Furthermore, if φ_1 *and* φ_2 *are any two nonzero linear forms defining the same hyperplane* H*, in the sense that* $H = \mathrm{Ker}\,\varphi_1 = \mathrm{Ker}\,\varphi_2$*, then there is some nonzero* $\alpha \in K$ *such that* $\varphi_2 = \alpha\varphi_1$.

Proof. First assume that $\varphi\colon E \to K$ is a nonzero linear form and that $H = \mathrm{Ker}\,\varphi$. Then there is a nonzero vector $u_0 \in E$ such that $\varphi(u_0) = \lambda_0 \neq 0$ for some $\lambda_0 \in K$, and so for every $\lambda \in K$, we have

$$\varphi(\lambda\lambda_0^{-1}u_0) = \lambda\lambda_0^{-1}\varphi(u_0) = \lambda\lambda_0^{-1}\lambda_0 = \lambda,$$

which means that φ is surjective onto K. It follows that in Theorem 6.16 we can define s by $s(\lambda_0) = u_0$, so the subspace $L = \mathrm{Im}\, s = Ku_0$ is a one-dimensional space and we have

$$E = \mathrm{Ker}\,\varphi \oplus L = H \oplus L,$$

so H is a hyperplane.

Conversely assume that H is a hyperplane, so that $E = H \oplus L$ where L is a subspace of dimension 1. If we pick a nonzero vector $u_0 \in L$, since L has dimension 1 and $E = H \oplus L$, every $u \in E$ can be written uniquely as $u = h + \lambda u_0$ for some $h \in H$ and some $\lambda \in K$. If we define the map $\varphi\colon E \to K$ by

$$\varphi(u + \lambda u_0) = \lambda,$$

we check immediately that φ is linear and that its kernel is H.

Assume that $H = \mathrm{Ker}\,\varphi_1 = \mathrm{Ker}\,\varphi_2$ for some nonzero linear forms φ_1 and φ_2. We just proved that $E = H \oplus Ku_0$ for some $u_0 \in E$ such that $\varphi_1(u_0) \neq 0$, and we must also have $\varphi_2(u_0) \neq 0$, since otherwise, as $H = \mathrm{Ker}\,\varphi_2$, the linear form φ_2 would be zero on E. Then observe that

$$\varphi_2 - \frac{\varphi_2(u_0)}{\varphi_1(u_0)}\varphi_1$$

is a linear form that vanishes on H since both φ_1 and φ_2 vanish on H, but also vanishes on Ku_0 since

$$\begin{aligned}\left(\varphi_2 - \frac{\varphi_2(u_0)}{\varphi_1(u_0)}\varphi_1\right)(\lambda u_0) &= \varphi_2(\lambda u_0) - \frac{\varphi_2(u_0)}{\varphi_1(u_0)}\varphi_1(\lambda u_0)\\ &= \lambda\varphi_2(u_0) - \lambda\frac{\varphi_2(u_0)}{\varphi_1(u_0)}\varphi_1(u_0) = \lambda\varphi_2(u_0) - \lambda\varphi_2(u_0) = 0\end{aligned}$$

for all $\lambda \in K$. Since $E = H \oplus Ku_0$, we deduce that $\varphi_2 - \alpha\varphi_2$ vanishes on E, with

$$\alpha = \frac{\varphi_2(u_0)}{\varphi_1(u_0)} \neq 0,$$

and so $\varphi_2 = \alpha\varphi_1$, as claimed. $\square$