any two projective spaces $\mathbf{P}(E)$ and $\mathbf{P}(F)$ of the same dimension $n \geq 2$, for any bijective function $f\colon \mathbf{P}(E) \to \mathbf{P}(F)$, if f maps any three distinct collinear points a, b, c to collinear points $f(a), f(b), f(c)$, then f is a projectivity. For more general fields, $f = \mathbf{P}(g)$ for some "semilinear" bijection $g\colon E \to F$. A map such as f (preserving collinearity of any three distinct points) is often called a *collineation*. For $K = \mathbb{R}$, collineations and projectivities coincide. For more details, see Samuel [142].

Before closing this section, we illustrate the power of Proposition 26.5 by proving two interesting results. We begin by characterizing perspectivities between lines.

Proposition 26.6. *Given any two distinct lines D and D' in the real projective plane $\mathbb{RP}^2$, a projectivity $f\colon D \to D'$ is a perspectivity iff $f(O) = O$, where O is the intersection of D and D'.*

Proof. If $f\colon D \to D'$ is a perspectivity, then by the very definition of f, we have $f(O) = O$. Conversely, let $f\colon D \to D'$ be a projectivity such that $f(O) = O$. Let a, b be any two distinct points on D also distinct from O, and let $a' = f(a)$ and $b' = f(b)$ on D'. Since f is a bijection and since a, b, O are pairwise distinct, $a' \neq b'$. Let c be the intersection of the lines $\langle a, a' \rangle$ and $\langle b, b' \rangle$, which by the assumptions on a, b, O, cannot be on D or D'. Then we can define the perspectivity $g\colon D \to D'$ of center c, and by the definition of c, we have

$$g(a) = a', \quad g(b) = b', \quad g(O) = O.$$

See Figure 26.12. However, f agrees with g on O, a, b, and since (O, a, b) is a projective frame for D, by Proposition 26.5, we must have $f = g$. $\square$

Using Proposition 26.6, we can give an elegant proof of a version of Desargues's theorem (in the plane).

Proposition 26.7. *(Desargues) Given two triangles (a, b, c) and (a', b', c') in $\mathbb{RP}^2$, where the points a, b, c, a', b', c' are pairwise distinct and the lines $A = \langle b, c \rangle$, $B = \langle a, c \rangle$, $C = \langle a, b \rangle$, $A' = \langle b', c' \rangle$, $B' = \langle a', c' \rangle$, $C' = \langle a', b' \rangle$ are pairwise distinct, if the lines $\langle a, a' \rangle$, $\langle b, b' \rangle$, and $\langle c, c' \rangle$ intersect in a common point d distinct from a, b, c, a', b', c', then the intersection points $p = \langle b, c \rangle \cap \langle b', c' \rangle$, $q = \langle a, c \rangle \cap \langle a', c' \rangle$, and $r = \langle a, b \rangle \cap \langle a', b' \rangle$ belong to a common line distinct from A, B, C, A', B', C'.*

Proof. In view of the assumptions on a, b, c, a', b', c', and d, the point r is on neither $\langle a, a' \rangle$ nor $\langle b, b' \rangle$, the point p is on neither $\langle b, b' \rangle$ nor $\langle c, c' \rangle$, and the point q is on neither $\langle a, a' \rangle$ nor $\langle c, c' \rangle$. It is also immediately shown that the line $\langle p, q \rangle$ is distinct from the lines A, B, C, A', B', C'. Let $f\colon \langle a, a' \rangle \to \langle b, b' \rangle$ be the perspectivity of center r and $g\colon \langle b, b' \rangle \to \langle c, c' \rangle$ be the perspectivity of center p. Let $h = g \circ f$. Since both $f(d) = d$ and $g(d) = d$, we also have