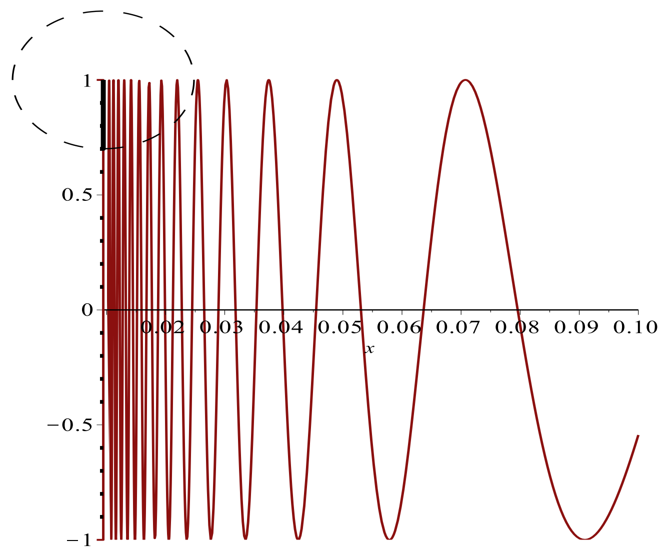

Figure 37.25: Let S be the graph of $f(x) = \sin(1/x)$ union the y-axis between -1 and 1. This space is connected, but not locally connected.

Proposition 37.22. *A topological space, E, is locally connected iff for every open subset, A, of E, the connected components of A are open.*

Proof. Assume that E is locally connected. Let A be any open subset of E and let C be one of the connected components of A. For any $a \in C \subseteq A$, there is some connected neighborhood, U, of a such that $U \subseteq A$ and since C is a connected component of A containing a, we must have $U \subseteq C$. This shows that for every $a \in C$, there is some open subset containing a contained in C, so C is open.

Conversely, assume that for every open subset, A, of E, the connected components of A are open. Then, for every $a \in E$ and every neighborhood, U, of a, since U contains some open set A containing a, the interior, $\overset{\circ}{U}$, of U is an open set containing a and its connected components are open. In particular, the connected component C containing a is a connected open set containing a and contained in U. $\square$

Proposition 37.22 shows that in a locally connected space, the connected open sets form a basis for the topology. It is easily seen that $\mathbb{R}^n$ is locally connected. Another very important property of surfaces and more generally, manifolds, is to be arcwise connected. The intuition is that any two points can be joined by a continuous arc of curve. This is formalized as follows.