Our hypotheses imply that $\theta > 0$. We can write

$$\begin{aligned}
w^\top u_i - b &= \eta + \theta - (\epsilon_i + \theta) & \epsilon_i + \theta > 0 \qquad & i \in E_\lambda \\
-w^\top v_j + b &= \eta + \theta - (\xi_j + \theta) & \xi_j + \theta > 0 \qquad & j \in E_\mu \\
w^\top u_i - b &\geq \eta + \theta & & i \notin E_\lambda \\
-w^\top v_j + b &\geq \eta + \theta & & j \notin E_\mu,
\end{aligned}$$

and by the choice of θ, either

$$w^\top u_i - b = \eta + \theta \quad \text{for some } i \notin E_\lambda$$

or

$$-w^\top v_j + b = \eta + \theta \quad \text{for some } j \notin E_\mu.$$

The original value of the objective function is

$$\omega(0) = \frac{1}{2} w^\top w + \frac{1}{2} b^2 - \nu\eta + \frac{1}{p+q}\Bigg(\sum_{i \in E_\lambda} \epsilon_i + \sum_{j \in E_\mu} \xi_j\Bigg),$$

and the new value is

$$\begin{aligned}
\omega(\theta) &= \frac{1}{2} w^\top w + \frac{1}{2} b^2 - \nu(\eta + \theta) + \frac{1}{p+q}\Bigg(\sum_{i \in E_\lambda} (\epsilon_i + \theta) + \sum_{j \in E_\mu} (\xi_j + \theta)\Bigg) \\
&= \frac{1}{2} w^\top w + \frac{1}{2} b^2 - \nu\eta + \frac{1}{p+q}\Bigg(\sum_{i \in E_\lambda} \epsilon_i + \sum_{j \in E_\mu} \xi_j\Bigg) - \bigg(\nu - \frac{p_{sf} + q_{sf}}{p+q}\bigg)\theta.
\end{aligned}$$

By Proposition 54.1,

$$\frac{p_{sf} + q_{sf}}{p+q} \leq \frac{p_f + q_f}{p+q} \leq \nu,$$

so

$$\nu - \frac{p_{sf} + q_{sf}}{p+q} \geq 0,$$

and so $\omega(\theta) \leq \omega(0)$. If the inequality is strict, then this contradicts the optimality of the original solution. Therefore, $\omega(\theta) = \omega(0)$ and $(w, b, \eta + \theta, \epsilon + \theta, \xi + \theta)$ is an optimal solution such that either

$$w^\top u_i - b = \eta + \theta \quad \text{for some } i \notin E_\lambda$$

or

$$-w^\top v_j + b = \eta + \theta \quad \text{for some } j \notin E_\mu,$$

as desired. $\square$

Proposition 54.6 cannot be strengthened to claim that there is some support vector u_{i_0} *and* some support vector v_{j_0}. We found examples for which the above condition fails for ν large enough.

The proof of Proposition 54.6 reveals that $(p_{sf} + q_{sf})/(p+q)$ is a critical value for ν. if this value is avoided we have the following corollary.