Given any cyclic group G, for any generator g of G, we can define a mapping $\varphi\colon \mathbb{Z} \to G$ by $\varphi(m) = g^m$. Since g generates G, this mapping is surjective. The mapping φ is clearly a group homomorphism, so let $H = \operatorname{Ker}\varphi$ be its kernel. By a previous observation, $H = n\mathbb{Z}$ for some $n \in \mathbb{Z}$, so by the first homomorphism theorem, we obtain an isomorphism

$$\overline{\varphi}\colon \mathbb{Z}/n\mathbb{Z} \longrightarrow G$$

from the quotient group $\mathbb{Z}/n\mathbb{Z}$ onto G. Obviously, if G has finite order, then $|G| = n$. In summary, we have the following result.

Proposition 2.16. *Every cyclic group G is either isomorphic to $\mathbb{Z}$, or to $\mathbb{Z}/n\mathbb{Z}$, for some natural number $n > 0$. In the first case, we say that G is an infinite cyclic group, and in the second case, we say that G is a cyclic group of order n.*

The quotient group $\mathbb{Z}/n\mathbb{Z}$ consists of the cosets $m+n\mathbb{Z} = \{m+nk \mid k \in \mathbb{Z}\}$, with $m \in \mathbb{Z}$, that is, of the equivalence classes of $\mathbb{Z}$ under the equivalence relation $\equiv$ defined such that

$$x \equiv y \quad \text{iff} \quad x - y \in nZ \quad \text{iff} \quad x \equiv y \pmod{n}.$$

We also denote the equivalence class $x + n\mathbb{Z}$ of x by $\overline{x}$, or if we want to be more precise by $[x]_n$. The group operation is given by

$$\overline{x} + \overline{y} = \overline{x + y}.$$

For every $x \in \mathbb{Z}$, there is a unique representative, $x \bmod n$ (the nonnegative remainder of the division of x by n) in the class $\overline{x}$ of x, such that $0 \leq x \bmod n \leq n-1$. For this reason, we often identity $\mathbb{Z}/n\mathbb{Z}$ with the set $\{0,\ldots,n-1\}$. To be more rigorous, we can give $\{0,\ldots,n-1\}$ a group structure by defining $+_n$ such that

$$x +_n y = (x + y) \bmod n.$$

Then, it is easy to see that $\{0,\ldots,n-1\}$ with the operation $+_n$ is a group with identity element 0 isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

We can also define a multiplication operation $\cdot$ on $\mathbb{Z}/n\mathbb{Z}$ as follows:

$$\overline{a} \cdot \overline{b} = \overline{ab} = \overline{ab \bmod n}.$$

Then, it is easy to check that $\cdot$ is abelian, associative, that 1 is an identity element for $\cdot$, and that $\cdot$ is distributive on the left and on the right with respect to addition. This makes $\mathbb{Z}/n\mathbb{Z}$ into a *commutative ring*. We usually suppress the dot and write $\overline{a}\,\overline{b}$ instead of $\overline{a} \cdot \overline{b}$.

Proposition 2.17. *Given any integer $n \geq 1$, for any $a \in \mathbb{Z}$, the residue class $\overline{a} \in \mathbb{Z}/n\mathbb{Z}$ is invertible with respect to multiplication iff* $\gcd(a,n) = 1$.