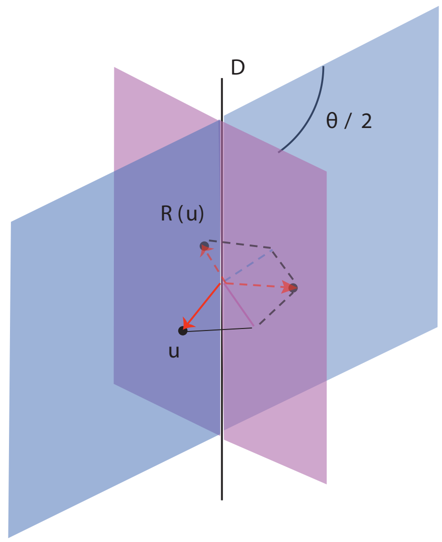

Figure 27.5: 3D rotation as the composition of two reflections.

rotation. The rotation R behaves like a two-dimensional rotation around the axis of rotation. Thus, the rotation R is the composition of two reflections about planes containing the axis of rotation D and forming an angle $\theta/2$. This is illustrated in Figure 27.5.

The measure of the angle of rotation θ can be determined through its cosine via the formula

$$\cos\theta = u \cdot R(u),$$

where u is any unit vector orthogonal to the direction of the axis of rotation. However, this does not determine $\theta \in [0, 2\pi[$ uniquely, since both θ and $2\pi - \theta$ are possible candidates. What is missing is an orientation of the plane (through the origin) orthogonal to the axis of rotation.

In the orthonormal basis of the lemma, a rotation is represented by a matrix of the form

$$R = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Remark: For an arbitrary rotation matrix A, since $a_{1\,1} + a_{2\,2} + a_{3\,3}$ (the *trace* of A) is the sum of the eigenvalues of A, and since these eigenvalues are $\cos\theta + i\sin\theta$, $\cos\theta - i\sin\theta$, and 1, for some $\theta \in [0, 2\pi[$, we can compute $\cos\theta$ from

$$1 + 2\cos\theta = a_{1\,1} + a_{2\,2} + a_{3\,3}.$$

It is also possible to determine the axis of rotation (see the problems).