is a bijection between $\mathbf{P}(E^*)$ and the set of hyperplanes in E, mapping the equivalence class $[f]_\sim = \{\lambda f \mid \lambda \neq 0\}$ of a nonnull linear form $f \in E^*$ to the hyperplane $H = \mathrm{Ker}\, f$. Furthermore, if $u \sim v$, which means that $u = \lambda v$ for some $\lambda \neq 0$, we have

$$f(u) = 0 \quad \text{iff} \quad f(v) = 0,$$

since $f(v) = \lambda f(u)$ and $\lambda \neq 0$. Thus, there is a bijection

$$\{\lambda f \mid \lambda \neq 0\} \mapsto \mathbf{P}(\mathrm{Ker}\, f)$$

mapping points in $\mathbf{P}(E^*)$ to hyperplanes in $\mathbf{P}(E)$. Any nonnull linear form f associated with some hyperplane $\mathbf{P}(H)$ in the above bijection (i.e., $H = \mathrm{Ker}\, f$) is called an *equation of the projective hyperplane* $\mathbf{P}(H)$. We also say that $f = 0$ *is the equation of the hyperplane* $\mathbf{P}(H)$.

Before ending this section, we give an example of a projective space where lines have a nontrivial geometric interpretation, namely as "pencils of lines." If $E = \mathbb{R}^3$, recall that the dual space E^* is the set of all linear maps $f\colon \mathbb{R}^3 \to \mathbb{R}$. As we have just explained, there is a bijection

$$p(f) \mapsto \mathbf{P}(\mathrm{Ker}\, f)$$

between $\mathbf{P}(E^*)$ and the set of lines in $\mathbf{P}(E)$, mapping every point $a^* = p(f)$ to the line $D_{a^*} = \mathbf{P}(\mathrm{Ker}\, f)$.

Is there a way to give a geometric interpretation in $\mathbf{P}(E)$ of a line Δ in $\mathbf{P}(E^*)$? Well, a line Δ in $\mathbf{P}(E^*)$ is defined by two distinct points $a^* = p(f)$ and $b^* = p(g)$, where $f, g \in E^*$ are two linearly independent linear forms. But f and g define two distinct planes $H_1 = \mathrm{Ker}\, f$ and $H_2 = \mathrm{Ker}\, g$ through the origin (in $E = \mathbb{R}^3$), and H_1 and H_2 define two distinct lines $D_1 = p(H_1)$ and $D_2 = p(H_2)$ in $\mathbf{P}(E)$. The line Δ in $\mathbf{P}(E^*)$ is of the form $\Delta = p(V)$, where

$$V = \{\lambda f + \mu g \mid \lambda, \mu \in \mathbb{R}\}$$

is the plane in E^* spanned by f, g. Every nonnull linear form $\lambda f + \mu g \in V$ defines a plane $H = \mathrm{Ker}\,(\lambda f + \mu g)$ in E, and since H_1 and H_2 (in E) are distinct, they intersect in a line L that is also contained in every plane H as above. Thus, the set of planes in E associated with nonnull linear forms in V is just the set of all planes containing the line L. Passing to $\mathbf{P}(E)$ using the projection p, the line L in E corresponds to the point $c = p(L)$ in $\mathbf{P}(E)$, which is just the intersection of the lines D_1 and D_2. Thus, every point of the line Δ in $\mathbf{P}(E^*)$ corresponds to a line in $\mathbf{P}(E)$ passing through c (the intersection of the lines D_1 and D_2), and this correspondence is bijective.

In summary, a line Δ in $\mathbf{P}(E^*)$ corresponds to the set of all lines in $\mathbf{P}(E)$ through some given point. Such sets of lines are called *pencils of lines* and are illustrated in Figure 26.6.

The above discussion can be generalized to higher dimensions and is discussed quite extensively in Section 26.12. In brief, letting $E = \mathbb{R}^{n+1}$, there is a bijection mapping points in $\mathbf{P}(E^*)$ to hyperplanes in $\mathbf{P}(E)$. A line in $\mathbf{P}(E^*)$ corresponds to a *pencil of hyperplanes* in