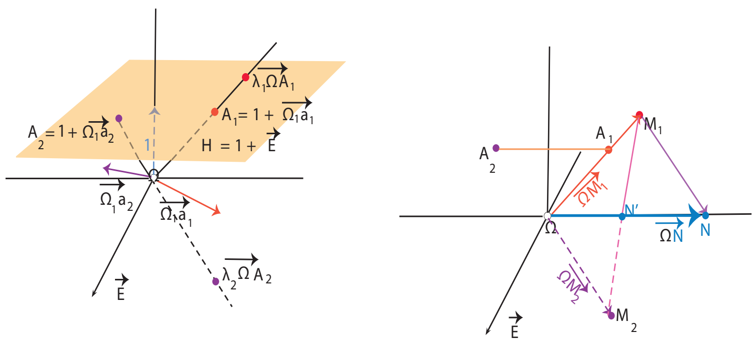

Figure 25.6: The geometric construction of $\widehat{\Omega}(\langle a_1, \lambda_1\rangle) + \widehat{\Omega}(\langle a_2, \lambda_2\rangle)$ for $\lambda_1 + \lambda_2 = 0$.

25.4 Extending Affine Maps to Linear Maps

Roughly, the vector space $\widehat{E}$ has the property that for any vector space $\overrightarrow{F}$ and any affine map $f\colon E \to \overrightarrow{F}$, there is a unique linear map $\widehat{f}\colon \widehat{E} \to \overrightarrow{F}$ extending $f\colon E \to \overrightarrow{F}$. As a consequence, given two affine spaces E and F, every affine map $f\colon E \to F$ extends uniquely to a linear map $\widehat{f}\colon \widehat{E} \to \widehat{F}$. First, we define rigorously the notion of homogenization of an affine space.

Definition 25.2. Given any affine space $\big(E, \overrightarrow{E}\big)$, a *homogenization (or linearization) of* $(E, \overrightarrow{E})$ is a triple $\langle \mathcal{E}, j, \omega\rangle$, where $\mathcal{E}$ is a vector space, $j\colon E \to \mathcal{E}$ is an injective affine map with associated injective linear map $i\colon \overrightarrow{E} \to \mathcal{E}$, $\omega\colon \mathcal{E} \to \mathbb{R}$ is a linear form such that $\omega^{-1}(0) = i\big(\overrightarrow{E}\big)$, $\omega^{-1}(1) = j(E)$, and for every vector space $\overrightarrow{F}$ and every affine map $f\colon E \to \overrightarrow{F}$ there is a unique linear map $\widehat{f}\colon \mathcal{E} \to \overrightarrow{F}$ extending f, i.e., $f = \widehat{f} \circ j$, as in the following diagram:

$$\begin{tikzcd} E \arrow[r, "j"] \arrow[dr, "f"'] & \mathcal{E} \arrow[d, "\widehat{f}"] \\ & \overrightarrow{F} \end{tikzcd}$$

Thus, $j(E) = \omega^{-1}(1)$ is an affine hyperplane with direction $i\big(\overrightarrow{E}\big) = \omega^{-1}(0)$. Note that we could have defined a homogenization of an affine space $(E, \overrightarrow{E})$, as a triple $\langle \mathcal{E}, j, H\rangle$, where $\mathcal{E}$ is a vector space, H is an affine hyperplane in $\mathcal{E}$, and $j\colon E \to \mathcal{E}$ is an injective affine map such that $j(E) = H$, and such that the universal property stated above holds. However, Definition 25.2 is more convenient for our purposes, since it makes the notion of weight more evident.

The obvious candidate for $\mathcal{E}$ is the vector space $\widehat{E}$ that we just constructed. The next proposition will show that $\widehat{E}$ indeed has the required extension property. As usual, objects