

Do this twice.

You will notice that the first two entries on row 1 and the first two entries on row 2 change, but the rest of the matrix looks the same, except that the dimension is reduced.

This suggests setting up a recurrence involving the entries  $u_k, v_k, x_k, y_k$  in the determinant

$$D_k = \begin{vmatrix} u_k & x_k & -3 & -4 & \cdots & -n+k-3 & -n+k-2 & -n+k-1 & -n+k \\ v_k & y_k & -1 & -1 & \cdots & -1 & -1 & -1 & -1 \\ 1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -2 & 1 \end{vmatrix},$$

starting with  $k = 0$ , with

$$u_0 = -1, \quad v_0 = -1, \quad x_0 = -2, \quad y_0 = -1,$$

and ending with  $k = n - 2$ , so that

$$d_n = D_{n-2} = \begin{vmatrix} u_{n-3} & x_{n-3} & -3 \\ v_{n-3} & y_{n-3} & -1 \\ 1 & -2 & 1 \end{vmatrix} = \begin{vmatrix} u_{n-2} & x_{n-2} \\ v_{n-2} & y_{n-2} \end{vmatrix}.$$

Prove that we have the recurrence relations

$$\begin{pmatrix} u_{k+1} \\ v_{k+1} \\ x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} 2 & -2 & 1 & -1 \\ 0 & 2 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_k \\ v_k \\ x_k \\ y_k \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -2 \\ -1 \end{pmatrix}.$$

These appear to be nasty affine recurrence relations, so we will use the trick to convert this affine map to a linear map.

(4) Consider the linear map given by

$$\begin{pmatrix} u_{k+1} \\ v_{k+1} \\ x_{k+1} \\ y_{k+1} \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & -2 & 1 & -1 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & -2 \\ 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_k \\ v_k \\ x_k \\ y_k \\ 1 \end{pmatrix},$$