and similarly, $\dim(P) + \dim(P^0) = n - 1$.

A linear system $P = \mathbf{P}(U)$ of hyperplanes in $\mathcal{H}(E)$ is called a *pencil of hyperplanes* if it corresponds to a projective line in $\mathbf{P}(E^*)$, which means that U is a subspace of dimension 2 of E^*. From $\dim(P) + \dim(P^0) = n - 1$, a pencil of hyperplanes P is the family of hyperplanes in $\mathcal{H}(E)$ containing some projective subspace $\mathbf{P}(V)$ of dimension $n-2$ (where $\mathbf{P}(V)$ is a projective subspace of $\mathbf{P}(E)$, and $\mathbf{P}(E)$ has dimension n). When $n = 2$, a pencil of hyperplanes in $\mathcal{H}(E)$, also called a *pencil of lines*, is the family of lines passing through a given point. When $n = 3$, a pencil of hyperplanes in $\mathcal{H}(E)$, also called a *pencil of planes*, is the family of planes passing through a given line.

When $n = 2$, the above duality takes a rather simple form. In this case (of a projective plane $\mathbf{P}(E)$), the duality is a bijection between points in $\mathbf{P}(E)$ and lines in $\mathbf{P}(E^*)$, represented by pencils of lines in $\mathcal{H}(E)$, with the following properties:

- A point a in $\mathbf{P}(E)$ maps to the line D_a in $\mathbf{P}(E^*)$ represented by the pencil of lines in $\mathcal{H}(E)$ containing a, also denoted by a^*. See Figure 26.36.

- A line D in $\mathbf{P}(E)$ maps to the point p_D in $\mathbf{P}(E^*)$ represented by the line D in $\mathcal{H}(E)$. See Figure 26.37.

- Two points a, b in $\mathbf{P}(E)$ map to lines D_a, D_b in $\mathbf{P}(E^*)$ represented by pencils of lines through a and b, and the intersection of D_a and D_b is the point $p_{\langle a,b\rangle}$ in $\mathbf{P}(E^*)$ corresponding to the line $\langle a, b\rangle$ belonging to both pencils. The point $p_{\langle a,b\rangle}$ is the image of the line $\langle a, b\rangle$ *via* duality. See Figure 26.38

- A line D in $\mathbf{P}(E)$ containing two points a, b maps to the intersection p_D of the lines D_a and D_b in $\mathbf{P}(E^*)$ which are the images of a and b under duality. This is because a, b map to lines D_a, D_b in $\mathbf{P}(E^*)$ represented by pencils of lines through a and b, and the intersection of D_a and D_b is the point p_D in $\mathbf{P}(E^*)$ corresponding to the line $D = \langle a, b\rangle$ belonging to both pencils. The point p_D is the image of the line $D = \langle a, b\rangle$ under duality. Once again, see Figure 26.38.

- If $a \in D$, where a is a point and D is a line in $\mathbf{P}(E)$, then $p_D \in D_a$ in $\mathbf{P}(E^*)$. This is because under duality, a is mapped to the line D_a in $\mathbf{P}(E^*)$ represented by the pencil of lines containing a, and D is mapped to the point $p_D \in \mathbf{P}(E^*)$ represented by the line D through a in this pencil, so $p_D \in D_a$.

The reader will discover that the dual of Desargues's theorem is its converse. This is a nice way of getting the converse for free! We will not spoil the reader's fun and let him discover the dual of Pappus's theorem.

In general, when $n \geq 2$, the above duality is a bijection between points in $\mathbf{P}(E)$ and hyperplanes in $\mathbf{P}(E^*)$, which are represented by linear systems of dimension $n-1$ in $\mathcal{H}(E)$, with the following properties: