It is easy to check that whenever we computed a square root, if we had chosen a negative sign instead of a positive sign, we would obtain the quaternion $-q$. However, both q and $-q$ determine the same rotation r_q.

The above discussion involving the cases $\mathrm{tr}(R) \neq -1$ and $\mathrm{tr}(R) = -1$ is reminiscent of the procedure for finding a logarithm of a rotation matrix using the Rodrigues formula (see Section 12.7). This is not surprising, because if

$$B = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}$$

and if we write $\theta = \sqrt{u_1^2 + u_2^2 + u_3^2}$ (with $0 \leq \theta \leq 2\pi$), then the Rodrigues formula says that

$$e^B = I + \frac{\sin\theta}{\theta} B + \frac{(1-\cos\theta)}{\theta^2} B^2, \quad \theta \neq 0,$$

with $e^0 = I$. It is easy to check that $\mathrm{tr}(e^B) = 1 + 2\cos\theta$. Then it is an easy exercise to check that the quaternion q corresponding to the rotation $R = e^B$ (with $B \neq 0$) is given by

$$q = \left[\cos\left(\frac{\theta}{2}\right), \sin\left(\frac{\theta}{2}\right)\left(\frac{u_1}{\theta}, \frac{u_2}{\theta}, \frac{u_3}{\theta}\right)\right].$$

So the method for finding the logarithm of a rotation R is essentially the same as the method for finding a quaternion defining R.

Remark: Geometrically, the group $\mathbf{SU}(2)$ is homeomorphic to the 3-sphere S^3 in $\mathbb{R}^4$,

$$S^3 = \{(x, y, z, t) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + t^2 = 1\}.$$

However, since the kernel of the surjective homomorphism $r\colon \mathbf{SU}(2) \to \mathbf{SO}(3)$ is $\{I, -I\}$, as a topological space, $\mathbf{SO}(3)$ is homeomorphic to the quotient of S^3 obtained by identifying antipodal points (x, y, z, t) and $-(x, y, z, t)$. This quotient space is the (real) projective space $\mathbb{RP}^3$, and it is more complicated than S^3. The space S^3 is simply-connected, but $\mathbb{RP}^3$ is not.

16.5 The Exponential Map $\exp\colon \mathfrak{su}(2) \to \mathbf{SU}(2)$

Given any matrix $A \in \mathfrak{su}(2)$, with

$$A = \begin{pmatrix} iu_1 & u_2 + iu_3 \\ -u_2 + iu_3 & -iu_1 \end{pmatrix},$$

it is easy to check that

$$A^2 = -\theta^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

with $\theta = \sqrt{u_1^2 + u_2^2 + u_3^2}$. Then we have the following formula whose proof is very similar to the proof of the formula given in Proposition 9.22.