Chapter 47

Linear Programming and Duality

47.1 Variants of the Farkas Lemma

If A is an $m \times n$ matrix and if $b \in \mathbb{R}^m$ is a vector, it is known from linear algebra that the linear system $Ax = b$ has no solution iff there is some linear form $y \in (\mathbb{R}^m)^*$ such that $yA = 0$ and $yb \neq 0$. This means that the linear from y vanishes on the columns $A^1, \ldots, A^n$ of A but does not vanish on b. Since the linear form y defines the linear hyperplane H of equation $yz = 0$ (with $z \in \mathbb{R}^m$), geometrically the equation $Ax = b$ has no solution iff there is a linear hyperplane H containing $A^1, \ldots, A^n$ and not containing b. This is a kind of separation theorem that says that the vectors $A^1, \ldots, A^n$ and b can be separated by some linear hyperplane H.

What we would like to do is to generalize this kind of criterion, first to a system $Ax = b$ subject to the constraints $x \geq 0$, and next to sets of inequality constraints $Ax \leq b$ and $x \geq 0$. There are indeed such criteria going under the name of *Farkas lemma*.

The key is a separation result involving polyhedral cones known as the Farkas–Minkowski proposition. We have the following fundamental separation lemma.

Proposition 47.1. *Let $C \subseteq \mathbb{R}^n$ be a closed nonempty (convex) cone. For any point $a \in \mathbb{R}^n$, if $a \notin C$, then there is a linear hyperplane H (through 0) such that*

1. *C lies in one of the two half-spaces determined by H.*
2. *$a \notin H$*
3. *a lies in the other half-space determined by H.*

We say that H strictly separates C and a.

Proposition 47.1, which is illustrated in Figure 47.1, is an easy consequence of another separation theorem that asserts that given any two nonempty closed convex sets A and B of $\mathbb{R}^n$ with A compact, there is a hyperplane H strictly separating A and B (which means that $A \cap H = \emptyset$, $B \cap H = \emptyset$, that A lies in one of the two half-spaces determined by H,