that is, in the basis $(e_1^*, \ldots, e_n^*)$, the inner product on $E^*$ is represented by the matrix $(g^{ij})$, the inverse of the matrix $(g_{ij})$.

The inner product on a finite vector space also yields a canonical isomorphism between the space $\mathrm{Hom}(E, E; K)$ of bilinear forms on $E$, and the space $\mathrm{Hom}(E, E)$ of linear maps from $E$ to itself. Using this isomorphism, we can define the trace of a bilinear form in an intrinsic manner. This technique is used in differential geometry, for example, to define the divergence of a differential one-form.

**Proposition 33.3.** *If $\langle -, - \rangle$ is an inner product on a finite vector space $E$ (over a field, $K$), then for every bilinear form $f\colon E \times E \to K$, there is a unique linear map $f^\natural\colon E \to E$ such that*

$$f(u, v) = \langle f^\natural(u), v \rangle, \quad \text{for all } u, v \in E.$$

*The map $f \mapsto f^\natural$ is a linear isomorphism between* $\mathrm{Hom}(E, E; K)$ *and* $\mathrm{Hom}(E, E)$.

*Proof.* For every $g \in \mathrm{Hom}(E, E)$, the map given by

$$f(u, v) = \langle g(u), v \rangle, \quad u, v \in E,$$

is clearly bilinear. It is also clear that the above defines a linear map from $\mathrm{Hom}(E, E)$ to $\mathrm{Hom}(E, E; K)$. This map is injective, because if $f(u, v) = 0$ for all $u, v \in E$, as $\langle -, - \rangle$ is an inner product, we get $g(u) = 0$ for all $u \in E$. Furthermore, both spaces $\mathrm{Hom}(E, E)$ and $\mathrm{Hom}(E, E; K)$ have the same dimension, so our linear map is an isomorphism. $\square$

If $(e_1, \ldots, e_n)$ is an orthonormal basis of $E$, then we check immediately that the trace of a linear map $g$ (which is independent of the choice of a basis) is given by

$$\mathrm{tr}(g) = \sum_{i=1}^{n} \langle g(e_i), e_i \rangle,$$

where $n = \dim(E)$.

**Definition 33.2.** We define the *trace of the bilinear form $f$* by

$$\mathrm{tr}(f) = \mathrm{tr}(f^\natural).$$

From Proposition 33.3, $\mathrm{tr}(f)$ is given by

$$\mathrm{tr}(f) = \sum_{i=1}^{n} f(e_i, e_i),$$

for any orthonormal basis $(e_1, \ldots, e_n)$ of $E$. We can also check directly that the above expression is independent of the choice of an orthonormal basis.