Proof. The proof is adapted from Rudin [141] (Section 12.9). By the Cauchy–Schwarz inequality, since

$$|\langle x, y\rangle| \leq \|x\| \, \|y\| \,,$$

we see that the sesquilinear map $(x, y) \mapsto \langle x, y\rangle$ on $E \times E$ is continuous. Let $\varphi\colon E \times E \to \mathbb{C}$ be the sesquilinear map given by

$$\varphi(u, v) = \langle f(u), v\rangle \quad \text{for all } u, v \in E.$$

Since f is continuous and the inner product $\langle -, -\rangle$ is continuous, this is a continuous map. By Proposition 48.10, there is a unique linear map $f^*\colon E \to E$ such that

$$\langle f(u), v\rangle = \varphi(u, v) = \langle u, f^*(v)\rangle \quad \text{for all } u, v \in E,$$

with $\|f^*\| = \|\varphi\|$.

We can also prove that $\|\varphi\| = \|f\|$. First, by definition of $\|\varphi\|$ we have

$$\begin{aligned}
\|\varphi\| &= \sup\{|\varphi(x, y)| \mid \|x\| \leq 1, \ \|y\| \leq 1\} \\
&= \sup\{|\langle f(x), y\rangle| \mid \|x\| \leq 1, \ \|y\| \leq 1\} \\
&\leq \sup\{\|f(x)\| \, \|y\| \mid \|x\| \leq 1, \|y\| \leq 1\} \\
&\leq \sup\{\|f(x)\| \mid \|x\| \leq 1\} \\
&= \|f\| \,.
\end{aligned}$$

In the other direction we have

$$\|f(x)\|^2 = \langle f(x), f(x)\rangle = \varphi(x, f(x)) \leq \|\varphi\| \, \|x\| \, \|f(x)\| \,,$$

and if $f(x) \neq 0$ we get $\|f(x)\| \leq \|\varphi\| \, \|x\|$. This inequality holds trivially if $f(x) = 0$, so we conclude that $\|f\| \leq \|\varphi\|$. Therefore we have

$$\|\varphi\| = \|f\| \,,$$

as claimed, and consequently $\|f^*\| = \|\varphi\| = \|f\|$. ☐

It is easy to show that the adjoint satisfies the following properties:

$$\begin{aligned}
(f + g)^* &= f^* + g^* \\
(\lambda f)^* &= \overline{\lambda} f^* \\
(f \circ g)^* &= g^* \circ f^* \\
f^{**} &= f.
\end{aligned}$$

One can also show that $\|f^* \circ f\| = \|f\|^2$ (see Rudin [141], Section 12.9).

As in the Hermitian case, given two Hilbert spaces E and F, the above results can be adapted to show that for any linear map $f\colon E \to F$, there is a unique linear map $f^*\colon F \to E$ such that

$$\langle f(u), v\rangle_2 = \langle u, f^*(v)\rangle_1$$

for all $u \in E$ and all $v \in F$. The linear map f^* is also called the adjoint of f.