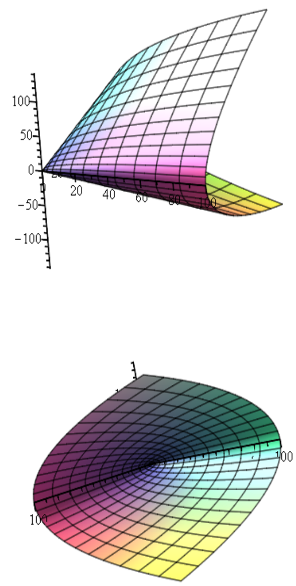

Figure 53.1: The parametric surface $\varphi_1(x_1, x_2) = (x_1^2, x_2^2, \sqrt{2}x_1x_2)$ where $-10 \leq x_1 \leq 10$ and $-10 \leq x_2 \leq 10$.

Example 53.3. Example 53.2 can be generalized as follows. Suppose we have a feature map $\varphi_1 \colon X \to \mathbb{R}^n$ and let $\kappa_1(x, y) = \langle \varphi_1(x), \varphi_1(y) \rangle$ be the corresponding kernel function (where $\langle -, - \rangle$ is the standard inner product on $\mathbb{R}^n$). Define the feature map $\varphi \colon X \to \mathbb{R}^n \times \mathbb{R}^n$ by its n^2 components

$$\varphi(x)_{(i,j)} = (\varphi_1(x))_i(\varphi_1(x))_j, \qquad 1 \leq i, j \leq n,$$

with the inner product on $\mathbb{R}^n \times \mathbb{R}^n$ given by

$$\langle u, v \rangle = \sum_{i,j=1}^{n} u_{(i,j)} v_{(i,j)}.$$

Then we have

$$\begin{aligned}
\langle \varphi(x), \varphi(y) \rangle &= \sum_{i,j=1}^{n} \varphi_{(i,j)}(x)\varphi_{(i,j)}(y) \\
&= \sum_{i,j=1}^{n} (\varphi_1(x))_i(\varphi_1(x))_j(\varphi_1(y))_i(\varphi_1(y))_j \\
&= \sum_{i=1}^{n} (\varphi_1(x))_i(\varphi_1(y))_i \sum_{j=1}^{n} (\varphi_1(x))_j(\varphi_1(y))_j \\
&= (\kappa_1(x, y))^2.
\end{aligned}$$