8. A very important example of vector space is the set of linear maps between two vector spaces to be defined in Section 11.1. Here is an example that will prepare us for the vector space of linear maps. Let X be any nonempty set and let E be a vector space. The set of all functions $f\colon X \to E$ can be made into a vector space as follows: Given any two functions $f\colon X \to E$ and $g\colon X \to E$, let $(f+g)\colon X \to E$ be defined such that

   $$(f+g)(x) = f(x) + g(x)$$

   for all $x \in X$, and for every $\lambda \in \mathbb{R}$, let $\lambda f\colon X \to E$ be defined such that

   $$(\lambda f)(x) = \lambda f(x)$$

   for all $x \in X$. The axioms of a vector space are easily verified.

Let E be a vector space. We would like to define the important notions of linear combination and linear independence.

Before defining these notions, we need to discuss a strategic choice which, depending how it is settled, may reduce or increase headaches in dealing with notions such as linear combinations and linear dependence (or independence). The issue has to do with using sets of vectors versus sequences of vectors.

3.3 Indexed Families; the Sum Notation $\sum_{i \in I} a_i$

Our experience tells us that *it is preferable to use sequences of vectors*; even better, indexed families of vectors. (We are not alone in having opted for sequences over sets, and we are in good company; for example, Artin [7], Axler [10], and Lang [109] use sequences. Nevertheless, some prominent authors such as Lax [113] use sets. We leave it to the reader to conduct a survey on this issue.)

Given a set A, recall that a *sequence* is an ordered n-tuple $(a_1, \ldots, a_n) \in A^n$ of elements from A, for some natural number n. The elements of a sequence need not be distinct and the order is important. For example, (a_1, a_2, a_1) and (a_2, a_1, a_1) are two distinct sequences in A^3. Their underlying set is $\{a_1, a_2\}$.

What we just defined are *finite* sequences, which can also be viewed as functions from $\{1, 2, \ldots, n\}$ to the set A; the ith element of the sequence $(a_1, \ldots, a_n)$ is the image of i under the function. This viewpoint is fruitful, because it allows us to define (countably) infinite sequences as functions $s\colon \mathbb{N} \to A$. But then, why limit ourselves to ordered sets such as $\{1, \ldots, n\}$ or $\mathbb{N}$ as index sets?

The main role of the index set is to tag each element uniquely, and the order of the tags is not crucial, although convenient. Thus, it is natural to define the notion of indexed family.