The problem is that the block matrix $(A_{ij})_{1\leq i\leq m, 1\leq j\leq n}$ is *not* equal to the original matrix $A$. First of all, the block matrix is $m \times n$ and its entries are matrices, but the matrix $A$ is $M \times N$ and its entries are scalars. But even if we think of the block matrix as an $M \times N$ matrix of scalars, some rows and some columns of the original matrix $A$ may have been *permuted* due to the choice of the partitions $S = S_1 \cup \cdots \cup S_m$ and $T = T_1 \cup \cdots \cup T_n$; see Example 6.3.

We propose to denote the block matrix $(A_{ij})_{1\leq i\leq m, 1\leq j\leq n}$ by $[A]$. This is not entirely satisfactory since all information about the partitions of $S$ and $T$ are lost, but at least this allows us to distinguish between $A$ and a block matrix arising from $A$.

To be completely rigorous we may proceed as follows. Let $[m] = \{1, \ldots, m\}$ and $[n] = \{1, \ldots, n\}$.

**Definition 6.9.** For any two finite sets of indices $S$ and $T$, an $S \times T$ *matrix* $A$ is an $S \times T$-indexed family with values in $K$, that is, a function

$$A\colon S \times T \to K.$$

Denote the space of $S \times T$ matrices with entries in $K$ by $\mathrm{M}_{S,T}(K)$.

An $S \times T$ matrix $A$ is an $S \times T$-indexed family $(a_{st})_{(s,t)\in S\times T}$, but the standard representation of a matrix by a rectangular array of scalars is not quite correct because it assumes that the rows are indexed by indices in the "canonical index set" $[m]$ and that the columns are indexed by indices in the "canonical index set" $[n]$. Also the index sets need not be ordered, but in practice they are, so if $S = \{s_1, \ldots, s_m\}$ and $T = \{t_1, \ldots, t_n\}$, we denote an $S \times T$ matrix $A$ by the rectangular array

$$A = \begin{pmatrix} a_{s_1 t_1} & \cdots & a_{s_1 t_n} \\ \vdots & \ddots & \vdots \\ a_{s_m t_1} & \cdots & a_{s_m t_n} \end{pmatrix}.$$

Even if the index sets are not ordered, the product of an $R \times S$ matrix $A$ and of an $S \times T$ matrix $B$ is well defined and $C = AB$ is an $R \times T$ matrix (where $R, S, T$ are finite index sets); see Proposition 6.13.

Then an $m \times n$ block matrix $X$ induced by two partitions $S = S_1 \cup \cdots \cup S_m$ and $T = T_1 \cup \cdots \cup T_n$ is an $[m] \times [n]$-indexed family

$$X\colon [m] \times [n] \to \prod_{(i,j)\in[m]\times[n]} \mathrm{M}_{S_i,T_j}(K),$$

such that $X(i,j) \in \mathrm{M}_{S_i,T_j}(K)$, which means that $X(i,j)$ is an $S_i \times T_j$ matrix with entries in $K$. The map $X$ also defines the $M \times N$ matrix $A = (a_{st})_{s\in S, t\in T}$, with

$$a_{st} = X(i,j)_{st},$$