Proposition 34.1. *Let $f\colon E^n \to F$ be a multilinear map. If f is alternating, then the following properties hold:*

(1) For all i, with $1 \leq i \leq n-1$,

$$f(\ldots, u_i, u_{i+1}, \ldots) = -f(\ldots, u_{i+1}, u_i, \ldots).$$

(2) For every permutation $\sigma \in \mathfrak{S}_n$,

$$f(u_{\sigma(1)}, \ldots, u_{\sigma(n)}) = \mathrm{sgn}(\sigma) f(u_1, \ldots, u_n).$$

(3) For all i, j, with $1 \leq i < j \leq n$,

$$f(\ldots, u_i, \ldots u_j, \ldots) = 0 \quad \textit{whenever } u_i = u_j.$$

Moreover, if our field K has characteristic different from 2, then every skew-symmetric multilinear map is alternating.

Proof. (1) By multilinearity applied twice, we have

$$\begin{aligned} f(\ldots, u_i + u_{i+1}, u_i + u_{i+1}, \ldots) = f(\ldots, u_i, u_i, \ldots) + f(\ldots, u_i, u_{i+1}, \ldots) \\ + f(\ldots, u_{i+1}, u_i, \ldots) + f(\ldots, u_{i+1}, u_{i+1}, \ldots). \end{aligned}$$

Since f is alternating, we get

$$0 = f(\ldots, u_i, u_{i+1}, \ldots) + f(\ldots, u_{i+1}, u_i, \ldots);$$

that is, $f(\ldots, u_i, u_{i+1}, \ldots) = -f(\ldots, u_{i+1}, u_i, \ldots)$.

(2) Clearly, the symmetric group, $\mathfrak{S}_n$, acts on $\mathrm{Alt}^n(E; F)$ on the left, *via*

$$\sigma \cdot f(u_1, \ldots, u_n) \;= f(u_{\sigma(1)}, \ldots, u_{\sigma(n)}).$$

Consequently, as $\mathfrak{S}_n$ is generated by the transpositions (permutations that swap exactly two elements), since for a transposition, (2) is simply (1), we deduce (2) by induction on the number of transpositions in σ.

(3) There is a permutation σ that sends u_i and u_j respectively to u_1 and u_2. By hypothesis $u_i = u_j$, so we have $u_{\sigma(1)} = u_{\sigma(2)}$, and as f is alternating we have

$$f(u_{\sigma(1)}, \ldots, u_{\sigma(n)}) = 0.$$

However, by (2),

$$f(u_1, \ldots, u_n) = \mathrm{sgn}(\sigma) f(u_{\sigma(1)}, \ldots, u_{\sigma(n)}) = 0.$$

Now when f is skew-symmetric, if σ is the transposition swapping u_i and $u_{i+1} = u_i$, as $\mathrm{sgn}(\sigma) = -1$, we get

$$f(\ldots, u_i, u_i, \ldots) = -f(\ldots, u_i, u_i, \ldots),$$