The following theorem inspired by the *Newton–Kantorovich theorem* gives sufficient conditions that guarantee that the sequence (x_k) constructed by a generalized Newton method converges to a zero of f close to x_0. Although quite technical, these conditions are not very surprising.

Theorem 41.1. *Let X be a Banach space, let $f\colon \Omega \to Y$ be differentiable on the open subset $\Omega \subseteq X$, and assume that there are constants $r, M, \beta > 0$ such that if we let*

$$B = \{x \in X \mid \|x - x_0\| \leq r\} \subseteq \Omega,$$

then

(1)
$$\sup_{k\geq 0}\sup_{x\in B} \left\|A_k^{-1}(x)\right\|_{\mathcal{L}(Y;X)} \leq M,$$

(2) $\beta < 1$ and
$$\sup_{k\geq 0}\sup_{x,x'\in B} \|f'(x) - A_k(x')\|_{\mathcal{L}(X;Y)} \leq \frac{\beta}{M}$$

(3)
$$\|f(x_0)\| \leq \frac{r}{M}(1-\beta).$$

Then the sequence (x_k) defined by

$$x_{k+1} = x_k - A_k^{-1}(x_\ell)(f(x_k)), \quad 0 \leq \ell \leq k$$

is entirely contained within B and converges to a zero a of f, which is the only zero of f in B. Furthermore, the convergence is geometric, which means that

$$\|x_k - a\| \leq \frac{\|x_1 - x_0\|}{1-\beta}\beta^k.$$

Proof. We follow Ciarlet [41] (Theorem 7.5.1, Section 7.5). The proof has three steps.

Step 1. We establish the following inequalities for all $k \geq 1$.

$$\|x_k - x_{k-1}\| \leq M\,\|f(x_{k-1})\| \tag{a}$$
$$\|x_k - x_0\| \leq r \quad (x_k \in B) \tag{b}$$
$$\|f(x_k)\| \leq \frac{\beta}{M}\,\|x_k - x_{k-1}\|\,. \tag{c}$$

We proceed by induction on k, starting with the base case $k = 1$. Since

$$x_1 = x_0 - A_0^{-1}(x_0)(f(x_0)),$$