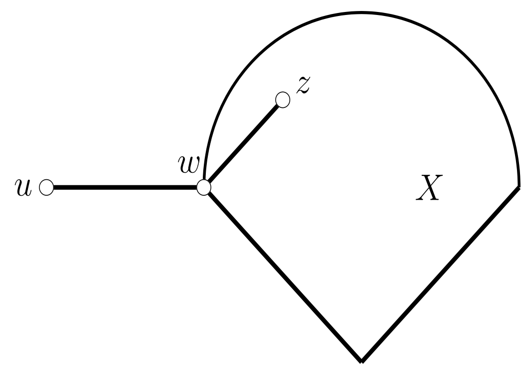

Figure 48.3: Inequality of Proposition 48.5.

It is immediately verified that each X_n is nonempty (by definition of d), convex, and that $X_{n+1} \subseteq X_n$. Also, by Proposition 48.3, (where $B = \{v \in E \mid \|u - v\| \leq d\}$, $C = \left\{v \in E \mid \|u - v\| \leq d + \frac{1}{n}\right\}$, and $A = X_n$), we have

$$\sup\{\|z - v\| \mid v, z \in X_n\} \leq \sqrt{12d/n},$$

and thus, $\bigcap_{n\geq 1} X_n$ contains at most one point; see Proposition 48.4(2). We will prove that $\bigcap_{n\geq 1} X_n$ contains exactly one point, namely, $p_X(u)$. For this, define a sequence $(w_n)_{n\geq 1}$ by picking some $w_n \in X_n$ for every $n \geq 1$. We claim that $(w_n)_{n\geq 1}$ is a Cauchy sequence. Given any $\epsilon > 0$, if we pick N such that

$$N > \frac{12d}{\epsilon^2},$$

since $(X_n)_{n\geq 1}$ is a monotonic decreasing sequence, which means that $X_{n+1} \subseteq X_n$ for all $n \geq 1$, for all $m, n \geq N$, we have

$$\|w_m - w_n\| \leq \sqrt{12d/N} < \epsilon,$$

as desired. Since E is complete, the sequence $(w_n)_{n\geq 1}$ has a limit w, and since $w_n \in X$ and X is closed, we must have $w \in X$. Also observe that

$$\|u - w\| \leq \|u - w_n\| + \|w_n - w\|,$$

and since w is the limit of $(w_n)_{n\geq 1}$ and

$$\|u - w_n\| \leq d + \frac{1}{n},$$

given any $\epsilon > 0$, there is some n large enough so that

$$\frac{1}{n} < \frac{\epsilon}{2} \quad \text{and} \quad \|w_n - w\| \leq \frac{\epsilon}{2},$$