If we define the scalar multiplication $\cdot\colon B \times (\rho_*(B) \otimes_A M) \to \rho_*(B) \otimes_A M$ by

$$\beta \cdot z = \mu_\beta(z), \quad \text{for all } \beta \in B \text{ and all } z \in \rho_*(B) \otimes_A M,$$

then it is easy to check that the axioms M1, M2, M3, M4 hold. Let us check M2 and M3. We have

$$\begin{aligned}
\mu_{\beta_1+\beta_2}(\beta' \otimes x) &= (\beta_1 + \beta_2)\beta' \otimes x \\
&= (\beta_1\beta' + \beta_2\beta') \otimes x \\
&= \beta_1\beta' \otimes x + \beta_2\beta' \otimes x \\
&= \mu_{\beta_1}(\beta' \otimes x) + \mu_{\beta_2}(\beta' \otimes x)
\end{aligned}$$

and

$$\begin{aligned}
\mu_{\beta_1\beta_2}(\beta' \otimes x) &= \beta_1\beta_2\beta' \otimes x \\
&= \mu_{\beta_1}(\beta_2\beta' \otimes x) \\
&= \mu_{\beta_1}(\mu_{\beta_2}(\beta' \otimes x)).
\end{aligned}$$

Definition 35.15. Given two rings A and B and a ring homomorphism $\rho\colon A \to B$, for any A-module M, with the scalar multiplication by elements of B given by

$$\beta \cdot (\beta' \otimes x) = (\beta\beta') \otimes x, \quad \beta, \beta' \in B,\ x \in M,$$

the tensor product $\rho_*(B) \otimes_A M$ is a B-module denoted by $\rho^*(M)$, or $M_{(B)}$ when ρ is the inclusion of A into B. The B-module $\rho^*(M)$ is sometimes called the *module induced from M by extension to B of the ring of scalars through ρ.*

Here is a specific example of Definition 35.15. Let $A = \mathbb{R}$, $B = \mathbb{C}$ and ρ be the inclusion map of $\mathbb{R}$ into $\mathbb{C}$, i.e. $\rho\colon \mathbb{R} \to \mathbb{C}$ with $\rho(a) = a$ for $a \in \mathbb{R}$. Let M be an $\mathbb{R}$-module. The field $\mathbb{C}$ is a $\mathbb{C}$-module, and when we restrict scalar multiplication by scalars $\lambda \in \mathbb{R}$, we obtain the $\mathbb{R}$-module $\rho_*(\mathbb{C})$ (which, as an abelian group, is just $\mathbb{C}$). Form $\rho_*(\mathbb{C}) \otimes_{\mathbb{R}} M$. This is an $\mathbb{R}$-module where typical elements have the form $\sum_{i=1}^n a_i(z_i \otimes m_i)$, $a_i \in \mathbb{R}$, $z_i \in \mathbb{C}$, and $m_i \in M$. Since

$$a_i(z_i \otimes m_i) = a_i z_i \otimes m_i$$

and since $a_i z_i \in \mathbb{C}$ and any element of $\mathbb{C}$ is obtained this way (let $a_i = 1$), the elements of $\rho_*(\mathbb{C}) \otimes_{\mathbb{R}} M$ can be written as

$$\sum_{i=1}^n z_i \otimes m_i, \quad z_i \in \mathbb{C},\ m_i \in M.$$

We want to make $\rho_*(\mathbb{C}) \otimes_{\mathbb{R}} M$ into a $\mathbb{C}$-module, denoted $\rho^*(M)$, and thus must describe how a complex number β acts on $\sum_{i=1}^n z_i \otimes m_i$. By linearity, it is enough to determine how $\beta = u + iv$ acts on a generator $z \otimes m$, where $z = x + iy$ and $m \in M$. The action is given by

$$\beta \cdot (z \otimes m) = \beta z \otimes m = (u + iv)(x + iy) \otimes m = (ux - vy + i(uy + vx)) \otimes m,$$