and

$$\begin{aligned}\varphi(\psi(u)) &= \varphi(\overline{1} \otimes u)\\ &= 1u\\ &= u,\end{aligned}$$

which shows that φ and ψ are mutual inverses. $\square$

We now develop the theory necessary to understand the structure of finitely generated modules over a PID.

35.4 Torsion Modules over a PID; The Primary Decomposition

We begin by considering modules over a product ring obtained from a direct decomposition, as in Definition 32.3. In this section and the next, we closely follow Bourbaki [26] (Chapter VII). Let A be a commutative ring and let $(\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$ be ideals in A such that there is an isomorphism $A \approx A/\mathfrak{b}_1 \times \cdots \times A/\mathfrak{b}_n$. From Theorem 32.16 part (b), there exist some elements $e_1, \ldots, e_n$ of A such that

$$\begin{aligned} e_i^2 &= e_i\\ e_i e_j &= 0, \quad i \neq j\\ e_1 + \cdots + e_n &= 1_A,\end{aligned}$$

and $\mathfrak{b}_i = (1_A - e_i)A$, for $i, j = 1, \ldots, n$.

Given an A-module M with $A \approx A/\mathfrak{b}_1 \times \cdots \times A/\mathfrak{b}_n$, let M_i be the subset of M annihilated by $\mathfrak{b}_i$; that is,

$$M_i = \{x \in M \mid bx = 0, \ \text{for all } \ b \in \mathfrak{b}_i\}.$$

Because $\mathfrak{b}_i$ is an ideal, each M_i is a submodule of M. Observe that if $\lambda, \mu \in A$, $b \in \mathfrak{b}_i$, and if $\lambda - \mu = b$, then for any $x \in M_i$, since $bx = 0$,

$$\lambda x = (\mu + b)x = \mu x + bx = \mu x,$$

so M_i can be viewed as a $A/\mathfrak{b}_i$- module.

Proposition 35.15. *Given a ring $A \approx A/\mathfrak{b}_1 \times \cdots \times A/\mathfrak{b}_n$ as above, the A-module M is the direct sum*

$$M = M_1 \oplus \cdots \oplus M_n,$$

where M_i is the submodule of M annihilated by $\mathfrak{b}_i$.