Example 2.4.

1. The map $\varphi\colon \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ given by $\varphi(m) = m \bmod n$ for all $m \in \mathbb{Z}$ is a homomorphism.

2. The map $\det\colon \mathbf{GL}(n, \mathbb{R}) \to \mathbb{R}$ is a homomorphism because $\det(AB) = \det(A)\det(B)$ for any two matrices A, B. Similarly, the map $\det\colon \mathbf{O}(n) \to \mathbb{R}$ is a homomorphism.

If $\varphi\colon G \to G'$ and $\psi\colon G' \to G''$ are group homomorphisms, then $\psi \circ \varphi\colon G \to G''$ is also a homomorphism. If $\varphi\colon G \to G'$ is a homomorphism of groups, and if $H \subseteq G$, $H' \subseteq G'$ are two subgroups, then it is easily checked that

$$\mathrm{Im}\ \varphi = \varphi(H) = \{\varphi(g) \mid g \in H\}$$

is a subgroup of G' and

$$\varphi^{-1}(H') = \{g \in G \mid \varphi(g) \in H'\}$$

is a subgroup of G. In particular, when $H' = \{e'\}$, we obtain the *kernel*, $\mathrm{Ker}\ \varphi$, of φ.

Definition 2.8. If $\varphi\colon G \to G'$ is a homomorphism of groups, and if $H \subseteq G$ is a subgroup of G, then the subgroup of G',

$$\mathrm{Im}\ \varphi = \varphi(H) = \{\varphi(g) \mid g \in H\},$$

is called the *image of H by φ*, and the subgroup of G,

$$\mathrm{Ker}\ \varphi = \{g \in G \mid \varphi(g) = e'\},$$

is called the *kernel* of φ.

Example 2.5.

1. The kernel of the homomorphism $\varphi\colon \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ is $n\mathbb{Z}$.

2. The kernel of the homomorphism $\det\colon \mathbf{GL}(n, \mathbb{R}) \to \mathbb{R}$ is $\mathbf{SL}(n, \mathbb{R})$. Similarly, the kernel of the homomorphism $\det\colon \mathbf{O}(n) \to \mathbb{R}$ is $\mathbf{SO}(n)$.

The following characterization of the injectivity of a group homomorphism is used all the time.

Proposition 2.9. *If $\varphi\colon G \to G'$ is a homomorphism of groups, then $\varphi\colon G \to G'$ is injective iff* $\mathrm{Ker}\ \varphi = \{e\}$. *(We also write* $\mathrm{Ker}\ \varphi = (0)$.*)*

Proof. Assume φ is injective. Since $\varphi(e) = e'$, if $\varphi(g) = e'$, then $\varphi(g) = \varphi(e)$, and by injectivity of φ we must have $g = e$, so $\mathrm{Ker}\ \varphi = \{e\}$.

Conversely, assume that $\mathrm{Ker}\ \varphi = \{e\}$. If $\varphi(g_1) = \varphi(g_2)$, then by multiplication on the left by $(\varphi(g_1))^{-1}$ we get

$$e' = (\varphi(g_1))^{-1}\varphi(g_1) = (\varphi(g_1))^{-1}\varphi(g_2),$$