with $\sigma(d) < \sigma(a_{11})$. Then, go back to Step 2a.

In Step 2b, if a_{11} does not divide a_{1k}, then first permute column 2 and column k (if $k \neq 2$). Then, if we write $a = a_{11}$ and $b = a_{1k}$, if d is a gcd of a and b and if x, y, s, t are determined as explained above, multiply on the right by the matrix

$$\begin{pmatrix} x & s & 0 & 0 & \cdots & 0 \\ y & t & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 \end{pmatrix}$$

to obtain a matrix of the form

$$\begin{pmatrix} d & 0 & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn} \end{pmatrix}$$

with $\sigma(d) < \sigma(a_{11})$. Then, go back to Step 2b. The other steps remain the same. Whenever we return to Step 2a or Step 2b, the σ-value of the $(1, 1)$-entry strictly decreases, so the whole procedure terminates. $\square$

We conclude this section by explaining how the rational canonical form of a matrix A can be obtained from the canonical form QDP^{-1} of $XI - A$.

Let $f \colon E \to E$ be a linear map over a K-vector space of dimension n. Recall from Theorem 36.3 (see Section 36.1) that as a $K[X]$-module, E_f is the image of the free module $E[X]$ by the map $\sigma \colon E[X] \to E_f$, where $E[X]$ consists of all linear combinations of the form

$$p_1e_1 + \cdots + p_ne_n,$$

where $(e_1, \ldots, e_n)$ is a basis of E and $p_1, \ldots, p_n \in K[X]$ are polynomials, and σ is given by

$$\sigma(p_1e_1 + \cdots + p_ne_n) = p_1(f)(e_1) + \cdots + p_n(f)(e_n).$$

Furthermore, the kernel of σ is equal to the image of the map $\psi \colon E[X] \to E[X]$, where

$$\psi(p_1e_1 + \cdots + p_ne_n) = Xp_1e_1 + \cdots + Xp_ne_n - (p_1f(e_1) + \cdots + p_n(e_n)).$$

The matrix A is the representation of a linear map f over the canonical basis $(e_1, \ldots, e_n)$ of $E = K^n$, and and $XI - A$ is the matrix of ψ with respect to the basis $(e_1, \ldots, e_n)$