42.5 Problems

Problem 42.1. Prove that the relation

$$A \succeq B$$

between any two $n \times n$ matrices (symmetric or not) iff $A - B$ is symmetric positive semidefinite is indeed a partial order.

Problem 42.2. (1) Prove that if A is symmetric positive definite, then so is A^{-1}.

(2) Prove that if C is a symmetric positive definite $m \times m$ matrix and A is an $m \times n$ matrix of rank n (and so $m \geq n$ and the map $x \mapsto Ax$ is injective), then $A^{\top} C A$ is symmetric positive definite.

Problem 42.3. Find the minimum of the function

$$Q(x_1, x_2) = \frac{1}{2}(2x_1^2 + x_2^2)$$

subject to the constraint

$$x_1 - x_2 = 3.$$

Problem 42.4. Consider the problem of minimizing the function

$$f(x) = \frac{1}{2} x^{\top} A x - x^{\top} b$$

in the case where we add an affine constraint of the form $C^{\top} x = t$, with $t \in \mathbb{R}^m$ and $t \neq 0$, and where C is an $n \times m$ matrix of rank $m \leq n$. As in Section 42.2, use a QR-decomposition

$$C = P \begin{pmatrix} R \\ 0 \end{pmatrix},$$

where P is an orthogonal $n \times n$ matrix and R is an $m \times m$ invertible upper triangular matrix, and write

$$x = P \begin{pmatrix} y \\ z \end{pmatrix},$$

to deduce that

$$R^{\top} y = t.$$

Give the details of the reduction of this constrained minimization problem to an unconstrained minimization problem involving the matrix $P^{\top} A P$.

Problem 42.5. Find the maximum and the minimum of the function

$$Q(x, y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

on the unit circle $x^2 + y^2 = 1$.