- The matrix of the transpose map $f^\top$ is equal to the transpose of the matrix of the map f (Proposition 11.14).
- For any $m \times n$ matrix A,
$$\mathrm{rk}(A) = \mathrm{rk}(A^\top).$$
- Characterization of the rank of a matrix in terms of a maximal invertible submatrix (Proposition 11.16).
- The *four fundamental subspaces*:
$$\mathrm{Im}\, f,\ \mathrm{Im}\, f^\top,\ \mathrm{Ker}\, f,\ \mathrm{Ker}\, f^\top.$$
- The *column space*, the *nullspace*, the *row space*, and the *left nullspace* (of a matrix).
- Criterion for the solvability of an equation of the form $Ax = b$ in terms of the left nullspace.

11.10 Problems

Problem 11.1. Prove the following properties of transposition:

$$\begin{aligned}(f+g)^\top &= f^\top + g^\top\\ (g \circ f)^\top &= f^\top \circ g^\top\\ \mathrm{id}_E^\top &= \mathrm{id}_{E^*}.\end{aligned}$$

Problem 11.2. Let $(u_1, \ldots, u_{n-1})$ be $n-1$ linearly independent vectors $u_i \in \mathbb{C}^n$. Prove that the hyperlane H spanned by $(u_1, \ldots, u_{n-1})$ is the nullspace of the linear form

$$x \mapsto \det(u_1, \ldots, u_{n-1}, x), \quad x \in \mathbb{C}^n.$$

Prove that if A is the $n \times n$ matrix whose columns are $(u_1, \ldots, u_{n-1}, x)$, and if $c_i = (-1)^{i+n} \det(A_{in})$ is the cofactor of $a_{in} = x_i$ for $i = 1, \ldots, n$, then H is defined by the equation

$$c_1 x_1 + \cdots + c_n x_n = 0.$$

Problem 11.3. (1) Let $\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the map defined by

$$\varphi((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = x_1 y_1 + \cdots + x_n y_n.$$

Prove that φ is a bilinear nondegenerate pairing. Deduce that $(\mathbb{R}^n)^*$ is isomorphic to $\mathbb{R}^n$.

Prove that $\varphi(x, x) = 0$ iff $x = 0$.