(1) v is a vertex of the Polyhedron $\mathcal{P}(A, b)$.

(2) v is a basic feasible solution of the Linear Program (P).

Proof. First, assume that v is a vertex of $\mathcal{P}(A, b)$, and let $\varphi(x) = cx - \mu$ be a linear form such that $cy < \mu$ for all $y \in \mathcal{P}(A, b)$ and $cv = \mu$. This means that v is the unique point of $\mathcal{P}(A, b)$ for which the objective function $x \mapsto cx$ has the maximum value μ on $\mathcal{P}(A, b)$, so by Theorem 45.4, since this maximum is achieved by some basic feasible solution, by uniqueness v must be a basic feasible solution.

Conversely, suppose v is a basic feasible solution of (P) corresponding to a subset $K \subseteq \{1, \ldots, n\}$ of size m. Let $\widehat{c} \in (\mathbb{R}^n)^*$ be the linear form defined by

$$\widehat{c}_j = \begin{cases} 0 & \text{if } j \in K \\ -1 & \text{if } j \notin K. \end{cases}$$

By construction $\widehat{c}\, v = 0$ and $\widehat{c}\, x \leq 0$ for any $x \geq 0$, hence the function $x \mapsto \widehat{c}\, x$ on $\mathcal{P}(A, b)$ has a maximum at v. Furthermore, $\widehat{c}\, x < 0$ for any $x \geq 0$ such that $x_j > 0$ for some $j \notin K$. However, by Proposition 45.5, the vector v is the only basic feasible solution such that $v_j = 0$ for all $j \notin K$, and therefore v is the only point of $\mathcal{P}(A, b)$ maximizing the function $x \mapsto \widehat{c}\, x$, so it is a vertex. ☐

In theory, to find an optimal solution we try all $\binom{n}{m}$ possible m-elements subsets K of $\{1, \ldots, n\}$ and solve for the corresponding unique solution x_K of $A_K x = b$. Then we check whether such a solution satisfies $x_K \geq 0$, compute cx_K, and return some feasible x_K for which the objective function is maximum. This is a totally impractical algorithm.

A practical algorithm is the *simplex algorithm*. Basically, the simplex algorithm tries to "climb" in the polyhderon $\mathcal{P}(A, b)$ from vertex to vertex along edges (using basic feasible solutions), trying to maximize the objective function. We present the simplex algorithm in the next chapter. The reader may also consult texts on linear programming. In particular, we recommend Matousek and Gardner [123], Chvatal [40], Papadimitriou and Steiglitz [134], Bertsimas and Tsitsiklis [21], Ciarlet [41], Schrijver [148], and Vanderbei [181].

Observe that Theorem 45.4 asserts that if a Linear Program (P) in standard form (where $Ax = b$ and A is an $m \times n$ matrix of rank m) has some feasible solution and is bounded above, then some basic feasible solution is an optimal solution. By Theorem 45.6, the polyhedron $\mathcal{P}(A, b)$ must have some vertex.

But suppose we only know that $\mathcal{P}(A, b)$ is nonempty; that is, we don't know that the objective function cx is bounded above. Does $\mathcal{P}(A, b)$ have some vertex?

The answer to the above question is *yes*, and this is important because the simplex algorithm needs an initial basic feasible solution to get started. Here we prove that if $\mathcal{P}(A, b)$ is nonempty, then it must contain a vertex. This proof still doesn't constructively yield a vertex, but we will see in the next chapter that the simplex algorithm always finds a vertex if there is one (provided that we use a pivot rule that prevents cycling).