for all i, j, $1 \leq i \leq m$, $1 \leq j \leq n$. The *adjoint A^* of A* is the matrix defined such that

$$A^* = \overline{(A^\top)} = \left(\overline{A}\right)^\top.$$

Proposition 14.15. *Let E be any Hermitian space of finite dimension n, and let $f\colon E \to E$ be any linear map. The following properties hold:*

(1) The linear map $f\colon E \to E$ is an isometry iff

$$f \circ f^* = f^* \circ f = \mathrm{id}.$$

(2) For every orthonormal basis $(e_1, \ldots, e_n)$ of E, if the matrix of f is A, then the matrix of f^* is the adjoint A^* of A, and f is an isometry iff A satisfies the identities

$$A\,A^* = A^*A = I_n,$$

where I_n denotes the identity matrix of order n, iff the columns of A form an orthonormal basis of $\mathbb{C}^n$, iff the rows of A form an orthonormal basis of $\mathbb{C}^n$.

Proof. (1) The proof is identical to that of Proposition 12.14 (1).

(2) If $(e_1, \ldots, e_n)$ is an orthonormal basis for E, let $A = (a_{i\,j})$ be the matrix of f, and let $B = (b_{i\,j})$ be the matrix of f^*. Since f^* is characterized by

$$f^*(u) \cdot v = u \cdot f(v)$$

for all $u, v \in E$, using the fact that if $w = w_1 e_1 + \cdots + w_n e_n$, we have $w_k = w \cdot e_k$, for all k, $1 \leq k \leq n$; letting $u = e_i$ and $v = e_j$, we get

$$b_{j\,i} = f^*(e_i) \cdot e_j = e_i \cdot f(e_j) = \overline{f(e_j) \cdot e_i} = \overline{a_{i\,j}},$$

for all i, j, $1 \leq i, j \leq n$. Thus, $B = A^*$. Now if X and Y are arbitrary matrices over the basis $(e_1, \ldots, e_n)$, denoting as usual the jth column of X by X^j, and similarly for Y, a simple calculation shows that

$$Y^*X = (X^j \cdot Y^i)_{1 \leq i,j \leq n}.$$

Then it is immediately verified that if $X = Y = A$, then $A^*A = A\,A^* = I_n$ iff the column vectors $(A^1, \ldots, A^n)$ form an orthonormal basis. Thus, from (1), we see that (2) is clear. $\square$

Proposition 12.14 shows that the inverse of an isometry f is its adjoint f^*. Proposition 12.14 also motivates the following definition.

Definition 14.9. A complex $n \times n$ matrix is a *unitary matrix* if

$$A\,A^* = A^*A = I_n.$$