where $\pi_1 \colon f(U) \times W \to f(U)$ is the first projection. Equivalently,

$$(f \circ \varphi^{-1})(y_1, \ldots, y_m, \ldots, y_n) = (y_1, \ldots, y_m).$$

$$\begin{tikzcd} U \subseteq A \arrow[r, "\varphi"] \arrow[dr, "f"'] & f(U) \times W \arrow[d, "\pi_1"] \\ & f(U) \subseteq \mathbb{R}^m \end{tikzcd}$$

Futhermore, the image of every open subset of A under f is an open subset of F. (The same result holds for $\mathbb{C}^n$ and $\mathbb{C}^m$). See Figure 39.7.

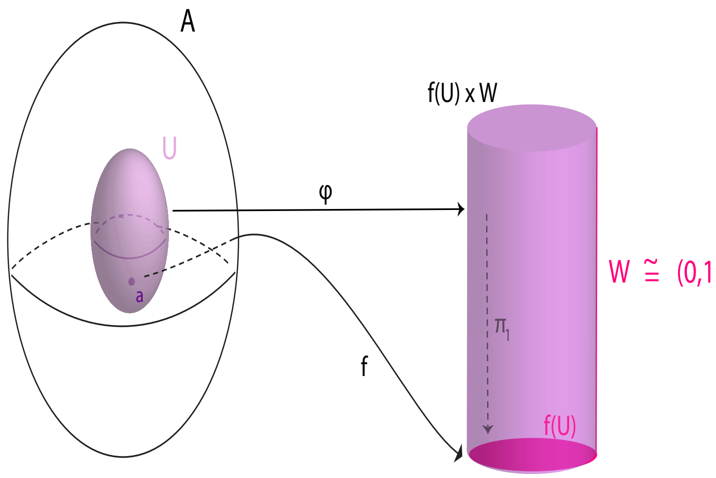

Figure 39.7: Let $n = 3$ and $m = 2$. The submersion maps the solid lavender egg in $\mathbb{R}^3$ onto the bottom pink circular face of the solid cylinder $f(U) \times W$.

Proposition 39.17. *Let A be an open subset of $\mathbb{R}^n$, and let $f \colon A \to \mathbb{R}^m$ be a function. For every $a \in A$, $f \colon A \to \mathbb{R}^m$ is an immersion at a iff there exists an open subset U of A containing a, an open subset V containing $f(a)$ such that $f(U) \subseteq V$, an open subset W containing 0 such that $W \subseteq \mathbb{R}^{m-n}$, and a diffeomorphism $\varphi \colon V \to U \times W$, such that,*

$$\varphi \circ f = in_1,$$

where $in_1 \colon U \to U \times W$ is the injection map such that $in_1(u) = (u, 0)$, or equivalently,

$$(\varphi \circ f)(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, \ldots, 0).$$

$$\begin{tikzcd} U \subseteq A \arrow[r, "f"] \arrow[dr, "in_1"'] & f(U) \subseteq V \arrow[d, "\varphi"] \\ & U \times W \end{tikzcd}$$

(The same result holds for $\mathbb{C}^n$ and $\mathbb{C}^m$). See Figure 39.8.