where $2 \leq k \leq n$.

(1) Prove the following properties:

(i) $P_k(x)$ is the characteristic polynomial of $A_k$, where $1 \leq k \leq n$.

(ii) $\lim_{x \to -\infty} P_k(x) = +\infty$, where $1 \leq k \leq n$.

(iii) If $P_k(x) = 0$, then $P_{k-1}(x)P_{k+1}(x) < 0$, where $1 \leq k \leq n-1$.

(iv) $P_k(x)$ has $k$ distinct real roots that separate the $k+1$ roots of $P_{k+1}(x)$, where $1 \leq k \leq n-1$.

(2) Given any real number $\mu > 0$, for every $k$, $1 \leq k \leq n$, define the function $sg_k(\mu)$ as follows:

$$sg_k(\mu) = \begin{cases} \text{sign of } P_k(\mu) & \text{if } P_k(\mu) \neq 0, \\ \text{sign of } P_{k-1}(\mu) & \text{if } P_k(\mu) = 0. \end{cases}$$

We encode the sign of a positive number as $+$, and the sign of a negative number as $-$. Then let $E(k, \mu)$ be the ordered list

$$E(k, \mu) = \langle +,\, sg_1(\mu),\, sg_2(\mu),\, \ldots,\, sg_k(\mu) \rangle\,,$$

and let $N(k, \mu)$ be the number changes of sign between consecutive signs in $E(k, \mu)$.

Prove that $sg_k(\mu)$ is well defined and that $N(k, \mu)$ is the number of roots $\lambda$ of $P_k(x)$ such that $\lambda < \mu$.

**Remark:** The above can be used to compute the eigenvalues of a (tridiagonal) symmetric matrix (the method of Givens-Householder).