We can now prove the following proposition.

Proposition 33.28. *Given a vector space E, if $(e_i)_{i\in I}$ is a basis for E, then the family of vectors*

$$\left(e_{i_1}^{\odot M(i_1)} \odot \cdots \odot e_{i_k}^{\odot M(i_k)}\right)_{\substack{M\in\mathbb{N}^{(I)},\, |M|=m,\\ \{i_1,\ldots,i_k\}=\mathrm{dom}(M)}}$$

is a basis of the symmetric m-th tensor power $\mathrm{S}^m(E)$.

Proof. The proof is very similar to that of Proposition 33.12. First assume that E has finite dimension n. In this case $I = \{1,\ldots,n\}$, and any multiset $M \in \mathbb{N}^{(I)}$ of size $|M| = m$ is of the form $M(m, \{1,\ldots,n\}, k_1,\ldots,k_n)$, with $k_i = M(i)$ and $k_1 + \cdots + k_n = m$.

For any nontrivial vector space F, for any family of vectors

$$(w_M)_{M\in\mathbb{N}^{(I)},\, |M|=m},$$

we show the existence of a symmetric multilinear map $h\colon \mathrm{S}^m(E) \to F$, such that for every $M \in \mathbb{N}^{(I)}$ with $|M| = m$, we have

$$h(e_{i_1}^{\odot M(i_1)} \odot \cdots \odot e_{i_k}^{\odot M(i_k)}) = w_M,$$

where $\{i_1,\ldots,i_k\} = \mathrm{dom}(M)$. We define the map $f\colon E^m \to F$ as follows: for any m vectors $v_1,\ldots,v_m \in E$ we can write $v_k = \sum_{i=1}^n u_{i,k} e_i$ for $k = 1,\ldots,m$ and we set

$$\begin{aligned} &f(v_1,\ldots,v_m)\\ &\quad = \sum_{k_1+\cdots+k_n=m} \left(\sum_{\substack{I_1\cup\cdots\cup I_n=\{1,\ldots,m\}\\ I_i\cap I_j=\emptyset,\, i\neq j,\, |I_j|=k_j}} \left(\prod_{i_1\in I_1} u_{1,i_1}\right) \cdots \left(\prod_{i_n\in I_n} u_{n,i_n}\right)\right) w_{M(m,\{1,\ldots,n\},k_1,\ldots,k_n)}. \end{aligned}$$

It is not difficult to verify that f is symmetric and multilinear. By the universal mapping property of the symmetric tensor product, the linear map $f_\odot\colon \mathrm{S}^m(E) \to F$ such that $f = f_\odot \circ \varphi$, is the desired map h. Then by Proposition 33.4, it follows that the family

$$\left(e_{i_1}^{\odot M(i_1)} \odot \cdots \odot e_{i_k}^{\odot M(i_k)}\right)_{\substack{M\in\mathbb{N}^{(I)},\, |M|=m,\\ \{i_1,\ldots,i_k\}=\mathrm{dom}(M)}}$$

is linearly independent. Using the commutativity of $\odot$, we can also show that these vectors generate $\mathrm{S}^m(E)$, and thus, they form a basis for $\mathrm{S}^m(E)$.

If I is infinite dimensional, then for any m vectors $v_1,\ldots,v_m \in F$ there is a finite subset J of I such that $v_k = \sum_{j\in J} u_{j,k} e_j$ for $k = 1,\ldots,m$, and if we write $n = |J|$, then the formula for $f(v_1,\ldots,v_m)$ is obtained by replacing the set $\{1,\ldots,n\}$ by J. The details are left as an exercise. $\square$