Example 12.15. If we apply the above `Matlab` function to the matrix

$$A = \begin{pmatrix} 4 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 4 \end{pmatrix},$$

we obtain

$$Q = \begin{pmatrix} 0.9701 & -0.2339 & 0.0619 & -0.0166 & 0.0046 \\ 0.2425 & 0.9354 & -0.2477 & 0.0663 & -0.0184 \\ 0 & 0.2650 & 0.9291 & -0.2486 & 0.0691 \\ 0 & 0 & 0.2677 & 0.9283 & -0.2581 \\ 0 & 0 & 0 & 0.2679 & 0.9634 \end{pmatrix}$$

and

$$R = \begin{pmatrix} 4.1231 & 1.9403 & 0.2425 & 0 & 0 \\ 0 & 3.7730 & 1.9956 & 0.2650 & 0 \\ 0 & 0 & 3.7361 & 1.9997 & 0.2677 \\ 0 & 0 & & 073.7324 & 2.0000 \\ 0 & 0 & 0 & 0 & 3.5956 \end{pmatrix}.$$

Remark: The `Matlab` function `qr`, called by `[Q, R] = qr(A)`, does not necessarily return an upper-triangular matrix whose diagonal entries are positive.

The QR-decomposition yields a rather efficient and numerically stable method for solving systems of linear equations. Indeed, given a system $Ax = b$, where A is an $n \times n$ invertible matrix, writing $A = QR$, since Q is orthogonal, we get

$$Rx = Q^{\top} b,$$

and since R is upper triangular, we can solve it by Gaussian elimination, by solving for the last variable x_n first, substituting its value into the system, then solving for x_{n-1}, etc. The QR-decomposition is also very useful in solving least squares problems (we will come back to this in Chapter 23), and for finding eigenvalues; see Chapter 18. It can be easily adapted to the case where A is a rectangular $m \times n$ matrix with independent columns (thus, $n \leq m$). In this case, Q is not quite orthogonal. It is an $m \times n$ matrix whose columns are orthogonal, and R is an invertible $n \times n$ upper triangular matrix with positive diagonal entries. For more on QR, see Strang [169, 170], Golub and Van Loan [80], Demmel [48], Trefethen and Bau [176], or Serre [156].

A somewhat surprising consequence of the QR-decomposition is a famous determinantal inequality due to Hadamard.