which shows that R presents the module $\mathbb{Z}/4\mathbb{Z}$.

Unfortunately a submodule of a free module of finite dimension is not necessarily finitely generated but, by Proposition 35.5, if A is a PID, then any submodule of a finitely generated module is finitely generated. This property actually characterizes Noetherian rings. To prove it, we need a slightly different version of Proposition 35.2.

Proposition 35.9. *Let $f\colon E \to F$ be a linear map between two A-modules E and F.*

(1) Given any set of generators $(v_1, \ldots, v_r)$ of $\mathrm{Im}(f)$, for any r vectors $u_1, \ldots, u_r \in E$ such that $f(u_i) = v_i$ for $i = 1, \ldots, r$, if U is the finitely generated submodule of E generated by $(u_1, \ldots, u_r)$, then the module E is the sum

$$E = \mathrm{Ker}\,(f) + U.$$

Consequently, if both $\mathrm{Ker}\,(f)$ and $\mathrm{Im}(f)$ are finitely generated, then E is finitely generated.

(2) If E is finitely generated, then so is $\mathrm{Im}(f)$.

Proof. (1) Pick any $w \in E$, write $f(w)$ over the generators $(v_1, \ldots, v_r)$ of $\mathrm{Im}(f)$ as $f(w) = a_1v_1 + \cdots + a_rv_r$, and let $u = a_1u_1 + \cdots + a_ru_r$. Observe that

$$\begin{aligned} f(w-u) &= f(w) - f(u) \\ &= a_1v_1 + \cdots + a_rv_r - (a_1f(u_1) + \cdots + a_rf(u_r)) \\ &= a_1v_1 + \cdots + a_rv_r - (a_1v_1 + \cdots + a_rv_r) \\ &= 0. \end{aligned}$$

Therefore, $h = w - u \in \mathrm{Ker}\,(f)$, and since $w = h + u$ with $h \in \mathrm{Ker}\,(f)$ and $u \in U$, we have $E = \mathrm{Ker}\,(f) + U$, as claimed. If $\mathrm{Ker}\,(f)$ is also finitely generated, by taking the union of a finite set of generators for $\mathrm{Ker}\,(f)$ and $(v_1, \ldots, v_r)$, we obtain a finite set of generators for E.

(2) If $(u_1, \ldots, u_n)$ generate E, it is obvious that $(f(u_1), \ldots, f(u_n))$ generate $\mathrm{Im}(f)$. $\square$

Theorem 35.10. *A ring A is Noetherian iff every submodule N of a finitely generated A-module M is itself finitely generated.*

Proof. First, assume that every submodule N of a finitely generated A-module M is itself finitely generated. The ring A is a module over itself and it is generated by the single element 1. Furthermore, every submodule of A is an ideal, so the hypothesis implies that every ideal in A is finitely generated, which shows that A is Noetherian.

Now, assume A is Noetherian. First, observe that it is enough to prove the theorem for the finitely generated free modules A^n (with $n \geq 1$). Indeed, assume that we proved for every $n \geq 1$ that every submodule of A^n is finitely generated. If M is any finitely generated A-module, then there is a surjection $\varphi\colon A^n \to M$ for some n (where n is the number of elements of a finite generating set for M). Given any submodule N of M, $L = \varphi^{-1}(N)$ is a