An affine subspace of codimension 1 is called a *hyperplane* (recall that a subspace F of a vector space E has codimension 1 iff there is some subspace G of dimension 1 such that $E = F \oplus G$, the direct sum of F and G, see Strang [170] or Lang [109]).

We say that two affine subspaces U and V are *parallel* if their directions are identical. Equivalently, since $\overrightarrow{U} = \overrightarrow{V}$, we have $U = a + \overrightarrow{U}$ and $V = b + \overrightarrow{U}$ for any $a \in U$ and any $b \in V$, and thus V is obtained from U by the translation $\overrightarrow{ab}$.

In general, when we talk about n points $a_1, \ldots, a_n$, we mean the sequence $(a_1, \ldots, a_n)$, and not the set $\{a_1, \ldots, a_n\}$ (the a_i's need not be distinct).

By Proposition 24.2, a line is specified by a point $a \in E$ and a nonzero vector $v \in \overrightarrow{E}$, i.e., a line is the set of all points of the form $a + \lambda v$, for $\lambda \in \mathbb{R}$.

We say that three points a, b, c are *collinear* if the vectors $\overrightarrow{ab}$ and $\overrightarrow{ac}$ are linearly dependent. If two of the points a, b, c are distinct, say $a \neq b$, then there is a unique $\lambda \in \mathbb{R}$ such that $\overrightarrow{ac} = \lambda \overrightarrow{ab}$, and we define the ratio $\frac{\overrightarrow{ac}}{\overrightarrow{ab}} = \lambda$.

A plane is specified by a point $a \in E$ and two linearly independent vectors $u, v \in \overrightarrow{E}$, i.e., a plane is the set of all points of the form $a + \lambda u + \mu v$, for $\lambda, \mu \in \mathbb{R}$.

We say that four points a, b, c, d are *coplanar* if the vectors $\overrightarrow{ab}, \overrightarrow{ac}$, and $\overrightarrow{ad}$ are linearly dependent. Hyperplanes will be characterized a little later.

Proposition 24.3. *Given an affine space $\langle E, \overrightarrow{E}, +\rangle$, for any family $(a_i)_{i\in I}$ of points in E, the set V of barycenters $\sum_{i\in I} \lambda_i a_i$ (where $\sum_{i\in I} \lambda_i = 1$) is the smallest affine subspace containing $(a_i)_{i\in I}$.*

Proof. If $(a_i)_{i\in I}$ is empty, then $V = \emptyset$, because of the condition $\sum_{i\in I} \lambda_i = 1$. If $(a_i)_{i\in I}$ is nonempty, then the smallest affine subspace containing $(a_i)_{i\in I}$ must contain the set V of barycenters $\sum_{i\in I} \lambda_i a_i$, and thus, it is enough to show that V is closed under affine combinations, which is immediately verified. $\square$

Given a nonempty subset S of E, the smallest affine subspace of E generated by S is often denoted by $\langle S\rangle$. For example, a line specified by two distinct points a and b is denoted by $\langle a, b\rangle$, or even (a, b), and similarly for planes, etc.

Remarks:

(1) Since it can be shown that the barycenter of n weighted points can be obtained by repeated computations of barycenters of two weighted points, a nonempty subset V of E is an affine subspace iff for every two points $a, b \in V$, the set V contains all barycentric combinations of a and b. If V contains at least two points, then V is an affine subspace iff for any two distinct points $a, b \in V$, the set V contains the line determined by a and b, that is, the set of all points $(1-\lambda)a + \lambda b$, $\lambda \in \mathbb{R}$.

(2) This result still holds if the field K has at least three distinct elements, but the proof is trickier!