We now consider the situation where E is a finite direct sum. Given a normed affine space $E = (E_1, a_1) \oplus \cdots \oplus (E_n, a_n)$ and a normed affine space F, given any open subset A of E, for any $c = (c_1, \ldots, c_n) \in A$, we define the continuous functions $i_j^c \colon E_j \to E$, such that

$$i_j^c(x) = (c_1, \ldots, c_{j-1}, x, c_{j+1}, \ldots, c_n).$$

For any function $f \colon A \to F$, we have functions $f \circ i_j^c \colon E_j \to F$, defined on $(i_j^c)^{-1}(A)$, which contains c_j. If $\mathrm{D}(f \circ i_j^c)(c_j)$ exists, we call it the *partial derivative of f w.r.t. its jth argument, at c*. We also denote this derivative by $\mathrm{D}_j f(c)$. Note that $\mathrm{D}_j f(c) \in \mathcal{L}(\overrightarrow{E_j}; \overrightarrow{F})$.

This notion is a generalization of the notion defined in Definition 39.4. In fact, when E is of dimension n, and a frame $(a_0, (u_1, \ldots, u_n))$ has been chosen, we can write $E = (E_1, a_1) \oplus \cdots \oplus (E_n, a_n)$, for some obvious (E_j, a_j) (as explained just after Proposition 39.9), and then

$$\mathrm{D}_j f(c)(\lambda u_j) = \lambda \partial_j f(c),$$

and the two notions are consistent.

The definition of i_j^c and of $\mathrm{D}_j f(c)$ also makes sense for a finite product $E_1 \times \cdots \times E_n$ of affine spaces E_i. We will use freely the notation $\partial_j f(c)$ instead of $\mathrm{D}_j f(c)$.

The notion $\partial_j f(c)$ introduced in Definition 39.4 is really that of the vector derivative, whereas $\mathrm{D}_j f(c)$ is the corresponding linear map. Although perhaps confusing, we identify the two notions. The following proposition holds.

Proposition 39.11. *Given a normed affine space $E = (E_1, a_1) \oplus \cdots \oplus (E_n, a_n)$, and a normed affine space F, given any open subset A of E, for any function $f \colon A \to F$, for every $c \in A$, if $\mathrm{D}f(c)$ exists, then each $\mathrm{D}_j f(c)$ exists, and*

$$\mathrm{D}f(c)(u_1, \ldots, u_n) = \mathrm{D}_1 f(c)(u_1) + \cdots + \mathrm{D}_n f(c)(u_n),$$

for every $u_i \in E_i$, $1 \leq i \leq n$. The same result holds for the finite product $E_1 \times \cdots \times E_n$.

Proof. Since every $c \in E$ can be written as $c = a + c - a$, where $a = (a_1, \ldots, a_n)$, defining $f_a \colon \overrightarrow{E} \to F$ such that, $f_a(u) = f(a + u)$, for every $u \in \overrightarrow{E}$, clearly, $\mathrm{D}f(c) = \mathrm{D}f_a(c - a)$, and thus, we can work with the function f_a whose domain is the vector space $\overrightarrow{E}$. The proposition is then a simple application of Theorem 39.6. ☐

39.3 Jacobian Matrices

If both E and F are of finite dimension, for any frame $(a_0, (u_1, \ldots, u_n))$ of E and any frame $(b_0, (v_1, \ldots, v_m))$ of F, every function $f \colon E \to F$ is determined by m functions $f_i \colon E \to \mathbb{R}$ (or $f_i \colon E \to \mathbb{C}$), where

$$f(x) = b_0 + f_1(x)v_1 + \cdots + f_m(x)v_m,$$