The exact same construction works for any field K, and we obtain a vector space denoted by $K^{(I)}$ and an injection $\iota\colon I \to K^{(I)}$.

The main reason why the free vector space $K^{(I)}$ over a set I is interesting is that it satisfies a *universal mapping property*. This means that for every vector space F (over the field K), any function $h\colon I \to F$, where F is *considered just a set*, has a unique linear extension $\overline{h}\colon K^{(I)} \to F$. By extension, we mean that $\overline{h}(i) = h(i)$ for all $i \in I$, or more rigorously that $h = \overline{h} \circ \iota$.

For example, if $I = \{\heartsuit, \diamondsuit, \spadesuit, \clubsuit\}$, $K = \mathbb{R}$, and $F = \mathbb{R}^3$, the function h given by

$$h(\heartsuit) = (1,1,1), \quad h(\diamondsuit) = (1,1,0), \quad h(\spadesuit) = (1,0,0), \quad h(\clubsuit) = (0,0-1)$$

has a unique linear extension $\overline{h}\colon \mathbb{R}^{(I)} \to \mathbb{R}^3$ to the free vector space $\mathbb{R}^{(I)}$, given by

$$\begin{aligned}\overline{h}(a\heartsuit + b\diamondsuit + c\spadesuit + d\clubsuit) &= a\overline{h}(\heartsuit) + b\overline{h}(\diamondsuit) + c\overline{h}(\spadesuit) + d\overline{h}(\clubsuit)\\ &= ah(\heartsuit) + bh(\diamondsuit) + ch(\spadesuit) + dh(\clubsuit)\\ &= a(1,1,1) + b(1,1,0) + c(1,0,0) + d(0,0,-1)\\ &= (a+b+c, a+b, a-d).\end{aligned}$$

To generalize the construction of a free vector space to infinite sets I, we observe that the formal linear combination $a\heartsuit + b\diamondsuit + c\spadesuit + d\clubsuit$ can be viewed as the function $f\colon I \to \mathbb{R}$ given by

$$f(\heartsuit) = a, \quad f(\diamondsuit) = b, \quad f(\spadesuit) = c, \quad f(\clubsuit) = d,$$

where $a, b, c, d \in \mathbb{R}$. More generally, we can replace $\mathbb{R}$ by any field K. If I is finite, then the set of all such functions is a vector space under pointwise addition and pointwise scalar multiplication. If I is infinite, since addition and scalar multiplication only makes sense for finite vectors, we require that our functions $f\colon I \to K$ take the value 0 except for possibly finitely many arguments. We can think of such functions as an infinite sequences $(f_i)_{i\in I}$ of elements f_i of K indexed by I, with only finitely many nonzero f_i. The formalization of this construction goes as follows.

Given any set I viewed as an index set, let $K^{(I)}$ be the set of all functions $f\colon I \to K$ such that $f(i) \neq 0$ only for finitely many $i \in I$. As usual, denote such a function by $(f_i)_{i\in I}$; it is a family of finite support. We make $K^{(I)}$ into a vector space by defining addition and scalar multiplication by

$$\begin{aligned}(f_i) + (g_i) &= (f_i + g_i)\\ \lambda(f_i) &= (\lambda f_i).\end{aligned}$$

The family $(e_i)_{i\in I}$ is defined such that $(e_i)_j = 0$ if $j \neq i$ and $(e_i)_i = 1$. It is a basis of the vector space $K^{(I)}$, so that every $w \in K^{(I)}$ can be uniquely written as a finite linear combination of the e_i. There is also an injection $\iota\colon I \to K^{(I)}$ such that $\iota(i) = e_i$ for every $i \in I$. Furthermore, it is easy to show that for any vector space F, and for any function