for some unique coordinates $(x_i)_{i\in I}$ of x.

To prove that f as defined by (†) is linear it suffices to prove that

$$f(\lambda x + \mu y) = \lambda f(x) + \mu f(y)$$

for all $x, y \in E$ and all $\lambda, \mu \in \mathbb{R}$. Since $(u_i)_{i\in I}$ is a basis of E, we have

$$x = \sum_{i\in I} x_i u_i, \quad y = \sum_{i\in I} y_i u_i,$$

for some unique coordinates $(x_i)_{i\in I}$ of x and $(y_i)_{i\in I}$ of y, and by (†) we have

$$f(x) = \sum_{i\in I} x_i v_i, \quad f(y) = \sum_{i\in I} y_i v_i,$$

and since

$$\lambda x + \mu y = \lambda\left(\sum_{i\in I} x_i u_i\right) + \mu\left(\sum_{i\in I} y_i u_i\right) = \sum_{i\in I}(\lambda x_i + \mu y_i) u_i,$$

by (†),

$$\begin{aligned} f(\lambda x + \mu y) &= f\left(\sum_{i\in I}(\lambda x_i + \mu y_i) u_i\right) = \sum_{i\in I}(\lambda x_i + \mu y_i) v_i \\ &= \lambda\left(\sum_{i\in I} x_i v_i\right) + \mu\left(\sum_{i\in I} y_i v_i\right) = \lambda f(x) + \mu f(y), \end{aligned}$$

proving that f is linear. The map f is unique by (†), and obviously, $f(u_i) = v_i$.

Now assume that f is injective. Let $(\lambda_i)_{i\in I}$ be any family of scalars, and assume that

$$\sum_{i\in I} \lambda_i v_i = 0.$$

Since $v_i = f(u_i)$ for every $i \in I$, we have

$$f\left(\sum_{i\in I} \lambda_i u_i\right) = \sum_{i\in I} \lambda_i f(u_i) = \sum_{i\in I} \lambda_i v_i = 0.$$

Since f is injective iff $\mathrm{Ker}\, f = (0)$, we have

$$\sum_{i\in I} \lambda_i u_i = 0,$$

and since $(u_i)_{i\in I}$ is a basis, we have $\lambda_i = 0$ for all $i \in I$, which shows that $(v_i)_{i\in I}$ is linearly independent. Conversely, assume that $(v_i)_{i\in I}$ is linearly independent. Since $(u_i)_{i\in I}$ is a basis of E, every vector $x \in E$ is a linear combination $x = \sum_{i\in I} \lambda_i u_i$ of $(u_i)_{i\in I}$. If

$$f(x) = f\left(\sum_{i\in I} \lambda_i u_i\right) = 0,$$