or equivalently

$$\|b - c\| = \inf_{v \in C} \|b - v\| > 0$$
$$\langle v - c, c - b \rangle \geq 0 \quad \text{for all } v \in C.$$

As a consequence, since $b \notin C$ and $c \in C$, we have $c - b \neq 0$, so

$$\langle v, c - b \rangle \geq \langle c, c - b \rangle > \langle b, c - b \rangle$$

because $\langle c, c - b \rangle - \langle b, c - b \rangle = \langle c - b, c - b \rangle > 0$, and if we pick $u = c - b$ and any α such that

$$\langle c, c - b \rangle > \alpha > \langle b, c - b \rangle,$$

the claim is satisfied.

We now prove the Farkas–Minkowski lemma. Assume that $b \notin C$. Since C is nonempty, convex, and closed, by the claim there is some $u \in V$ and some $\alpha \in \mathbb{R}$ such that

$$\begin{aligned} &\langle v, u \rangle > \alpha \quad \text{for every } v \in C \\ &\langle b, u \rangle < \alpha. \end{aligned}$$

But C is a polyhedral cone containing 0, so we must have $\alpha < 0$. Then for every $v \in C$, since C a polyhedral cone if $v \in C$ then $\lambda v \in C$ for all $\lambda > 0$, so by the above

$$\langle v, u \rangle > \frac{\alpha}{\lambda} \quad \text{for every } \lambda > 0,$$

which implies that

$$\langle v, u \rangle \geq 0.$$

Since $a_i \in C$ for $i = 1, \ldots, m$, we proved that

$$\langle a_i, u \rangle \geq 0 \quad i = 1, \ldots, m \quad \text{and} \quad \langle b, u \rangle < \alpha < 0,$$

which proves Farkas lemma. ☐

Remark: Observe that the claim established during the proof of Theorem 48.12 shows that the affine hyperplane $H_{u,\alpha}$ of equation $\langle v, u \rangle = \alpha$ for all $v \in V$ separates strictly C and $\{b\}$.

48.4 Summary

The main concepts and results of this chapter are listed below:

- Hilbert space.
- Projection lemma.