Chapter 29

The Geometry of Bilinear Forms; Witt's Theorem; The Cartan–Dieudonné Theorem

29.1 Bilinear Forms

In this chapter, we study the structure of a K-vector space E endowed with a nondegenerate bilinear form $\varphi\colon E \times E \to K$ (for any field K), which can be viewed as a kind of generalized inner product. Unlike the case of an inner product, there may be nonzero vectors $u \in E$ such that $\varphi(u,u) = 0$ so the map $u \mapsto \varphi(u,u)$ can no longer be interpreted as a notion of square length (also, $\varphi(u,u)$ may not be real and positive!). However, the notion of orthogonality survives: we say that $u, v \in E$ are orthogonal iff $\varphi(u,v) = 0$. Under some additional conditions on φ, it is then possible to split E into orthogonal subspaces having some special properties. It turns out that the special cases where φ is symmetric (or Hermitian) or skew-symmetric (or skew-Hermitian) can be handled uniformly using a deep theorem due to Witt (the Witt decomposition theorem (1936)).

We begin with the very general situation of a bilinear form $\varphi\colon E \times F \to K$, where K is an arbitrary field, possibly of characteristic 2. Actually, even though at first glance this may appear to be an unnecessary abstraction, it turns out that this situation arises in attempting to prove properties of a bilinear map $\varphi\colon E \times E \to K$, because it may be necessary to restrict φ to different subspaces U and V of E. This general approach was pioneered by Chevalley [37], E. Artin [6], and Bourbaki [24]. The third source was a major source of inspiration, and many proofs are taken from it. Other useful references include Snapper and Troyer [162], Berger [12], Jacobson [98], Grove [83], Taylor [174], and Berndt [14].

Definition 29.1. Given two vector spaces E and F over a field K, a map $\varphi\colon E \times F \to K$ is a *bilinear form* iff the following conditions hold: For all $u, u_1, u_2 \in E$, all $v, v_1, v_2 \in F$, for