37.10 The Contraction Mapping Theorem

If (E, d) is a nonempty complete metric space, every map, $f\colon E \to E$, for which there is some k such that $0 \leq k < 1$ and

$$d(f(x), f(y)) \leq kd(x, y)$$

for all $x, y \in E$, has the very important property that it has a unique fixed point, that is, there is a unique, $a \in E$, such that $f(a) = a$. A map as above is called a *contraction mapping*. Furthermore, the fixed point of a contraction mapping can be computed as the limit of a fast converging sequence.

The fixed point property of contraction mappings is used to show some important theorems of analysis, such as the implicit function theorem and the existence of solutions to certain differential equations. It can also be used to show the existence of fractal sets defined in terms of iterated function systems. Since the proof is quite simple, we prove the fixed point property of contraction mappings. First, observe that a contraction mapping is (uniformly) continuous.

Proposition 37.54. *If (E, d) is a nonempty complete metric space, every contraction mapping, $f\colon E \to E$, has a unique fixed point. Furthermore, for every $x_0 \in E$, defining the sequence, (x_n), such that $x_{n+1} = f(x_n)$, the sequence, (x_n), converges to the unique fixed point of f.*

Proof. First we prove that f has at most one fixed point. Indeed, if $f(a) = a$ and $f(b) = b$, since

$$d(a, b) = d(f(a), f(b)) \leq kd(a, b)$$

and $0 \leq k < 1$, we must have $d(a, b) = 0$, that is, $a = b$.

Next, we prove that (x_n) is a Cauchy sequence. Observe that

$$\begin{aligned}
d(x_2, x_1) &\leq kd(x_1, x_0), \\
d(x_3, x_2) &\leq kd(x_2, x_1) \leq k^2 d(x_1, x_0), \\
\vdots & \qquad \vdots \\
d(x_{n+1}, x_n) &\leq kd(x_n, x_{n-1}) \leq \cdots \leq k^n d(x_1, x_0).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
d(x_{n+p}, x_n) &\leq d(x_{n+p}, x_{n+p-1}) + d(x_{n+p-1}, x_{n+p-2}) + \cdots + d(x_{n+1}, x_n) \\
&\leq (k^{p-1} + k^{p-2} + \cdots + k + 1)k^n d(x_1, x_0) \\
&\leq \frac{k^n}{1-k}\, d(x_1, x_0).
\end{aligned}$$