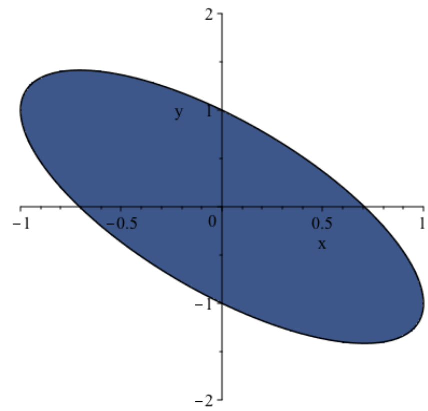

Figure 9.6: The unit closed unit ball $\{(u_1, u_2) \in \mathbb{R}^2 \mid \|(u_1, u_2)\| \leq 1\}$, where $\|(u_1, u_2)\| = \left((u_1 + u_2)^2 + u_1^2\right)^{1/2}$.

is known as *Hölder's inequality*. For $p = 2$, it is the *Cauchy–Schwarz inequality*.

Actually, if we define the *Hermitian inner product* $\langle -, - \rangle$ on $\mathbb{C}^n$ by

$$\langle u, v \rangle = \sum_{i=1}^{n} u_i \overline{v}_i,$$

where $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$, then

$$|\langle u, v \rangle| \leq \sum_{i=1}^{n} |u_i \overline{v}_i| = \sum_{i=1}^{n} |u_i v_i|,$$

so Hölder's inequality implies the following inequalities.

Corollary 9.2. *(Hölder's inequalities) For any real numbers p, q, such that $p, q \geq 1$ and*

$$\frac{1}{p} + \frac{1}{q} = 1,$$

(with $q = +\infty$ if $p = 1$ and $p = +\infty$ if $q = 1$), we have the inequalities

$$\sum_{i=1}^{n} |u_i v_i| \leq \left(\sum_{i=1}^{n} |u_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |v_i|^q\right)^{1/q}$$

and

$$|\langle u, v \rangle| \leq \|u\|_p \|v\|_q, \qquad u, v \in \mathbb{C}^n.$$