The basis

$$((f-\lambda\mathrm{id})^{n-1}(u),(f-\lambda\mathrm{id})^{n-2}(u),\ldots,(f-\lambda\mathrm{id})(u),u),$$

provided by Proposition 36.16 is known as a *Jordan chain*. Note that $(f-\lambda\mathrm{id})^{n-1}(u)$ is an eigenvector for f. To construct the Jordan chain, we must find u which is a generalized eigenvector of f. This is done by first finding an eigenvector x_1 of f and recursively solving the system $(f-\lambda\mathrm{id})x_{i+1}=x_i$ for $i \leq 1 \leq n-1$. For example suppose $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ where $f(x,y,z)=(x+y+z,y+z,z)$. In terms of the standard basis, the matrix representation for f is $M=\begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}$. By using M, it is readily verified that the minimal polynomial f equals the characteristic polynomial, namely $(X-1)^3$. Thus f has the eigenvalue $\lambda=1$ with repeated three times. To find the eigenvector x_1 associated with $\lambda=1$, we solve the system $(M-I)x_1=0$, or equivalently

$$\begin{pmatrix}0&1&1\\0&0&1\\0&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Thus $y=z=0$ with $x=1$ solves this system to provide the eigenvector $x_1=\begin{pmatrix}1\\0\\0\end{pmatrix}$. We next solve the system $(M-I)x_2=x_1$, namely

$$\begin{pmatrix}0&1&1\\0&0&1\\0&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix},$$

which implies that $z=0$ and $y=1$. Hence $x_2=\begin{pmatrix}1\\1\\0\end{pmatrix}$ will work. To finish constructing our Jordan chain, we must solve the system $(M-I)x_3=x_2$, namely

$$\begin{pmatrix}0&1&1\\0&0&1\\0&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\1\\0\end{pmatrix},$$

from which we see that $z=1$, $y=0$, and $x_3=\begin{pmatrix}1\\0\\1\end{pmatrix}$. By setting $x_3=u$, we form the basis

$$((f-\lambda\mathrm{id})^2(u),(f-\lambda\mathrm{id})^1(u),\ldots,(f-\lambda\mathrm{id})(u),u)=(x_1,x_2,x_3).$$