Proposition 30.5. *If A is an integral domain, for any $a, b \in A$ with $a, b \neq 0$, we have $(a) = (b)$ iff there exists some invertible $d \in A$ such that $b = ad$.*

An invertible element $u \in A$ is also called a *unit*.

Given two ideals $\mathfrak{I}$ and $\mathfrak{J}$, their sum

$$\mathfrak{I} + \mathfrak{J} = \{a + b \mid a \in \mathfrak{I},\ b \in \mathfrak{J}\}$$

is clearly an ideal. Given any nonempty subset J of A, the set

$$\{a_1x_1 + \cdots + a_nx_n \mid x_1, \ldots, x_n \in A,\, a_1, \ldots, a_n \in J,\, n \geq 1\}$$

is easily seen to be an ideal, and in fact, it is the smallest ideal containing J. It is usually denoted by (J).

Ideals play a very important role in the study of rings. They tend to show up everywhere. For example, they arise naturally from homomorphisms.

Proposition 30.6. *Given any ring homomorphism $h\colon A \to B$, the kernel $\mathrm{Ker}\, h = \{a \in A \mid h(a) = 0\}$ of h is an ideal.*

Proof. Given $a, b \in A$, we have $a, b \in \mathrm{Ker}\, h$ iff $h(a) = h(b) = 0$, and since h is a homomorphism, we get

$$h(b - a) = h(b) - h(a) = 0,$$

and

$$h(ax) = h(a)h(x) = 0$$

for all $x \in A$, which shows that $\mathrm{Ker}\, h$ is an ideal. □

There is a sort of converse property. Given a ring A and an ideal $\mathfrak{I} \subseteq A$, we can define the quotient ring $A/\mathfrak{I}$, and there is a surjective homomorphism $\pi\colon A \to A/\mathfrak{I}$ whose kernel is precisely $\mathfrak{I}$.

Proposition 30.7. *Given any ring A and any ideal $\mathfrak{I} \subseteq A$, the equivalence relation $\equiv_{\mathfrak{I}}$ defined by $a \equiv_{\mathfrak{I}} b$ iff $b - a \in \mathfrak{I}$ is a congruence, which means that if $a_1 \equiv_{\mathfrak{I}} b_1$ and $a_2 \equiv_{\mathfrak{I}} b_2$, then*

1. $a_1 + a_2 \equiv_{\mathfrak{I}} b_1 + b_2$, *and*
2. $a_1a_2 \equiv_{\mathfrak{I}} b_1b_2$.

Then, the set $A/\mathfrak{I}$ of equivalence classes modulo $\mathfrak{I}$ is a ring under the operations

$$\begin{aligned} [a] + [b] &= [a + b] \\ [a][b] &= [ab]. \end{aligned}$$

The map $\pi\colon A \to A/\mathfrak{I}$ such that $\pi(a) = [a]$ is a surjective homomorphism whose kernel is precisely $\mathfrak{I}$.