Note that a real hypersurface may have points other than real points, or no real points at all. For example,

$$x^2 + y^2 - z^2 = 0$$

contains real and complex points such as $(1, i, 0)$ and $(1, -i, 0)$, and

$$x^2 + y^2 + z^2 = 0$$

contains only complex points. When $m = 2$ (where m is the total degree of P), a hypersurface is called a *quadric*, and when $m = 2$ and $n = 2$, a *conic*. When $m = 1$, a hypersurface is just a hyperplane.

Given any homogeneous polynomial $P(x_1, \ldots, x_{n+1})$ over $\mathbb{R}$ of total degree m, since $\mathbb{R} \subseteq \mathbb{C}$, P viewed as a homogeneous polynomial over $\mathbb{C}$ defines a hypersurface $V(P)_{\mathbb{C}}$ in $\widetilde{E}_{\mathbb{C}}$, and also a hypersurface $V(P)$ in $\mathbf{P}(E)$. It is clear that $V(P)$ is naturally embedded in $V(P)_{\mathbb{C}}$, and $V(P)_{\mathbb{C}}$ is called the *complexification of* $V(P)$.

We now show how certain real quadrics without real points can be used to define orthogonality and angles.

26.15 Similarity Structures on a Projective Space

We begin with a real Euclidean plane $\big(E, \overrightarrow{E}\big)$. We will show that the angle of two lines D_1 and D_2 can be expressed as a certain cross-ratio involving the lines D_1, D_2 and also two lines D_I and D_J joining the intersection point $D_1 \cap D_2$ of D_1 and D_2 to two complex points at infinity I and J called the *circular points*. However, there is a slight problem, which is that we haven't yet defined the angle of two lines! Recall that we define the (oriented) angle $\widehat{u_1 u_2}$ of two unit vectors u_1, u_2 as the equivalence class of pairs of unit vectors under the equivalence relation defined such that

$$\langle u_1, u_2 \rangle \equiv \langle u_3, u_4 \rangle$$

iff there is some rotation r such that $r(u_1) = u_3$ and $r(u_2) = u_4$. The set of (oriented) angles of vectors is a group isomorphic to the group $\mathbf{SO}(2)$ of plane rotations. If the Euclidean plane is oriented, the measure of the angle of two vectors is defined up to $2k\pi$ ($k \in \mathbb{Z}$). The angle of two vectors has a measure that is either θ or $2\pi - \theta$, where $\theta \in [0, 2\pi[$, depending on the orientation of the plane. The problem with lines is that they are not oriented: A line is defined by a point a and a vector u, but also by a and $-u$. Given any two lines D_1 and D_2, if r is a rotation of angle θ such that $r(D_1) = D_2$, note that the rotation $-r$ of angle $\theta + \pi$ also maps D_1 onto D_2. Thus, in order to define the (oriented) angle $\widehat{D_1 D_2}$ of two lines D_1, D_2, we define an equivalence relation on pairs of lines as follows:

$$\langle D_1, D_2 \rangle \equiv \langle D_3, D_4 \rangle$$