Since $(|c_k|^2)_{k\in K}$ is summable with sum c, by Proposition A.1(1) we know that for every $\epsilon > 0$, there is some finite subset I of K such that

$$\sum_{j\in J} |c_j|^2 < \epsilon^2$$

for every finite subset J of K such that $I \cap J = \emptyset$. Since

$$\Big\|\sum_{j\in J} c_j u_j\Big\|^2 = \sum_{j\in J} |c_j|^2,$$

we get

$$\Big\|\sum_{j\in J} c_j u_j\Big\| < \epsilon.$$

This proves that $(c_k u_k)_{k\in K}$ is a Cauchy family, which, by Proposition A.1(1), implies that $(c_k u_k)_{k\in K}$ is summable since E is complete. Thus, the Fourier series $\sum_{k\in K} c_k u_k$ is summable, with its sum denoted $u \in V$.

Since $\sum_{k\in K} c_k u_k$ is summable with sum u, for every $\epsilon > 0$, there is some finite subset I_1 of K such that

$$\Big\|u - \sum_{j\in J} c_j u_j\Big\| < \epsilon$$

for every finite subset J of K such that $I_1 \subseteq J$. By the triangle inequality, for every finite subset I of K,

$$\Big\|u - v\Big\| \leq \Big\|u - \sum_{i\in I} c_i u_i\Big\| + \Big\|\sum_{i\in I} c_i u_i - v\Big\|.$$

By (2), every $w \in V$ is the sum of its Fourier series $\sum_{k\in K} \lambda_k u_k$, and for every $\epsilon > 0$, there is some finite subset I_2 of K such that

$$\Big\|w - \sum_{j\in J} \lambda_j u_j\Big\| < \epsilon$$

for every finite subset J of K such that $I_2 \subseteq J$. By the triangle inequality, for every finite subset I of K,

$$\Big\|v - \sum_{i\in I} \lambda_i u_i\Big\| \leq \|v - w\| + \Big\|w - \sum_{i\in I} \lambda_i u_i\Big\|.$$

Letting $I = I_1 \cup I_2$, since we showed in (2) that

$$\Big\|v - \sum_{i\in I} c_i u_i\Big\| \leq \Big\|v - \sum_{i\in I} \lambda_i u_i\Big\|$$