Since $E_k^k = E_k$, we obtain

$$(E_1^k)^{-1} \cdots (E_{k-1}^k)^{-1} (E_k^k)^{-1} = (I + P_k \Lambda_{k-1}) E_k^{-1}.$$

However, by definition

$$I + \Lambda_k = (I + P_k \Lambda_{k-1}) E_k^{-1},$$

which proves that

$$I + \Lambda_k = (E_1^k)^{-1} \cdots (E_{k-1}^k)^{-1} (E_k^k)^{-1}, \tag{\dagger}$$

and finishes the induction step for the proof of this formula.

If we apply Equation $(*)$ again with $k+1$ in place of k, we have

$$(E_1^k)^{-1} \cdots (E_k^k)^{-1} = I + \mathcal{E}_1^k + \cdots + \mathcal{E}_k^k,$$

and together with $(\dagger)$, we obtain,

$$\Lambda_k = \mathcal{E}_1^k + \cdots + \mathcal{E}_k^k,$$

also finishing the induction step for the proof of this formula. For $k = n-1$ in $(\dagger)$, we obtain the desired equation: $L = I + \Lambda_{n-1}$. $\square$

8.7 Dealing with Roundoff Errors; Pivoting Strategies

Let us now briefly comment on the choice of a pivot. Although theoretically, any pivot can be chosen, the possibility of roundoff errors implies that it is not a good idea to pick very small pivots. The following example illustrates this point. Consider the linear system

$$\begin{array}{rcrcl} 10^{-4}x & + & y & = & 1 \\ x & + & y & = & 2. \end{array}$$

Since 10^{-4} is nonzero, it can be taken as pivot, and we get

$$\begin{array}{rcrcl} 10^{-4}x & + & y & = & 1 \\ & & (1 - 10^4)y & = & 2 - 10^4. \end{array}$$

Thus, the exact solution is

$$x = \frac{10^4}{10^4 - 1}, \quad y = \frac{10^4 - 2}{10^4 - 1}.$$

However, if roundoff takes place on the fourth digit, then $10^4 - 1 = 9999$ and $10^4 - 2 = 9998$ will be rounded off both to 9990, and then the solution is $x = 0$ and $y = 1$, very far from the exact solution where $x \approx 1$ and $y \approx 1$. The problem is that we picked a very small pivot. If instead we permute the equations, the pivot is 1, and after elimination we get the system

$$\begin{array}{rcrcl} x & + & y & = & 2 \\ & & (1 - 10^{-4})y & = & 1 - 2 \times 10^{-4}. \end{array}$$