It follows that

$$\lambda - a_n = \delta \pm \sqrt{\delta^2 + b_{n-1}^2},$$

and from this it is easy to see that the eigenvalue closer to a_n is given by

$$\mu = a_n - \frac{\mathrm{sign}(\delta) b_{n-1}^2}{(|\delta| + \sqrt{\delta^2 + b_{n-1}^2})}.$$

If $\delta = 0$, then we pick arbitrarily one of the two eigenvalues. Observe that the Wilkinson shift applied to the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is either $+1$ or -1, and in one step, deflation occurs and the algorithm terminates successfully.

We now discuss double shifts, which are intended to deal with pairs of complex conjugate eigenvalues.

Let us assume that A is a real matrix. For any complex number σ_k with nonzero imaginary part, a *double shift* consists of the following steps:

$$\begin{aligned}
A_k - \sigma_k I &= Q_k R_k \\
A_{k+1} &= R_k Q_k + \sigma_k I \\
A_{k+1} - \overline{\sigma}_k I &= Q_{k+1} R_{k+1} \\
A_{k+2} &= R_{k+1} Q_{k+1} + \overline{\sigma}_k I.
\end{aligned}$$

From the computation made for a single shift, we have $A_{k+1} = Q_k^* A_k Q_k$ and $A_{k+2} = Q_{k+1}^* A_{k+1} Q_{k+1}$, so we obtain

$$A_{k+2} = Q_{k+1}^* Q_k^* A_k Q_k Q_{k+1}.$$

The matrices Q_k are complex, so we would expect that the A_k are also complex, but remarkably we can keep the products $Q_k Q_{k+1}$ real, and so the A_k also real. This is highly desirable to avoid complex arithmetic, which is more expensive.

Observe that since

$$Q_{k+1} R_{k+1} = A_{k+1} - \overline{\sigma}_k I = R_k Q_k + (\sigma_k - \overline{\sigma}_k) I,$$

we have

$$\begin{aligned}
Q_k Q_{k+1} R_{k+1} R_k &= Q_k (R_k Q_k + (\sigma_k - \overline{\sigma}_k) I) R_k \\
&= Q_k R_k Q_k R_k + (\sigma_k - \overline{\sigma}_k) Q_k R_k \\
&= (A_k - \sigma_k I)^2 + (\sigma_k - \overline{\sigma}_k)(A_k - \sigma_k I) \\
&= A_k^2 - 2(\Re \sigma_k) A_k + |\sigma_k|^2 I.
\end{aligned}$$