Definition 34.7. The operator $*$ from $\bigwedge^k V$ to $\bigwedge^{n-k} V$ defined by Proposition 34.15 is called the *Hodge $*$-operator*.

Obseve that the Hodge $*$-operator is linear.

The Hodge $*$-operator is defined in terms of the orthonormal basis elements of $\bigwedge V$ as follows: For any increasing sequence $(i_1, \ldots, i_k)$ of elements $i_p \in \{1, \ldots, n\}$, if $(j_1, \ldots, j_{n-k})$ is the increasing sequence of elements $j_q \in \{1, \ldots, n\}$ such that

$$\{i_1, \ldots, i_k\} \cup \{j_1, \ldots, j_{n-k}\} = \{1, \ldots, n\},$$

then

$$*(e_{i_1} \wedge \cdots \wedge e_{i_k}) = \mathrm{sign}(i_1, \ldots i_k, j_1, \ldots, j_{n-k})\, e_{j_1} \wedge \cdots \wedge e_{j_{n-k}}.$$

In particular, for $k = 0$ and $k = n$, we have

$$\begin{aligned} *(1) &= e_1 \wedge \cdots \wedge e_n \\ *(e_1 \wedge \cdots \wedge e_n) &= 1. \end{aligned}$$

For example, if $n = 3$, we have

$$\begin{aligned} *e_1 &= e_2 \wedge e_3 \\ *e_2 &= -e_1 \wedge e_3 \\ *e_3 &= e_1 \wedge e_2 \\ *(e_1 \wedge e_2) &= e_3 \\ *(e_1 \wedge e_3) &= -e_2 \\ *(e_2 \wedge e_3) &= e_1. \end{aligned}$$

The Hodge $*$-operators $*\colon \bigwedge^k V \to \bigwedge^{n-k} V$ induce a linear map $*\colon \bigwedge(V) \to \bigwedge(V)$. We also have Hodge $*$-operators $*\colon \bigwedge^k V^* \to \bigwedge^{n-k} V^*$.

The following proposition shows that the linear map $*\colon \bigwedge(V) \to \bigwedge(V)$ is an isomorphism.

Proposition 34.16. *If V is any oriented vector space of dimension n, for every k with $0 \leq k \leq n$, we have*

(i) $** = (-\mathrm{id})^{k(n-k)}$.

(ii) $\langle x, y \rangle_\wedge = *(x \wedge *y) = *(y \wedge *x)$, *for all* $x, y \in \bigwedge^k V$.

Proof. (1) Let $(e_i)_{i=1}^n$ is an orthonormal basis of V. It is enough to check the identity on basis elements. We have

$$*(e_{i_1} \wedge \cdots \wedge e_{i_k}) = \mathrm{sign}(i_1, \ldots i_k, j_1, \ldots, j_{n-k})\, e_{j_1} \wedge \cdots \wedge e_{j_{n-k}}$$