**Proposition 16.7.** *For every matrix $A \in \mathfrak{su}(2)$, with*

$$A = \begin{pmatrix} iu_1 & u_2 + iu_3 \\ -u_2 + iu_3 & -iu_1 \end{pmatrix},$$

*if we write $\theta = \sqrt{u_1^2 + u_2^2 + u_3^2}$, then*

$$e^A = \cos\theta I + \frac{\sin\theta}{\theta} A, \quad \theta \neq 0,$$

*and $e^0 = I$.*

Therefore, by the discussion at the end of the previous section, $e^A$ is a unit quaternion representing the rotation of angle $2\theta$ and axis $(u_1, u_2, u_3)$ (or $I$ when $\theta = k\pi$, $k \in \mathbb{Z}$). The above formula shows that we may assume that $0 \leq \theta \leq \pi$. Proposition 16.7 shows that the exponential yields a map $\exp\colon \mathfrak{su}(2) \to \mathbf{SU}(2)$. It is an analog of the exponential map $\exp\colon \mathfrak{so}(3) \to \mathbf{SO}(3)$.

**Remark:** Because $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ are real vector spaces of dimension 3, they are isomorphic, and it is easy to construct an isomorphism. In fact, $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ are isomorphic as Lie algebras, which means that there is a linear isomorphism preserving the the Lie bracket $[A, B] = AB - BA$. However, as observed earlier, the groups $\mathbf{SU}(2)$ and $\mathbf{SO}(3)$ are *not isomorphic*.

An equivalent, but often more convenient, formula is obtained by assuming that $u = (u_1, u_2, u_3)$ is a unit vector, equivalently $\det(A) = 1$, in which case $A^2 = -I$, so we have

$$e^{\theta A} = \cos\theta I + \sin\theta A.$$

Using the quaternion notation, this is read as

$$e^{\theta A} = [\cos\theta, \sin\theta\, u].$$

**Proposition 16.8.** *The exponential map* $\exp\colon \mathfrak{su}(2) \to \mathbf{SU}(2)$ *is surjective*

*Proof.* We give an algorithm to find the logarithm $A \in \mathfrak{su}(2)$ of a unit quaternion

$$q = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$$

with $\alpha = a + bi$ and $\beta = c + id$.

If $q = I$ (*i.e.* $a = 1$), then $A = 0$. If $q = -I$ (*i.e.* $a = -1$), then

$$A = \pm\pi \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$