$h\colon I \to F$, there is a unique linear map $\overline{h}\colon K^{(I)} \to F$ such that $h = \overline{h} \circ \iota$, as in the following diagram.

$$\begin{tikzcd} I \arrow[r, "\iota"] \arrow[dr, "h"'] & K^{(I)} \arrow[d, "\overline{h}"] \\ & F \end{tikzcd}$$

Definition 33.5. The vector space $(K^{(I)}, \iota)$ constructed as above from a set I is called the *free vector space generated by* I (or over I). The commutativity of the above diagram is called the *universal mapping property* of the free vector space $(K^{(I)}, \iota)$ over I.

Using the proof technique of Proposition 33.5, it is not hard to prove that any two vector spaces satisfying the above universal mapping property are isomorphic.

We can now return to the construction of tensor products. For simplicity consider two vector spaces E_1 and E_2. Whatever $E_1 \otimes E_2$ and $\varphi\colon E_1 \times E_2 \to E_1 \otimes E_2$ are, since φ is supposed to be bilinear, we must have

$$\begin{aligned} \varphi(u_1 + u_2, v_1) &= \varphi(u_1, v_1) + \varphi(u_2, v_1) \\ \varphi(u_1, v_1 + v_2) &= \varphi(u_1, v_1) + \varphi(u_1, v_2) \\ \varphi(\lambda u_1, v_1) &= \lambda\varphi(u_1, v_1) \\ \varphi(u_1, \mu v_1) &= \mu\varphi(u_1, v_1) \end{aligned}$$

for all $u_1, u_2 \in E_1$, all $v_1, v_2 \in E_2$, and all $\lambda, \mu \in K$. Since $E_1 \otimes E_2$ must satisfy the universal mapping property of Definition 33.4, we may want to define $E_1 \otimes E_2$ as the free vector space $K^{(E_1 \times E_2)}$ generated by $I = E_1 \times E_2$ and let φ be the injection of $E_1 \times E_2$ into $K^{(E_1 \times E_2)}$. The problem is that in $K^{(E_1 \times E_2)}$, vectors such that

$$(u_1 + u_2, v_1) \quad \textit{and} \quad (u_1, v_1) + (u_2, v_2)$$

are different, when they should really be the same, since φ is bilinear. Since $K^{(E_1 \times E_2)}$ is free, there are no relations among the generators and this vector space is too big for our purpose.

The remedy is simple: take the quotient of the free vector space $K^{(E_1 \times E_2)}$ by the subspace N generated by the vectors of the form

$$\begin{aligned} &(u_1 + u_2, v_1) - (u_1, v_1) - (u_2, v_1) \\ &(u_1, v_1 + v_2) - (u_1, v_1) - (u_1, v_2) \\ &(\lambda u_1, v_1) - \lambda(u_1, v_1) \\ &(u_1, \mu v_1) - \mu(u_1, v_1). \end{aligned}$$

Then, if we let $E_1 \otimes E_2$ be the quotient space $K^{(E_1 \times E_2)}/N$ and let φ be the quotient map, this forces φ to be bilinear. Checking that $(K^{(E_1 \times E_2)}/N, \varphi)$ satisfies the universal mapping property is straightforward. Here is the detailed construction.