inequality constraints which are convex but not necessarily differentiable. In fact, it is fair to say that the theory of extended real-valued convex functions and the notions of subgradient and subdifferential developed in this chapter constitute the machinery needed to extend the Lagrangian framework to convex functions that are not necessarily differentiable.

This chapter relies heavily on Rockafellar [138]. Some of the results in this chapter are also discussed in Bertsekas [16, 19, 17]. It should be noted that Bertsekas has developed a framework to discuss duality that he refers to as the *min common/max crossing* framework, for short MC/MC. Although this framework is elegant and interesting in its own right, the fact that Bertsekas relies on it to prove properties of subdifferentials makes it little harder for a reader to "jump in."

51.1 Extended Real-Valued Convex Functions

We extend the ordering on $\mathbb{R}$ by setting

$$-\infty < x < +\infty, \quad \text{for all } x \in \mathbb{R}.$$

Definition 51.2. A (total) function $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$ is called an *extended real-valued function*. For any $x \in \mathbb{R}^n$, we say that $f(x)$ is *finite* if $f(x) \in \mathbb{R}$ (equivalently, $f(x) \neq \pm\infty$). The function f is *finite* if $f(x)$ is finite for all $x \in \mathbb{R}^n$.

Adapting slightly Definition 40.8, given a function $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$, the *epigraph* of f is the subset of $\mathbb{R}^{n+1}$ given by

$$\mathbf{epi}(f) = \{(x, y) \in \mathbb{R}^{n+1} \mid f(x) \leq y\}.$$

See Figure 51.1.

If S is a nonempty subset of $\mathbb{R}^n$, the epigraph of the restriction of f to S is defined as

$$\mathbf{epi}(f|S) = \{(x, y) \in \mathbb{R}^{n+1} \mid f(x) \leq y,\ x \in S\}.$$

Observe the following facts:

1. For any $x \in S$, if $f(x) = -\infty$, then $\mathbf{epi}(f)$ contains the "vertical line" $\{(x, y) \mid y \in \mathbb{R}\}$ in $\mathbb{R}^{n+1}$.

2. For any $x \in S$, if $f(x) \in \mathbb{R}$, then $\mathbf{epi}(f)$ contains the ray $\{(x, y)\} \mid f(x) \leq y\}$ in $\mathbb{R}^{n+1}$.

3. For any $x \in S$, if $f(x) = +\infty$, then $\mathbf{epi}(f)$ does not contain any point (x, y), with $y \in \mathbb{R}$.

4. We have $\mathbf{epi}(f) = \emptyset$ iff f corresponds to the partial function undefined everywhere; that is, $f(x) = +\infty$ for all $x \in \mathbb{R}^n$.