Proposition 6.2. *Given any two vector spaces, E_1 and E_2, for every pair of linear maps, $f\colon D \to E_1$ and $g\colon D \to E_2$, there is a unique linear map, $f \times g\colon D \to E_1 \coprod E_2$, such that $\pi_1 \circ (f \times g) = f$ and $\pi_2 \circ (f \times g) = g$, as in the following diagram:*

$$\begin{tikzcd} & E_1 \\ D \arrow[ur, "f"] \arrow[r, "f\times g"] \arrow[dr, "g"'] & E_1 \coprod E_2 \arrow[u, "\pi_1"'] \arrow[d, "\pi_2"] \\ & E_2 \end{tikzcd}$$

Proof. Define
$$(f \times g)(w) = \{\langle 1, f(w)\rangle, \langle 2, g(w)\rangle\},$$
for every $w \in D$. It is immediately verified that $f \times g$ is the unique linear map with the required properties. $\square$

Remark: It is a peculiarity of linear algebra that direct sums and products of finite families are isomorphic. However, this is no longer true for products and sums of infinite families.

When U, V are subspaces of a vector space E, letting $i_1\colon U \to E$ and $i_2\colon V \to E$ be the inclusion maps, if $U \coprod V$ is isomomorphic to E under the map $i_1 + i_2$ given by Proposition 6.1, we say that E *is a direct sum* of U and V, and we write $E = U \coprod V$ (with a slight abuse of notation, since E and $U \coprod V$ are only isomorphic). It is also convenient to define the sum $U_1 + \cdots + U_p$ and the internal direct sum $U_1 \oplus \cdots \oplus U_p$ of any number of subspaces of E.

Definition 6.2. Given $p \geq 2$ vector spaces $E_1, \ldots, E_p$, the product $F = E_1 \times \cdots \times E_p$ can be made into a vector space by defining addition and scalar multiplication as follows:

$$\begin{aligned} (u_1, \ldots, u_p) + (v_1, \ldots, v_p) &= (u_1 + v_1, \ldots, u_p + v_p) \\ \lambda(u_1, \ldots, u_p) &= (\lambda u_1, \ldots, \lambda u_p), \end{aligned}$$

for all $u_i, v_i \in E_i$ and all $\lambda \in \mathbb{R}$. The zero vector of $E_1 \times \cdots \times E_p$ is the p-tuple

$$(\underbrace{0, \ldots, 0}_{p}),$$

where the ith zero is the zero vector of E_i.

With the above addition and multiplication, the vector space $F = E_1 \times \cdots \times E_p$ is called the *direct product* of the vector spaces $E_1, \ldots, E_p$.

As a special case, when $E_1 = \cdots = E_p = \mathbb{R}$, we find again the vector space $F = \mathbb{R}^p$. The *projection maps* $pr_i\colon E_1 \times \cdots \times E_p \to E_i$ given by

$$pr_i(u_1, \ldots, u_p) = u_i$$