on $\mathbb{R}^n$. As an exercise, the reader should write similar formulae for the Taylor–MacLaurin formula of order 2.

Another application of Taylor's formula is the derivation of a formula which gives the m-th derivative of the composition of two functions, usually known as "Faà di Bruno's formula." This formula is useful when dealing with geometric continuity of splines curves and surfaces.

Proposition 39.27. *Given any normed affine space E, for any function $f\colon \mathbb{R} \to \mathbb{R}$ and any function $g\colon \mathbb{R} \to E$, for any $a \in \mathbb{R}$, letting $b = f(a)$, $f^{(i)}(a) = \mathrm{D}^i f(a)$, and $g^{(i)}(b) = \mathrm{D}^i g(b)$, for any $m \geq 1$, if $f^{(i)}(a)$ and $g^{(i)}(b)$ exist for all i, $1 \leq i \leq m$, then $(g\circ f)^{(m)}(a) = \mathrm{D}^m(g\circ f)(a)$ exists and is given by the following formula:*

$$(g \circ f)^{(m)}(a) = \sum_{0\leq j\leq m} \sum_{\substack{i_1+i_2+\cdots+i_m=j\\ i_1+2i_2+\cdots+mi_m=m\\ i_1,i_2,\cdots,i_m\geq 0}} \frac{m!}{i_1!\cdots i_m!}\, g^{(j)}(b) \left(\frac{f^{(1)}(a)}{1!}\right)^{i_1} \cdots \left(\frac{f^{(m)}(a)}{m!}\right)^{i_m}.$$

When $m = 1$, the above simplifies to the familiar formula

$$(g \circ f)'(a) = g'(b)f'(a),$$

and for $m = 2$, we have

$$(g \circ f)^{(2)}(a) = g^{(2)}(b)(f^{(1)}(a))^2 + g^{(1)}(b)f^{(2)}(a).$$

39.8 Vector Fields, Covariant Derivatives, Lie Brackets

In this section, we briefly consider vector fields and covariant derivatives of vector fields. Such derivatives play an important role in continuous mechanics. Given a normed affine space $(E, \overrightarrow{E})$, a *vector field over* $(E, \overrightarrow{E})$ is a function $X\colon E \to \overrightarrow{E}$. Intuitively, a vector field assigns a vector to every point in E. Such vectors could be forces, velocities, accelerations, etc.

Given two vector fields X, Y defined on some open subset Ω of E, for every point $a \in \Omega$, we would like to define the derivative of X with respect to Y at a. This is a type of directional derivative that gives the variation of X as we move along Y, and we denote it by $\mathrm{D}_Y X(a)$. The derivative $\mathrm{D}_Y X(a)$ is defined as follows.

Definition 39.20. Let $(E, \overrightarrow{E})$ be a normed affine space. Given any open subset Ω of E, given any two vector fields X and Y defined over Ω, for any $a \in \Omega$, the *covariant derivative (or Lie derivative) of X w.r.t. the vector field Y at a*, denoted by $\mathrm{D}_Y X(a)$, is the limit (if it exists)

$$\lim_{t\to 0,\, t\in U} \frac{X(a + tY(a)) - X(a)}{t},$$

where $U = \{t \in \mathbb{R} \mid a + tY(a) \in \Omega,\ t \neq 0\}$.