for some $\lambda_j \in \mathbb{R}$ and some finite subset J of I. By taking the inner product with u_i for any $i \in J$, and using the the bilinearity of the inner product and the fact that $u_i \cdot u_j = 0$ whenever $i \neq j$, we get

$$\begin{aligned} 0 = u_i \cdot 0 = u_i \cdot \left(\sum_{j \in J} \lambda_j u_j \right) \\ = \sum_{j \in J} \lambda_j (u_i \cdot u_j) = \lambda_i (u_i \cdot u_i), \end{aligned}$$

so

$$\lambda_i (u_i \cdot u_i) = 0, \qquad \text{for all } i \in J,$$

and since $u_i \neq 0$ and an inner product is positive definite, $u_i \cdot u_i \neq 0$, so we obtain

$$\lambda_i = 0, \qquad \text{for all } i \in J,$$

which shows that the family $(u_i)_{i \in I}$ is linearly independent. $\square$

We leave the following simple result as an exercise.

Proposition 12.5. *Given a Euclidean space E, any two vectors $u, v \in E$ are orthogonal iff*

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2.$$

See Figure 12.2 for a geometrical interpretation.

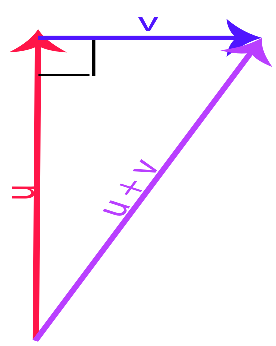

Figure 12.2: The sum of the lengths of the two sides of a right triangle is equal to the length of the hypotenuse; i.e. the Pythagorean theorem.

One of the most useful features of orthonormal bases is that they afford a very simple method for computing the coordinates of a vector over any basis vector. Indeed, assume that $(e_1, \ldots, e_m)$ is an orthonormal basis. For any vector

$$x = x_1 e_1 + \cdots + x_m e_m,$$