and since $u_{k+1} = u_k - \rho_k \nabla J_{u_k}$ we get

$$\langle \nabla J_{u_{k+1}}, \nabla J_{u_k} \rangle = 0,$$

which shows that two consecutive descent directions are orthogonal.

Since $u_{k+1} = u_k - \rho_k \nabla J_{u_k}$ and we assumed that that $u_{k+1} \neq u_k$, we have $\rho_k \neq 0$, and we also get

$$\langle \nabla J_{u_{k+1}}, u_{k+1} - u_k \rangle = 0.$$

By the inequality of Theorem 49.8(1) we have

$$J(u_k) - J(u_{k+1}) \geq \frac{\alpha}{2} \left\| u_k - u_{k+1} \right\|^2.$$

Step 2. Show that $\lim_{k \mapsto \infty} \| u_k - u_{k+1} \| = 0$.

It follows from the inequality proven in Step 1 that the sequence $(J(u_k))_{k \geq 0}$ is decreasing and bounded below (by $J(u)$, where u is the minimum of J), so it converges and we conclude that

$$\lim_{k \mapsto \infty} (J(u_k) - J(u_{k+1})) = 0,$$

which combined with the preceding inequality shows that

$$\lim_{k \mapsto \infty} \| u_k - u_{k+1} \| = 0.$$

Step 3. Show that $\| \nabla J_{u_k} \| \leq \left\| \nabla J_{u_k} - \nabla J_{u_{k+1}} \right\|$.

Using the orthogonality of consecutive descent directions, by Cauchy–Schwarz we have

$$\begin{aligned} \| \nabla J_{u_k} \|^2 &= \langle \nabla J_{u_k}, \nabla J_{u_k} - \nabla J_{u_{k+1}} \rangle \\ &\leq \| \nabla J_{u_k} \| \left\| \nabla J_{u_k} - \nabla J_{u_{k+1}} \right\|, \end{aligned}$$

so that

$$\| \nabla J_{u_k} \| \leq \left\| \nabla J_{u_k} - \nabla J_{u_{k+1}} \right\|.$$

Step 4. Show that $\lim_{k \mapsto \infty} \| \nabla J_{u_k} \| = 0$.

Since the sequence $(J(u_k))_{k \geq 0}$ is decreasing and the functional J is coercive, the sequence $(u_k)_{k \geq 0}$ must be bounded. By hypothesis, the derivative dJ is of J is continuous, so it is uniformly continuous over compact subsets of $\mathbb{R}^n$; here we are using the fact that $\mathbb{R}^n$ is finite dimensional. Hence, we deduce that for every $\epsilon > 0$, there is some $\delta > 0$ such that if $\| u_k - u_{k+1} \| < \delta$ then

$$\left\| dJ_{u_k} - dJ_{u_{k+1}} \right\|_2 < \epsilon.$$