iff λ is a zero of $\chi_f(X)$. Therefore, the minimal and the characteristic polynomials have the same zeros (in K), except for multiplicities.

Proof. First assume that $m(\lambda) = 0$ (with $\lambda \in K$, and writing m instead of m_f). If so, using polynomial division, m can be factored as

$$m = (X - \lambda)q,$$

with $\deg(q) < \deg(m)$. Since m is the minimal polynomial, $q(f) \neq 0$, so there is some nonzero vector $v \in E$ such that $u = q(f)(v) \neq 0$. But then, because m is the minimal polynomial,

$$\begin{aligned} 0 &= m(f)(v) \\ &= (f - \lambda \mathrm{id})(q(f)(v)) \\ &= (f - \lambda \mathrm{id})(u), \end{aligned}$$

which shows that λ is an eigenvalue of f.

Conversely, assume that $\lambda \in K$ is an eigenvalue of f. This means that for some $u \neq 0$, we have $f(u) = \lambda u$. Now it is easy to show that

$$m(f)(u) = m(\lambda)u,$$

and since m is the minimal polynomial of f, we have $m(f)(u) = 0$, so $m(\lambda)u = 0$, and since $u \neq 0$, we must have $m(\lambda) = 0$. ☐

Proposition 31.2. *Let $f \colon E \to E$ be a linear map on some finite-dimensional vector space E. If f diagonalizable, then its minimal polynomial is a product of distinct factors of degree 1.*

Proof. If we assume that f is diagonalizable, then its eigenvalues are all in K, and if $\lambda_1, \ldots, \lambda_k$ are the distinct eigenvalues of f, and then by Proposition 31.1, the minimal polynomial m of f must be a product of powers of the polynomials $(X - \lambda_i)$. Actually, we claim that

$$m = (X - \lambda_1) \cdots (X - \lambda_k).$$

For this we just have to show that m annihilates f. However, for any eigenvector u of f, one of the linear maps $f - \lambda_i \mathrm{id}$ sends u to 0, so

$$m(f)(u) = (f - \lambda_1 \mathrm{id}) \circ \cdots \circ (f - \lambda_k \mathrm{id})(u) = 0.$$

Since E is spanned by the eigenvectors of f, we conclude that

$$m(f) = 0.$$ ☐

It turns out that the converse of Proposition 31.2 is true, but this will take a little work to establish it.