**Proposition 16.9.** *Any section $s\colon \mathbf{SO}(3) \to \mathbf{SU}(2)$ of $\rho$ is neither a homomorphism nor continuous.*

Intuitively, this means that there is no "nice and simple " way to pick the sign of the quaternion representing a rotation.

The following proof is due to Marcel Berger.

*Proof.* Let $\Gamma$ be the subgroup of $\mathbf{SU}(2)$ consisting of all quaternions of the form $q = [a, (b, 0, 0)]$. Then, using the formula for the rotation matrix $R_q$ corresponding to $q$ (and the fact that $a^2 + b^2 = 1$), we get

$$R_q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2a^2 - 1 & -2ab \\ 0 & 2ab & 2a^2 - 1 \end{pmatrix}.$$

Since $a^2 + b^2 = 1$, we may write $a = \cos\theta, b = \sin\theta$, and we see that

$$R_q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & -\sin 2\theta \\ 0 & \sin 2\theta & \cos 2\theta \end{pmatrix},$$

a rotation of angle $2\theta$ around the $x$-axis. Thus, both $\Gamma$ and its image are isomorphic to $\mathbf{SO}(2)$, which is also isomorphic to $\mathbf{U}(1) = \{w \in \mathbb{C} \mid |w| = 1\}$. By identifying $\mathbf{i}$ and $i$, and identifying $\Gamma$ and its image to $\mathbf{U}(1)$, if we write $w = \cos\theta + i\sin\theta \in \Gamma$, the restriction of the map $\rho$ to $\Gamma$ is given by $\rho(w) = w^2$.

We claim that any section $s$ of $\rho$ is not a homomorphism. Consider the restriction of $s$ to $\mathbf{U}(1)$. Then since $\rho \circ s = \mathrm{id}$ and $\rho(w) = w^2$, for $-1 \in \rho(\Gamma) \approx \mathbf{U}(1)$, we have

$$-1 = \rho(s(-1)) = (s(-1))^2.$$

On the other hand, if $s$ is a homomorphism, then

$$(s(-1))^2 = s((-1)^2) = s(1) = 1,$$

contradicting $(s(-1))^2 = -1$.

We also claim that $s$ is not continuous. Assume that $s(1) = 1$, the case where $s(1) = -1$ being analogous. Then $s$ is a bijection inverting $\rho$ on $\Gamma$ whose restriction to $\mathbf{U}(1)$ must be given by

$$s(\cos\theta + i\sin\theta) = \cos(\theta/2) + \mathbf{i}\sin(\theta/2), \quad -\pi \le \theta < \pi.$$

If $\theta$ tends to $\pi$, that is $z = \cos\theta + i\sin\theta$ tends to $-1$ in the upper-half plane, then $s(z)$ tends to $\mathbf{i}$, but if $\theta$ tends to $-\pi$, that is $z$ tends to $-1$ in the lower-half plane, then $s(z)$ tends to $-\mathbf{i}$, which shows that $s$ is not continuous. $\square$