Chapter 52

Dual Ascent Methods; ADMM

This chapter is devoted to the presentation of one of the best methods known at the present for solving optimization problems involving equality constraints. In fact, this method can also handle more general constraints, namely, membership in a convex set. It can also be used to solve a range of problems arising in machine learning including *lasso minimization*, *elastic net regression*, *support vector machine (SVM)*, and *ν-SV regression*. In order to obtain a good understanding of this method, called the *alternating direction method of multipliers*, for short *ADMM*, we review two precursors of ADMM, the *dual ascent method* and the *method of multipliers*.

ADMM is not a new method. In fact, it was developed in the 1970's. It has been revived as a very effective method to solve problems in statistical and machine learning dealing with very large data because it is well suited to distributed (convex) optimization. An extensive presentation of ADMM, its variants, and its applications, is given in the excellent paper by Boyd, Parikh, Chu, Peleato and Eckstein [28]. This paper is essentially a book on the topic of ADMM, and our exposition is deeply inspired by it.

In this chapter, we consider the problem of minimizing a convex function J (not necessarily differentiable) under the equality constraints $Ax = b$. In Section 52.1 we discuss the dual ascent method. It is essentially gradient descent applied to the dual function G, but since G is maximized, gradient descent becomes gradient ascent.

In order to make the minimization step of the dual ascent method more robust, one can use the trick of adding the penalty term $(\rho/2)\left\|Au - b\right\|_2^2$ to the Lagrangian. We obtain the *augmented Lagrangian*

$$L_\rho(u, \lambda) = J(u) + \lambda^\top(Au - b) + (\rho/2)\left\|Au - b\right\|_2^2,$$

with $\lambda \in \mathbb{R}^m$, and where $\rho > 0$ is called the *penalty parameter*. We obtain the minimization Problem (P_ρ),

$$\begin{aligned} &\text{minimize} && J(u) + (\rho/2)\left\|Au - b\right\|_2^2 \\ &\text{subject to} && Au = b, \end{aligned}$$