the following determinant:

$$V(x_1,\dots,x_n) = \begin{vmatrix} 1 & 1 & \dots & 1 \\ 0 & x_2 - x_1 & \dots & x_n - x_1 \\ 0 & x_2(x_2 - x_1) & \dots & x_n(x_n - x_1) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & x_2^{n-2}(x_2 - x_1) & \dots & x_n^{n-2}(x_n - x_1) \end{vmatrix}.$$

Now expanding this determinant according to the first column and using multilinearity, we can factor $(x_i - x_1)$ from the column of index $i-1$ of the matrix obtained by deleting the first row and the first column, and thus

$$V(x_1,\dots,x_n) = (x_2 - x_1)(x_3 - x_1)\cdots(x_n - x_1)V(x_2,\dots,x_n),$$

which establishes the induction step.

Example 7.3. The determinant of upper triangular matrices and more generally of block matrices that are block upper triangular has a remarkable form. Recall that an $n\times n$ matrix $A = (a_{ij})$ is upper-triangular if it is of the form

$$A = \begin{pmatrix} a_{11} & \times & \times & \cdots & \times \\ 0 & a_{22} & \times & \cdots & \times \\ 0 & 0 & \ddots & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & a_{nn} \end{pmatrix},$$

that is, $a_{ij} = 0$ for all $i > j$, $1 \leq i, j \leq n$. Using $n-1$ times Laplace expansion with respect to the first column we obain

$$\det(A) = a_{11}a_{22}\cdots a_{nn}.$$

Similarly, if A is an $n\times n$ block matrix which is *block upper triangular*,

$$A = \begin{pmatrix} A_{11} & \times & \times & \cdots & \times \\ 0 & A_{22} & \times & \cdots & \times \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & A_{pp} \end{pmatrix},$$

where each A_{ii} is an $n_i\times n_i$ matrix, with $n_1 + \cdots + n_p = n$, each block $\times$ above the diagonal in position (i,j) for $i < j$ is an $n_i \times n_j$ matrix, and each block in position (i,j) for $i > j$ is the $n_i \times n_j$ zero matrix, then it can be shown by induction on $p \geq 1$ that

$$\det(A) = \det(A_{11})\det(A_{22})\cdots\det(A_{pp}).$$