The second equation yields

$$L(u, \mu) = J(u) + \sum_{i=1}^{m} \mu_i \varphi_i(u) \leq J(u) = J(u) + \sum_{i=1}^{m} \lambda_i \varphi_i(u) = L(u, \lambda),$$

that is,

$$L(u, \mu) \leq L(u, \lambda) \quad \text{for all } \mu \in \mathbb{R}^m_+ \tag{$*_2$}$$

(since $\varphi_i(u) \leq 0$ as $u \in U$), and since the function $v \mapsto J(v) + \sum_{i=1} \lambda_i \varphi_i(v) = L(v, \lambda)$ is convex as a sum of convex functions, by Theorem 40.13(4), the first equation is a sufficient condition for the existence of minimum. Consequently,

$$L(u, \lambda) \leq L(v, \lambda) \quad \text{for all } v \in \Omega, \tag{$*_3$}$$

and $(*_2)$ and $(*_3)$ show that (u, λ) is a saddle point of L. □

To recap what we just proved, under some mild hypotheses, the set of solutions of the Minimization Problem (P)

$$\begin{aligned} &\text{minimize} && J(v) \\ &\text{subject to} && \varphi_i(v) \leq 0, \quad i = 1, \ldots, m \end{aligned}$$

coincides with the set of first arguments of the saddle points of the Lagrangian

$$L(v, \mu) = J(v) + \sum_{i=1}^{m} \mu_i \varphi_i(v),$$

and for any optimum $u \in U$ of Problem (P), we have $J(u) = L(u, \lambda)$.

Therefore, if we knew some particular second argument λ of these saddle points, then the *constrained* Problem (P) would be replaced by the *unconstrained* Problem (P_λ):

$$\begin{aligned} &\text{find } u_\lambda \in \Omega \text{ such that} \\ &L(u_\lambda, \lambda) = \inf_{v \in \Omega} L(v, \lambda). \end{aligned}$$

How do we find such an element $\lambda \in \mathbb{R}^m_+$*?*

For this, remember that for a saddle point (u_λ, λ), by Proposition 50.14, we have

$$L(u_\lambda, \lambda) = \inf_{v \in \Omega} L(v, \lambda) = \sup_{\mu \in \mathbb{R}^m_+} \inf_{v \in \Omega} L(v, \mu),$$

so we are naturally led to introduce the function $G \colon \mathbb{R}^m_+ \to \mathbb{R}$ given by

$$G(\mu) = \inf_{v \in \Omega} L(v, \mu) \quad \mu \in \mathbb{R}^m_+,$$