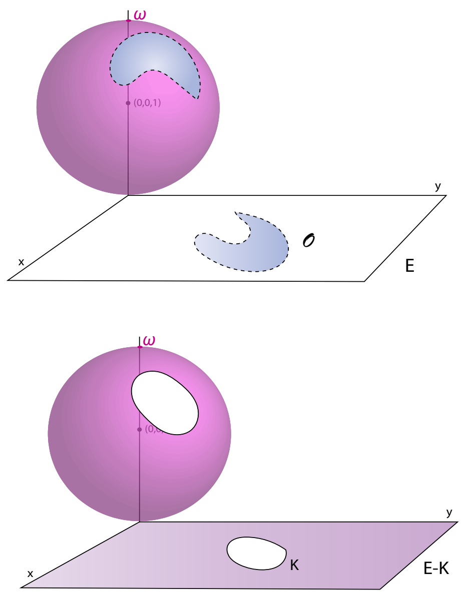

Figure 37.37: The two types of open sets associated with the Alexandroff compactification of the xy-plane. The first type of open set does not include ω, i.e. the north pole, while the second type of open set contains ω.

Definition 37.33. A topological space E is called *second-countable* if there is a countable basis for its topology, i.e., if there is a countable family, $(U_i)_{i\geq 0}$, of open sets such that every open set of E is a union of open sets U_i.

It is easily seen that $\mathbb{R}^n$ is second-countable and more generally, that every normed vector space of finite dimension is second-countable. More generally, a metric space is second-countable if and only if it is separable, a very useful property that holds for all of the spaces that we will consider in practice.

Definition 37.34. A topological space E is *separable* if it contains some countable subset S which is dense in X, that is, $\overline{S} = E$.

Observe that by Proposition 37.4, a subset S of E is dense in E if and only if every nonempty open subset of E contains some element of S.

The (metric) space $\mathbb{R}$ is separable because $\mathbb{Q}$ is a countable dense subset of $\mathbb{R}$. Similarly, $\mathbb{C}$ is separable. In general, $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$, so $\mathbb{R}^n$ is separable, and similarly, every finite-dimensional normed vector space over $\mathbb{R}$ (or $\mathbb{C}$) is separable. For metric spaces, we have the following useful result.

Proposition 37.37. *If E is a metric space, then E is second-countable iff E is separable.*