of the Cartan–Dieudonné theorem can easily be shown: every affine isometry in $\mathbf{Is}(n, \mathbb{C})$ can be written as the composition of at most $2n-1$ isometries if it has a fixed point, or else as the composition of at most $2n+1$ isometries, where all these isometries are affine hyperplane reflections except for possibly one affine Hermitian reflection. We also prove that every rigid motion in $\mathbf{SE}(n, \mathbb{C})$ is the composition of at most $2n-2$ flips (for $n \geq 3$).

Definition 28.2. Given any two nontrivial Hermitian affine spaces E and F of the same finite dimension n, a function $f\colon E \to F$ is *an affine isometry (or rigid map)* iff it is an affine map and

$$\left\|\overrightarrow{f(a)f(b)}\right\| = \left\|\overrightarrow{ab}\right\|,$$

for all $a, b \in E$. When $E = F$, an affine isometry $f\colon E \to E$ is also called a *rigid motion*.

Thus, an affine isometry is an affine map that preserves the distance. This is a rather strong requirement, but unlike the Euclidean case, not strong enough to force f to be an affine map.

The following simple Proposition is left as an exercise.

Proposition 28.9. *Given any two nontrivial Hermitian affine spaces E and F of the same finite dimension n, an affine map $f\colon E \to F$ is an affine isometry iff its associated linear map $\overrightarrow{f}\colon \overrightarrow{E} \to \overrightarrow{F}$ is an isometry. An affine isometry is a bijection.*

As in the Euclidean case, given an affine isometry $f\colon E \to E$, if $\overrightarrow{f}$ is a rotation, we call f a *proper (or direct) affine isometry*, and if $\overrightarrow{f}$ is a an improper linear isometry, we call f a *an improper (or skew) affine isometry*. It is easily shown that the set of affine isometries $f\colon E \to E$ forms a group, and those for which $\overrightarrow{f}$ is a rotation is a subgroup. The group of affine isometries, or rigid motions, is a subgroup of the affine group $\mathbf{GA}(E, \mathbb{C})$ denoted as $\mathbf{Is}(E, \mathbb{C})$ (or $\mathbf{Is}(n, \mathbb{C})$ when $E = \mathbb{C}^n$). The subgroup of $\mathbf{Is}(E, \mathbb{C})$ consisting of the direct rigid motions is also a subgroup of $\mathbf{SA}(E, \mathbb{C})$, and it is denoted as $\mathbf{SE}(E, \mathbb{C})$ (or $\mathbf{SE}(n, \mathbb{C})$, when $E = \mathbb{C}^n$). The translations are the affine isometries f for which $\overrightarrow{f} = \mathrm{id}$, the identity map on $\overrightarrow{E}$. The following Proposition is the counterpart of Proposition 14.14 for isometries between Hermitian vector spaces.

Proposition 28.10. *Given any two nontrivial Hermitian affine spaces E and F of the same finite dimension n, for every function $f\colon E \to F$, the following properties are equivalent:*

(1) f is an affine map and $\left\|\overrightarrow{f(a)f(b)}\right\| = \left\|\overrightarrow{ab}\right\|$, for all $a, b \in E$.

(2) $\left\|\overrightarrow{f(a)f(b)}\right\| = \left\|\overrightarrow{ab}\right\|$, and there is some $\Omega \in E$ such that

$$f(\Omega + i\overrightarrow{ab}) = f(\Omega) + i(\overrightarrow{f(\Omega)f(\Omega + \overrightarrow{ab})}),$$

for all $a, b \in E$.