Proof. (Second proof) Let $(\mathfrak{A}_i)_{i\geq 1}$ be an ascending sequence of ideals in $A[X]$. Consider the doubly indexed family $(L_i(\mathfrak{A}_j))$ of ideals in A. Since A is noetherian, by the maximal property, this family has a maximal element $L_p(\mathfrak{A}_q)$. Since the $L_i(\mathfrak{A}_j)$'s form an ascending sequence when either i or j is fixed, we have $L_i(\mathfrak{A}_j) = L_p(\mathfrak{A}_q)$ for all i and j with $i \geq p$ and $j \geq q$, and thus, $L_i(\mathfrak{A}_q) = L_i(\mathfrak{A}_j)$ for all i and j with $i \geq p$ and $j \geq q$. On the other hand, for any fixed i, the a.c.c. shows that there exists some integer $n(i)$ so that $L_i(\mathfrak{A}_j) = L_i(\mathfrak{A}_{n(i)})$ for all $j \geq n(i)$. Since $L_i(\mathfrak{A}_q) = L_i(\mathfrak{A}_j)$ when $i \geq p$ and $j \geq q$, we may take $n(i) = q$ if $i \geq p$. This shows that there is some n_0 so that $n(i) \leq n_0$ for all $i \geq 0$, and thus, we have $L_i(\mathfrak{A}_j) = L_i(\mathfrak{A}_{n(0)})$ for every i and for every $j \geq n(0)$. By Lemma 32.19, we get $\mathfrak{A}_j = \mathfrak{A}_{n(0)}$ for every $j \geq n(0)$, establishing the fact that $A[X]$ satisfies the a.c.c. $\square$

Using induction, we immediately obtain the following important result.

Corollary 32.21. *If A is a noetherian ring, then $A[X_1, \ldots, X_n]$ is also a noetherian ring.*

Since a field K is obviously noetherian (since it has only two ideals, (0) and K), we also have:

Corollary 32.22. *If K is a field, then $K[X_1, \ldots, X_n]$ is a noetherian ring.*

32.4 Futher Readings

The material of this Chapter is thoroughly covered in Lang [109], Artin [7], Mac Lane and Birkhoff [118], Bourbaki [25, 26], Malliavin [119], Zariski and Samuel [194], and Van Der Waerden [179].