The reader should try the above procedure on some concrete examples for 2×2 and 3×3 matrices.

Remarks:

(1) Because the diagonal entries of R are positive, it can be shown that Q and R are unique. More generally, if A is invertible and if $A = Q_1R_1 = Q_2R_2$ are two QR-decompositions for A, then
$$R_1R_2^{-1} = Q_1^\top Q_2.$$
The matrix $Q_1^\top Q_2$ is orthogonal and it is easy to see that $R_1R_2^{-1}$ is upper triangular. But an upper triangular matrix which is orthogonal must be a diagonal matrix D with diagonal entries ± 1, so $Q_2 = Q_1D$ and $R_1 = DR_2$.

(2) The QR-decomposition holds even when A is not invertible. In this case, R has some zero on the diagonal. However, a different proof is needed. We will give a nice proof using Householder matrices (see Proposition 13.4, and also Strang [169, 170], Golub and Van Loan [80], Trefethen and Bau [176], Demmel [48], Kincaid and Cheney [102], or Ciarlet [41]).

For better numerical stability, it is preferable to use the modified Gram–Schmidt method to implement the QR-factorization method. Here is a `Matlab` program implementing QR-factorization using modified Gram–Schmidt.

```
function [Q,R] = qrv4(A)
n = size(A,1);
for i = 1:n
    Q(:,i) = A(:,i);
    for j = 1:i-1
        R(j,i) = Q(:,j)'*Q(:,i);
        Q(:,i) = Q(:,i) - R(j,i)*Q(:,j);
    end
    R(i,i) = sqrt(Q(:,i)'*Q(:,i));
    Q(:,i) = Q(:,i)/R(i,i);
end
end
```

Example 12.13. Consider the matrix
$$A = \begin{pmatrix} 0 & 0 & 5 \\ 0 & 4 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

To determine the QR-decomposition of A, we first use the Gram-Schmidt orthonormalization