Remark: One may wonder whether axiom (V4) is really needed. Could it be derived from the other axioms? The answer is **no**. For example, one can take $E = \mathbb{R}^n$ and define $\cdot \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$\lambda \cdot (x_1, \ldots, x_n) = (0, \ldots, 0)$$

for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$ and all $\lambda \in \mathbb{R}$. Axioms (V0)–(V3) are all satisfied, but (V4) fails. Less trivial examples can be given using the notion of a basis, which has not been defined yet.

The field K itself can be viewed as a vector space over itself, addition of vectors being addition in the field, and multiplication by a scalar being multiplication in the field.

Example 3.1.

1. The fields $\mathbb{R}$ and $\mathbb{C}$ are vector spaces over $\mathbb{R}$.

2. The groups $\mathbb{R}^n$ and $\mathbb{C}^n$ are vector spaces over $\mathbb{R}$, with scalar multiplication given by

$$\lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n),$$

   for any $\lambda \in \mathbb{R}$ and with $(x_1, \ldots, x_n) \in \mathbb{R}^n$ or $(x_1, \ldots, x_n) \in \mathbb{C}^n$, and $\mathbb{C}^n$ is a vector space over $\mathbb{C}$ with scalar multiplication as above, but with $\lambda \in \mathbb{C}$.

3. The ring $\mathbb{R}[X]_n$ of polynomials of degree at most n with real coefficients is a vector space over $\mathbb{R}$, and the ring $\mathbb{C}[X]_n$ of polynomials of degree at most n with complex coefficients is a vector space over $\mathbb{C}$, with scalar multiplication $\lambda \cdot P(X)$ of a polynomial

$$P(X) = a_m X^m + a_{m-1} X^{m-1} + \cdots + a_1 X + a_0$$

   (with $a_i \in \mathbb{R}$ or $a_i \in \mathbb{C}$) by the scalar λ (in $\mathbb{R}$ or $\mathbb{C}$), with $m \leq n$, given by

$$\lambda \cdot P(X) = \lambda a_m X^m + \lambda a_{m-1} X^{m-1} + \cdots + \lambda a_1 X + \lambda a_0.$$

4. The ring $\mathbb{R}[X]$ of all polynomials with real coefficients is a vector space over $\mathbb{R}$, and the ring $\mathbb{C}[X]$ of all polynomials with complex coefficients is a vector space over $\mathbb{C}$, with the same scalar multiplication as above.

5. The ring of $n \times n$ matrices $\mathrm{M}_n(\mathbb{R})$ is a vector space over $\mathbb{R}$.

6. The ring of $m \times n$ matrices $\mathrm{M}_{m,n}(\mathbb{R})$ is a vector space over $\mathbb{R}$.

7. The ring $\mathcal{C}(a, b)$ of continuous functions $f \colon (a, b) \to \mathbb{R}$ is a vector space over $\mathbb{R}$, with the scalar multiplication λf of a function $f \colon (a, b) \to \mathbb{R}$ by a scalar $\lambda \in \mathbb{R}$ given by

$$(\lambda f)(x) = \lambda f(x), \qquad \text{for all } x \in (a, b).$$