Let $f_{k+1} = h_{k+1} \circ f_k \circ h_{k+1}$, and finish the proof.

(3) Prove that given any symmetric $n \times n$-matrix A, there are $n-2$ matrices $H_1, \ldots, H_{n-2}$, Householder matrices or the identity, such that

$$B = H_{n-2} \cdots H_1 A H_1 \cdots H_{n-2}$$

is a symmetric tridiagonal matrix.

(4) Write a computer program implementing the above method.

Problem 13.7. Recall from Problem 6.6 that an $n \times n$ matrix H is *upper Hessenberg* if $h_{jk} = 0$ for all (j, k) such that $j - k \geq 0$. Adapt the proof of Problem 13.6 to prove that given any $n \times n$-matrix A, there are $n - 2 \geq 1$ matrices $H_1, \ldots, H_{n-2}$, Householder matrices or the identity, such that

$$B = H_{n-2} \cdots H_1 A H_1 \cdots H_{n-2}$$

is upper Hessenberg.

Problem 13.8. The purpose of this problem is to prove that given any linear map $f \colon E \to E$, where E is a Euclidean space of dimension $n \geq 2$, given an orthonormal basis $(e_1, \ldots, e_n)$, there are isometries g_i, h_i, hyperplane reflections or the identity, such that the matrix of

$$g_n \circ \cdots \circ g_1 \circ f \circ h_1 \circ \cdots \circ h_n$$

is a lower bidiagonal matrix, which means that the nonzero entries (if any) are on the main descending diagonal and on the diagonal below it.

(1) Let U'_k be the subspace spanned by $(e_1, \ldots, e_k)$ and U''_k be the subspace spanned by $(e_{k+1}, \ldots, e_n)$, $1 \leq k \leq n - 1$. Proceed by induction For the base case, proceed as follows.

Let $v_1 = f^*(e_1)$ and $r_{1,1} = \|v_1\|$. Find an isometry h_1 (reflection or id) such that

$$h_1(f^*(e_1)) = r_{1,1} e_1.$$

Observe that $h_1(f^*(e_1)) \in U'_1$, so that

$$\langle h_1(f^*(e_1)), e_j \rangle = 0$$

for all $j, 2 \leq j \leq n$, and conclude that

$$\langle e_1, f \circ h_1(e_j) \rangle = 0$$

for all $j, 2 \leq j \leq n$.

Next let

$$u_1 = f \circ h_1(e_1) = u'_1 + u''_1,$$

where $u'_1 \in U'_1$ and $u''_1 \in U''_1$, and let $r_{2,1} = \|u''_1\|$. Find an isometry g_1 (reflection or id) such that

$$g_1(u''_1) = r_{2,1} e_2.$$