on the left, $i > j$, and on the right, $i < j$. The index i is the index of the row that is *changed* by the multiplication. For example, if $m = 3$ and we want to add twice row 1 to row 3, since $\beta = 2$, $j = 1$ and $i = 3$, we form

$$E_{3,1;2} = I + 2e_{31} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix},$$

and calculate

$$\begin{aligned} E_{3,1;2}B &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & \cdots & \cdots b_{1n} \\ b_{21} & b_{22} & \cdots & \cdots & \cdots b_{2n} \\ b_{31} & b_{32} & \cdots & \cdots & \cdots b_{3n} \end{pmatrix} \\ &= \begin{pmatrix} b_{11} & b_{12} & \cdots & \cdots & \cdots b_{1n} \\ b_{21} & b_{22} & \cdots & \cdots & \cdots b_{2n} \\ 2b_{11} + b_{31} & 2b_{12} + b_{32} & \cdots & \cdots & \cdots 2b_{1n} + b_{3n} \end{pmatrix}. \end{aligned}$$

Observe that the inverse of $E_{i,j;\beta} = I + \beta e_{i\,j}$ is $E_{i,j;-\beta} = I - \beta e_{i\,j}$ and that $\det(E_{i,j;\beta}) = 1$. Therefore, during Step 3 (the elimination step), the matrix A is multiplied on the left by a product E_k of matrices of the form $E_{i,k;\beta_{i,k}}$, with $i > k$.

Consequently, we see that

$$A_{k+1} = E_k P_k A_k,$$

and then

$$A_k = E_{k-1} P_{k-1} \cdots E_1 P_1 A.$$

This justifies the claim made earlier that $A_k = M_k A$ for some invertible matrix M_k; we can pick

$$M_k = E_{k-1} P_{k-1} \cdots E_1 P_1,$$

a product of invertible matrices.

The fact that $\det(P(i,k)) = -1$ and that $\det(E_{i,j;\beta}) = 1$ implies immediately the fact claimed above: We always have

$$\det(A_k) = \pm \det(A).$$

Furthermore, since

$$A_k = E_{k-1} P_{k-1} \cdots E_1 P_1 A$$

and since Gaussian elimination stops for $k = n$, the matrix

$$A_n = E_{n-1} P_{n-1} \cdots E_2 P_2 E_1 P_1 A$$

is upper-triangular. Also note that if we let $M = E_{n-1} P_{n-1} \cdots E_2 P_2 E_1 P_1$, then $\det(M) = \pm 1$, and

$$\det(A) = \pm \det(A_n).$$

The matrices $P(i,k)$ and $E_{i,j;\beta}$ are called *elementary matrices*. We can summarize the above discussion in the following theorem: