Proposition 30.29. *The Hermite interpolation problem has a unique solution of degree $\leq n$, where $n = n_1 + \cdots + n_{m+1} + m$.*

Proof. First, we prove that the Hermite interpolation problem has at most one solution. Assume that P and Q are two distinct solutions of degree $\leq n$. Then, by Proposition 30.26 and the criterion following it, $P - Q$ has among its roots α_1 of multiplicity at least $n_1+1, \ldots$, α_{m+1} of multiplicity at least $n_{m+1}+1$. However, by Theorem 30.23, we should have

$$n_1 + 1 + \cdots + n_{m+1} + 1 = n_1 + \cdots + n_{m+1} + m + 1 \leq n,$$

which is a contradiction, since $n = n_1 + \cdots + n_{m+1} + m$. Thus, $P = Q$. We are left with proving the existence of a Hermite interpolant. A quick way to do so is to use Proposition 7.12, which tells us that given a square matrix A over a field K, the following properties hold:

For every column vector B, there is a unique column vector X such that $AX = B$ iff the only solution to $AX = 0$ is the trivial vector $X = 0$ iff $D(A) \neq 0$.

If we let $P = y_0 + y_1 X + \cdots + y_n X^n$, the Hermite interpolation problem yields a linear system of equations in the unknowns $(y_0, \ldots, y_n)$ with some associated $(n+1)\times(n+1)$ matrix A. Now, the system $AY = 0$ has a solution iff P has among its roots α_1 of multiplicity at least $n_1 + 1, \ldots$, α_{m+1} of multiplicity at least $n_{m+1} + 1$. By the previous argument, since P has degree $\leq n$, we must have $P = 0$, that is, $Y = 0$. This concludes the proof. ☐

Proposition 30.29 shows the existence of unique polynomials $H^i_j(X)$ of degree $\leq n$ such that $\mathrm{D}^i H^i_j(\alpha_j) = 1$ and $\mathrm{D}^k H^i_j(\alpha_l) = 0$, for $k \neq i$ or $l \neq j$, $1 \leq j, l \leq m+1$, $0 \leq i, k \leq n_j$. The polynomials H^i_j are called *Hermite basis polynomials*.

One problem with Proposition 30.29 is that it does not give an explicit way of computing the Hermite basis polynomials. We first show that this can be done explicitly in the special cases $n_1 = \ldots = n_{m+1} = 1$, and $n_1 = \ldots = n_{m+1} = 2$, and then suggest a method using a generalized Newton interpolant.

Assume that $n_1 = \ldots = n_{m+1} = 1$. We try $H^0_j = (a(X - \alpha_j) + b)L^2_j$, and $H^1_j = (c(X - \alpha_j) + d)L^2_j$, where L_j is the Lagrange interpolant determined earlier. Since

$$\mathrm{D}H^0_j = aL^2_j + 2(a(X - \alpha_j) + b)L_j \mathrm{D}L_j,$$

requiring that $H^0_j(\alpha_j) = 1$, $H^0_j(\alpha_k) = 0$, $\mathrm{D}H^0_j(\alpha_j) = 0$, and $\mathrm{D}H^0_j(\alpha_k) = 0$, for $k \neq j$, implies $b = 1$ and $a = -2\mathrm{D}L_j(\alpha_j)$. Similarly, from the requirements $H^1_j(\alpha_j) = 0$, $H^1_j(\alpha_k) = 0$, $\mathrm{D}H^1_j(\alpha_j) = 1$, and $\mathrm{D}H^1_j(\alpha_k) = 0$, $k \neq j$, we get $c = 1$ and $d = 0$.

Thus, we have the Hermite polynomials

$$H^0_j = (1 - 2\mathrm{D}L_j(\alpha_j)(X - \alpha_j))L^2_j, \qquad H^1_j = (X - \alpha_j)L^2_j.$$