We can now prove that if a linear program has a feasible solution and is bounded, then it has an optimal solution.

Proposition 45.1. *Let (P_2) be a linear program in standard form, with equality constraint $Ax = b$. If $\mathcal{P}(A, b)$ is nonempty and bounded above, and if μ is the least upper bound of the set $\{cx \in \mathbb{R} \mid x \in \mathcal{P}(A, b)\}$, then there is some $p \in \mathcal{P}(A, b)$ such that*

$$cp = \mu,$$

that is, the objective function $x \mapsto cx$ has a maximum value μ on $\mathcal{P}(A, b)$ which is achieved by some optimum solution $p \in \mathcal{P}(A, b)$.

Proof. Since $\mu = \sup\{cx \in \mathbb{R} \mid x \in \mathcal{P}(A, b)\}$, there is a sequence $(x^{(k)})_{k \geq 0}$ of vectors $x^{(k)} \in \mathcal{P}(A, b)$ such that $\lim_{k \mapsto \infty} cx^{(k)} = \mu$. In particular, if we write $x^{(k)} = (x_1^{(k)}, \ldots, x_n^{(k)})$ we have $x_j^{(k)} \geq 0$ for $j = 1, \ldots, n$ and for all $k \geq 0$. Let $\widetilde{A}$ be the $(m+1) \times n$ matrix

$$\widetilde{A} = \begin{pmatrix} c \\ A \end{pmatrix},$$

and consider the sequence $(\widetilde{A}x^{(k)})_{k \geq 0}$ of vectors $\widetilde{A}x^{(k)} \in \mathbb{R}^{m+1}$. We have

$$\widetilde{A}x^{(k)} = \begin{pmatrix} c \\ A \end{pmatrix} x^{(k)} = \begin{pmatrix} cx^{(k)} \\ Ax^{(k)} \end{pmatrix} = \begin{pmatrix} cx^{(k)} \\ b \end{pmatrix},$$

since by hypothesis $x^{(k)} \in \mathcal{P}(A, b)$, and the constraints are $Ax = b$ and $x \geq 0$. Since by hypothesis $\lim_{k \mapsto \infty} cx^{(k)} = \mu$, the sequence $(\widetilde{A}x^{(k)})_{k \geq 0}$ converges to the vector $\begin{pmatrix} \mu \\ b \end{pmatrix}$. Now, observe that each vector $\widetilde{A}x^{(k)}$ can be written as the convex combination

$$\widetilde{A}x^{(k)} = \sum_{j=1}^{n} x_j^{(k)} \widetilde{A}^j,$$

with $x_j^{(k)} \geq 0$ and where $\widetilde{A}^j \in \mathbb{R}^{m+1}$ is the jth column of $\widetilde{A}$. Therefore, $\widetilde{A}x^{(k)}$ belongs to the polyheral cone

$$C = \operatorname{cone}(\widetilde{A}^1, \ldots, \widetilde{A}^n) = \{\widetilde{A}x \mid x \in \mathbb{R}^n,\ x \geq 0\},$$

and since by Proposition 44.2 this cone is closed, $\lim_{k \geq \infty} \widetilde{A}x^{(k)} \in C$, which means that there is some $u \in \mathbb{R}^n$ with $u \geq 0$ such that

$$\begin{pmatrix} \mu \\ b \end{pmatrix} = \lim_{k \geq \infty} \widetilde{A}x^{(k)} = \widetilde{A}u = \begin{pmatrix} cu \\ Au \end{pmatrix},$$

that is, $cu = \mu$ and $Au = b$. Hence, u is an optimal solution of (P_2). $\square$

The next question is, how do we find such an optimal solution? It turns out that for linear programs in standard form where the constraints are of the form $Ax = b$ and $x \geq 0$, there are always optimal solutions of a special type called *basic feasible solutions*.