Problem 3.4. (1) Prove that the axioms of vector spaces imply that

$$\begin{aligned}\alpha \cdot 0 &= 0\\ 0 \cdot v &= 0\\ \alpha \cdot (-v) &= -(\alpha \cdot v)\\ (-\alpha) \cdot v &= -(\alpha \cdot v),\end{aligned}$$

for all $v \in E$ and all $\alpha \in K$, where E is a vector space over K.

(2) For every $\lambda \in \mathbb{R}$ and every $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, define λx by

$$\lambda x = \lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n).$$

Recall that every vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ can be written uniquely as

$$x = x_1 e_1 + \cdots + x_n e_n,$$

where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, with a single 1 in position i. For any operation $\cdot \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, if $\cdot$ satisfies the Axiom (V1) of a vector space, then prove that for any $\alpha \in \mathbb{R}$, we have

$$\alpha \cdot x = \alpha \cdot (x_1 e_1 + \cdots + x_n e_n) = \alpha \cdot (x_1 e_1) + \cdots + \alpha \cdot (x_n e_n).$$

Conclude that $\cdot$ is completely determined by its action on the one-dimensional subspaces of $\mathbb{R}^n$ spanned by $e_1, \ldots, e_n$.

(3) Use (2) to define operations $\cdot \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ that satisfy the Axioms (V1–V3), but for which Axiom V4 fails.

(4) For any operation $\cdot \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, prove that if $\cdot$ satisfies the Axioms (V2–V3), then for every rational number $r \in \mathbb{Q}$ and every vector $x \in \mathbb{R}^n$, we have

$$r \cdot x = r(1 \cdot x).$$

In the above equation, $1 \cdot x$ is some vector $(y_1, \ldots, y_n) \in \mathbb{R}^n$ not necessarily equal to $x = (x_1, \ldots, x_n)$, and

$$r(1 \cdot x) = (r y_1, \ldots, r y_n),$$

as in Part (2).

Use (4) to conclude that any operation $\cdot \colon \mathbb{Q} \times \mathbb{R}^n \to \mathbb{R}^n$ that satisfies the Axioms (V1–V3) is completely determined by the action of 1 on the one-dimensional subspaces of $\mathbb{R}^n$ spanned by $e_1, \ldots, e_n$.

Problem 3.5. Let A_1 be the following matrix:

$$A_1 = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & -1 \\ -3 & -5 & 1 \end{pmatrix}.$$

Prove that the columns of A_1 are linearly independent. Find the coordinates of the vector $x = (6, 2, -7)$ over the basis consisting of the column vectors of A_1.