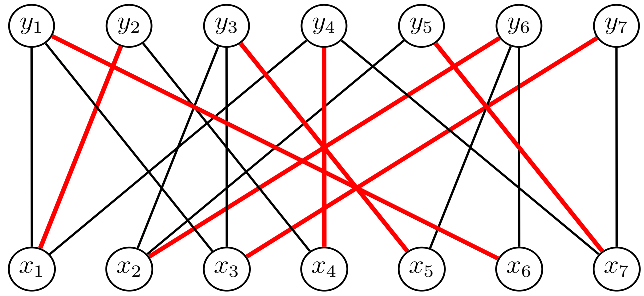

Figure 7.4: A perfect matching in the bipartite graph G.

bijections $\pi\colon \{x_1, \ldots, x_m\} \to \{y_1, \ldots, y_m\}$, we see that *the permanent* $\mathrm{per}(A)$ *of the* $(0, 1)$-*matrix* A *representing the bipartite graph* G *counts the number of perfect matchings in* G.

In a famous paper published in 1979, Leslie Valiant proves that computing the permanent is a #P-complete problem. Such problems are suspected to be intractable. It is known that if a polynomial-time algorithm existed to solve a #P-complete problem, then we would have $P = NP$, which is believed to be very unlikely.

Another combinatorial interpretation of the permanent can be given in terms of systems of distinct representatives. Given a finite set S, let $(A_1, \ldots, A_n)$ be any sequence of nonempty subsets of S (not necessarily distinct). A *system of distinct representatives* (for short *SDR*) of the sets $A_1, \ldots, A_n$ is a sequence of n distinct elements $(a_1, \ldots, a_n)$, with $a_i \in A_i$ for $i = 1, \ldots, n$. The number of SDR's of a sequence of sets plays an important role in combinatorics. Now, if $S = \{1, 2, \ldots, n\}$ and if we associate to any sequence $(A_1, \ldots, A_n)$ of nonempty subsets of S the matrix $A = (a_{ij})$ defined such that $a_{ij} = 1$ if $j \in A_i$ and $a_{ij} = 0$ otherwise, then *the permanent* $\mathrm{per}(A)$ *counts the number of SDR's of the sets* $A_1, \ldots, A_n$.

This interpretation of permanents in terms of SDR's can be used to prove bounds for the permanents of various classes of matrices. Interested readers are referred to van Lint and Wilson [180] (Chapters 11 and 12). In particular, a proof of a theorem known as *Van der Waerden conjecture* is given in Chapter 12. This theorem states that for any $n \times n$ matrix A with nonnegative entries in which all row-sums and column-sums are 1 (doubly stochastic matrices), we have

$$\mathrm{per}(A) \geq \frac{n!}{n^n},$$

with equality for the matrix in which all entries are equal to $1/n$.

7.9 Summary

The main concepts and results of this chapter are listed below: