is expressed in matrix form by

$$y_{\mathcal{V}} = M_{\mathcal{U},\mathcal{V}}(f)\, x_{\mathcal{U}}.$$

When $\mathcal{U} = \mathcal{V}$, we abbreviate $M_{\mathcal{U},\mathcal{V}}(f)$ as $M_{\mathcal{U}}(f)$.

The above notation seems reasonable, but it has the slight disadvantage that in the expression $M_{\mathcal{U},\mathcal{V}}(f)x_{\mathcal{U}}$, the input argument $x_{\mathcal{U}}$ which is fed to the matrix $M_{\mathcal{U},\mathcal{V}}(f)$ does not appear next to the subscript $\mathcal{U}$ in $M_{\mathcal{U},\mathcal{V}}(f)$. We could have used the notation $M_{\mathcal{V},\mathcal{U}}(f)$, and some people do that. But then, we find a bit confusing that $\mathcal{V}$ comes before $\mathcal{U}$ when f maps from the space E with the basis $\mathcal{U}$ to the space F with the basis $\mathcal{V}$. So, we prefer to use the notation $M_{\mathcal{U},\mathcal{V}}(f)$.

Be aware that other authors such as Meyer [125] use the notation $[f]_{\mathcal{U},\mathcal{V}}$, and others such as Dummit and Foote [54] use the notation $M_{\mathcal{U}}^{\mathcal{V}}(f)$, instead of $M_{\mathcal{U},\mathcal{V}}(f)$. This gets worse! You may find the notation $M_{\mathcal{V}}^{\mathcal{U}}(f)$ (as in Lang [109]), or ${}_{\mathcal{U}}[f]_{\mathcal{V}}$, or other strange notations.

Definition 4.2 shows that the function which associates to a linear map $f\colon E \to F$ the matrix $M(f)$ w.r.t. the bases $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_m)$ has the property that matrix multiplication corresponds to composition of linear maps. This allows us to transfer properties of linear maps to matrices. Here is an illustration of this technique:

Proposition 4.1. *(1) Given any matrices $A \in \mathrm{M}_{m,n}(K)$, $B \in \mathrm{M}_{n,p}(K)$, and $C \in \mathrm{M}_{p,q}(K)$, we have*

$$(AB)C = A(BC);$$

that is, matrix multiplication is associative.

(2) Given any matrices $A, B \in \mathrm{M}_{m,n}(K)$, and $C, D \in \mathrm{M}_{n,p}(K)$, for all $\lambda \in K$, we have

$$\begin{aligned}(A + B)C &= AC + BC\\ A(C + D) &= AC + AD\\ (\lambda A)C &= \lambda(AC)\\ A(\lambda C) &= \lambda(AC),\end{aligned}$$

so that matrix multiplication $\cdot\colon \mathrm{M}_{m,n}(K) \times \mathrm{M}_{n,p}(K) \to \mathrm{M}_{m,p}(K)$ is bilinear.

Proof. (1) Every $m \times n$ matrix $A = (a_{i\,j})$ defines the function $f_A\colon K^n \to K^m$ given by

$$f_A(x) = Ax,$$

for all $x \in K^n$. It is immediately verified that f_A is linear and that the matrix $M(f_A)$ representing f_A over the canonical bases in K^n and K^m is equal to A. Then Formula (4) proves that

$$M(f_A \circ f_B) = M(f_A)M(f_B) = AB,$$

so we get

$$M((f_A \circ f_B) \circ f_C) = M(f_A \circ f_B)M(f_C) = (AB)C$$