$P(k,i)\,\mathcal{E}_j^{k-1}\,P(k,i)$ is. However, since

$$\mathcal{E}_j^{k-1} = \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & \ell_{j+1j}^{(k-1)} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \ell_{nj}^{(k-1)} & 0 & \cdots & 0 \end{pmatrix},$$

and because $k+1 \leq i \leq n$ and $j \leq k-1$, multiplying $\mathcal{E}_j^{k-1}$ on the right by $P(k,i)$ will permute *columns* i and k, which are columns of zeros, so

$$P(k,i)\,\mathcal{E}_j^{k-1}\,P(k,i) = P(k,i)\,\mathcal{E}_j^{k-1},$$

and thus,

$$(E_j^k)^{-1} = I + P(k,i)\,\mathcal{E}_j^{k-1}.$$

But since

$$(E_j^k)^{-1} = I + \mathcal{E}_j^k,$$

we deduce that

$$\mathcal{E}_j^k = P(k,i)\,\mathcal{E}_j^{k-1}.$$

We also know that multiplying $\mathcal{E}_j^{k-1}$ on the left by $P(k,i)$ will permute *rows* i and k, which shows that $\mathcal{E}_j^k$ has the desired form, as claimed. Since all $\mathcal{E}_j^k$ are strictly lower triangular, all $(E_j^k)^{-1} = I + \mathcal{E}_j^k$ are lower triangular, so the product

$$L = (E_1^{n-1})^{-1} \cdots (E_{n-1}^{n-1})^{-1}$$

is also lower triangular.

Step 3. Express L as $L = I + \Lambda_{n-1}$, with $\Lambda_{n-1} = \mathcal{E}_1^1 + \cdots + \mathcal{E}_{n-1}^{n-1}$.

From Step 1 of Part (3), we know that

$$L = (E_1^{n-1})^{-1} \cdots (E_{n-1}^{n-1})^{-1}.$$

We prove by induction on k that

$$\begin{aligned} I + \Lambda_k &= (E_1^k)^{-1} \cdots (E_k^k)^{-1} \\ \Lambda_k &= \mathcal{E}_1^k + \cdots + \mathcal{E}_k^k, \end{aligned}$$

for $k = 1, \ldots, n-1$.