Remark: When we use the term "canonical isomorphism," we mean that such an isomorphism is defined independently of any choice of bases. For example, if E is a finite dimensional vector space and $(e_1, \ldots, e_n)$ is any basis of E, we have the dual basis $(e_1^*, \ldots, e_n^*)$ of E^* (where, $e_i^*(e_j) = \delta_{i\,j}$), and thus the map $e_i \mapsto e_i^*$ is an isomorphism between E and E^*. This isomorphism is *not* canonical.

On the other hand, if $\langle -, - \rangle$ is an inner product on E, then Proposition 33.1 shows that the nondegenerate pairing $\langle -, - \rangle$ on $E \times E$ induces a canonical isomorphism between E and E^*. This isomorphism is often denoted $\flat \colon E \to E^*$, and we usually write $u^\flat$ for $\flat(u)$, with $u \in E$. Schematically, $u^\flat = \langle u, - \rangle$. The inverse of $\flat$ is denoted $\sharp \colon E^* \to E$, and given any linear form $\omega \in E^*$, we usually write $\omega^\sharp$ for $\sharp(\omega)$. Schematically, $\omega = \langle \omega^\sharp, - \rangle$.

Given any basis, $(e_1, \ldots, e_n)$ of E (not necessarily orthonormal), let (g_{ij}) be the $n \times n$-matrix given by $g_{ij} = \langle e_i, e_j \rangle$ (the *Gram* matrix of the inner product). Recall that the *dual basis* $(e_1^*, \ldots, e_n^*)$ of E^* consists of the coordinate forms $e_i^* \in E^*$, which are characterized by the following properties:

$$e_i^*(e_j) = \delta_{ij}, \quad 1 \leq i, j \leq n.$$

The inverse of the Gram matrix (g_{ij}) is often denoted by (g^{ij}) (by raising the indices).

The tradition of raising and lowering indices is pervasive in the literature on tensors. It is indeed useful to have some notational convention to distinguish between vectors and linear forms (also called *one-forms* or *covectors*). The usual convention is that coordinates of vectors are written using superscripts, as in $u = \sum_{i=1}^n u^i e_i$, and coordinates of one-forms are written using subscripts, as in $\omega = \sum_{i=1}^n \omega_i e_i^*$. Actually, since vectors are indexed with subscripts, one-forms are indexed with superscripts, so e_i^* should be written as e^i.

The motivation is that summation signs can then be omitted, according to the *Einstein summation convention*. According to this convention, whenever a summation variable (such as i) appears both as a subscript and a superscript in an expression, it is assumed that it is involved in a summation. For example the sum $\sum_{i=1}^n u^i e_i$ is abbreviated as

$$u^i e_i,$$

and the sum $\sum_{i=1}^n \omega_i e^i$ is abbreviated as

$$\omega_i e^i.$$

In this text we will not use the Einstein summation convention, which we find somewhat confusing, and we will also write e_i^* instead of e^i.

The maps $\flat$ and $\sharp$ can be described explicitly in terms of the Gram matrix of the inner product and its inverse.

Proposition 33.2. *For any vector space E, given a basis $(e_1, \ldots, e_n)$ for E and its dual basis $(e_1^*, \ldots, e_n^*)$ for E^*, for any inner product $\langle -, - \rangle$ on E, if (g_{ij}) is its Gram matrix, with*