Analogously to the case of Euclidean spaces of finite dimension, the Hermitian product induces a canonical bijection (i.e., independent of the choice of bases) between the vector space E and the space E^*. This is one of the places where conjugation shows up, but in this case, troubles are minor.

Given a Hermitian space E, for any vector $u \in E$, let $\varphi^l_u \colon E \to \mathbb{C}$ be the map defined such that

$$\varphi^l_u(v) = \overline{u \cdot v}, \quad \text{for all } v \in E.$$

Similarly, for any vector $v \in E$, let $\varphi^r_v \colon E \to \mathbb{C}$ be the map defined such that

$$\varphi^r_v(u) = u \cdot v, \quad \text{for all } u \in E.$$

Since the Hermitian product is linear in its first argument u, the map φ^r_v is a linear form in E^*, and since it is semilinear in its second argument v, the map φ^l_u is also a linear form in E^*. Thus, we have two maps $\flat^l \colon E \to E^*$ and $\flat^r \colon E \to E^*$, defined such that

$$\flat^l(u) = \varphi^l_u, \quad \text{and} \quad \flat^r(v) = \varphi^r_v.$$

Proposition 14.5. *The equations $\varphi^l_u = \varphi^r_u$ and $\flat^l = \flat^r$ hold.*

Proof. Indeed, for all $u, v \in E$, we have

$$\begin{aligned} \flat^l(u)(v) &= \varphi^l_u(v) \\ &= \overline{u \cdot v} \\ &= v \cdot u \\ &= \varphi^r_u(v) \\ &= \flat^r(u)(v). \end{aligned}$$

□

Therefore, we use the notation φ_u for both φ^l_u and φ^r_u, and $\flat$ for both $\flat^l$ and $\flat^r$.

Theorem 14.6. *Let E be a Hermitian space E. The map $\flat \colon E \to E^*$ defined such that*

$$\flat(u) = \varphi^l_u = \varphi^r_u \quad \textit{for all } u \in E$$

is semilinear and injective. When E is also of finite dimension, the map $\flat \colon \overline{E} \to E^*$ is a canonical isomorphism.

Proof. That $\flat \colon E \to E^*$ is a semilinear map follows immediately from the fact that $\flat = \flat^r$, and that the Hermitian product is semilinear in its second argument. If $\varphi_u = \varphi_v$, then $\varphi_u(w) = \varphi_v(w)$ for all $w \in E$, which by definition of φ_u and φ_v means that

$$w \cdot u = w \cdot v$$

for all $w \in E$, which by semilinearity on the right is equivalent to

$$w \cdot (v - u) = 0 \quad \text{for all } w \in E,$$

which implies that $u = v$, since the Hermitian product is positive definite. Thus, $\flat \colon E \to E^*$ is injective. Finally, when E is of finite dimension n, E^* is also of dimension n, and then $\flat \colon E \to E^*$ is bijective. Since $\flat$ is semilinar, the map $\flat \colon \overline{E} \to E^*$ is an isomorphism. □