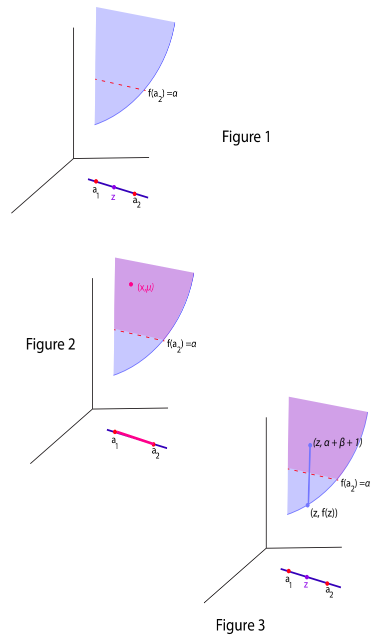

Figure 51.17: Figure (1) illustrates $\mathbf{epi}(f)$, where $\mathbf{epi}(f)$ is contained in a vertical plane of affine dimension 2. Figure (2) illustrates the magenta open subset $\{(x, \mu) \in \mathbb{R}^2 \mid x \in \mathrm{int}(P),\ \alpha < \mu\}$ of $\mathbf{epi}(f)$. Figure (3) illustrates the vertical line segment $\{(z, \mu) \in \mathbb{R}^2 \mid f(z) \leq \mu \leq \alpha + \beta + 1\}$.

Proof. By Proposition 51.14, for any $x \in \mathbf{relint}(\mathrm{dom}(f))$, we have $(x, \mu) \in \mathbf{relint}(\mathbf{epi}(f))$ for all $\mu \in \mathbb{R}$ such that $f(x) < \mu$. Since by definition of $\mathbf{epi}(f)$ we have $(x, f(x)) \in \mathbf{epi}(f) - \mathbf{relint}(\mathbf{epi}(f))$, by Minkowski's theorem (Theorem 51.11), there is a supporting hyperplane $\mathcal{H}$ to $\mathbf{epi}(f)$ through $(x, f(x))$. Since $x \in \mathbf{relint}(\mathrm{dom}(f))$ and f is proper, the hyperplane $\mathcal{H}$ is not a vertical hyperplane. By Definition 51.14, the function f is subdifferentiable at any $x \in \mathbf{relint}(\mathrm{dom}(f))$, and the subgradient inequality shows that if we pick some $x \in \mathbf{relint}(\mathrm{dom}(f))$ and if we let $\varphi(z) = f(x) + \langle z - x, u\rangle$, then φ is an affine form such that $f(z) \geq \varphi(z)$ for all $z \in \mathbb{R}^n$. $\square$

Intuitively, a proper convex function can't decrease faster than an affine function. It is surprising how much work it takes to prove such an "obvious" fact.

Remark: Consider the proper convex function $f\colon \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ given by

$$f(x) = \begin{cases} -\sqrt{x} & \text{if } x \geq 0 \\ +\infty & \text{if } x < 0. \end{cases}$$