The following theorem shows the relationship between the rank of f and the rank of $f^\top$.

Theorem 11.12. *Given a linear map $f\colon E \to F$, the following properties hold.*

(a) The dual $(\operatorname{Im} f)^*$ of $\operatorname{Im} f$ is isomorphic to $\operatorname{Im} f^\top = f^\top(F^*)$; that is,

$$(\operatorname{Im} f)^* \cong \operatorname{Im} f^\top.$$

(b) If F is finite dimensional, then $\operatorname{rk}(f) = \operatorname{rk}(f^\top)$.

Proof. (a) Consider the linear maps

$$E \xrightarrow{p} \operatorname{Im} f \xrightarrow{j} F,$$

where $E \xrightarrow{p} \operatorname{Im} f$ is the surjective map induced by $E \xrightarrow{f} F$, and $\operatorname{Im} f \xrightarrow{j} F$ is the injective inclusion map of $\operatorname{Im} f$ into F. By definition, $f = j \circ p$. To simplify the notation, let $I = \operatorname{Im} f$. By Proposition 11.8, since $E \xrightarrow{p} I$ is surjective, $I^* \xrightarrow{p^\top} E^*$ is injective, and since $\operatorname{Im} f \xrightarrow{j} F$ is injective, $F^* \xrightarrow{j^\top} I^*$ is surjective. Since $f = j \circ p$, we also have

$$f^\top = (j \circ p)^\top = p^\top \circ j^\top,$$

and since $F^* \xrightarrow{j^\top} I^*$ is surjective, and $I^* \xrightarrow{p^\top} E^*$ is injective, we have an isomorphism between $(\operatorname{Im} f)^*$ and $f^\top(F^*)$.

(b) We already noted that Part (a) of Theorem 11.4 shows that $\dim(F) = \dim(F^*)$, for every vector space F of finite dimension. Consequently, $\dim(\operatorname{Im} f) = \dim((\operatorname{Im} f)^*)$, and thus, by Part (a) we have $\operatorname{rk}(f) = \operatorname{rk}(f^\top)$.

Remark: When both E and F are finite-dimensional, there is also a simple proof of (b) that doesn't use the result of Part (a). By Theorem 11.4(c)

$$\dim(\operatorname{Im} f) + \dim((\operatorname{Im} f)^0) = \dim(F),$$

and by Theorem 6.16

$$\dim(\operatorname{Ker} f^\top) + \dim(\operatorname{Im} f^\top) = \dim(F^*).$$

Furthermore, by Proposition 11.11, we have

$$\operatorname{Ker} f^\top = (\operatorname{Im} f)^0,$$

and since F is finite-dimensional $\dim(F) = \dim(F^*)$, so we deduce

$$\dim(\operatorname{Im} f) + \dim((\operatorname{Im} f)^0) = \dim((\operatorname{Im} f)^0) + \dim(\operatorname{Im} f^\top),$$

which yields $\dim(\operatorname{Im} f) = \dim(\operatorname{Im} f^\top)$; that is, $\operatorname{rk}(f) = \operatorname{rk}(f^\top)$. $\square$