running `Hessenberg1` we find

$$H = \begin{pmatrix} 1.0000 & -5.3852 & 0 & 0 \\ -5.3852 & 15.2069 & -1.6893 & -0.0000 \\ -0.0000 & -1.6893 & -0.2069 & -0.0000 \\ 0 & -0.0000 & 0.0000 & 0.0000 \end{pmatrix}$$

$$Q = \begin{pmatrix} 1.0000 & 0 & 0 & 0 \\ 0 & -0.3714 & -0.5571 & -0.7428 \\ 0 & 0.8339 & 0.1516 & -0.5307 \\ 0 & 0.4082 & -0.8165 & 0.4082 \end{pmatrix}.$$

An important property of (upper) Hessenberg matrices is that if some subdiagonal entry $H_{p+1p} = 0$, then H is of the form

$$H = \begin{pmatrix} H_{11} & H_{12} \\ 0 & H_{22} \end{pmatrix},$$

where both H_{11} and H_{22} are upper Hessenberg matrices (with H_{11} a $p \times p$ matrix and H_{22} a $(n-p) \times (n-p)$ matrix), and the eigenvalues of H are the eigenvalues of H_{11} and H_{22}. For example, in the matrix

$$H = \begin{pmatrix} * & * & * & * & * \\ h_{21} & * & * & * & * \\ 0 & h_{32} & * & * & * \\ 0 & 0 & h_{43} & * & * \\ 0 & 0 & 0 & h_{54} & * \end{pmatrix},$$

if $h_{43} = 0$, then we have the block matrix

$$H = \begin{pmatrix} * & * & * & * & * \\ h_{21} & * & * & * & * \\ 0 & h_{32} & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & h_{54} & * \end{pmatrix}.$$

Then the list of eigenvalues of H is the concatenation of the list of eigenvalues of H_{11} and the list of the eigenvalues of H_{22}. This is easily seen by induction on the dimension of the block H_{11}.

More generally, every upper Hessenberg matrix can be written in such a way that it has diagonal blocks that are Hessenberg blocks whose subdiagonal is not zero.

Definition 18.2. An upper Hessenberg $n \times n$ matrix H is *unreduced* if $h_{i+1i} \neq 0$ for $i = 1, \ldots, n-1$. A Hessenberg matrix which is not unreduced is said to be *reduced*.