(over $K[X]$). What Theorem 36.20 tells us is that there are $K[X]$-bases $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ of E_f with respect to which the matrix of ψ is D. Then

$$\psi(u_i) = v_i, \quad i = 1, \ldots, n-m,$$
$$\psi(u_{n-m+i}) = q_i v_{n-m+i}, \quad i = 1, \ldots, m,$$

and because $\mathrm{Im}(\psi) = \mathrm{Ker}\,(\sigma)$, this implies that

$$\sigma(v_i) = 0, \quad i = 1, \ldots, n-m.$$

Consequently, $w_1 = \sigma(v_{n-m+1}), \ldots, w_m = \sigma(v_n)$ span E_f as a $K[X]$-module, with $w_i \in E$, and we have

$$M(f) = K[X]w_1 \oplus \cdots \oplus K[X]w_m,$$

where $K[X]w_i \approx K[X]/(q_i)$ as a cyclic $K[X]$-module. Since $\mathrm{Im}(\psi) = \mathrm{Ker}\,(\sigma)$, we have

$$0 = \sigma(\psi(u_{n-m+i})) = \sigma(q_i v_{n-m+i}) = q_i \sigma(v_{n-m+i}) = q_i w_i,$$

so as a K-vector space, the cyclic subspace $Z(w_i; f) = K[X]w_i$ has q_i as annihilator, and by a remark from Section 36.1, it has the basis (over K)

$$(w_i, f(w_i), \ldots, f^{n_i - 1}(w_i)), \quad n_i = \deg(q_i).$$

Furthermore, over this basis, the restriction of f to $Z(w_i; f)$ is represented by the companion matrix of q_i. By putting all these bases together, we obtain a block matrix which is the canonical rational form of f (and A).

Now, $XI - A = QDP^{-1}$ is the matrix of ψ with respect to the canonical basis $(e_1, \ldots, e_n)$ (over $K[X]$), and D is the matrix of ψ with respect to the bases $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ (over $K[X]$), which tells us that the columns of Q consist of the coordinates (in $K[X]$) of the basis vectors $(v_1, \ldots, v_n)$ with respect to the basis $(e_1, \ldots, e_n)$. Therefore, the coordinates (in K) of the vectors $(w_1, \ldots, w_m)$ spanning E_f over $K[X]$, where $w_i = \sigma(v_{n-m+i})$, are obtained by substituting the matrix A for X in the coordinates of the columns vectors of Q, and evaluating the resulting expressions.

Since

$$D = Q^{-1}(XI - A)P,$$

the matrix D is obtained from A by a sequence of elementary row operations whose product is Q^{-1} and a sequence of elementary column operations whose product is P. Therefore, to compute the vectors $w_1, \ldots, w_m$ from A, we simply have to figure out how to construct Q from the sequence of elementary row operations that yield Q^{-1}. The trick is to use column operations to gather a product of row operations in reverse order.

Indeed, if Q^{-1} is the product of elementary row operations

$$Q^{-1} = E_k \cdots E_2 E_1,$$