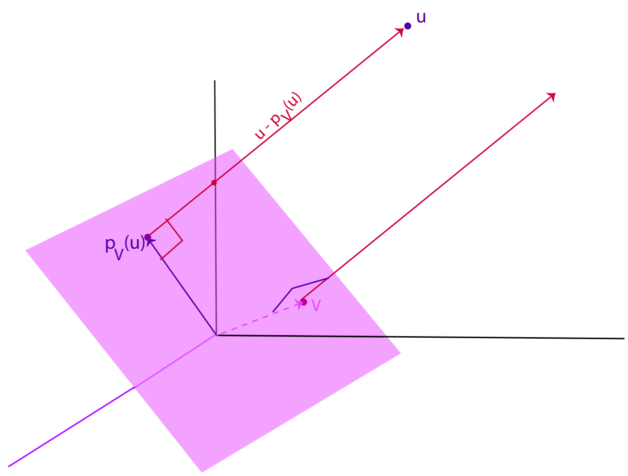

Figure 48.7: Let V be the pink plane. The vector $u - p_V(u)$ is perpendicular to any $v \in V$.

problem has a solution, but it does! The problem can be restated as follows: Is there some $x \in \mathbb{R}^n$ such that

$$\|Ax - b\| = \inf_{y \in \mathbb{R}^n} \|Ay - b\|,$$

or equivalently, is there some $z \in \mathrm{Im}\,(A)$ such that

$$\|z - b\| = d(b, \mathrm{Im}\,(A)),$$

where $\mathrm{Im}\,(A) = \{Ay \in \mathbb{R}^m \mid y \in \mathbb{R}^n\}$, the image of the linear map induced by A. Since $\mathrm{Im}\,(A)$ is a closed subspace of $\mathbb{R}^m$, because we are in finite dimension, Proposition 48.7 tells us that there is a unique $z \in \mathrm{Im}\,(A)$ such that

$$\|z - b\| = \inf_{y \in \mathbb{R}^n} \|Ay - b\|,$$

and thus the problem always has a solution since $z \in \mathrm{Im}\,(A)$, and since there is at least some $x \in \mathbb{R}^n$ such that $Ax = z$ (by definition of $\mathrm{Im}\,(A)$). Note that such an x is not necessarily unique. Furthermore, Proposition 48.7 also tells us that $z \in \mathrm{Im}\,(A)$ is the solution of the equation

$$\langle z - b, w \rangle = 0 \quad \text{for all } w \in \mathrm{Im}\,(A),$$

or equivalently, that $x \in \mathbb{R}^n$ is the solution of

$$\langle Ax - b, Ay \rangle = 0 \quad \text{for all } y \in \mathbb{R}^n,$$

which is equivalent to

$$\langle A^\top (Ax - b), y \rangle = 0 \quad \text{for all } y \in \mathbb{R}^n,$$

and thus, since the inner product is positive definite, to $A^\top (Ax - b) = 0$, i.e.,

$$A^\top A x = A^\top b.$$