Definition 29.4. Given a bilinear map $\varphi\colon E \times F \to K$, for every $u \in E$, let $l_\varphi(u)$ be the linear form in F^* given by

$$l_\varphi(u)(y) = \varphi(u, y) \quad \text{for all } y \in F,$$

and for every $v \in F$, let $r_\varphi(v)$ be the linear form in E^* given by

$$r_\varphi(v)(x) = \varphi(x, v) \quad \text{for all } x \in E.$$

Because φ is bilinear, the maps $l_\varphi\colon E \to F^*$ and $r_\varphi\colon F \to E^*$ are linear.

Definition 29.5. A bilinear map $\varphi\colon E \times F \to K$ is said to be *nondegenerate* iff the following conditions hold:

(1) For every $u \in E$, if $\varphi(u, v) = 0$ for all $v \in F$, then $u = 0$, and

(2) For every $v \in F$, if $\varphi(u, v) = 0$ for all $u \in E$, then $v = 0$.

The following proposition shows the importance of l_φ and r_φ.

Proposition 29.1. *Given a bilinear map $\varphi\colon E \times F \to K$, the following properties hold:*

(a) The map l_φ is injective iff Property (1) of Definition 29.5 holds.

(b) The map r_φ is injective iff Property (2) of Definition 29.5 holds.

(c) The bilinear form φ is nondegenerate and iff l_φ and r_φ are injective.

(d) If the bilinear form φ is nondegenerate and if E and F have finite dimensions, then $\dim(E) = \dim(F)$, and $l_\varphi\colon E \to F^*$ and $r_\varphi\colon F \to E^*$ are linear isomorphisms.

Proof. (a) Assume that (1) of Definition 29.5 holds. If $l_\varphi(u) = 0$, then $l_\varphi(u)$ is the linear form whose value is 0 for all y; that is,

$$l_\varphi(u)(y) = \varphi(u, y) = 0 \quad \text{for all } y \in F,$$

and by (1) of Definition 29.5, we must have $u = 0$. Therefore, l_φ is injective. Conversely, if l_φ is injective, and if

$$l_\varphi(u)(y) = \varphi(u, y) = 0 \quad \text{for all } y \in F,$$

then $l_\varphi(u)$ is the zero form, and by injectivity of l_φ, we get $u = 0$; that is, (1) of Definition 29.5 holds.

(b) The proof is obtained by swapping the arguments of φ.

(c) This follows from (a) and (b).

(d) If E and F are finite dimensional, then $\dim(E) = \dim(E^*)$ and $\dim(F) = \dim(F^*)$. Since φ is nondegenerate, $l_\varphi\colon E \to F^*$ and $r_\varphi\colon F \to E^*$ are injective, so $\dim(E) \leq \dim(F^*) = \dim(F)$ and $\dim(F) \leq \dim(E^*) = \dim(E)$, which implies that

$$\dim(E) = \dim(F),$$

and thus, $l_\varphi\colon E \to F^*$ and $r_\varphi\colon F \to E^*$ are bijective. $\square$