show that there is some nonzero vector $e_1 \in E$ such that $\varphi(e_1, e_1) \neq 0$ since otherwise $\varphi$ would vanish for all $u, v \in E$. We claim that the set

$$H = \{v \in E \mid \varphi(e_1, v) = 0\}$$

has dimension $n - 1$, and that $e_1 \notin H$.

This is because

$$H = \mathrm{Ker}\,(l_\varphi(e_1)),$$

where $l_\varphi(e_1)$ is the linear form in $E^*$ determined by $e_1$. Since $\varphi(e_1, e_1) \neq 0$, we have $e_1 \notin H$, the linear form $l_\varphi(e_1)$ is not the zero form, and thus its kernel is a hyperplane $H$ (a subspace of dimension $n - 1$). Since $\dim(H) = n - 1$ and $e_1 \notin H$, we have the direct sum

$$E = H \oplus K e_1.$$

By the induction hypothesis applied to $H$, we get a basis $(e_2, \ldots, e_n)$ of vectors in $H$ such that $\varphi(e_i, e_j) = 0$, for all $i \neq j$ with $2 \leq i, j \leq n$. Since $\varphi(e_1, v) = 0$ for all $v \in H$ and since $\varphi$ is symmetric, we also have $\varphi(v, e_1) = 0$ for all $v \in H$, so we obtain a basis $(e_1, \ldots, e_n)$ of $E$ such that $\varphi(e_i, e_j) = 0$, for all $i \neq j$. □

If $E$ and $F$ are finite-dimensional vector spaces and if $(e_1, \ldots, e_m)$ is a basis of $E$ and $(f_1, \ldots, f_n)$ is a basis of $F$ then the bilinearity of $\varphi$ yields

$$\varphi\left(\sum_{i=1}^{m} x_i e_i, \sum_{j=1}^{n} y_j f_j\right) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i \varphi(e_i, f_j) y_j.$$

This shows that $\varphi$ is completely determined by the $n \times m$ matrix $M = (m_{ij})$ with $m_{ij} = \varphi(e_j, f_i)$, and in matrix form, we have

$$\varphi(x, y) = x^\top M^\top y = y^\top M x,$$

where $x$ and $y$ are the column vectors associated with $(x_1, \ldots, x_m) \in K^m$ and $(y_1, \ldots, y_n) \in K^n$. As in Section 12.1, we are committing the slight abuse of notation of letting $x$ denote both the vector $x = \sum_{i=1}^n x_i e_i$ and the column vector associated with $(x_1, \ldots, x_n)$ (and similarly for $y$).

**Definition 29.6.** *If* $(e_1, \ldots, e_m)$ is a basis of $E$ and $(f_1, \ldots, f_n)$ is a basis of $F$, for any bilinear form $\varphi \colon E \times F \to K$, the $n \times m$ matrix $M = (m_{ij})$ given by $m_{ij} = \varphi(e_j, f_i)$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$ is called the *matrix of* $\varphi$ *with respect to the bases* $(e_1, \ldots, e_m)$ *and* $(f_1, \ldots, f_n)$.

The following fact is easily proved.

**Proposition 29.5.** *If* $m = \dim(E) = \dim(F) = n$, *then* $\varphi$ *is nondegenerate iff the matrix* $M$ *is invertible iff* $\det(M) \neq 0$.