Since f is an isometry, we must have $\varphi(f(u), f(v)) = \varphi(u, v)$ for all $u, v \in E$, which means that

$$\begin{aligned}\varphi(u, v) &= \varphi(f(u), f(v)) \\ &= \varphi(u + \psi(u)w, v + \psi(v)w) \\ &= \varphi(u, v) + \psi(u)\varphi(w, v) + \psi(v)\varphi(u, w) + \psi(u)\psi(v)\varphi(w, w) \\ &= \varphi(u, v) + \psi(u)\varphi(w, v) - \psi(v)\varphi(w, u),\end{aligned}$$

which yields

$$\psi(u)\varphi(w, v) = \psi(v)\varphi(w, u) \quad \text{for all } u, v \in E.$$

Since φ is nondegenerate, we can pick some v_0 such that $\varphi(w, v_0) \neq 0$, and we get $\psi(u)\varphi(w, v_0) = \psi(v_0)\varphi(w, u)$ for all $u \in E$; that is,

$$\psi(u) = \lambda\varphi(w, u) \quad \text{for all } u \in E,$$

for some $\lambda \in K$. Therefore, f is of the form

$$f(u) = u + \lambda\varphi(w, u)w, \quad \text{for all } u \in E.$$

It is also clear that every f of the above form is a symplectic map. If $\lambda = 0$, then $f = \mathrm{id}$. Otherwise, if $\lambda \neq 0$, then $f(u) = u$ iff $\varphi(w, u) = 0$ iff $u \in (Kw)^{\perp} = H$, where H is a hyperplane. Thus, f fixes every vector in the hyperplane H. Note that since φ is alternating, $\varphi(w, w) = 0$, which means that $w \in H$.

In summary, we have characterized all the symplectic maps that leave every vector in some hyperplane fixed, and we make the following definition.

Definition 29.20. Given a nondegenerate alternating form φ on a space E, a *symplectic transvection (of direction w)* is a linear map f of the form

$$f(u) = u + \lambda\varphi(w, u)w, \quad \text{for all } u \in E,$$

for some nonzero $w \in E$ and some $\lambda \in K$. If $\lambda \neq 0$, the subspace of vectors left fixed by f is the hyperplane $H = (Kw)^{\perp}$. The map f is also denoted $\tau_{w,\lambda}$.

Observe that

$$\tau_{w,\lambda} \circ \tau_{w,\mu} = \tau_{w,\lambda+\mu}$$

and $\tau_{w,\lambda} = \mathrm{id}$ iff $\lambda = 0$. The above shows that $\det(\tau_{w,\lambda}) = 1$, since when $\lambda \neq 0$, we have $\tau_{w,\lambda} = (\tau_{w,\lambda/2})^2$.

Our next goal is to show that if u and v are any two nonzero vectors in E, then there is a simple symplectic map f such that $f(u) = v$.

Proposition 29.36. *Given any two nonzero vectors $u, v \in E$, there is a symplectic map f such that $f(u) = v$, and f is either a symplectic transvection, or the composition of two symplectic transvections.*