(1) *For any $x \in \mathbb{R}^n$, if there is some $y \in \partial h(x)$ such that $-y \in N_C(x)$, that is, $-y$ is normal to C at x, then h attains its minimum on C at x.*

(2) *If* $\mathbf{relint}(\mathrm{dom}(h)) \cap \mathbf{relint}(C) \neq \emptyset$*, then the converse of (1) holds. This means that if h attains its minimum on C at x, then there is some $y \in \partial h(x)$ such that $-y \in N_C(x)$.*

Proposition 51.38 is proven in Rockafellar [138] (Theorem 27.4). The proof is actually quite simple.

Proof. (1) By Proposition 51.34, h attains its minimum on C at x iff

$$0 \in \partial(h + I_C)(x).$$

By Proposition 51.23, since

$$\partial h(x) + \partial I_C(x) \subseteq \partial(h + I_C)(x),$$

if $0 \in \partial h(x) + \partial I_C(x)$, then h attains its minimum on C at x. But we saw in Section 51.2 that $\partial I_C(x) = N_C(x)$, the normal cone to C at x. Then the condition $0 \in \partial h(x) + \partial I_C(x)$ says that there is some $y \in \partial h(x)$ such that $y + z = 0$ for some $z \in N_C(x)$, and this is equivalent to $-y \in N_C(x)$.

(2) By definition of I_C, the condition $\mathbf{relint}(\mathrm{dom}(h)) \cap \mathbf{relint}(C) \neq \emptyset$ is the hypothesis of Proposition 51.23 to have

$$\partial(h + I_C)(x) = \partial h(x) + \partial I_C(x).$$

If h attains its minimum on C at x, then by Proposition 51.34 we have $0 \in \partial(h + I_C)(x)$, so $0 \in \partial h(x) + \partial I_C(x) = \partial h(x) + N_C(x)$, and by the reasoning of Part (1), this means that there is some $y \in \partial h(x)$ such that $-y \in N_C(x)$. ☐

Remark: A *polyhedral function* is a convex function whose epigraph is a polyhedron. It is easy to see that Proposition 51.38(2) also holds in the following cases

(1) C is a $\mathcal{H}$-polyhedron and $\mathbf{relint}(\mathrm{dom}(h)) \cap C \neq \emptyset$

(2) h is polyhedral and $\mathrm{dom}(h) \cap \mathbf{relint}(C) \neq \emptyset$.

(3) Both h and C are polyhedral, and $\mathrm{dom}(h) \cap C \neq \emptyset$.