and

$$\Lambda_k = (I + \Lambda'_k)E_k^{-1} - I = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & 0 \\ \lambda'^{(k-1)}_{21} & 0 & 0 & 0 & 0 & \vdots & \vdots & 0 \\ \lambda'^{(k-1)}_{31} & \lambda'^{(k-1)}_{32} & \ddots & 0 & 0 & \vdots & \vdots & 0 \\ \vdots & \vdots & \ddots & 0 & 0 & \vdots & \vdots & \vdots \\ \lambda'^{(k-1)}_{k1} & \lambda'^{(k-1)}_{k2} & \cdots & \lambda'^{(k-1)}_{k\,(k-1)} & 0 & \cdots & \cdots & 0 \\ \lambda'^{(k-1)}_{k+11} & \lambda'^{(k-1)}_{k+12} & \cdots & \lambda'^{(k-1)}_{k+1\,(k-1)} & \ell^{(k)}_{k+1k} & \ddots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda'^{(k-1)}_{n1} & \lambda'^{(k-1)}_{n2} & \cdots & \lambda'^{(k-1)}_{n\,k-1} & \ell^{(k)}_{nk} & \cdots & \cdots & 0 \end{pmatrix},$$

with $P_k = I$ or $P_k = P(k, i)$ for some $i > k$. This means that in assembling L, row k and row i of Λ_{k-1} need to be permuted when a pivoting step permuting row k and row i of A_k is required. Then

$$\begin{aligned} I + \Lambda_k &= (E_1^k)^{-1} \cdots (E_k^k)^{-1} \\ \Lambda_k &= \mathcal{E}_1^k + \cdots + \mathcal{E}_k^k, \end{aligned}$$

for $k = 1, \ldots, n-1$, and therefore

$$L = I + \Lambda_{n-1}.$$

The proof of Theorem 8.5, which is very technical, is given in Section 8.6.

We emphasize again that Part (3) of Theorem 8.5 shows the remarkable fact that in assembling the matrix L while performing Gaussian elimination with pivoting, the only change to the algorithm is to make the same transposition on the rows of Λ_{k-1} that we make on the rows of A (really A_k) during a pivoting step involving row k and row i. We can also assemble P by starting with the identity matrix and applying to P the same row transpositions that we apply to A and Λ. Here is an example illustrating this method.

Example 8.4. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & -3 & 4 \\ 4 & 8 & 12 & -8 \\ 2 & 3 & 2 & 1 \\ -3 & -1 & 1 & -4 \end{pmatrix}.$$

We set $P_0 = I_4$, and we can also set $\Lambda_0 = 0$. The first step is to permute row 1 and row 2, using the pivot 4. We also apply this permutation to P_0:

$$A'_1 = \begin{pmatrix} 4 & 8 & 12 & -8 \\ 1 & 2 & -3 & 4 \\ 2 & 3 & 2 & 1 \\ -3 & -1 & 1 & -4 \end{pmatrix} \quad P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$