The proof also shows that x minimizes $\|Ax - b\|_2^2$ iff $\overrightarrow{pb} = b - Ax$ is orthogonal to U, which can be expressed by saying that $b - Ax$ is orthogonal to every column of A. However, this is equivalent to

$$A^\top(b - Ax) = 0, \quad \text{i.e.,} \quad A^\top Ax = A^\top b.$$

Finally, it turns out that the minimum norm least squares solution x^+ can be found in terms of the pseudo-inverse A^+ of A, which is itself obtained from any SVD of A.

Definition 23.1. Given any nonzero $m \times n$ matrix A of rank r, if $A = VDU^\top$ is an SVD of A such that

$$D = \begin{pmatrix} \Lambda & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{pmatrix},$$

with

$$\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_r)$$

an $r \times r$ diagonal matrix consisting of the nonzero singular values of A, then if we let D^+ be the $n \times m$ matrix

$$D^+ = \begin{pmatrix} \Lambda^{-1} & 0_{r,m-r} \\ 0_{n-r,r} & 0_{n-r,m-r} \end{pmatrix},$$

with

$$\Lambda^{-1} = \mathrm{diag}(1/\lambda_1, \ldots, 1/\lambda_r),$$

the *pseudo-inverse* of A is defined by

$$A^+ = UD^+V^\top.$$

If $A = 0_{m,n}$ is the zero matrix, we set $A^+ = 0_{n,m}$. Observe that D^+ is obtained from D by inverting the nonzero diagonal entries of D, leaving all zeros in place, and then transposing the matrix. For example, given the matrix

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

its pseudo-inverse is

$$D^+ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The pseudo-inverse of a matrix is also known as the *Moore–Penrose pseudo-inverse*.

Actually, it seems that A^+ depends on the specific choice of U and V in an SVD (U, D, V) for A, but the next theorem shows that this is not so.