(3) Assume b is perturbed by a small amount δb (note that δb is a vector). Find the new solution of the system

$$A(x + \delta x) = b + \delta b,$$

where δx is also a vector. In the case where $b = (0, \ldots, 0, 1)$, and $\delta b = (0, \ldots, 0, \epsilon)$, show that

$$|(\delta x)_1| = 2^{n-1}|\epsilon|.$$

(where $(\delta x)_1$ is the first component of δx).

(4) Prove that $(A - I)^n = 0$.

Problem 3.13. An $n \times n$ matrix N is *nilpotent* if there is some integer $r \geq 1$ such that $N^r = 0$.

(1) Prove that if N is a nilpotent matrix, then the matrix $I - N$ is invertible and

$$(I - N)^{-1} = I + N + N^2 + \cdots + N^{r-1}.$$

(2) Compute the inverse of the following matrix A using (1):

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Problem 3.14. (1) Let A be an $n \times n$ matrix. If A is invertible, prove that for any $x \in \mathbb{R}^n$, if $Ax = 0$, then $x = 0$.

(2) Let A be an $m \times n$ matrix and let B be an $n \times m$ matrix. Prove that $I_m - AB$ is invertible iff $I_n - BA$ is invertible.

Hint. If for all $x \in \mathbb{R}^n$, $Mx = 0$ implies that $x = 0$, then M is invertible.

Problem 3.15. Consider the following $n \times n$ matrix, for $n \geq 3$:

$$B = \begin{pmatrix} 1 & -1 & -1 & -1 & \cdots & -1 & -1 \\ 1 & -1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & -1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & -1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & -1 & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 & -1 \end{pmatrix}$$