Corollary 49.10. *If $J\colon \mathbb{R}^n \to \mathbb{R}$ is a quadratic function given by*

$$J(v) = \frac{1}{2}\langle Av, v\rangle - \langle b, v\rangle$$

then

$$\langle \nabla^2 J_u(w), w\rangle \leq \lambda_n \left\|w\right\|^2$$

where λ_n is the largest eigenvalue of A;

The above fact will be useful later on.

Similarly, given a quadratic functional J defined on a Hilbert space V, where

$$J(v) = \frac{1}{2}a(v, v) - h(v),$$

by Theorem 49.8 (4), the functional J is elliptic iff there is some $\alpha > 0$ such that

$$\langle \nabla^2 J_u(v), v\rangle = a(v, v) \geq \alpha \left\|v\right\|^2 \quad \text{for all } v \in V.$$

This is precisely the hypothesis $(*_\alpha)$ used in Theorem 49.4.

49.5 Iterative Methods for Unconstrained Problems

We will now describe methods for solving unconstrained minimization problems, that is, finding the minimum (or minima) of a functions J over the whole space V. These methods are *iterative*, which means that given some *initial* vector u_0, we construct a sequence $(u_k)_{k\geq 0}$ that converges to a minimum u of the function J.

The key step is define u_{k+1} from u_k, and a first idea is to reduce the problem to a simpler problem, namely the minimization of a function of a *single (real) variable*. For this, we need two perform two steps:

(1) Find a *descent direction* at u_k, which is a some nonzero vector d_k which is usually determined from the gradient of J at various points. The descent direction d_k must satisfy the inequality $\langle \nabla J_{u_k}, d_k\rangle < 0$.

(2) *Exact line search*: Find the minimum of the restriction of the function J along the line through u_k and parallel to the direction d_k. This means finding a real $\rho_k \in \mathbb{R}$ (depending on u_k and d_k) such that

$$J(u_k + \rho_k d_k) = \inf_{\rho\in\mathbb{R}} J(u_k + \rho d_k).$$

Typically, $\rho_k > 0$. This problem only succeeds if ρ_k is *unique*, in which case we set

$$u_{k+1} = u_k + \rho_k d_k.$$

This step is often called a *line search* or *line minimization*, and ρ_k is called the *stepsize* parameter. See Figure 49.1.