and thus, $e' = a_h e$. Since $a_h = f(e') = f(a_h e) = a_h f(e)$, and since $a_h \neq 0$, we must have $f(e) = 1$.

Next, we claim that

$$F = Ae \oplus f^{-1}(0)$$

and

$$M = Ae' \oplus (M \cap f^{-1}(0)),$$

with $e' = a_h e$.

Indeed, every $x \in F$ can be written as

$$x = f(x)e + (x - f(x)e),$$

and since $f(e) = 1$, we have $f(x - f(x)e) = f(x) - f(x)f(e) = f(x) - f(x) = 0$. Thus, $F = Ae + f^{-1}(0)$. Similarly, for any $x \in M$, we have $f(x) = ra_h$, for some $r \in A$, and thus,

$$x = f(x)e + (x - f(x)e) = ra_h e + (x - f(x)e) = re' + (x - f(x)e),$$

we still have $x - f(x)e \in f^{-1}(0)$, and clearly, $x - f(x)e = x - ra_h e = x - re' \in M$, since $e' \in M$. Thus, $M = Ae' + (M \cap f^{-1}(0))$.

To prove that we have a direct sum, it is enough to prove that $Ae \cap f^{-1}(0) = \{0\}$. For any $x = re \in Ae$, if $f(x) = 0$, then $f(re) = rf(e) = r = 0$, since $f(e) = 1$ and, thus, $x = 0$. Therefore, the sums are direct sums.

We can now prove that M is a free module by induction on the size, q, of a maximal linearly independent family for M.

If $q = 0$, the result is trivial. Otherwise, since

$$M = Ae' \oplus (M \cap f^{-1}(0)),$$

it is clear that $M \cap f^{-1}(0)$ is a submodule of F and that every maximal linearly independent family in $M \cap f^{-1}(0)$ has at most $q - 1$ elements. By the induction hypothesis, $M \cap f^{-1}(0)$ is a free module, and by adding e' to a basis of $M \cap f^{-1}(0)$, we obtain a basis for M, since the sum is direct.

The second part is shown by induction on the dimension n of F.

The case $n = 0$ is trivial. Otherwise, since

$$F = Ae \oplus f^{-1}(0),$$

and since, by the previous argument, $f^{-1}(0)$ is also free, $f^{-1}(0)$ has dimension $n - 1$. By the induction hypothesis applied to its submodule $M \cap f^{-1}(0)$, there is a basis $(e_2, \ldots, e_n)$ of $f^{-1}(0)$, some $q \leq n$, and some nonzero elements $a_2, \ldots, a_q \in A$, such that, $(a_2 e_2, \ldots, a_q e_q)$ is a basis of $M \cap f^{-1}(0)$, and a_i divides a_{i+1} for all i, with $2 \leq i \leq q - 1$. Let $e_1 = e$, and $a_1 = a_h$, as above. It is clear that $(e_1, \ldots, e_n)$ is a basis of F, and that that $(a_1 e_1, \ldots, a_q e_q)$