33.6 Tensor Algebras

Our goal is to define a vector space $T(V)$ obtained by taking the direct sum of the tensor products

$$\underbrace{V \otimes \cdots \otimes V}_{m},$$

and to define a multiplication operation on $T(V)$ which makes $T(V)$ into an algebraic structure called an algebra. The algebra $T(V)$ satisfies a universal property stated in Proposition 33.19, which makes it the "free algebra" generated by the vector space V.

Definition 33.8. The tensor product

$$\underbrace{V \otimes \cdots \otimes V}_{m}$$

is also denoted as

$$\bigotimes^{m} V \quad \text{or} \quad V^{\otimes m}$$

and is called the *m-th tensor power of* V (with $V^{\otimes 1} = V$, and $V^{\otimes 0} = K$).

We can pack all the tensor powers of V into the "big" vector space

$$T(V) = \bigoplus_{m \geq 0} V^{\otimes m},$$

denoted $T^{\bullet}(V)$ or $\bigotimes V$ to avoid confusion with the tangent bundle.

This is an interesting object because we can define a multiplication operation on it which makes it into an *algebra*.

When V is of finite dimension n, we can pick some basis $(e_1 \ldots, e_n)$ of V, and then every tensor $\omega \in T(V)$ can be expressed as a linear combination of terms of the form $e_{i_1} \otimes \cdots \otimes e_{i_k}$, where $(i_1, \ldots, i_k)$ is any sequence of elements from the set $\{1, \ldots, n\}$. We can think of the tensors $e_{i_1} \otimes \cdots \otimes e_{i_k}$ as monomials in the noncommuting variables $e_1, \ldots, e_n$. Thus the space $T(V)$ corresponds to the algebra of polynomials with coefficients in K in n *noncommuting variables*.

Let us review the definition of an algebra over a field. Let K denote any (commutative) field, although for our purposes, we may assume that $K = \mathbb{R}$ (and occasionally, $K = \mathbb{C}$). Since we will only be dealing with associative algebras with a multiplicative unit, we only define algebras of this kind.

Definition 33.9. Given a field K, a K*-algebra* is a K-vector space A together with a bilinear operation $\cdot \colon A \times A \to A$, called *multiplication*, which makes A into a ring with unity 1 (or 1_A, when we want to be very precise). This means that $\cdot$ is associative and that there is a multiplicative identity element 1 so that $1 \cdot a = a \cdot 1 = a$, for all $a \in A$. Given two