Then we can recursively apply the QR algorithm to H_{11} and H_{22}.

In particular, if $(H_k)_{nn-1} = 0$ or is very small, then $(H_k)_{nn}$ is a good approximation of an eigenvalue, so we can delete the last row and the last column of H_k and apply the QR algorithm to this submatrix. This process is called *deflation*. If $(H_k)_{n-1n-2} = 0$ or is very small, then the 2×2 "corner block"

$$\begin{pmatrix} (H_k)_{n-1n-1} & (H_k)_{n-1n} \\ (H_k)_{nn-1} & (H_k)_{nn} \end{pmatrix}$$

appears, and its eigenvalues can be computed immediately by solving a quadratic equation. Then we deflate H_k by deleting its last two rows and its last two columns and apply the QR algorithm to this submatrix.

Thus it would seem desirable to modify the basic QR algorithm so that the above situations arises, and this is what shifts are designed for. More precisely, under the hypotheses of Theorem 18.1, it can be shown (see Ciarlet [41], Section 6.3) that the entry $(A_k)_{ij}$ with $i > j$ converges to 0 as $|\lambda_i/\lambda_j|^k$ converges to 0. Also, if we let r_i be defined by

$$r_1 = \left|\frac{\lambda_2}{\lambda_1}\right|, \quad r_i = \max\left\{\left|\frac{\lambda_i}{\lambda_{i-1}}\right|, \left|\frac{\lambda_{i+1}}{\lambda_i}\right|\right\}, \quad 2 \leq i \leq n-1, \quad r_n = \left|\frac{\lambda_n}{\lambda_{n-1}}\right|,$$

then there is a constant C (independent of k) such that

$$|(A_k)_{ii} - \lambda_i| \leq C r_i^k, \quad 1 \leq i \leq n.$$

In particular, if H is upper Hessenberg, then the entry $(H_k)_{i+1i}$ converges to 0 as $|\lambda_{i+1}/\lambda_i|^k$ converges to 0. Thus if we pick σ_k close to λ_i, we expect that $(H_k - \sigma_k I)_{i+1i}$ converges to 0 as $|(\lambda_{i+1} - \sigma_k)/(\lambda_i - \sigma_k)|^k$ converges to 0, and this ratio is much smaller than 1 as σ_k is closer to λ_i. Typically, we apply a shift to accelerate convergence to λ_n (so $i = n-1$). In this case, both $(H_k - \sigma_k I)_{nn-1}$ and $|(H_k - \sigma_k I)_{nn} - \lambda_n|$ converge to 0 as $|(\lambda_n - \sigma_k)/(\lambda_{n-1} - \sigma_k)|^k$ converges to 0.

A *shift* is the following modified QR-steps (switching back to an arbitrary matrix A, since the shift technique applies in general). Pick some σ_k, hopefully close to some eigenvalue of A (in general, λ_n), and QR-factor $A_k - \sigma_k I$ as

$$A_k - \sigma_k I = Q_k R_k,$$

and then form

$$A_{k+1} = R_k Q_k + \sigma_k I.$$

Since

$$\begin{aligned} A_{k+1} &= R_k Q_k + \sigma_k I \\ &= Q_k^* Q_k R_k Q_k + Q_k^* Q_k \sigma_k \\ &= Q_k^* (Q_k R_k + \sigma_k I) Q_k \\ &= Q_k^* A_k Q_k, \end{aligned}$$