Chapter 48

Basics of Hilbert Spaces

Most of the “deep” results about the existence of minima of real-valued functions proven in Chapter 49 rely on two fundamental results of Hilbert space theory:

(1) The projection lemma, which is a result about nonempty, closed, convex subsets of a Hilbert space V.

(2) The Riesz representation theorem, which allows us to express a continuous linear form on a Hilbert space V in terms of a vector in V and the inner product on V.

The correctness of the Karush–Kuhn–Tucker conditions appearing in Lagrangian duality follows from a version of the Farkas–Minkowski proposition, which also follows from the projection lemma.

Thus, we feel that it is indispensable to review some basic results of Hilbert space theory, although in most applications considered here the Hilbert space in question will be finite-dimensional. However, in optimization theory, there are many problems where we seek to find a *function* minimizing some type of energy functional (often given by a bilinear form), in which case we are dealing with an infinite dimensional Hilbert space, so it necessary to develop tools to deal with the more general situation of infinite-dimensional Hilbert spaces.

48.1 The Projection Lemma

Given a Hermitian space $\langle E, \varphi \rangle$, we showed in Section 14.1 that the function $\| \, \| \colon E \to \mathbb{R}$ defined such that $\|u\| = \sqrt{\varphi(u, u)}$, is a norm on E. Thus, E is a normed vector space. If E is also complete, then it is a very interesting space.

Recall that completeness has to do with the convergence of Cauchy sequences. A normed vector space $\langle E, \| \, \| \rangle$ is automatically a metric space under the metric d defined such that $d(u, v) = \|v - u\|$ (see Chapter 37 for the definition of a normed vector space and of a metric space, or Lang [111, 112], or Dixmier [51]). Given a metric space E with metric d, a sequence