In the case of a real Hilbert space, there is an intuitive geometric interpretation of the condition

$$\langle u - p_X(u), z - p_X(u)\rangle \leq 0$$

for all $z \in X$. If we restate the condition as

$$\langle u - p_X(u), p_X(u) - z\rangle \geq 0$$

for all $z \in X$, this says that the absolute value of the measure of the angle between the vectors $u - p_X(u)$ and $p_X(u) - z$ is at most $\pi/2$. See Figure 48.5. This makes sense, since X is convex, and points in X must be on the side opposite to the "tangent space" to X at $p_X(u)$, which is orthogonal to $u - p_X(u)$. Of course, this is only an intuitive description, since the notion of tangent space has not been defined!

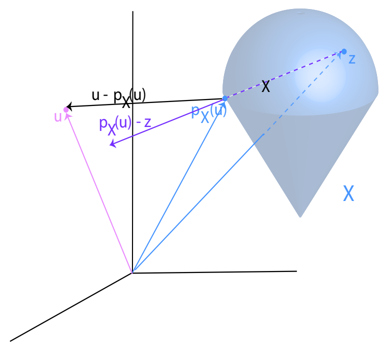

Figure 48.5: Let X be the solid blue ice cream cone. The acute angle between the black vector $u - p_X(u)$ and the purple vector $p_X(u) - z$ is less than $\pi/2$.

If X is a closed subspace of E, then Condition $(**)$ says that the vector $u - p_X(u)$ is *orthogonal* to X, in the sense that $u - p_X(u)$ is orthogonal to every vector $z \in X$.

The map $p_X \colon E \to X$ is continuous as shown below.

Proposition 48.6. *Let E be a Hilbert space. For any nonempty convex and closed subset $X \subseteq E$, the map $p_X \colon E \to X$ is continuous. In fact, p_X satisfies the Lipschitz condition*

$$\|p_X(v) - p_X(u)\| \leq \|v - u\| \quad \textit{for all } u, v \in E.$$

Proof. For any two vectors $u, v \in E$, let $x = p_X(u) - u$, $y = p_X(v) - p_X(u)$, and $z = v - p_X(v)$. Clearly, (as illustrated in Figure 48.6),

$$v - u = x + y + z,$$

and from Proposition 48.5(2), we also have

$$\Re \langle x, y\rangle \geq 0 \quad \text{and} \quad \Re \langle z, y\rangle \geq 0,$$