As the right hand side is clearly symmetric, we get a linear map $\eta_{\odot}\colon \mathrm{S}^n(E) \to E^{\otimes n}$ making the following diagram commute.

$$\begin{tikzcd} E^n \arrow[r, "\iota_{\odot}"] \arrow[dr, "\eta"'] & \mathrm{S}^n(E) \arrow[d, "\eta_{\odot}"] \\ & E^{\otimes n} \end{tikzcd}$$

Clearly, $\eta_{\odot}(\mathrm{S}^n(E))$ is the set of symmetrized tensors in $E^{\otimes n}$. If we consider the map $S = \eta_{\odot} \circ \pi\colon E^{\otimes n} \longrightarrow E^{\otimes n}$ where π is the surjection $\pi\colon E^{\otimes n} \to \mathrm{S}^n(E)$, it is easy to check that $S \circ S = S$. Therefore, S is a projection, and by linear algebra, we know that

$$E^{\otimes n} = S(E^{\otimes n}) \oplus \mathrm{Ker}\, S = \eta_{\odot}(\mathrm{S}^n(E)) \oplus \mathrm{Ker}\, S.$$

It turns out that $\mathrm{Ker}\, S = E^{\otimes n} \cap \mathfrak{I} = \mathrm{Ker}\, \pi$, where $\mathfrak{I}$ is the two-sided ideal of $T(E)$ generated by all tensors of the form $u \otimes v - v \otimes u \in E^{\otimes 2}$ (for example, see Knapp [104], Appendix A). Therefore, $\eta_{\odot}$ is injective,

$$E^{\otimes n} = \eta_{\odot}(\mathrm{S}^n(E)) \oplus (E^{\otimes n} \cap \mathfrak{I}) = \eta_{\odot}(\mathrm{S}^n(E)) \oplus \mathrm{Ker}\, \pi,$$

and the symmetric tensor power $\mathrm{S}^n(E)$ is naturally embedded into $E^{\otimes n}$.

33.11 Symmetric Algebras

As in the case of tensors, we can pack together all the symmetric powers $\mathrm{S}^n(V)$ into an algebra.

Definition 33.20. Given a vector space V, the space

$$\mathrm{S}(V) = \bigoplus_{m \geq 0} \mathrm{S}^m(V),$$

is called the *symmetric tensor algebra of* V.

We could adapt what we did in Section 33.6 for general tensor powers to symmetric tensors but since we already have the algebra $T(V)$, we can proceed faster. If $\mathfrak{I}$ is the two-sided ideal generated by all tensors of the form $u \otimes v - v \otimes u \in V^{\otimes 2}$, we set

$$\mathrm{S}^{\bullet}(V) = T(V)/\mathfrak{I}.$$

Observe that since the ideal $\mathfrak{I}$ is generated by elements in $V^{\otimes 2}$, every tensor in $\mathfrak{I}$ is a linear combination of tensors of the form $\omega_1 \otimes (u \otimes v - v \otimes u) \otimes \omega_2$, with $\omega_1 \in V^{\otimes n_1}$ and $\omega_2 \in V^{\otimes n_2}$ for some $n_1, n_2 \in \mathbb{N}$, which implies that

$$\mathfrak{I} = \bigoplus_{m \geq 0} (\mathfrak{I} \cap V^{\otimes m}).$$