Sometimes, we say *global* maximum (or minimum) to stress that a maximum (or a minimum) is not simply a local maximum (or minimum).

Theorem 40.13. *Given any normed vector space E, let U be any nonempty convex subset of E.*

(1) For any convex function $J\colon U \to \mathbb{R}$, for any $u \in U$, if J has a local minimum at u in U, then J has a (global) minimum at u in U.

(2) Any strict convex function $J\colon U \to \mathbb{R}$ has at most one minimum (in U), and if it does, then it is a strict minimum (in U).

(3) Let $J\colon \Omega \to \mathbb{R}$ be any function defined on some open subset Ω of E with $U \subseteq \Omega$ and assume that J is convex on U. For any point $u \in U$, if $dJ(u)$ exists, then J has a minimum in u with respect to U iff

$$dJ(u)(v-u) \geq 0 \quad \text{for all } v \in U.$$

(4) If the convex subset U in (3) is open, then the above condition is equivalent to

$$dJ(u) = 0.$$

Proof. (1) Let $v = u + w$ be any arbitrary point in U. Since J is convex, for all t with $0 \leq t \leq 1$, we have

$$J(u+tw) = J(u+t(v-u)) = J((1-t)u+tv) \leq (1-t)J(u) + tJ(v),$$

which yields

$$J(u+tw) - J(u) \leq t(J(v) - J(u)).$$

Because J has a local minimum at u, there is some t_0 with $0 < t_0 < 1$ such that

$$0 \leq J(u+t_0w) - J(u) \leq t_0(J(v) - J(u)),$$

which implies that $J(v) - J(u) \geq 0$.

(2) If J is strictly convex, the above reasoning with $w \neq 0$ shows that there is some t_0 with $0 < t_0 < 1$ such that

$$0 \leq J(u+t_0w) - J(u) < t_0(J(v) - J(u)),$$

which shows that u is a strict global minimum (in U), and thus that it is unique.

(3) We already know from Theorem 40.9 that the condition $dJ(u)(v-u) \geq 0$ for all $v \in U$ is necessary (even if J is not convex). Conversely, because J is convex, careful inspection of the proof of Part (1) of Proposition 40.11 shows that only the fact that $dJ(u)$ exists is needed to prove that

$$J(v) - J(u) \geq dJ(u)(v-u) \quad \text{for all } v \in U,$$