If w is an optimal solution, then $\delta = 1/\left\|w\right\|$ is the shortest distance from the support vectors to the separating hyperplane $H_{w,b}$ of equation $w^\top x - b = 0$. If we consider the two hyperplanes $H_{w,b+1}$ and $H_{w,b-1}$ of equations

$$w^\top x - b - 1 = 0 \quad \text{and} \quad w^\top x - b + 1 = 0,$$

then $H_{w,b+1}$ and $H_{w,b-1}$ are two hyperplanes parallel to the hyperplane $H_{w,b}$ and the distance between them is 2δ. Furthermore, $H_{w,b+1}$ contains the support vectors u_i, $H_{w,b-1}$ contains the support vectors v_j, and there are no data points u_i or v_j in the open region between these two hyperplanes containing the separating hyperplane $H_{w,b}$ (called a "slab" by Boyd and Vandenberghe; see [29], Section 8.6). This situation is illustrated in Figure 50.14.

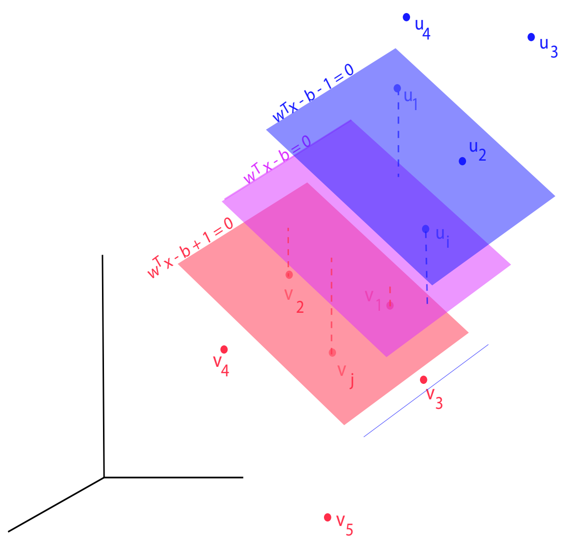

Figure 50.14: In $\mathbb{R}^3$, the solution to Hard Margin SVM_{h2} is the purple plane sandwiched between the red plane $w^\top x - b + 1 = 0$ and the blue plane $w^\top x - b - 1 = 0$, each of which contains the appropriate support vectors u_i and v_j.

Even if $p = 1$ and $q = 2$, a solution is not obvious. In the plane, there are four possibilities:

(1) If u_1 is on the segment (v_1, v_2), there is no solution.

(2) If the projection h of u_1 onto the line determined by v_1 and v_2 is between v_1 and v_2, that is $h = (1-\alpha)v_1 + \alpha_2 v_2$ with $0 \leq \alpha \leq 1$, then it is the line parallel to $v_2 - v_1$ and equidistant to u and both v_1 and v_2, as illustrated in Figure 50.15.

(3) If the projection h of u_1 onto the line determined by v_1 and v_2 is to the right of v_2, that is $h = (1-\alpha)v_1 + \alpha_2 v_2$ with $\alpha > 1$, then it is the bisector of the line segment (u_1, v_2).

(4) If the projection h of u_1 onto the line determined by v_1 and v_2 is to the left of v_1, that is $h = (1-\alpha)v_1 + \alpha_2 v_2$ with $\alpha < 0$, then it is the bisector of the line segment (u_1, v_1).