which is a vector space, but now the problem is that $V(P)$ is not necessarily well defined!. For example, if $P(x,y,z) = -x^2 + 1$, we have

$$P(1,0,0) = 0 \quad \text{and} \quad P(2,0,0) = -3,$$

and yet $(2,0,0) = 2(1,0,0)$, so that $P(x,y,z)$ takes different values depending on the representative chosen in the equivalence class $[1,0,0]$. Thus, we are led to restrict ourselves to homogeneous polynomials. Actually, this is usually an advantage more than a disadvantage, because homogeneous polynomials tend to be well behaved.

What are the curves $V(P)$? One way to "see" such curves is to go back to the hyperplane model of $\mathbb{RP}^2$ in terms of the plane H of equation $z = 1$ in $\mathbb{R}^3$. Then the trace of $V(P)$ on H is the circle of equation

$$ax^2 + ay^2 + bx + cy + d = 0.$$

Thus, we may think of $\mathbf{P}(E)$ as a projective space of circles. However, there are some problems. For example, $V(P)$ may be empty! This happens, for instance, for $P(x,y,z) = x^2 + y^2 + z^2$, since the equation

$$x^2 + y^2 + z^2 = 0$$

has only the trivial solution $(0,0,0)$, which does not correspond to any point in $\mathbb{RP}^2$. Indeed, only nonnull vectors in $\mathbb{R}^3$ yield points in $\mathbb{RP}^2$. It is also possible that $V(P)$ is reduced to a single point, for instance when $P(x,y,z) = x^2 + y^2$, since the only homogeneous solution of

$$x^2 + y^2 = 0$$

is $(0,0,1)$. Also, note that the map

$$[P] \mapsto V(P)$$

is not injective. For instance, $P = x^2 + y^2$ and $Q = x^2 + 2y^2$ define the same degenerate circle reduced to the point $(0,0,1)$. We also accept as circles the union of two lines, as in the case

$$(bx + cy + dz)z = 0,$$

where $a = 0$, and even a double line, as in the case

$$z^2 = 0,$$

where $a = b = c = 0$.

A clean way to resolve most of these problems is to switch to homogeneous polynomials over the complex field $\mathbb{C}$ and to consider curves in $\mathbb{CP}^2$. This is what is done in algebraic geometry (see Fulton [66] or Harris [87]). If $P(x,y,z)$ is a homogeneous polynomial over $\mathbb{C}$ of degree 2 (plus the null polynomial), it is easy to show that $V(P)$ is always nonempty, and in fact infinite. It can also be shown that $V(P) = V(Q)$ implies that $Q = \lambda P$ for some $\lambda \in \mathbb{C}$, with $\lambda \neq 0$ (see Samuel [142], Section 1.6, Theorem 10). Another advantage of switching to