and use this to conclude that if $U^2 = S$, then $b^2 + c^2 + d^2 = 1$. Then show that

$$\exp^{-1}(R) = \left\{ (2k+1)\pi \begin{pmatrix} 0 & -d & c \\ d & 0 & -b \\ -c & b & 0 \end{pmatrix},\ k \in \mathbb{Z} \right\},$$

where (b, c, d) is any unit vector such that for the corresponding skew symmetric matrix U, we have $U^2 = S$.

(4) To find a skew symmetric matrix U so that $U^2 = S = \frac{1}{2}(R - I)$ as in (3), we can solve the system

$$\begin{pmatrix} b^2 - 1 & bc & bd \\ bc & c^2 - 1 & cd \\ bd & cd & d^2 - 1 \end{pmatrix} = S.$$

We immediately get b^2, c^2, d^2, and then, since one of b, c, d is nonzero, say b, if we choose the positive square root of b^2, we can determine c and d from bc and bd.

Implement a computer program in `Matlab` to solve the above system.

Problem 17.9. It was shown in Proposition 15.15 that the exponential map is a map $\exp\colon \mathfrak{so}(n) \to \mathbf{SO}(n)$, where $\mathfrak{so}(n)$ is the vector space of real $n \times n$ skew-symmetric matrices. Use the spectral theorem to prove that the map $\exp\colon \mathfrak{so}(n) \to \mathbf{SO}(n)$ is surjective.

Problem 17.10. Let $\mathfrak{u}(n)$ be the space of (complex) $n \times n$ skew-Hermitian matrices ($B^* = -B$) and let $\mathfrak{su}(n)$ be its subspace consisting of skew-Hermitian matrice with zero trace ($\mathrm{tr}(B) = 0$).

(1) Prove that if $B \in \mathfrak{u}(n)$, then $e^B \in \mathbf{U}(n)$, and if if $B \in \mathfrak{su}(n)$, then $e^B \in \mathbf{SU}(n)$. Thus we have well-defined maps $\exp\colon \mathfrak{u}(n) \to \mathbf{U}(n)$ and $\exp\colon \mathfrak{su}(n) \to \mathbf{SU}(n)$.

(2) Prove that the map $\exp\colon \mathfrak{u}(n) \to \mathbf{U}(n)$ is surjective.

(3) Prove that the map $\exp\colon \mathfrak{su}(n) \to \mathbf{SU}(n)$ is surjective.

Problem 17.11. Recall that a matrix $B \in \mathrm{M}_n(\mathbb{R})$ is skew-symmetric if $B^\top = -B$. Check that the set $\mathfrak{so}(n)$ of skew-symmetric matrices is a vector space of dimension $n(n-1)/2$, and thus is isomorphic to $\mathbb{R}^{n(n-1)/2}$.

(1) Given a rotation matrix

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

where $0 < \theta < \pi$, prove that there is a skew symmetric matrix B such that

$$R = (I - B)(I + B)^{-1}.$$