Since
$$\dim(\mathrm{Im}\, f) = n - \dim(\mathrm{Ker}\, f)$$
and
$$\dim(\mathrm{Im}\, f^*) = n - \dim((\mathrm{Im}\, f^*)^{\perp}),$$
from
$$\mathrm{Ker}\, f = (\mathrm{Im}\, f^*)^{\perp}$$
we also have
$$\dim(\mathrm{Ker}\, f) = \dim((\mathrm{Im}\, f^*)^{\perp}),$$
from which we obtain
$$\dim(\mathrm{Im}\, f) = \dim(\mathrm{Im}\, f^*).$$
Since
$$\dim(\mathrm{Ker}\,(f^* \circ f)) + \dim(\mathrm{Im}\,(f^* \circ f)) = \dim(E),$$
$\mathrm{Ker}\,(f^* \circ f) = \mathrm{Ker}\, f$ and $\mathrm{Ker}\, f = (\mathrm{Im}\, f^*)^{\perp}$, we get
$$\dim((\mathrm{Im}\, f^*)^{\perp}) + \dim(\mathrm{Im}\,(f^* \circ f)) = \dim(E).$$
Since
$$\dim((\mathrm{Im}\, f^*)^{\perp}) + \dim(\mathrm{Im}\, f^*) = \dim(E),$$
we deduce that
$$\dim(\mathrm{Im}\, f^*) = \dim(\mathrm{Im}\,(f^* \circ f)).$$
A similar proof shows that
$$\dim(\mathrm{Im}\, f) = \dim(\mathrm{Im}\,(f \circ f^*)).$$
Consequently, f, f^*, $f^* \circ f$, and $f \circ f^*$ have the same rank. $\square$

22.2 Singular Value Decomposition for Square Matrices

We will now prove that every square matrix has an SVD. Stronger results can be obtained if we first consider the polar form and then derive the SVD from it (there are uniqueness properties of the polar decomposition). For our purposes, uniqueness results are not as important so we content ourselves with existence results, whose proofs are simpler. Readers interested in a more general treatment are referred to Gallier [72].

The early history of the singular value decomposition is described in a fascinating paper by Stewart [165]. The SVD is due to Beltrami and Camille Jordan independently (1873, 1874). Gauss is the grandfather of all this, for his work on least squares (1809, 1823) (but Legendre also published a paper on least squares!). Then come Sylvester, Schmidt, and