But by definition of the operator norm and using the Cauchy–Schwarz inequality

$$\begin{aligned}\left\|dJ_{u_k} - dJ_{u_{k+1}}\right\|_2 &= \sup_{\|w\|=1} |dJ_{u_k}(w) - dJ_{u_{k+1}}(w)| \\ &= \sup_{\|w\|=1} |\langle \nabla J_{u_k} - \nabla J_{u_{k+1}}, w\rangle| \\ &\leq \left\|\nabla J_{u_k} - \nabla J_{u_{k+1}}\right\|.\end{aligned}$$

But we also have

$$\begin{aligned}\left\|\nabla J_{u_k} - \nabla J_{u_{k+1}}\right\|^2 &= \langle \nabla J_{u_k} - \nabla J_{u_{k+1}}, \nabla J_{u_k} - \nabla J_{u_{k+1}}\rangle \\ &= dJ_{u_k}(\nabla J_{u_k} - \nabla J_{u_{k+1}}) - dJ_{u_{k+1}}(\nabla J_{u_k} - \nabla J_{u_{k+1}}) \\ &\leq \left\|dJ_{u_k} - dJ_{u_{k+1}}\right\|_2^2,\end{aligned}$$

and so

$$\left\|dJ_{u_k} - dJ_{u_{k+1}}\right\|_2 = \left\|\nabla J_{u_k} - \nabla J_{u_{k+1}}\right\|.$$

It follows that since

$$\lim_{k\mapsto\infty} \|u_k - u_{k+1}\| = 0$$

then

$$\lim_{k\mapsto\infty} \left\|\nabla J_{u_k} - \nabla J_{u_{k+1}}\right\| = \lim_{k\mapsto\infty} \left\|dJ_{u_k} - dJ_{u_{k+1}}\right\|_2 = 0,$$

and using the fact that

$$\|\nabla J_{u_k}\| \leq \left\|\nabla J_{u_k} - \nabla J_{u_{k+1}}\right\|,$$

we obtain

$$\lim_{k\mapsto\infty} \|\nabla J_{u_k}\| = 0.$$

Step 5. Finally we can prove the convergence of the sequence $(u_k)_{k\geq 0}$.

Since J is elliptic and since $\nabla J_u = 0$ (since u is the minimum of J over $\mathbb{R}^n$), we have

$$\begin{aligned}\alpha \|u_k - u\|^2 &\leq \langle \nabla J_{u_k} - \nabla J_u, u_k - u\rangle \\ &= \langle \nabla J_{u_k}, u_k - u\rangle \\ &\leq \|\nabla J_{u_k}\| \|u_k - u\|.\end{aligned}$$

Hence, we obtain

$$\|u_k - u\| \leq \frac{1}{\alpha} \|\nabla J_{u_k}\|, \tag{b}$$

and since we showed that

$$\lim_{k\mapsto\infty} \|\nabla J_{u_k}\| = 0,$$

we see that the sequence $(u_k)_{k\geq 0}$ converges to the mininum u. ∎