By subtracting Equation (3) from Equation (1) we get

$$v^* K_S u = A - iB.$$

Then

$$u^* K_S^* v = \overline{v^* K_S u} = \overline{A - iB} = A + iB = u^* K_S v,$$

for all $u, v \in \mathbb{C}^*$, which implies $K_S^* = K_S$. $\square$

If the map $\kappa\colon X \times X \to \mathbb{R}$ is real-valued, then we have the following criterion for κ to be a positive definite kernel that only involves real vectors.

Proposition 53.3. *If $\kappa\colon X \times X \to \mathbb{R}$, then κ is a positive definite kernel iff for any finite subset $S = \{x_1, \ldots, x_p\}$ of X, the $p \times p$ real matrix K_S given by*

$$K_S = (\kappa(x_k, x_j))_{1 \leq j,k \leq p}$$

is symmetric, that is, $K_S^\top = K_S$, and

$$u^\top K_S\, u = \sum_{j,k=1}^{p} \kappa(x_j, x_k) u_j u_k \geq 0, \qquad \textit{for all } u \in \mathbb{R}^p.$$

Proof. If κ is a real-valued positive definite kernel, then the proposition is a trivial consequence of Proposition 53.2.

For the converse assume that κ is symmetric and that it satisfies the second condition of the proposition. We need to show that κ is a positive definite kernel with respect to complex vectors. If we write $u_k = a_k + ib_k$, then

$$\begin{aligned} u^* K_S\, u &= \sum_{j,k=1}^{p} \kappa(x_j, x_k)(a_j + ib_j)(a_k - ib_k) \\ &= \sum_{j,k=1}^{p} (a_j a_k + b_j b_k)\kappa(x_j, x_k) + i \sum_{j,k=1}^{p} (b_j a_k - a_j b_k)\kappa(x_j, x_k) \\ &= \sum_{j,k=1}^{p} (a_j a_k + b_j b_k)\kappa(x_j, x_k) + i \sum_{1 \leq j < k \leq p} b_j a_k(\kappa(x_j, x_k) - \kappa(x_k, x_j)). \end{aligned}$$

Thus $u^* K_S u$ is real iff K_S is symmetric. $\square$

Consequently we make the following definition.

Definition 53.3. Let X be a nonempty set. A function $\kappa\colon X \times X \to \mathbb{R}$ is a *(real) positive definite kernel* if $\kappa(x, y) = \kappa(y, x)$ for all $x, y \in X$, and for every finite subset $S = \{x_1, \ldots, x_p\}$ of X, if K_S is the $p \times p$ real symmetric matrix

$$K_S = (\kappa(x_i, x_j))_{1 \leq i,j \leq p},$$