so we get

$$|\langle v, w_\ell\rangle - \langle v, w_m\rangle| \leq \epsilon/2 + \epsilon/2 = \epsilon \quad \text{for all } \ell, m \geq \ell_0.$$

This proves that the sequence $(\langle v, w_\ell\rangle)_{\ell\geq 0}$ is a Cauchy sequence, and thus it converges.

Define the function $g\colon V \to \mathbb{R}$ by

$$g(v) = \lim_{\ell\mapsto\infty} \langle v, w_\ell\rangle, \quad \text{for all } v \in V.$$

Since

$$|\langle v, w_\ell\rangle| \leq \|v\| \, \|w_\ell\| \leq C \, \|v\| \quad \text{for all } \ell \geq 0,$$

we have

$$|g(v)| \leq C \, \|v\| \, ,$$

so g is a continuous linear map. By the Riesz representation theorem (Proposition 48.9), there is a unique $u \in V$ such that

$$g(v) = \langle v, u\rangle \quad \text{for all } v \in V,$$

which shows that

$$\lim_{\ell\mapsto\infty} \langle v, w_\ell\rangle = \langle v, u\rangle \quad \text{for all } v \in V,$$

namely the subsequence $(w_\ell)_{\ell\geq 0}$ of the sequence $(u_k)_{k\geq 0}$ converges weakly to $u \in V$.

Step 2. We prove that the "weak limit" u of the sequence $(w_\ell)_{\ell\geq 0}$ belongs to U.

Consider the projection $p_U(u)$ of $u \in V$ onto the closed convex set U. Since $w_\ell \in U$, by Proposition 48.5(2) and the fact that U is convex and closed, we have

$$\langle p_U(u) - u, w_\ell - p_U(u)\rangle \geq 0 \quad \text{for all } \ell \geq 0.$$

The weak convergence of the sequence $(w_\ell)_{\ell\geq 0}$ to u implies that

$$\begin{aligned} 0 \leq \lim_{\ell\mapsto\infty} \langle p_U(u) - u, w_\ell - p_U(u)\rangle &= \langle p_U(u) - u, u - p_U(u)\rangle \\ &= - \|p_U(u) - u\| \leq 0, \end{aligned}$$

so $\|p_U(u) - u\| = 0$, which means that $p_U(u) = u$, and so $u \in U$.

Step 3. We prove that

$$J(v) \leq \liminf_{\ell\mapsto\infty} J(z_\ell)$$

for every sequence $(z_\ell)_{\ell\geq 0}$ converging weakly to some element $v \in V$.

Since J is assumed to be differentiable and convex, by Proposition 40.11(1) we have

$$J(v) + \langle \nabla J_v, z_\ell - v\rangle \leq J(z_\ell) \quad \text{for all } \ell \geq 0,$$