Proposition 34.8 shows that *geometrically every nonzero wedge $u_1 \wedge \cdots \wedge u_n$ corresponds to some oriented version of an n-dimensional subspace of E.*

34.3 Some Useful Isomorphisms for Exterior Powers

We can show the following property of the exterior tensor product, using the proof technique of Proposition 33.13.

Proposition 34.9. *We have the following isomorphism:*

$$\bigwedge^n (E \oplus F) \cong \bigoplus_{k=0}^n \bigwedge^k (E) \otimes \bigwedge^{n-k} (F).$$

34.4 Duality for Exterior Powers

In this section *all vector spaces are assumed to have finite dimension*. We define a nondegenerate pairing $\bigwedge^n(E^*) \times \bigwedge^n(E) \longrightarrow K$ as follows: Consider the multilinear map

$$(E^*)^n \times E^n \longrightarrow K$$

given by

$$\begin{aligned}(v_1^*, \ldots, v_n^*, u_1, \ldots, u_n) &\mapsto \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)\, v_{\sigma(1)}^*(u_1) \cdots v_{\sigma(n)}^*(u_n) = \det(v_j^*(u_i)) \\ &= \begin{vmatrix} v_1^*(u_1) & \cdots & v_1^*(u_n) \\ \vdots & \ddots & \vdots \\ v_n^*(u_1) & \cdots & v_n^*(u_n) \end{vmatrix}.\end{aligned}$$

It is easily checked that this expression is alternating w.r.t. the u_i's and also w.r.t. the v_j^*. For any fixed $(v_1^*, \ldots, v_n^*) \in (E^*)^n$, we get an alternating multilinear map

$$l_{v_1^*, \ldots, v_n^*} \colon (u_1, \ldots, u_n) \mapsto \det(v_j^*(u_i))$$

from E^n to K. The map $l_{v_1^*, \ldots, v_n^*}$ extends uniquely to a linear map $L_{v_1^*, \ldots, v_n^*} \colon \bigwedge^n(E) \to K$ making the following diagram commute:

$$\begin{tikzcd} E^n \arrow[r, "\iota_\wedge"] \arrow[dr, "l_{v_1^*, \ldots, v_n^*}"'] & \bigwedge^n(E) \arrow[d, "L_{v_1^*, \ldots, v_n^*}"] \\ & K. \end{tikzcd}$$

We also have the alternating multilinear map

$$(v_1^*, \ldots, v_n^*) \mapsto L_{v_1^*, \ldots, v_n^*}$$