is a positive definite kernel. Let $f\colon X \to \mathbb{R}$ be the function given by

$$f(x) = e^{-\frac{\|x\|^2}{2\sigma^2}}.$$

Then by Proposition 53.6(3),

$$\kappa_2(x, y) = f(x)f(y) = e^{-\frac{\|x\|^2}{2\sigma^2}} e^{-\frac{\|y\|^2}{2\sigma^2}} = e^{-\frac{\|x\|_X^2 + \|y\|_X^2}{2\sigma^2}}$$

is a positive definite kernel. By Proposition 53.5, the function $\kappa_1\kappa_2$ is a positive definite kernel, that is

$$\kappa_1(x, y)\kappa_2(x, y) = e^{\frac{\langle x,y\rangle_X}{\sigma^2}} e^{-\frac{\|x\|_X^2 + \|y\|_X^2}{2\sigma^2}} = e^{\frac{\langle x,y\rangle_X}{\sigma^2} - \frac{\|x\|_X^2 + \|y\|_X^2}{2\sigma^2}} = e^{-\frac{\|x-y\|_X^2}{2\sigma^2}}$$

is a positive definite kernel. ☐

Definition 53.4. The positive definite kernel

$$\kappa(x, y) = e^{-\frac{\|x-y\|_X^2}{2\sigma^2}}$$

is called a *Gaussian kernel*.

This kernel requires a feature map in an infinite-dimensional space because it is an infinite sum of distinct kernels.

Remark: If κ_1 is a positive definite kernel, the proof of Proposition 53.7(3) is immediately adapted to show that

$$\kappa(x, y) = e^{-\frac{\kappa_1(x,x) + \kappa_1(y,y) - 2\kappa_1(x,y)}{2\sigma^2}}$$

is a positive definite kernel.

Next we prove that every positive definite kernel arises from a feature map in a Hilbert space which is a function space.

53.3 Hilbert Space Representation of a Positive Definite Kernel

The following result shows how to construct a so-called *reproducing kernel Hilbert space*, for short RKHS, from a positive definite kernel.

Theorem 53.8. *Let $\kappa\colon X \times X \to \mathbb{C}$ be a positive definite kernel on a nonempty set X. For every $x \in X$, let $\kappa_x\colon X \to \mathbb{C}$ be the function given by*

$$\kappa_x(y) = \kappa(x, y), \qquad y \in X.$$