by $a + u$. However, since

$$\langle a, \lambda\rangle \mathbin{\widehat{+}} u = \langle a + \lambda^{-1}u, \lambda\rangle,$$

we will refrain from writing $\lambda a \mathbin{\widehat{+}} u$ as $\lambda a + u$, because we find it too confusing. From Proposition 25.1, for every $a \in E$, every element of $\widehat{E}$ can be written uniquely as $u \mathbin{\widehat{+}} \lambda a$. We also denote

$$\lambda a \mathbin{\widehat{+}} (-\mu) b$$

by

$$\lambda a \mathbin{\widehat{-}} \mu b.$$

We can now justify rigorously the programming trick of the introduction of an extra coordinate to distinguish between points and vectors. First, we make a few observations. Given any family $(a_i)_{i\in I}$ of points in E, and any family $(\lambda_i)_{i\in I}$ of scalars in $\mathbb{R}$, it is easily shown by induction on the size of I that the following holds:

(1) If $\sum_{i\in I} \lambda_i = 0$, then

$$\sum_{i\in I} \langle a_i, \lambda_i\rangle = \overrightarrow{\sum_{i\in I} \lambda_i a_i},$$

where

$$\overrightarrow{\sum_{i\in I} \lambda_i a_i} = \sum_{i\in I} \lambda_i \overrightarrow{b a_i}$$

for any $b \in E$, which, by Proposition 24.1, is a vector independent of b, or

(2) If $\sum_{i\in I} \lambda_i \neq 0$, then

$$\sum_{i\in I} \langle a_i, \lambda_i\rangle = \left\langle \sum_{i\in I} \frac{\lambda_i}{\sum_{i\in I} \lambda_i} a_i, \sum_{i\in I} \lambda_i \right\rangle.$$

Thus, we see how barycenters reenter the scene quite naturally, and that in $\widehat{E}$, we can make sense of $\sum_{i\in I}\langle a_i, \lambda_i\rangle$, regardless of the value of $\sum_{i\in I} \lambda_i$. When $\sum_{i\in I} \lambda_i = 1$, the element $\sum_{i\in I}\langle a_i, \lambda_i\rangle$ belongs to the hyperplane $\omega^{-1}(1)$, and thus it is a point. When $\sum_{i\in I} \lambda_i = 0$, the linear combination of points $\sum_{i\in I} \lambda_i a_i$ is a vector, and when $I = \{1, \ldots, n\}$, we allow ourselves to write

$$\lambda_1 a_1 \mathbin{\widehat{+}} \cdots \mathbin{\widehat{+}} \lambda_n a_n,$$

where some of the occurrences of $\widehat{+}$ can be replaced by $\widehat{-}$, as

$$\lambda_1 a_1 + \cdots + \lambda_n a_n,$$

where the occurrences of $\widehat{-}$ (if any) are replaced by $-$.

In fact, we have the following slightly more general property, which is left as an exercise.