Let H_0 be the subspace of the vector space $\mathbb{C}^X$ of functions from X to $\mathbb{C}$ spanned by the family of functions $(\kappa_x)_{\in X}$, and let $\varphi\colon X \to H_0$ be the map given by $\varphi(x) = \kappa_x$. There is a Hermitian inner product $\langle -, - \rangle$ on H_0 such that

$$\kappa(x, y) = \langle \varphi(x), \varphi(y) \rangle, \qquad \textit{for all } x, y \in X.$$

The completion H of H_0 is a Hilbert space, and the map $\eta\colon H \to \mathbb{C}^X$ given by

$$\eta(f)(x) = \langle f, \kappa_x \rangle, \qquad x \in X,$$

is linear and injective, so H can be identified with a subspace of $\mathbb{C}^X$. We also have

$$\kappa(x, y) = \langle \varphi(x), \varphi(y) \rangle, \qquad \textit{for all } x, y \in X.$$

For all $f \in H_0$ and all $x \in X$,

$$\langle f, \kappa_x \rangle = f(x), \tag{$*$}$$

a property known as the **reproducing property***.*

Proof.

Step 1. Define a candidate inner product.

For any two linear combinations $f = \sum_{j=1}^{p} \alpha_j \kappa_{x_j}$ and $g = \sum_{k=1}^{q} \beta_k \kappa_{y_k}$ in H_0, with $x_j, y_k \in X$ and $\alpha_j, \beta_k \in \mathbb{C}$, define $\langle f, g \rangle$ by

$$\langle f, g \rangle = \sum_{j=1}^{p} \sum_{k=1}^{q} \alpha_j \overline{\beta_k} \kappa(x_j, y_k). \tag{$\dagger$}$$

At first glance, the above expression appears to depend on the expression of f and g as linear combinations, but since $\kappa(x_j, y_k) = \overline{\kappa(y_k, x_j)}$, observe that

$$\sum_{k=1}^{q} \overline{\beta_k} f(y_k) = \sum_{j=1}^{p} \sum_{k=1}^{q} \alpha_j \overline{\beta_k} \kappa(x_j, y_k) = \sum_{j=1}^{p} \alpha_j \overline{g(x_j)}, \tag{$*$}$$

and since the first and the third term are equal for all linear combinations representing f and g, we conclude that ($\dagger$) depends only on f and g and not on their representation as a linear combination.

Step 2. Prove that the inner product defined by ($\dagger$) is Hermitian semidefinite.

Obviously ($\dagger$) defines a Hermitian sequilinear form. For every $f \in H_0$, we have

$$\langle f, f \rangle = \sum_{j,k=1}^{p} \alpha_j \overline{\alpha_k} \kappa(x_j, x_k) \geq 0,$$

since κ is a positive definite kernel.