43.4 Summary

The main concepts and results of this chapter are listed below:

- Schur complements.
- The matrix inversion lemma.
- Symmetric positive definite matrices and Schur complements.
- Symmetric positive semidefinite matrices and Schur complements.

43.5 Problems

Problem 43.1. Prove that maximizing the function $g(\lambda)$ given by

$$g(\lambda) = c_0 + \lambda c_1 - (b_0 + \lambda b_1)^\top (A_0 + \lambda A_1)^+ (b_0 + \lambda b_1),$$

subject to

$$A_0 + \lambda A_1 \succeq 0, \quad b_0 + \lambda b_1 \in \mathrm{range}(A_0 + \lambda A_1),$$

with A_0, A_1 some $n \times n$ symmetric positive semidefinite matrices, $b_0, b_1 \in \mathbb{R}^n$, and $c_0, c_1 \in \mathbb{R}$, is equivalent to maximizing γ subject to the constraints

$$\begin{aligned} &\lambda \geq 0 \\ &\begin{pmatrix} A_0 + \lambda A_1 & b_0 + \lambda b_1 \\ (b_0 + \lambda b_1)^\top & c_0 + \lambda c_1 - \gamma \end{pmatrix} \succeq 0. \end{aligned}$$

Problem 43.2. Let $a_1, \ldots, a_m$ be m vectors in $\mathbb{R}^n$ and assume that they span $\mathbb{R}^n$.

(1) Prove that the matrix

$$\sum_{k=1}^m a_k a_k^\top$$

is symmetric positive definite.

(2) Define the matrix X by

$$X = \left(\sum_{k=1}^m a_k a_k^\top \right)^{-1}.$$

Prove that

$$\begin{pmatrix} \sum_{k=1}^m a_k a_k^\top & a_i \\ a_i^\top & 1 \end{pmatrix} \succeq 0, \quad i = 1, \ldots, m.$$

Deduce that

$$a_i^\top X a_i \leq 1, \quad 1 \leq i \leq m.$$