for all $t \in I$ is well-defined. By applying the chain rule, we see that $\varphi$ is differentiable at $t = 0$, and we get

$$\varphi'(0) = dJ_u(v).$$

Without loss of generality, assume that $u$ is a local minimum. Then we have

$$\varphi'(0) = \lim_{t \mapsto 0_-} \frac{\varphi(t) - \varphi(0)}{t} \leq 0$$

and

$$\varphi'(0) = \lim_{t \mapsto 0_+} \frac{\varphi(t) - \varphi(0)}{t} \geq 0,$$

which shows that $\varphi'(0) = dJ_u(v) = 0$. As $v \in E$ is arbitrary, we conclude that $dJ_u = 0$. $\square$

**Definition 40.2.** A point $u \in \Omega$ such that $J'(u) = 0$ is called a *critical point* of $J$.

If $E = \mathbb{R}^n$, then the condition $dJ_u = 0$ is equivalent to the system

$$\begin{aligned} \frac{\partial J}{\partial x_1}(u_1, \ldots, u_n) &= 0 \\ &\vdots \\ \frac{\partial J}{\partial x_n}(u_1, \ldots, u_n) &= 0. \end{aligned}$$

The condition of Proposition 40.1 is only a *necessary* condition for the existence of an extremum, but not a sufficient condition.

Here are some counter-examples. If $f \colon \mathbb{R} \to \mathbb{R}$ is the function given by $f(x) = x^3$, since $f'(x) = 3x^2$, we have $f'(0) = 0$, but 0 is neither a minimum nor a maximum of $f$ as evidenced by the graph shown in Figure 40.1.

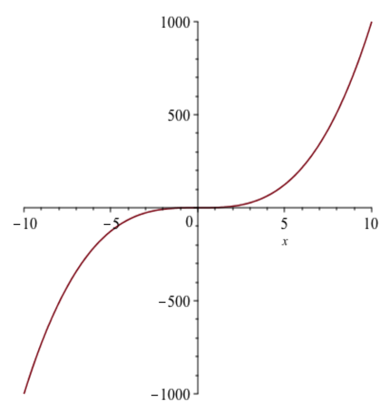


Figure 40.1: The graph of $f(x) = x^3$. Note that $x = 0$ is a saddle point and not a local extremum.