which yields the equations

$$\begin{aligned} -dy + cz &= 0 \\ cx - by &= 0 \\ dx - bz &= 0. \end{aligned}$$

This linear system has the nontrivial solution (b, c, d) and the matrix of this system is

$$\begin{pmatrix} 0 & -d & c \\ c & -b & 0 \\ d & 0 & -b \end{pmatrix}.$$

Since $(b, c, d) \neq (0, 0, 0)$, this matrix always has a 2×2 submatrix which is nonsingular, so it has rank 2, and consequently its kernel is the one-dimensional space spanned by (b, c, d). Therefore, r_q has the eigenvalue 1 with multiplicity 1. If we had $\det(r_q) = -1$, then the eigenvalues of r_q would be either $(-1, 1, 1)$ or $(-1, e^{i\theta}, e^{-i\theta})$ with $\theta \neq k2\pi$ (with $k \in \mathbb{Z}$), contradicting the fact that 1 is an eigenvalue with multiplicity 1. Therefore, r_q is a rotation; in fact, its axis is determined by (b, c, d). $\square$

In summary, $q \mapsto r_q$ is a map r from $\mathbf{SU}(2)$ to $\mathbf{SO}(3)$.

Theorem 16.3. *The map* $r\colon \mathbf{SU}(2) \to \mathbf{SO}(3)$ *is a homomorphism whose kernel is* $\{I, -I\}$.

Proof. This map is a homomorphism, because if $q_1, q_2 \in \mathbf{SU}(2)$, then

$$\begin{aligned} r_{q_2}(r_{q_1}(x, y, z)) &= \varphi^{-1}(q_2\varphi(r_{q_1}(x, y, z))q_2^*) \\ &= \varphi^{-1}(q_2\varphi(\varphi^{-1}(q_1\varphi(x, y, z)q_1^*))q_2^*) \\ &= \varphi^{-1}((q_2q_1)\varphi(x, y, z)(q_2q_1)^*) \\ &= r_{q_2q_1}(x, y, z). \end{aligned}$$

The computation that showed that if $(b, c, d) \neq (0, 0, 0)$, then r_q has the eigenvalue 1 with multiplicity 1 implies the following: if $r_q = I_3$, namely r_q has the eigenvalue 1 with multiplicity 3, then $(b, c, d) = (0, 0, 0)$. But then $a = \pm 1$, and so $q = \pm I_2$. Therefore, the kernel of the homomorphism $r\colon \mathbf{SU}(2) \to \mathbf{SO}(3)$ is $\{I, -I\}$. $\square$

Remark: Perhaps the quickest way to show that r maps $\mathbf{SU}(2)$ into $\mathbf{SO}(3)$ is to observe that the map r is continuous. Then, since it is known that $\mathbf{SU}(2)$ is connected, its image by r lies in the connected component of I, namely $\mathbf{SO}(3)$.

Proposition 16.2 showed that if $u = (b, c, d) \neq (0, 0, 0)$, then r_q is a rotation whose axis is determined by $u = (b, c, d)$. The angle θ of this rotation can also be determined. The following result is proven in Gallier [72] (Chapter 9).