Part (a) of Theorem 11.4 shows that

$$\dim(E) \leq \dim(E^*).$$

When E is of finite dimension n and  $(u_1, \ldots, u_n)$  is a basis of E, by part (c), the family  $(u_1^*, \ldots, u_n^*)$  is a basis of the dual space  $E^*$ , called the *dual basis* of  $(u_1, \ldots, u_n)$ . This fact was also proven directly in Theorem 3.23.

Define the function  $\mathcal{E}$  ( $\mathcal{E}$  for equations) from subspaces of E to subspaces of  $E^*$  and the function  $\mathcal{Z}$  ( $\mathcal{Z}$  for zeros) from subspaces of  $E^*$  to subspaces of E by

$$\mathcal{E}(V) = V^0, \quad V \subseteq E$$
  
 $\mathcal{Z}(U) = U^0, \quad U \subseteq E^*.$ 

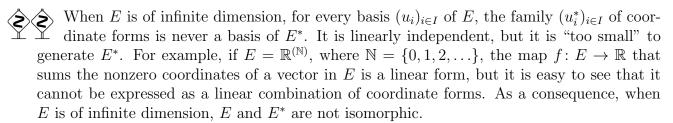
By Parts (c) and (d) of Theorem 11.4,

$$(\mathcal{Z} \circ \mathcal{E})(V) = V^{00} = V$$
$$(\mathcal{E} \circ \mathcal{Z})(U) = U^{00} = U,$$

so  $\mathcal{Z} \circ \mathcal{E} = \operatorname{id}$  and  $\mathcal{E} \circ \mathcal{Z} = \operatorname{id}$ , and the maps  $\mathcal{E}$  and  $\mathcal{Z}$  are inverse bijections. These maps set up a *duality* between subspaces of E and subspaces of  $E^*$ . In particular, every subspace  $V \subseteq E$  of dimension m is the set of common zeros of the space of linear forms (equations)  $V^0$ , which has dimension n-m. This confirms the claim we made about the dimension of the subspace defined by a set of linear equations.



One should be careful that this bijection does not extend to subspaces of  $E^*$  of infinite dimension.



We now discuss some applications of the duality theorem.

**Problem 1**. Suppose that V is a subspace of  $\mathbb{R}^n$  of dimension m and that  $(v_1, \ldots, v_m)$  is a basis of V. The problem is to find a basis of  $V^0$ .

We first extend  $(v_1, \ldots, v_m)$  to a basis  $(v_1, \ldots, v_n)$  of  $\mathbb{R}^n$ , and then by part (c) of Theorem 11.4, we know that  $(v_{m+1}^*, \ldots, v_n^*)$  is a basis of  $V^0$ .

**Example 11.6.** For example, suppose that V is the subspace of  $\mathbb{R}^4$  spanned by the two linearly independent vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix},$$