Definition 47.2. Given any Linear Program (P)

$$\begin{aligned} &\text{maximize} \quad cx \\ &\text{subject to} \quad Ax \leq b \text{ and } x \geq 0, \end{aligned}$$

with A an $m \times n$ matrix, the *dual* (D) of (P) is the following optimization problem:

$$\begin{aligned} &\text{minimize} \quad yb \\ &\text{subject to} \quad yA \geq c \text{ and } y \geq 0, \end{aligned}$$

where $y \in (\mathbb{R}^m)^*$.

The variables $y_1, \ldots, y_m$ are called the *dual variables*. The original Linear Program (P) is called the *primal* linear program and the original variables $x_1, \ldots, x_n$ are the *primal variables*.

Here is an explicit example of a linear program and its dual.

Example 47.1. Consider the linear program illustrated by Figure 47.3

$$\begin{aligned} &\text{maximize} \quad 2x_1 + 3x_2 \\ &\text{subject to} \\ &\qquad 4x_1 + 8x_2 \leq 12 \\ &\qquad 2x_1 + x_2 \leq 3 \\ &\qquad 3x_1 + 2x_2 \leq 4 \\ &\qquad x_1 \geq 0,\ x_2 \geq 0. \end{aligned}$$

Its dual linear program is illustrated in Figure 47.4

$$\begin{aligned} &\text{minimize} \quad 12y_1 + 3y_2 + 4y_3 \\ &\text{subject to} \\ &\qquad 4y_1 + 2y_2 + 3y_3 \geq 2 \\ &\qquad 8y_1 + y_2 + 2y_3 \geq 3 \\ &\qquad y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0. \end{aligned}$$

It can be checked that $(x_1, x_2) = (1/2, 5/4)$ is an optimal solution of the primal linear program, with the maximum value of the objective function $2x_1 + 3x_2$ equal to $19/4$, and that $(y_1, y_2, y_3) = (5/16, 0, 1/4)$ is an optimal solution of the dual linear program, with the minimum value of the objective function $12y_1 + 3y_2 + 4y_3$ also equal to $19/4$.

Observe that in the Primal Linear Program (P), we are looking for a *vector* $x \in \mathbb{R}^n$ maximizing the form cx, and that the constraints are determined by the action of the *rows* of the matrix A on x. On the other hand, in the Dual Linear Program (D), we are looking for a *linear form* $y \in (\mathbb{R}^*)^m$ minimizing the form yb, and the constraints are determined by