We now prove (b).

Proposition 49.16. *Assume that $\nabla J_{u_i} \neq 0$ for $i = 0, \ldots, k$, and let $\Delta_\ell = u_{\ell+1} - u_\ell$, for $\ell = 0, \ldots, k$. Then $\Delta_\ell \neq 0$ for $\ell = 0, \ldots, k$, and*

$$\langle A\Delta_\ell, \Delta_i \rangle = 0, \quad 0 \leq i < \ell \leq k.$$

The vectors $\Delta_0, \ldots, \Delta_k$ are linearly independent.

Proof. Since J is a quadratic functional we have

$$\nabla J_{v+w} = A(v+w) - b = Av - b + Aw = \nabla J_v + Aw.$$

It follows that

$$\nabla J_{u_{\ell+1}} = \nabla J_{u_\ell + \Delta_\ell} = \nabla J_{u_\ell} + A\Delta_\ell, \quad 0 \leq \ell \leq k. \tag{$*_1$}$$

By Proposition 49.15, since

$$\langle \nabla J_{u_i}, \nabla J_{u_j} \rangle = 0, \quad 0 \leq i \neq j \leq k,$$

we get

$$0 = \langle \nabla J_{u_\ell+1}, \nabla J_{u_\ell} \rangle = \left\| \nabla J_{u_\ell} \right\|^2 + \langle A\Delta_\ell, \nabla J_{u_\ell} \rangle, \quad \ell = 0, \ldots, k,$$

and since by hypothesis $\nabla J_{u_i} \neq 0$ for $i = 0, \ldots, k$, we deduce that

$$\Delta_\ell \neq 0, \quad \ell = 0, \ldots, k.$$

If $k \geq 1$, for $i = 0, \ldots, \ell - 1$ and $\ell \leq k$ we also have

$$\begin{aligned} 0 = \langle \nabla J_{u_{\ell+1}}, \nabla J_{u_i} \rangle &= \langle \nabla J_{u_\ell}, \nabla J_{u_i} \rangle + \langle A\Delta_\ell, \nabla J_{u_i} \rangle \\ &= \langle A\Delta_\ell, \nabla J_{u_i} \rangle. \end{aligned}$$

Since $\Delta_j = u_{j+1} - u_j \in \mathcal{G}_j$ and $\mathcal{G}_j$ is spanned by $(\nabla J_{u_0}, \nabla J_{u_1}, \ldots, \nabla J_{u_j})$, we obtain

$$\langle A\Delta_\ell, \Delta_j \rangle = 0, \quad 0 \leq j < \ell \leq k.$$

For the last statement of the proposition, let $w_0, w_1, \ldots, w_k$ be any $k+1$ nonzero vectors such that

$$\langle Aw_i, w_j \rangle = 0, \quad 0 \leq i < j \leq k.$$

We claim that $w_0, w_1, \ldots, w_k$ are linearly independent.

If we have a linear dependence $\sum_{i=0}^{k} \lambda_i w_i = 0$, then we have

$$0 = \left\langle A\left(\sum_{i=0}^{k} \lambda_i w_i\right), w_j \right\rangle = \sum_{i=0}^{k} \lambda_i \langle Aw_i, w_j \rangle = \lambda_j \langle Aw_j, w_j \rangle,$$

where we form these inner products for $j = 0, \ldots, k$, in that order. Since A is symmetric positive definite (because J is a quadratic elliptic functional) and $w_j \neq 0$, we have $\langle Aw_j, w_j \rangle > 0$, and so $\lambda_j = 0$ for $j = 0, \ldots, k$. Therefore the vectors $w_0, w_1, \ldots, w_k$ are linearly independent. $\square$