which is (4).

Finally, assume (4). Because $\| \ \|$ is a matrix norm,

$$\|B^k\| \leq \|B\|^k,$$

and since $\|B\| < 1$, we deduce that (1) holds. $\square$

The following proposition is needed to study the rate of convergence of iterative methods.

Proposition 10.2. *For every square matrix $B \in \mathrm{M}_n(\mathbb{C})$ and every matrix norm $\| \ \|$, we have*

$$\lim_{k \mapsto \infty} \|B^k\|^{1/k} = \rho(B).$$

Proof. We know from Proposition 9.6 that $\rho(B) \leq \|B\|$, and since $\rho(B) = (\rho(B^k))^{1/k}$, we deduce that

$$\rho(B) \leq \|B^k\|^{1/k} \quad \text{for all } k \geq 1.$$

Now let us prove that for every $\epsilon > 0$, there is some integer $N(\epsilon)$ such that

$$\|B^k\|^{1/k} \leq \rho(B) + \epsilon \quad \text{for all } k \geq N(\epsilon).$$

Together with the fact that

$$\rho(B) \leq \|B^k\|^{1/k} \quad \text{for all } k \geq 1,$$

we deduce that $\lim_{k \mapsto \infty} \|B^k\|^{1/k}$ exists and that

$$\lim_{k \mapsto \infty} \|B^k\|^{1/k} = \rho(B).$$

For any given $\epsilon > 0$, let B_ϵ be the matrix

$$B_\epsilon = \frac{B}{\rho(B) + \epsilon}.$$

Since $\rho(B_\epsilon) < 1$, Theorem 10.1 implies that $\lim_{k \mapsto \infty} B_\epsilon^k = 0$. Consequently, there is some integer $N(\epsilon)$ such that for all $k \geq N(\epsilon)$, we have

$$\|B_\epsilon^k\| = \frac{\|B^k\|}{(\rho(B) + \epsilon)^k} \leq 1,$$

which implies that

$$\|B^k\|^{1/k} \leq \rho(B) + \epsilon,$$

as claimed. $\square$

We now apply the above results to the convergence of iterative methods.