Proposition 29.29. *Let φ be an ϵ-Hermitian form on E, assume that φ is nondegenerate and satisfies property (T), and let U be any totally isotropic subspace of E of finite dimension* $\dim(U) = r \geq 1$.

(1) *If U' is any totally isotropic subspace of dimension r and if $U' \cap U^{\perp} = (0)$, then $U \oplus U'$ is nondegenerate, and for any basis $(u_1, \ldots, u_r)$ of U, there is a basis $(u'_1, \ldots, u'_r)$ of U' such that $\varphi(u_i, u'_j) = \delta_{ij}$, for all $i, j = 1, \ldots, r$.*

(2) *If W is any totally isotropic subspace of dimension at most r and if $W \cap U^{\perp} = (0)$, then there exists a totally isotropic subspace U' with $\dim(U') = r$ such that $W \subseteq U'$ and $U' \cap U^{\perp} = (0)$.*

Proof. (1) Let φ' be the restriction of φ to $U \times U'$. Since $U' \cap U^{\perp} = (0)$, for any $v \in U'$, if $\varphi(u, v) = 0$ for all $u \in U$, then $v = 0$. Thus, φ' is nondegenerate (we only have to check on the left since φ is ϵ-Hermitian). Then, the assertion about bases follows from the version of Proposition 29.3 for sesquilinear forms. Since U is totally isotropic, $U \subseteq U^{\perp}$, and since $U' \cap U^{\perp} = (0)$, we must have $U' \cap U = (0)$, which show that we have a direct sum $U \oplus U'$.

It remains to prove that $U + U'$ is nondegenerate. Observe that

$$H = (U + U') \cap (U + U')^{\perp} = (U + U') \cap U^{\perp} \cap U'^{\perp}.$$

Since U is totally isotropic, $U \subseteq U^{\perp}$, and since $U' \cap U^{\perp} = (0)$, we have

$$(U + U') \cap U^{\perp} = (U \cap U^{\perp}) + (U' \cap U^{\perp}) = U + (0) = U,$$

thus $H = U \cap U'^{\perp}$. Since φ' is nondegenerate, $U \cap U'^{\perp} = (0)$, so $H = (0)$ and $U + U'$ is nondegenerate.

(2) We proceed by descending induction on $s = \dim(W)$. The base case $s = r$ is trivial. For the induction step, it suffices to prove that if $s < r$, then there is a totally isotropic subspace W' containing W such that $\dim(W') = s + 1$ and $W' \cap U^{\perp} = (0)$.

Since $s = \dim(W) < \dim(U)$, the restriction of φ to $U \times W$ is degenerate. Since $W \cap U^{\perp} = (0)$, we must have $U \cap W^{\perp} \neq (0)$. We claim that

$$W^{\perp} \not\subseteq W + U^{\perp}.$$

If we had

$$W^{\perp} \subseteq W + U^{\perp},$$

then because U and W are finite-dimensional and φ is nondegenerate, by Proposition 29.13, $U^{\perp\perp} = U$ and $W^{\perp\perp} = W$, so by taking orthogonals, $W^{\perp} \subseteq W + U^{\perp}$ would yield

$$(W + U^{\perp})^{\perp} \subseteq W^{\perp\perp},$$

that is,

$$W^{\perp} \cap U \subseteq W,$$