Remark: If the Jacobian matrix $\mathrm{Jac}(\varphi)(v) = \big((\partial\varphi_i/\partial x_j)(v)\big)$ has rank m for all $v \in U$ (which is equivalent to the linear independence of the linear forms $d\varphi_i(v)$), then we say that $0 \in \mathbb{R}^m$ is a *regular value* of φ. In this case, it is known that

$$U = \{v \in \Omega \mid \varphi(v) = 0\}$$

is a *smooth submanifold of dimension* $n - m$ *of* $\mathbb{R}^n$. Furthermore, the set

$$T_vU = \{w \in \mathbb{R}^n \mid d\varphi_i(v)(w) = 0,\ 1 \leq i \leq m\} = \bigcap_{i=1}^{m} \mathrm{Ker}\, d\varphi_i(v)$$

is the *tangent space* to U at v (a vector space of dimension $n - m$). Then, the condition

$$dJ(v) + \mu_1 d\varphi_1(v) + \cdots + \mu_m d\varphi_m(v) = 0$$

implies that $dJ(v)$ vanishes on the tangent space T_vU. Conversely, if $dJ(v)(w) = 0$ for all $w \in T_vU$, this means that $dJ(v)$ is orthogonal (in the sense of Definition 11.3) to T_vU. Since (by Theorem 11.4 (b)) the orthogonal of T_vU is the space of linear forms spanned by $d\varphi_1(v), \ldots, d\varphi_m(v)$, it follows that $dJ(v)$ must be a linear combination of the $d\varphi_i(v)$. Therefore, when 0 is a regular value of φ, Theorem 40.2 asserts that if $u \in U$ is a local extremum of J, then $dJ(u)$ must vanish on the tangent space T_uU. We can say even more. The subset $Z(J)$ of Ω given by

$$Z(J) = \{v \in \Omega \mid J(v) = J(u)\}$$

(the *level set of level* $J(u)$) is a hypersurface in Ω, and if $dJ(u) \neq 0$, the zero locus of $dJ(u)$ is the tangent space $T_uZ(J)$ to $Z(J)$ at u (a vector space of dimension $n - 1$), where

$$T_uZ(J) = \{w \in \mathbb{R}^n \mid dJ(u)(w) = 0\}.$$

Consequently, Theorem 40.2 asserts that

$$T_uU \subseteq T_uZ(J);$$

this is a geometric condition.

We now return to the general situation where E_1 and E_2 may be infinite-dimensional normed vector spaces (with E_1 a Banach space) and we state and prove the following general result about the method of Lagrange multipliers.

Theorem 40.4. *(Necessary condition for a constrained extremum) Let $\Omega \subseteq E_1 \times E_2$ be an open subset of a product of normed vector spaces, with E_1 a Banach space (E_1 is complete), let $\varphi\colon \Omega \to E_2$ be a C^1-function (which means that $d\varphi(\omega)$ exists and is continuous for all $\omega \in \Omega$), and let*

$$U = \{(u_1, u_2) \in \Omega \mid \varphi(u_1, u_2) = 0\}.$$