Here is an example taken from Artin [7] (Chapter 12, Section 4). Let F be the free $\mathbb{Z}$-module $\mathbb{Z}^2$, and let M be the lattice generated by the columns of the matrix

$$R = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}.$$

The columns (u_1, u_2) of R are linearly independent, but they are not a basis of $\mathbb{Z}^2$. For example, in order to obtain e_1 as a linear combination of these columns, we would need to solve the linear system

$$\begin{aligned} 2x - y &= 1 \\ x + 2y &= 0. \end{aligned}$$

From the second equation, we get $x = -2y$, which yields

$$-5y = 1.$$

But, $y = -1/5$ is not an integer. We leave it as an exercise to check that

$$\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix},$$

which means that

$$\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix},$$

so $R = QDP^{-1}$ with

$$Q = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

The new basis (u_1', u_2') for $\mathbb{Z}^2$ consists of the columns of Q and the new basis for M consists of the columns $(u_1', 5u_2')$ of QD, where

$$QD = \begin{pmatrix} 1 & 0 \\ 3 & 5 \end{pmatrix}.$$

A picture of the lattice and its generators (u_1, u_2) and of the same lattice with the new basis $(u_1', 5u_2')$ is shown in Figure 35.1, where the lattice points are displayed as stars.

The invariant factor decomposition of a finitely generated module M over a PID A given by Theorem 35.31 says that

$$M_{\mathrm{tor}} \approx A/\mathfrak{a}_{r+1} \oplus \cdots \oplus A/\mathfrak{a}_m,$$

a direct sum of cyclic modules, with $(0) \neq \mathfrak{a}_{r+1} \subseteq \cdots \subseteq \mathfrak{a}_m \neq A$. Using the Chinese Remainder Theorem (Theorem 32.15), we can further decompose each module $A/\alpha_i A$ into a direct sum of modules of the form $A/p^n A$, where p is a prime in A.