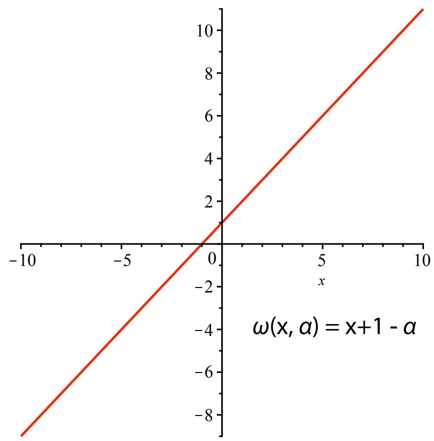


Figure 51.9: Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be the affine form $\varphi(x) = x + 1$. Let $\omega\colon \mathbb{R}^2 \to \mathbb{R}$ be the affine form $\omega(x, \alpha) = x + 1 - \alpha$. The hyperplane $\mathcal{H} = \omega^{-1}(0)$ is the red line with equation $x - \alpha + 1 = 0$.

We say that $\mathcal{H}$ is the *hyperplane (in $\mathbb{R}^{n+1}$) induced by the affine form $\varphi\colon \mathbb{R}^n \to \mathbb{R}$*. Also recall the notion of supporting hyperplane to a convex set.

**Definition 51.12.** If $C$ is a nonempty convex set in $\mathbb{R}^n$ and $x$ is a vector in $C$, an affine hyperplane $H$ is a *supporting hyperplane to $C$ at $x$* if

(1) $x \in H$.

(2) Either $C \subseteq H_+$ or $C \subseteq H_-$.

See Figure 51.10. Equivalently, there is some nonconstant affine form $\varphi$ such that $\varphi(z) = \langle z, u\rangle - c$ for all $z \in \mathbb{R}^n$, for some nonzero $u \in \mathbb{R}^n$ and some $c \in \mathbb{R}$, such that

(1) $\langle x, u\rangle = c$.

(2) $\langle z, u\rangle \leq c$ for all $z \in C$

The notion of vector normal to a convex set is defined as follows.

**Definition 51.13.** Given a nonempty convex set $C$ in $\mathbb{R}^n$, for any $a \in C$, a vector $u \in \mathbb{R}^n$ is *normal to $C$ at $a$* if

$$\langle z - a, u\rangle \leq 0 \quad \text{for all } z \in C.$$

In other words, $u$ does not make an acute angle with any line segment in $C$ with $a$ as endpoint. The set of all vectors $u$ normal to $C$ is called the *normal cone* to $C$ at $a$ and is denoted by $N_C(a)$. See Figure 51.11.

It is easy to check that the normal cone to $C$ at $a$ is a convex cone. Also, if the hyperplane $H$ defined by an affine form $\varphi(z) = \langle z, u\rangle - c$ with $u \neq 0$ is a supporting hyperplane to $C$ at