Theorem 5.1. *If H is an $n \times n$ Hadamard matrix, then either $n = 1, 2$, or $n = 4k$ for some positive integer k.*

Sylvester introduced a family of Hadamard matrices and proved that there are Hadamard matrices of dimension $n = 2^m$ for all $m \geq 1$ using the following construction.

Proposition 5.2. *(Sylvester, 1867) If H is a Hadamard matrix of dimension n, then the block matrix of dimension $2n$,*

$$\begin{pmatrix} H & H \\ H & -H \end{pmatrix},$$

is a Hadamard matrix.

If we start with

$$H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

we obtain an infinite family of symmetric Hadamard matrices usually called *Sylvester–Hadamard* matrices and denoted by H_{2^m}. The Sylvester–Hadamard matrices H_2, H_4 and H_8 are shown on the previous page.

In 1893, Hadamard gave examples of Hadamard matrices for $n = 12$ and $n = 20$. At the present, Hadamard matrices are known for all $n = 4k \leq 1000$, *except for* $n = 668, 716$, *and* 892.

Hadamard matrices have various applications to error correcting codes, signal processing, and numerical linear algebra; see Seberry, Wysocki and Wysocki [154] and Tropp [177]. For example, there is a code based on H_{32} that can correct 7 errors in any 32-bit encoded block, and can detect an eighth. This code was used on a Mariner spacecraft in 1969 to transmit pictures back to the earth.

For every $m \geq 0$, the piecewise affine functions $\mathrm{plf}((H_{2^m})_i)$ associated with the 2^m rows of the Sylvester–Hadamard matrix H_{2^m} are functions on $[0, 1]$ known as the *Walsh functions*. It is customary to index these 2^m functions by the integers $0, 1, \ldots, 2^m - 1$ in such a way that the Walsh function $\mathrm{Wal}(k, t)$ is equal to the function $\mathrm{plf}((H_{2^m})_i)$ associated with the Row i of H_{2^m} that contains k changes of signs between consecutive groups of $+1$ and consecutive groups of -1. For example, the fifth row of H_8, namely

$$\begin{pmatrix} 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \end{pmatrix},$$

has five consecutive blocks of $+1$s and -1s, four sign changes between these blocks, and thus is associated with $\mathrm{Wal}(4, t)$. In particular, Walsh functions corresponding to the rows of H_8 (from top down) are:

$$\mathrm{Wal}(0, t),\ \mathrm{Wal}(7, t),\ \mathrm{Wal}(3, t),\ \mathrm{Wal}(4, t),\ \mathrm{Wal}(1, t),\ \mathrm{Wal}(6, t),\ \mathrm{Wal}(2, t),\ \mathrm{Wal}(5, t).$$

Because of the connection between Sylvester–Hadamard matrices and Walsh functions, Sylvester–Hadamard matrices are called *Walsh–Hadamard matrices* by some authors. For