18.3 Making the QR Method More Efficient Using Shifts

To improve efficiency and cope with pairs of complex conjugate eigenvalues in the case of real matrices, the following steps are taken:

(1) Initially reduce the matrix A to upper Hessenberg form, as $A = UHU^*$. Then apply the QR-algorithm to H (actually, to its unreduced Hessenberg blocks). It is easy to see that the matrices H_k produced by the QR algorithm remain upper Hessenberg.

(2) To accelerate convergence, use *shifts*, and to deal with pairs of complex conjugate eigenvalues, use *double shifts*.

(3) Instead of computing a QR-factorization explicitly while doing a shift, perform an *implicit shift* which computes $A_{k+1} = Q_k^* A_k Q_k$ without having to compute a QR-factorization (of $A_k - \sigma_k I$), and similarly in the case of a double shift. This is the most intricate modification of the basic QR algorithm and we will not discuss it here. This method is usually referred as *bulge chasing*. Details about this technique for real matrices can be found in Demmel [48] (Section 4.4.8) and Golub and Van Loan [80] (Section 7.5). Watkins discusses the QR algorithm with shifts as a bulge chasing method in the more general case of complex matrices [187, 188].

Let us repeat an important remark made in the previous section. If we start with a matrix H in upper Hessenberg form, if at any stage of the QR algorithm we find that some subdiagonal entry $(H_k)_{p+1p} = 0$ *or is very small*, then H_k is of the form

$$H_k = \begin{pmatrix} H_{11} & H_{12} \\ 0 & H_{22} \end{pmatrix},$$

where both H_{11} and H_{22} are upper Hessenberg matrices (with H_{11} a $p \times p$ matrix and H_{22} a $(n-p) \times (n-p)$ matrix), and the eigenvalues of H_k are the eigenvalues of H_{11} and H_{22}. For example, in the matrix

$$H = \begin{pmatrix} * & * & * & * & * \\ h_{21} & * & * & * & * \\ 0 & h_{32} & * & * & * \\ 0 & 0 & h_{43} & * & * \\ 0 & 0 & 0 & h_{54} & * \end{pmatrix},$$

if $h_{43} = 0$, then we have the block matrix

$$H = \begin{pmatrix} * & * & * & * & * \\ h_{21} & * & * & * & * \\ 0 & h_{32} & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & h_{54} & * \end{pmatrix}.$$