3.20 hold for modules. However, the other propositions do not generalize to modules. The definition of an isomorphism generalizes to modules. As a consequence, a module is free iff it is isomorphic to a module of the form $A^{(I)}$.

Section 3.8 generalizes to modules. Given a submodule N of a module M, we can define the quotient module M/N.

If $\mathfrak{a}$ is an ideal in A and if M is an A-module, we define $\mathfrak{a}M$ as the set of finite sums of the form

$$a_1 m_1 + \cdots + a_k m_k, \quad a_i \in \mathfrak{a},\ m_i \in M.$$

It is immediately verified that $\mathfrak{a}M$ is a submodule of M.

Interestingly, the part of Theorem 3.11 that asserts that any two bases of a vector space have the same cardinality holds for modules. One way to prove this fact is to "pass" to a vector space by a quotient process.

Theorem 35.1. *For any free module M, any two bases of M have the same cardinality.*

Proof sketch. We give the argument for finite bases, but it also holds for infinite bases. The trick is to pick any maximal ideal $\mathfrak{m}$ in A (whose existence is guaranteed by Theorem C.3). Then, $A/\mathfrak{m}$ is a field, and $M/\mathfrak{m}M$ can be made into a vector space over $A/\mathfrak{m}$; we leave the details as an exercise. If $(u_1, \ldots, u_n)$ is a basis of M, then it is easy to see that the image of this basis is a basis of the vector space $M/\mathfrak{m}M$. By Theorem 3.11, the number n of elements in any basis of $M/\mathfrak{m}M$ is an invariant, so any two bases of M must have the same number of elements. $\square$

Definition 35.4. The common number of elements in any basis of a free module is called the *dimension* (or *rank*) of the free module.

One should realize that the notion of linear independence in a module is a little tricky. According to the definition, the one-element sequence (u) consisting of a single nonzero vector is linearly independent if for all $\lambda \in A$, if $\lambda u = 0$ then $\lambda = 0$. However, there are free modules that contain nonzero vectors that are not linearly independent! For example, the ring $A = \mathbb{Z}/6\mathbb{Z}$ viewed as a module over itself has the basis (1), but the zero-divisors, such as 2 or 4, are not linearly independent. Using language introduced in Definition 35.5, a free module may have torsion elements. There are also nonfree modules such that every nonzero vector is linearly independent, such as $\mathbb{Q}$ over $\mathbb{Z}$.

All definitions from Section 4.1 about matrices apply to free modules, and so do all the propositions. Similarly, all definitions from Section 6.1 about direct sums and direct products apply to modules. All propositions that do not involve extending bases still hold. The important Proposition 6.15 survives in the following form.