Let $S_1^{n-1}$ be the unit sphere with respect to the norm $\| \|_1$, namely

$$S_1^{n-1} = \{x \in E \mid \|x\|_1 = 1\}.$$

Now $S_1^{n-1}$ is a closed and bounded subset of a finite-dimensional vector space, so by Heine–Borel (or equivalently, by Bolzano–Weiertrass), $S_1^{n-1}$ is compact. On the other hand, it is a well known result of analysis that any continuous real-valued function on a nonempty compact set has a minimum and a maximum, and that they are achieved. Using these facts, we can prove the following important theorem:

**Theorem 9.5.** *If $E$ is any real or complex vector space of finite dimension, then any two norms on $E$ are equivalent.*

*Proof.* It is enough to prove that any norm $\| \|$ is equivalent to the 1-norm. We already proved that the function $x \mapsto \|x\|$ is continuous with respect to the norm $\| \|_1$, and we observed that the unit sphere $S_1^{n-1}$ is compact. Now we just recalled that because the function $f : x \mapsto \|x\|$ is continuous and because $S_1^{n-1}$ is compact, the function $f$ has a minimum $m$ and a maximum $M$, and because $\|x\|$ is never zero on $S_1^{n-1}$, we must have $m > 0$. Consequently, we just proved that if $\|x\|_1 = 1$, then

$$0 < m \leq \|x\| \leq M,$$

so for any $x \in E$ with $x \neq 0$, we get

$$m \leq \|x / \|x\|_1\| \leq M,$$

which implies

$$m \|x\|_1 \leq \|x\| \leq M \|x\|_1 .$$

Since the above inequality holds trivially if $x = 0$, we just proved that $\| \|$ and $\| \|_1$ are equivalent, as claimed. ☐

**Remark:** Let $P$ be a $n \times n$ symmetric positive definite matrix. It is immediately verified that the map $x \mapsto \|x\|_P$ given by

$$\|x\|_P = (x^\top P x)^{1/2}$$

is a norm on $\mathbb{R}^n$ called a *quadratic norm*. Using some convex analysis (the Löwner–John ellipsoid), it can be shown that *any* norm $\| \|$ on $\mathbb{R}^n$ can be approximated by a quadratic norm in the sense that there is a quadratic norm $\| \|_P$ such that

$$\|x\|_P \leq \|x\| \leq \sqrt{n}\, \|x\|_P \qquad \text{for all } x \in \mathbb{R}^n;$$

see Boyd and Vandenberghe [29], Section 8.4.1.

Next we will consider norms on matrices.