(1) *Suppose the functions* $\varphi_i \colon \Omega \to \mathbb{R}$ *are continuous, and that for every* $\mu \in \mathbb{R}^m_+$, *the Problem* (P_μ)*:*

$$\begin{aligned} &\text{minimize} && L(v, \mu) \\ &\text{subject to} && v \in \Omega, \end{aligned}$$

has a unique solution u_μ, *so that*

$$L(u_\mu, \mu) = \inf_{v \in \Omega} L(v, \mu) = G(\mu),$$

and the function $\mu \mapsto u_\mu$ *is continuous (on* $\mathbb{R}^m_+$*). Then the function* G *is differentiable for all* $\mu \in \mathbb{R}^m_+$, *and*

$$G'_\mu(\xi) = \sum_{i=1}^m \xi_i \varphi_i(u_\mu) \quad \textit{for all } \xi \in \mathbb{R}^m.$$

If λ *is any solution of Problem* (D)*:*

$$\begin{aligned} &\text{maximize} && G(\mu) \\ &\text{subject to} && \mu \in \mathbb{R}^m_+, \end{aligned}$$

then the solution u_λ *of the corresponding problem* (P_λ) *is a solution of Problem* (P).

(2) *Assume Problem* (P) *has some solution* $u \in U$, *and that* Ω *is convex (open), the functions* φ_i $(1 \leq i \leq m)$ *and* J *are convex and differentiable at* u, *and that the constraints are qualified. Then Problem* (D) *has a solution* $\lambda \in \mathbb{R}^m_+$, *and* $J(u) = G(\lambda)$*; that is, the duality gap is zero.*

Proof. (1) Our goal is to prove that for any solution λ of Problem (D), the pair (u_λ, λ) is a saddle point of L. By Theorem 50.15(1), the point $u_\lambda \in U$ is a solution of Problem (P).

Since $\lambda \in \mathbb{R}^m_+$ is a solution of Problem (D), by definition of $G(\lambda)$ and since u_λ satisfies Problem (P_λ), we have

$$G(\lambda) = \inf_{v \in \Omega} L(v, \lambda) = L(u_\lambda, \lambda),$$

which is one of the two equations characterizing a saddle point. In order to prove the second equation characterizing a saddle point,

$$\sup_{\mu \in \mathbb{R}^m_+} L(u_\mu, \mu) = L(u_\lambda, \lambda),$$

we will begin by proving that the function G is differentiable for all $\mu \in \mathbb{R}^m_+$, in order to be able to apply Theorem 40.9 to conclude that since G has a maximum at λ, that is, $-G$ has minimum at λ, then $-G'_\lambda(\mu - \lambda) \geq 0$ for all $\mu \in \mathbb{R}^m_+$. In fact, we prove that

$$G'_\mu(\xi) = \sum_{i=1}^m \xi_i \varphi_i(u_\mu) \quad \text{for all } \xi \in \mathbb{R}^m. \tag{$*_{\text{deriv}}$}$$