for all $\lambda \in \mathbb{C}$. In particular, the above holds for $\lambda = \langle u - p_V(u), v\rangle$, which yields

$$|\langle u - p_V(u), v\rangle| \leq 0,$$

and thus, $\langle u - p_V(u), v\rangle = 0$. See Figure 48.7. As a consequence, $u - p_V(u) \in V^{\perp}$ for all $u \in E$. Since $u = p_V(u) + u - p_V(u)$ for every $u \in E$, we have $E = V + V^{\perp}$. On the other hand, since $\langle -, -\rangle$ is positive definite, $V \cap V^{\perp} = \{0\}$, and thus $E = V \oplus V^{\perp}$.

We already proved in Proposition 48.6 that $p_V \colon E \to V$ is continuous. Also, since

$$p_V(\lambda u + \mu v) - (\lambda p_V(u) + \mu p_V(v)) = p_V(\lambda u + \mu v) - (\lambda u + \mu v) + \lambda(u - p_V(u)) + \mu(v - p_V(v)),$$

for all $u, v \in E$, and since the left-hand side term belongs to V, and from what we just showed, the right-hand side term belongs to $V^{\perp}$, we have

$$p_V(\lambda u + \mu v) - (\lambda p_V(u) + \mu p_V(v)) = 0,$$

showing that p_V is linear.

(2) This is basically obvious from (1). We proved in (1) that $u - p_V(u) \in V^{\perp}$, which is exactly the condition

$$\langle u - p_V(u), z\rangle = 0$$

for all $z \in V$. Conversely, if $w \in V$ satisfies the condition

$$\langle u - w, z\rangle = 0$$

for all $z \in V$, since $w \in V$, every vector $z \in V$ is of the form $y - w$, with $y = z + w \in V$, and thus, we have

$$\langle u - w, y - w\rangle = 0$$

for all $y \in V$, which implies the condition of Proposition 48.5(2):

$$\Re \langle u - w, y - w\rangle \leq 0$$

for all $y \in V$. By Proposition 48.5, $w = p_V(u)$ is the projection of u onto V. $\square$

Remark: If $p_V \colon E \to V$ is linear, then V is a subspace of E. It follows that if V is a closed convex subset of E, then $p_V \colon E \to V$ is linear iff V is a subspace of E.

Example 48.3. Let us illustrate the power of Proposition 48.7 on the following "least squares" problem. Given a real $m \times n$-matrix A and some vector $b \in \mathbb{R}^m$, we would like to solve the linear system

$$Ax = b$$

in the least-squares sense, which means that we would like to find some solution $x \in \mathbb{R}^n$ that minimizes the Euclidean norm $\|Ax - b\|$ of the error $Ax - b$. It is actually not clear that the