where $\mathrm{tr}(h)$ is the *trace* of h, where h is viewed as the linear map given by the matrix, (a_{ij}). Actually, since $c_{1,1}$ is defined independently of any basis, $c_{1,1}$ provides an intrinsic definition of the trace of a linear map $h \in \mathrm{Hom}(V, V)$.

Remark: Using the Einstein summation convention, if

$$\alpha = a^{i_1,\ldots,i_r}_{j_1,\ldots,j_s} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s},$$

then

$$c_{k,l}(\alpha) = a^{i_1,\ldots,i_{k-1},i_{k+1}\ldots,i_r}_{j_1,\ldots,j_{l-1},j_{l+1},\ldots,j_s} e_{i_1} \otimes \cdots \otimes \widehat{e_{i_k}} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes \widehat{e^{j_l}} \otimes \cdots \otimes e^{j_s}.$$

If E and F are two K-algebras, we know that their tensor product $E \otimes F$ exists as a vector space. We can make $E \otimes F$ into an algebra as well. Indeed, we have the multilinear map

$$E \times F \times E \times F \longrightarrow E \otimes F$$

given by $(a, b, c, d) \mapsto (ac) \otimes (bd)$, where ac is the product of a and c in E and bd is the product of b and d in F. By the universal mapping property, we get a linear map,

$$E \otimes F \otimes E \otimes F \longrightarrow E \otimes F.$$

Using the isomorphism

$$E \otimes F \otimes E \otimes F \cong (E \otimes F) \otimes (E \otimes F),$$

we get a linear map

$$(E \otimes F) \otimes (E \otimes F) \longrightarrow E \otimes F,$$

and thus a bilinear map,

$$(E \otimes F) \times (E \otimes F) \longrightarrow E \otimes F$$

which is our multiplication operation in $E \otimes F$. This multiplication is determined by

$$(a \otimes b) \cdot (c \otimes d) = (ac) \otimes (bd).$$

In summary we have the following proposition.

Proposition 33.22. *Given two K-algebra E and F, the operation on $E \otimes F$ defined on generators by*

$$(a \otimes b) \cdot (c \otimes d) = (ac) \otimes (bd)$$

makes $E \otimes F$ into a K-algebra.

We now turn to symmetric tensors.