This representation is many-to-one: the sequences $(\pm w_1, \ldots, \pm w_n)$ describe the same polygon. To every sequence of complex numbers $(w_1, \ldots, w_n)$, we associate the pair of vectors (a, b), with $a, b \in \mathbb{R}^n$, such that if $w_k = a_k + ib_k$, then

$$a = (a_1, \ldots, a_n), \quad b = (b_1, \ldots, b_n).$$

The mapping

$$(w_1, \ldots, w_n) \mapsto (a, b)$$

is clearly a bijection, so we can also represent polygons by pairs of vectors $(a, b) \in \mathbb{R}^n \times \mathbb{R}^n$.

(1) Prove that a polygon P represented by a pair of vectors $(a, b) \in \mathbb{R}^n \times \mathbb{R}^n$ is closed iff $a \cdot b = 0$ and $\|a\|_2 = \|b\|_2$.

(2) Given a polygon P represented by a pair of vectors $(a, b) \in \mathbb{R}^n \times \mathbb{R}^n$, the length $l(P)$ of the polygon P is defined by $l(P) = |w_1|^2 + \cdots + |w_n|^2$, with $w_k = a_k + ib_k$. Prove that

$$l(P) = \|a\|_2^2 + \|b\|_2^2.$$

Deduce from (a) and (b) that every closed polygon of length 2 with n edges is represented by a $n \times 2$ matrix A such that $A^\top A = I$.

Remark: The space of all a $n \times 2$ real matrices A such that $A^\top A = I$ is a space known as the *Stiefel manifold* $S(2, n)$.

(3) Recall that in $\mathbb{R}^2$, the rotation of angle θ specified by the matrix

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is expressed in terms of complex numbers by the map

$$z \mapsto ze^{i\theta}.$$

Let P be a polygon represented by a pair of vectors $(a, b) \in \mathbb{R}^n \times \mathbb{R}^n$. Prove that the polygon $R_\theta(P)$ obtained by applying the rotation R_θ to every vertex $w_k^2 = (a_k + ib_k)^2$ of P is specified by the pair of vectors

$$(\cos(\theta/2)a - \sin(\theta/2)b,\ \sin(\theta/2)a + \cos(\theta/2)b) = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \end{pmatrix} \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}.$$

(4) The reflection ρ_x about the x-axis corresponds to the map

$$z \mapsto \overline{z},$$