Proof. Let $E_{\mathbb{C}}$ be the complexification of E, $\langle -, - \rangle_{\mathbb{C}}$ the complexification of the inner product $\langle -, - \rangle$ on E, and $f_{\mathbb{C}} \colon E_{\mathbb{C}} \to E_{\mathbb{C}}$ the complexification of $f \colon E \to E$. By definition of $f_{\mathbb{C}}$ and $\langle -, - \rangle_{\mathbb{C}}$, if f is self-adjoint, we have

$$\begin{aligned}\langle f_{\mathbb{C}}(u_1 + iv_1), u_2 + iv_2 \rangle_{\mathbb{C}} &= \langle f(u_1) + if(v_1), u_2 + iv_2 \rangle_{\mathbb{C}} \\ &= \langle f(u_1), u_2 \rangle + \langle f(v_1), v_2 \rangle + i(\langle u_2, f(v_1) \rangle - \langle f(u_1), v_2 \rangle) \\ &= \langle u_1, f(u_2) \rangle + \langle v_1, f(v_2) \rangle + i(\langle f(u_2), v_1 \rangle - \langle u_1, f(v_2) \rangle) \\ &= \langle u_1 + iv_1, f(u_2) + if(v_2) \rangle_{\mathbb{C}} \\ &= \langle u_1 + iv_1, f_{\mathbb{C}}(u_2 + iv_2) \rangle_{\mathbb{C}},\end{aligned}$$

which shows that $f_{\mathbb{C}}$ is also self-adjoint with respect to $\langle -, - \rangle_{\mathbb{C}}$.

As we pointed out earlier, f and $f_{\mathbb{C}}$ have the same characteristic polynomial $\det(zI - f_{\mathbb{C}}) = \det(zI - f)$, which is a polynomial with real coefficients. Proposition 17.5 shows that the zeros of $\det(zI - f_{\mathbb{C}}) = \det(zI - f)$ are all real, and for each real zero λ of $\det(zI - f)$, the linear map $\lambda \mathrm{id} - f$ is singular, which means that there is some nonzero $u \in E$ such that $f(u) = \lambda u$. Therefore, all the eigenvalues of f are real. ☐

Proposition 17.7. *Given a Hermitian space E, for any linear map $f \colon E \to E$, if f is skew-self-adjoint, then f has eigenvalues that are pure imaginary or zero, and if f is unitary, then f has eigenvalues of absolute value 1.*

Proof. If f is skew-self-adjoint, $f^* = -f$, and then by the definition of the adjoint map, for any eigenvalue λ and any eigenvector u associated with λ, we have

$$\lambda \langle u, u \rangle = \langle \lambda u, u \rangle = \langle f(u), u \rangle = \langle u, f^*(u) \rangle = \langle u, -f(u) \rangle = -\langle u, \lambda u \rangle = -\overline{\lambda} \langle u, u \rangle,$$

and since $u \neq 0$ and $\langle -, - \rangle$ is positive definite, $\langle u, u \rangle \neq 0$, so

$$\lambda = -\overline{\lambda},$$

which shows that $\lambda = ir$ for some $r \in \mathbb{R}$.

If f is unitary, then f is an isometry, so for any eigenvalue λ and any eigenvector u associated with λ, we have

$$|\lambda|^2 \langle u, u \rangle = \lambda \overline{\lambda} \langle u, u \rangle = \langle \lambda u, \lambda u \rangle = \langle f(u), f(u) \rangle = \langle u, u \rangle,$$

and since $u \neq 0$, we obtain $|\lambda|^2 = 1$, which implies

$$|\lambda| = 1.$$

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