Proof. Since $K[X]$ is an integral domain, for all nonzero polynomials $p, q \in K[X]$, $\deg(pq) = \deg(p) + \deg(q)$, and thus, $(p) \neq K[X]$ iff $\deg(p) \geq 1$. Assume that $p \in K[X]$ is irreducible. Since every ideal in $K[X]$ is a principal ideal, every ideal in $K[X]$ is of the form (q), for some $q \in K[X]$. If $(p) \subseteq (q)$, with $\deg(q) \geq 1$, then q divides p, and since $p \in K[X]$ is irreducible, this implies that $p = \lambda q$ for some $\lambda \neq 0$ in K, and so, $(p) = (q)$. Thus, (p) is a maximal ideal. Conversely, assume that (p) is a maximal ideal. Then, as we showed above, $\deg(p) \geq 1$, and if q divides p, with $\deg(q) \geq 1$, then $(p) \subseteq (q)$, and since (p) is a maximal ideal, this implies that $(p) = (q)$, which means that $p = \lambda q$ for some $\lambda \neq 0$ in K, and so, p is irreducible. □

Let $p \in K[X]$ be irreducible. Then, for every nonzero polynomial $g \in K[X]$, either p and g are relatively prime, or p divides g. Indeed, if d is any gcd of p and g, if d is a constant, then p and g are relatively prime, and if not, because p is irreducible, we have $d = \lambda p$ for some $\lambda \neq 0$ in K, and thus, p divides g. As a consequence, if $p, q \in K[X]$ are both irreducible, then either p and q are relatively prime, or $p = \lambda q$ for some $\lambda \neq 0$ in K. In particular, if $p, q \in K[X]$ are both irreducible monic polynomials and $p \neq q$, then p and q are relatively prime.

We now prove the (unique) factorization of polynomials into irreducible factors.

Theorem 30.17. *Given any field K, for every nonzero polynomial*

$$f = a_d X^d + a_{d-1} X^{d-1} + \cdots + a_0$$

of degree $d = \deg(f) \geq 1$ in $K[X]$, there exists a unique set $\{\langle p_1, k_1\rangle, \ldots, \langle p_m, k_m\rangle\}$ such that

$$f = a_d p_1^{k_1} \cdots p_m^{k_m},$$

where the $p_i \in K[X]$ are distinct irreducible monic polynomials, the k_i are (not necessarily distinct) integers, and $m \geq 1$, $k_i \geq 1$.

Proof. First, we prove the existence of such a factorization by induction on $d = \deg(f)$. Clearly, it is enough to prove the result for monic polynomials f of degree $d = \deg(f) \geq 1$. If $d = 1$, then $f = X + a_0$, which is an irreducible monic polynomial.

Assume $d \geq 2$, and assume the induction hypothesis for all monic polynomials of degree $< d$. Consider the set S of all monic polynomials g such that $\deg(g) \geq 1$ and g divides f. Since $f \in S$, the set S is nonempty, and thus, S contains some monic polynomial p_1 of minimal degree. Since $\deg(p_1) \geq 1$, the monic polynomial p_1 must be irreducible. Otherwise we would have $p_1 = g_1 g_2$, for some monic polynomials g_1, g_2 such that $\deg(p_1) > \deg(g_1) \geq 1$ and $\deg(p_1) > \deg(g_2) \geq 1$, and since p_1 divide f, then g_1 would divide f, contradicting the minimality of the degree of p_1. Thus, we have $f = p_1 q$, for some irreducible monic polynomial p_1, with q also monic. Since $\deg(p_1) \geq 1$, we have $\deg(q) < \deg(f)$, and we can apply the induction hypothesis to q. Thus, we obtain a factorization of the desired form.