By putting $(*_3)$ and $(*_5)$ together we obtain

$$\sum_{i=1}^{m} \xi_i \varphi_i(u_{\mu+\xi}) \leq G(\mu+\xi) - G(\mu) \leq \sum_{i=1}^{m} \xi_i \varphi_i(u_\mu). \qquad (*_6)$$

Consequently there is some $\theta \in [0,1]$ such that

$$\begin{aligned} G(\mu+\xi) - G(\mu) &= (1-\theta)\left(\sum_{i=1}^{m} \xi_i \varphi_i(u_\mu)\right) + \theta\left(\sum_{i=1}^{m} \xi_i \varphi_i(u_{\mu+\xi})\right) \\ &= \sum_{i=1}^{m} \xi_i \varphi_i(u_\mu) + \theta\left(\sum_{i=1}^{m} \xi_i (\varphi_i(u_{\mu+\xi}) - \varphi_i(u_\mu))\right). \end{aligned}$$

Since by hypothesis the functions $\mu \mapsto u_\mu$ (from $\mathbb{R}^m_+$ to Ω) and $\varphi_i \colon \Omega \to \mathbb{R}$ are continuous, for any $\mu \in \mathbb{R}^m_+$ we can write

$$G(\mu+\xi) - G(\mu) = \sum_{i=1}^{m} \xi_i \varphi_i(u_\mu) + \|\xi\| \, \epsilon(\xi), \quad \text{with } \lim_{\xi \mapsto 0} \epsilon(\xi) = 0, \qquad (*_7)$$

for any $\| \ \|$ norm on $\mathbb{R}^m$. Equation $(*_7)$ show that G is differentiable for any $\mu \in \mathbb{R}^m_+$, and that

$$G'_\mu(\xi) = \sum_{i=1}^{m} \xi_i \varphi_i(u_\mu) \quad \text{for all } \xi \in \mathbb{R}^m. \qquad (*_8)$$

Actually there is a small problem, namely that the notion of derivative was defined for a function defined on an *open* set, but $\mathbb{R}^m_+$ is not open. The difficulty only arises to ensure that the derivative is unique, but in our case we have a unique expression for the derivative so there is no problem as far as defining the derivative. There is still a potential problem, which is that we would like to apply Theorem 40.9 to conclude that since G has a maximum at λ, that is, $-G$ has a minimum at λ, then

$$-G'_\lambda(\mu - \lambda) \geq 0 \quad \text{for all } \mu \in \mathbb{R}^m_+, \qquad (*_9)$$

but the hypotheses of Theorem 40.9 require the domain of the function to be open. Fortunately, close examination of the proof of Theorem 40.9 shows that the proof still holds with $U = \mathbb{R}^m_+$. Therefore, $(*_8)$ holds, Theorem 40.9 is valid, which in turn implies

$$G'_\lambda(\mu - \lambda) \leq 0 \quad \text{for all } \mu \in \mathbb{R}^m_+, \qquad (*_{10})$$

which, using the expression for G'_λ given in $(*_8)$ gives

$$\sum_{i=1}^{m} \mu_i \varphi_i(u_\lambda) \leq \sum_{i=1}^{m} \lambda_i \varphi_i(u_\lambda), \quad \text{for all } \mu \in \mathbb{R}^m_+. \qquad (*_{11})$$