When $E = \mathbb{R}^2$, the only existence of $\dfrac{\partial^2 f}{\partial x \partial y}(a)$ and $\dfrac{\partial^2 f}{\partial y \partial x}(a)$ is not sufficient to insure the existence of $\mathrm{D}^2 f(a)$.

When E if of finite dimension n and $(a_0, (e_1, \ldots, e_n))$ is a frame for E, if $\mathrm{D}^2 f(a)$ exists, for every $u = u_1 e_1 + \cdots + u_n e_n$ and $v = v_1 e_1 + \cdots + v_n e_n$ in $\overrightarrow{E}$, since $\mathrm{D}^2 f(a)$ is a symmetric bilinear form, we have

$$\mathrm{D}^2 f(a)(u, v) = \sum_{i=1, j=1}^{n} u_i v_j \frac{\partial^2 f}{\partial x_i \partial x_j}(a),$$

which can be written in matrix form as:

$$\mathrm{D}^2 f(a)(u, v) = U^\top \begin{pmatrix} \dfrac{\partial^2 f}{\partial x_1^2}(a) & \dfrac{\partial^2 f}{\partial x_1 \partial x_2}(a) & \ldots & \dfrac{\partial^2 f}{\partial x_1 \partial x_n}(a) \\ \dfrac{\partial^2 f}{\partial x_1 \partial x_2}(a) & \dfrac{\partial^2 f}{\partial x_2^2}(a) & \ldots & \dfrac{\partial^2 f}{\partial x_2 \partial x_n}(a) \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f}{\partial x_1 \partial x_n}(a) & \dfrac{\partial^2 f}{\partial x_2 \partial x_n}(a) & \ldots & \dfrac{\partial^2 f}{\partial x_n^2}(a) \end{pmatrix} V$$

where U is the column matrix representing u, and V is the column matrix representing v, over the frame $(a_0, (e_1, \ldots, e_n))$.

Definition 39.15. The above symmetric matrix is called the *Hessian of f at a*.

Example 39.10. Consider the function f defined on real invertible 2×2 matrices such that $ad - bc > 0$ given by

$$f(a, b, c, d) = \log(ad - bc).$$

We immediately verify that the Jacobian matrix of f is given by

$$df_{a,b,c,d} = \frac{1}{ad - bc} \begin{pmatrix} d & -c & -b & a \end{pmatrix}.$$

It is easily checked that if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix},$$

then

$$df_A(X) = \mathrm{tr}(A^{-1}X) = \frac{1}{ad - bc} \mathrm{tr}\left(\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \right).$$

Computing second-order derivatives, we find that the Hessian matrix of f is given by

$$Hf(A) = \frac{1}{(ad - bc)^2} \begin{pmatrix} -d^2 & cd & bd & -bc \\ cd & -c^2 & -ad & ac \\ bd & -ad & -b^2 & ab \\ -bc & ac & ab & -a^2 \end{pmatrix}.$$