10.7 Problems

Problem 10.1. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix}.$$

Prove that $\rho(J) = 0$ and $\rho(\mathcal{L}_1) = 2$, so

$$\rho(J) < 1 < \rho(\mathcal{L}_1),$$

where J is Jacobi's matrix and $\mathcal{L}_1$ is the matrix of Gauss–Seidel.

Problem 10.2. Consider the matrix

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & 2 \end{pmatrix}.$$

Prove that $\rho(J) = \sqrt{5}/2$ and $\rho(\mathcal{L}_1) = 1/2$, so

$$\rho(\mathcal{L}_1) < \rho(J),$$

where where J is Jacobi's matrix and $\mathcal{L}_1$ is the matrix of Gauss–Seidel.

Problem 10.3. Consider the following linear system:

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 19 \\ 19 \\ -3 \\ -12 \end{pmatrix}.$$

(1) Solve the above system by Gaussian elimination.

(2) Compute the sequences of vectors $u_k = (u_1^k, u_2^k, u_3^k, u_4^k)$ for $k = 1, \ldots, 10$, using the methods of Jacobi, Gauss–Seidel, and relaxation for the following values of ω: $\omega = 1.1, 1.2, \ldots, 1.9$. In all cases, the initial vector is $u_0 = (0, 0, 0, 0)$.

Problem 10.4. Recall that a complex or real $n \times n$ matrix A is *strictly row diagonally dominant* if $|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|$ for $i = 1, \ldots, n$.

(1) Prove that if A is strictly row diagonally dominant, then Jacobi's method converges.

(2) Prove that if A is strictly row diagonally dominant, then Gauss–Seidel's method converges.