equivalence relation $\sim$ defined such that for all $u, v \in E - \{0\}$,

$$u \sim v \quad \text{iff} \quad v = \lambda u, \text{ for some } \lambda \in K - \{0\}.$$

The *canonical projection* $p\colon (E - \{0\}) \to \mathbf{P}(E)$ is the function associating the equivalence class $[u]_\sim$ modulo $\sim$ to $u \neq 0$. The *dimension* $\dim(\mathbf{P}(E))$ of $\mathbf{P}(E)$ is defined as follows: If E is of infinite dimension, then $\dim(\mathbf{P}(E)) = \dim(E)$, and if E has finite dimension, $\dim(E) = n \geq 1$ then $\dim(\mathbf{P}(E)) = n - 1$.

Mathematically, a projective space $\mathbf{P}(E)$ is a set of equivalence classes of vectors in E. The spirit of projective geometry is to view an equivalence class $p(u) = [u]_\sim$ as an "atomic" object, forgetting the internal structure of the equivalence class. For this reason, it is customary to call an equivalence class $a = [u]_\sim$ a *point* (the entire equivalence class $[u]_\sim$ is collapsed into a single object viewed as a point).

Remarks:

(1) If we view E as an affine space, then for any nonnull vector $u \in E$, since

$$[u]_\sim = \{\lambda u \mid \lambda \in K,\, \lambda \neq 0\},$$

letting

$$Ku = \{\lambda u \mid \lambda \in K\}$$

denote the subspace of dimension 1 spanned by u, the map

$$[u]_\sim \mapsto Ku$$

from $\mathbf{P}(E)$ to the set of one-dimensional subspaces of E is clearly a bijection, and since subspaces of dimension 1 correspond to lines through the origin in E, we can view $\mathbf{P}(E)$ as the set of lines in E passing through the origin. So, the projective space $\mathbf{P}(E)$ can be viewed as the set obtained from E when lines through the origin are treated as points.

However, this is a somewhat deceptive view. Indeed, depending on the structure of the vector space E, a line (through the origin) in E may be a fairly complex object, and treating a line just as a point is really a mental game. For example, E may be the vector space of real homogeneous polynomials $P(x, y, z)$ of degree 2 in three variables x, y, z (plus the null polynomial), and a "line" (through the origin) in E corresponds to an algebraic curve of degree 2. Lots of details need to be filled in, but roughly speaking, the curve defined by P is the "zero locus of P," i.e., the set of points $(x, y, z) \in \mathbf{P}(\mathbb{R}^3)$ (or perhaps in $\mathbf{P}(\mathbb{C}^3)$) for which $P(x, y, z) = 0$. We will come back to this point in Section 26.4 after having introduced homogeneous coordinates.

More generally, E may be a vector space of homogeneous polynomials of degree m in 3 or more variables (plus the null polynomial), and the lines in E correspond to