The new solution turns out to be $x + \Delta x = (9.2, -12.6, 4.5, -1.1)$, where

$$\Delta x = (9.2, -12.6, 4.5, -1.1) - (1, 1, 1, 1) = (8.2, -13.6, 3.5, -2.1).$$

Then a relative error of the data in terms of the one-norm,

$$\frac{\|\Delta b\|_1}{\|b\|_1} = \frac{0.4}{119} = \frac{4}{1190} \approx \frac{1}{300},$$

produces a relative error in the input

$$\frac{\|\Delta x\|_1}{\|x\|_1} = \frac{27.4}{4} \approx 7.$$

So a relative error of the order $1/300$ in the data produces a relative error of the order $7/1$ in the solution, which represents an amplification of the relative error of the order 2100.

Now let us perturb the matrix slightly, obtaining the new system

$$\begin{pmatrix} 10 & 7 & 8.1 & 7.2 \\ 7.08 & 5.04 & 6 & 5 \\ 8 & 5.98 & 9.98 & 9 \\ 6.99 & 4.99 & 9 & 9.98 \end{pmatrix} \begin{pmatrix} x_1 + \Delta x_1 \\ x_2 + \Delta x_2 \\ x_3 + \Delta x_3 \\ x_4 + \Delta x_4 \end{pmatrix} = \begin{pmatrix} 32 \\ 23 \\ 33 \\ 31 \end{pmatrix}.$$

This time the solution is $x + \Delta x = (-81, 137, -34, 22)$. Again a small change in the data alters the result rather drastically. Yet the original system is symmetric, has determinant 1, and has integer entries. The problem is that the matrix of the system is badly conditioned, a concept that we will now explain.

Given an invertible matrix A, first assume that we perturb b to $b + \Delta b$, and let us analyze the change between the two exact solutions x and $x + \Delta x$ of the two systems

$$\begin{aligned} Ax &= b \\ A(x + \Delta x) &= b + \Delta b. \end{aligned}$$

We also assume that we have some norm $\| \ \|$ and we use the *subordinate* matrix norm on matrices. From

$$\begin{aligned} Ax &= b \\ Ax + A\Delta x &= b + \Delta b, \end{aligned}$$

we get

$$\Delta x = A^{-1}\Delta b,$$

and we conclude that

$$\begin{aligned} \|\Delta x\| &\leq \left\|A^{-1}\right\| \|\Delta b\| \\ \|b\| &\leq \|A\| \|x\|. \end{aligned}$$