and B lies in the other half-space determined by H); see Gallier [72] (Chapter 7, Corollary 7.4 and Proposition 7.3). This proof is nontrivial and involves a geometric version of the Hahn–Banach theorem.

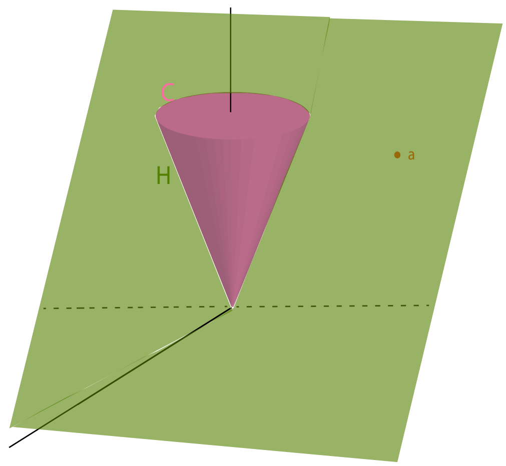

Figure 47.1: In $\mathbb{R}^3$, the olive green hyperplane H separates the cone C from the orange point a.

The Farkas–Minkowski proposition is Proposition 47.1 applied to a polyhedral cone

$$C = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid \lambda_i \geq 0,\ i = 1, \ldots, n\}$$

where $\{a_1, \ldots, a_n\}$ is a *finite* number of vectors $a_i \in \mathbb{R}^n$. By Proposition 44.2, any polyhedral cone is closed, so Proposition 47.1 applies and we obtain the following separation lemma.

Proposition 47.2. *(Farkas–Minkowski) Let $C \subseteq \mathbb{R}^n$ be a nonempty polyhedral cone $C = \mathrm{cone}(\{a_1, \ldots, a_n\})$. For any point $b \in \mathbb{R}^n$, if $b \notin C$, then there is a linear hyperplane H (through 0) such that*

1. *C lies in one of the two half-spaces determined by H.*

2. *$b \notin H$*

3. *b lies in the other half-space determined by H.*

Equivalently, there is a nonzero linear form $y \in (\mathbb{R}^n)^*$ such that

1. *$ya_i \geq 0$ for $i = 1, \ldots, n$.*

2. *$yb < 0$.*