Since the series $\sum_{k=1}^{\infty} \left\| r^k \right\|$ converges, the partial sums form a Cauchy sequence, and this immediately implies that for any $\epsilon > 0$ we can find $N > 0$ such that

$$\rho(\left\| r^{k+1} \right\| + \cdots + \left\| r^{k+p} \right\|) < \epsilon, \quad \text{for all } k, p + k \geq N,$$

so the sequence (λ^k) is also a Cauchy sequence, thus it converges.

Step 5. Prove that the sequence (p^k) converges to p^*.

For this, we need two more inequalities. Following Boyd et al. [28], we need to prove that

$$p^{k+1} - p^* \leq -(\lambda^{k+1})^\top r^{k+1} - \rho(B(z^{k+1} - z^k))^\top(-r^{k+1} + B(z^{k+1} - z^*)) \tag{A2}$$

and

$$p^* - p^{k+1} \leq (\lambda^*)^\top r^{k+1}. \tag{A3}$$

Since we proved that the sequence (r^k) and $B(z^{k+1} - z^k)$ converge to 0, and that the sequence (λ^{k+1}) converges, from

$$(\lambda^{k+1})^\top r^{k+1} + \rho(B(z^{k+1} - z^k))^\top(-r^{k+1} + B(z^{k+1} - z^*)) \leq p^* - p^{k+1} \leq (\lambda^*)^\top r^{k+1},$$

we deduce that in the limit, p^{k+1} converges to p^*.

Step 6. Prove (A3).

Since (x^*, y^*, λ^*) is a saddle point, we have

$$L_0(x^*, z^*, \lambda^*) \leq L_0(x^{k+1}, z^{k+1}, \lambda^*).$$

Since $Ax^* + Bz^* = c$, we have $L_0(x^*, z^*, \lambda^*) = p^*$, and since $p^{k+1} = f(x^{k+1}) + g(z^{k+1})$, we have

$$L_0(x^{k+1}, z^{k+1}, \lambda^*) = p^{k+1} + (\lambda^*)^\top r^{k+1},$$

so we obtain

$$p^* \leq p^{k+1} + (\lambda^*)^\top r^{k+1},$$

which yields (A3).

Step 7. Prove (A2).

By Proposition 51.34, z^{k+1} minimizes $L_\rho(x^{k+1}, z, \lambda^k)$ iff

$$\begin{aligned} 0 &\in \partial g(z^{k+1}) + B^\top \lambda^k + \rho B^\top(Ax^{k+1} + Bz^{k+1} - c) \\ &= \partial g(z^{k+1}) + B^\top \lambda^k + \rho B^\top r^{k+1} \\ &= \partial g(z^{k+1}) + B^\top \lambda^{k+1}, \end{aligned}$$

since $r^{k+1} = Ax^{k+1} + Bz^{k+1} - c$ and $\lambda^{k+1} = \lambda^k + \rho(Ax^{k+1} + Bz^{k+1} - c)$.

In summary, we have

$$0 \in \partial g(z^{k+1}) + B^\top \lambda^{k+1}, \tag{$\dagger_1$}$$