Since this isomorphism is used often, we record it as the following proposition.

Proposition 33.8. *Given a tensor product $E_1 \otimes \cdots \otimes E_n$,, there is a canonical isomorphism*

$$\mathrm{L}(E_1, \ldots, E_n; K) \cong (E_1 \otimes \cdots \otimes E_n)^*$$

between the vector space of multilinear maps $\mathcal{L}(E_1, \ldots, E_n; K)$ and the dual $(E_1 \otimes \cdots \otimes E_n)^*$ of the tensor product $E_1 \otimes \cdots \otimes E_n$.

The fact that the map $\varphi\colon E_1 \times \cdots \times E_n \to E_1 \otimes \cdots \otimes E_n$ is multilinear, can also be expressed as follows:

$$\begin{aligned} u_1 \otimes \cdots \otimes (v_i + w_i) \otimes \cdots \otimes u_n &= (u_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes u_n) + (u_1 \otimes \cdots \otimes w_i \otimes \cdots \otimes u_n), \\ u_1 \otimes \cdots \otimes (\lambda u_i) \otimes \cdots \otimes u_n &= \lambda(u_1 \otimes \cdots \otimes u_i \otimes \cdots \otimes u_n). \end{aligned}$$

Of course, this is just what we wanted!

Definition 33.6. Tensors in $E_1 \otimes \cdots \otimes E_n$ are called *n-tensors*, and tensors of the form $u_1 \otimes \cdots \otimes u_n$, where $u_i \in E_i$ are called *simple (or decomposable) n-tensors*. Those n-tensors that are not simple are often called *compound n-tensors*.

Not only do tensor products act on spaces, but they also act on linear maps (they are functors).

Proposition 33.9. *Given two linear maps $f\colon E \to E'$ and $g\colon F \to F'$, there is a unique linear map*

$$f \otimes g\colon E \otimes F \to E' \otimes F'$$

such that

$$(f \otimes g)(u \otimes v) = f(u) \otimes g(v),$$

for all $u \in E$ and all $v \in F$.

Proof. We can define $h\colon E \times F \to E' \otimes F'$ by

$$h(u, v) = f(u) \otimes g(v).$$

It is immediately verified that h is bilinear, and thus it induces a unique linear map

$$f \otimes g\colon E \otimes F \to E' \otimes F'$$

making the following diagram commutes

$$\begin{tikzcd} E \times F \arrow[r, "\iota_\otimes"] \arrow[dr, "h"'] & E \otimes F \arrow[d, "f \otimes g"] \\ & E' \otimes F', \end{tikzcd}$$

such that $(f \otimes g)(u \otimes v) = f(u) \otimes g(v)$, for all $u \in E$ and all $v \in F$. ∎