Proof. We proceed by induction on n. When $n = 1$, every isometry $f \in \mathbf{O}(E)$ is either the identity or $-\mathrm{id}$, but $-\mathrm{id}$ is a reflection about $H = \{0\}$. When $n \geq 2$, we have $\mathrm{id} = s \circ s$ for every reflection s. Let us now consider the case where $n \geq 2$ and f is not the identity. There are two subcases.

Case 1. The map f admits 1 as an eigenvalue, i.e., there is some nonnull vector w such that $f(w) = w$. In this case, let H be the hyperplane orthogonal to w, so that $E = H \oplus \mathbb{R}w$. We claim that $f(H) \subseteq H$. Indeed, if

$$v \cdot w = 0$$

for any $v \in H$, since f is an isometry, we get

$$f(v) \cdot f(w) = v \cdot w = 0,$$

and since $f(w) = w$, we get

$$f(v) \cdot w = f(v) \cdot f(w) = 0,$$

and thus $f(v) \in H$. Furthermore, since f is not the identity, f is not the identity of H. Since H has dimension $n - 1$, by the induction hypothesis applied to H, there are at most $k \leq n - 1$ reflections $s_1, \ldots, s_k$ about some hyperplanes $H_1, \ldots, H_k$ in H, such that the restriction of f to H is the composition $s_k \circ \cdots \circ s_1$. Each s_i can be extended to a reflection in E as follows: If $H = H_i \oplus L_i$ (where $L_i = H_i^{\perp}$, the orthogonal complement of H_i in H), $L = \mathbb{R}w$, and $F_i = H_i \oplus L$, since H and L are orthogonal, F_i is indeed a hyperplane, $E = F_i \oplus L_i = H_i \oplus L \oplus L_i$, and for every $u = h + \lambda w \in H \oplus L = E$, since

$$s_i(h) = p_{H_i}(h) - p_{L_i}(h),$$

we can define s_i on E such that

$$s_i(h + \lambda w) = p_{H_i}(h) + \lambda w - p_{L_i}(h),$$

and since $h \in H$, $w \in L$, $F_i = H_i \oplus L$, and $H = H_i \oplus L_i$, we have

$$s_i(h + \lambda w) = p_{F_i}(h + \lambda w) - p_{L_i}(h + \lambda w),$$

which defines a reflection about $F_i = H_i \oplus L$. Now, since f is the identity on $L = \mathbb{R}w$, it is immediately verified that $f = s_k \circ \cdots \circ s_1$, with $k \leq n - 1$. See Figure 27.1.

Case 2. The map f does not admit 1 as an eigenvalue, i.e., $f(u) \neq u$ for all $u \neq 0$. Pick any $w \neq 0$ in E, and let H be the hyperplane orthogonal to $f(w) - w$. Since f is an isometry, we have $\|f(w)\| = \|w\|$, and by Lemma 13.2, we know that $s(w) = f(w)$, where s is the reflection about H, and we claim that $s \circ f$ leaves w invariant. Indeed, since $s^2 = \mathrm{id}$, we have

$$s(f(w)) = s(s(w)) = w.$$

See Figure 27.2.