**Definition 44.6.** Given any (nonempty) subset $S$ of $\mathbb{R}^n$, the smallest convex set containing $S$ is denoted by $\mathrm{conv}(S)$ and called the *convex hull of* $S$ (it is the intersection of all convex sets containing $S$).

It is essential not only to have a good understanding of $\mathrm{conv}(S)$, but to also have good methods for computing it. We have the following simple but crucial result.

**Proposition 44.1.** *For any family $S = (a_i)_{i\in I}$ of points in $\mathbb{R}^n$, the set $V$ of convex combinations $\sum_{i\in I} \lambda_i a_i$ (where $\sum_{i\in I} \lambda_i = 1$ and $\lambda_i \geq 0$) is the convex hull* $\mathrm{conv}(S)$ *of $S = (a_i)_{i\in I}$.*

It is natural to wonder whether Proposition 44.1 can be sharpened in two directions: (1) Is it possible to have a fixed bound on the number of points involved in the convex combinations? (2) Is it necessary to consider convex combinations of all points, or is it possible to consider only a subset with special properties?

The answer is yes in both cases. In Case 1, Carathéodory's theorem asserts that it is enough to consider convex combinations of $n+1$ points. For example, in the plane $\mathbb{R}^2$, the convex hull of a set $S$ of points is the union of all triangles (interior points included) with vertices in $S$. In Case 2, the theorem of Krein and Milman asserts that a convex set that is also compact is the convex hull of its extremal points (given a convex set $S$, a point $a \in S$ is *extremal* if $S - \{a\}$ is also convex).

We will not prove these theorems here, but we invite the reader to consult Gallier [73] or Berger [12].

Convex sets also arise as half-spaces cut out by affine hyperplanes.

**Definition 44.7.** An *affine form* $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ is defined by some linear form $c \in (\mathbb{R}^n)^*$ and some scalar $\beta \in \mathbb{R}$ so that

$$\varphi(x) = cx + \beta \quad \text{for all } x \in \mathbb{R}^n.$$

If $c \neq 0$, the affine form $\varphi$ specified by $(c, \beta)$ defines the *affine hyperplane* (for short *hyperplane*) $H(\varphi)$ given by

$$H(\varphi) = \{x \in \mathbb{R}^n \mid \varphi(x) = 0\} = \{x \in \mathbb{R}^n \mid cx + \beta = 0\},$$

and the two *(closed) half-spaces*

$$\begin{aligned} H_+(\varphi) &= \{x \in \mathbb{R}^n \mid \varphi(x) \geq 0\} = \{x \in \mathbb{R}^n \mid cx + \beta \geq 0\}, \\ H_-(\varphi) &= \{x \in \mathbb{R}^n \mid \varphi(x) \leq 0\} = \{x \in \mathbb{R}^n \mid cx + \beta \leq 0\}. \end{aligned}$$

When $\beta = 0$, we call $H$ a *linear hyperplane*.

Both $H_+(\varphi)$ and $H_-(\varphi)$ are convex and $H = H_+(\varphi) \cap H_-(\varphi)$.

For example, $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ with $\varphi(x, y) = 2x + y + 3$ is an affine form defining the line given by the equation $y = -2x - 3$. Another example of an affine form is $\varphi\colon \mathbb{R}^3 \to \mathbb{R}$