Otherwise, there is some column, say j, such that a_{11} does not divide some entry a_{ij}, so add the jth column to the first column. This yields a matrix of the form

$$M = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ b_{2j} & & & \\ \vdots & & Y & \\ b_{mj} & & & \end{pmatrix}$$

where the ith entry in column 1 is nonzero, so go back to Step 2a,

Again, since the σ-value of the $(1,1)$-entry strictly decreases whenever we reenter Step 2a and Step 2b, such a sequence must terminate with a matrix of the form

$$M' = \begin{pmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & Y & \\ 0 & & & \end{pmatrix}$$

where α_1 divides every entry in Y. Then, we apply the induction hypothesis to Y. $\square$

If the PID A is the polynomial ring $K[X]$ where K is a field, the α_i are nonzero polynomials, so we can apply row operations to normalize their leading coefficients to be 1. We obtain the following theorem.

Theorem 36.19. *(Smith Normal Form) If M is an $m \times n$ matrix over the polynomial ring $K[X]$, where K is a field, then there exist some invertible $n \times n$ matrix P and some invertible $m \times m$ matrix Q, where P and Q are products of elementary matrices with entries in $K[X]$, and a $m \times n$ matrix D of the form*

$$D = \begin{pmatrix} q_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & q_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & q_r & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}$$

for some nonzero monic polynomials $q_i \in k[X]$, such that

(1) $q_1 \mid q_2 \mid \cdots \mid q_r$, *and*

(2) $M = QDP^{-1}$.