is an affine bijection between E_H and the affine hyperplane $w + H$ in E, where $w \in E - H$ is any fixed vector. Choosing w as an origin in E_H, we know that $\widehat{E_H} = H \widehat{+} Kw$, and since $E = H \oplus Kw$, it is obvious how to define a linear bijection between $\widehat{E_H} = H \widehat{+} Kw$ and $E = H \oplus Kw$. As a consequence the projective spaces $\widetilde{E_H}$ and $\mathbf{P}(E)$ are isomorphic, i.e., there is a projectivity between them.

Proposition 26.17. *Given any affine space $\big(E, \overrightarrow{E}\big)$, for every projective space $\mathbf{P}(F)$ (where F is some vector space), every hyperplane H in F, and every map $f\colon E \to \mathbf{P}(F)$ such that $f(E) \subseteq F_H$ and f is affine (F_H being viewed as an affine patch), there is a unique projective map $\widetilde{f}\colon \widetilde{E} \to \mathbf{P}(F)$ such that*

$$f = \widetilde{f} \circ i \quad \textit{and} \quad \mathbf{P}\big(\overrightarrow{f}\big) = \widetilde{f} \circ \mathbf{P}(\overrightarrow{i}),$$

(where $\overrightarrow{i}\colon \overrightarrow{E} \to \overrightarrow{E}$ and $\overrightarrow{f}\colon \overrightarrow{E} \to H$ are the linear maps associated with the affine maps $i\colon E \to \widetilde{E}$ and $f\colon E \to \mathbf{P}(F)$), as in the following diagram:

$$\begin{tikzcd}
E \arrow[r, "i"] \arrow[dr, "f"'] & \mathcal{E}_{\mathcal{H}} \subseteq \mathbf{P}(\mathcal{E}) \supseteq \mathbf{P}(\mathcal{H}) \arrow[d, "\widetilde{f}"] & \mathbf{P}\big(\overrightarrow{E}\big) \arrow[l, "\mathbf{P}(\overrightarrow{i})"'] \arrow[dl, "\mathbf{P}(\overrightarrow{f})"] \\
& F_H \subseteq \mathbf{P}(F) \supseteq \mathbf{P}(H) &
\end{tikzcd}$$

Proof. The existence of $\widetilde{f}$ is a consequence of Proposition 25.6, where we observe that $\widehat{F_H}$ is isomorphic to F. Just take the projective map $\mathbf{P}\big(\widehat{f}\big)\colon \widetilde{E} \to \mathbf{P}(F)$, where $\widehat{f}\colon \widehat{E} \to F$ is the unique linear map extending f. It remains to prove its uniqueness.

As explained in the proof of Proposition 26.16, the affine patch F_H is affinely isomorphic to some affine hyperplane of the form $w + H$ for some $w \in F - H$. If we pick any $a \in E$, since by hypothesis $f(a) \in F_H$, we may assume that $w \in F - H$ is chosen so that $f(a) = [w]$, and we have $F = Kw \oplus H$. Since $f\colon E \to F_H$ is affine, for any $a \in E$ and any $u \in \overrightarrow{E}$, we have

$$f(a+u) = f(a) + \overrightarrow{f}(u) = w + \overrightarrow{f}(u),$$

where $\overrightarrow{f}\colon \overrightarrow{E} \to H$ is a linear map, and where $f(a)$ is viewed as the vector w.

Assume that $\widetilde{f}\colon \widetilde{E} \to \mathbf{P}(F)$ exists with the desired property. Then there is some linear map $g\colon \widehat{E} \to F$ such that $\widetilde{f} = \mathbf{P}(g)$. Our goal is to prove that $g = \mu \widehat{f}$ for some nonzero $\mu \in K$. First, we prove that g vanishes on $\operatorname{Ker} \overrightarrow{f}$.

Since $f = \widetilde{f} \circ i$, we must have $f(a) = [w] = [g(a)]$, and thus $g(a) = \mu w$, for some $\mu \neq 0$. Also, for every $u \in \overrightarrow{E}$,

$$\begin{aligned} f(a+u) &= [w] + \overrightarrow{f}(u) = \big[w + \overrightarrow{f}(u)\big] = [g(a+u)] \\ &= [g(a) + g(u)] = [\mu w + g(u)], \end{aligned}$$