where $\widehat{A}^j = \sum_{k\in K^*} \gamma^j_k \widehat{A}^k$, or using the notations of Section 46.3,

$$\widehat{c}_j - \widehat{c}_{K^*}\widehat{A}^{-1}_{K^*}\widehat{A}^j \leq 0 \quad \text{for all } j \in N^*.$$

The above inequalities can be written as

$$\widehat{c}_{N^*} - \widehat{c}_{K^*}\widehat{A}^{-1}_{K^*}\widehat{A}_{N^*} \leq 0^\top_n,$$

or equivalently as

$$\widehat{c}_{K^*}\widehat{A}^{-1}_{K^*}\widehat{A}_{N^*} \geq \widehat{c}_{N^*}. \tag{$*_1$}$$

The value of the objective function for the optimal solution $\widehat{u}^*$ is $\widehat{c}\,\widehat{u}^* = \widehat{c}_{K^*}\widehat{u}^*_{K^*}$, and since $\widehat{u}^*_{K^*}$ satisfies the equation $\widehat{A}_{K^*}\widehat{u}^*_{K^*} = b$, the value of the objective function is

$$\widehat{c}_{K^*}\,\widehat{u}^*_{K^*} = \widehat{c}_{K^*}\widehat{A}^{-1}_{K^*}b. \tag{$*_2$}$$

Then if we let $y^* = \widehat{c}_{K^*}\widehat{A}^{-1}_{K^*}$, obviously we have $y^*b = \widehat{c}_{K^*}\widehat{u}_{K^*}$, so if we can prove that y^* is a feasible solution of the Dual Linear program (D), by weak duality, y^* is an optimal solution of (D). We have

$$y^*\widehat{A}_{K^*} = \widehat{c}_{K^*}\widehat{A}^{-1}_{K^*}\widehat{A}_{K^*} = \widehat{c}_{K^*}, \tag{$*_3$}$$

and by $(*_1)$ we get

$$y^*\widehat{A}_{N^*} = \widehat{c}_{K^*}\widehat{A}^{-1}_{K^*}\widehat{A}_{N^*} \geq \widehat{c}_{N^*}. \tag{$*_4$}$$

Let P be the $(n+m)\times(n+m)$ permutation matrix defined so that

$$\widehat{A}\,P = \begin{pmatrix} A & I_m \end{pmatrix} P = \begin{pmatrix} \widehat{A}_{K^*} & \widehat{A}_{N^*} \end{pmatrix}.$$

Then we also have

$$\widehat{c}\,P = \begin{pmatrix} c & 0^\top_m \end{pmatrix} P = \begin{pmatrix} \widehat{c}_{K^*} & \widehat{c}_{N^*} \end{pmatrix}.$$

Using Equations $(*_3)$ and $(*_4)$ we obtain

$$y^*\begin{pmatrix} \widehat{A}_{K^*} & \widehat{A}_{N^*} \end{pmatrix} \geq \begin{pmatrix} \widehat{c}_{K^*} & \widehat{c}_{N^*} \end{pmatrix},$$

that is,

$$y^*\begin{pmatrix} A & I_m \end{pmatrix} P \geq \begin{pmatrix} c & 0^\top_m \end{pmatrix} P,$$

which is equivalent to

$$y^*\begin{pmatrix} A & I_m \end{pmatrix} \geq \begin{pmatrix} c & 0^\top_m \end{pmatrix},$$

that is

$$y^*A \geq c, \quad y \geq 0,$$

and these are exactly the conditions that say that y^* is a feasible solution of the Dual Program (D).