Furthermore, the initial shape of the string is known at $t = 0$, as well as the distribution of the initial velocities along the string; in other words, there are two functions $u_{i,0}$ and $u_{i,1}$ such that

$$\begin{aligned} u(x,0) &= u_{i,0}(x), \quad 0 \leq x \leq L, \\ \frac{\partial u}{\partial t}(x,0) &= u_{i,1}(x), \quad 0 \leq x \leq L. \end{aligned}$$

For example, if the string is simply released from its given starting position, we have $u_{i,1} = 0$. Lastly, because the ends of the string are fixed, we must have

$$u(0,t) = u(L,t) = 0, \quad t \geq 0.$$

Consequently, we look for a function $u\colon \mathbb{R}_+ \times [0,L] \to \mathbb{R}$ satisfying the following conditions:

$$\begin{aligned} \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) &= f(x,t), \quad 0 < x < L,\ t > 0, \\ u(0,t) &= u(L,t) = 0, \quad t \geq 0 \quad \text{(boundary condition)}, \\ u(x,0) &= u_{i,0}(x), \quad 0 \leq x \leq L \quad \text{(intitial condition)}, \\ \frac{\partial u}{\partial t}(x,0) &= u_{i,1}(x), \quad 0 \leq x \leq L \quad \text{(intitial condition)}. \end{aligned}$$

This is an example of a *time-dependent boundary-value problem*, with two *initial conditions*.

To simplify the problem, assume that $f = 0$, which amounts to neglecting the effect of gravity. In this case, our PDE becomes

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = 0, \quad 0 < x < L,\ t > 0,$$

Let us try our trick of multiplying by a test function v depending only on x, C^1 on $[0,L]$, and such that $v(0) = v(L) = 0$, and integrate by parts. We get the equation

$$\int_0^L \frac{\partial^2 u}{\partial t^2}(x,t)v(x)dx - c^2 \int_0^L \frac{\partial^2 u}{\partial x^2}(x,t)v(x)dx = 0.$$

For the first term, we get

$$\begin{aligned} \int_0^L \frac{\partial^2 u}{\partial t^2}(x,t)v(x)dx &= \int_0^L \frac{\partial^2}{\partial t^2}[u(x,t)v(x)]dx \\ &= \frac{d^2}{dt^2}\int_0^L u(x,t)v(x)dx \\ &= \frac{d^2}{dt^2}\langle u, v\rangle, \end{aligned}$$