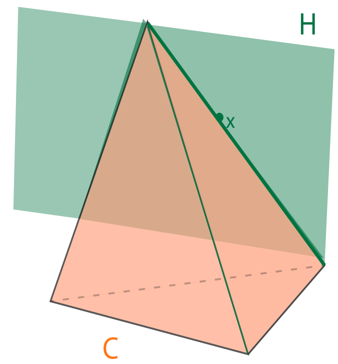

Figure 51.10: Let C be the solid peach tetrahedron in $\mathbb{R}^3$. The green plane H is a supporting hyperplane to the point x since $x \in H$ and $C \subseteq H_+$, i.e. H only intersects C on the edge containing x and so the tetrahedron lies in "front" of H.

x, since $\langle z, u\rangle \leq c$ for all $z \in C$ and $\langle x, u\rangle = c$, we have $\langle z - x, u\rangle \leq 0$ for all $z \in C$, *which means that u is normal to C at x*. This concept is illustrated by Figure 51.12.

The notion of subgradient can be motived as follows. A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is differentiable at $x \in \mathbb{R}^n$ if

$$f(x + y) = f(x) + df_x(y) + \epsilon(y) \left\|y\right\|_2,$$

for all $y \in \mathbb{R}^n$ in some nonempty subset containing x, where $df_x\colon \mathbb{R}^n \to \mathbb{R}$ is a linear form, and ϵ is some function such that $\lim_{\|y\| \mapsto 0} \epsilon(y) = 0$. Furthermore,

$$df_x(y) = \langle y, \nabla f_x\rangle \quad \text{for all } y \in \mathbb{R}^n,$$

where ∇f_x is the *gradient* of f at x, so

$$f(x + y) = f(x) + \langle y, \nabla f_x\rangle + \epsilon(y) \left\|y\right\|_2.$$

If we assume that f is convex, it makes sense to replace the equality sign by the inequality sign $\geq$ in the above equation and to drop the "error term" $\epsilon(y) \left\|y\right\|_2$, so a vector u is a subgradient of f at x if

$$f(x + y) \geq f(x) + \langle y, u\rangle \quad \text{for all } y \in \mathbb{R}^n.$$

Thus we are led to the following definition.

Definition 51.14. Let $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$ be a convex function. For any $x \in \mathbb{R}^n$, a *subgradient of f at x* is any vector $u \in \mathbb{R}^n$ such that

$$f(z) \geq f(x) + \langle z - x, u\rangle, \quad \text{for all } z \in \mathbb{R}^n. \tag{$*_{\text{subgrad}}$}$$

The above inequality is called the *subgradient inequality*. The set of all subgradients of f at x is denoted $\partial f(x)$ and is called the *subdifferential* of f at x. If $\partial f(x) \neq \emptyset$, then we say that *$f$ is subdifferentiable at x*.