Thus, an affine isometry is an affine map that preserves the distance. This is a rather strong requirement. In fact, we will show that for any function $f\colon E \to F$, the assumption that

$$\|\overrightarrow{f(a)f(b)}\| = \|\overrightarrow{ab}\|,$$

for all $a, b \in E$, forces f to be an affine map.

Remark: Sometimes, an affine isometry is defined as a *bijective* affine isometry. When E and F are of finite dimension, the definitions are equivalent.

The following simple lemma is left as an exercise.

Proposition 27.6. *Given any two nontrivial Euclidean affine spaces E and F of the same finite dimension n, an affine map $f\colon E \to F$ is an affine isometry iff its associated linear map $\overrightarrow{f}\colon \overrightarrow{E} \to \overrightarrow{F}$ is an isometry. An affine isometry is a bijection.*

Let us now consider affine isometries $f\colon E \to E$. If $\overrightarrow{f}$ is a rotation, we call f a *proper (or direct) affine isometry*, and if $\overrightarrow{f}$ is an improper linear isometry, we call f an *improper (or skew) affine isometry*. It is easily shown that the set of affine isometries $f\colon E \to E$ forms a group, and those for which $\overrightarrow{f}$ is a rotation is a subgroup. The group of affine isometries, or rigid motions, is a subgroup of the affine group $\mathbf{GA}(E)$, denoted by $\mathbf{Is}(E)$ (or $\mathbf{Is}(n)$ when $E = \mathbb{E}^n$). In Snapper and Troyer [162] the group of rigid motions is denoted by $\mathbf{Mo}(E)$. Since we denote the group of affine bijections as $\mathbf{GA}(E)$, perhaps we should denote the group of affine isometries by $\mathbf{IA}(E)$ (or $\mathbf{EA}(E)$!). The subgroup of $\mathbf{Is}(E)$ consisting of the direct rigid motions is also a subgroup of $\mathbf{SA}(E)$, and it is denoted by $\mathbf{SE}(E)$ (or $\mathbf{SE}(n)$, when $E = \mathbb{E}^n$). The translations are the affine isometries f for which $\overrightarrow{f} = \mathrm{id}$, the identity map on $\overrightarrow{E}$. The following lemma is the counterpart of Lemma 12.12 for isometries between Euclidean vector spaces.

Proposition 27.7. *Given any two nontrivial Euclidean affine spaces E and F of the same finite dimension n, for every function $f\colon E \to F$, the following properties are equivalent:*

(1) f is an affine map and $\|\overrightarrow{f(a)f(b)}\| = \|\overrightarrow{ab}\|$, for all $a, b \in E$.

(2) $\|\overrightarrow{f(a)f(b)}\| = \|\overrightarrow{ab}\|$, for all $a, b \in E$.

Proof. Obviously, (1) implies (2). In order to prove that (2) implies (1), we proceed as follows. First, we pick some arbitrary point $\Omega \in E$. We define the map $g\colon \overrightarrow{E} \to \overrightarrow{F}$ such that

$$g(u) = \overrightarrow{f(\Omega)f(\Omega + u)}$$

for all $u \in E$. Since

$$f(\Omega) + g(u) = f(\Omega) + \overrightarrow{f(\Omega)f(\Omega + u)} = f(\Omega + u)$$