The next proposition is needed to compare the rate of convergence of iterative methods. It shows that *asymptotically, the error vector* $e_k = B^k e_0$ *behaves at worst like* $(\rho(B))^k$.

Proposition 10.4. *Let* $\| \; \|$ *be any vector norm, let* $B \in \mathrm{M}_n(\mathbb{C})$ *be a matrix such that* $I - B$ *is invertible, and let* $\widetilde{u}$ *be the unique solution of* $u = Bu + c$.

(1) If (u_k) *is any sequence defined iteratively by*

$$u_{k+1} = Bu_k + c, \quad k \in \mathbb{N},$$

then

$$\lim_{k \mapsto \infty} \left[\sup_{\|u_0 - \widetilde{u}\| = 1} \|u_k - \widetilde{u}\|^{1/k} \right] = \rho(B).$$

(2) Let B_1 *and* B_2 *be two matrices such that* $I - B_1$ *and* $I - B_2$ *are invertible, assume that both* $u = B_1 u + c_1$ *and* $u = B_2 u + c_2$ *have the same unique solution* $\widetilde{u}$, *and consider any two sequences* (u_k) *and* (v_k) *defined inductively by*

$$\begin{aligned} u_{k+1} &= B_1 u_k + c_1 \\ v_{k+1} &= B_2 v_k + c_2, \end{aligned}$$

with $u_0 = v_0$. *If* $\rho(B_1) < \rho(B_2)$, *then for any* $\epsilon > 0$, *there is some integer* $N(\epsilon)$, *such that for all* $k \geq N(\epsilon)$, *we have*

$$\sup_{\|u_0 - \widetilde{u}\| = 1} \left[\frac{\|v_k - \widetilde{u}\|}{\|u_k - \widetilde{u}\|} \right]^{1/k} \geq \frac{\rho(B_2)}{\rho(B_1) + \epsilon}.$$

Proof. Let $\| \; \|$ be the subordinate matrix norm. Recall that

$$u_k - \widetilde{u} = B^k e_0,$$

with $e_0 = u_0 - \widetilde{u}$. For every $k \in \mathbb{N}$, we have

$$(\rho(B))^k = \rho(B^k) \leq \|B^k\| = \sup_{\|e_0\| = 1} \|B^k e_0\|,$$

which implies

$$\rho(B) = \sup_{\|e_0\| = 1} \|B^k e_0\|^{1/k} = \|B^k\|^{1/k},$$

and Statement (1) follows from Proposition 10.2.

Because $u_0 = v_0$, we have

$$\begin{aligned} u_k - \widetilde{u} &= B_1^k e_0 \\ v_k - \widetilde{u} &= B_2^k e_0, \end{aligned}$$