10.2 Convergence of Iterative Methods

Recall that iterative methods for solving a linear system $Ax = b$ (with $A \in \mathrm{M}_n(\mathbb{C})$ invertible) consists in finding some matrix B and some vector c, such that $I - B$ is invertible, and the unique solution $\widetilde{x}$ of $Ax = b$ is equal to the unique solution $\widetilde{u}$ of $u = Bu + c$. Then starting from *any* vector u_0, compute the sequence (u_k) given by

$$u_{k+1} = Bu_k + c, \quad k \in \mathbb{N},$$

and say that the iterative method is *convergent* iff

$$\lim_{k \mapsto \infty} u_k = \widetilde{u},$$

for *every* initial vector u_0.

Here is a fundamental criterion for the convergence of any iterative methods based on a matrix B, called the *matrix of the iterative method*.

Theorem 10.3. *Given a system $u = Bu + c$ as above, where $I - B$ is invertible, the following statements are equivalent:*

(1) The iterative method is convergent.

(2) $\rho(B) < 1$.

(3) $\|B\| < 1$, for some subordinate matrix norm $\| \;\|$.

Proof. Define the vector e_k (*error vector*) by

$$e_k = u_k - \widetilde{u},$$

where $\widetilde{u}$ is the unique solution of the system $u = Bu + c$. Clearly, the iterative method is convergent iff

$$\lim_{k \mapsto \infty} e_k = 0.$$

We claim that

$$e_k = B^k e_0, \quad k \geq 0,$$

where $e_0 = u_0 - \widetilde{u}$.

This is proven by induction on k. The base case $k = 0$ is trivial. By the induction hypothesis, $e_k = B^k e_0$, and since $u_{k+1} = Bu_k + c$, we get

$$u_{k+1} - \widetilde{u} = Bu_k + c - \widetilde{u},$$

and because $\widetilde{u} = B\widetilde{u} + c$ and $e_k = B^k e_0$ (by the induction hypothesis), we obtain

$$u_{k+1} - \widetilde{u} = Bu_k - B\widetilde{u} = B(u_k - \widetilde{u}) = Be_k = BB^k e_0 = B^{k+1} e_0,$$

proving the induction step. Thus, the iterative method converges iff

$$\lim_{k \mapsto \infty} B^k e_0 = 0.$$

Consequently, our theorem follows by Theorem 10.1. $\square$