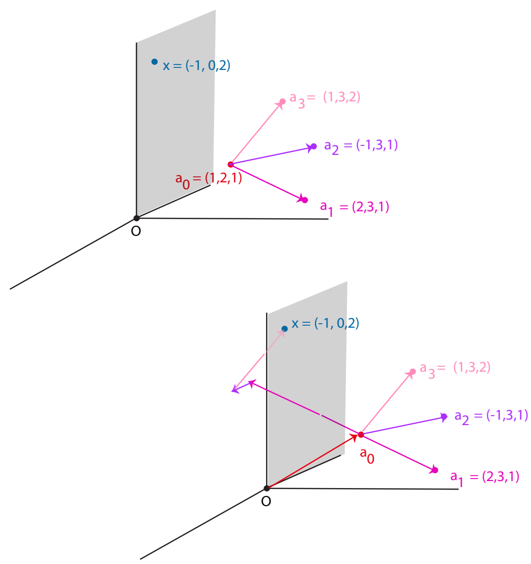

Figure 24.15: The affine frame (a_0, a_1, a_2, a_3) for $\mathbb{A}^3$. The coordinates for $x = (-1, 0, 2)$ are $x_1 = -8/3$, $x_2 = -1/3$, $x_3 = 1$, while the barycentric coordinates for x are $\lambda_0 = 3$, $\lambda_1 = -8/3$, $\lambda_2 = -1/3$, $\lambda_3 = 1$.

Proposition 24.6. *Given an affine space $\langle E, \overrightarrow{E}, +\rangle$, let $(a_i)_{i\in I}$ be a family of points in E. The family $(a_i)_{i\in I}$ is affinely dependent iff there is a family $(\lambda_i)_{i\in I}$ such that $\lambda_j \neq 0$ for some $j \in I$, $\sum_{i\in I} \lambda_i = 0$, and $\sum_{i\in I} \lambda_i \overrightarrow{xa_i} = 0$ for every $x \in E$.*

Proof. By Proposition 24.5, the family $(a_i)_{i\in I}$ is affinely dependent iff the family of vectors $(\overrightarrow{a_ia_j})_{j\in(I-\{i\})}$ is linearly dependent for some $i \in I$. For any $i \in I$, the family $(\overrightarrow{a_ia_j})_{j\in(I-\{i\})}$ is linearly dependent iff there is a family $(\lambda_j)_{j\in(I-\{i\})}$ such that $\lambda_j \neq 0$ for some j, and such that

$$\sum_{j\in(I-\{i\})} \lambda_j \overrightarrow{a_ia_j} = 0.$$

Then, for any $x \in E$, we have

$$\begin{aligned}\sum_{j\in(I-\{i\})} \lambda_j \overrightarrow{a_ia_j} &= \sum_{j\in(I-\{i\})} \lambda_j (\overrightarrow{xa_j} - \overrightarrow{xa_i}) \\ &= \sum_{j\in(I-\{i\})} \lambda_j \overrightarrow{xa_j} - \Bigg(\sum_{j\in(I-\{i\})} \lambda_j\Bigg) \overrightarrow{xa_i},\end{aligned}$$

and letting $\lambda_i = -\left(\sum_{j\in(I-\{i\})} \lambda_j\right)$, we get $\sum_{i\in I} \lambda_i \overrightarrow{xa_i} = 0$, with $\sum_{i\in I} \lambda_i = 0$ and $\lambda_j \neq 0$ for some $j \in I$. The converse is obvious by setting $x = a_i$ for some i such that $\lambda_i \neq 0$, since $\sum_{i\in I} \lambda_i = 0$ implies that $\lambda_j \neq 0$, for some $j \neq i$. $\square$