We conclude that $d(x_{n+p}, x_n)$ converges to 0 when n goes to infinity, which shows that (x_n) is a Cauchy sequence. Since E is complete, the sequence (x_n) has a limit, a. Since f is continuous, the sequence $(f(x_n))$ converges to $f(a)$. But $x_{n+1} = f(x_n)$ converges to a and so $f(a) = a$, the unique fixed point of f. $\square$

Note that no matter how the starting point x_0 of the sequence (x_n) is chosen, (x_n) converges to the unique fixed point of f. Also, the convergence is fast, since

$$d(x_n, a) \leq \frac{k^n}{1-k}\, d(x_1, x_0).$$

The Hausdorff distance between compact subsets of a metric space provides a very nice illustration of some of the theorems on complete and compact metric spaces just presented.

Definition 37.40. Given a metric space, (X, d), for any subset, $A \subseteq X$, for any, $\epsilon \geq 0$, define the *ϵ-hull* of A as the set

$$V_\epsilon(A) = \{x \in X,\ \exists a \in A \mid d(a, x) \leq \epsilon\}.$$

See Figure 37.46. Given any two nonempty bounded subsets, A, B of X, define $D(A, B)$, *the Hausdorff distance between A and B*, by

$$D(A, B) = \inf\{\epsilon \geq 0 \mid A \subseteq V_\epsilon(B) \text{ and } B \subseteq V_\epsilon(A)\}.$$

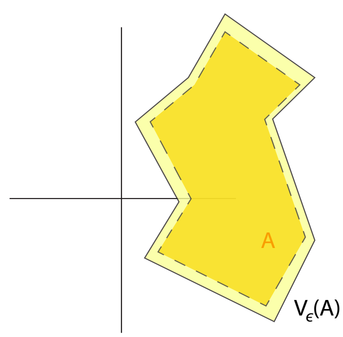

Figure 37.46: The ϵ-hull of a polygonal region A of $\mathbb{R}^2$

Note that since we are considering nonempty bounded subsets, $D(A, B)$ is well defined (i.e., not infinite). However, D is not necessarily a distance function. It is a distance function if we restrict our attention to nonempty compact subsets of X (actually, it is also a metric on closed and bounded subsets). We let $\mathcal{K}(X)$ denote the set of all nonempty compact subsets of X. The remarkable fact is that D is a distance on $\mathcal{K}(X)$ and that if X is complete or compact, then so is $\mathcal{K}(X)$. The following theorem is taken from Edgar [55].