projective space, also denoted by $\widetilde{E}$, has some very interesting properties. In fact, it satisfies a universal property, but before we can say what it is, we have to take a closer look at $\widetilde{E}$.

Since the vector space $\widehat{E}$ is the disjoint union of elements of the form $\langle a, \lambda \rangle$, where $a \in E$ and $\lambda \in K - \{0\}$, and elements of the form $u \in \overrightarrow{E}$, observe that if $\sim$ is the equivalence relation on $\widehat{E}$ used to define the projective space $\mathbf{P}(\widehat{E})$, then the equivalence class $[\langle a, \lambda \rangle]_\sim$ of a weighted point contains the special representative $a = \langle a, 1 \rangle$, and the equivalence class $[u]_\sim$ of a nonzero vector $u \in \overrightarrow{E}$ is just a point of the projective space $\mathbf{P}(\overrightarrow{E})$. Thus, there is a bijection

$$\mathbf{P}(\widehat{E}) \longleftrightarrow E \cup \mathbf{P}(\overrightarrow{E})$$

between $\mathbf{P}(\widehat{E})$ and the disjoint union $E \cup \mathbf{P}(\overrightarrow{E})$, which allows us to view E as being embedded in $\mathbf{P}(\widehat{E})$. The points of $\mathbf{P}(\widehat{E})$ in $\mathbf{P}(\overrightarrow{E})$ will be called *points at infinity*, and the projective hyperplane $\mathbf{P}(\overrightarrow{E})$ is called the *hyperplane at infinity*. We will also denote the point $[u]_\sim$ of $\mathbf{P}(\overrightarrow{E})$ (where $u \neq 0$) by u_∞.

Thus, we can think of $\widetilde{E} = \mathbf{P}(\widehat{E})$ as the projective completion of the affine space E obtained by adding points at infinity forming the hyperplane $\mathbf{P}(\overrightarrow{E})$. As we commented in Section 26.2 when we presented the hyperplane model of $\mathbf{P}(E)$, the notion of point at infinity is really an affine notion. But even if a vector space E doesn't arise from the completion of an affine space, there is an affine structure on the complement of any hyperplane $\mathbf{P}(H)$ in the projective space $\mathbf{P}(E)$. In the case of $\widetilde{E}$, the complement E of the projective hyperplane $\mathbf{P}(\overrightarrow{E})$ is indeed an affine space. This is a general property that is needed in order to figure out the universal property of $\widetilde{E}$.

Proposition 26.16. *Given a vector space E and a hyperplane H in E, the complement $E_H = \mathbf{P}(E) - \mathbf{P}(H)$ of the projective hyperplane $\mathbf{P}(H)$ in the projective space $\mathbf{P}(E)$ can be given an affine structure such that the associated vector space of E_H is H. The affine structure on E_H depends only on H, and under this affine structure, E_H is isomorphic to an affine hyperplane in E.*

Proof. Since H is a hyperplane in E, there is some $w \in E - H$ such that $E = Kw \oplus H$. Thus, every vector u in $E - H$ can be written in a unique way as $\lambda w + h$, where $\lambda \neq 0$ and $h \in H$. As a consequence, for every point $[u]$ in E_H, the equivalence class $[u]$ contains a representative of the form $w + \lambda^{-1} h$, with $\lambda \neq 0$. Then we see that the map $\varphi\colon (w + H) \to E_H$, defined such that

$$\varphi(w + h) = [w + h],$$

is a bijection. In order to define an affine structure on E_H, we define $+\colon E_H \times H \to E_H$ as follows: For every point $[w + h_1] \in E_H$ and every $h_2 \in H$, we let

$$[w + h_1] + h_2 = [w + h_1 + h_2].$$

The axioms of an affine space are immediately verified. Now, $w + H$ is an affine hyperplane is E, and under the affine structure just given to E_H, the map $\varphi\colon (w + H) \to E_H$ is an affine