$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

However, there is a problem when the origin of the coordinate system belongs to the plane (a, b, c), since in this case, the matrix is not invertible! What we should really be doing is to solve the system

$$\lambda_0 \overrightarrow{Oa} + \lambda_1 \overrightarrow{Ob} + \lambda_2 \overrightarrow{Oc} = \overrightarrow{Ox},$$

where O is any point **not** in the plane (a, b, c). An alternative is to use certain well-chosen cross products.

It can be shown that barycentric coordinates correspond to various ratios of areas and volumes; see the problems.

24.7 Affine Maps

Corresponding to linear maps we have the notion of an affine map. An affine map is defined as a map preserving affine combinations.

Definition 24.6. Given two affine spaces $\langle E, \overrightarrow{E}, +\rangle$ and $\langle E', \overrightarrow{E'}, +'\rangle$, a function $f\colon E \to E'$ is an *affine map* iff for every family $((a_i, \lambda_i))_{i\in I}$ of weighted points in E such that $\sum_{i\in I} \lambda_i = 1$, we have

$$f\left(\sum_{i\in I} \lambda_i a_i\right) = \sum_{i\in I} \lambda_i f(a_i).$$

In other words, f preserves barycenters.

Affine maps can be obtained from linear maps as follows. For simplicity of notation, the same symbol $+$ is used for both affine spaces (instead of using both $+$ and $+'$).

Proposition 24.7. *Given any point $a \in E$, any point $b \in E'$, and any linear map $h\colon \overrightarrow{E} \to \overrightarrow{E'}$, the map $f\colon E \to E'$ defined such that*

$$f(a + v) = b + h(v)$$

is an affine map.

Proof. Indeed, for any family $(\lambda_i)_{i\in I}$ of scalars with $\sum_{i\in I} \lambda_i = 1$ and any family $(v_i)_{i\in I}$, since

$$\sum_{i\in I} \lambda_i (a + v_i) = a + \sum_{i\in I} \lambda_i \overrightarrow{a(a + v_i)} = a + \sum_{i\in I} \lambda_i v_i$$