since $\langle u_j, u_i \rangle = 0$ for all $i \neq j$ and $c_i = \langle v, u_i \rangle / \|u_i\|^2$. As a consequence, we have

$$\begin{aligned}
\|v\|^2 &= \Big\| v - \sum_{i \in I} c_i u_i + \sum_{i \in I} c_i u_i \Big\|^2 \\
&= \Big\| v - \sum_{i \in I} c_i u_i \Big\|^2 + \Big\| \sum_{i \in I} c_i u_i \Big\|^2 \\
&= \Big\| v - \sum_{i \in I} c_i u_i \Big\|^2 + \sum_{i \in I} |c_i|^2,
\end{aligned}$$

since the u_i are pairwise orthogonal, that is,

$$\|v\|^2 = \Big\| v - \sum_{i \in I} c_i u_i \Big\|^2 + \sum_{i \in I} |c_i|^2.$$

Thus,

$$\sum_{i \in I} |c_i|^2 \leq \|v\|^2,$$

as claimed.

(2) We prove the chain of implications $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$.

$(a) \Rightarrow (b)$: If $v \in V$, since V is the closure of the subspace spanned by $(u_k)_{k \in K}$, for every $\epsilon > 0$, there is some finite subset I of K and some family $(\lambda_i)_{i \in I}$ of complex numbers, such that

$$\Big\| v - \sum_{i \in I} \lambda_i u_i \Big\| < \epsilon.$$

Now for every finite subset J of K such that $I \subseteq J$, we have

$$\begin{aligned}
\Big\| v - \sum_{i \in I} \lambda_i u_i \Big\|^2 &= \Big\| v - \sum_{j \in J} c_j u_j + \sum_{j \in J} c_j u_j - \sum_{i \in I} \lambda_i u_i \Big\|^2 \\
&= \Big\| v - \sum_{j \in J} c_j u_j \Big\|^2 + \Big\| \sum_{j \in J} c_j u_j - \sum_{i \in I} \lambda_i u_i \Big\|^2,
\end{aligned}$$

since $I \subseteq J$ and the u_j (with $j \in J$) are orthogonal to $v - \sum_{j \in J} c_j u_j$ by the argument in (1), which shows that

$$\Big\| v - \sum_{j \in J} c_j u_j \Big\| \leq \Big\| v - \sum_{i \in I} \lambda_i u_i \Big\| < \epsilon,$$

and thus, that the family $(c_k u_k)_{k \in K}$ is summable with sum v, so that

$$v = \sum_{k \in K} c_k u_k.$$