As in the case of a sum, $U_1 \oplus U_2 = U_2 \oplus U_1$.

If the map a is injective, then by Proposition 3.17 we have $\operatorname{Ker} a = \{(\underbrace{0, \ldots, 0}_{p})\}$ where each 0 is the zero vector of E, which means that if $u_i \in U_i$ for $i = 1, \ldots, p$ and if

$$u_1 + \cdots + u_p = 0,$$

then $(u_1, \ldots, u_p) = (0, \ldots, 0)$, that is, $u_1 = 0, \ldots, u_p = 0$.

Proposition 6.3. *If the map $a \colon U_1 \times \cdots \times U_p \to E$ is injective, then every $u \in U_1 + \cdots + U_p$ has a unique expression as a sum*

$$u = u_1 + \cdots + u_p,$$

with $u_i \in U_i$, for $i = 1, \ldots, p$.

Proof. If

$$u = v_1 + \cdots + v_p = w_1 + \cdots + w_p,$$

with $v_i, w_i \in U_i$, for $i = 1, \ldots, p$, then we have

$$w_1 - v_1 + \cdots + w_p - v_p = 0,$$

and since $v_i, w_i \in U_i$ and each U_i is a subspace, $w_i - v_i \in U_i$. The injectivity of a implies that $w_i - v_i = 0$, that is, $w_i = v_i$ for $i = 1, \ldots, p$, which shows the uniqueness of the decomposition of u. □

Proposition 6.4. *If the map $a \colon U_1 \times \cdots \times U_p \to E$ is injective, then any p nonzero vectors $u_1, \ldots, u_p$ with $u_i \in U_i$ are linearly independent.*

Proof. To see this, assume that

$$\lambda_1 u_1 + \cdots + \lambda_p u_p = 0$$

for some $\lambda_i \in \mathbb{R}$. Since $u_i \in U_i$ and U_i is a subspace, $\lambda_i u_i \in U_i$, and the injectivity of a implies that $\lambda_i u_i = 0$, for $i = 1, \ldots, p$. Since $u_i \neq 0$, we must have $\lambda_i = 0$ for $i = 1, \ldots, p$; that is, $u_1, \ldots, u_p$ with $u_i \in U_i$ and $u_i \neq 0$ are linearly independent. □

Observe that if a is injective, then we must have $U_i \cap U_j = (0)$ whenever $i \neq j$. However, this condition is generally not sufficient if $p \geq 3$. For example, if $E = \mathbb{R}^2$ and U_1 the line spanned by $e_1 = (1, 0)$, U_2 is the line spanned by $d = (1, 1)$, and U_3 is the line spanned by $e_2 = (0, 1)$, then $U_1 \cap U_2 = U_1 \cap U_3 = U_2 \cap U_3 = \{(0, 0)\}$, but $U_1 + U_2 = U_1 + U_3 = U_2 + U_3 = \mathbb{R}^2$, so $U_1 + U_2 + U_3$ is not a direct sum. For example, d is expressed in two different ways as

$$d = (1, 1) = (1, 0) + (0, 1) = e_1 + e_2.$$