the complex field $\mathbb{C}$ is that the theory of intersection is cleaner. Thus, any two circles that do not contain a common line always intersect in four points, some of which might be multiple points (as in the case of tangent circles). This may seem surprising, since in the real plane, two circles intersect in at most two points. Where are the other two points? They turn out to be the points $(1, i, 0)$ and $(1, -i, 0)$, as one can immediately verify. We can think of them as complex points at infinity! Not only are they at infinity, but they are not real. No wonder we cannot see them! We will come back to these points, called the *circular points*, in Section 26.14.

Going back to the vector space E of circles over $\mathbb{R}$, it is worth saying that it can be shown that if $V(P) = V(Q)$ contains at least two points (in which case, $V(P)$ is actually infinite), then $Q = \lambda P$ for some $\lambda \in \mathbb{R}$ with $\lambda \neq 0$ (see Tisseron [175], Theorem 3.6.1 and Theorem 4.7). Thus, even over $\mathbb{R}$, the mapping

$$[P] \mapsto V(P)$$

is injective whenever $V(P)$ is neither empty nor reduced to a single point. Note that the projective space $\mathbf{P}(E)$ of circles has dimension 3. In fact, it is easy to show that three distinct points that are not collinear determine a unique circle (see Samuel [142], Section 1.6).

In a similar vein, we can define the *projective space of conics* $\mathbf{P}(E)$ where E is the vector space (over $\mathbb{R}$) consisting of all homogeneous polynomials of degree 2 in x, y, z,

$$ax^2 + by^2 + cxy + dxz + eyz + fz^2$$

(plus the null polynomial). The curves $V(P)$ are indeed conics, perhaps degenerate. To see this, we can use the hyperplane model of $\mathbb{RP}^2$. The trace of $V(P)$ on the plane of equation $z = 1$ is the conic of equation

$$ax^2 + by^2 + cxy + dx + ey + f = 0.$$

Another way to see that $V(P)$ is a conic is to observe that in $\mathbb{R}^3$,

$$ax^2 + by^2 + cxy + dxz + eyz + fz^2 = 0$$

defines a cone with vertex $(0, 0, 0)$, and since its section by the plane $z = 1$ is a conic, all of its sections by planes are conics. See Figure 26.10 for schematic illustration of a projective conic embedded in $\mathbb{RP}^2$.

The mapping

$$[P] \mapsto V(P)$$

is still injective when E is defined over the ground field $\mathbb{C}$ (Samuel [142], Section 1.6, Theorem 10), or if $V(P)$ has at least two points when E is defined over $\mathbb{R}$ (Tisseron [175], Theorem 3.6.1 and Theorem 4.7). Note that the projective space $\mathbf{P}(E)$ of conics has dimension 5. In fact, it can be shown that five distinct points, no four of which are collinear, determine a