(5) Prove that $\nu_m \leq 2$. Prove that $\nu_m = 2$ iff the underlying graph of G contains a nontrivial connected bipartite component.

Hint. Use Problem 20.4(3).

(6) Prove that if G is connected, then $\nu_2 > 0$.

Problem 20.6. Let G be a graph with a set of nodes V with $m \geq 2$ elements, without isolated nodes. Let $\mathrm{vol}(G) = \sum_{v \in V} d_v$ and let

$$\overline{x} = \frac{\sum_v d_v x(v)}{\mathrm{vol}(G)}.$$

Prove that

$$\nu_2 = \min_{x \neq 0} \frac{\displaystyle\sum_{u \sim v} (x(u) - x(v))^2}{\displaystyle\sum_v d_v (x(v) - \overline{x})^2}.$$

Problem 20.7. Let G be a connected bipartite graph. Prove that if ν is an eigenvalue of L_{sym}, then $2 - \nu$ is also an eigenvalue of L_{sym}.

Problem 20.8. Prove Proposition 20.7.