and a vector $b \in \mathbb{R}^m$, the set $U = \{x \in \mathbb{R}^n \mid Ax = b\}$ of solutions of the system $Ax = b$ is an affine space, but not a vector space (linear space) in general.

Use coordinate systems only when needed!

This chapter proceeds as follows. We take advantage of the fact that almost every affine concept is the counterpart of some concept in linear algebra. We begin by defining affine spaces, stressing the physical interpretation of the definition in terms of points (particles) and vectors (forces). Corresponding to linear combinations of vectors, we define affine combinations of points (barycenters), realizing that we are forced to restrict our attention to families of scalars adding up to 1. Corresponding to linear subspaces, we introduce affine subspaces as subsets closed under affine combinations. Then, we characterize affine subspaces in terms of certain vector spaces called their directions. This allows us to define a clean notion of parallelism. Next, corresponding to linear independence and bases, we define affine independence and affine frames. We also define convexity. Corresponding to linear maps, we define affine maps as maps preserving affine combinations. We show that every affine map is completely defined by the image of one point and a linear map. Then, we investigate briefly some simple affine maps, the translations and the central dilatations. At this point, we give a glimpse of affine geometry. We prove the theorems of Thales, Pappus, and Desargues. After this, the definition of affine hyperplanes in terms of affine forms is reviewed. The section ends with a closer look at the intersection of affine subspaces.

Our presentation of affine geometry is far from being comprehensive, and it is biased toward the algorithmic geometry of curves and surfaces. For more details, the reader is referred to Pedoe [136], Snapper and Troyer [162], Berger [11, 12], Coxeter [44], Samuel [142], Tisseron [175], Fresnel [65], Vienne [185], and Hilbert and Cohn-Vossen [92].

Suppose we have a particle moving in 3D space and that we want to describe the trajectory of this particle. If one looks up a good textbook on dynamics, such as Greenwood [82], one finds out that the particle is modeled as a point, and that the position of this point x is determined with respect to a "frame" in $\mathbb{R}^3$ by a vector. Curiously, the notion of a frame is rarely defined precisely, but it is easy to infer that a frame is a pair $(O, (e_1, e_2, e_3))$ consisting of an origin O (which is a point) together with a basis of three vectors (e_1, e_2, e_3). For example, the standard frame in $\mathbb{R}^3$ has origin $O = (0, 0, 0)$ and the basis of three vectors $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$. The position of a point x is then defined by the "unique vector" from O to x.

But wait a minute, this definition seems to be defining frames and the position of a point without defining what a point is! Well, let us identify points with elements of $\mathbb{R}^3$. If so, given any two points $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$, there is a unique *free vector*, denoted by $\overrightarrow{ab}$, from a to b, the vector $\overrightarrow{ab} = (b_1 - a_1, b_2 - a_2, b_3 - a_3)$. Note that

$$b = a + \overrightarrow{ab},$$