nonnegative real number $\|A\|$ which behaves a lot like the absolute value $|x|$ of a real number x. Then our goal is to find some low-rank matrix A' that minimizes the norm

$$\|A - A'\|^2,$$

over all matrices A' of rank at most k, for some given $k \ll \min\{m, n\}$.

Some advantages of a low-rank approximation are:

1. Fewer elements are required to represent A; namely, $k(m + n)$ instead of mn. Thus less storage and fewer operations are needed to reconstruct A.

2. Often, the process for obtaining the decomposition exposes the underlying structure of the data. Thus, it may turn out that "most" of the significant data are concentrated along some directions called *principal directions*.

Low-rank decompositions of a set of data have a multitude of applications in engineering, including computer science (especially computer vision), statistics, and machine learning. As we will see later in Chapter 23, the *singular value decomposition* (SVD) provides a very satisfactory solution to the low-rank approximation problem. Still, in many cases, the data sets are so large that another ingredient is needed: *randomization*. However, as a first step, linear algebra often yields a good initial solution.

We will now be more precise as to what kinds of operations are allowed on vectors. In the early 1900, the notion of a *vector space* emerged as a convenient and unifying framework for working with "linear" objects and we will discuss this notion in the next few sections.

3.2 Vector Spaces

For every $n \geq 1$, let $\mathbb{R}^n$ be the set of n-tuples $x = (x_1, \ldots, x_n)$. Addition can be extended to $\mathbb{R}^n$ as follows:

$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$.

We can also define an operation $\cdot\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ as follows:

$\lambda \cdot (x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n)$.

The resulting algebraic structure has some interesting properties, those of a vector space.

However, keep in mind that vector spaces are not just algebraic objects; they are also geometric objects.

Vector spaces are defined as follows.