and write

$$\rho = \sqrt{\alpha^2 + \beta^2},$$

then

$$\Gamma = \rho \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

which corresponds to a similarity. Observe that h is an involution, that is, $h^2 = \mathrm{id}$ iff $\theta = \pi/2$.

(2) *Parabolic homographies*. In this case, we must have $(a+d)^2 - 4(ad - bc) = 0$. The matrix A is not diagonalizable and it has a Jordan form of the form

$$\Gamma = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

In the affine line $y = 1$, a parabolic homography behaves like the translation by $1/\lambda$.

(3) *Hyperbolic homographies*. In this case, $(a+d)^2 - 4(ad - bc) > 0$, so A has two distinct nonzero reals eigenvalues λ and μ, and in a basis of eigenvectors it is represented by the diagonal matrix

$$\Gamma = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}.$$

If P and Q are the distinct fixed points of the the homography h, it is not hard to show that for every $M \in \mathbb{RP}^1$ such that $M \neq P, Q$, we have

$$[P, Q, M, h(M)] = k$$

where $k = \lambda/\mu$. For example, see Sidler [161] (Chapter 3, Proposition 3.3.1), and Berger [11] (Lemma 6.6.3). It can also be shown that h is an involution ($h^2 = \mathrm{id}$) with two distinct fixed points P and Q iff $a + d = 0$ iff $k = -1$ in the above equation; see Sidler [161] (Chapter 3, Proposition 3.3.2), and Samuel [142] (Section 2.4).

We now classify the homographies of $\mathbb{RP}^2$. Since the characteristic polynomial of a 3×3 real matrix A has degree 3 and since every real polynomial of degree 3 has at least one real zero, A has some real eigenvalue. Since $\mathbb{C}$ is algebraically closed, every complex polynomial of degree 3 has three zeros (counted with multiplicity), in which case, all three eigenvalues of a 3×3 complex matrix A belong to $\mathbb{C}$. Thus we have the following useful fact.

Proposition 26.22. *Every homography of the real projective plane $\mathbb{RP}^2$ or of the complex projective plane $\mathbb{CP}^2$ has at least one fixed point.*

Here is the classification of the homographies of $\mathbb{RP}^2$ based on the real Jordan form of a 3×3 matrix. Most details are left as exercises. We denote by Γ the 3×3 matrix representing the real Jordan form of the matrix of the linear map representing the homography h.