Definition 33.4. A *tensor product* of $n \geq 2$ vector spaces $E_1, \ldots, E_n$ is a vector space T together with a multilinear map $\varphi\colon E_1 \times \cdots \times E_n \to T$, such that for every vector space F and for every multilinear map $f\colon E_1 \times \cdots \times E_n \to F$, there is a unique linear map $f_\otimes\colon T \to F$ with

$$f(u_1, \ldots, u_n) = f_\otimes(\varphi(u_1, \ldots, u_n)),$$

for all $u_1 \in E_1, \ldots, u_n \in E_n$, or for short

$$f = f_\otimes \circ \varphi.$$

Equivalently, there is a unique linear map $f_\otimes$ such that the following diagram commutes.

$$\begin{tikzcd} E_1 \times \cdots \times E_n \arrow[r, "\varphi"] \arrow[dr, "f"'] & T \arrow[d, "f_\otimes"] \\ & F \end{tikzcd}$$

The above property is called the *universal mapping property* of the tensor product (T, φ).

We show that any two tensor products (T_1, φ_1) and (T_2, φ_2) for $E_1, \ldots, E_n$, are isomorphic.

Proposition 33.5. *Given any two tensor products (T_1, φ_1) and (T_2, φ_2) for $E_1, \ldots, E_n$, there is an isomorphism $h\colon T_1 \to T_2$ such that*

$$\varphi_2 = h \circ \varphi_1.$$

Proof. Focusing on (T_1, φ_1), we have a multilinear map $\varphi_2\colon E_1 \times \cdots \times E_n \to T_2$, and thus there is a unique linear map $(\varphi_2)_\otimes\colon T_1 \to T_2$ with

$$\varphi_2 = (\varphi_2)_\otimes \circ \varphi_1$$

as illustrated by the following commutative diagram.

$$\begin{tikzcd} E_1 \times \cdots \times E_n \arrow[r, "\varphi_1"] \arrow[dr, "\varphi_2"'] & T_1 \arrow[d, "(\varphi_2)_\otimes"] \\ & T_2 \end{tikzcd}$$

Similarly, focusing now on on (T_2, φ_2), we have a multilinear map $\varphi_1\colon E_1 \times \cdots \times E_n \to T_1$, and thus there is a unique linear map $(\varphi_1)_\otimes\colon T_2 \to T_1$ with

$$\varphi_1 = (\varphi_1)_\otimes \circ \varphi_2$$