permutation $\sigma\colon [n] \to [n]$ such that, for some sequence $(i_1, i_2, \ldots, i_k)$ of distinct elements of $[n]$ with $2 \leq k \leq n$,

$$\sigma(i_1) = i_2, \sigma(i_2) = i_3, \ldots, \sigma(i_{k-1}) = i_k,\ \sigma(i_k) = i_1,$$

and $\sigma(j) = j$, for $j \in [n] - \{i_1, \ldots, i_k\}$. The set $\{i_1, \ldots, i_k\}$ is called the *domain* of the cyclic permutation, and the cyclic permutation is usually denoted by $(i_1\, i_2\, \ldots\, i_k)$.

If $\tau$ is a transposition, clearly, $\tau \circ \tau = \mathrm{id}$. Also, a cyclic permutation of order 2 is a transposition, and for a cyclic permutation $\sigma$ of order $k$, we have $\sigma^k = \mathrm{id}$. Clearly, the composition of two permutations is a permutation and every permutation has an inverse which is also a permutation. Therefore, the set of permutations on $[n]$ is a *group* often denoted $\mathfrak{S}_n$. It is easy to show by induction that the group $\mathfrak{S}_n$ has $n!$ elements. We will also use the terminology product of permutations (or transpositions), as a synonym for composition of permutations.

A permutation $\sigma$ on $n$ elements, say $\sigma(i) = k_i$ for $i = 1, \ldots, n$, can be represented in functional notation by the $2 \times n$ array

$$\begin{pmatrix} 1 & \cdots & i & \cdots & n \\ k_1 & \cdots & k_i & \cdots & k_n \end{pmatrix}$$

known as *Cauchy two-line notation*. For example, we have the permutation $\sigma$ denoted by

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 3 & 6 & 5 & 1 \end{pmatrix}.$$

A more concise notation often used in computer science and in combinatorics is to represent a permutation by its image, namely by the sequence

$$\sigma(1)\ \sigma(2)\ \cdots\ \sigma(n)$$

written as a row vector without commas separating the entries. The above is known as the *one-line notation*. For example, in the one-line notation, our previous permutation $\sigma$ is represented by

$$2\ 4\ 3\ 6\ 5\ 1.$$

The reason for not enclosing the above sequence within parentheses is avoid confusion with the notation for cycles, for which is it customary to include parentheses.

The following proposition shows the importance of cyclic permutations and transpositions.

**Proposition 7.1.** *For every $n \geq 2$, for every permutation $\pi\colon [n] \to [n]$, there is a partition of $[n]$ into $r$ subsets called the orbits of $\pi$, with $1 \leq r \leq n$, where each set $J$ in this partition is either a singleton $\{i\}$, or it is of the form*

$$J = \{i, \pi(i), \pi^2(i), \ldots, \pi^{r_i - 1}(i)\},$$