example, they are used to prove the convergence of the ADMM method discussed in Chapter 52.

One should note that the notion of subdifferential is not just a gratuitous mathematical generalization. The remarkable fact that the optimization problem

$$\begin{aligned} &\text{minimize} \quad J(u) \\ &\text{subject to} \quad u \in C, \end{aligned}$$

where C is a closed convex set in $\mathbb{R}^n$ can be rewritten as

$$\begin{aligned} &\text{minimize} \quad J(u) + I_C(z) \\ &\text{subject to} \quad u - z = 0, \end{aligned}$$

where I_C is the indicator function of C, forces us to deal with functions such as $J(u) + I_C(z)$ which are not differentiable, even if J is. ADMM can cope with this situation (under certain conditions), and subdifferentials cannot be avoided in justifying its convergence. However, it should be said that the subdifferential $\partial f(x)$ is a theoretical tool that is never computed in practice (except in very special simple cases).

To define subgradients we need to review (affine) hyperplanes.

Recall that an *affine form* $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ is a function of the form

$$\varphi(x) = h(x) + c, \quad x \in \mathbb{R}^n,$$

where $h\colon \mathbb{R}^n \to \mathbb{R}$ is a linear form and $c \in \mathbb{R}$ is some constant. An *affine hyperplane* $H \subseteq \mathbb{R}^n$ is the kernel of any nonconstant affine form $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ (which means that the linear form h defining φ is not the zero linear form),

$$H = \varphi^{-1}(0) = \{x \in \mathbb{R}^n \mid \varphi(x) = 0\}.$$

Any two nonconstant affine forms φ and ψ defining the same (affine) hyperplane H, in the sense that $H = \varphi^{-1}(0) = \psi^{-1}(0)$, must be proportional, which means that there is some nonzero $\alpha \in \mathbb{R}$ such that $\psi = \alpha\varphi$.

A nonconstant affine form φ also defines the two *half spaces* H_+ and H_- given by

$$H_+ = \{x \in \mathbb{R}^n \mid \varphi(x) \geq 0\}, \quad H_- = \{x \in \mathbb{R}^n \mid \varphi(x) \leq 0\}.$$

Clearly, $H_+ \cap H_- = H$, their common boundary. See Figure 51.8. The choice of sign is somewhat arbitrary, since the affine form $\alpha\varphi$ with $\alpha < 0$ defines the half spaces with H_- and H_+ (the half spaces are swapped).

By the duality induced by the Euclidean inner product on $\mathbb{R}^n$, a linear form $h\colon \mathbb{R}^n \to \mathbb{R}$ corresponds to a *unique* vector $u \in \mathbb{R}^n$ such that

$$h(x) = \langle x, u\rangle \quad \text{for all } x \in \mathbb{R}^n.$$