Chapter 34

Exterior Tensor Powers and Exterior Algebras

34.1 Exterior Tensor Powers

In this chapter we consider *alternating* (also called *skew-symmetric*) multilinear maps and *exterior tensor powers* (also called *alternating tensor powers*), denoted $\bigwedge^n(E)$. In many respects alternating multilinear maps and exterior tensor powers can be treated much like symmetric tensor powers, except that $\mathrm{sgn}(\sigma)$ needs to be inserted in front of the formulae valid for symmetric powers.

Roughly speaking, we are now in the world of determinants rather than in the world of permanents. However, there are also some fundamental differences, one of which being that the exterior tensor power $\bigwedge^n(E)$ is the trivial vector space (0) when E is finite-dimensional and when $n > \dim(E)$. This chapter provides the firm foundations for understanding differential forms.

As in the case of symmetric tensor powers, since we already have the tensor algebra $T(V)$, we can proceed rather quickly. But first let us review some basic definitions and facts.

Definition 34.1. Let $f\colon E^n \to F$ be a multilinear map. We say that f *alternating* iff for all $u_i \in E$, $f(u_1, \ldots, u_n) = 0$ whenever $u_i = u_{i+1}$, for some i with $1 \leq i \leq n-1$; that is, $f(u_1, \ldots, u_n) = 0$ whenever two adjacent arguments are identical. We say that f is *skew-symmetric* (or *anti-symmetric*) iff

$$f(u_{\sigma(1)}, \ldots, u_{\sigma(n)}) = \mathrm{sgn}(\sigma) f(u_1, \ldots, u_n),$$

for every permutation $\sigma \in \mathfrak{S}_n$, and all $u_i \in E$.

For $n = 1$, we agree that every linear map $f\colon E \to F$ is alternating. The vector space of all multilinear alternating maps $f\colon E^n \to F$ is denoted $\mathrm{Alt}^n(E; F)$. Note that $\mathrm{Alt}^1(E; F) = \mathrm{Hom}(E, F)$. The following basic proposition shows the relationship between alternation and skew-symmetry.