presenting the module M, then we have the relations

$$\begin{aligned} 2e_1 + e_2 &= 0 \\ -e_1 + 2e_2 &= 0. \end{aligned}$$

From the first equation, we get $e_2 = -2e_1$, and substituting into the second equation we get

$$-5e_1 = 0.$$

It follows that the generator e_2 can be eliminated and M is generated by the single generator e_1 satisfying the relation

$$5e_1 = 0,$$

which shows that $M \approx \mathbb{Z}/5\mathbb{Z}$.

The above example shows that many different matrices can present the same module. Here are some useful rules for manipulating a relation matrix without changing the isomorphism class of the module M it presents.

Proposition 35.8. *If R is an $m \times n$ matrix presenting an A-module M, then the matrices S of the form listed below present the same module (a module isomorphic to M):*

(1) $S = QRP^{-1}$, where Q is a $m \times m$ invertible matrix and P a $n \times n$ invertible matrix (both over A).

(2) S is obtained from R by deleting a column of zeros.

(3) The jth column of R is e_i, and S is obtained from R by deleting the ith row and the jth column.

Proof. (1) By definition, we have an isomorphism $M \approx A^m/RA^n$, where we denote by RA^n the image of A^n by the linear map defined by R. Going from R to QRP^{-1} corresponds to making a change of basis in A^m and a change of basis in A^n, and this yields a quotient module isomorphic to M.

(2) A zero column does not contribute to the span of the columns of R, so it can be eliminated.

(3) If the jth column of R is e_i, then when taking the quotient A^m/RA^n, the generator e_i goes to zero. This means that the generator e_i is redundant, and when we delete it, we get a matrix of relations in which the ith row of R and the jth column of R are deleted. $\square$

The matrices P and Q are often products of elementary operations. One should be careful that rows of zeros cannnot be eliminated. For example, the 2×1 matrix

$$R_1 = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$