lines $\langle a, a' \rangle$, $\langle b, b' \rangle$, and $\langle c, c' \rangle$ intersect in a common point d distinct from $a, b, c,\ a', b', c'$, then the intersection points $p = \langle b, c \rangle \cap \langle b', c' \rangle$, $q = \langle a, c \rangle \cap \langle a', c' \rangle$, and $r = \langle a, b \rangle \cap \langle a', b' \rangle$ belong to a common line distinct from $A, B, C,\ A', B', C'$.

Proof. First, it is immediately shown that the line $\langle p, q \rangle$ is distinct from the lines A, B, C, A', B', C'. Let us assume that $\mathbf{P}(E)$ has dimension $n \geq 3$. If the seven points d, a, b, c, a', b', c' generate a projective subspace of dimension 3, then by Proposition 26.1, the intersection of the two planes $\langle a, b, c \rangle$ and $\langle a', b', c' \rangle$ is a line, and thus p, q, r are collinear.

If $\mathbf{P}(E)$ has dimension $n = 2$ or the seven points d, a, b, c, a', b', c' generate a projective subspace of dimension 2, we use the following argument. In the projective plane X generated by the seven points d, a, b, c, a', b', c', choose the projective line $\Delta = \langle p, r \rangle$ as the line at infinity. Then in the affine plane $Y = X - \Delta$, the lines $\langle b, c \rangle$ and $\langle b', c' \rangle$ are parallel, and the lines $\langle a, b \rangle$ and $\langle a', b' \rangle$ are parallel, and the lines $\langle a, a' \rangle$, $\langle b, b' \rangle$, and $\langle c, c' \rangle$ are either parallel or concurrent. Then by the converse of the affine version of Desargues's theorem (Proposition 24.13), the lines $\langle a, c \rangle$ and $\langle a', c' \rangle$ are parallel, which means that their intersection q belongs to the line at infinity $\Delta = \langle p, r \rangle$, and thus that p, q, r are collinear. $\square$

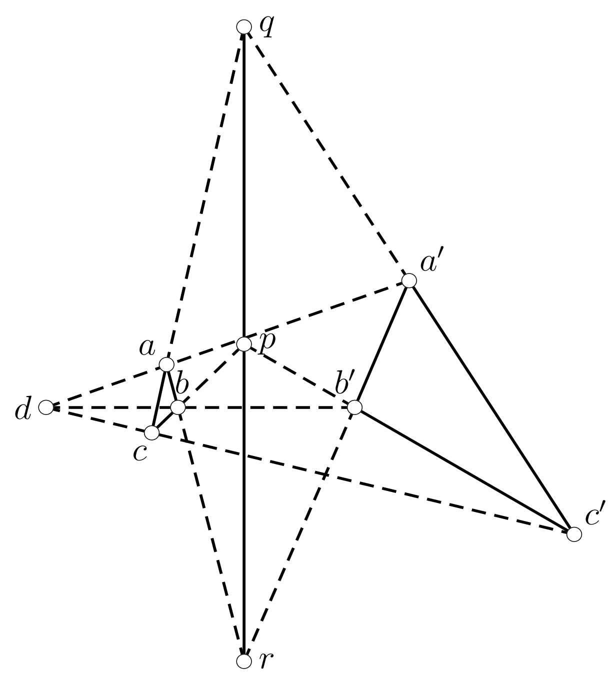

Figure 26.21: Desargues's theorem (projective version).

The converse of Desargues's theorem also holds. Using the projective completion of an affine space, it is easy to state an improved affine version of Desargues's theorem. The reader will have to figure out how to deal with the case where some of the points p, q, r go