Example 14.4. Let $\mathcal{C}[a, b]$ be the set of complex-valued continuous functions $f\colon [a, b] \to \mathbb{C}$ under the Hermitian form

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}dx.$$

It is easy to check that this Hermitian form is positive definite. Thus, $\mathcal{C}[a, b]$ is a Hermitian space.

Example 14.5. Let $E = \mathrm{M}_n(\mathbb{C})$ be the vector space of complex $n \times n$ matrices. If we view a matrix $A \in \mathrm{M}_n(\mathbb{C})$ as a "long" column vector obtained by concatenating together its columns, we can define the Hermitian product of two matrices $A, B \in \mathrm{M}_n(\mathbb{C})$ as

$$\langle A, B\rangle = \sum_{i,j=1}^{n} a_{ij}\overline{b}_{ij},$$

which can be conveniently written as

$$\langle A, B\rangle = \mathrm{tr}(A^\top \overline{B}) = \mathrm{tr}(B^* A).$$

Since this can be viewed as the standard Hermitian product on $\mathbb{C}^{n^2}$, it is a Hermitian product on $\mathrm{M}_n(\mathbb{C})$. The corresponding norm

$$\|A\|_F = \sqrt{\mathrm{tr}(A^* A)}$$

is the Frobenius norm (see Section 9.2).

If E is finite-dimensional and if $\varphi\colon E \times E \to \mathbb{R}$ is a sequilinear form on E, given any basis $(e_1, \ldots, e_n)$ of E, we can write $x = \sum_{i=1}^n x_i e_i$ and $y = \sum_{j=1}^n y_j e_j$, and we have

$$\varphi(x, y) = \varphi\left(\sum_{i=1}^n x_i e_i, \sum_{j=1}^n y_j e_j\right) = \sum_{i,j=1}^n x_i \overline{y}_j \varphi(e_i, e_j).$$

If we let $G = (g_{ij})$ be the matrix given by $g_{ij} = \varphi(e_j, e_i)$, and if x and y are the column vectors associated with $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$, then we can write

$$\varphi(x, y) = x^\top G^\top \, \overline{y} = y^* G x,$$

where $\overline{y}$ corresponds to $(\overline{y}_1, \ldots, \overline{y}_n)$. As in Section 12.1, we are committing the slight abuse of notation of letting x denote both the vector $x = \sum_{i=1}^n x_i e_i$ and the column vector associated with $(x_1, \ldots, x_n)$ (and similarly for y). The "correct" expression for $\varphi(x, y)$ is

$$\varphi(x, y) = \mathbf{y}^* G \mathbf{x} = \mathbf{x}^\top G^\top \overline{\mathbf{y}}.$$

Observe that in $\varphi(x, y) = y^* G x$, the matrix involved is the transpose of the matrix $(\varphi(e_i, e_j))$. The reason for this is that we want G to be positive definite when φ is positive definite, not $G^\top$.