The fourth case shows that the sign of the affine form in $(*)$ is positive, and thus λ_1/α_1, $\lambda_2/\alpha_2, \lambda_3/\alpha_3 > 0$, which implies that the scalars in each of the pairs (α_1, λ_1), (α_2, λ_2) and (α_3, λ_3), must have the same sign. □

The generalization to any dimension $n \geq 2$ is immediate: the scalars in each pair (α_i, λ_i) must have the same sign for $i = 1, \ldots, n+2$.

In dimension 2, since $\alpha_3 = 1 - \alpha_1 - \alpha_2$ and $\lambda_3 = 1 - \lambda_1 - \lambda_2$, there are four cases to consider:

(1) $\alpha_1, \lambda_1, \alpha_2, \lambda_2 < 0$. In this case, $\alpha_3, \lambda_3 > 1$ so α_3, λ_3 also have the same sign.

(2) $\alpha_1, \lambda_1 < 0$ and $\alpha_2, \lambda_2 > 0$. In this case, since $\alpha_3 = 1 - \alpha_1 - \alpha_2$ and $\lambda_3 = 1 - \lambda_1 - \lambda_2$, we must have either both $\alpha_1 + \alpha_2 < 1$ and $\lambda_1 + \lambda_2 < 1$, or both $\alpha_1 + \alpha_2 > 1$ and $\lambda_1 + \lambda_2 > 1$, in order for α_3 and λ_3 to have the same sign.

(3) $\alpha_1, \lambda_1 > 0$ and $\alpha_2, \lambda_2 < 0$. As in the previous case, since $\alpha_3 = 1 - \alpha_1 - \alpha_2$ and $\lambda_3 = 1 - \lambda_1 - \lambda_2$, we must have either both $\alpha_1 + \alpha_2 < 1$ and $\lambda_1 + \lambda_2 < 1$, or both $\alpha_1 + \alpha_2 > 1$ and $\lambda_1 + \lambda_2 > 1$, in order for α_3 and λ_3 to have the same sign.

(4) $\alpha_1, \lambda_1, \alpha_2, \lambda_2 > 0$. As in the previous case, since $\alpha_3 = 1 - \alpha_1 - \alpha_2$ and $\lambda_3 = 1 - \lambda_1 - \lambda_2$, we must have either both $\alpha_1 + \alpha_2 < 1$ and $\lambda_1 + \lambda_2 < 1$, or both $\alpha_1 + \alpha_2 > 1$ and $\lambda_1 + \lambda_2 > 1$, in order for α_3 and λ_3 to have the same sign.

Since $\alpha_3 = 1 - \alpha_1 - \alpha_2$ and $\lambda_3 = 1 - \lambda_1 - \lambda_2$, we can write

$$
\begin{aligned}
p_4 &= \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 = p_3 + \alpha_1(p_1 - p_3) + \alpha_2(p_2 - p_3) \\
q_4 &= \lambda_1 q_1 + \lambda_2 q_2 + \lambda_3 q_3 = q_3 + \lambda_1(q_1 - q_3) + \lambda_2(q_2 - q_3).
\end{aligned}
$$

In the affine frame $(p_3, (p_1 - p_3, p_2 - p_3))$, points have coordinates (α_1, α_2), and in the affine frame $(q_3, (q_1 - q_3, q_2 - q_3))$, points have coordinates (λ_1, λ_2). In the first affine frame, the line $\langle p_1, p_2 \rangle$ is given by the equation $\alpha_1 + \alpha_2 = 1$, and in the second affine frame, the line $\langle q_1, q_2 \rangle$ is given by the equation $\lambda_1 + \lambda_2 = 1$. The open half plane containing p_3 and bounded by the line $\langle p_1, p_2 \rangle$ corresponds to the points of coordinates (α_1, α_2) satisfying $\alpha_1 + \alpha_2 < 1$, and the other open half plane not containing p_3 corresponds to the points of coordinates (α_1, α_2) satisfying $\alpha_1 + \alpha_2 > 1$. Similarly, the open half plane containing q_3 and bounded by the line $\langle q_1, q_2 \rangle$ corresponds to the points of coordinates (λ_1, λ_2) satisfying $\lambda_1 + \lambda_2 < 1$, and the other open half plane not containing q_3 corresponds to the points of coordinates (λ_1, λ_2) satisfying $\lambda_1 + \lambda_2 > 1$.

Then, the above conditions have the following interpretation in terms of regions in the affine plane $z = 1$:

(1) When $\alpha_1 < 0$ and $\alpha_2 < 0$, the point p_4 lies in quadrant III (with respect to the affine frames $(p_3, (p_1 - p_3, p_2 - p_3))$). Under the mapping f, the point q_4 is also mapped to quadrant III (with respect to the affine frame $(q_3, (q_1 - q_3, q_2 - q_3))$); see Figure 26.14.