Proposition 37.31. *Let E be a topological space and let F be a topological Hausdorff space. For every compact subset, A, of E, for every continuous map, $f\colon E \to F$, the subspace $f(A)$ is compact.*

Proof. Let $(U_i)_{i\in I}$ be an open cover of $f(A)$. We claim that $(f^{-1}(U_i))_{i\in I}$ is an open cover of A, which is easily checked. Since A is compact, there is a finite open subcover, $(f^{-1}(U_j))_{j\in J}$, of A, and thus, $(U_j)_{j\in J}$ is an open subcover of $f(A)$. $\square$

As a corollary of Proposition 37.31, if E is compact, F is Hausdorff, and $f\colon E \to F$ is continuous and bijective, then f is a homeomorphism. Indeed, it is enough to show that f^{-1} is continuous, which is equivalent to showing that f maps closed sets to closed sets. However, closed sets are compact and Proposition 37.31 shows that compact sets are mapped to compact sets, which, by Proposition 37.26, are closed.

Another important corollary of Proposition 37.31 is the following result.

Proposition 37.32. *If E is a compact nonempty topological space and if $f\colon E \to \mathbb{R}$ is a continuous function, then there are points $a, b \in E$ such that $f(a)$ is the minimum of $f(E)$ and $f(b)$ is the maximum of $f(E)$.*

Proof. The set $f(E)$ is a compact subset of $\mathbb{R}$ and thus, a closed and bounded set which contains its greatest lower bound and its least upper bound. $\square$

The following property also holds.

Proposition 37.33. *Let (E, d) be a metric space. For any nonempty subset A of E, if A is compact, then for every open subset U such that $A \subseteq U$, there is some $r > 0$ such that $V_r(A) \subseteq U$.*

Proof. The function $x \mapsto d(x, E - U)$ is continuous and $d(x, E - U) > 0$ for $x \in A$ (since $A \subseteq U$). By Proposition 37.32, there is some $a \in A$ such that

$$d(a, E - U) = \inf_{x\in A} d(x, E - U).$$

But $d(a, E - U) = r > 0$, which implies that $V_r(A) \subseteq U$. $\square$

Another useful notion is that of local compactness. Indeed manifolds and surfaces are locally compact.

Definition 37.31. A topological space E is *locally compact* if it is Hausdorff and for every $a \in E$, there is some compact neighborhood K of a. See Figure 37.33.

From Proposition 37.30, every compact space is locally compact but the converse is false. For example, $\mathbb{R}$ is locally compact but not compact. In fact it can be shown that a normed vector space of finite dimension is locally compact.