and

$$\lim_{x \to a, x \in A \cap (a, +\infty)} f(x) = f(a_+)$$

both exist, and either $f(a_-) \neq f(a)$, or $f(a_+) \neq f(a)$.

Note that it is possible that $f(a_-) = f(a_+)$, but f is still discontinuous at a if this common value differs from $f(a)$. Functions defined on a nonempty subset of $\mathbb{R}$, and that are continuous, except for some points of discontinuity of the first kind, play an important role in analysis.

We now turn to connectivity properties of topological spaces.

37.4 Connected Sets

Connectivity properties of topological spaces play a very important role in understanding the topology of surfaces. This section gathers the facts needed to have a good understanding of the classification theorem for compact surfaces (with boundary). The main references are Ahlfors and Sario [2] and Massey [121, 122]. For general background on topology, geometry, and algebraic topology, we also highly recommend Bredon [30] and Fulton [67].

Definition 37.21. A topological space $(E, \mathcal{O})$ is *connected* if the only subsets of E that are both open and closed are the empty set and E itself. Equivalently, $(E, \mathcal{O})$ is connected if E cannot be written as the union $E = U \cup V$ of two disjoint nonempty open sets U, V, or if E cannot be written as the union $E = U \cup V$ of two disjoint nonempty closed sets. A subset, $S \subseteq E$, is *connected* if it is connected in the subspace topology on S induced by $(E, \mathcal{O})$. See Figure 37.22. A connected open set is called a *region* and a closed set is a *closed region* if its interior is a connected (open) set.

The definition of connectivity is meant to capture the fact that a connected space S is "one piece." Given the metric space $(\mathbb{R}^n, \| \|_2)$, the quintessential examples of connected spaces are $B_0(a, \rho)$ and $B(a, \rho)$. In particular, the following standard proposition characterizing the connected subsets of $\mathbb{R}$ can be found in most topology texts (for example, Munkres [131], Schwartz [150]). For the sake of completeness, we give a proof.

Proposition 37.16. *A subset of the real line, $\mathbb{R}$, is connected iff it is an interval, i.e., of the form $[a, b]$, $(a, b]$, where $a = -\infty$ is possible, $[a, b)$, where $b = +\infty$ is possible, or (a, b), where $a = -\infty$ or $b = +\infty$ is possible.*

Proof. Assume that A is a connected nonempty subset of $\mathbb{R}$. The cases where $A = \emptyset$ or A consists of a single point are trivial. Otherwise, we show that whenever $a, b \in A$, $a < b$, then the entire interval $[a, b]$ is a subset of A. Indeed, if this was not the case, there would be some $c \in (a, b)$ such that $c \notin A$, and then we could write $A = ((-\infty, c) \cap A) \cup ((c, +\infty) \cap A)$, where $(-\infty, c) \cap A$ and $(c, +\infty) \cap A$ are nonempty and disjoint open subsets of A, contradicting the fact that A is connected. It follows easily that A must be an interval.