26.5 Projective Maps

Given two nontrivial vector spaces E and F and a linear map $f: E \to F$, observe that for every $u, v \in (E - \operatorname{Ker} f)$, if $v = \lambda u$ for some $\lambda \in K - \{0\}$, then $f(v) = \lambda f(u)$, and thus f restricted to $(E - \operatorname{Ker} f)$ induces a function $\mathbf{P}(f): (\mathbf{P}(E) - \mathbf{P}(\operatorname{Ker} f)) \to \mathbf{P}(F)$ defined such that

$$\mathbf{P}(f)([u]_\sim) = [f(u)]_\sim,$$

as in the following commutative diagram:

$$\begin{tikzcd} E - \operatorname{Ker} f \arrow[r, "f"] \arrow[d, "p"'] & F - \{0\} \arrow[d, "p"] \\ \mathbf{P}(E) - \mathbf{P}(\operatorname{Ker} f) \arrow[r, "\mathbf{P}(f)"'] & \mathbf{P}(F) \end{tikzcd}$$

When f is injective, i.e., when $\operatorname{Ker} f = \{0\}$, then $\mathbf{P}(f): \mathbf{P}(E) \to \mathbf{P}(F)$ is indeed a well-defined function. The above discussion motivates the following definition.

Definition 26.5. Given two nontrivial vector spaces E and F, any linear map $f: E \to F$ induces a partial map $\mathbf{P}(f): \mathbf{P}(E) \to \mathbf{P}(F)$ called a *projective map*, such that if $\operatorname{Ker} f = \{u \in E \mid f(u) = 0\}$ is the kernel of f, then $\mathbf{P}(f): (\mathbf{P}(E) - \mathbf{P}(\operatorname{Ker} f)) \to \mathbf{P}(F)$ is a total map defined such that

$$\mathbf{P}(f)([u]_\sim) = [f(u)]_\sim,$$

as in the following commutative diagram:

$$\begin{tikzcd} E - \operatorname{Ker} f \arrow[r, "f"] \arrow[d, "p"'] & F - \{0\} \arrow[d, "p"] \\ \mathbf{P}(E) - \mathbf{P}(\operatorname{Ker} f) \arrow[r, "\mathbf{P}(f)"'] & \mathbf{P}(F) \end{tikzcd}$$

If f is injective, i.e., when $\operatorname{Ker} f = \{0\}$, then $\mathbf{P}(f): \mathbf{P}(E) \to \mathbf{P}(F)$ is a total function called a *projective transformation*, and when f is bijective, we call $\mathbf{P}(f)$ a *projectivity, or projective isomorphism, or homography*. The set of projectivities $\mathbf{P}(f): \mathbf{P}(E) \to \mathbf{P}(E)$ is a group called the *projective (linear) group*, and is denoted by $\mathbf{PGL}(E)$.

One should realize that if a linear map $f: E \to F$ is not injective, then the projective map $\mathbf{P}(f): \mathbf{P}(E) \to \mathbf{P}(F)$ is only a *partial map*, i.e., it is undefined on $\mathbf{P}(\operatorname{Ker} f)$. In particular, if $f: E \to F$ is the null map (i.e., $\operatorname{Ker} f = E$), the domain of $\mathbf{P}(f)$ is empty and $\mathbf{P}(f)$ is the partial function undefined everywhere. We might want to require in Definition 26.5 that f not be the null map to avoid this degenerate case. Projective maps are often defined only when they are induced by bijective linear maps.