Proposition 11.9 yields another proof of part (b) of the duality theorem (theorem 11.4) that does not involve the existence of bases (in infinite dimension).

Proposition 11.10. *For any vector space E and any subspace V of E, we have $V^{00} = V$.*

Proof. We begin by observing that $V^0 = V^{000}$. This is because, for any subspace U of E^*, we have $U \subseteq U^{00}$, so $V^0 \subseteq V^{000}$. Furthermore, $V \subseteq V^{00}$ holds, and for any two subspaces M, N of E, if $M \subseteq N$ then $N^0 \subseteq N^0$, so we get $V^{000} \subseteq V^0$. Write $V_1 = V^{00}$, so that $V_1^0 = V^{000} = V^0$. We wish to prove that $V_1 = V$.

Since $V \subseteq V_1 = V^{00}$, the canonical projection $p_1 \colon E \to E/V_1$ factors as $p_1 = f \circ p$ as in the diagram below,

$$\begin{tikzcd} E \arrow[r, "p"] \arrow[dr, "p_1"'] & E/V \arrow[d, "f"] \\ & E/V_1 \end{tikzcd}$$

where $p \colon E \to E/V$ is the canonical projection onto E/V and $f \colon E/V \to E/V_1$ is the quotient map induced by p_1, with $f(\overline{u}_{E/V}) = p_1(u) = \overline{u}_{E/V_1}$, for all $u \in E$ (since $V \subseteq V_1$, if $u - u' = v \in V$, then $u - u' = v \in V_1$, so $p_1(u) = p_1(u')$). Since p_1 is surjective, so is f. We wish to prove that f is actually an isomorphism, and for this, it is enough to show that f is injective. By transposing all the maps, we get the commutative diagram

$$\begin{tikzcd} E^* & (E/V)^* \arrow[l, "p^\top"'] \\ & (E/V_1)^*, \arrow[ul, "p_1^\top"] \arrow[u, "f^\top"'] \end{tikzcd}$$

but by Proposition 11.9, the maps $p^\top \colon (E/V)^* \to V^0$ and $p_1^\top \colon (E/V_1)^* \to V_1^0$ are isomorphism, and since $V^0 = V_1^0$, we have the following diagram where both $p^\top$ and $p_1^\top$ are isomorphisms:

$$\begin{tikzcd} V^0 & (E/V)^* \arrow[l, "p^\top"'] \\ & (E/V_1)^*. \arrow[ul, "p_1^\top"] \arrow[u, "f^\top"'] \end{tikzcd}$$

Therefore, $f^\top = (p^\top)^{-1} \circ p_1^\top$ is an isomorphism. We claim that this implies that f is injective.

If f is not injective, then there is some $x \in E/V$ such that $x \neq 0$ and $f(x) = 0$, so for every $\varphi \in (E/V_1)^*$, we have $f^\top(\varphi)(x) = \varphi(f(x)) = 0$. However, there is linear form $\psi \in (E/V)^*$ such that $\psi(x) = 1$, so $\psi \neq f^\top(\varphi)$ for all $\varphi \in (E/V_1)^*$, contradicting the fact that $f^\top$ is surjective. To find such a linear form ψ, pick any supplement W of Kx in E/V, so that $E/V = Kx \oplus W$ (W is a hyperplane in E/V not containing x), and define ψ to be zero