and since $\varphi(u, v) = 1$, we get

$$\lambda = -\overline{\lambda} \quad \text{for all } \lambda \in K.$$

For $\lambda = 1$, we get $1 = -1$, which means that K has characterictic 2. But then

$$\lambda = -\overline{\lambda} = \overline{\lambda} \quad \text{for all } \lambda \in K,$$

so the automorphism $\lambda \mapsto \overline{\lambda}$ is the identity. $\square$

The definition of the linear maps l_φ and r_φ requires a small twist due to the automorphism $\lambda \mapsto \overline{\lambda}$.

Definition 29.9. Given a vector space E over a field K with an involutive automorphism $\lambda \mapsto \overline{\lambda}$, we define the K-vector space $\overline{E}$ as E with its abelian group structure, but with scalar multiplication given by

$$(\lambda, u) \mapsto \overline{\lambda} u.$$

Given two K-vector spaces E and F, a *semilinear map* $f\colon E \to F$ is a function, such that for all $u, v \in E$, for all $\lambda \in K$, we have

$$\begin{aligned} f(u + v) &= f(u) + f(v) \\ f(\lambda u) &= \overline{\lambda} f(u). \end{aligned}$$

Because $\overline{\overline{\lambda}} = \lambda$, observe that a function $f\colon E \to F$ is semilinear iff it is a linear map $f\colon \overline{E} \to F$. The K-vector spaces E and $\overline{E}$ are isomorphic, since any basis $(e_i)_{i \in I}$ of E is also a basis of $\overline{E}$.

The maps l_φ and r_φ are defined as follows:

For every $u \in E$, let $l_\varphi(u)$ be the linear form in F^* defined so that

$$l_\varphi(u)(y) = \overline{\varphi(u, y)} \quad \text{for all } y \in F,$$

and for every $v \in F$, let $r_\varphi(v)$ be the linear form in E^* defined so that

$$r_\varphi(v)(x) = \varphi(x, v) \quad \text{for all } x \in E.$$

The reader should check that because we used $\overline{\varphi(u, y)}$ in the definition of $l_\varphi(u)(y)$, the function $l_\varphi(u)$ is indeed a linear form in F^*. It is also easy to check that l_φ is a linear map $l_\varphi\colon \overline{E} \to F^*$, and that r_φ is a linear map $r_\varphi\colon \overline{F} \to E^*$ (equivalently, $l_\varphi\colon E \to F^*$ and $r_\varphi\colon F \to E^*$ are semilinear).

The notion of a nondegenerate sesquilinear form is identical to the notion for bilinear forms. For the convenience of the reader, we repeat the definition.

Definition 29.10. A sesquilinear map $\varphi\colon E \times F \to K$ is said to be *nondegenerate* iff the following conditions hold: