Proposition 53.5. *(I. Schur) If $\kappa_1 \colon X \times X \to \mathbb{C}$ and $\kappa_2 \colon X \times X \to \mathbb{C}$ are two positive definite kernels, then the function $\kappa \colon X \times X \to \mathbb{C}$ given by $\kappa(x, y) = \kappa_1(x, y)\kappa_2(x, y)$ for all $x, y \in X$ is also a positive definite kernel.*

Proof. It suffices to prove that if $A = (a_{jk})$ and $B = (b_{jk})$ are two Hermitian positive semidefinite $p \times p$ matrices, then so is their pointwise product $C = A \circ B = (a_{jk}b_{jk})$ (also known as Hadamard or Schur product). Recall that a Hermitian positive semidefinite matrix A can be diagonalized as $A = U\Lambda U^*$, where Λ is a diagonal matrix with nonnegative entries and U is a unitary matrix. Let $\Lambda^{1/2}$ be the diagonal matrix consisting of the positive square roots of the diagonal entries in Λ. Then we have

$$A = U\Lambda U^* = U\Lambda^{1/2}\Lambda^{1/2}U^* = U\Lambda^{1/2}(U\Lambda^{1/2})^*.$$

Thus if we set $R = U\Lambda^{1/2}$, we have

$$A = RR^*,$$

which means that

$$a_{jk} = \sum_{h=1}^{p} r_{jh}\overline{r_{kh}}.$$

Then for any $u \in \mathbb{C}^p$, we have

$$\begin{aligned} u^*(A \circ B)u &= \sum_{j,k=1}^{p} a_{jk}b_{jk}u_j\overline{u_k} \\ &= \sum_{j,k=1}^{p}\sum_{h=1}^{p} r_{jh}\overline{r_{kh}}b_{jk}u_j\overline{u_k} \\ &= \sum_{h=1}^{p}\sum_{j,k=1}^{p} b_{jk}u_jr_{jh}\overline{u_kr_{kh}}. \end{aligned}$$

Since B is positive semidefinite, for each fixed h, we have

$$\sum_{j,k=1}^{p} b_{jk}u_jr_{jh}\overline{u_kr_{kh}} = \sum_{j,k=1}^{p} b_{jk}z_j\overline{z_k} \geq 0,$$

as we see by letting $z = (u_1r_{1h}, \ldots, u_pr_{ph})$, $\square$

In contrast, the ordinary product AB of two symmetric positive semidefinite matrices A and B may not be symmetric positive semidefinite; see Section 8.9 for an example.

Here are other ways of obtaining new positive definite kernels from old ones.

Proposition 53.6. *Let $\kappa_1 \colon X \times X \to \mathbb{C}$ and $\kappa_2 \colon X \times X \to \mathbb{C}$ be two positive definite kernels, $f \colon X \to \mathbb{C}$ be a function, $\psi \colon X \to \mathbb{R}^N$ be a function, $\kappa_3 \colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{C}$ be a positive definite kernel, and $a \in \mathbb{R}$ be any positive real number. Then the following functions are positive definite kernels:*