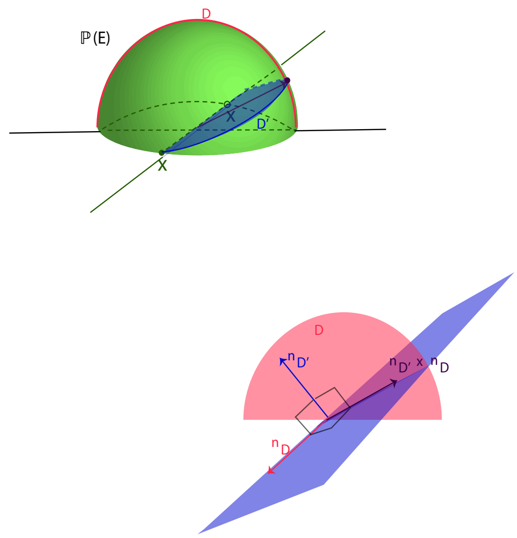

Figure 26.8: The intersection of two projective lines in the projective plane $\mathbf{P}(E)$ is the cross product of the normals for the two corresponding planes in $\mathbb{R}^3$.

are zero because they have two equal rows, and since by expanding these determinants with respect to their first row using the Laplace expansion formula we get

$$0 = \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix} = \alpha(\beta\gamma' - \beta'\gamma) + \beta(\gamma\alpha' - \gamma'\alpha) + \gamma(\alpha\beta' - \alpha'\beta)$$

and

$$0 = \begin{vmatrix} \alpha' & \beta' & \gamma' \\ \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{vmatrix} = \alpha'(\beta\gamma' - \beta'\gamma) + \beta'(\gamma\alpha' - \gamma'\alpha) + \gamma'(\alpha\beta' - \alpha'\beta),$$

which confirms that the point

$$Q = (\beta\gamma' - \beta'\gamma \colon \gamma\alpha' - \gamma'\alpha \colon \alpha\beta' - \alpha'\beta)$$

satisfies both equations in $(*)$, and thus belongs to both lines D and D'. Since the matrix

$$\begin{pmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \end{pmatrix}$$

has rank 2, at least one of the coordinates of Q is nonzero, so Q is indeed a point in the projective plane, and it is the intersection of the lines D and D'.

The result that we just proved yields the following criterion for three lines D, D', D'' in a projective plane to pass through a common point (to be concurrent). In a projective plane,