Observe that the minimal polynomial m_f of f always belongs to $S_f(u, W)$, so this is a nontrivial set. Also, if $W = (0)$, then $S_f(u, (0))$ is just the annihilator of f. The crucial property of $S_f(u, W)$ is that it is an ideal.

Proposition 31.4. *If W is an invariant subspace for f, then for each $u \in E$, the f-conductor $S_f(u, W)$ is an ideal in $K[X]$.*

We leave the proof as a simple exercise, using the fact that if W invariant under f, then W is invariant under every polynomial $q(f)$ in $S_f(u, W)$.

Since $S_f(u, W)$ is an ideal, it is generated by a unique monic polynomial q of smallest degree, and because the minimal polynomial m_f of f is in $S_f(u, W)$, the polynomial q divides m.

Definition 31.3. The unique monic polynomial which generates $S_f(u, W)$ is called the *conductor of u into W.*

Example 31.1. For example, suppose $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ where $f(x, y) = (x, 0)$. Observe that $W = \{(x, 0) \in \mathbb{R}^2\}$ is invariant under f. By representing f as $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, we see that $m_f(X) = \chi_f(X) = X^2 - X$. Let $u = (0, y)$. Then $S_f(u, W) = (X)$ and we say X is the conductor of u into W.

Proposition 31.5. *Let $f\colon E \to E$ be a linear map on a finite-dimensional space E and assume that the minimal polynomial m of f is of the form*

$$m = (X - \lambda_1)^{r_1} \cdots (X - \lambda_k)^{r_k},$$

where the eigenvalues $\lambda_1, \ldots, \lambda_k$ of f belong to K. If W is a proper subspace of E which is invariant under f, then there is a vector $u \in E$ with the following properties:

(a) $u \notin W$;

(b) $(f - \lambda \mathrm{id})(u) \in W$, for some eigenvalue λ of f.

Proof. Observe that (a) and (b) together assert that the conductor of u into W is a polynomial of the form $X - \lambda_i$. Pick any vector $v \in E$ not in W, and let g be the conductor of v into W, i.e. $g(f)(v) \in W$. Since g divides m and $v \notin W$, the polynomial g is not a constant, and thus it is of the form

$$g = (X - \lambda_1)^{s_1} \cdots (X - \lambda_k)^{s_k},$$

with at least some $s_i > 0$. Choose some index j such that $s_j > 0$. Then $X - \lambda_j$ is a factor of g, so we can write

$$g = (X - \lambda_j)q. \tag{*}$$