

Proposition 43.1. *If the matrix D is invertible, then*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & 0 \\ D^{-1}C & I \end{pmatrix}.$$

The above expression can be checked directly and has the advantage of requiring only the invertibility of D .

Remark: If A is invertible, then we can use the Schur complement $D - CA^{-1}B$ of A to obtain the following factorization of M :

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} I & A^{-1}B \\ 0 & I \end{pmatrix}.$$

If $D - CA^{-1}B$ is invertible, we can invert all three matrices above, and we get another formula for the inverse of M in terms of $(D - CA^{-1}B)$, namely,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}.$$

If A, D and both Schur complements $A - BD^{-1}C$ and $D - CA^{-1}B$ are all invertible, by comparing the two expressions for M^{-1} , we get the (non-obvious) formula

$$(A - BD^{-1}C)^{-1} = A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1}.$$

Using this formula, we obtain another expression for the inverse of M involving the Schur complements of A and D (see Horn and Johnson [95]):

Proposition 43.2. *If A, D and both Schur complements $A - BD^{-1}C$ and $D - CA^{-1}B$ are all invertible, then*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}.$$

If we set $D = I$ and change B to $-B$, we get

$$(A + BC)^{-1} = A^{-1} - A^{-1}B(I - CA^{-1}B)^{-1}CA^{-1},$$

a formula known as the *matrix inversion lemma* (see Boyd and Vandenberghe [29], Appendix C.4, especially C.4.3).