By symmetry, we also obtain $d(y, A) \leq d(x, y) + d(x, A)$, and thus

$$|d(x, A) - d(y, A)| \leq d(x, y),$$

as claimed. $\square$

Definition 37.6. Let (E, d) be a metric space. For any nonempty subset A of E, and any $r > 0$, let

$$V_r(A) = \{x \in E \mid d(x, A) < r\}.$$

Proposition 37.3. *Let (E, d) be a metric space. For any nonempty subset A of E, and any $r > 0$, the set $V_r(A)$ is an open set containing A.*

Proof. For any $y \in E$ such that $d(x, y) < r - d(x, A)$, by Proposition 37.2 we have

$$d(y, A) \leq d(x, A) + d(x, y) \leq d(x, A) + r - d(x, A) = r,$$

so $V_r(A)$ contains the open ball $B_0(x, r - d(x, A))$, which means that it is open. Obviously, $A \subseteq V_r(A)$. $\square$

37.2 Topological Spaces

Motivated by Proposition 37.1, a topological space is defined in terms of a family of sets satisfying the properties of open sets stated in that proposition.

Definition 37.7. Given a set E, a *topology on E (or a topological structure on E)*, is defined as a family $\mathcal{O}$ of subsets of E called *open sets*, and satisfying the following three properties:

(1) For every finite family $(U_i)_{1 \leq i \leq n}$ of sets $U_i \in \mathcal{O}$, we have $U_1 \cap \cdots \cap U_n \in \mathcal{O}$, i.e., $\mathcal{O}$ is closed under finite intersections.

(2) For every arbitrary family $(U_i)_{i \in I}$ of sets $U_i \in \mathcal{O}$, we have $\bigcup_{i \in I} U_i \in \mathcal{O}$, i.e., $\mathcal{O}$ is closed under arbitrary unions.

(3) $\emptyset \in \mathcal{O}$, and $E \in \mathcal{O}$, i.e., $\emptyset$ and E belong to $\mathcal{O}$.

A set E together with a topology $\mathcal{O}$ on E is called a *topological space*. Given a topological space $(E, \mathcal{O})$, a subset F of E is a *closed set* if $F = E - U$ for some open set $U \in \mathcal{O}$, i.e., F is the complement of some open set.

It is possible that an open set is also a closed set. For example, $\emptyset$ and E are both open and closed. When a topological space contains a proper nonempty subset U which is both open and closed, the space E is said to be *disconnected*.