Recalling that $e = b - B\lambda$, since

$$G(\lambda) = \frac{1}{2}(B\lambda - b)^\top A(B\lambda - b) + \lambda^\top f,$$

we can also write

$$G(\lambda) = \frac{1}{2}e^\top Ae + \lambda^\top f.$$

This expression often represents the total potential energy of a system. Again, the optimal solution is the one that minimizes the potential energy (and thus maximizes $-G(\lambda)$).

(2) It is immediately verified that the equations of Proposition 42.3 are equivalent to the equations stating that the partial derivatives of the Lagrangian $L(x, \lambda)$ are null:

$$\begin{aligned} \frac{\partial L}{\partial x_i} &= 0, \quad i = 1, \ldots, m, \\ \frac{\partial L}{\partial \lambda_j} &= 0, \quad j = 1, \ldots, n. \end{aligned}$$

Thus, the constrained minimum of $Q(x)$ subject to $B^\top x = f$ is an extremum of the Lagrangian $L(x, \lambda)$. As we showed in Proposition 42.3, this extremum corresponds to simultaneously minimizing $L(x, \lambda)$ with respect to x and maximizing $L(x, \lambda)$ with respect to λ. Geometrically, such a point is a *saddle point* for $L(x, \lambda)$. Saddle points are discussed in Section 50.7.

(3) The Lagrange multipliers sometimes have a natural physical meaning. For example, in the spring-mass system they correspond to node displacements. In some general sense, Lagrange multipliers are correction terms needed to satisfy equilibrium equations and the price paid for the constraints. For more details, see Strang [169].

Going back to the constrained minimization of $Q(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$ subject to

$$2x_1 - x_2 = 5,$$

the Lagrangian is

$$L(x_1, x_2, \lambda) = \frac{1}{2}\left(x_1^2 + x_2^2\right) + \lambda(2x_1 - x_2 - 5),$$

and the equations stating that the Lagrangian has a saddle point are

$$\begin{aligned} x_1 + 2\lambda &= 0, \\ x_2 - \lambda &= 0, \\ 2x_1 - x_2 - 5 &= 0. \end{aligned}$$

We obtain the solution $(x_1, x_2, \lambda) = (2, -1, -1)$.