Chapter 10

Iterative Methods for Solving Linear Systems

10.1 Convergence of Sequences of Vectors and Matrices

In Chapter 8 we discussed some of the main methods for solving systems of linear equations. These methods are *direct methods*, in the sense that they yield exact solutions (assuming infinite precision!).

Another class of methods for solving linear systems consists in approximating solutions using *iterative methods*. The basic idea is this: Given a linear system $Ax = b$ (with A a square invertible matrix in $\mathrm{M}_n(\mathbb{C})$), find another matrix $B \in \mathrm{M}_n(\mathbb{C})$ and a vector $c \in \mathbb{C}^n$, such that

1. The matrix $I - B$ is invertible

2. The unique solution $\widetilde{x}$ of the system $Ax = b$ is *identical* to the unique solution $\widetilde{u}$ of the system
$$u = Bu + c,$$

and then starting from any vector u_0, compute the sequence (u_k) given by

$$u_{k+1} = Bu_k + c, \quad k \in \mathbb{N}.$$

Under certain conditions (to be clarified soon), the sequence (u_k) converges to a limit $\widetilde{u}$ which is the unique solution of $u = Bu + c$, and thus of $Ax = b$.

Consequently, it is important to find conditions that ensure the convergence of the above sequences and to have tools to compare the "rate" of convergence of these sequences. Thus, we begin with some general results about the convergence of sequences of vectors and matrices.