which implies that $B^\top D = I$, and so

$$\det(A) = \det(B)\det(D) = \det(B^\top)\det(D) = \det(B^\top D) = \det(I) = 1,$$

as claimed $\square$

Proposition 29.43. *Let φ be a nondegenerate symmetric bilinear form on a space E of dimension n, and let f be any isometry $f \in \mathbf{O}(\varphi)$ such that $f(u)-u$ is nonzero and isotropic for every nonisotropic vector $u \in E$. Then, E is an Artinian space of dimension $n = 2m$, and f is a rotation ($f \in \mathbf{SO}(\varphi)$).*

Proof. We follow E. Artin's proof (see [6], Chapter III, Section 4). First, consider the case $n = 2$. Since we are assuming that E has some nonzero isotropic vector, by Proposition 29.26, E is an Artinian plane and there is a basis in which φ is represented by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

we have $\varphi((x_1, x_2), (x_1, x_2)) = 2x_1x_2$, and the matrices representing isometries are of the form

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{pmatrix}, \quad \lambda \in K - \{0\}.$$

In the second case,

$$\begin{pmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{pmatrix}\begin{pmatrix} \lambda \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda \\ 1 \end{pmatrix},$$

but $u = (\lambda, 1)$ is a nonisotropic vector such that $f(u) - u = 0$. Therefore, we must be in the first case, and $\det(f) = +1$.

Let us now assume that $n \geq 3$. We are going to prove that $f(y) - y$ is isotropic for all nonzero isotropic vectors y. Let y be any nonzero isotropic vector. Since $n \geq 3$, the orthogonal space $(Ky)^\perp$ has dimension at least 2, and we know that $\mathrm{rad}(Ky) = \mathrm{rad}((Ky)^\perp)$, a space of dimension at most 1, which implies that $(Ky)^\perp$ contains some nonisotropic vector, say x. We have $\varphi(x, y) = 0$, so $\varphi(x + \epsilon y, x + \epsilon y) = \varphi(x, x) \neq 0$, for $\epsilon = \pm 1$. Then, by hypothesis, the vectors $f(x) - x, f(x + y) - (x + y) = f(x) - x + (f(y) - y)$, and $f(x - y) - (x - y) = f(x) - x - (f(y) - y)$ are isotropic. The last two vectors can be written as $f(x) - x + \epsilon(f(y) - y)$ with $\epsilon = \pm 1$, so we have

$$\begin{aligned} 0 = \varphi(f(x) - x + \epsilon(f(y) - y), f(x) - x + \epsilon(f(y) - y)) \\ = 2\epsilon\varphi(f(x) - x, f(y) - y)) + \epsilon^2\varphi(f(y) - y, f(y) - y). \end{aligned}$$

If we write the two equations corresponding to $\epsilon = \pm 1$, and then add them up, we get

$$\varphi(f(y) - y, f(y) - y) = 0.$$