Proposition 26.8. *With respect to the basis $\mathcal{P} = (p_1, p_2, p_3)$, the matrix $A_{\mathcal{P}}$ of the unique homography h of $\mathbb{RP}^2$ mapping the projective frame $([p_1], [p_2], [p_3], [p_4])$ to the projective frame $[(q_1], [q_2], [q_3], [q_4])$ is given by*

$$A_{\mathcal{P}} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix} \begin{pmatrix} \frac{\lambda_1}{\alpha_1} & 0 & 0 \\ 0 & \frac{\lambda_2}{\alpha_2} & 0 \\ 0 & 0 & \frac{\lambda_3}{\alpha_3} \end{pmatrix}.$$

Proof. Let $u_1 = \alpha_1 p_1$, $u_2 = \alpha_2 p_2$, $u_3 = \alpha_3 p_3$, and let $v_1 = \lambda_1 q_1$, $v_2 = \lambda_2 q_2$, $v_3 = \lambda_3 q_3$, so that

$$p_4 = u_1 + u_2 + u_3$$

and

$$q_4 = v_1 + v_2 + v_3.$$

Because p_1, p_2, p_3 are linearly independent and since $\alpha_i \neq 0$ for $i = 1, 2, 3$, the vectors (u_1, u_2, u_2) are also linearly independent, so there is a unique linear map $f \colon \mathbb{R}^3 \to \mathbb{R}^3$, such that

$$f(u_i) = v_i \quad i = 1, \ldots, 3,$$

and by linearity

$$f(p_4) = f(u_1 + u_2 + u_3) = f(u_1) + f(u_2) + f(u_3) = v_1 + v_2 + v_3 = q_4.$$

With respect to the basis $\mathcal{P} = (p_1, p_2, p_3)$, we have

$$f(p_i) = \frac{1}{\alpha_i} v_i = \frac{\lambda_i}{\alpha_i} q_i, \quad i = 1, \ldots, 3,$$

so with respect to the basis $\mathcal{P}$, the matrix of f is

$$A_{\mathcal{P}} = \begin{pmatrix} \frac{\lambda_1}{\alpha_1} x_1 & \frac{\lambda_2}{\alpha_2} x_2 & \frac{\lambda_3}{\alpha_3} x_3 \\ \frac{\lambda_1}{\alpha_1} y_1 & \frac{\lambda_2}{\alpha_2} y_2 & \frac{\lambda_3}{\alpha_3} y_3 \\ \frac{\lambda_1}{\alpha_1} z_1 & \frac{\lambda_2}{\alpha_2} z_2 & \frac{\lambda_3}{\alpha_3} z_3 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix} \begin{pmatrix} \frac{\lambda_1}{\alpha_1} & 0 & 0 \\ 0 & \frac{\lambda_2}{\alpha_2} & 0 \\ 0 & 0 & \frac{\lambda_3}{\alpha_3} \end{pmatrix},$$

as claimed. $\square$

If we assume that we pick the coordinates of (p_1, p_2, p_3, p_4) and (q_1, q_2, q_3, q_4) with respect to the canonical basis $\mathcal{E}$, then the coordinates $\alpha_1, \alpha_2, \alpha_3$ and $\lambda_1, \lambda_2, \lambda_3$ are solutions of the systems

$$\begin{pmatrix} p_1^x & p_2^x & p_3^x \\ p_1^y & p_2^y & p_3^y \\ p_1^z & p_2^z & p_3^z \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} p_4^x \\ p_4^y \\ p_4^z \end{pmatrix}$$