Case 3'. $f(u) - u$ is nonzero and isotropic for all nonisotropic $u \in E$. In this case, what saves us is that E must be an Artinian space of dimension $n = 2m$ and that f must be a rotation ($f \in \mathbf{SO}(\varphi)$).

If we acccept this fact proved in Proposition 29.43 then pick any hyperplane reflection τ. Then, since f is a rotation, $g = \tau \circ f$ is *not* a rotation because $\det(g) = \det(\tau)\det(f) = (-1)(+1) = -1$, so $g(u) - u$ is either 0 or not isotropic for some nonisotropic $u \in E$ (otherwise, g would be a rotation), we are back to either Case 1 or Case 2, and using the induction hypothesis, we get

$$\tau \circ f = \tau_k \circ \ldots, \tau_1,$$

where each τ_i is a hyperplane reflection, and $k \leq 2m$. Since $\tau \circ f$ is not a rotation, actually $k \leq 2m - 1$, and then $f = \tau \circ \tau_k \circ \ldots, \tau_1$, the composition of at most $k + 1 \leq 2m$ hyperplane reflections.

Therefore, except for the fact that in Case 3', E must be an Artinian space of dimension $n = 2m$ and that f must be a rotation, which has not been proven yet, we proved the following theorem.

Theorem 29.41. *(Cartan–Dieudonné, strong form) Let φ be a nondegenerate symmetric bilinear form on a K-vector space E of dimension n ($\mathrm{char}(K) \neq 2$). Then, every isometry $f \in \mathbf{O}(\varphi)$ with $f \neq \mathrm{id}$ is the composition of at most n hyperplane reflections.*

To fill in the gap, we need two propositions.

Proposition 29.42. *Let (E, φ) be an Artinian space of dimension $2m$, and let U be a totally isotropic subspace of dimension m. For any isometry $f \in \mathbf{O}(\varphi)$, if $f(U) = U$, then $\det(f) = 1$ (f is a rotation).*

Proof. We know that we can find a basis $(u_1, \ldots, u_m, v_1, \ldots, v_m)$ of E such $(u_1, \ldots, u_m)$ is a basis of U and φ is represented by the matrix

$$\begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}.$$

Since $f(U) = U$, the matrix representing f is of the form

$$A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}.$$

The condition $A^\top A_{m,m} A = A_{m,m}$ translates as

$$\begin{pmatrix} B^\top & 0 \\ C^\top & D^\top \end{pmatrix} \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix} \begin{pmatrix} B & C \\ 0 & D \end{pmatrix} = \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}$$

that is,

$$\begin{pmatrix} B^\top & 0 \\ C^\top & D^\top \end{pmatrix} \begin{pmatrix} 0 & D \\ B & C \end{pmatrix} = \begin{pmatrix} 0 & B^\top D \\ D^\top B & C^\top D + D^\top C \end{pmatrix} = \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix},$$