$K^+ = (K - \{k^-\}) \cup \{j^+\}$ is a basis. This is because the coefficient $\gamma_{k^-}^{j^+}$ associated with the column A^{k^-} is nonzero (in fact, $\gamma_{k^-}^{j^+} > 0$), so Equation $(*)$, namely

$$A^{j^+} = \gamma_{k^-}^{j^+} A^{k^-} + \sum_{k \in K - \{k^-\}} \gamma_k^{j^+} A^k,$$

yields the equation

$$A^{k^-} = (\gamma_{k^-}^{j^+})^{-1} A^{j^+} - \sum_{k \in K - \{k^-\}} (\gamma_{k^-}^{j^+})^{-1} \gamma_k^{j^+} A^k,$$

and these equations imply that the subspaces spanned by the vectors $(A^k)_{k \in K}$ and the vectors $(A^k)_{k \in K^+}$ are identical. However, K is a basis of dimension m so this subspace has dimension m, and since K^+ also has m elements, it must be a basis. Therefore, $u^+ = u(\theta^{j^+})$ is a basic feasible solution.

The above case is the most common one, but other situations may arise. In what follows, we discuss all eventualities.

Case (A).

We have $c_j - \sum_{k \in K} \gamma_k^j c_k \leq 0$ for all $j \notin K$. Then it turns out that u is an *optimal solution*. Otherwise, we are in Case (B).

Case (B).

We have $c_j - \sum_{k \in K} \gamma_k^j c_k > 0$ for some $j \notin K$ (not necessarily unique). There are three subcases.

Case (B1).

If for some $j \notin K$ as above we also have $\gamma_k^j \leq 0$ for all $k \in K$, since $u_k \geq 0$ for all $k \in K$, this places no restriction on θ, and the objective function is *unbounded above*. This is demonstrated by Example 46.3 with $K = \{3, 4\}$ and $j = 2$ since $\gamma_{\{3,4\}}^2 = (-1, 0)$.

Case (B2).

There is some index $j^+ \notin K$ such that simultaneously

(1) $c_{j^+} - \sum_{k \in K} \gamma_k^{j^+} c_k > 0$, which means that the objective function can potentially be increased;

(2) There is some $k \in K$ such that $\gamma_k^{j^+} > 0$, and for every $k \in K$, if $\gamma_k^{j^+} > 0$ then $u_k > 0$, which implies that $\theta^{j^+} > 0$.

If we pick $\theta = \theta^{j^+}$ where

$$\theta^{j^+} = \min\left\{ \frac{u_k}{\gamma_k^{j^+}} \;\middle|\; \gamma_k^{j^+} > 0,\ k \in K \right\} > 0,$$