We define the dimension $\dim(\mathcal{A})$ of $\mathcal{A}$ as the dimension $\dim(U)$ of $U$.

(1) If $\mathcal{A} = a + U$, why is $a \in \mathcal{A}$?

What are affine subspaces of dimension 0? What are affine subspaces of dimension 1 (begin with $\mathbb{R}^2$)? What are affine subspaces of dimension 2 (begin with $\mathbb{R}^3$)?

Prove that any nonempty affine subspace is closed under affine combinations.

(2) Prove that if $\mathcal{A} = a + U$ is any nonempty affine subspace, then $\mathcal{A} = b + U$ for any $b \in \mathcal{A}$.

(3) Let $\mathcal{A}$ be any nonempty subset of $\mathbb{R}^n$ closed under affine combinations. For any $a \in \mathcal{A}$, prove that

$$U_a = \{x - a \in \mathbb{R}^n \mid x \in \mathcal{A}\}$$

is a (linear) subspace of $\mathbb{R}^n$ such that

$$\mathcal{A} = a + U_a.$$

Prove that $U_a$ does not depend on the choice of $a \in \mathcal{A}$; that is, $U_a = U_b$ for all $a, b \in \mathcal{A}$. In fact, prove that

$$U_a = U = \{y - x \in \mathbb{R}^n \mid x, y \in \mathcal{A}\}, \quad \text{for all } a \in \mathcal{A},$$

and so

$$\mathcal{A} = a + U, \quad \text{for any } a \in \mathcal{A}.$$

**Remark:** The subspace $U$ is called the *direction* of $\mathcal{A}$.

(4) Two nonempty affine subspaces $\mathcal{A}$ and $\mathcal{B}$ are said to be *parallel* iff they have the same direction. Prove that that if $\mathcal{A} \neq \mathcal{B}$ and $\mathcal{A}$ and $\mathcal{B}$ are parallel, then $\mathcal{A} \cap \mathcal{B} = \emptyset$.

**Remark:** The above shows that affine subspaces behave quite differently from linear subspaces.

**Problem 6.11.** (Affine frames and affine maps) For any vector $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$, let $\widehat{v} \in \mathbb{R}^{n+1}$ be the vector $\widehat{v} = (v_1, \ldots, v_n, 1)$. Equivalently, $\widehat{v} = (\widehat{v}_1, \ldots, \widehat{v}_{n+1}) \in \mathbb{R}^{n+1}$ is the vector defined by

$$\widehat{v}_i = \begin{cases} v_i & \text{if } 1 \leq i \leq n, \\ 1 & \text{if } i = n+1. \end{cases}$$

(1) For any $m+1$ vectors $(u_0, u_1, \ldots, u_m)$ with $u_i \in \mathbb{R}^n$ and $m \leq n$, prove that if the $m$ vectors $(u_1 - u_0, \ldots, u_m - u_0)$ are linearly independent, then the $m+1$ vectors $(\widehat{u}_0, \ldots, \widehat{u}_m)$ are linearly independent.

(2) Prove that if the $m+1$ vectors $(\widehat{u}_0, \ldots, \widehat{u}_m)$ are linearly independent, then for any choice of $i$, with $0 \leq i \leq m$, the $m$ vectors $u_j - u_i$ for $j \in \{0, \ldots, m\}$ with $j - i \neq 0$ are linearly independent.