The Courant–Fischer theorem yields the following useful result about perturbing the eigenvalues of a symmetric matrix due to Hermann Weyl.

Proposition 17.28. *Given two $n \times n$ symmetric matrices A and $B = A + \Delta A$, if $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$ are the eigenvalues of A and $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_n$ are the eigenvalues of B, then*

$$|\alpha_k - \beta_k| \leq \rho(\Delta A) \leq \|\Delta A\|_2, \quad k = 1, \ldots, n.$$

Proof. Let $\mathcal{V}_k$ be defined as in the Courant–Fischer theorem and let V_k be the subspace spanned by the k eigenvectors associated with $\alpha_1, \ldots, \alpha_k$. By the Courant–Fischer theorem applied to B, we have

$$\begin{aligned}
\beta_k &= \min_{W \in \mathcal{V}_k} \max_{x \in W, x \neq 0} \frac{x^\top B x}{x^\top x} \\
&\leq \max_{x \in V_k} \frac{x^\top B x}{x^\top x} \\
&= \max_{x \in V_k} \left(\frac{x^\top A x}{x^\top x} + \frac{x^\top \Delta\, A x}{x^\top x} \right) \\
&\leq \max_{x \in V_k} \frac{x^\top A x}{x^\top x} + \max_{x \in V_k} \frac{x^\top \Delta A\, x}{x^\top x}.
\end{aligned}$$

By Proposition 17.23, we have

$$\alpha_k = \max_{x \in V_k} \frac{x^\top A x}{x^\top x},$$

so we obtain

$$\begin{aligned}
\beta_k &\leq \max_{x \in V_k} \frac{x^\top A x}{x^\top x} + \max_{x \in V_k} \frac{x^\top \Delta A\, x}{x^\top x} \\
&= \alpha_k + \max_{x \in V_k} \frac{x^\top \Delta A\, x}{x^\top x} \\
&\leq \alpha_k + \max_{x \in \mathbb{R}^n} \frac{x^\top \Delta A\, x}{x^\top x}.
\end{aligned}$$

Now by Proposition 17.23 and Proposition 9.9, we have

$$\max_{x \in \mathbb{R}^n} \frac{x^\top \Delta A\, x}{x^\top x} = \max_i \lambda_i(\Delta A) \leq \rho(\Delta A) \leq \|\Delta A\|_2,$$

where $\lambda_i(\Delta A)$ denotes the ith eigenvalue of ΔA, which implies that

$$\beta_k \leq \alpha_k + \rho(\Delta A) \leq \alpha_k + \|\Delta A\|_2.$$

By exchanging the roles of A and B, we also have

$$\alpha_k \leq \beta_k + \rho(\Delta A) \leq \beta_k + \|\Delta A\|_2,$$

and thus,

$$|\alpha_k - \beta_k| \leq \rho(\Delta A) \leq \|\Delta A\|_2, \quad k = 1, \ldots, n,$$

as claimed. ∎