Definition 31.5. If $\lambda \in K$ is an eigenvalue of f, we define a *generalized eigenvector* of f as a nonzero vector $u \in E$ such that

$$(\lambda \mathrm{id} - f)^r(u) = 0, \quad \text{for some } r \geq 1.$$

The *index* of λ is defined as the smallest $r \geq 1$ such that

$$\mathrm{Ker}\,(\lambda \mathrm{id} - f)^r = \mathrm{Ker}\,(\lambda \mathrm{id} - f)^{r+1}.$$

It is clear that $\mathrm{Ker}\,(\lambda \mathrm{id} - f)^i \subseteq \mathrm{Ker}\,(\lambda \mathrm{id} - f)^{i+1}$ for all $i \geq 1$. By Theorem 31.11(d), if $\lambda = \lambda_i$, the index of λ_i is equal to r_i.

31.5 Jordan Decomposition

Recall that a linear map $g\colon E \to E$ is said to be *nilpotent* if there is some positive integer r such that $g^r = 0$. Another important consequence of Theorem 31.11 is that f can be written as the sum of a diagonalizable and a nilpotent linear map (which commute). For example $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the $\mathbb{R}$-linear map $f(x, y) = (x, x + y)$ with standard matrix representation $X_f = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. A basic calculation shows that $m_f(x) = \chi_f(x) = (x - 1)^2$. By Theorem 31.6 we know that f is not diagonalizable over $\mathbb{R}$. But since the eigenvalue $\lambda_1 = 1$ of f does belong to $\mathbb{R}$, we may use the projection construction inherent within Theorem 31.11 to write $f = D + N$, where D is a diagonalizable linear map and N is a nilpotent linear map. The proof of Theorem 31.10 implies that

$$p_1^{r_1} = (x - 1)^2, \qquad g_1 = 1 = h_1, \qquad \pi_1 = g_1(f)h_1(f) = \mathrm{id}.$$

Then

$$D = \lambda_1 \pi_1 = \mathrm{id}, \qquad N = f - D = f(x, y) - \mathrm{id}(x, y) = (x, x + y) - (x, y) = (0, y),$$

which is equivalent to the matrix decomposition

$$X_f = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

This example suggests that the diagonal summand of f is related to the projection constructions associated with the proof of the primary decomposition theorem. If we write

$$D = \lambda_1 \pi_1 + \cdots + \lambda_k \pi_k,$$

where π_i is the projection from E onto the subspace W_i defined in the proof of Theorem 31.10, since

$$\pi_1 + \cdots + \pi_k = \mathrm{id},$$