Theorem 35.17. *(Primary Decomposition Theorem) Let M be a torsion-module over a PID. For every irreducible element $p \in P$, let M_p be the submodule of M annihilated by some power of p. Then, M is the (possibly infinite) direct sum*

$$M = \bigoplus_{p \in P} M_p.$$

Proof. Since M is a torsion-module, for every $x \in M$, there is some $\alpha \in A$ such that $x \in M(\alpha)$. By Proposition 35.16, if $\alpha = up_1^{n_1} \cdots p_r^{n_r}$ is a factorization of α into prime factors (where u is a unit), then the module $M(\alpha)$ is the direct sum

$$M(\alpha) = M(p_1^{n_1}) \oplus \cdots \oplus M(p_r^{n_r}).$$

This means that x can be written as

$$x = \sum_{p \in P} x_p, \quad x_p \in M_p,$$

with only finitely many x_p nonzero. If

$$\sum_{p \in P} x_p = \sum_{p \in P} y_p$$

for all $p \in P$, with only finitely many x_p and y_p nonzero, then x_p and y_p are annihilated by some common nonzero element $a \in A$, so $x_p, y_p \in M(a)$. By Proposition 35.16, we must have $x_p = y_p$ for all p, which proves that we have a direct sum. $\square$

It is clear that if p and p' are two irreducible elements such that $p = up'$ for some unit u, then $M_p = M_{p'}$. Therefore, M_p only depends on the ideal (p).

Definition 35.10. Given a torsion-module M over a PID, the modules M_p associated with irreducible elements in P are called the *p-primary components* of M.

The p-primary components of a torsion module uniquely determine the module, as shown by the next proposition.

Proposition 35.18. *Two torsion modules M and N over a PID are isomorphic iff for every every irreducible element $p \in P$, the p-primary components M_p and N_p of M and N are isomorphic.*

Proof. Let $f \colon M \to N$ be an isomorphism. For any $p \in P$, we have $x \in M_p$ iff $p^k x = 0$ for some $k \geq 1$, so

$$0 = f(p^k x) = p^k f(x),$$

which shows that $f(x) \in N_p$. Therefore, f restricts to a linear map $f \mid M_p$ from M_p to N_p. Since f is an isomorphism, we also have a linear map $f^{-1} \colon M \to N$, and our previous