Proof. Since $M = A + N$ and A is Hermitian, $A^* = A$, so we get

$$M^* + N = A^* + N^* + N = A + N + N^* = M + N^* = (M^* + N)^*,$$

which shows that $M^* + N$ is indeed Hermitian.

Because A is Hermitian positive definite, the function

$$v \mapsto (v^* A v)^{1/2}$$

from $\mathbb{C}^n$ to $\mathbb{R}$ is a vector norm $\| \; \|$, and let $\| \; \|$ also denote its subordinate matrix norm. We prove that

$$\|M^{-1}N\| < 1,$$

which by Theorem 10.1 proves that $\rho(M^{-1}N) < 1$. By definition

$$\|M^{-1}N\| = \|I - M^{-1}A\| = \sup_{\|v\|=1} \|v - M^{-1}Av\|,$$

which leads us to evaluate $\|v - M^{-1}Av\|$ when $\|v\| = 1$. If we write $w = M^{-1}Av$, using the facts that $\|v\| = 1$, $v = A^{-1}Mw$, $A^* = A$, and $A = M - N$, we have

$$\begin{aligned}
\|v - w\|^2 &= (v - w)^* A (v - w) \\
&= \|v\|^2 - v^* A w - w^* A v + w^* A w \\
&= 1 - w^* M^* w - w^* M w + w^* A w \\
&= 1 - w^* (M^* + N) w.
\end{aligned}$$

Now since we assumed that $M^* + N$ is positive definite, if $w \neq 0$, then $w^*(M^* + N)w > 0$, and we conclude that

$$\text{if} \quad \|v\| = 1, \quad \text{then} \quad \|v - M^{-1}Av\| < 1.$$

Finally, the function

$$v \mapsto \|v - M^{-1}Av\|$$

is continuous as a composition of continuous functions, therefore it achieves its maximum on the compact subset $\{v \in \mathbb{C}^n \mid \|v\| = 1\}$, which proves that

$$\sup_{\|v\|=1} \|v - M^{-1}Av\| < 1,$$

and completes the proof. $\square$

Now as in the previous sections, we assume that A is written as $A = D - E - F$, with D invertible, possibly in block form. The next theorem provides a sufficient condition (which turns out to be also necessary) for the relaxation method to converge (and thus, for the method of Gauss–Seidel to converge). This theorem is known as the *Ostrowski-Reich theorem*.