sum of two nonzero proper submodules of M. For a proof, see Bourbaki [26] (Chapter VII, Section 4, No. 8, Proposition 8). Theorem 35.38 shows that a finitely generated module over a PID is a direct sum of indecomposable modules.

In Chapter 36 we apply the structure theorems for finitely generated (torsion) modules to the $K[X]$-module E_f associated with an endomorphism f on a vector space E. First, we need to understand the process of extension of the ring of scalars.

35.6 Extension of the Ring of Scalars

The need to extend the ring of scalars arises, in particular when dealing with eigenvalues. First we need to define how to restrict scalar multiplication to a subring. The situation is that we have two rings A and B, a B-module M, and a ring homomorphism $\rho\colon A \to B$. The special case that arises often is that A is a subring of B (B could be a field) and ρ is the inclusion map. Then we can make M into an A-module by defining the scalar multiplication $\cdot\colon A \times M \to M$ as follows.

Definition 35.14. Given two rings A and B and a ring homomorphism $\rho\colon A \to B$, any B-module M can made into an A-module denoted by $\rho_*(M)$, by defining scalar multiplication by any element of A as follows:

$$a \cdot x = \rho(a)x, \quad \text{for all } a \in A \text{ and all } x \in M.$$

In particular, viewing B as a B-module, we obtain the A-module $\rho_*(B)$.

If M and N are two B-modules and if $f\colon M \to N$ is a B-linear map, the map f is a homomorphism $f\colon \rho_*(M) \to \rho_*(N)$ of the abelian groups $\rho_*(M)$ and $\rho_*(N)$. This map is also A-linear, because for all $u \in M$ and all $a \in A$, by definition of the scalar multiplication by elements of A, we have

$$f(a \cdot u) = f(\rho(a)u) = \rho(a)f(u) = a \cdot f(u).$$

The map $f\colon \rho_*(M) \to \rho_*(N)$ viewed as an A-linear map is denoted by $\rho_*(f)$. As homomorphisms of abelian groups, the maps $f\colon M \to N$ and $\rho_*(f)\colon \rho_*(M) \to \rho_*(N)$ are identical, but f is a B-linear map whereas $\rho_*(f)$ is an A-linear map.

Now we can describe the process of scalar extension. Given any A-module M, we make $\rho_*(B) \otimes_A M$ into a (left) B-module as follows: for every $\beta \in B$, let $\mu_\beta\colon \rho_*(B) \times M \to \rho_*(B) \otimes_A M$ be given by

$$\mu_\beta(\beta', x) = (\beta\beta') \otimes x.$$

The map μ_β is bilinear so it induces a linear map $\mu_\beta\colon \rho_*(B) \otimes_A M \to \rho_*(B) \otimes_A M$ such that

$$\mu_\beta(\beta' \otimes x) = (\beta\beta') \otimes x.$$