Proposition 48.5. *(Projection lemma) Let E be a Hilbert space and let $X \subseteq E$ be any nonempty convex and closed subset.*

(1) For any $u \in E$, there is a unique vector $p_X(u) \in X$ such that

$$\|u - p_X(u)\| = \inf_{v \in X} \|u - v\| = d(u, X).$$

See Figure 48.2.

(2) The vector $p_X(u)$ is the unique vector $w \in E$ satisfying the following property (see Figure 48.3):

$$w \in X \quad \text{and} \quad \Re \langle u - w, z - w \rangle \leq 0 \quad \text{for all } z \in X. \tag{$*$}$$

(3) If X is a nonempty closed subspace of E, then the vector $p_X(u)$ is the unique vector $w \in E$ satisfying the following property:

$$w \in X \quad \text{and} \quad \langle u - w, z \rangle = 0 \quad \text{for all } z \in X. \tag{$**$}$$

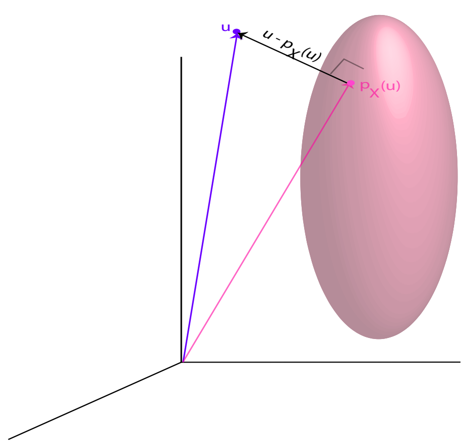

Figure 48.2: Let X be the solid pink ellipsoid. The projection of the purple point u onto X is the magenta point $p_X(u)$.

Proof. (1) Let $d = \inf_{v \in X} \|u - v\| = d(u, X)$. We define a sequence X_n of subsets of X as follows: for every $n \geq 1$,

$$X_n = \left\{ v \in X \mid \|u - v\| \leq d + \frac{1}{n} \right\}.$$