such that $z \in \bigwedge^2 W$, we calculate $u^* \lrcorner z$, where $u^* \in \bigwedge^1 E^*$. The multilinearity of $\lrcorner$ implies it is enough to calculate $u^* \lrcorner z$ for $u^* \in \{e_1^*, e_2^*, e_3^*, e_4^*\}$. Proposition 34.18 (4) implies that

$$\begin{aligned}
e_1^* \lrcorner z &= e_1^* \lrcorner (e_1 \wedge e_2 + e_2 \wedge e_3) = e_1^* \lrcorner e_1 \wedge e_2 = -e_2 \\
e_2^* \lrcorner z &= e_2^* \lrcorner (e_1 \wedge e_2 + e_2 \wedge e_3) = e_1 - e_3 \\
e_3^* \lrcorner z &= e_3^* \lrcorner (e_1 \wedge e_2 + e_2 \wedge e_3) = e_3^* \lrcorner e_2 \wedge e_3 = e_2 \\
e_4^* \lrcorner z &= e_4^* \lrcorner (e_1 \wedge e_2 + e_2 \wedge e_3) = 0.
\end{aligned}$$

Thus $W$ is the two-dimensional vector space generated by the basis $\{e_2, e_1 - e_3\}$. This is not surprising since $z = -e_2 \wedge (e_1 - e_3)$ and is in fact decomposable. As this example demonstrates, the action of the left hook provides a way of extracting a basis of $W$ from $z$.

Proposition 34.25 implies the following corollary.

**Corollary 34.26.** *Any nonzero $z \in \bigwedge^p E$ is decomposable iff the smallest subspace $W$ of $E$ such that $z \in \bigwedge^p W$ has dimension $p$. Furthermore, if $z = u_1 \wedge \cdots \wedge u_p$ is decomposable, then $(u_1, \ldots, u_p)$ is a basis of the smallest subspace $W$ of $E$ such that $z \in \bigwedge^p W$*

*Proof.* If $\dim(W) = p$, then for any basis $(e_1, \ldots, e_p)$ of $W$ we know that $\bigwedge^p W$ has $e_1 \wedge \cdots \wedge e_p$ has a basis, and thus has dimension 1. Since $z \in \bigwedge^p W$, we have $z = \lambda e_1 \wedge \cdots \wedge e_p$ for some nonzero $\lambda$, so $z$ is decomposable.

Conversely assume that $z \in \bigwedge^p W$ is nonzero and decomposable. Then, $z = u_1 \wedge \cdots \wedge u_p$, and since $z \neq 0$, by Proposition 34.8 $(u_1, \ldots, u_p)$ are linearly independent. Then for any $v_i^* = u_1^* \wedge \cdots u_{i-1}^* \wedge u_{i+1}^* \wedge \cdots \wedge u_p^*$ (where $u_i^*$ is omitted), we have

$$v_i^* \lrcorner z = (u_1^* \wedge \cdots u_{i-1}^* \wedge u_{i+1}^* \wedge \cdots \wedge u_p^*) \lrcorner (u_1 \wedge \cdots \wedge u_p) = \pm u_i,$$

so by Proposition 34.25 we have $u_i \in W$ for $i = 1, \ldots, p$. This shows that $\dim(W) \geq p$, but since $z = u_1 \wedge \cdots \wedge u_p$, we have $\dim(W) = p$, which means that $(u_1, \ldots, u_p)$ is a basis of $W$. ∎

Finally we are ready to state and prove the criterion for decomposability with respect to left hooks.

**Proposition 34.27.** *Any nonzero $z \in \bigwedge^p E$ is decomposable iff*

$$(u^* \lrcorner z) \wedge z = 0, \qquad \text{for all } u^* \in \textstyle\bigwedge^{p-1} E^*.$$

*Proof.* First assume that $z \in \bigwedge^p E$ is decomposable. If so, by Corollary 34.26, the smallest subspace $W$ of $E$ such that $z \in \bigwedge^p W$ has dimension $p$, so we have $z = e_1 \wedge \cdots \wedge e_p$ where $e_1, \ldots, e_p$ form a basis of $W$. By Proposition 34.25, for every $u^* \in \bigwedge^{p-1} E^*$, we have $u^* \lrcorner z \in W$, so each $u^* \lrcorner z$ is a linear combination of the $e_i$'s, say

$$u^* \lrcorner z = \alpha_1 e_1 + \cdots + \alpha_p e_p,$$