It can be shown (and you may use these facts without proof) that $\otimes$ is associative and that

$$\begin{aligned}(A \otimes B)(C \otimes D) &= AC \otimes BD \\ (A \otimes B)^\top &= A^\top \otimes B^\top,\end{aligned}$$

whenever AC and BD are well defined.

Given any $n \times n$ matrix X, let $\mathrm{vec}(X)$ be the vector in $\mathbb{R}^{n^2}$ obtained by concatenating the *rows* of X.

(1) Prove that $AX = Y$ iff

$$(A \otimes I_n)\mathrm{vec}(X) = \mathrm{vec}(Y)$$

and $XA = Y$ iff

$$(I_n \otimes A^\top)\mathrm{vec}(X) = \mathrm{vec}(Y).$$

Deduce that $AX + XA = Y$ iff

$$((A \otimes I_n) + (I_n \otimes A^\top))\mathrm{vec}(X) = \mathrm{vec}(Y).$$

In the case where $n = 2$ and if we write

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

check that

$$A \otimes I_2 + I_2 \otimes A^\top = \begin{pmatrix} 2a & c & b & 0 \\ b & a+d & 0 & b \\ c & 0 & a+d & c \\ 0 & c & b & 2d \end{pmatrix}.$$

The problem is determine when the matrix $(A \otimes I_n) + (I_n \otimes A^\top)$ is invertible.

Remark: The equation $AX + XA = Y$ is a special case of the equation $AX + XB = C$ (sometimes written $AX - XB = C$), called the *Sylvester equation*, where A is an $m \times m$ matrix, B is an $n \times n$ matrix, and X, C are $m \times n$ matrices; see Higham [91] (Appendix B).

(2) In the case where $n = 2$, prove that

$$\det(A \otimes I_2 + I_2 \otimes A^\top) = 4(a + d)^2(ad - bc).$$

(3) Let A and B be any two $n \times n$ complex matrices. Use Schur factorizations $A = UT_1U^*$ and $B = VT_2V^*$ (where U and V are unitary and T_1, T_2 are upper-triangular) to prove that if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A and $\mu_1, \ldots, \mu_n$ are the eigenvalues of B, then the scalars $\lambda_i\mu_j$ are the eigenvalues of $A \otimes B$, for $i, j = 1, \ldots, n$.

Hint. Check that $U \otimes V$ is unitary and that $T_1 \otimes T_2$ is upper triangular.