

matrix A is similar to its transpose A^\top . The proof requires advanced concepts (the Jordan form or similarity invariants).

If $\mathcal{U} = (u_1, \dots, u_n)$ and $\mathcal{V} = (v_1, \dots, v_n)$ are two bases of E , the change of basis matrix

$$P = P_{\mathcal{V}, \mathcal{U}} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

from (u_1, \dots, u_n) to (v_1, \dots, v_n) is the matrix whose j th column consists of the coordinates of v_j over the basis (u_1, \dots, u_n) , which means that

$$v_j = \sum_{i=1}^n a_{ij} u_i.$$

It is natural to extend the matrix notation and to express the vector $\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ in E^n as the

product of a matrix times the vector $\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$ in E^n , namely as

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix},$$

but notice that the matrix involved is not P , but its *transpose* P^\top .

This observation has the following consequence: if $\mathcal{U} = (u_1, \dots, u_n)$ and $\mathcal{V} = (v_1, \dots, v_n)$ are two bases of E and if

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = A \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix},$$

that is,

$$v_i = \sum_{j=1}^n a_{ij} u_j,$$

for any vector $w \in E$, if

$$w = \sum_{i=1}^n x_i u_i = \sum_{k=1}^n y_k v_k = \sum_{k=1}^n y_k \left(\sum_{j=1}^n a_{kj} u_j \right) = \sum_{j=1}^n \left(\sum_{k=1}^n a_{kj} y_k \right) u_j,$$