Prove that

$$P(X) = \beta_1 L_1(X) + \cdots + \beta_{m+1} L_{m+1}(X)$$

is the unique polynomial of degree at most m such that

$$P(\alpha_i) = \beta_i, \quad 1 \leq i \leq m+1.$$

(3) Prove that $L_1(X), \ldots, L_{m+1}(X)$ are linearly independent, and that they form a basis of all polynomials of degree at most m.

How is 1 (the constant polynomial 1) expressed over the basis $(L_1(X), \ldots, L_{m+1}(X))$?

Give the expression of every polynomial $P(X)$ of degree at most m over the basis $(L_1(X), \ldots, L_{m+1}(X))$.

(4) Prove that the dual basis $(L_1^*, \ldots, L_{m+1}^*)$ of the basis $(L_1(X), \ldots, L_{m+1}(X))$ consists of the linear forms L_i^* given by

$$L_i^*(P) = P(\alpha_i),$$

for every polynomial P of degree at most m; this is simply *evaluation at* α_i.