and then $u_{k+1} = u_k - \rho_k d_k$. In fact, by $(*_2)$, since

$$\Delta_k = \sum_{i=0}^{k} \delta_i^k \, \nabla J_{u_i} = \delta_k^k \left(\sum_{i=0}^{k-1} \frac{\delta_i^k}{\delta_k^k} \, \nabla J_{u_i} + \nabla J_{u_k} \right),$$

we must have

$$\Delta_k = \delta_k^k d_k \quad \text{and} \quad \rho_k = -\delta_k^k. \tag{$*_4$}$$

Remarkably, the coefficients λ_i^k and the descent directions d_k can be computed easily using the following formulae.

Proposition 49.17. *Assume that $\nabla J_{u_i} \neq 0$ for $i = 0, \ldots, k$. If we write*

$$d_\ell = \sum_{i=0}^{\ell-1} \lambda_i^\ell \, \nabla J_{u_i} + \nabla J_{u_\ell}, \quad 0 \leq \ell \leq k,$$

then we have

$$(\dagger) \quad \begin{cases} \lambda_i^k = \dfrac{\left\|\nabla J_{u_k}\right\|^2}{\left\|\nabla J_{u_i}\right\|^2}, & 0 \leq i \leq k-1, \\ d_0 = \nabla J_{u_0} \\ d_\ell = \nabla J_{u_\ell} + \dfrac{\left\|\nabla J_{u_\ell}\right\|^2}{\left\|\nabla J_{u_{\ell-1}}\right\|^2} \, d_{\ell-1}, & 1 \leq \ell \leq k. \end{cases}$$

Proof. Since by $(*_4)$ we have $\Delta_k = \delta_k^k d_k$, $\delta_k^k \neq 0$, (by Proposition 49.16) we have

$$\langle A d_\ell, \Delta_i \rangle = 0, \quad 0 \leq i < \ell \leq k.$$

By $(*_1)$ we have $\nabla J_{u_{\ell+1}} = \nabla J_{u_\ell} + A\Delta_\ell$, and since A is a symmetric matrix, we have

$$0 = \langle A d_k, \Delta_\ell \rangle = \langle d_k, A\Delta_\ell \rangle = \langle d_k, \nabla J_{u_{\ell+1}} - \nabla J_{u_\ell} \rangle,$$

for $\ell = 0, \ldots, k-1$. Since

$$d_k = \sum_{i=0}^{k-1} \lambda_i^k \, \nabla J_{u_i} + \nabla J_{u_k},$$

we have

$$\left\langle \sum_{i=0}^{k-1} \lambda_i^k \, \nabla J_{u_i} + \nabla J_{u_k}, \nabla J_{u_{\ell+1}} - \nabla J_{u_\ell} \right\rangle = 0, \quad 0 \leq \ell \leq k-1.$$

Since by Proposition 49.15 the gradients ∇J_{u_i} are pairwise orthogonal, the above equations yield

$$\begin{aligned} -\lambda_{k-1}^k \left\|\nabla J_{u_{k-1}}\right\|^2 + \left\|\nabla J_{u_k}\right\|^2 &= 0 \\ -\lambda_\ell^k \left\|\nabla J_{u_\ell}\right\|^2 + \lambda_{\ell+1}^k \left\|\nabla J_{u_{\ell+1}}\right\|^2 &= 0, \quad 0 \leq \ell \leq k-2, \quad k \geq 2, \end{aligned}$$