Prove that every matrix satisfying the above constraints is of the form

$$\begin{pmatrix} a+b-c & -a+c+e & -b+c+d \\ -a-b+c+d+e & a & b \\ c & d & e \end{pmatrix},$$

with $a, b, c, d, e \in \mathbb{R}$. Find a basis for this subspace. (Use the method to find a basis for the kernel of a matrix).

Problem 8.8. If A is an $n \times n$ matrix and B is any $n \times n$ invertible matrix, prove that A is symmetric positive definite iff $B^\top AB$ is symmetric positive definite.

Problem 8.9. (1) Consider the matrix

$$A_4 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

Find three matrices of the form $E_{2,1;\beta_1}, E_{3,2;\beta_2}, E_{4,3;\beta_3}$, such that

$$E_{4,3;\beta_3}E_{3,2;\beta_2}E_{2,1;\beta_1}A_4 = U_4$$

where U_4 is an upper triangular matrix. Compute

$$M = E_{4,3;\beta_3}E_{3,2;\beta_2}E_{2,1;\beta_1}$$

and check that

$$MA_4 = U_4 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & 0 & 5/4 \end{pmatrix}.$$

(2) Now consider the matrix

$$A_5 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Find four matrices of the form $E_{2,1;\beta_1}, E_{3,2;\beta_2}, E_{4,3;\beta_3}, E_{5,4;\beta_4}$, such that

$$E_{5,4;\beta_4}E_{4,3;\beta_3}E_{3,2;\beta_2}E_{2,1;\beta_1}A_5 = U_5$$

where U_5 is an upper triangular matrix. Compute

$$M = E_{5,4;\beta_4}E_{4,3;\beta_3}E_{3,2;\beta_2}E_{2,1;\beta_1}$$