where the coefficients g_i are polynomials in $A[X_1, \ldots, X_{n-1}]$. Now, for every $(\alpha_1, \ldots, \alpha_{n-1}) \in A_1 \times \cdots \times A_{n-1}$, $f(\alpha_1, \ldots, \alpha_{n-1}, X_n)$ determines a polynomial $h(X_n) \in A[X_n]$, and since A_n is infinite and $h(\alpha_n) = f(\alpha_1, \ldots, \alpha_{n-1}, \alpha_n) = 0$ for all $\alpha_n \in A_n$, by the induction hypothesis, we have $g_i(\alpha_1, \ldots, \alpha_{n-1}) = 0$. Now, since $A_1, \ldots, A_{n-1}$ are infinite, using the induction hypothesis again, we get $g_i = 0$, which shows that f is the null polynomial. The second part of the proposition follows immediately from the first, by letting $A_i = A$. □

When A is an infinite integral domain, in particular an infinite field, since the map $f \mapsto f_A$ is injective, we identify the polynomial f with the polynomial function f_A, and we write f_A simply as f.

The following proposition can be very useful to show polynomial identities.

Proposition 30.28. *Let A be an infinite integral domain and $f, g_1, \ldots, g_m \in A[X_1, \ldots, X_n]$ be polynomials. If the g_i are nonnull polynomials and if*

$$f(\alpha_1, \ldots, \alpha_n) = 0 \textit{ whenever } g_i(\alpha_1, \ldots, \alpha_n) \neq 0 \textit{ for all } i,\ 1 \leq i \leq m,$$

for every $(\alpha_1, \ldots, \alpha_n) \in A^n$, then

$$f = 0,$$

i.e., f is the null polynomial.

Proof. If f is not the null polynomial, since the g_i are nonnull and A is an integral domain, then the product $fg_1 \cdots g_m$ is nonnull. By Proposition 30.27, only the null polynomial maps to the zero function, and thus there must be some $(\alpha_1, \ldots, \alpha_n) \in A^n$, such that

$$f(\alpha_1, \ldots, \alpha_n)g_1(\alpha_1, \ldots, \alpha_n) \cdots g_m(\alpha_1, \ldots, \alpha_n) \neq 0,$$

but this contradicts the hypothesis. □

Proposition 30.28 is often called the *principle of extension of algebraic identities*. Another perhaps more illuminating way of stating this proposition is as follows: For any polynomial $g \in A[X_1, \ldots, X_n]$, let

$$V(g) = \{(\alpha_1, \ldots, \alpha_n) \in A^n \mid g(\alpha_1, \ldots, \alpha_n) = 0\},$$

the set of zeros of g. Note that $V(g_1) \cup \cdots \cup V(g_m) = V(g_1 \cdots g_m)$. Then, Proposition 30.28 can be stated as:

If $f(\alpha_1, \ldots, \alpha_n) = 0$ for every $(\alpha_1, \ldots, \alpha_n) \in A^n - V(g_1 \cdots g_m)$, then $f = 0$.

In other words, if the algebraic identity $f(\alpha_1, \ldots, \alpha_n) = 0$ holds on the complement of $V(g_1) \cup \cdots \cup V(g_m) = V(g_1 \cdots g_m)$, then $f(\alpha_1, \ldots, \alpha_n) = 0$ holds everywhere in A^n. With this second formulation, we understand better the terminology "principle of extension of algebraic identities."