- Normalized graph Laplacian.
- Spectral graph theory.
- Graph clustering using normalized cuts.

20.6 Problems

Problem 20.1. Find the unnormalized Laplacian of the graph representing a triangle and of the graph representing a square.

Problem 20.2. Consider the complete graph K_m on $m \geq 2$ nodes.

(1) Prove that the normalized Laplacian L_{sym} of K is

$$L_{\mathrm{sym}} = \begin{pmatrix} 1 & -1/(m-1) & \dots & -1/(m-1) & -1/(m-1) \\ -1/(m-1) & 1 & \dots & -1/(m-1) & -1/(m-1) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ -1/(m-1) & -1/(m-1) & \dots & 1 & -1/(m-1) \\ -1/(m-1) & -1/(m-1) & \dots & -1/(m-1) & 1 \end{pmatrix}.$$

(2) Prove that the characteristic polynomial of L_{sym} is

$$\begin{vmatrix} \lambda - 1 & 1/(m-1) & \dots & 1/(m-1) & 1/(m-1) \\ 1/(m-1) & \lambda - 1 & \dots & 1/(m-1) & 1/(m-1) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1/(m-1) & 1/(m-1) & \dots & \lambda - 1 & 1/(m-1) \\ 1/(m-1) & 1/(m-1) & \dots & 1/(m-1) & \lambda - 1 \end{vmatrix} = \lambda \left(\lambda - \frac{m}{m-1} \right)^{m-1}.$$

Hint. First subtract the second column from the first, factor $\lambda - m/(m-1)$, and then add the first row to the second. Repeat this process. You will end up with the determinant

$$\begin{vmatrix} \lambda - 1/(m-1) & 1 \\ 1/(m-1) & \lambda - 1 \end{vmatrix}.$$

Problem 20.3. Consider the complete bipartite graph $K_{m,n}$ on $m + n \geq 3$ nodes, with edges between each of the first $m \geq 1$ nodes to each of the last $n \geq 1$ nodes. Prove that the eigenvalues of the normalized Laplacian L_{sym} of $K_{m,n}$ are 0, 1 with multiplicity $m + n - 2$, and 2.

Problem 20.4. Let G be a graph with a set of nodes V with $m \geq 2$ elements, without isolated nodes, and let $L_{\mathrm{sym}} = D^{-1/2} L D^{-1/2}$ be its normalized Laplacian (with L its unnormalized Laplacian).