim(E) = n$, there is a basis $(e_1, \ldots, e_n)$ of E such that*

$$\Phi\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{p} x_i^2 - \sum_{i=p+1}^{p+q} x_i^2,$$

with $0 \leq p, q$ and $p + q \leq n$.

The proof of Proposition 29.11 is the same as the real case of Proposition 29.6. Sylvester's inertia law (Proposition 29.7) also holds for Hermitian forms: p and q only depend on φ.

29.3 Orthogonality

In this section we assume that we are dealing with a sesquilinear form $\varphi\colon E \times F \to K$. We allow the automorphism $\lambda \mapsto \overline{\lambda}$ to be the identity, in which case φ is a bilinear form. This way, we can deal with properties shared by bilinear forms and sesquilinear forms in a uniform fashion. Orthogonality is such a property.

Definition 29.12. Given a sesquilinear form $\varphi\colon E \times F \to K$, we say that two vectors $u \in E$ and $v \in F$ are *orthogonal* (or *conjugate*) if $\varphi(u, v) = 0$. Two subsets $E' \subseteq E$ and $F' \subseteq F$ are *orthogonal* if $\varphi(u, v) = 0$ for all $u \in E'$ and all $v \in F'$. Given a subspace U of E, the *right orthogonal space* of U, denoted $U^{\perp}$, is the subspace of F given by

$$U^{\perp} = \{v \in F \mid \varphi(u, v) = 0 \quad \text{for all } u \in U\},$$

and given a subspace V of F, the *left orthogonal space* of V, denoted $V^{\perp}$, is the subspace of E given by

$$V^{\perp} = \{u \in E \mid \varphi(u, v) = 0 \quad \text{for all } v \in V\}.$$

When E and F are distinct, there is little chance of confusing the right orthogonal subspace $U^{\perp}$ of a subspace U of E and the left orthogonal subspace $V^{\perp}$ of a subspace V of F. However, if $E = F$, then $\varphi(u, v) = 0$ *does not necessarily imply* that $\varphi(v, u) = 0$, that is, orthogonality is not necessarily symmetric. Thus, if both U and V are subsets of E, there is a notational ambiguity if $U = V$. In this case, we may write $U^{\perp_r}$ for the right orthogonal and $U^{\perp_l}$ for the left orthogonal.

The above discussion brings up the following point: When is orthogonality symmetric?