(2) *Conversely, if the restriction of J to U is convex and if there exist vectors $\lambda \in \mathbb{R}^m_+$ and $\nu \in \mathbb{R}^p$ such that the KKT conditions hold, then the function J has a (global) minimum at u with respect to U.*

The Lagrangian $L(v, \lambda, \nu)$ of Problem (P') is defined as

$$L(v, \mu, \nu) = J(v) + \sum_{i=1}^{m} \mu_i \varphi_i(v) + \sum_{j=1}^{p} \nu_i \psi_j(v),$$

where $v \in \Omega$, $\mu \in \mathbb{R}^m_+$, and $\nu \in \mathbb{R}^p$.

The function $G\colon \mathbb{R}^m_+ \times \mathbb{R}^p \to \mathbb{R}$ given by

$$G(\mu, \nu) = \inf_{v \in \Omega} L(v, \mu, \nu) \quad \mu \in \mathbb{R}^m_+,\ \nu \in \mathbb{R}^p$$

is called the *Lagrange dual function* (or *dual function*), and the *Dual Problem (D') is*

$$\begin{aligned} &\text{maximize} && G(\mu, \nu) \\ &\text{subject to} && \mu \in \mathbb{R}^m_+,\ \nu \in \mathbb{R}^p. \end{aligned}$$

Observe that the Lagrange multipliers ν *are not restricted* to be nonnegative.

Theorem 50.15 and Theorem 50.17 are immediately generalized to Problem (P'). We only state the new version of 50.17, leaving the new version of Theorem 50.15 as an exercise.

Theorem 50.19. *Consider the minimization problem (P'):*

$$\begin{aligned} &\text{minimize} && J(v) \\ &\text{subject to} && \varphi_i(v) \leq 0, \quad i = 1, \ldots, m \\ & && \psi_j(v) = 0, \quad j = 1, \ldots, p. \end{aligned}$$

where the functions J, φ_i are defined on some open subset Ω of a finite-dimensional Euclidean vector space V (more generally, a real Hilbert space V), and the functions ψ_j are affine.

(1) *Suppose the functions $\varphi_i\colon \Omega \to \mathbb{R}$ are continuous, and that for every $\mu \in \mathbb{R}^m_+$ and every $\nu \in \mathbb{R}^p$, the Problem $(P_{\mu,\nu})$:*

$$\begin{aligned} &\text{minimize} && L(v, \mu, \nu) \\ &\text{subject to} && v \in \Omega, \end{aligned}$$

has a unique solution $u_{\mu,\nu}$, so that

$$L(u_{\mu,\nu}, \mu, \nu) = \inf_{v \in \Omega} L(v, \mu, \nu) = G(\mu, \nu),$$