Consequently we have the following commutative diagrams:

$$
\begin{tikzcd}
 & \widehat{E}_2 \arrow[d, "\widehat{\varphi_1}"] \\
E \arrow[ur, "\varphi_2"] \arrow[r, "\varphi_1"] \arrow[dr, "\varphi_2"'] & \widehat{E}_1 \arrow[d, "\widehat{\varphi_2}"] \\
 & \widehat{E}_2
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \widehat{E}_1 \arrow[d, "\widehat{\varphi_2}"] \\
E \arrow[ur, "\varphi_1"] \arrow[r, "\varphi_2"] \arrow[dr, "\varphi_1"'] & \widehat{E}_2 \arrow[d, "\widehat{\varphi_1}"] \\
 & \widehat{E}_1.
\end{tikzcd}
$$

However, $\mathrm{id}_{\widehat{E}_1}$ and $\mathrm{id}_{\widehat{E}_2}$ are uniformly continuous functions making the following diagrams commute

$$
\begin{tikzcd}
E \arrow[r, "\varphi_1"] \arrow[dr, "\varphi_1"'] & \widehat{E}_1 \arrow[d, "\mathrm{id}_{\widehat{E}_1}"] \\
 & \widehat{E}_1
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[r, "\varphi_2"] \arrow[dr, "\varphi_2"'] & \widehat{E}_2 \arrow[d, "\mathrm{id}_{\widehat{E}_2}"] \\
 & \widehat{E}_2,
\end{tikzcd}
$$

so by the uniqueness of extensions we must have

$$\widehat{\varphi_1} \circ \widehat{\varphi_2} = \mathrm{id}_{\widehat{E}_1} \quad \text{and} \quad \widehat{\varphi_2} \circ \widehat{\varphi_1} = \mathrm{id}_{\widehat{E}_2}.$$

This proves that $\widehat{\varphi_1}$ and $\widehat{\varphi_2}$ are mutual inverses. Now, since $\varphi_2 = \widehat{\varphi_2} \circ \varphi_1$, we have

$$\widehat{\varphi_2}|\varphi_1(E) = \varphi_2 \circ \varphi_1^{-1},$$

and since φ_1^{-1} and φ_2 are isometries, so is $\widehat{\varphi_2}|\varphi_1(E)$. But we saw earlier that $\widehat{\varphi_2}$ is the uniform continuous extension of $\widehat{\varphi_2}|\varphi_1(E)$ and $\varphi_1(E)$ is dense in $\widehat{E}_1$, so for any two elements $\alpha, \beta \in \widehat{E}_1$, if (a_n) and (b_n) are sequences in $\varphi_1(E)$ converging to α and β, we have

$$\widehat{d}_2((\widehat{\varphi_2}|\varphi_1(E))(a_n), ((\widehat{\varphi_2}|\varphi_1(E))(b_n)) = \widehat{d}_1(a_n, b_n),$$

and by passing to the limit we get

$$\widehat{d}_2(\widehat{\varphi_2}(\alpha), \widehat{\varphi_2}(\beta)) = \widehat{d}_1(\alpha, \beta),$$

which shows that $\widehat{\varphi_2}$ is an isometry (similarly, $\widehat{\varphi_1}$ is an isometry). $\square$

Remarks:

1. Except for Step 8 and Step 9, the proof of Theorem 37.53 is the proof given in Schwartz [149] (Chapter XI, Section 4, Theorem 1), and Kormogorov and Fomin [105] (Chapter 2, Section 7, Theorem 4).

2. The construction of $\widehat{E}$ relies on the completeness of $\mathbb{R}$, and so it cannot be used to construct $\mathbb{R}$ from $\mathbb{Q}$. However, this construction can be modified to yield a construction of $\mathbb{R}$ from $\mathbb{Q}$.

We show in Section 37.12 that Theorem 37.53 yields a construction of the completion of a normed vector space.