and $\alpha_1 \mid \alpha_2 \mid \cdots \mid \alpha_r$. *Furthermore, the ideals* $\alpha_1 A, \ldots, \alpha_r A$ *are the invariant factors of* $f(F)$ *with respect* F'.

Proof. Let F_0 be the kernel of f. Since $M' = f(F)$ is a submodule of the free module F', it is free, and similarly F_0 is free as a submodule of the free module F (by Proposition 35.23). By Proposition 35.2, we have

$$F = F_0 \oplus F_1,$$

where F_1 is a free module, and the restriction of f to F_1 is an isomorphism onto $M' = f(F)$. Proposition 35.32 applied to F' and M' yields a basis $(e_1', \ldots, e_m')$ of F' such that $(\alpha_1 e_1', \ldots, \alpha_r e_r')$ is a basis of M', where $\alpha_1 A, \ldots, \alpha_r A$ are the invariant factors for M' with respect to F'. Since the restriction of f to F_1 is and isomorphism, there is a basis $(e_1, \ldots, e_r)$ of F_1 such that

$$f(e_i) = \alpha_i e_i', \quad i = 1, \ldots, r.$$

We can extend this basis to a basis of F by picking a basis of F_0 (a free module), which yields the desired result. ☐

The matrix version of Proposition 35.34 is the following proposition.

Proposition 35.35. *If* X *is an* $m \times n$ *matrix of rank* r *over a PID* A, *then there exist some invertible* $n \times n$ *matrix* P, *some invertible* $m \times m$ *matrix* Q, *and a* $m \times n$ *matrix* D *of the form*

$$D = \begin{pmatrix} \alpha_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \alpha_r & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}$$

for some nonzero $\alpha_i \in A$, *such that*

(1) $\alpha_1 \mid \alpha_2 \mid \cdots \mid \alpha_r$,

(2) $X = QDP^{-1}$, *and*

(3) The α_i*s are uniquely determined up to a unit.*

The ideals $\alpha_1 A, \ldots, \alpha_r A$ are called the *invariant factors* of the matrix X. Recall that two $m \times n$ matrices X and Y are *equivalent* iff

$$Y = QXP^{-1},$$

for some invertible matrices, P and Q. Then, Proposition 35.35 implies the following fact.