For every $\epsilon > 0$, there is some $\eta > 0$, such that, for every $x \in A$,

$$\text{if } \|x - a\|_E \leq \eta, \text{ then } \|f(x) - b\|_F \leq \epsilon.$$

We have the following result relating continuity at a point and the previous notion.

Proposition 37.14. *Let $(E, \mathcal{O}_E)$ and $(F, \mathcal{O}_F)$ be two topological spaces, and let $f \colon E \to F$ be a function. For any $a \in E$, the function f is continuous at a iff $f(x)$ approaches $f(a)$ when x approaches a (with values in E).*

Proof. Left as a trivial exercise. $\square$

Another important proposition relating the notion of convergence of a sequence to continuity, is stated without proof.

Proposition 37.15. *Let $(E, \mathcal{O}_E)$ and $(F, \mathcal{O}_F)$ be two topological spaces, and let $f \colon E \to F$ be a function.*

(1) If f is continuous, then for every sequence $(x_n)_{n \in \mathbb{N}}$ in E, if (x_n) converges to a, then $(f(x_n))$ converges to $f(a)$.

(2) If E is a metric space, and $(f(x_n))$ converges to $f(a)$ whenever (x_n) converges to a, for every sequence $(x_n)_{n \in \mathbb{N}}$ in E, then f is continuous.

A special case of Definition 37.20 will be used when E and F are (nontrivial) normed vector spaces with norms $\| \, \|_E$ and $\| \, \|_F$. Let U be any nonempty open subset of E. We showed earlier that E has no isolated points and that every set $\{v\}$ is closed, for every $v \in E$. Since E is nontrivial, for every $v \in U$, there is a nontrivial open ball contained in U (an open ball not reduced to its center). Then, for every $v \in U$, $A = U - \{v\}$ is open and nonempty, and clearly, $v \in \overline{A}$. For any $v \in U$, if $f(x)$ approaches b when x approaches v with values in $A = U - \{v\}$, we say that $f(x)$ *approaches b when x approaches v with values $\neq v$ in U*. This is denoted by

$$\lim_{x \to v, x \in U, x \neq v} f(x) = b.$$

Remark: Variations of the above case show up in the following case: $E = \mathbb{R}$, and F is some arbitrary topological space. Let A be some nonempty subset of $\mathbb{R}$, and let $f \colon A \to F$ be some function. For any $a \in A$, we say that f *is continuous on the right at a* if

$$\lim_{x \to a, x \in A \cap [a, +\infty)} f(x) = f(a).$$

We can define *continuity on the left* at a in a similar fashion.

Let us consider another variation. Let A be some nonempty subset of $\mathbb{R}$, and let $f \colon A \to F$ be some function. For any $a \in A$, we say that f *has a discontinuity of the first kind at a* if

$$\lim_{x \to a, x \in A \cap \, (-\infty, a)} f(x) = f(a_-)$$