If f is a proper function on $\mathbb{R}$, then its effective domain being convex is an interval whose relative interior is an open interval (a, b). In Proposition 51.16, we can pick $y = 1$ so $\langle y, u\rangle = u$, and for any $x \in (a, b)$, since the limits $f'_-(x) = -f'(x; -1)$ and $f'_+(x) = f'(x; 1)$ exist, with $f'_-(x) \leq f'_+(x)$, we deduce that $\partial f(x) = [f'_-(x), f'_+(x)]$. The numbers $\alpha \in [f'_-(x), f'_+(x)]$ are the slopes of nonvertical lines in $\mathbb{R}^2$ passing through $(x, f(x))$ that are supporting lines to the epigraph $\mathbf{epi}(f)$ of f.

Example 51.10. If f is the celebrated **ReLU** function (ramp function) from deep learning defined so that

$$\mathrm{ReLU}(x) = \max\{x, 0\} = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } x \geq 0, \end{cases}$$

then $\partial\,\mathrm{ReLU}(0) = [0, 1]$. See Figure 51.20. The function ReLU is differentiable for $x \neq 0$, with $\mathrm{ReLU}'(x) = 0$ if $x < 0$ and $\mathrm{ReLU}'(x) = 1$ if $x > 0$.

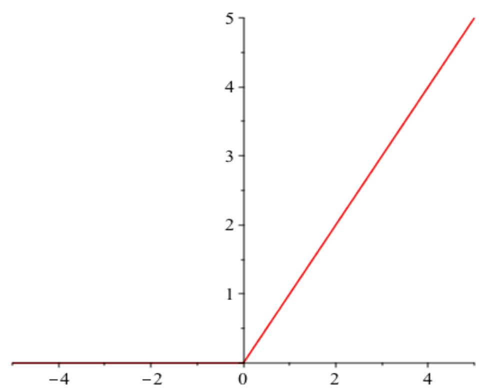

Figure 51.20: The graph of the ReLU function.

Proposition 51.16 has several interesting consequences.

Proposition 51.17. *Let $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$ be a convex function. For any $x \in \mathbb{R}^n$, if $f(x)$ is finite and if f is subdifferentiable at x, then f is proper. If f is not subdifferentiable at x, then there is some $y \neq 0$ such that*

$$f'(x; y) = -f'(x; -y) = -\infty.$$

Proposition 51.17 is proven in Rockafellar [138] (Theorem 23.3). It confirms that improper convex functions are rather pathological objects, because if a convex function is subdifferentiable for some x such that $f(x)$ is finite, then f must be proper. This is because if $f(x)$ is finite, then the subgradient inequality implies that f majorizes an affine function, which is proper.

The next theorem is one of the most important results about the connection between one-sided directional derivatives and subdifferentials. It sharpens the result of Theorem 51.14.