with the space $E^* = \mathrm{Hom}(E, \mathbb{R})$ of all linear maps from $E$ to $\mathbb{R}$. A continuous bilinear map $\varphi\colon E \times E \to \mathbb{R}$ in $\mathcal{L}_2(E, E; \mathbb{R})$ yields a map $\Phi$ from $E$ to $E'$ given by

$$\Phi(u) = \varphi_u,$$

where $\varphi_u \in E'$ is the linear form defined by

$$\varphi_u(v) = \varphi(u, v).$$

It is easy to check that $\varphi_u$ is continuous and that the map $\Phi$ is continuous. Then we say that $\varphi$ is *nondegenerate* iff $\Phi\colon E \to E'$ is an isomorphism of Banach spaces, which means that $\Phi$ is invertible and that both $\Phi$ and $\Phi^{-1}$ are continuous linear maps. Given a function $J\colon \Omega \to \mathbb{R}$ differentiable on $\Omega$ as before (where $\Omega$ is an open subset of $E$), if $\mathrm{D}^2 J(u)$ exists for some $u \in \Omega$, we say that $u$ is a *nondegenerate critical point* if $dJ(u) = 0$ and if $\mathrm{D}^2 J(u)$ is nondegenerate. Of course, $\mathrm{D}^2 J(u)$ is positive definite if $\mathrm{D}^2 J(u)(w, w) > 0$ for all $w \in E - \{0\}$.

Using the above definition, Proposition 40.7 can be generalized to a nondegenerate positive definite bilinear form (on a Banach space) and Theorem 40.8 can also be generalized to the situation where $J\colon \Omega \to \mathbb{R}$ is defined on an open subset of a Banach space. For details and proofs, see Cartan [34] (Part I Chapter 8) and Avez [9] (Chapter 8 and Chapter 10).

In the next section we make use of convexity; both on the domain $\Omega$ and on the function $J$ itself.

## 40.3 Using Convexity to Find Extrema

We begin by reviewing the definition of a convex set and of a convex function.

**Definition 40.7.** Given any real vector space $E$, we say that a subset $C$ of $E$ is *convex* if either $C = \emptyset$ or if for every pair of points $u, v \in C$, the line segment connecting $u$ and $v$ is contained in $C$, i.e.,

$$(1 - \lambda)u + \lambda v \in C \quad \text{for all } \lambda \in \mathbb{R} \text{ such that } 0 \leq \lambda \leq 1.$$

Given any two points $u, v \in E$, the *line segment* $[u, v]$ is the set

$$[u, v] = \{(1 - \lambda)u + \lambda v \in E \mid \lambda \in \mathbb{R},\ 0 \leq \lambda \leq 1\}.$$

Clearly, a nonempty set $C$ is convex iff $[u, v] \subseteq C$ whenever $u, v \in C$. See Figure 40.4 for an example of a convex set.

**Definition 40.8.** If $C$ is a nonempty convex subset of $E$, a function $f\colon C \to \mathbb{R}$ is *convex* (on $C$) if for every pair of points $u, v \in C$,

$$f((1 - \lambda)u + \lambda v) \leq (1 - \lambda)f(u) + \lambda f(v) \quad \text{for all } \lambda \in \mathbb{R} \text{ such that } 0 \leq \lambda \leq 1;$$