Proposition 37.34. *Given a locally compact topological space, E, for every $a \in E$, for every neighborhood, N, of a, there exists a compact neighborhood, U, of a, such that $U \subseteq N$.*

Proof. For any $a \in E$, there is some compact neighborhood, V, of a. By Proposition 37.30, every neigborhood of a relative to V contains some compact neighborhood U of a relative to V. But every neighborhood of a relative to V is a neighborhood of a relative to E and every neighborhood N of a in E yields a neighborhood, $V \cap N$, of a in V and thus, for every neighborhood, N, of a, there exists a compact neighborhood, U, of a such that $U \subseteq N$. $\square$

When E is a metric space, the subsets $V_r(A)$ defined in Definition 37.6 have the following property.

Proposition 37.35. *Let (E, d) be a metric space. If E is locally compact, then for any nonempty compact subset A of E, there is some $r > 0$ such that $\overline{V_r(A)}$ is compact.*

Proof. Since E is locally compact, for every $x \in A$, there is some compact subset V_x whose interior $\overset{\circ}{V}_x$ contains x. The family of open subsets $\overset{\circ}{V}_x$ is an open cover A, and since A is compact, it has a finite subcover $\{\overset{\circ}{V}_{x_1}, \ldots, \overset{\circ}{V}_{x_n}\}$. Then $U = V_{x_1} \cup \cdots \cup V_{x_n}$ is compact (as a finite union of compact subsets), and it contains an open subset containing A (the union of the $\overset{\circ}{V}_{x_i}$). By Proposition 37.33, there is some $r > 0$ such that $V_r(A) \subseteq \overset{\circ}{U}$, and thus $\overline{V_r(A)} \subseteq U$. Since U is compact and $\overline{V_r(A)}$ is closed, $\overline{V_r(A)}$ is compact. $\square$

It is much harder to deal with noncompact manifolds than it is to deal with compact manifolds. However, manifolds are locally compact and it turns out that there are various ways of embedding a locally compact Hausdorff space into a compact Hausdorff space. The most economical construction consists in adding just one point. This construction, known as the *Alexandroff compactification*, is technically useful, and we now describe it and sketch the proof that it achieves its goal.

To help the reader's intuition, let us consider the case of the plane, $\mathbb{R}^2$. If we view the plane, $\mathbb{R}^2$, as embedded in 3-space, $\mathbb{R}^3$, say as the xy plane of equation $z = 0$, we can consider the sphere, Σ, of radius 1 centered on the z-axis at the point $(0, 0, 1)$ and tangent to the xOy plane at the origin (sphere of equation $x^2 + y^2 + (z - 1)^2 = 1$). If N denotes the north pole on the sphere, i.e., the point of coordinates $(0, 0, 2)$, then any line, D, passing through the north pole and not tangent to the sphere (i.e., not parallel to the xOy plane) intersects the xOy plane in a unique point, M, and the sphere in a unique point, P, other than the north pole, N. This, way, we obtain a bijection between the xOy plane and the punctured sphere Σ, i.e., the sphere with the north pole N deleted. This bijection is called a *stereographic projection*. See Figure 37.35.

The Alexandroff compactification of the plane puts the north pole back on the sphere, which amounts to adding a single point at infinity ∞ to the plane. Intuitively, as we travel away from the origin O towards infinity (in any direction!), we tend towards an ideal point at infinity ∞. Imagine that we "bend" the plane so that it gets wrapped around the sphere,