Chapter 12

Euclidean Spaces

Rien n'est beau que le vrai.

—Hermann Minkowski

12.1 Inner Products, Euclidean Spaces

So far the framework of vector spaces allows us to deal with ratios of vectors and linear combinations, but there is no way to express the notion of angle or to talk about orthogonality of vectors. A Euclidean structure allows us to deal with *metric notions* such as angles, orthogonality, and length (or distance).

This chapter covers the bare bones of Euclidean geometry. Deeper aspects of Euclidean geometry are investigated in Chapter 13. One of our main goals is to give the basic properties of the transformations that preserve the Euclidean structure, rotations and reflections, since they play an important role in practice. Euclidean geometry is the study of properties invariant under certain affine maps called *rigid motions*. Rigid motions are the maps that preserve the distance between points.

We begin by defining inner products and Euclidean spaces. The Cauchy–Schwarz inequality and the Minkowski inequality are shown. We define orthogonality of vectors and of subspaces, orthogonal bases, and orthonormal bases. We prove that every finite-dimensional Euclidean space has orthonormal bases. The first proof uses duality and the second one the Gram–Schmidt orthogonalization procedure. The QR-decomposition for invertible matrices is shown as an application of the Gram–Schmidt procedure. Linear isometries (also called orthogonal transformations) are defined and studied briefly. We conclude with a short section in which some applications of Euclidean geometry are sketched. One of the most important applications, the method of least squares, is discussed in Chapter 23.

For a more detailed treatment of Euclidean geometry see Berger [11, 12], Snapper and Troyer [162], or any other book on geometry, such as Pedoe [136], Coxeter [44], Fresnel [65], Tisseron [175], or Cagnac, Ramis, and Commeau [32]. Serious readers should consult Emil