A real matrix $Q \in \mathrm{M}_n(\mathbb{R})$ is *orthogonal* if

$$QQ^\top = Q^\top Q = I.$$

Given any matrix $A = (a_{ij}) \in \mathrm{M}_n(\mathbb{C})$, the *trace* $\mathrm{tr}(A)$ of A is the sum of its diagonal elements

$$\mathrm{tr}(A) = a_{11} + \cdots + a_{nn}.$$

It is easy to show that the trace is a linear map, so that

$$\mathrm{tr}(\lambda A) = \lambda \mathrm{tr}(A)$$

and

$$\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B).$$

Moreover, if A is an $m \times n$ matrix and B is an $n \times m$ matrix, it is not hard to show that

$$\mathrm{tr}(AB) = \mathrm{tr}(BA).$$

We also review eigenvalues and eigenvectors. We content ourselves with definition involving matrices. A more general treatment will be given later on (see Chapter 15).

Definition 9.4. Given any square matrix $A \in \mathrm{M}_n(\mathbb{C})$, a complex number $\lambda \in \mathbb{C}$ is an *eigenvalue* of A if there is some *nonzero* vector $u \in \mathbb{C}^n$, such that

$$Au = \lambda u.$$

If λ is an eigenvalue of A, then the *nonzero* vectors $u \in \mathbb{C}^n$ such that $Au = \lambda u$ are called *eigenvectors of A associated with λ*; together with the zero vector, these eigenvectors form a subspace of $\mathbb{C}^n$ denoted by $E_\lambda(A)$, and called the *eigenspace associated with λ*.

Remark: Note that Definition 9.4 *requires an eigenvector to be nonzero*. A somewhat unfortunate consequence of this requirement is that the set of eigenvectors is *not* a subspace, since the zero vector is missing! On the positive side, whenever eigenvectors are involved, there is no need to say that they are nonzero. In contrast, even if we allow 0 to be an eigenvector, in order for a scalar λ to be an eigenvalue, there must be a *nonzero vector u* such that $Au = \lambda u$. Without this restriction, since $A0 = \lambda 0 = 0$ for all λ, every scalar would be an eigenvector, which would make the definition of an eigenvalue trivial and useless. The fact that eigenvectors are nonzero is implicitly used in all the arguments involving them, so it seems preferable (but perhaps not as elegant) to stipulate that eigenvectors should be nonzero.

If A is a square real matrix $A \in \mathrm{M}_n(\mathbb{R})$, then we restrict Definition 9.4 to real eigenvalues $\lambda \in \mathbb{R}$ and real eigenvectors. However, it should be noted that although every complex