then we have

$$u^\top K_S\, u = \sum_{i,j=1}^{p} \kappa(x_i, x_j) u_i u_j \geq 0, \qquad \text{for all } u \in \mathbb{R}^p.$$

Among other things, the next proposition shows that a positive definite kernel satisfies the Cauchy–Schwarz inequality.

Proposition 53.4. *A Hermitian* 2×2 *matrix*

$$A = \begin{pmatrix} a & \overline{b} \\ b & d \end{pmatrix}$$

is positive semidefinite if and only if $a \geq 0$, $d \geq 0$, *and* $ad - |b|^2 \geq 0$.

Let $\kappa\colon X \times X \to \mathbb{C}$ *be a positive definite kernel. For all* $x, y \in X$, *we have*

$$|\kappa(x, y)|^2 \leq \kappa(x, x)\kappa(y, y).$$

Proof. For all $x, y \in \mathbb{C}$, we have

$$\begin{aligned} \begin{pmatrix} \overline{x} & \overline{y} \end{pmatrix} \begin{pmatrix} a & \overline{b} \\ b & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \overline{x} & \overline{y} \end{pmatrix} \begin{pmatrix} ax + \overline{b}y \\ bx + dy \end{pmatrix} \\ &= a|x|^2 + bx\overline{y} + \overline{bx\overline{y}} + d|y|^2. \end{aligned}$$

If A is positive semidefinite, then we already know that $a \geq 0$ and $d \geq 0$. If $a = 0$, then we must have $b = 0$, since otherwise we can make $bx\overline{y} + \overline{bx\overline{y}}$, which is twice the real part of $bx\overline{y}$, as negative as we want. In this case, $ad - |b|^2 = 0$.

If $a > 0$, then

$$a|x|^2 + bx\overline{y} + \overline{bx\overline{y}} + d|y|^2 = a\left| x + \frac{\overline{b}}{a} y \right|^2 + \frac{|y|^2}{a}(ad - |b|^2).$$

If $ad - |b|^2 < 0$, we can pick $y \neq 0$ and $x = -(\overline{b}y)/a$, so that the above expression is negative. Therefore, $ad - |b|^2 \geq 0$. The converse is trivial.

If $x = y$, the inequality $|\kappa(x, y)|^2 \leq \kappa(x, x)\kappa(y, y)$ is trivial. If $x \neq y$, the inequality follows by applying the criterion for being positive semidefinite to the matrix

$$\begin{pmatrix} \kappa(x, x) & \overline{\kappa(x, y)} \\ \kappa(x, y) & \kappa(y, y) \end{pmatrix},$$

as claimed. ∎

The following property due to I. Schur (1911) shows that the pointwise product of two positive definite kernels is also a positive definite kernel.