(2) Prove that the eigenvalues of a skew-symmetric matrix are either 0 or pure imaginary (that is, of the form $i\mu$ for $\mu \in \mathbb{R}$.).

Let $C\colon \mathfrak{so}(n) \to \mathrm{M}_n(\mathbb{R})$ be the function (called the *Cayley transform* of B) given by

$$C(B) = (I - B)(I + B)^{-1}.$$

Prove that if B is skew-symmetric, then $I - B$ and $I + B$ are invertible, and so C is well-defined. Prove that

$$(I + B)(I - B) = (I - B)(I + B),$$

and that

$$(I + B)(I - B)^{-1} = (I - B)^{-1}(I + B).$$

Prove that

$$(C(B))^\top C(B) = I$$

and that

$$\det C(B) = +1,$$

so that $C(B)$ is a rotation matrix. Furthermore, show that $C(B)$ does not admit -1 as an eigenvalue.

(3) Let $\mathbf{SO}(n)$ be the group of $n \times n$ rotation matrices. Prove that the map

$$C\colon \mathfrak{so}(n) \to \mathbf{SO}(n)$$

is bijective onto the subset of rotation matrices that do not admit -1 as an eigenvalue. Show that the inverse of this map is given by

$$B = (I + R)^{-1}(I - R) = (I - R)(I + R)^{-1},$$

where $R \in \mathbf{SO}(n)$ does not admit -1 as an eigenvalue.

Problem 17.12. Please refer back to Problem 4.6. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A (not necessarily distinct). Using Schur's theorem, A is similar to an upper triangular matrix B, that is, $A = PBP^{-1}$ with B upper triangular, and we may assume that the diagonal entries of B in descending order are $\lambda_1, \ldots, \lambda_n$.

(1) If the E_{ij} are listed according to total order given by

$$(i, j) < (h, k) \quad \text{iff} \quad \begin{cases} i = h \text{ and } j > k \\ \text{or } i < h. \end{cases}$$

prove that R_B is an upper triangular matrix whose diagonal entries are

$$(\underbrace{\lambda_n, \ldots, \lambda_1, \ldots, \lambda_n, \ldots, \lambda_1}_{n^2}),$$