## 36.3 The Rational Canonical Form, Second Version

Let us now translate the Elementary Divisors Decomposition Theorem, Theorem 35.38, in terms of $E_f$. We obtain the following result.

**Theorem 36.14.** *(Cyclic Decomposition Theorem, Second Version) Let $f\colon E \to E$ be an endomorphism on a $K$-vector space of dimension $n$. Then, $E$ is the direct sum of of cyclic subspaces $E_j = Z(u_j; f)$ for $f$, such that the minimal polynomial of $E_j$ is of the form $p_i^{n_{i,j}}$, for some irreducible monic polynomials $p_1, \ldots, p_t \in K[X]$ and some positive integers $n_{i,j}$, such that for each $i = 1, \ldots, t$, there is a sequence of integers*

$$1 \leq \underbrace{n_{i,1}, \ldots, n_{i,1}}_{m_{i,1}} < \underbrace{n_{i,2}, \ldots, n_{i,2}}_{m_{i,2}} < \cdots < \underbrace{n_{i,s_i}, \ldots, n_{i,s_i}}_{m_{i,s_i}},$$

*with $s_i \geq 1$, and where $n_{i,j}$ occurs $m_{i,j} \geq 1$ times, for $j = 1, \ldots, s_i$. Furthermore, the monic polynomials $p_i$ and the integers $r, t, n_{i,j}, s_i, m_{i,j}$ are uniquely determined.*

Note that there are $\mu = \sum m_{i,j}$ cyclic subspaces $Z(u_j; f)$. Using bases for the cyclic subspaces $Z(u_j; f)$ as in Theorem 36.6, we get the following theorem.

**Theorem 36.15.** *(Rational Canonical Form, Second Version) Let $f\colon E \to E$ be an endomorphism on a $K$-vector space of dimension $n$. There exist $t$ distinct irreducible monic polynomials $p_1, \ldots, p_t \in K[X]$ and some positive integers $n_{i,j}$, such that for each $i = 1, \ldots, t$, there is a sequence of integers*

$$1 \leq \underbrace{n_{i,1}, \ldots, n_{i,1}}_{m_{i,1}} < \underbrace{n_{i,2}, \ldots, n_{i,2}}_{m_{i,2}} < \cdots < \underbrace{n_{i,s_i}, \ldots, n_{i,s_i}}_{m_{i,s_i}},$$

*with $s_i \geq 1$, and where $n_{i,j}$ occurs $m_{i,j} \geq 1$ times, for $j = 1, \ldots, s_i$, and there is a basis of $E$ such that the matrix $M$ of $f$ is a block matrix of the form*

$$M = \begin{pmatrix} M_1 & 0 & \cdots & 0 & 0 \\ 0 & M_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & M_{\mu-1} & 0 \\ 0 & 0 & \cdots & 0 & M_\mu \end{pmatrix},$$

*where each $M_j$ is the companion matrix of some $p_i^{n_{i,j}}$, and $\mu = \sum m_{i,j}$. The monic polynomials $p_1, \ldots, p_t$ and the integers $r, t, n_{i,j}, s_i, m_{i,j}$ are uniquely determined*

The polynomials $p_i^{n_{i,j}}$ are called the *elementary divisors* of $f$ (and $M$). These polynomials are factors of the minimal polynomial.

**Example 1 continued:** Recall that $f(x, y, z, w) = (x + w, y + z, y + z, x + w)$ has two nontrivial invariant factors $q_1 = x(x - 2) = q_2$. Thus the elementary factors of $f$ are $p_1 = x = p_2$ and $p_3 = x - 2 = p_4$. Theorem 36.14 implies that

$$\mathbb{R}^4 = E_1 \oplus E_2 \oplus E_3 \oplus E_4,$$