for any arbitrary $v \in \mathbb{R}^{n-r}$. It follows that the $n-r$ columns of the matrix

$$N = \begin{pmatrix} -F \\ I_{n-r} \end{pmatrix}$$

form a basis of the kernel of A. This is because N contains the identity matrix I_{n-r} as a submatrix, so the columns of N are linearly independent. In summary, if $N^1, \ldots, N^{n-r}$ are the columns of N, then the general solution of the equation $Ax = b$ is given by

$$x = \begin{pmatrix} d \\ 0_{n-r} \end{pmatrix} + x_{r+1}N^1 + \cdots + x_n N^{n-r},$$

where $x_{r+1}, \ldots, x_n$ are the free variables; that is, the nonpivot variables.

Going back to our example from Kumpel and Thorpe [107], we see that

$$N = \begin{pmatrix} -F \\ I_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & -2 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since earlier we permuted the second and the third column, row 2 and row 3 need to be swapped so the general solution in terms of the original variables is given by

$$x = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ 0 \\ 3 \\ 0 \\ 1 \end{pmatrix}.$$

In the general case where the columns corresponding to pivots are mixed with the columns corresponding to free variables, we find the special solution as follows. Let $i_1 < \cdots < i_r$ be the indices of the columns corresponding to pivots. Assign b'_k to the pivot variable x_{i_k} for $k = 1, \ldots, r$, and set all other variables to 0. To find a basis of the kernel, we form the $n-r$ vectors N^k obtained as follows. Let $j_1 < \cdots < j_{n-r}$ be the indices of the columns corresponding to free variables. For every column j_k corresponding to a free variable $(1 \leq k \leq n-r)$, form the vector N^k defined so that the entries $N^k_{i_1}, \ldots, N^k_{i_r}$ are equal to the negatives of the first r entries in column j_k (flip the sign of these entries); let $N^k_{j_k} = 1$, and set all other entries to zero. Schematically, if the column of index j_k (corresponding to the free variable x_{j_k}) is

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_r \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$