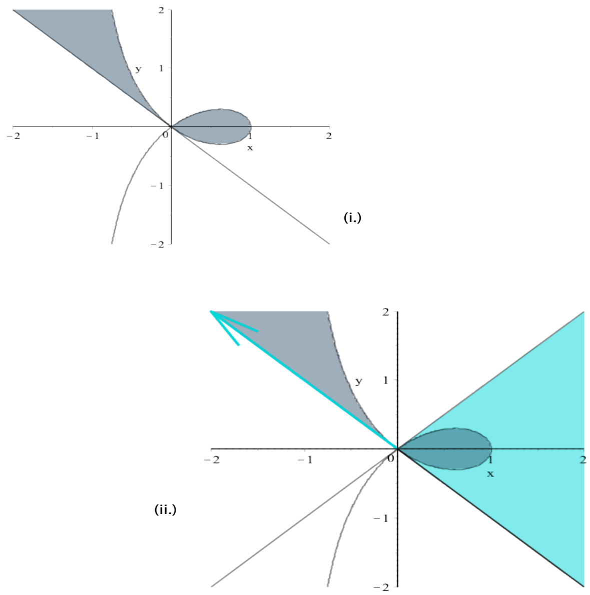

Figure 50.4: Figure (i.) illustrates U as the shaded gray region which lies between the line $y = -x$ and nodal cubic. Figure (ii.) shows the cone of feasible directions, $C(0)$, as the union of the turquoise triangular cone and the turquoise directional ray $(-1, 1)$.

Proposition 50.1. *Let U be any nonempty subset of a normed vector space V.*

(1) For any $u \in U$, the cone $C(u)$ of feasible directions at u is closed.

(2) Let $J\colon \Omega \to \mathbb{R}$ be a function defined on an open subset Ω containing U. If J has a local minimum with respect to the set U at a point $u \in U$, and if J'_u exists at u, then

$$J'_u(v - u) \geq 0 \quad \text{for all } v \in u + C(u).$$

Proof. (1) Let $(w_n)_{n\geq 0}$ be a sequence of vectors $w_n \in C(u)$ converging to a limit $w \in V$. We may assume that $w \neq 0$, since $0 \in C(u)$ by definition, and thus we may also assume that $w_n \neq 0$ for all $n \geq 0$. By definition, for every $n \geq 0$, there is a sequence $(u_k^n)_{k\geq 0}$ of vectors in V and some $w_n \neq 0$ such that

(1) $u_k^n \in U$ and $u_k^n \neq u$ for all $k \geq 0$, and $\lim_{k\mapsto\infty} u_k^n = u$.

(2) There is a sequence $(\delta_k^n)_{k\geq 0}$ of vectors $\delta_k^n \in V$ such that

$$u_k^n = u + \|u_k^n - u\| \frac{w_n}{\|w_n\|} + \|u_k^n - u\| \, \delta_k^n, \quad \lim_{k\mapsto\infty} \delta_k^n = 0, \;\; w_n \neq 0.$$

Let $(\epsilon_n)_{n\geq 0}$ be a sequence of real numbers $\epsilon_n > 0$ such that $\lim_{n\mapsto\infty} \epsilon_n = 0$ (for example, $\epsilon_n = 1/(n+1)$). Due to the convergence of the sequences (u_k^n) and (δ_k^n) for every fixed n,