Theorem 51.18. *Let $f\colon \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ be a proper convex function. For any $x \notin \mathrm{dom}(f)$, we have $\partial f(x) = \emptyset$. For any $x \in \mathbf{relint}(\mathrm{dom}(f))$, we have $\partial f(x) \neq \emptyset$, the map $y \mapsto f'(x; y)$ is convex, closed and proper, and*

$$f'(x; y) = \sup_{u\in\partial f(x)} \langle y, u\rangle = \delta^*(y|\partial f(x)) \quad \textit{for all } y \in \mathbb{R}^n.$$

The subdifferential $\partial f(x)$ is nonempty and bounded (also closed and convex) if and only if $x \in \mathrm{int}(\mathrm{dom}(f))$, in which case $f'(x; y)$ is finite for all $y \in \mathbb{R}^n$.

Theorem 51.18 is proven in Rockafellar [138] (Theorem 23.4). If we write

$$\mathrm{dom}(\partial f) = \{x \in \mathbb{R}^n \mid \partial f(x) \neq \emptyset\},$$

then Theorem 51.18 implies that

$$\mathbf{relint}(\mathrm{dom}(f)) \subseteq \mathrm{dom}(\partial f) \subseteq \mathrm{dom}(f).$$

However, $\mathrm{dom}(\partial f)$ is not necessarily convex as shown by the following counterexample.

Example 51.11. Consider the proper convex function defined on $\mathbb{R}^2$ given by

$$f(x, y) = \max\{g(x), |y|\},$$

where

$$g(x) = \begin{cases} 1 - \sqrt{x} & \text{if } x \geq 0 \\ +\infty & \text{if } x < 0. \end{cases}$$

See Figure 51.21. It is easy to see that $\mathrm{dom}(f) = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0\}$, but $\mathrm{dom}(\partial f) = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0\} - \{(0, y) \mid -1 < y < 1\}$, which is not convex.

The following theorem is important because it tells us when a convex function is differentiable in terms of its subdifferential, as shown in Rockafellar [138] (Theorem 25.1).

Theorem 51.19. *Let f be a convex function on $\mathbb{R}^n$, and let $x \in \mathbb{R}^n$ such that $f(x)$ is finite. If f is differentiable at x then $\partial f(x) = \{\nabla f_x\}$ (where ∇f_x is the gradient of f at x) and we have*

$$f(z) \geq f(x) + \langle z - x, \nabla f_x\rangle \quad \textit{for all } z \in \mathbb{R}^n.$$

Conversely, if $\partial f(x)$ consists of a single vector, then $\partial f(x) = \{\nabla f_x\}$ and f is differentiable at x.

The first direction is easy to prove. Indeed, if f is differentiable at x, then

$$f'(x; y) = \langle y, \nabla f_x\rangle \quad \text{for all } y \in \mathbb{R}^n,$$

so by Proposition 51.16, a vector u is a subgradient at x iff

$$\langle y, \nabla f_x\rangle \geq \langle y, u\rangle \quad \text{for all } y \in \mathbb{R}^n,$$

so $\langle y, \nabla f_x - u\rangle \geq 0$ for all y, which implies that $u = \nabla f_x$.

We obtain the following corollary.