we get the $n \times m$ system

$$\frac{\partial J}{\partial x_1}(u) + \lambda_1 \frac{\partial \varphi_1}{\partial x_1}(u) + \cdots + \lambda_m \frac{\partial \varphi_m}{\partial x_1}(u) = 0$$
$$\vdots$$
$$\frac{\partial J}{\partial x_n}(u) + \lambda_1 \frac{\partial \varphi_1}{\partial x_n}(u) + \cdots + \lambda_m \frac{\partial \varphi_m}{\partial x_n}(u) = 0,$$

and it is important to note that the matrix of this system is the *transpose* of the Jacobian matrix of φ at u. If we write $\mathrm{Jac}(\varphi)(u) = \big((\partial \varphi_i / \partial x_j)(u)\big)$ for the Jacobian matrix of φ (at u), then the above system is written in matrix form as

$$\nabla J(u) + (\mathrm{Jac}(\varphi)(u))^\top \lambda = 0,$$

where λ is viewed as a column vector, and the Lagrangian is equal to

$$L(u, \lambda) = J(u) + (\varphi_1(u), \ldots, \varphi_m(u))\lambda.$$

The beauty of the Lagrangian is that the constraints $\{\varphi_i(v) = 0\}$ have been incorporated into the function $L(v, \lambda)$, and that the necessary condition for the existence of a constrained local extremum of J is reduced to the necessary condition for the existence of a local extremum of the *unconstrained* L.

However, one should be careful to check that the assumptions of Theorem 40.2 are satisfied (in particular, the linear independence of the linear forms $d\varphi_i$).

Example 40.1. For example, let $J\colon \mathbb{R}^3 \to \mathbb{R}$ be given by

$$J(x, y, z) = x + y + z^2$$

and $g\colon \mathbb{R}^3 \to \mathbb{R}$ by

$$g(x, y, z) = x^2 + y^2.$$

Since $g(x, y, z) = 0$ iff $x = y = 0$, we have $U = \{(0, 0, z) \mid z \in \mathbb{R}\}$ and the restriction of J to U is given by

$$J(0, 0, z) = z^2,$$

which has a minimum for $z = 0$. However, a "blind" use of Lagrange multipliers would require that there is some λ so that

$$\frac{\partial J}{\partial x}(0, 0, z) = \lambda \frac{\partial g}{\partial x}(0, 0, z), \quad \frac{\partial J}{\partial y}(0, 0, z) = \lambda \frac{\partial g}{\partial y}(0, 0, z), \quad \frac{\partial J}{\partial z}(0, 0, z) = \lambda \frac{\partial g}{\partial z}(0, 0, z),$$

and since

$$\frac{\partial g}{\partial x}(x, y, z) = 2x, \quad \frac{\partial g}{\partial y}(x, y, z) = 2y, \quad \frac{\partial g}{\partial z}(0, 0, z) = 0,$$