affine frame $(a_0, (v_1, v_2, v_2))$. With respect to this affine frame, every point $x \in E$ is represented by its coordinates (x_1, x_2, x_3), where $a = a_0 + x_1 v_1 + x_2 v_2 + x_3 v_3$. A vector $u \in \overrightarrow{E}$ is also represented by its coordinates (u_1, u_2, u_3) over the basis (v_1, v_2, v_2). One way to distinguish between points and vectors is to add a fourth coordinate, and to agree that points are represented by (row) vectors $(x_1, x_2, x_3, 1)$ whose fourth coordinate is 1, and that vectors are represented by (row) vectors $(v_1, v_2, v_3, 0)$ whose fourth coordinate is 0. This "programming trick" actually works very well. Of course, we are opening the door for strange elements such as $(x_1, x_2, x_3, 5)$, where the fourth coordinate is neither 1 nor 0.

The question is, can we make sense of such elements, and of such a construction? The answer is yes. We will present a construction in which an affine space $\big(E, \overrightarrow{E}\big)$ is embedded in a vector space $\widehat{E}$, in which $\overrightarrow{E}$ is embedded as a hyperplane passing through the origin, and E itself is embedded as an affine hyperplane, defined as $\omega^{-1}(1)$, for some linear form $\omega\colon \widehat{E} \to \mathbb{R}$. In the case of an affine space E of dimension 2, we can think of $\widehat{E}$ as the vector space $\mathbb{R}^3$ of dimension 3 in which $\overrightarrow{E}$ corresponds to the xy-plane, and E corresponds to the plane of equation $z = 1$, parallel to the xy-plane and passing through the point on the z-axis of coordinates $(0, 0, 1)$. The construction of the vector space $\widehat{E}$ is presented in some detail in Berger [11]. Berger explains the construction in terms of vector fields. We prefer a more geometric and simpler description in terms of simple geometric transformations, translations, and dilatations.

Remark: Readers with a good knowledge of geometry will recognize the first step in embedding an affine space into a projective space. We will also show that the homogenization $\widehat{E}$ of an affine space $\big(E, \overrightarrow{E}\big)$, satisfies a universal property with respect to the extension of affine maps to linear maps. As a consequence, the vector space $\widehat{E}$ is unique up to isomorphism, and its actual construction is not so important. However, it is quite useful to visualize the space $\widehat{E}$, in order to understand well rational curves and rational surfaces.

As usual, for simplicity, it is assumed that all vector spaces are defined over the field $\mathbb{R}$ of real numbers, and that all families of scalars (points and vectors) are finite. The extension to arbitrary fields and to families of finite support is immediate. We begin by defining two very simple kinds of geometric (affine) transformations. Given an affine space $\big(E, \overrightarrow{E}\big)$, every $u \in \overrightarrow{E}$ induces a mapping $t_u\colon E \to E$, called a *translation*, and defined such that $t_u(a) = a + u$ for every $a \in E$. Clearly, the set of translations is a vector space isomorphic to $\overrightarrow{E}$. Thus, we will use the same notation u for both the vector u and the translation t_u. Given any point a and any scalar $\lambda \in \mathbb{R}$, we define the mapping $H_{a,\lambda}\colon E \to E$, called *dilatation (or central dilatation, or homothety) of center a and ratio λ*, and defined such that

$$H_{a,\lambda}(x) = a + \lambda \overrightarrow{ax},$$

for every $x \in E$. We have $H_{a,\lambda}(a) = a$, and when $\lambda \neq 0$ and $x \neq a$, $H_{a,\lambda}(x)$ is on the line defined by a and x, and is obtained by "scaling" $\overrightarrow{ax}$ by λ. The effect is a uniform dilatation