Because $p_F + p_G = \mathrm{id}$, note that we also have

$$s(u) = p_F(u) - p_G(u)$$

and

$$s(u) = u - 2p_G(u),$$

$s^2 = \mathrm{id}$, s is the identity on F, and $s = -\mathrm{id}$ on G.

We now assume that E is a Euclidean space of *finite* dimension.

Definition 13.2. Let E be a Euclidean space of finite dimension n. For any two subspaces F and G, if F and G form a direct sum $E = F \oplus G$ and F and G are orthogonal, i.e., $F = G^{\perp}$, the *orthogonal symmetry (or reflection) with respect to F and parallel to G* is the linear map $s\colon E \to E$ defined such that

$$s(u) = 2p_F(u) - u = p_F(u) - p_G(u),$$

for every $u \in E$. When F is a hyperplane, we call s a *hyperplane symmetry with respect to F (or reflection about F)*, and when G is a plane (and thus $\dim(F) = n - 2$), we call s a *flip about F*.

A reflection about a hyperplane F is shown in Figure 13.1.

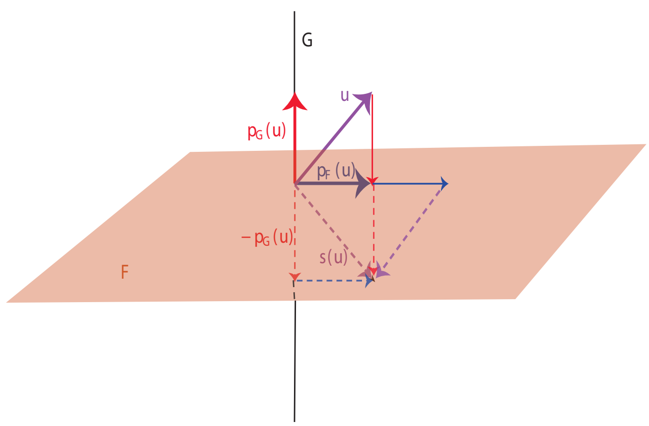

Figure 13.1: A reflection about the peach hyperplane F. Note that u is purple, $p_F(u)$ is blue and $p_G(u)$ is red.

For any two vectors $u, v \in E$, it is easily verified using the bilinearity of the inner product that

$$\|u + v\|^2 - \|u - v\|^2 = 4(u \cdot v). \qquad (*)$$

In particular, if $u \cdot v = 0$, then $\|u + v\| = \|u - v\|$. Then since

$$u = p_F(u) + p_G(u)$$