Theorem 17.14 implies that if $\lambda_1, \dots, \lambda_p$ are the distinct real eigenvalues of f, and E_i is the eigenspace associated with λ_i, then

$$E = E_1 \oplus \cdots \oplus E_p,$$

where E_i and E_j are orthogonal for all $i \neq j$.

Remark: Another way to prove that a self-adjoint map has a real eigenvalue is to use a little bit of calculus. We learned such a proof from Herman Gluck. The idea is to consider the real-valued function $\Phi \colon E \to \mathbb{R}$ defined such that

$$\Phi(u) = \langle f(u), u \rangle$$

for every $u \in E$. This function is C^∞, and if we represent f by a matrix A over some orthonormal basis, it is easy to compute the gradient vector

$$\nabla \Phi(X) = \left(\frac{\partial \Phi}{\partial x_1}(X), \dots, \frac{\partial \Phi}{\partial x_n}(X) \right)$$

of Φ at X. Indeed, we find that

$$\nabla \Phi(X) = (A + A^\top)X,$$

where X is a column vector of size n. But since f is self-adjoint, $A = A^\top$, and thus

$$\nabla \Phi(X) = 2AX.$$

The next step is to find the maximum of the function Φ on the sphere

$$S^{n-1} = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 = 1\}.$$

Since S^{n-1} is compact and Φ is continuous, and in fact C^∞, Φ takes a maximum at some X on S^{n-1}. But then it is well known that at an extremum X of Φ we must have

$$d\Phi_X(Y) = \langle \nabla \Phi(X), Y \rangle = 0$$

for all tangent vectors Y to S^{n-1} at X, and so $\nabla \Phi(X)$ is orthogonal to the tangent plane at X, which means that

$$\nabla \Phi(X) = \lambda X$$

for some $\lambda \in \mathbb{R}$. Since $\nabla \Phi(X) = 2AX$, we get

$$2AX = \lambda X,$$

and thus $\lambda/2$ is a real eigenvalue of A (i.e., of f).

Next we consider skew-self-adjoint maps.