(1) The first phase, called the *damped Newton phase*, occurs while $\|\nabla J_{u_k}\|_2 \geq \eta$. During this phase, the procedure can choose a step size $\rho_k = t < 1$, and there is some constant $\gamma > 0$ such that

$$J(u_{k+1}) - J(u_k) \leq -\gamma.$$

(2) The second phase, called the *quadratically convergent phase* or *pure Newton phase*, occurs while $\|\nabla J_{u_k}\|_2 < \eta$. During this phase, the step size $\rho_k = t = 1$ is always chosen, and we have

$$\frac{L}{2m^2}\left\|\nabla J_{u_{k+1}}\right\|_2 \leq \left(\frac{L}{2m^2}\left\|\nabla J_{u_k}\right\|_2\right)^2. \qquad (*_1)$$

If we denote the minimal value of f by p^*, then the number of damped Newton steps is at most

$$\frac{J(u_0) - p^*}{\gamma}.$$

Equation $(*_1)$ and the fact that $\eta \leq m^2/L$ shows that if $\|\nabla J_{u_k}\|_2 < \eta$, then $\left\|\nabla J_{u_{k+1}}\right\|_2 < \eta$. It follows by induction that for all $\ell \geq k$, we have

$$\frac{L}{2m^2}\left\|\nabla J_{u_{\ell+1}}\right\|_2 \leq \left(\frac{L}{2m^2}\left\|\nabla J_{u_\ell}\right\|_2\right)^2, \qquad (*_2)$$

and thus (since $\eta \leq m^2/L$ and $\|\nabla J_{u_k}\|_2 < \eta$, we have $(L/m^2)\left\|\nabla J_{u_k}\right\|_2 < (L/m^2)\eta \leq 1$), so

$$\frac{L}{2m^2}\left\|\nabla J_{u_\ell}\right\|_2 \leq \left(\frac{L}{2m^2}\left\|\nabla J_{u_k}\right\|_2\right)^{2^{\ell-k}} \leq \left(\frac{1}{2}\right)^{2^{\ell-k}}, \quad \ell \geq k. \qquad (*_3)$$

It is shown in Boyd and Vandenberghe [29] (Section 9.1.2) that the hypothesis $mI \preceq \nabla^2 J(x)$ implies that

$$J(x) - p^* \leq \frac{1}{2m}\left\|\nabla J_x\right\|_2^2 \quad x \in \Omega.$$

As a consequence, by $(*_3)$, we have

$$J(u_\ell) - p^* \leq \frac{1}{2m}\left\|\nabla J_{u_\ell}\right\|_2^2 \leq \frac{2m^3}{L^2}\left(\frac{1}{2}\right)^{2^{\ell-k+1}}. \qquad (*_4)$$

Equation $(*_4)$ shows that the convergence during the quadratically convergence phase is very fast. If we let

$$\epsilon_0 = \frac{2m^3}{L^2},$$

then Equation $(*_4)$ implies that we must have $J(u_\ell) - p^* \leq \epsilon$ after no more than

$$\log_2 \log_2(\epsilon_0/\epsilon)$$