we have

$$f = f\pi_1 + \cdots + f\pi_k,$$

and so we get

$$N = f - D = (f - \lambda_1 \mathrm{id})\pi_1 + \cdots + (f - \lambda_k \mathrm{id})\pi_k.$$

We claim that $N = f - D$ is a nilpotent operator. Since by construction the π_i are polynomials in f, they commute with f, using the properties of the π_i, we get

$$N^r = (f - \lambda_1 \mathrm{id})^r \pi_1 + \cdots + (f - \lambda_k \mathrm{id})^r \pi_k.$$

Therefore, if $r = \max\{r_i\}$, we have $(f - \lambda_k \mathrm{id})^r = 0$ for $i = 1, \ldots, k$, which implies that

$$N^r = 0.$$

It remains to show that D is diagonalizable. Since N is a polynomial in f, it commutes with f, and thus with D. From

$$D = \lambda_1 \pi_1 + \cdots + \lambda_k \pi_k,$$

and

$$\pi_1 + \cdots + \pi_k = \mathrm{id},$$

we see that

$$\begin{aligned} D - \lambda_i \mathrm{id} &= \lambda_1 \pi_1 + \cdots + \lambda_k \pi_k - \lambda_i(\pi_1 + \cdots + \pi_k) \\ &= (\lambda_1 - \lambda_i)\pi_1 + \cdots + (\lambda_{i-1} - \lambda_i)\pi_{i-1} + (\lambda_{i+1} - \lambda_i)\pi_{i+1} + \cdots + (\lambda_k - \lambda_i)\pi_k. \end{aligned}$$

Since the projections π_j with $j \neq i$ vanish on W_i, the above equation implies that $D - \lambda_i \mathrm{id}$ vanishes on W_i and that $(D - \lambda_j \mathrm{id})(W_i) \subseteq W_i$, and thus that the minimal polynomial of D is

$$(X - \lambda_1) \cdots (X - \lambda_k).$$

Since the λ_i are distinct, by Theorem 31.6, the linear map D is diagonalizable.

In summary we have shown that when all the eigenvalues of f belong to K, there exist a diagonalizable linear map D and a nilpotent linear map N such that

$$\begin{aligned} f &= D + N \\ DN &= ND, \end{aligned}$$

and N and D are polynomials in f.

Definition 31.6. A decomposition of f as $f = D + N$ as above is called a *Jordan decomposition*.

In fact, we can prove more: the maps D and N are uniquely determined by f.