If f is differentiable on A, then f defines a function $\mathrm{D}f\colon A \to \mathcal{L}(\overrightarrow{E}; \overrightarrow{F})$. It turns out that the continuity of the partial derivatives on A is a necessary and sufficient condition for $\mathrm{D}f$ to exist and to be continuous on A.

If $f\colon [a, b] \to \mathbb{R}$ is a function which is continuous on $[a, b]$ and differentiable on (a, b), then there is some c with $a < c < b$ such that

$$f(b) - f(a) = (b - a)f'(c).$$

This result is known as the *mean value theorem* and is a generalization of *Rolle's theorem*, which corresponds to the case where $f(a) = f(b)$.

Unfortunately, the mean value theorem fails for vector-valued functions. For example, the function $f\colon [0, 2\pi] \to \mathbb{R}^2$ given by

$$f(t) = (\cos t, \sin t)$$

is such that $f(2\pi) - f(0) = (0, 0)$, yet its derivative $f'(t) = (-\sin t, \cos t)$ does not vanish in $(0, 2\pi)$.

A suitable generalization of the mean value theorem to vector-valued functions is possible if we consider an inequality (an upper bound) instead of an equality. This generalized version of the mean value theorem plays an important role in the proof of several major results of differential calculus.

If E is an affine space (over $\mathbb{R}$ or $\mathbb{C}$), given any two points $a, b \in E$, the *closed segment* $[a, b]$ is the set of all points $a + \lambda(b - a)$, where $0 \leq \lambda \leq 1$, $\lambda \in \mathbb{R}$, and the *open segment* (a, b) is the set of all points $a + \lambda(b - a)$, where $0 < \lambda < 1$, $\lambda \in \mathbb{R}$.

Proposition 39.12. *Let E and F be two normed affine spaces, let A be an open subset of E, and let $f\colon A \to F$ be a continuous function on A. Given any $a \in A$ and any $h \neq 0$ in $\overrightarrow{E}$, if the closed segment $[a, a + h]$ is contained in A, if $f\colon A \to F$ is differentiable at every point of the open segment $(a, a + h)$, and*

$$\sup_{x \in (a, a+h)} \|\mathrm{D}f(x)\| \leq M,$$

for some $M \geq 0$, then

$$\|f(a + h) - f(a)\| \leq M\|h\|.$$

As a corollary, if $L\colon \overrightarrow{E} \to \overrightarrow{F}$ is a continuous linear map, then

$$\|f(a + h) - f(a) - L(h)\| \leq M\|h\|,$$

where $M = \sup_{x \in (a, a+h)} \|\mathrm{D}f(x) - L\|$.

The above proposition is sometimes called the "mean value theorem." Proposition 39.12 can be used to show the following important result.