(d) *For every subspace U of finite dimension m of E^*, the orthogonal U^0 of U in E is of finite codimension m, so that*

$$\operatorname{codim}(U^0) = \dim(U).$$

Furthermore, $U^{00} = U$.

Proof. (a) Assume that

$$\sum_{i \in I} \lambda_i u_i^* = 0,$$

for a family $(\lambda_i)_{i \in I}$ (of scalars in K). Since $(\lambda_i)_{i \in I}$ has finite support, there is a finite subset J of I such that $\lambda_i = 0$ for all $i \in I - J$, and we have

$$\sum_{j \in J} \lambda_j u_j^* = 0.$$

Applying the linear form $\sum_{j \in J} \lambda_j u_j^*$ to each u_j ($j \in J$), by Definition 11.2, since $u_i^*(u_j) = 1$ if $i = j$ and 0 otherwise, we get $\lambda_j = 0$ for all $j \in J$, that is $\lambda_i = 0$ for all $i \in I$ (by definition of J as the support). Thus, $(u_i^*)_{i \in I}$ is linearly independent.

(b) Clearly, we have $V \subseteq V^{00}$. If $V \neq V^{00}$, then let $(u_i)_{i \in I \cup J}$ be a basis of V^{00} such that $(u_i)_{i \in I}$ is a basis of V (where $I \cap J = \emptyset$). Since $V \neq V^{00}$, $u_{j_0} \in V^{00}$ for some $j_0 \in J$ (and thus, $j_0 \notin I$). Since $u_{j_0} \in V^{00}$, u_{j_0} is orthogonal to every linear form in V^0. Now, we have $u_{j_0}^*(u_i) = 0$ for all $i \in I$, and thus $u_{j_0}^* \in V^0$. However, $u_{j_0}^*(u_{j_0}) = 1$, contradicting the fact that u_{j_0} is orthogonal to every linear form in V^0. Thus, $V = V^{00}$.

(c) Let $J = I - \{1, \ldots, m\}$. Every linear form $f^* \in V^0$ is orthogonal to every u_j, for $j \in J$, and thus, $f^*(u_j) = 0$, for all $j \in J$. For such a linear form $f^* \in V^0$, let

$$g^* = f^*(u_1)u_1^* + \cdots + f^*(u_m)u_m^*.$$

We have $g^*(u_i) = f^*(u_i)$, for every i, $1 \leq i \leq m$. Furthermore, by definition, g^* vanishes on all u_j, where $j \in J$. Thus, f^* and g^* agree on the basis $(u_i)_{i \in I}$ of E, and so, $g^* = f^*$. This shows that $(u_1^*, \ldots, u_m^*)$ generates V^0, and since it is also a linearly independent family, $(u_1^*, \ldots, u_m^*)$ is a basis of V^0. It is then obvious that $\dim(V^0) = \operatorname{codim}(V)$, and by part (b), we have $V^{00} = V$.

(d) Let $(u_1^*, \ldots, u_m^*)$ be a basis of U. Note that the map $h \colon E \to K^m$ defined such that

$$h(v) = (u_1^*(v), \ldots, u_m^*(v))$$

for every $v \in E$, is a linear map, and that its kernel $\operatorname{Ker} h$ is precisely U^0. Then, by Proposition 6.16,

$$E \approx \operatorname{Ker}(h) \oplus \operatorname{Im} h = U^0 \oplus \operatorname{Im} h,$$

and since $\dim(\operatorname{Im} h) \leq m$, we deduce that U^0 is a subspace of E of finite codimension at most m, and by (c), we have $\dim(U^{00}) = \operatorname{codim}(U^0) \leq m = \dim(U)$. However, it is clear that $U \subseteq U^{00}$, which implies $\dim(U) \leq \dim(U^{00})$, and so $\dim(U^{00}) = \dim(U) = m$, and we must have $U = U^{00}$. $\square$