If $u = 0$, then $f_m(0) = f_n(0) = 0$ for all m, n, so the sequence $(f_n(0))$ is a Cauchy sequence in F converging to 0. If $u \neq 0$, by replacing ϵ by $\epsilon / \|u\|$, we see that the sequence $(f_n(u))$ is a Cauchy sequence in F. Since F is complete, the sequence $(f_n(u))$ has a limit which we denote by $f(u)$. This defines our candidate limit function f by

$$f(u) = \lim_{n \mapsto \infty} f_n(u).$$

It remains to prove that

1. f is linear.

2. f is continous.

3. f is the limit of (f_n) for the operator norm.

Step 2. The function f is linear.

Recall that in a normed vector space, addition and multiplication by a fixed scalar are continuous (since $\|u + v\| \leq \|u\| + \|v\|$ and $\|\lambda u\| \leq |\lambda| \|u\|$). Thus by definition of f and since the f_n are linear we have

$$\begin{aligned}
f(u + v) &= \lim_{n \mapsto \infty} f_n(u + v) && \text{by definition of } f \\
&= \lim_{n \mapsto \infty} (f_n(u) + f_n(v)) && \text{by linearity of } f_n \\
&= \lim_{n \mapsto \infty} f_n(u) + \lim_{n \mapsto \infty} f_n(v) && \text{since } + \text{ is continuous} \\
&= f(u) + f(v) && \text{by definition of } f.
\end{aligned}$$

Similarly,

$$\begin{aligned}
f(\lambda u) &= \lim_{n \mapsto \infty} f_n(\lambda u) && \text{by definition of } f \\
&= \lim_{n \mapsto \infty} \lambda f_n(u) && \text{by linearity of } f_n \\
&= \lambda \lim_{n \mapsto \infty} f_n(u) && \text{by continuity of scalar multiplication} \\
&= \lambda f(u) && \text{by definition of } f.
\end{aligned}$$

Therefore, f is linear.

Step 3. The function f is continuous.

Since $(f_n)_{n \geq 1}$ is a Cauchy sequence, for every $\epsilon > 0$, there is some $N > 0$ such that $\|f_m - f_n\| < \epsilon$ for all $m, n \geq N$. Since $f_m = f_n + f_m - f_n$, we get $\|f_m\| \leq \|f_n\| + \|f_m - f_n\|$, which implies that

$$\|f_m\| \leq \|f_n\| + \epsilon \quad \text{for all } m, n \geq N. \tag{$*_2$}$$