Problem 39.3. If $f\colon \mathrm{M}_n(\mathbb{R}) \to \mathrm{M}_n(\mathbb{R})$ and $g\colon \mathrm{M}_n(\mathbb{R}) \to \mathrm{M}_n(\mathbb{R})$ are differentiable matrix functions, prove that

$$d(fg)_A(B) = df_A(B)g(A) + f(A)dg_A(B),$$

for all $A, B \in \mathrm{M}_n(\mathbb{R})$.

Problem 39.4. Recall that $\mathfrak{so}(3)$ denotes the vector space of real skew-symmetric $n \times n$ matrices ($B^\top = -B$). Let $C\colon \mathfrak{so}(n) \to \mathrm{M}_n(\mathbb{R})$ be the function given by

$$C(B) = (I - B)(I + B)^{-1}.$$

(1) Prove that if B is skew-symmetric, then $I - B$ and $I + B$ are invertible, and so C is well-defined. Prove that

(2) Prove that

$$dC(B)(A) = -[I + (I - B)(I + B)^{-1}]A(I + B)^{-1} = -2(I + B)^{-1}A(I + B)^{-1}.$$

(3) Prove that $dC(B)$ is injective for every skew-symmetric matrix B.

Problem 39.5. Prove that

$$\begin{aligned}&d^m C_B(H_1, \ldots, H_m)\\ &\quad = 2(-1)^m \sum_{\pi \in \mathfrak{S}_m} (I + B)^{-1} H_{\pi(1)} (I + B)^{-1} H_{\pi(2)} (I + B)^{-1} \cdots (I + B)^{-1} H_{\pi(m)} (I + B)^{-1}.\end{aligned}$$

Problem 39.6. Consider the function g defined for all $A \in \mathbf{GL}(n, \mathbb{R})$, that is, all $n \times n$ real invertible matrices, given by

$$g(A) = \det(A).$$

(1) Prove that

$$dg_A(X) = \det(A)\mathrm{tr}(A^{-1}X),$$

for all $n \times n$ real matrices X.

(2) Consider the function f defined for all $A \in \mathbf{GL}^+(n, \mathbb{R})$, that is, $n \times n$ real invertible matrices of positive determinants, given by

$$f(A) = \log g(A) = \log \det(A).$$

Prove that

$$\begin{aligned} df_A(X) &= \mathrm{tr}(A^{-1}X)\\ \mathrm{D}^2 f(A)(X_1, X_2) &= -\mathrm{tr}(A^{-1}X_1A^{-1}X_2),\end{aligned}$$

for all $n \times n$ real matrices X, X_1, X_2.

(3) Prove that

$$\mathrm{D}^m f(A)(X_1, \ldots, X_m) = (-1)^{m-1} \sum_{\sigma \in \mathfrak{S}_{m-1}} \mathrm{tr}(A^{-1}X_1A^{-1}X_{\sigma(1)+1}A^{-1}X_{\sigma(2)+1} \cdots A^{-1}X_{\sigma(m-1)+1})$$

for any $m \geq 1$, where $X_1, \ldots X_m$ are any $n \times n$ real matrices.