

The beginning of the proof of Proposition 17.11 actually shows that for every linear map $f: E \rightarrow E$ there is some subspace W such that $f(W) \subseteq W$, where W has dimension 1 or 2. In general, it doesn't seem possible to prove that W^\perp is invariant under f . *However, this happens when f is normal.*

We can finally prove our first main theorem.

Theorem 17.12. (Main spectral theorem) *Given a Euclidean space E of dimension n , for every normal linear map $f: E \rightarrow E$, there is an orthonormal basis (e_1, \dots, e_n) such that the matrix of f w.r.t. this basis is a block diagonal matrix of the form*

$$\begin{pmatrix} A_1 & & \dots & \\ & A_2 & & \\ & \vdots & \ddots & \\ & & \dots & A_p \end{pmatrix}$$

such that each block A_j is either a one-dimensional matrix (i.e., a real scalar) or a two-dimensional matrix of the form

$$A_j = \begin{pmatrix} \lambda_j & -\mu_j \\ \mu_j & \lambda_j \end{pmatrix},$$

where $\lambda_j, \mu_j \in \mathbb{R}$, with $\mu_j > 0$.

Proof. We proceed by induction on the dimension n of E as follows. If $n = 1$, the result is trivial. Assume now that $n \geq 2$. First, since \mathbb{C} is algebraically closed (i.e., every polynomial has a root in \mathbb{C}), the linear map $f_{\mathbb{C}}: E_{\mathbb{C}} \rightarrow E_{\mathbb{C}}$ has some eigenvalue $z = \lambda + i\mu$ (where $\lambda, \mu \in \mathbb{R}$). Let $w = u + iv$ be some eigenvector of $f_{\mathbb{C}}$ for $\lambda + i\mu$ (where $u, v \in E$). We can now apply Proposition 17.11.

If $\mu = 0$, then either u or v is an eigenvector of f for $\lambda \in \mathbb{R}$. Let W be the subspace of dimension 1 spanned by $e_1 = u/\|u\|$ if $u \neq 0$, or by $e_1 = v/\|v\|$ otherwise. It is obvious that $f(W) \subseteq W$ and $f^*(W) \subseteq W$. The orthogonal W^\perp of W has dimension $n - 1$, and by Proposition 17.9, we have $f(W^\perp) \subseteq W^\perp$. But the restriction of f to W^\perp is also normal, and we conclude by applying the induction hypothesis to W^\perp .

If $\mu \neq 0$, then $\langle u, v \rangle = 0$ and $\langle u, u \rangle = \langle v, v \rangle$, and if W is the subspace spanned by $u/\|u\|$ and $v/\|v\|$, then $f(W) = W$ and $f^*(W) = W$. We also know that the restriction of f to W has the matrix

$$\begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix}$$

with respect to the basis $(u/\|u\|, v/\|v\|)$. If $\mu < 0$, we let $\lambda_1 = \lambda$, $\mu_1 = -\mu$, $e_1 = u/\|u\|$, and $e_2 = v/\|v\|$. If $\mu > 0$, we let $\lambda_1 = \lambda$, $\mu_1 = \mu$, $e_1 = v/\|v\|$, and $e_2 = u/\|u\|$. In all cases, it is easily verified that the matrix of the restriction of f to W w.r.t. the orthonormal basis (e_1, e_2) is

$$A_1 = \begin{pmatrix} \lambda_1 & -\mu_1 \\ \mu_1 & \lambda_1 \end{pmatrix},$$