Definition 49.4 is a natural extension of the notion of a quadratic functional on $\mathbb{R}^n$. Indeed, by Proposition 48.10, there is a unique continuous self-adjoint linear map $A\colon V \to V$ such that

$$a(u,v) = \langle Au, v\rangle \quad \text{for all } u, v \in V,$$

and by the Riesz representation theorem (Proposition 48.9), there is a unique $b \in V$ such that

$$h(v) = \langle b, v\rangle \quad \text{for all } v \in V.$$

Consequently, J can be written as

$$J(v) = \frac{1}{2}\langle Av, v\rangle - \langle b, v\rangle \quad \text{for all } v \in V. \tag{1}$$

Since a is bilinear and h is linear, by Propositions 39.3 and 39.5, observe that the derivative of J is given by

$$dJ_u(v) = a(u,v) - h(v) \quad \text{for all } v \in V,$$

or equivalently by

$$dJ_u(v) = \langle Au, v\rangle - \langle b, v\rangle = \langle Au - b, v\rangle, \quad \text{for all } v \in V.$$

Thus the gradient of J is given by

$$\nabla J_u = Au - b, \tag{2}$$

just as in the case of a quadratic function of the form $J(v) = (1/2)v^\top Av - b^\top v$, where A is a symmetric $n \times n$ matrix and $b \in \mathbb{R}^n$. To find the second derivative $\mathrm{D}^2 J_u$ of J at u we compute

$$dJ_{u+v}(w) - dJ_u(w) = a(u+v, w) - h(w) - (a(u,w) - h(w)) = a(v,w),$$

so

$$\mathrm{D}^2 J_u(v,w) = a(v,w) = \langle Av, w\rangle,$$

which yields

$$\nabla^2 J_u = A. \tag{3}$$

We will also make use of the following formula.

Proposition 49.3. *If J is a quadratic functional, then*

$$J(u + \rho v) = \frac{\rho^2}{2} a(v,v) + \rho(a(u,v) - h(v)) + J(u).$$