If H is any nondegenerate hyperplane in E, then $D = H^{\perp}$ is a nondegenerate line and we have

$$E = H \stackrel{\perp}{\oplus} H^{\perp}.$$

For any nonzero vector $u \in D = H^{\perp}$ Consider the map τ_u given by

$$\tau_u(v) = v - 2\frac{\varphi(v,u)}{\varphi(u,u)}u \quad \text{for all } v \in E.$$

If we replace u by λu with $\lambda \neq 0$, we have

$$\tau_{\lambda u}(v) = v - 2\frac{\varphi(v,\lambda u)}{\varphi(\lambda u,\lambda u)}\lambda u = v - 2\frac{\lambda\varphi(v,u)}{\lambda^2\varphi(u,u)}\lambda u = v - 2\frac{\varphi(v,u)}{\varphi(u,u)}u,$$

which shows that τ_u depends only on the line D, and thus only the hyperplane H. Therefore, denote by τ_H the linear map τ_u determined as above by any nonzero vector $u \in H^{\perp}$. Note that if $v \in H$, then

$$\tau_H(v) = v,$$

and if $v \in D$, then

$$\tau_H(v) = -v.$$

A simple computation shows that

$$\varphi(\tau_H(u), \tau_H(v)) = \varphi(u,v) \quad \text{for all } u, v \in E,$$

so $\tau_H \in \mathbf{O}(\varphi)$, and by picking a basis consisting of u and vectors in H, that $\det(\tau_H) = -1$. It is also clear that $\tau_H^2 = \mathrm{id}$.

Definition 29.21. If H is any nondegenerate hyperplane in E, for any nonzero vector $u \in H^{\perp}$, the linear map τ_H given by

$$\tau_H(v) = v - 2\frac{\varphi(v,u)}{\varphi(u,u)}u \quad \text{for all } v \in E$$

is an involutive isometry of E called the *reflection through (or about) the hyperplane H*.

Remarks:

1. It can be shown that if $f \in \mathbf{O}(\varphi)$ leaves every vector in some hyperplane H fixed, then either $f = \mathrm{id}$ or $f = \tau_H$; see Taylor [174] (Chapter 11). Thus, there is no analog to symplectic transvections in the orthogonal group.

2. If $K = \mathbb{R}$ and φ is the usual Euclidean inner product, the matrices corresponding to hyperplane reflections are called *Householder matrices*.

Our goal is to prove that $\mathbf{O}(\varphi)$ is generated by the hyperplane reflections. The following proposition is needed.