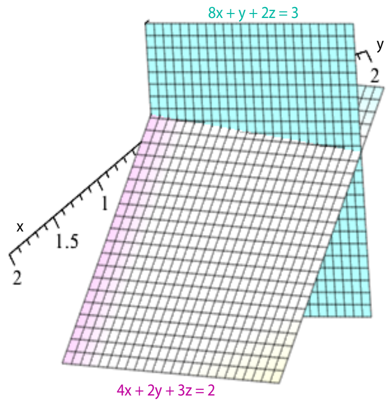

Figure 47.4: The $\mathcal{H}$-polyhedron for the dual linear program of Example 47.1 is the spacial region "above" the pink plane and in "front" of the blue plane. Note $y_1 \to x$, $y_2 \to y$, and $y_3 \to z$.

What happens if x^* is an optimal solution of (P) and if y^* is an optimal solution of (D)? We have $cx^* \leq y^*b$, but is there a "duality gap," that is, is it possible that $cx^* < y^*b$?

The answer is **no**, this is the *strong duality theorem*. Actually, the strong duality theorem asserts more than this.

Theorem 47.7. *(Strong Duality for Linear Programming) Let (P) be any linear program*

$$\begin{aligned} &\text{maximize} && cx \\ &\text{subject to} && Ax \leq b \ \textit{and} \ x \geq 0, \end{aligned}$$

with A an $m \times n$ matrix. The Primal Problem (P) has a feasible solution and is bounded above iff the Dual Problem (D) has a feasible solution and is bounded below. Furthermore, if (P) has a feasible solution and is bounded above, then for every optimal solution x^* of (P) and every optimal solution y^* of (D), we have

$$cx^* = y^*b.$$

Proof. If (P) has a feasible solution and is bounded above, then we know from Proposition 45.1 that (P) has some optimal solution. Let x^* be any optimal solution of (P). First we will show that (D) has a feasible solution v.

Let $\mu = cx^*$ be the maximum of the objective function $x \mapsto cx$. Then for any $\epsilon > 0$, the system of inequalities

$$Ax \leq b, \quad x \geq 0, \quad cx \geq \mu + \epsilon$$

has no solution, since otherwise μ would not be the maximum value of the objective function cx. We would like to apply Farkas II, so first we transform the above system of inequalities