Proposition 30.25. *Let A be any ring. For every nonnull polynomial $f \in A[X]$, $\alpha \in A$ is a simple root of f iff α is a root of f and α is not a root of f'.*

Proof. Since $\alpha \in A$ is a root of f, we have $f = (X - \alpha)g$ for some $g \in A[X]$. Now, α is a simple root of f iff $g(\alpha) \neq 0$. However, we have $f' = g + (X - \alpha)g'$, and so $f'(\alpha) = g(\alpha)$. Thus, α is a simple root of f iff $f'(\alpha) \neq 0$. $\square$

We can improve the previous proposition as follows.

Proposition 30.26. *Let A be any ring. For every nonnull polynomial $f \in A[X]$, let $\alpha \in A$ be a root of multiplicity $k \geq 1$ of f. Then, α is a root of multiplicity at least $k - 1$ of f'. If A is a field of characteristic zero, then α is a root of multiplicity $k - 1$ of f'.*

Proof. Since $\alpha \in A$ is a root of multiplicity k of f, we have $f = (X - \alpha)^k g$ for some $g \in A[X]$ and $g(\alpha) \neq 0$. Since

$$f' = k(X - \alpha)^{k-1} g + (X - \alpha)^k g' = (X - \alpha)^{k-1}(kg + (X - \alpha)g'),$$

it is clear that the multiplicity of α w.r.t. f' is at least $k-1$. Now, $(kg+(X-\alpha)g')(\alpha) = kg(\alpha)$, and if A is of characteristic zero, since $g(\alpha) \neq 0$, then $kg(\alpha) \neq 0$. Thus, α is a root of multiplicity $k - 1$ of f'. $\square$

As a consequence, we obtain the following test for the existence of a root of multiplicity k for a polynomial f:

Given a field K of characteristic zero, for any nonnull polynomial $f \in K[X]$, any $\alpha \in K$ is a root of multiplicity $k \geq 1$ of f iff α is a root of $f, \mathrm{D}^1 f, \mathrm{D}^2 f, \ldots, \mathrm{D}^{k-1} f$, but not a root of $\mathrm{D}^k f$.

We can now return to polynomial functions and tie up some loose ends. Given a ring A, recall that every polynomial $f \in A[X_1, \ldots, X_n]$ induces a function $f_A \colon A^n \to A$ defined such that $f_A(\alpha_1, \ldots, \alpha_n) = f(\alpha_1, \ldots, \alpha_n)$, for every $(\alpha_1, \ldots, \alpha_n) \in A^n$. We now give a sufficient condition for the mapping $f \mapsto f_A$ to be injective.

Proposition 30.27. *Let A be an integral domain. For every polynomial $f \in A[X_1, \ldots, X_n]$, if $A_1, \ldots, A_n$ are n infinite subsets of A such that $f(\alpha_1, \ldots, \alpha_n) = 0$ for all $(\alpha_1, \ldots, \alpha_n) \in A_1 \times \cdots \times A_n$, then $f = 0$, i.e., f is the null polynomial. As a consequence, if A is an infinite integral domain, then the map $f \mapsto f_A$ is injective.*

Proof. We proceed by induction on n. Assume $n = 1$. If $f \in A[X_1]$ is nonnull, let $m = \deg(f)$ be its degree. Since A_1 is infinite and $f(\alpha_1) = 0$ for all $\alpha_1 \in A_1$, then f has an infinite number of roots. But since f is of degree m, this contradicts Theorem 30.23. Thus, $f = 0$.

If $n \geq 2$, we can view $f \in A[X_1, \ldots, X_n]$ as a polynomial

$$f = g_m X_n^m + g_{m-1} X_n^{m-1} + \cdots + g_0,$$