where $x_j \neq 0$ for some j. If we compute

$$\det(A^1, \ldots, x_1 A^1 + \cdots + x_j A^j + \cdots + x_n A^n, \ldots, A^n) = \det(A^1, \ldots, 0, \ldots, A^n) = 0,$$

where 0 occurs in the j-th position. By multilinearity, all terms containing two identical columns A^k for $k \neq j$ vanish, and we get

$$\det(A^1, \ldots, x_1 A^1 + \cdots + x_j A^j + \cdots + x_n A^n, \ldots, A^n) = x_j \det(A^1, \ldots, A^n) = 0.$$

Since $x_j \neq 0$ and K is a field, we must have $\det(A^1, \ldots, A^n) = 0$.

Conversely, we show that if the columns $A^1, \ldots, A^n$ of A are linearly independent, then $\det(A^1, \ldots, A^n) \neq 0$. If the columns $A^1, \ldots, A^n$ of A are linearly independent, then they form a basis of K^n, and we can express the standard basis $(e_1, \ldots, e_n)$ of K^n in terms of $A^1, \ldots, A^n$. Thus, we have

$$\begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix} = \begin{pmatrix} b_{1\,1} & b_{1\,2} & \ldots & b_{1\,n} \\ b_{2\,1} & b_{2\,2} & \ldots & b_{2\,n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n\,1} & b_{n\,2} & \ldots & b_{n\,n} \end{pmatrix} \begin{pmatrix} A^1 \\ A^2 \\ \vdots \\ A^n \end{pmatrix},$$

for some matrix $B = (b_{i\,j})$, and by Proposition 7.8, we get

$$\det(e_1, \ldots, e_n) = \det(B) \det(A^1, \ldots, A^n),$$

and since $\det(e_1, \ldots, e_n) = 1$, this implies that $\det(A^1, \ldots, A^n) \neq 0$ (and $\det(B) \neq 0$). For the second assertion, recall that the rank of a matrix is equal to the maximum number of linearly independent columns, and the conclusion is clear. $\square$

We now characterize when a system of linear equations of the form $Ax = b$ has a unique solution.

Proposition 7.12. *Given an $n \times n$-matrix A over a field K, the following properties hold:*

(1) For every column vector b, there is a unique column vector x such that $Ax = b$ iff the only solution to $Ax = 0$ is the trivial vector $x = 0$, iff $\det(A) \neq 0$.

(2) If $\det(A) \neq 0$, *the unique solution of $Ax = b$ is given by the expressions*

$$x_j = \frac{\det(A^1, \ldots, A^{j-1}, b, A^{j+1}, \ldots, A^n)}{\det(A^1, \ldots, A^{j-1}, A^j, A^{j+1}, \ldots, A^n)},$$

known as Cramer's rules.

(3) The system of linear equations $Ax = 0$ has a nonzero solution iff $\det(A) = 0$.