Remark: When writing a program implementing row reduction, we may stop when the last column of the matrix A is reached. In this case, the test whether the system $Ax = b$ is solvable is that the row-reduced matrix A' has no zero row of index $i > r$ such that $b'_i \neq 0$ (where r is the number of pivots, and b' is the row-reduced right-hand side).

If we have a *homogeneous system* $Ax = 0$, which means that $b = 0$, of course $x = 0$ is always a solution, but Theorem 8.16 implies that if the system $Ax = 0$ has more variables than equations, then it has some nonzero solution (we call it a *nontrivial solution*).

Proposition 8.17. *Given any homogeneous system $Ax = 0$ of m equations in n variables, if $m < n$, then there is a nonzero vector $x \in \mathbb{R}^n$ such that $Ax = 0$.*

Proof. Convert the matrix A to a reduced row echelon matrix A'. We know that $Ax = 0$ iff $A'x = 0$. If r is the number of pivots of A', we must have $r \leq m$, so by Theorem 8.16 we may assign arbitrary values to $n - r > 0$ nonpivot variables and we get nontrivial solutions. $\square$

Theorem 8.16 can also be used to characterize when a square matrix is invertible. First, note the following simple but important fact:

If a square $n \times n$ matrix A is a row reduced echelon matrix, then either A is the identity or the bottom row of A is zero.

Proposition 8.18. *Let A be a square matrix of dimension n. The following conditions are equivalent:*

(a) The matrix A can be reduced to the identity by a sequence of elementary row operations.

(b) The matrix A is a product of elementary matrices.

(c) The matrix A is invertible.

(d) The system of homogeneous equations $Ax = 0$ has only the trivial solution $x = 0$.

Proof. First we prove that (a) implies (b). If (a) can be reduced to the identity by a sequence of row operations $E_1, \ldots, E_p$, this means that $E_p \cdots E_1 A = I$. Since each E_i is invertible, we get

$$A = E_1^{-1} \cdots E_p^{-1},$$

where each E_i^{-1} is also an elementary row operation, so (b) holds. Now if (b) holds, since elementary row operations are invertible, A is invertible and (c) holds. If A is invertible, we already observed that the homogeneous system $Ax = 0$ has only the trivial solution $x = 0$, because from $Ax = 0$, we get $A^{-1}Ax = A^{-1}0$; that is, $x = 0$. It remains to prove that (d) implies (a) and for this we prove the contrapositive: if (a) does not hold, then (d) does not hold.

Using our basic observation about reducing square matrices, if A does not reduce to the identity, then A reduces to a row echelon matrix A' whose bottom row is zero. Say $A' = PA$,