The proof that if the set $\{cx \in \mathbb{R} \mid x \in \mathcal{P}(A, b)\}$ is nonempty and bounded above, then there is an optimal solution $p \in \mathcal{P}(A, b)$, is not as trivial as it might seem. It relies on the fact that a polyhedral cone is closed, a fact that was shown in Section 44.3.

We also use a trick that makes the proof simpler, which is that a Linear Program (P) with inequality constraints $Ax \leq b$

$$\begin{aligned} &\text{maximize} \quad cx \\ &\text{subject to} \quad Ax \leq b \text{ and } x \geq 0, \end{aligned}$$

is equivalent to the Linear Program (P_2) with equality constraints

$$\begin{aligned} &\text{maximize} \quad \widehat{c}\,\widehat{x} \\ &\text{subject to} \quad \widehat{A}\widehat{x} = b \text{ and } \widehat{x} \geq 0, \end{aligned}$$

where $\widehat{A}$ is an $m \times (n+m)$ matrix, $\widehat{c}$ is a linear form in $(\mathbb{R}^{n+m})^*$, and $\widehat{x} \in \mathbb{R}^{n+m}$, given by

$$\widehat{A} = \begin{pmatrix} A & I_m \end{pmatrix}, \quad \widehat{c} = \begin{pmatrix} c & 0_m^\top \end{pmatrix}, \quad \text{and} \quad \widehat{x} = \begin{pmatrix} x \\ z \end{pmatrix},$$

with $x \in \mathbb{R}^n$ and $z \in \mathbb{R}^m$.

Indeed, $\widehat{A}\widehat{x} = b$ and $\widehat{x} \geq 0$ iff

$$Ax + z = b, \quad x \geq 0,\ z \geq 0,$$

iff

$$Ax \leq b, \quad x \geq 0,$$

and $\widehat{c}\,\widehat{x} = cx$.

Definition 45.5. The variables z are called *slack variables*, and a linear program of the form (P_2) is called a linear program in *standard form*.

The result of converting the linear program of Example 45.4 to standard form is the program shown in Example 45.5.

Example 45.5.

$$\begin{aligned} &\text{maximize} \quad \frac{1}{6}x_1 + x_2 \\ &\text{subject to} \\ &\qquad x_2 - x_1 + z_1 = 1 \\ &\qquad x_1 + 6x_2 + z_2 = 15 \\ &\qquad 4x_1 - x_2 + z_3 = 10 \\ &\qquad x_1 \geq 0,\ x_2 \geq 0,\ z_1 \geq 0,\ z_2 \geq 0,\ z_3 \geq 0. \end{aligned}$$