Problem 15.13. Consider the following real tridiagonal symmetric $n \times n$ matrix

$$A = \begin{pmatrix} c & 1 & 0 & & \\ 1 & c & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & c & 1 \\ & & 0 & 1 & c \end{pmatrix}.$$

(1) Using Problem 10.6, prove that the eigenvalues of the matrix A are given by

$$\lambda_k = c + 2\cos\left(\frac{k\pi}{n+1}\right), \quad k = 1, \ldots, n.$$

(2) Find a condition on c so that A is positive definite. It is satisfied by $c = 4$?

Problem 15.14. Let A be an $m \times n$ matrix and B be an $n \times m$ matrix (over $\mathbb{C}$).

(1) Prove that

$$\det(I_m - AB) = \det(I_n - BA).$$

Hint. Consider the matrices

$$X = \begin{pmatrix} I_m & A \\ B & I_n \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} I_m & 0 \\ -B & I_n \end{pmatrix}.$$

(2) Prove that

$$\lambda^n \det(\lambda I_m - AB) = \lambda^m \det(\lambda I_n - BA).$$

Hint. Consider the matrices

$$X = \begin{pmatrix} \lambda I_m & A \\ B & I_n \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} I_m & 0 \\ -B & \lambda I_n \end{pmatrix}.$$

Deduce that AB and BA have the same nonzero eigenvalues with the same multiplicity.

Problem 15.15. The purpose of this problem is to prove that the characteristic polynomial of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 2 & 3 & 4 & 5 & \cdots & n+1 \\ 3 & 4 & 5 & 6 & \cdots & n+2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ n & n+1 & n+2 & n+3 & \cdots & 2n-1 \end{pmatrix}$$

is

$$P_A(\lambda) = \lambda^{n-2}\left(\lambda^2 - n^2\lambda - \frac{1}{12}n^2(n^2-1)\right).$$