Now that we have defined $\widehat{E}$ and investigated the relationship between affine frames in E and bases in $\widehat{E}$, we can give another construction of a vector space $\mathcal{F}$ from E and $\overrightarrow{E}$ that will allow us to "visualize" in a much more intuitive fashion the structure of $\widehat{E}$ and of its operations $\widehat{+}$ and $\cdot$.

25.3 Another Construction of $\widehat{E}$

One would probably wish that we could start with this construction of $\mathcal{F}$ first, and then define $\widehat{E}$ using the isomorphism $\widehat{\Omega}\colon \widehat{E} \to \mathcal{F}$ defined below. Unfortunately, we first need the vector space structure on $\widehat{E}$ to show that $\widehat{\Omega}$ is linear!

Definition 25.1. Given any affine space $\big(E, \overrightarrow{E}\big)$, we define the vector space $\mathcal{F}$ as the direct sum $\overrightarrow{E} \oplus \mathbb{R}$, where $\mathbb{R}$ denotes the field $\mathbb{R}$ considered as a vector space (over itself). Denoting the unit vector in $\mathbb{R}$ by 1, since $\mathcal{F} = \overrightarrow{E} \oplus \mathbb{R}$, every vector $v \in \mathcal{F}$ can be written as $v = u + \lambda 1$, for some unique $u \in \overrightarrow{E}$ and some unique $\lambda \in \mathbb{R}$. Then, for any choice of an origin Ω_1 in E, we define the map $\widehat{\Omega}\colon \widehat{E} \to \mathcal{F}$, as follows:

$$\widehat{\Omega}(\theta) = \begin{cases} \lambda(1 + \overrightarrow{\Omega_1 a}) & \text{if } \theta = \langle a, \lambda\rangle, \text{ where } a \in E \text{ and } \lambda \neq 0;\\ u & \text{if } \theta = u, \text{ where } u \in \overrightarrow{E}. \end{cases}$$

The idea is that, once again, viewing $\mathcal{F}$ as an affine space under its canonical structure, E is embedded in $\mathcal{F}$ as the hyperplane $H = 1 + \overrightarrow{E}$, with direction $\overrightarrow{E}$, the hyperplane $\overrightarrow{E}$ in $\mathcal{F}$. Then, every point $a \in E$ is in bijection with the point $A = 1 + \overrightarrow{\Omega_1 a}$, in the hyperplane H. If we denote the origin 0 of the canonical affine space $\mathcal{F}$ by Ω, the map $\widehat{\Omega}$ maps a point $\langle a, \lambda\rangle \in E$ to a point in $\mathcal{F}$, as follows: $\widehat{\Omega}(\langle a, \lambda\rangle)$ is the point on the line passing through both the origin Ω of $\mathcal{F}$ and the point $A = 1 + \overrightarrow{\Omega_1 a}$ in the hyperplane $H = 1 + \overrightarrow{E}$, such that

$$\widehat{\Omega}(\langle a, \lambda\rangle) = \lambda\overrightarrow{\Omega A} = \lambda(1 + \overrightarrow{\Omega_1 a}).$$

The following proposition shows that $\widehat{\Omega}$ is an isomorphism of vector spaces.

Proposition 25.4. *Given any affine space $(E, \overrightarrow{E})$, for any choice Ω_1 of an origin in E, the map $\widehat{\Omega}\colon \widehat{E} \to \mathcal{F}$ is a linear isomorphism between $\widehat{E}$ and the vector space $\mathcal{F}$ of Definition 25.1. The inverse of $\widehat{\Omega}$ is given by*

$$\widehat{\Omega}^{-1}(u + \lambda 1) = \begin{cases} \langle \Omega_1 + \lambda^{-1} u, \lambda\rangle) & \text{if } \lambda \neq 0;\\ u & \text{if } \lambda = 0. \end{cases}$$

Proof. It is a straightforward verification. We check that $\widehat{\Omega}$ is invertible, leaving the verification that it is linear as an exercise. We have

$$\langle a, \lambda\rangle \mapsto \lambda 1 + \lambda\overrightarrow{\Omega_1 a} \mapsto \langle \Omega_1 + \overrightarrow{\Omega_1 a}, \lambda\rangle = \langle a, \lambda\rangle$$