Proof. Everything is straightforward. For example, if $a_1 \equiv_{\mathfrak{I}} b_1$ and $a_2 \equiv_{\mathfrak{I}} b_2$, then $b_1 - a_1 \in \mathfrak{I}$ and $b_2 - a_2 \in \mathfrak{I}$. Since $\mathfrak{I}$ is an ideal, we get

$$(b_1 - a_1)b_2 = b_1b_2 - a_1b_2 \in \mathfrak{I}$$

and

$$(b_2 - a_2)a_1 = a_1b_2 - a_1a_2 \in \mathfrak{I}.$$

Since $\mathfrak{I}$ is an ideal, and thus, an additive group, we get

$$b_1b_2 - a_1a_2 \in \mathfrak{I},$$

i.e., $a_1a_2 \equiv_{\mathfrak{I}} b_1b_2$. The equality $\mathrm{Ker}\,\pi = \mathfrak{I}$ holds because $\mathfrak{I}$ is an ideal. ☐

Example 30.1.

1. In the ring $\mathbb{Z}$, for every $p \in \mathbb{Z}$, the subroup $p\mathbb{Z}$ is an ideal, and $\mathbb{Z}/p\mathbb{Z}$ is a ring, the ring of residues modulo p. This ring is a field iff p is a prime number.

2. The quotient of the polynomial ring $\mathbb{R}[X]$ by a prime ideal $\mathfrak{I}$ is an integral domain.

3. The quotient of the polynomial ring $\mathbb{R}[X]$ by a maximal ideal $\mathfrak{I}$ is a field. For example, if $\mathfrak{I} = (X^2 + 1)$, the principal ideal generated by $X^2 + 1$ (which is indeed a maximal ideal since $X^2 + 1$ has no real roots), then $\mathbb{R}[X]/(X^2 + 1) \cong \mathbb{C}$.

The following proposition yields a characterization of prime ideals and maximal ideals in terms of quotients.

Proposition 30.8. *Given a ring A, for any ideal $\mathfrak{I} \subseteq A$, the following properties hold.*

(1) The ideal $\mathfrak{I}$ is a prime ideal iff $A/\mathfrak{I}$ is an integral domain.

(2) The ideal $\mathfrak{I}$ is a maximal ideal iff $A/\mathfrak{I}$ is a field.

Proof. (1) Assume that $\mathfrak{I}$ is a prime ideal. Since $\mathfrak{I}$ is prime, $\mathfrak{I} \neq A$, and thus, $A/\mathfrak{I}$ is not the trivial ring (0). If $[a][b] = 0$, since $[a][b] = [ab]$, we have $ab \in \mathfrak{I}$, and since $\mathfrak{I}$ is prime, then either $a \in \mathfrak{I}$ or $b \in \mathfrak{I}$, so that either $[a] = 0$ or $[b] = 0$. Thus, $A/\mathfrak{I}$ is an integral domain.

Conversely, assume that $A/\mathfrak{I}$ is an integral domain. Since $A/\mathfrak{I}$ is not the trivial ring, $\mathfrak{I} \neq A$. Assume that $ab \in \mathfrak{I}$. Then, we have

$$\pi(ab) = \pi(a)\pi(b) = 0,$$

which implies that either $\pi(a) = 0$ or $\pi(b) = 0$, since $A/\mathfrak{I}$ is an integral domain (where $\pi \colon A \to A/\mathfrak{I}$ is the quotient map). Thus, either $a \in \mathfrak{I}$ or $b \in \mathfrak{I}$, and $\mathfrak{I}$ is a prime ideal.