If $\nabla J_x \neq 0$, we have $\lambda(x) \neq 0$, so $\langle \nabla J_x, d_{\rm nt}\rangle < 0$, and $d_{\rm nt}$ is indeed a descent direction. The number $\langle \nabla J_x, d_{\rm nt}\rangle$ is the constant that shows up during a backtracking line search.

A nice feature of the Newton step and of the Newton decrement is that they are affine invariant. This means that if T is an invertible matrix and if we define g by $g(y) = J(Ty)$, if the Newton step associated with J is denoted by $d_{J,{\rm nt}}$ and similarly the Newton step associated with g is denoted by $d_{g,{\rm nt}}$, then it is shown in Boyd and Vandenberghe [29] (Section 9.5.1) that

$$d_{g,{\rm nt}} = T^{-1} d_{J,{\rm nt}},$$

and so

$$x + d_{J,{\rm nt}} = T(y + d_{g,{\rm nt}}).$$

A similar properties applies to the Newton decrement.

Newton's method consists of the following steps: Given a starting point $u_0 \in {\rm dom}(J)$ and a tolerance $\epsilon > 0$ do:

repeat

(1) Compute the Newton step and decrement
$d_{{\rm nt},k} = -(\nabla^2 J(u_k))^{-1} \nabla J_{u_k}$ and $\lambda(u_k)^2 = (\nabla J_{u_k})^\top (\nabla^2 J(u_k))^{-1} \nabla J_{u_k}$.

(2) Stopping criterion. **quit** if $\lambda(u_k)^2/2 \leq \epsilon$.

(3) Line Search. Perform an exact or backtracking line search to find ρ_k.

(4) Update. $u_{k+1} = u_k + \rho_k d_{{\rm nt},k}$.

Observe that this is essentially the descent procedure of Section 49.8 using the Newton step as search direction, except that the stopping criterion is checked just after computing the search direction, rather than after the update (a very minor difference).

The convergence of Newton's method is thoroughly analyzed in Boyd and Vandenberghe [29] (Section 9.5.3). This analysis is made under the following assumptions:

(1) The function $J \colon \Omega \to \mathbb{R}$ is a convex function defined on some open subset Ω of $\mathbb{R}^n$ which is twice differentiable and its Hessian $\nabla^2 J(x)$ is symmetric positive definite for all $x \in \Omega$. This implies that there are two constants $m > 0$ and $M > 0$ such that $mI \preceq \nabla^2 J(x) \preceq MI$ for all $x \in \Omega$, which means that the eigenvalues of $\nabla^2 J(x)$ belong to $[m, M]$.

(2) The Hessian is Lipschitzian, which means that there is some $L \geq 0$ such that

$$\left\| \nabla^2 J(x) - \nabla^2 J(y) \right\|_2 \leq L \left\| x, y \right\|_2 \quad \text{for all } x, y \in \Omega.$$

It turns out that the iterations of Newton's method fall into two phases, depending whether $\left\| \nabla J_{u_k} \right\|_2 \geq \eta$ or $\left\| \nabla J_{u_k} \right\|_2 < \eta$, where η is a number which depends on m, L, and the constant α used in the backtracking line search, and $\eta \leq m^2/L$.