because the 5×6 matrix obtained by deleting the first row in the matrix of the determinant has rank 5. Indeed, this is the matrix of the linear system determining the six coefficients of the conic passign through p_1, p_2, p_3, p_4, p_5 (up to a scalar), and since this conic is unique, this matrix must have rank 5.

It is also interesting to see what are lines in the space of circles or in the space of conics. In both cases we get pencils (of circles and conics, respectively). For more details, see Samuel [142], Sidler [161], Tisseron [175], Lehmann and Bkouche [115], Pedoe [136], Coxeter [45, 46], and Veblen and Young [183, 184].

The generalization of the space of projective conics is the space of *projective quadrics* $\mathbf{P}(E)$, where E is the vector space (over a field K, typically $K = \mathbb{R}$ or $K = \mathbb{C}$) consisting of all homogeneous polynomials $P(x_1, \ldots, x_{N+1})$ of degree 2 in the variables $x_1, \ldots, x_{N+1}$, with $N \geq 3$ (plus the null polynomial). The zero locus $V(P)$ of P is defined just as before as

$$V(P) = \{(x_1\colon \cdots : x_{N+1}) \in \mathbb{P}^N_K \mid P(x_1, \ldots, x_{N+1}) = 0\}.$$

If the field K is algebraically closed, in particular if $K = \mathbb{C}$, then $V(P) = V(Q)$ implies that there is some nonzero $\lambda \in K$ such that $Q = \lambda P$; see Berger [12] (Chapter 14, Theorem 14.1.6.2).

Another situation where the map $[P] \mapsto V(P)$ is injective involves the notion of simple (or regular) point of a quadric. For any $a = (a_1\colon \cdots : a_{N+1}) \in \mathbb{P}^N_K$, let $P_{x_i}(a)$ be the partial derivative of P at a given by

$$P_{x_i}(a) = \frac{\partial P}{\partial x_i}(a_1, \ldots, a_{N+1}).$$

Strictly speaking, $P_{x_i}(a)$ depends on the representative $(a_1, \ldots, a_{N+1}) \in K^{N+1}$ chosen for the point a, but since P is homogeneous of degree 2, for any nonzero $\lambda \in K$,

$$\frac{\partial P}{\partial x_i}(\lambda a_1, \ldots, \lambda a_{N+1}) = \lambda \frac{\partial P}{\partial x_i}(a_1, \ldots, a_{N+1}).$$

Thus $P_{x_i}(a)$ is defined up to a nonzero scalar. In particular, whether or not $P_{x_i}(a) = 0$ depends only the point $a = (a_1\colon \cdots : a_{N+1}) \in \mathbb{P}^N_K$. Then the point $a \in V(P)$ is said to be *simple* (or *regular*) if

$$P_{x_i}(a) \neq 0 \quad \text{for some } i,\ 1 \leq i \leq N+1.$$

Otherwise, if $P_{x_1}(a) = \cdots = P_{x_{N+1}}(a) = 0$, we say that $a \in V(P)$ is a *singular* point. If $a \in V(P)$ is a regular point, then the *tangent hyperplane* $T_a V(P)$ to $V(P)$ at a is the hyperplane given by the equation

$$P_{x_1}(a)x_1 + \cdots + P_{x_{N+1}}(a)x_{N+1} = 0.$$

It can be shown that if the field K is not the field $\mathbf{F}_2 = \{0, 1\}$ and if the quadric $V(P)$ contains some regular point, then $V(P) = V(Q)$ implies that there is some nonzero $\lambda \in K$ such that $Q = \lambda P$; see Samuel [142] (Chapter 3, Theorem 46).