Proposition 27.8. *Let E be any affine space of finite dimension. For every affine map $f\colon E \to E$, let $\mathrm{Fix}(f) = \{a \in E \mid f(a) = a\}$ be the set of fixed points of f. The following properties hold:*

(1) If f has some fixed point a, so that $\mathrm{Fix}(f) \neq \emptyset$, then $\mathrm{Fix}(f)$ is an affine subspace of E such that

$$\mathrm{Fix}(f) = a + E\big(1, \overrightarrow{f}\big) = a + \mathrm{Ker}\left(\overrightarrow{f} - \mathrm{id}\right),$$

where $E\big(1, \overrightarrow{f}\big)$ is the eigenspace of the linear map $\overrightarrow{f}$ for the eigenvalue 1.

(2) The affine map f has a unique fixed point iff $E\big(1, \overrightarrow{f}\big) = \mathrm{Ker}\left(\overrightarrow{f} - \mathrm{id}\right) = \{0\}$.

Proof. (1) Since the identity

$$\overrightarrow{\Omega f(b)} - \overrightarrow{\Omega b} = \overrightarrow{\Omega f(\Omega)} + \overrightarrow{f}(\overrightarrow{\Omega b}) - \overrightarrow{\Omega b}$$

holds for all $\Omega, b \in E$, if $f(a) = a$, then $\overrightarrow{af(a)} = 0$, and thus, letting $\Omega = a$, for any $b \in E$ we have

$$\overrightarrow{af(b)} - \overrightarrow{ab} = \overrightarrow{af(a)} + \overrightarrow{f}(\overrightarrow{ab}) - \overrightarrow{ab} = \overrightarrow{f}(\overrightarrow{ab}) - \overrightarrow{ab},$$

and so

$$f(b) = b$$

iff

$$\overrightarrow{af(b)} - \overrightarrow{ab} = 0$$

iff

$$\overrightarrow{f}(\overrightarrow{ab}) - \overrightarrow{ab} = 0$$

iff

$$\overrightarrow{ab} \in E\big(1, \overrightarrow{f}\big) = \mathrm{Ker}\left(\overrightarrow{f} - \mathrm{id}\right),$$

which proves that

$$\mathrm{Fix}(f) = a + E\big(1, \overrightarrow{f}\big) = a + \mathrm{Ker}\left(\overrightarrow{f} - \mathrm{id}\right).$$

(2) Again, fix some origin Ω. Some a satisfies $f(a) = a$ iff

$$\overrightarrow{\Omega f(a)} - \overrightarrow{\Omega a} = 0$$

iff

$$\overrightarrow{\Omega f(\Omega)} + \overrightarrow{f}(\overrightarrow{\Omega a}) - \overrightarrow{\Omega a} = 0,$$

which can be rewritten as

$$\big(\overrightarrow{f} - \mathrm{id}\big)(\overrightarrow{\Omega a}) = -\overrightarrow{\Omega f(\Omega)}.$$

We have $E\big(1, \overrightarrow{f}\big) = \mathrm{Ker}\left(\overrightarrow{f} - \mathrm{id}\right) = \{0\}$ iff $\overrightarrow{f} - \mathrm{id}$ is injective, and since $\overrightarrow{E}$ has finite dimension, $\overrightarrow{f} - \mathrm{id}$ is also surjective, and thus, there is indeed some $a \in E$ such that

$$\big(\overrightarrow{f} - \mathrm{id}\big)(\overrightarrow{\Omega a}) = -\overrightarrow{\Omega f(\Omega)},$$