we ensure that for $\rho \in [a, b]$ we have

$$T(\rho)^{1/2} = (M^2\rho^2 - 2\alpha\rho + 1)^{1/2} \leq (\max\{T(a), T(b)\})^{1/2} = \beta < 1.$$

Then by induction we get

$$\|u_{k+1} - u\| \leq \beta^{k+1} \|u_0 - u\|,$$

which proves convergence. $\square$

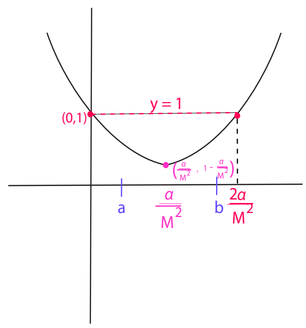

Figure 49.7: The parabola $T(\rho)$ used in the proof of Proposition 49.14.

Remarks: In the proof of Proposition 49.14, it is the fact that V is complete which plays a crucial role. If J is twice differentiable, the hypothesis

$$\|\nabla J_v - \nabla J_u\| \leq M \|v - u\| \quad \text{for all } u, v \in V$$

can be expressed as

$$\sup_{v \in V} \left\|\nabla^2 J_v\right\| \leq M.$$

In the case of a quadratic elliptic functional defined over $\mathbb{R}^n$,

$$J(v) = \langle Av, v\rangle - \langle b, v\rangle,$$

the upper bound $2\alpha/M^2$ can be improved. In this case we have

$$\nabla J_v = Av - b,$$

and we know that $\alpha = \lambda_1$ and $M = \lambda_n$ do the job, where λ_1 is the smallest eigenvalue of A and λ_n is the largest eigenvalue of A. Hence we can pick a, b such that

$$0 < a \leq \rho_k \leq b < \frac{2\lambda_1}{\lambda_n^2}.$$