Given a set A, a multiset with elements from A is a generalization of the concept of a set that allows multiple instances of elements from A to occur. For example, if $A = \{a, b, c, d\}$, the following are multisets:

$$M_1 = \{a, a, b\}, \;\; M_2 = \{a, a, b, b, c\}, \;\; M_3 = \{a, a, b, b, c, d, d, d\}.$$

Here is another way to represent multisets as tables showing the multiplicities of the elements in the multiset:

$$M_1 = \begin{pmatrix} a & b & c & d \\ 2 & 1 & 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} a & b & c & d \\ 2 & 2 & 1 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} a & b & c & d \\ 2 & 2 & 1 & 3 \end{pmatrix}.$$

The above are just graphs of functions from the set $A = \{a, b, c, d\}$ to $\mathbb{N}$. This suggests the following definition.

Definition 33.17. A finite *multiset* M over a set A is a function $M\colon A \to \mathbb{N}$ such that $M(a) \neq 0$ for finitely many $a \in A$. The *multiplicity* of an element $a \in A$ in M is $M(a)$. The set of all multisets over A is denoted by $\mathbb{N}^{(A)}$, and we let $\mathrm{dom}(M) = \{a \in A \mid M(a) \neq 0\}$, which is a finite set. The set $\mathrm{dom}(M)$ is the set of elements in A that actually occur in M. For any multiset $M \in \mathbb{N}^{(A)}$, note that $\sum_{a\in A} M(a)$ makes sense, since $\sum_{a\in A} M(a) = \sum_{a\in \mathrm{dom}(A)} M(a)$, and $\mathrm{dom}(M)$ is finite; this sum is the total number of elements in the multiset A and is called the *size* of M. Let $|M| = \sum_{a\in A} M(a)$.

Going back to our symmetric tensors, we can view the tensors of the form $u_1 \odot \cdots \odot u_n$ as multisets of size n over the set E.

Theorem 33.24 implies the following proposition.

Proposition 33.25. *There is a canonical isomorphism*

$$\mathrm{Hom}(\mathrm{S}^n(E), F) \cong \mathrm{Sym}^n(E; F),$$

between the vector space of linear maps $\mathrm{Hom}(\mathrm{S}^n(E), F)$ *and the vector space of symmetric multilinear maps* $\mathrm{Sym}^n(E; F)$ *given by the linear map* $- \circ \varphi$ *defined by* $h \mapsto h \circ \varphi$*, with* $h \in \mathrm{Hom}(\mathrm{S}^n(E), F)$*.*

Proof. The map $h \circ \varphi$ is clearly symmetric multilinear. By Theorem 33.24, for every symmetric multilinear map $f \in \mathrm{Sym}^n(E; F)$ there is a unique linear map $f_\odot \in \mathrm{Hom}(\mathrm{S}^n(E), F)$ such that $f = f_\odot \circ \varphi$, so the map $- \circ \varphi$ is bijective. Its inverse is the map $f \mapsto f_\odot$. $\square$

In particular, when $F = K$, we get the following important fact.

Proposition 33.26. *There is a canonical isomorphism*

$$(\mathrm{S}^n(E))^* \cong \mathrm{Sym}^n(E; K).$$