4. If d is a square-free positive integer and if $d \geq 2$, the ring $\mathbb{Z}[\sqrt{d}]$ is an integral domain. Similarly, if $d \geq 1$ is a square-free positive integer, the ring $\mathbb{Z}[\sqrt{-d}]$ is an integral domain. Finding the invertible elements of these rings is a very interesting problem.

5. The ring of $n \times n$ matrices $\mathrm{M}_n(\mathbb{R})$ has zero divisors.

A homomorphism between rings is a mapping preserving addition and multiplication (and 0 and 1).

Definition 2.18. Given two rings A and B, a *homomorphism between A and B* is a function $h\colon A \to B$ satisfying the following conditions for all $x, y \in A$:

$$\begin{aligned} h(x+y) &= h(x) + h(y) \\ h(xy) &= h(x)h(y) \\ h(0) &= 0 \\ h(1) &= 1. \end{aligned}$$

Actually, because B is a group under addition, $h(0) = 0$ follows from

$$h(x+y) = h(x) + h(y).$$

Example 2.8.

1. If A is a ring, for any integer $n \in \mathbb{Z}$, for any $a \in A$, we define $n \cdot a$ by

   $$n \cdot a = \underbrace{a + \cdots + a}_{n}$$

   if $n \geq 0$ (with $0 \cdot a = 0$) and

   $$n \cdot a = -(-n) \cdot a$$

   if $n < 0$. Then, the map $h\colon \mathbb{Z} \to A$ given by

   $$h(n) = n \cdot 1_A$$

   is a ring homomorphism (where 1_A is the multiplicative identity of A).

2. Given any real $\lambda \in \mathbb{R}$, the evaluation map $\eta_\lambda\colon \mathbb{R}[X] \to \mathbb{R}$ defined by

   $$\eta_\lambda(f(X)) = f(\lambda)$$

   for every polynomial $f(X) \in \mathbb{R}[X]$ is a ring homomorphism.

Definition 2.19. A ring homomorphism $h\colon A \to B$ is an *isomorphism* iff there is a ring homomorphism $g\colon B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. An isomorphism from a ring to itself is called an *automorphism*.