for every $v \in \overrightarrow{E}$ is a linear map $\overrightarrow{f} : \overrightarrow{E} \to \overrightarrow{E'}$. Indeed, we can write

$$a + \lambda v = \lambda(a + v) + (1 - \lambda)a,$$

since $a + \lambda v = a + \lambda\overrightarrow{a(a + v)} + (1 - \lambda)\overrightarrow{aa}$, and also

$$a + u + v = (a + u) + (a + v) - a,$$

since $a + u + v = a + \overrightarrow{a(a + u)} + \overrightarrow{a(a + v)} - \overrightarrow{aa}$. Since f preserves barycenters, we get

$$f(a + \lambda v) = \lambda f(a + v) + (1 - \lambda)f(a).$$

If we recall that $x = \sum_{i\in I} \lambda_i a_i$ is the barycenter of a family $((a_i, \lambda_i))_{i\in I}$ of weighted points (with $\sum_{i\in I} \lambda_i = 1$) iff

$$\overrightarrow{bx} = \sum_{i\in I} \lambda_i \overrightarrow{ba_i} \quad \text{for every } b \in E,$$

we get

$$\overrightarrow{f(a)f(a + \lambda v)} = \lambda\overrightarrow{f(a)f(a + v)} + (1 - \lambda)\overrightarrow{f(a)f(a)} = \lambda\overrightarrow{f(a)f(a + v)},$$

showing that $\overrightarrow{f}(\lambda v) = \lambda\overrightarrow{f}(v)$. We also have

$$f(a + u + v) = f(a + u) + f(a + v) - f(a),$$

from which we get

$$\overrightarrow{f(a)f(a + u + v)} = \overrightarrow{f(a)f(a + u)} + \overrightarrow{f(a)f(a + v)},$$

showing that $\overrightarrow{f}(u + v) = \overrightarrow{f}(u) + \overrightarrow{f}(v)$. Consequently, $\overrightarrow{f}$ is a linear map. For any other point $b \in E$, since

$$b + v = a + \overrightarrow{ab} + v = a + \overrightarrow{a(a + v)} - \overrightarrow{aa} + \overrightarrow{ab},$$

$b + v = (a + v) - a + b$, and since f preserves barycenters, we get

$$f(b + v) = f(a + v) - f(a) + f(b),$$

which implies that

$$\begin{aligned}
\overrightarrow{f(b)f(b + v)} &= \overrightarrow{f(b)f(a + v)} - \overrightarrow{f(b)f(a)} + \overrightarrow{f(b)f(b)}, \\
&= \overrightarrow{f(a)f(b)} + \overrightarrow{f(b)f(a + v)}, \\
&= \overrightarrow{f(a)f(a + v)}.
\end{aligned}$$

Thus, $\overrightarrow{f(b)f(b + v)} = \overrightarrow{f(a)f(a + v)}$, which shows that the definition of $\overrightarrow{f}$ does not depend on the choice of $a \in E$. The fact that $\overrightarrow{f}$ is unique is obvious: We must have $\overrightarrow{f}(v) = \overrightarrow{f(a)f(a + v)}$. $\square$