Then Proposition 34.34 yields the following result.

**Proposition 34.36.** *If $(e_1, \ldots, e_p)$ is any basis of $E$, then every element $\omega \in \mathrm{Alt}^n(E; F)$ can be written in a unique way as*

$$\omega = \sum_I e_I^* \cdot f_I, \qquad f_I \in F,$$

*where the $e_I^*$ are defined as in Section 34.2.*

## 34.11 Problems

**Problem 34.1.** Complete the induction argument used in the proof of Proposition 34.1 (2).

**Problem 34.2.** Prove Proposition 34.2.

**Problem 34.3.** Prove Proposition 34.9.

**Problem 34.4.** Show that the pairing given by $(*)$ in Section 34.4 is nondegenerate.

**Problem 34.5.** Let $\mathfrak{I}_a$ be the two-sided ideal generated by all tensors of the form $u \otimes u \in V^{\otimes 2}$. Prove that

$$\bigwedge^m(V) \cong V^{\otimes m}/(\mathfrak{I}_a \cap V^{\otimes m}).$$

**Problem 34.6.** Complete the induction proof of Proposition 34.12.

**Problem 34.7.** Prove the following lemma: If $V$ is a vector space with $\dim(V) \leq 3$, then $\alpha \wedge \alpha = 0$ whenever $\alpha \in \bigwedge(V)$.

**Problem 34.8.** Prove Proposition 34.13.

**Problem 34.9.** Given two graded algebras $E$ and $F$, define $E \mathbin{\widehat{\otimes}} F$ to be the vector space $E \otimes F$, but with a skew-commutative multiplication given by

$$(a \otimes b) \wedge (c \otimes d) = (-1)^{\deg(b)\deg(c)}(ac) \otimes (bd),$$

where $a \in E^m, b \in F^p$, $c \in E^n, d \in F^q$. Show that

$$\bigwedge(E \oplus F) \cong \bigwedge(E) \mathbin{\widehat{\otimes}} \bigwedge(F).$$

**Problem 34.10.** If $\langle -, - \rangle$ denotes the inner product on $V$, recall that we defined an inner product on $\bigwedge^k V$, also denoted $\langle -, - \rangle$, by setting

$$\langle u_1 \wedge \cdots \wedge u_k, v_1 \wedge \cdots \wedge v_k \rangle = \det(\langle u_i, v_j \rangle),$$

for all $u_i, v_i \in V$, and extending $\langle -, - \rangle$ by bilinearity.

Show that if $(e_1, \ldots, e_n)$ is an orthonormal basis of $V$, then the basis of $\bigwedge^k V$ consisting of the $e_I$ (where $I = \{i_1, \ldots, i_k\}$, with $1 \leq i_1 < \cdots < i_k \leq n$) is also an orthonormal basis of $\bigwedge^k V$.