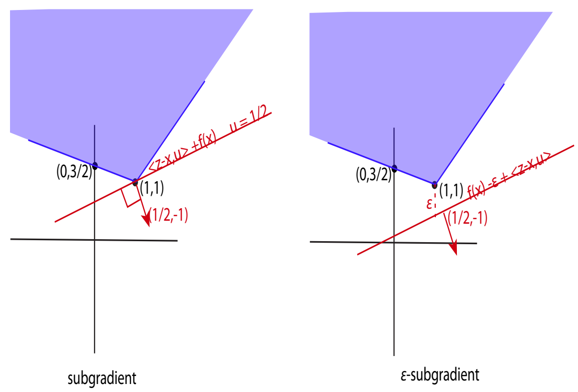

Figure 51.23: Let $f\colon \mathbb{R} \to \mathbb{R} \cup \{-\infty, +\infty\}$ be the piecewise function defined by $f(x) = x + 1$ for $x \geq 1$ and $f(x) = -\frac{1}{2}x + \frac{3}{2}$ for $x < 1$. Its epigraph is the shaded blue region in $\mathbb{R}^2$. The line $\frac{1}{2}(x-1)+1$ (with normal $(\frac{1}{2}, -1)$ is a supporting hyperplane to the graph of $f(x)$ at $(1,1)$ while the line $\frac{1}{2}(x-1)+1-\epsilon$ is the hyperplane associated with the ϵ-subgradient at $x = 1$ and shows that $u = \frac{1}{2} \in \partial_\epsilon f(x)$.

The set $\partial_\epsilon f(x)$ can be defined in terms of the conjugate of the function h_x given by

$$h_x(y) = f(x+y) - f(x), \quad \text{for all } y \in \mathbb{R}^n.$$

Proposition 51.32. *Let $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be any proper convex function. For any $\epsilon > 0$, if h_x is given by*

$$h_x(y) = f(x+y) - f(x), \quad \textit{for all } y \in \mathbb{R}^n,$$

then

$$h_x^*(y) = f^*(y) + f(x) - \langle x, y\rangle \quad \textit{for all } y \in \mathbb{R}^n$$

and

$$\partial_\epsilon f(x) = \{u \in \mathbb{R}^n \mid h_x^*(u) \leq \epsilon\}.$$

Proof. We have

$$\begin{aligned}
h_x^*(y) &= \sup_{z\in\mathbb{R}^n} (\langle y, z\rangle - h_x(z)) \\
&= \sup_{z\in\mathbb{R}^n} (\langle y, z\rangle - f(x+z) + f(x)) \\
&= \sup_{x+z\in\mathbb{R}^n} (\langle y, x+z\rangle - f(x+z) - \langle y, x\rangle + f(x)) \\
&= f^*(y) + f(x) - \langle x, y\rangle.
\end{aligned}$$

Observe that $u \in \partial_\epsilon f(x)$ iff for every $y \in \mathbb{R}^n$,

$$f(x+y) \geq f(x) - \epsilon + \langle y, u\rangle$$