If a complex $n \times n$ matrix A is expressed in terms of its Jordan decomposition as $A = D + N$, since D and N commute, by Proposition 9.21, the exponential of A is given by

$$e^A = e^D e^N,$$

and since N is an $n \times n$ nilpotent matrix, $N^{n-1} = 0$, so we obtain

$$e^A = e^D \left(I + \frac{N}{1!} + \frac{N^2}{2!} + \cdots + \frac{N^{n-1}}{(n-1)!} \right).$$

In particular, the above applies if A is a Jordan matrix. This fact can be used to solve (at least in theory) systems of first-order linear differential equations. Such systems are of the form

$$\frac{dX}{dt} = AX, \tag{$*$}$$

where A is an $n \times n$ matrix and X is an n-dimensional vector of functions of the parameter t.

It can be shown that the columns of the matrix e^{tA} form a basis of the vector space of solutions of the system of linear differential equations $(*)$; see Artin [7] (Chapter 4). Furthermore, for any matrix B and any invertible matrix P, if $A = PBP^{-1}$, then the system $(*)$ is equivalent to

$$P^{-1}\frac{dX}{dt} = BP^{-1}X,$$

so if we make the change of variable $Y = P^{-1}X$, we obtain the system

$$\frac{dY}{dt} = BY. \tag{$**$}$$

Consequently, if B is such that the exponential e^{tB} can be easily computed, we obtain an explicit solution Y of $(**)$, and $X = PY$ is an explicit solution of $(*)$. This is the case when B is a Jordan form of A. In this case, it suffices to consider the Jordan blocks of B. Then we have

$$J_r(\lambda) = \lambda I_r + \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} = \lambda I_r + N,$$

and the powers N^k are easily computed.

For example, if

$$B = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix} = 3I_3 + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$