may not be any obvious y feasible for (D). Preferably we would like to find such a y very cheaply.

There is a trick to deal with this situation. We pick some very large positive number M and add to the set of equations $Ax = b$ the new equation

$$x_1 + \cdots + x_n + x_{n+1} = M,$$

with the new variable x_{n+1} constrained to be nonnegative. If the Program (P) has a feasible solution, such an M exists. In fact it can shown that for any basic feasible solution $u = (u_1, \ldots, u_n)$, each $|u_i|$ is bounded by some expression depending only on A and b; see Papadimitriou and Steiglitz [134] (Lemma 2.1). The proof is not difficult and relies on the fact that the inverse of a matrix can be expressed in terms of certain determinants (the adjugates). Unfortunately, this bound contains $m!$ as a factor, which makes it quite impractical.

Having added the new equation above, we obtain the new set of equations

$$\begin{pmatrix} A & 0_n \\ \mathbf{1}_n^\top & 1 \end{pmatrix} \begin{pmatrix} x \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} b \\ M \end{pmatrix},$$

with $x \geq 0, x_{n+1} \geq 0$, and the new objective function given by

$$\begin{pmatrix} c & 0 \end{pmatrix} \begin{pmatrix} x \\ x_{n+1} \end{pmatrix} = cx.$$

The dual of the above linear program is

$$\begin{aligned} &\text{minimize} \quad yb + y_{m+1}M \\ &\text{subject to} \quad yA^j + y_{m+1} \geq c_j \quad j = 1, \ldots, n \\ &\qquad\qquad\quad y_{m+1} \geq 0. \end{aligned}$$

If $c_j > 0$ for some j, observe that the linear form $\widetilde{y}$ given by

$$\widetilde{y}_i = \begin{cases} 0 & \text{if } 1 \leq i \leq m \\ \max_{1 \leq j \leq n}\{c_j\} > 0 \end{cases}$$

is a feasible solution of the new dual program. In practice, we can choose M to be a number close to the largest integer representable on the computer being used.

Here is an example of the primal-dual algorithm given in the Math 588 class notes of T. Molla [128].

Example 47.3. Consider the following linear program in standard form:

$$\begin{aligned} &\text{Maximize} \quad -x_1 - 3x_2 - 3x_3 - x_4 \\ &\text{subject to} \quad \begin{pmatrix} 3 & 4 & -3 & 1 \\ 3 & -2 & 6 & -1 \\ 6 & 4 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \text{ and } x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$