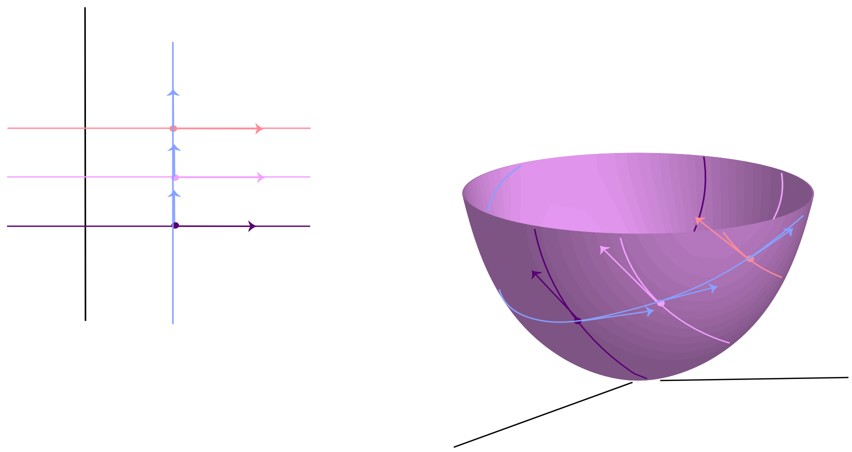

Figure 39.5: The parametric surface $x = u, y = v, z = u^2 + v^2$.

the scalar product of $\mathrm{grad} f(a)$ and v.

Example 39.5. Consider the quadratic function $f\colon \mathbb{R}^n \to \mathbb{R}$ given by

$$f(x) = x^\top A x, \quad x \in \mathbb{R}^n,$$

where A is a real $n \times n$ symmetric matrix. We claim that

$$df_u(h) = 2u^\top A h \quad \text{for all } u, h \in \mathbb{R}^n.$$

Since A is symmetric, we have

$$\begin{aligned} f(u+h) &= (u^\top + h^\top)A(u+h) \\ &= u^\top A u + u^\top A h + h^\top A u + h^\top A h \\ &= u^\top A u + 2u^\top A h + h^\top A h, \end{aligned}$$

so we have

$$f(u+h) - f(u) - 2u^\top A h = h^\top A h.$$

If we write

$$\epsilon(h) = \frac{h^\top A h}{\|h\|}$$

for $h \notin 0$ where $\| \ \|$ is the 2-norm, by Cauchy–Schwarz we have

$$|\epsilon(h)| \leq \frac{\|h\| \, \|Ah\|}{\|h\|} \leq \frac{\|h\|^2 \, \|A\|}{\|h\|} = \|h\| \, \|A\| \,,$$

which shows that $\lim_{h \mapsto 0} \epsilon(h) = 0$. Therefore,

$$df_u(h) = 2u^\top A h \quad \text{for all } u, h \in \mathbb{R}^n,$$