can also be used to determine the rank of a matrix when other methods such as Gaussian elimination produce very small pivots. One of the main applications of the SVD is the computation of the pseudo-inverse. Pseudo-inverses are the key to the solution of various optimization problems, in particular the method of least squares. This topic is discussed in the next chapter (Chapter 23). Applications of the material of this chapter can be found in Strang [170, 169]; Ciarlet [41]; Golub and Van Loan [80], which contains many other references; Demmel [48]; and Trefethen and Bau [176].

22.5 Ky Fan Norms and Schatten Norms

The singular values of a matrix can be used to define various norms on matrices which have found recent applications in quantum information theory and in spectral graph theory. Following Horn and Johnson [96] (Section 3.4) we can make the following definitions:

Definition 22.5. For any matrix $A \in \mathrm{M}_{m,n}(\mathbb{C})$, let $q = \min\{m, n\}$, and if $\sigma_1 \geq \cdots \geq \sigma_q$ are the singular values of A, for any k with $1 \leq k \leq q$, let

$$N_k(A) = \sigma_1 + \cdots + \sigma_k,$$

called the *Ky Fan k-norm* of A.

More generally, for any $p \geq 1$ and any k with $1 \leq k \leq q$, let

$$N_{k;p}(A) = (\sigma_1^p + \cdots + \sigma_k^p)^{1/p},$$

called the *Ky Fan p-k-norm* of A. When $k = q$, $N_{q;p}$ is also called the *Schatten p-norm*.

Observe that when $k = 1$, $N_1(A) = \sigma_1$, and the Ky Fan norm N_1 is simply the *spectral norm* from Chapter 9, which is the subordinate matrix norm associated with the Euclidean norm. When $k = q$, the Ky Fan norm N_q is given by

$$N_q(A) = \sigma_1 + \cdots + \sigma_q = \mathrm{tr}((A^* A)^{1/2})$$

and is called the *trace norm* or *nuclear norm*. When $p = 2$ and $k = q$, the Ky Fan $N_{q;2}$ norm is given by

$$N_{k;2}(A) = (\sigma_1^2 + \cdots + \sigma_q^2)^{1/2} = \sqrt{\mathrm{tr}(A^* A)} = \|A\|_F\,,$$

which is the *Frobenius norm* of A.

It can be shown that N_k and $N_{k;p}$ are unitarily invariant norms, and that when $m = n$, they are matrix norms; see Horn and Johnson [96] (Section 3.4, Corollary 3.4.4 and Problem 3).