Proposition 25.2. *Given any affine space $(E, \overrightarrow{E})$, for any family $(a_i)_{i\in I}$ of points in E, any family $(\lambda_i)_{i\in I}$ of scalars in $\mathbb{R}$, and any family $(v_j)_{j\in J}$ of vectors in $\overrightarrow{E}$, with $I \cap J = \emptyset$, the following properties hold:*

(1) If $\sum_{i\in I} \lambda_i = 0$, then

$$\sum_{i\in I} \langle a_i, \lambda_i\rangle \,\widehat{+}\, \sum_{j\in J} v_j = \overrightarrow{\sum_{i\in I} \lambda_i a_i} + \sum_{j\in J} v_j,$$

where

$$\overrightarrow{\sum_{i\in I} \lambda_i a_i} = \sum_{i\in I} \lambda_i \overrightarrow{b a_i}$$

for any $b \in E$, which, by Proposition 24.1, is a vector independent of b, or

(2) If $\sum_{i\in I} \lambda_i \neq 0$, then

$$\sum_{i\in I} \langle a_i, \lambda_i\rangle \,\widehat{+}\, \sum_{j\in J} v_j = \left\langle \sum_{i\in I} \frac{\lambda_i}{\sum_{i\in I} \lambda_i} a_i + \sum_{j\in J} \frac{v_j}{\sum_{i\in I} \lambda_i}, \sum_{i\in I} \lambda_i \right\rangle.$$

Proof. By induction on the size of I and the size of J. □

The above formulae show that we have some kind of extended barycentric calculus. Operations on weighted points and vectors were introduced by H. Grassmann, in his book published in 1844! This calculus will be helpful in dealing with rational curves.

25.2 Affine Frames of E and Bases of $\widehat{E}$

There is also a nice relationship between affine frames in $(E, \overrightarrow{E})$ and bases of $\widehat{E}$, stated in the following proposition.

Proposition 25.3. *Given any affine space $(E, \overrightarrow{E})$, for any affine frame $(a_0, (\overrightarrow{a_0 a_1}, \ldots, \overrightarrow{a_0 a_m}))$ for E, the family $(\overrightarrow{a_0 a_1}, \ldots, \overrightarrow{a_0 a_m}, a_0)$ is a basis for $\widehat{E}$, and for any affine frame $(a_0, \ldots, a_m)$ for E, the family $(a_0, \ldots, a_m)$ is a basis for $\widehat{E}$. Furthermore, given any element $\langle x, \lambda\rangle \in \widehat{E}$, if*

$$x = a_0 + x_1 \overrightarrow{a_0 a_1} + \cdots + x_m \overrightarrow{a_0 a_m}$$

over the affine frame $(a_0, (\overrightarrow{a_0 a_1}, \ldots, \overrightarrow{a_0 a_m}))$ in E, then the coordinates of $\langle x, \lambda\rangle$ over the basis $(\overrightarrow{a_0 a_1}, \ldots, \overrightarrow{a_0 a_m}, a_0)$ in $\widehat{E}$ are

$$(\lambda x_1, \ldots, \lambda x_m, \lambda).$$

For any vector $v \in \overrightarrow{E}$, if

$$v = v_1 \overrightarrow{a_0 a_1} + \cdots + v_m \overrightarrow{a_0 a_m}$$