Proposition 11.19. *If* $\dim(E)$ *is finite, then we have*

$$\mathrm{Ker}\,(f^{\top\top}) = \mathrm{eval}_E(\mathrm{Ker}\,(f)).$$

Proof. Indeed, if E is finite-dimensional, the map $\mathrm{eval}_E\colon E \to E^{**}$ is an isomorphism, so every $\varphi \in E^{**}$ is of the form $\varphi = \mathrm{eval}_E(u)$ for some $u \in E$, the map $\mathrm{eval}_F\colon F \to F^{**}$ is injective, and we have

$$\begin{aligned} f^{\top\top}(\varphi) = 0 \quad & \text{iff} \quad f^{\top\top}(\mathrm{eval}_E(u)) = 0 \\ & \text{iff} \quad \mathrm{eval}_F(f(u)) = 0 \\ & \text{iff} \quad f(u) = 0 \\ & \text{iff} \quad u \in \mathrm{Ker}\,(f) \\ & \text{iff} \quad \varphi \in \mathrm{eval}_E(\mathrm{Ker}\,(f)), \end{aligned}$$

which proves that $\mathrm{Ker}\,(f^{\top\top}) = \mathrm{eval}_E(\mathrm{Ker}\,(f))$. □

Remarks: If $\dim(E)$ is finite, following an argument of Dan Guralnik, the fact that $\mathrm{rk}(f) = \mathrm{rk}(f^{\top})$ can be proven using Proposition 11.19.

Proof. We know from Proposition 11.11 applied to $f^{\top}\colon F^* \to E^*$ that

$$\mathrm{Ker}\,(f^{\top\top}) = (\mathrm{Im}\, f^{\top})^0,$$

and we showed in Proposition 11.19 that

$$\mathrm{Ker}\,(f^{\top\top}) = \mathrm{eval}_E(\mathrm{Ker}\,(f)).$$

It follows (since eval_E is an isomorphism) that

$$\dim((\mathrm{Im}\, f^{\top})^0) = \dim(\mathrm{Ker}\,(f^{\top\top})) = \dim(\mathrm{Ker}\,(f)) = \dim(E) - \dim(\mathrm{Im}\, f),$$

and since

$$\dim(\mathrm{Im}\, f^{\top}) + \dim((\mathrm{Im}\, f^{\top})^0) = \dim(E),$$

we get

$$\dim(\mathrm{Im}\, f^{\top}) = \dim(\mathrm{Im}\, f).$$ □

As indicated by Dan Guralnik, if $\dim(E)$ is finite, the above result can be used to prove the following result.

Proposition 11.20. *If* $\dim(E)$ *is finite, then for any linear map* $f\colon E \to F$*, we have*

$$\mathrm{Im}\, f^{\top} = (\mathrm{Ker}\,(f))^0.$$