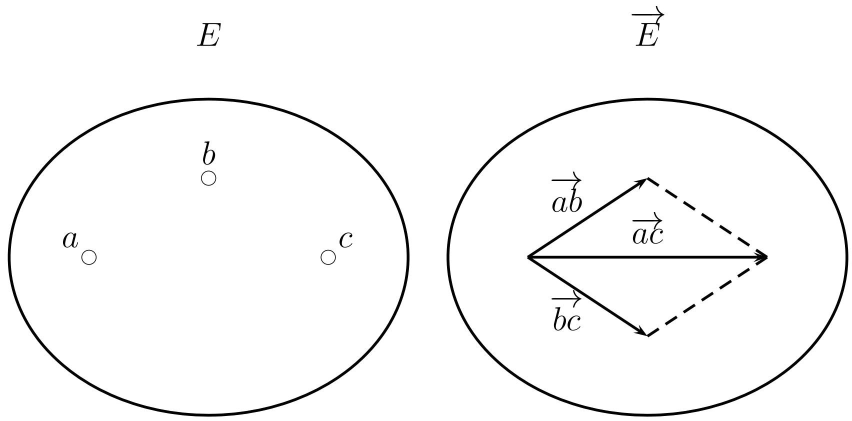

Figure 24.8: Points and corresponding vectors in affine geometry.

risk of boring certain readers, we give another example showing what goes wrong if we are not careful in defining linear combinations of points.

Consider $\mathbb{R}^2$ as an affine space, under its natural coordinate system with origin $O = (0, 0)$ and basis vectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$. Given any two points $a = (a_1, a_2)$ and $b = (b_1, b_2)$, it is natural to define the affine combination $\lambda a + \mu b$ as the point of coordinates

$$(\lambda a_1 + \mu b_1, \lambda a_2 + \mu b_2).$$

Thus, when $a = (-1, -1)$ and $b = (2, 2)$, the point $a + b$ is the point $c = (1, 1)$.

Let us now consider the new coordinate system with respect to the origin $c = (1, 1)$ (and the same basis vectors). This time, the coordinates of a are $(-2, -2)$, the coordinates of b are $(1, 1)$, and the point $a+b$ is the point d of coordinates $(-1, -1)$. However, it is clear that the point d is identical to the origin $O = (0, 0)$ of the first coordinate system. This situation is illustrated in Figure 24.9.

Thus, $a + b$ corresponds to two different points depending on which coordinate system is used for its computation!

This shows that some extra condition is needed in order for affine combinations to make sense. It turns out that if the scalars sum up to 1, the definition is intrinsic, as the following proposition shows.

Proposition 24.1. *Given an affine space E, let $(a_i)_{i\in I}$ be a family of points in E, and let $(\lambda_i)_{i\in I}$ be a family of scalars. For any two points $a, b \in E$, the following properties hold:*

(1) If $\sum_{i\in I} \lambda_i = 1$, then

$$a + \sum_{i\in I} \lambda_i \overrightarrow{aa_i} = b + \sum_{i\in I} \lambda_i \overrightarrow{ba_i}.$$