there is some integer $n \geq 1$ so that

$$\mathfrak{A}_i = \mathfrak{A}_n \quad \text{for all} \quad i \geq n+1.$$

We say that A satisfies the *maximum condition* if every nonempty collection C of ideals in A has a maximal element, i.e., there is some ideal $\mathfrak{A} \in C$ which is not contained in any other ideal in C.

Proposition 32.17. *A ring A satisfies the a.c.c if and only if it satisfies the maximum condition.*

Proof. Suppose that A does not satisfy the a.c.c. Then, there is an infinite strictly ascending sequence of ideals

$$\mathfrak{A}_1 \subset \mathfrak{A}_2 \subset \cdots \subset \mathfrak{A}_i \subset \cdots,$$

and the collection $C = \{\mathfrak{A}_i\}$ has no maximal element.

Conversely, assume that A satisfies the a.c.c. Let C be a nonempty collection of ideals Since C is nonempty, we may pick some ideal $\mathfrak{A}_1$ in C. If $\mathfrak{A}_1$ is not maximal, then there is some ideal $\mathfrak{A}_2$ in C so that

$$\mathfrak{A}_1 \subset \mathfrak{A}_2.$$

Using this process, if C has no maximal element, we can define by induction an infinite strictly increasing sequence

$$\mathfrak{A}_1 \subset \mathfrak{A}_2 \subset \cdots \subset \mathfrak{A}_i \subset \cdots.$$

However, the a.c.c. implies that such a sequence cannot exist. Therefore, C has a maximal element. ☐

Having shown that the a.c.c. condition is equivalent to the maximal condition, we now prove that the a.c.c. condition is equivalent to the fact that every ideal is finitely generated.

Proposition 32.18. *A ring A satisfies the a.c.c if and only if every ideal is finitely generated.*

Proof. Assume that every ideal is finitely generated. Consider an ascending sequence of ideals

$$\mathfrak{A}_1 \subseteq \mathfrak{A}_2 \subseteq \cdots \subseteq \mathfrak{A}_i \subseteq \cdots.$$

Observe that $\mathfrak{A} = \bigcup_i \mathfrak{A}_i$ is also an ideal. By hypothesis, $\mathfrak{A}$ has a finite generating set $\{a_1, \ldots, a_n\}$. By definition of $\mathfrak{A}$, each a_i belongs to some $\mathfrak{A}_{j_i}$, and since the $\mathfrak{A}_i$ form an ascending chain, there is some m so that $a_i \in \mathfrak{A}_m$ for $i = 1, \ldots, n$. But then,

$$\mathfrak{A}_i = \mathfrak{A}_m$$

for all $i \geq m+1$, and the a.c.c. holds.

Conversely, assume that the a.c.c. holds. Let $\mathfrak{A}$ be any ideal in A and consider the family C of subideals of $\mathfrak{A}$ that are finitely generated. The family C is nonempty, since (0) is a subideal of $\mathfrak{A}$. By Proposition 32.17, the family C has some maximal element, say $\mathfrak{B}$. For