C.3 Existence of Maximal Ideals Containing a Given Proper Ideal

Let A be a commutative ring with identity element. Recall that an ideal $\mathfrak{A}$ in A is a *proper ideal* if $\mathfrak{A} \neq A$. The following theorem holds:

Theorem C.3. *Given any proper ideal, $\mathfrak{A} \subseteq A$, there is a maximal ideal, $\mathfrak{B}$, containing $\mathfrak{A}$.*

Proof. Let $\mathcal{I}$ be the set of all proper ideals, $\mathfrak{B}$, in A that contain $\mathfrak{A}$. The set $\mathcal{I}$ is nonempty, since $\mathfrak{A} \in \mathcal{I}$. We claim that $\mathcal{I}$ is inductive. Consider any chain $(\mathfrak{A}_i)_{i\in I}$ of ideals $\mathfrak{A}_i$ in A. One can easily check that $\mathfrak{B} = \bigcup_{i\in I} \mathfrak{A}_i$ is an ideal. Furthermore, $\mathfrak{B}$ is a proper ideal, since otherwise, the identity element 1 would belong to $\mathfrak{B} = A$, and so, we would have $1 \in \mathfrak{A}_i$ for some i, which would imply $\mathfrak{A}_i = A$, a contradiction. Also, $\mathfrak{B}$ is obviously an upper bound for all the $\mathfrak{A}_i$'s. By Zorn's lemma (Lemma C.1), the set $\mathcal{I}$ has a maximal element, say $\mathfrak{B}$, and $\mathfrak{B}$ is a maximal ideal containing $\mathfrak{A}$. $\square$