has no eigenvalue in the field K is quite different from the case where f has some eigenvalue in K. In the first case, h has no fixed point. It turns out that this implies that $\dim(E)$ is even and there is a simple description of the matrices representing an involution. If h has some fixed point, then f is an involution of E, so it has the eigenvalues $+1$ and -1, and E is the direct sum of the corresponding eigenspaces E_1 and E_{-1}. Then h can be described in terms of $\mathbb{P}(E_1)$ and $\mathbb{P}(E_{-1})$. For details, we refer the reader to Vienne [185] (Chapter IV, Propositions 11 and 12).

26.12 Duality in Projective Geometry

We now consider duality in projective geometry. Given a vector space E of finite dimension $n+1$, recall that its *dual space* E^* is the vector space of all linear forms $f\colon E \to K$ and that E^* is isomorphic to E. We also have a canonical isomorphism between E and its bidual E^{**}, which allows us to identify E and E^{**}.

Let $\mathcal{H}(E)$ denote the set of hyperplanes in $\mathbf{P}(E)$. In Section 26.3 we observed that the map

$$p(f) \mapsto \mathbf{P}(\mathrm{Ker}\, f)$$

is a bijection between $\mathbf{P}(E^*)$ and $\mathcal{H}(E)$, in which the equivalence class $p(f) = \{\lambda f \mid \lambda \neq 0\}$ of a nonnull linear form $f \in E^*$ is mapped to the hyperplane $\mathbf{P}(\mathrm{Ker}\, f)$. Using the above bijection between $\mathbf{P}(E^*)$ and $\mathcal{H}(E)$, a projective subspace $\mathbf{P}(U)$ of $\mathbf{P}(E^*)$ (where U is a subspace of E^*) can be identified with a subset of $\mathcal{H}(E)$, namely the family

$$\{\mathbf{P}(H) \mid H = \mathrm{Ker}\, f,\ f \in U - \{0\}\}$$

consisting of the projective hyperplanes in $\mathcal{H}(E)$ corresponding to nonnull linear forms in U. Such subsets of $\mathcal{H}(E)$ are called *linear systems (of hyperplanes)*.

The bijection between $\mathbf{P}(E^*)$ and $\mathcal{H}(E)$ allows us to view $\mathcal{H}(E)$ as a projective space, and linear systems as projective subspaces of $\mathcal{H}(E)$. In the projective space $\mathcal{H}(E)$, a point is a hyperplane in $\mathbf{P}(E)$! The duality between subspaces of E and subspaces of E^* (reviewed below) and the fact that there is a bijection between $\mathbf{P}(E^*)$ and $\mathcal{H}(E)$ yields a powerful duality between the set of projective subspaces of $\mathbf{P}(E)$ and the set of linear systems in $\mathcal{H}(E)$ (or equivalently, the set of projective subspaces of $\mathbf{P}(E^*)$).

The idea of duality in projective geometry goes back to Gergonne and Poncelet, in the early nineteenth century. However, Poncelet had a more restricted type of duality in mind (polarity with respect to a conic or a quadric), whereas Gergonne had the more general idea of the duality between points and lines (or points and planes). This more general duality arises from a specific pairing between E and E^* (a nonsingular bilinear form). Here we consider the pairing $\langle -,- \rangle\colon E^* \times E \to K$, defined such that

$$\langle f, v\rangle = f(v),$$