Remark: When $\pi = \mathrm{id}_n$ is the identity permutation, we can agree that the composition of 0 transpositions is the identity. The second part of Proposition 7.1 shows that the transpositions generate the group of permutations $\mathfrak{S}_n$.

In writing a permutation π as a composition $\pi = \sigma_1 \circ \ldots \circ \sigma_s$ of cyclic permutations, it is clear that the order of the σ_i does not matter, since their domains are disjoint. Given a permutation written as a product of transpositions, we now show that the parity of the number of transpositions is an invariant.

Definition 7.2. For every $n \geq 2$, since every permutation $\pi\colon [n] \to [n]$ defines a partition of r subsets over which π acts either as the identity or as a cyclic permutation, let $\epsilon(\pi)$, called the *signature* of π, be defined by $\epsilon(\pi) = (-1)^{n-r}$, where r is the number of sets in the partition.

If τ is a transposition exchanging i and j, it is clear that the partition associated with τ consists of $n-1$ equivalence classes, the set $\{i, j\}$, and the $n-2$ singleton sets $\{k\}$, for $k \in [n] - \{i, j\}$, and thus, $\epsilon(\tau) = (-1)^{n-(n-1)} = (-1)^1 = -1$.

Proposition 7.2. *For every $n \geq 2$, for every permutation $\pi\colon [n] \to [n]$, for every transposition τ, we have*

$$\epsilon(\tau \circ \pi) = -\epsilon(\pi).$$

Consequently, for every product of transpositions such that $\pi = \tau_m \circ \ldots \circ \tau_1$, we have

$$\epsilon(\pi) = (-1)^m,$$

which shows that the parity of the number of transpositions is an invariant.

Proof. Assume that $\tau(i) = j$ and $\tau(j) = i$, where $i < j$. There are two cases, depending whether i and j are in the same equivalence class J_l of R_π, or if they are in distinct equivalence classes. If i and j are in the same class J_l, then if

$$J_l = \{i_1, \ldots, i_p, \ldots i_q, \ldots i_k\},$$

where $i_p = i$ and $i_q = j$, since

$$\tau(\pi(\pi^{-1}(i_p))) = \tau(i_p) = \tau(i) = j = i_q$$

and

$$\tau(\pi(i_{q-1})) = \tau(i_q) = \tau(j) = i = i_p,$$

it is clear that J_l splits into two subsets, one of which is $\{i_p, \ldots, i_{q-1}\}$, and thus, the number of classes associated with $\tau \circ \pi$ is $r+1$, and $\epsilon(\tau \circ \pi) = (-1)^{n-r-1} = -(-1)^{n-r} = -\epsilon(\pi)$. If i and j are in distinct equivalence classes J_l and J_m, say

$$\{i_1, \ldots, i_p, \ldots i_h\}$$