Hence

$$A \subseteq V_{\epsilon_1}(B) \subseteq V_{\epsilon_1}(V_{\epsilon_2}(C)) \subseteq V_{\epsilon_1+\epsilon_2}(C),$$

and thus the triangle inequality follows.

Next we need to prove that if (X, d) is complete, then $(\mathcal{K}(X), D)$ is also complete. First we show that if (A_n) is a sequence of nonempty compact sets converging to a nonempty compact set A in the Hausdorff metric, then

$$A = \{x \in X \mid \text{there is a sequence, } (x_n), \text{ with } x_n \in A_n \text{ converging to } x\}.$$

Indeed, if (x_n) is a sequence with $x_n \in A_n$ converging to x and (A_n) converges to A then, for every $\epsilon > 0$, there is some x_n such that $d(x_n, x) \leq \epsilon/2$ and there is some $a_n \in A$ such that $d(a_n, x_n) \leq \epsilon/2$ and thus, $d(a_n, x) \leq \epsilon$, which shows that $x \in \overline{A}$. Since A is compact, it is closed, and $x \in A$. See Figure 37.48.

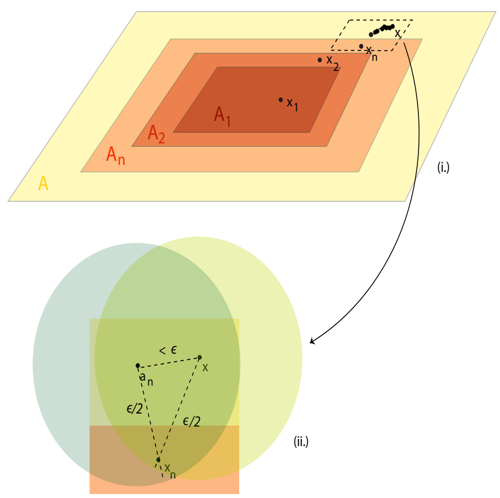

Figure 37.48: Let (A_n) be the sequence of parallelograms converging to A, the large pale yellow parallelogram. Figure $(ii.)$ expands the dashed region and shows why $d(a_n, x) < \epsilon$.

Conversely, since (A_n) converges to A, for every $x \in A$, for every $n \geq 1$, there is some $x_n \in A_n$ such that $d(x_n, x) \leq 1/n$ and the sequence (x_n) converges to x.

Now let (A_n) be a Cauchy sequence in $\mathcal{K}(X)$. It can be proven that (A_n) converges to the set

$$A = \{x \in X \mid \text{there is a sequence, } (x_n), \text{ with } x_n \in A_n \text{ converging to } x\},$$

and that A is nonempty and compact. To prove that A is compact, one proves that it is totally bounded and complete. Details are given in Edgar [55].