affine patch), there is a unique projective map $\widetilde{f}\colon \mathbf{P}(\mathcal{E}) \to \mathbf{P}(F)$ such that

$$f = \widetilde{f} \circ i \quad \text{and} \quad \mathbf{P}\big(\overrightarrow{f}\big) = \widetilde{f} \circ \mathbf{P}(\overrightarrow{i})$$

(where $\overrightarrow{i} : \overrightarrow{E} \to \mathcal{H}$ and $\overrightarrow{f} : \overrightarrow{E} \to H$ are the linear maps associated with the affine maps $i\colon E \to \mathbf{P}(\mathcal{E})$ and $f\colon E \to \mathbf{P}(F)$), as in the following diagram:

$$\begin{tikzcd}
E \arrow[r, "i"] \arrow[dr, "f"'] & \mathcal{E}_{\mathcal{H}} \subseteq \mathbf{P}(\mathcal{E}) \supseteq \mathbf{P}(\mathcal{H}) \arrow[d, "\widetilde{f}"] & \mathbf{P}\big(\overrightarrow{E}\big) \arrow[l, "\mathbf{P}(\overrightarrow{i})"'] \arrow[dl, "\mathbf{P}\big(\overrightarrow{f}\big)"] \\
 & F_H \subseteq \mathbf{P}(F) \supseteq \mathbf{P}(H) &
\end{tikzcd}$$

The points of $\mathbf{P}(\mathcal{E})$ in $\mathbf{P}(\mathcal{H})$ are called *points at infinity*, and the projective hyperplane $\mathbf{P}(\mathcal{H})$ is called the *hyperplane at infinity*. We will also denote the point $[u]_\sim$ of $\mathbf{P}(\mathcal{H})$ (where $u \neq 0$) by u_∞. As usual, objects defined by a universal property are unique up to isomorphism. We leave the proof as an exercise.

The importance of the notion of projective completion stems from the fact that every affine map $f\colon E \to F$ extends *in a unique way* to a projective map $\widetilde{f}\colon \mathbf{P}(\mathcal{E}) \to \mathbf{P}(\mathcal{F})$, where $\langle \mathbf{P}(\mathcal{E}), \mathbf{P}(\mathcal{H}_E), i_E\rangle$ is a projective completion of E and $\langle \mathbf{P}(\mathcal{F}), \mathbf{P}(\mathcal{H}_F), i_F\rangle$ is a projective completion of F, provided that the restriction of $\widetilde{f}$ to $\mathbf{P}\big(\overrightarrow{E}\big)$ agrees with $\mathbf{P}\big(\overrightarrow{f}\big)$, as illustrated in the following commutative diagram:

$$\begin{tikzcd}
E \arrow[r, "f"] \arrow[d, "i_E"'] & F \arrow[d, "i_F"] \\
\mathbf{P}(\mathcal{E}) \arrow[r, "\widetilde{f}"'] & \mathbf{P}(\mathcal{F}).
\end{tikzcd}$$

We will now show that $\big\langle \widetilde{E}, \mathbf{P}\big(\overrightarrow{E}\big), i\big\rangle$ is the projective completion of E, where $i\colon E \to \widetilde{E}$ is the injection of E into $\widetilde{E} = E \cup \mathbf{P}\big(\overrightarrow{E}\big)$. For example, if $E = \mathbb{A}^1_K$ is an affine line, its projective completion $\widetilde{\mathbb{A}^1_K}$ is isomorphic to the projective line $\mathbf{P}(K^2)$, and they both can be identified with $\mathbb{A}^1_K \cup \{\infty\}$, the result of adding a point at infinity (∞) to $\mathbb{A}^1_K$. In general, the projective completion $\widetilde{\mathbb{A}^m_K}$ of the affine space $\mathbb{A}^m_K$ is isomorphic to $\mathbf{P}(K^{m+1})$. Thus, $\widetilde{\mathbb{A}^m}$ is isomorphic to $\mathbb{RP}^m$, and $\widetilde{\mathbb{A}^m_{\mathbb{C}}}$ is isomorphic to $\mathbb{CP}^m$.

First, let us observe that if E is a vector space and H is a hyperplane in E, then the homogenization $\widehat{E_H}$ of the affine patch E_H (the complement of the projective hyperplane $\mathbf{P}(H)$ in $\mathbf{P}(E)$) is isomorphic to E. The proof is rather simple and uses the fact that there