Proof. Let $(e_1, \ldots, e_n)$ be the standard basis of $\mathbb{R}^n$ (a similar proof applies to $\mathbb{C}^n$). In view of Proposition 9.3, it is enough to prove the proposition for the norm

$$\|x\|_\infty = \max\{|x_i| \mid 1 \leq i \leq n\}.$$

We have,

$$\|f(v) - f(u)\| = \|f(v-u)\| = \left\| f(\sum_{1 \leq i \leq n} (v_i - u_i)e_i) \right\| = \left\| \sum_{1 \leq i \leq n} (v_i - u_i) f(e_i) \right\|,$$

and so,

$$\|f(v) - f(u)\| \leq \left(\sum_{1 \leq i \leq n} \|f(e_i)\| \right) \max_{1 \leq i \leq n} |v_i - u_i| = \left(\sum_{1 \leq i \leq n} \|f(e_i)\| \right) \|v - u\|_\infty.$$

By the argument used in Proposition 37.56 to prove that (3) implies (4), f is continuous. ☐

Actually, we proved in Theorem 9.5 that if E is a vector space of finite dimension, then any two norms are equivalent, so that they define the same topology. This fact together with Proposition 37.57 prove the following:

Theorem 37.58. *If E is a vector space of finite dimension (over $\mathbb{R}$ or $\mathbb{C}$), then all norms are equivalent (define the same topology). Furthermore, for any normed vector space F, every linear map $f\colon E \to F$ is continuous.*

If E is a normed vector space of infinite dimension, a linear map $f\colon E \to F$ may not be continuous. As an example, let E be the infinite vector space of all polynomials over $\mathbb{R}$. Let

$$\|P(X)\| = \max_{0 \leq x \leq 1} |P(x)|.$$

We leave as an exercise to show that this is indeed a norm. Let $F = \mathbb{R}$, and let $f\colon E \to F$ be the map defined such that, $f(P(X)) = P(3)$. It is clear that f is linear. Consider the sequence of polynomials

$$P_n(X) = \left(\frac{X}{2}\right)^n.$$

It is clear that $\|P_n\| = \left(\frac{1}{2}\right)^n$, and thus, the sequence P_n has the null polynomial as a limit. However, we have

$$f(P_n(X)) = P_n(3) = \left(\frac{3}{2}\right)^n,$$

and the sequence $f(P_n(X))$ diverges to $+\infty$. Consequently, in view of Proposition 37.15 (1), f is not continuous.