Theorem 54.7. *For every optimal solution* $(w, b, \eta, \epsilon, \xi)$ *of Problem* (SVM_{s3}) *with* $w \neq 0$ *and* $\eta > 0$, *if*

$$(p_{sf} + q_{sf})/(p+q) < \nu < 1,$$

then some u_{i_0} *or some* v_{j_0} *is a support vector.*

The proof proceeds by contradiction using Proposition 54.6 (for a very similar proof, see the proof of Theorem 54.3).

54.12 Solving SVM (SVM_{s3}) Using ADMM

In order to solve (SVM_{s3}) using ADMM we need to write the matrix corresponding to the constraints in equational form,

$$\begin{aligned} \sum_{i=1}^{p} \lambda_i + \sum_{j=1}^{q} \mu_j &= K_m \\ \lambda_i + \alpha_i &= K_s, \quad i = 1, \ldots, p \\ \mu_j + \beta_j &= K_s, \quad j = 1, \ldots, q \end{aligned}$$

with $K_m = (p+q)K_s\nu$. This is the $(p+q+1) \times 2(p+q)$ matrix A given by

$$A = \begin{pmatrix} \mathbf{1}_p^\top & \mathbf{1}_q^\top & 0_p^\top & 0_q^\top \\ I_p & 0_{p,q} & I_p & 0_{p,q} \\ 0_{q,p} & I_q & 0_{q,p} & I_q \end{pmatrix}.$$

We leave it as an exercise to prove that A has rank $p+q+1$. The right-hand side is

$$c = \begin{pmatrix} K_m \\ K_s \mathbf{1}_{p+q} \end{pmatrix}.$$

The symmetric positive semidefinite $(p+q)\times(p+q)$ matrix P defining the quadratic functional is

$$P = X^\top X + \begin{pmatrix} \mathbf{1}_p\mathbf{1}_p^\top & -\mathbf{1}_p\mathbf{1}_q^\top \\ -\mathbf{1}_q\mathbf{1}_p^\top & \mathbf{1}_q\mathbf{1}_q^\top \end{pmatrix}, \quad \text{with} \quad X = \begin{pmatrix} -u_1 & \cdots & -u_p & v_1 & \cdots & v_q \end{pmatrix},$$

and

$$q = 0_{p+q}.$$

Since there are $2(p+q)$ Lagrange multipliers $(\lambda, \mu, \alpha, \beta)$, the $(p+q) \times (p+q)$ matrix P must be augmented with zero's to make it a $2(p+q) \times 2(p+q)$ matrix P_a given by

$$P_a = \begin{pmatrix} P & 0_{p+q,p+q} \\ 0_{p+q,p+q} & 0_{p+q,p+q} \end{pmatrix},$$