Equivalently, in terms of gradients, the above conditions are expressed as

$$\nabla J_u + \sum_{i=1}^m \lambda_i(u) \nabla(\varphi_i)_u = 0,$$

and

$$\sum_{i=1}^m \lambda_i(u)\varphi_i(u) = 0, \quad \lambda_i(u) \geq 0, \quad i = 1, \ldots, m.$$

(2) *Conversely, if the restriction of J to U is convex and if there exist scalars $(\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m_+$ such that the KKT conditions hold, then the function J has a (global) minimum at u with respect to U.*

Proof. (1) It suffices to prove that if the convex constraints are qualified according to Definition 50.6, then they are qualified according to Definition 50.5, since in this case we can apply Theorem 50.5.

If $v \in \Omega$ is a vector such that Condition (b) of Definition 50.6 holds and if $v \neq u$, for any $i \in I(u)$, since $\varphi_i(u) = 0$ and since φ_i is convex, by Proposition 40.11(1),

$$\varphi_i(v) \geq \varphi_i(u) + (\varphi'_i)_u(v - u) = (\varphi'_i)_u(v - u),$$

so if we let $w = v - u$ then

$$(\varphi'_i)_u(w) \leq \varphi_i(v),$$

which shows that the nonaffine constraints φ_i for $i \in I(u)$ are qualified according to Definition 50.5, by Condition (b) of Definition 50.6.

If $v = u$, then the constraints φ_i for which $\varphi_i(u) = 0$ must be affine (otherwise, Condition (b)(ii) of Definition 50.6 would be false), and in this case we can pick $w = 0$.

(2) Let v be any arbitrary point in the convex subset U. Since $\varphi_i(v) \leq 0$ and $\lambda_i \geq 0$ for $i = 1, \ldots, m$, we have $\sum_{i=1}^m \lambda_i \varphi_i(v) \leq 0$, and using the fact that

$$\sum_{i=1}^m \lambda_i(u)\varphi_i(u) = 0, \quad \lambda_i(u) \geq 0, \quad i = 1, \ldots, m,$$

we have $\lambda_i = 0$ if $i \notin I(u)$ and $\varphi_i(u) = 0$ if $i \in I(u)$, so we have

$$\begin{aligned}
J(u) &\leq J(u) - \sum_{i=1}^m \lambda_i \varphi_i(v) \\
&\leq J(u) - \sum_{i \in I(u)} \lambda_i(\varphi_i(v) - \varphi_i(u)) && \lambda_i = 0 \text{ if } i \notin I(u),\ \varphi_i(u) = 0 \text{ if } i \in I(u) \\
&\leq J(u) - \sum_{i \in I(u)} \lambda_i (\varphi'_i)_u(v - u) && \text{(by Proposition 40.11)(1)} \\
&\leq J(u) + J'_u(v - u) && \text{(by the KKT conditions)} \\
&\leq J(v) && \text{(by Proposition 40.11)(1),}
\end{aligned}$$

and this shows that u is indeed a (global) minimum of J over U. $\square$