Problem 17.2. Prove that the formula

$$\langle u_1 + iv_1, u_2 + iv_2 \rangle_{\mathbb{C}} = \langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle + i(\langle v_1, u_2 \rangle - \langle u_1, v_2 \rangle)$$

defines a Hermitian form on $E_{\mathbb{C}}$ that is positive definite and that $\langle -, - \rangle_{\mathbb{C}}$ agrees with $\langle -, - \rangle$ on real vectors.

Problem 17.3. Given any linear map $f \colon E \to E$, prove the map $f_{\mathbb{C}}^*$ defined such that

$$f_{\mathbb{C}}^*(u + iv) = f^*(u) + if^*(v)$$

for all $u, v \in E$ is the adjoint of $f_{\mathbb{C}}$ w.r.t. $\langle -, - \rangle_{\mathbb{C}}$.

Problem 17.4. Let A be a real symmetric $n \times n$ matrix whose eigenvalues are nonnegative. Prove that for every $p > 0$, there is a real symmetric matrix S whose eigenvalues are nonnegative such that $S^p = A$.

Problem 17.5. Let A be a real symmetric $n \times n$ matrix whose eigenvalues are positive.

(1) Prove that there is a real symmetric matrix S such that $A = e^S$.

(2) Let S be a real symmetric $n \times n$ matrix. Prove that $A = e^S$ is a real symmetric $n \times n$ matrix whose eigenvalues are positive.

Problem 17.6. Let A be a complex matrix. Prove that if A can be diagonalized with respect to an orthonormal basis, then A is normal.

Problem 17.7. Let $f \colon \mathbb{C}^n \to \mathbb{C}^n$ be a linear map.

(1) Prove that if f is diagonalizable and if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of f, then $\lambda_1^2, \ldots, \lambda_n^2$ are the eigenvalues of f^2, and if $\lambda_i^2 = \lambda_j^2$ implies that $\lambda_i = \lambda_j$, then f and f^2 have the same eigenspaces.

(2) Let f and g be two real self-adjoint linear maps $f, g \colon \mathbb{R}^n \to \mathbb{R}^n$. Prove that if f and g have nonnegative eigenvalues (f and g are positve semidefinite) and if $f^2 = g^2$, then $f = g$.

Problem 17.8. (1) Let $\mathfrak{so}(3)$ be the space of 3×3 skew symmetric matrices

$$\mathfrak{so}(3) = \left\{ \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}.$$

For any matrix

$$A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \in \mathfrak{so}(3),$$

if we let $\theta = \sqrt{a^2 + b^2 + c^2}$, recall from Section 12.7 (the Rodrigues formula) that the exponential map $\exp \colon \mathfrak{so}(3) \to \mathbf{SO}(3)$ is given by

$$e^A = I_3 + \frac{\sin\theta}{\theta} A + \frac{(1 - \cos\theta)}{\theta^2} A^2, \quad \text{if } \theta \neq 0,$$