else if $-1 \leq b < 0$, then

$$g(x_1, x_2) = -\sqrt{1 - x_1^2 - x_2^2}.$$

Assuming $0 < b \leq 1$, We have

$$\frac{\partial f}{\partial x}(x, g(x)) = (2x_1 \;\; 2x_2),$$

and

$$\left(\frac{\partial f}{\partial y}(x, g(x))\right)^{-1} = \frac{1}{2\sqrt{1 - x_1^2 - x_2^2}},$$

so according to the theorem,

$$dg_x = -\frac{1}{\sqrt{1 - x_1^2 - x_2^2}}(x_1 \;\; x_2),$$

which matches the derivative of g computed directly.

Observe that the functions $(x_1, x_2) \mapsto \sqrt{1 - x_1^2 - x_2^2}$ and $(x_1, x_2) \mapsto -\sqrt{1 - x_1^2 - x_2^2}$ are two differentiable parametrizations of the sphere, but the union of their ranges does not cover the entire sphere. Since $b \neq 0$, none of the points on the unit circle in the (x_1, x_2)-plane are covered. Our function f views b as lying on the x_3-axis. In order to cover the entire sphere using this method, we need four more maps, which correspond to b lying on the x_1-axis or on the x_2 axis. Then we get the additional (implicit) maps $(x_2, x_3) \mapsto \pm\sqrt{1 - x_2^2 - x_3^2}$ and $(x_1, x_3) \mapsto \pm\sqrt{1 - x_1^2 - x_3^2}$.

The implicit function theorem plays an important role in the calculus of variations.

We now consider another very important notion, that of a (local) diffeomorphism.

Definition 39.8. Given two topological spaces E and F, and an open subset A of E, we say that a function $f \colon A \to F$ is a *local homeomorphism from A to F* if for every $a \in A$, there is an open set $U \subseteq A$ containing a and an open set V containing $f(a)$ such that f is a homeomorphism from U to $V = f(U)$. If B is an open subset of F, we say that $f \colon A \to F$ is a *(global) homeomorphism from A to B* if f is a homeomorphism from A to $B = f(A)$. If E and F are normed affine spaces, we say that $f \colon A \to F$ is a *local diffeomorphism from A to F* if for every $a \in A$, there is an open set $U \subseteq A$ containing a and an open set V containing $f(a)$ such that f is a bijection from U to V, f is a C^1-function on U, and f^{-1} is a C^1-function on $V = f(U)$. We say that $f \colon A \to F$ is a *(global) diffeomorphism from A to B* if f is a homeomorphism from A to $B = f(A)$, f is a C^1-function on A, and f^{-1} is a C^1-function on B.

Note that a local diffeomorphism is a local homeomorphism. Also, as a consequence of Proposition 39.8, if f is a diffeomorphism on A, then $\mathrm{D}f(a)$ is a linear isomorphism for every $a \in A$. The following theorem can be shown. In fact, there is a fairly simple proof using Theorem 39.14.