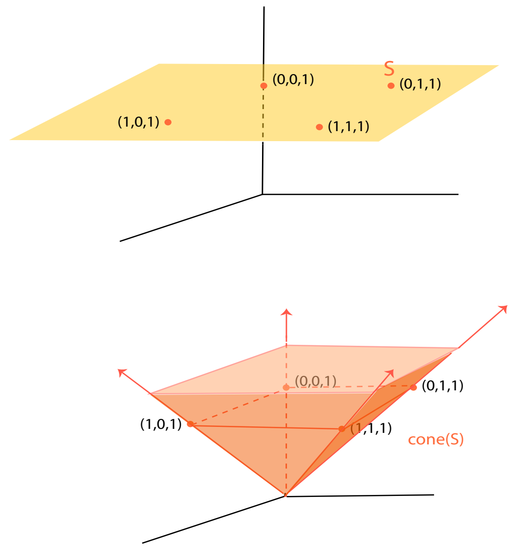

Figure 44.3: Let $S = \{(0,0,1),(1,0,1),(1,1,1),(0,1,1)\}$. The polyhedral cone, cone(S), is the solid "pyramid" with apex at the origin and square cross sections.

Note that if some nonzero vector u belongs to a cone C, then $\lambda u \in C$ for all $\lambda \geq 0$, that is, the *ray* $\{\lambda u \mid \lambda \geq 0\}$ belongs to C.

Remark: The cones (and polyhedral cones) of Definition 44.9 are *always convex*. For this reason, we use the simpler terminology cone instead of convex cone. However, there are more general kinds of cones (see Definition 50.1) that are not convex (for example, a union of polyhedral cones or the linear cone generated by the curve in Figure 44.4), and if we were dealing with those we would refer to the cones of Definition 44.9 as convex cones.

Definition 44.10. An $\mathcal{H}$-*polyhedron*, for short a *polyhedron*, is any subset $\mathcal{P} = \bigcap_{i=1}^{s} C_i$ of $\mathbb{R}^n$ defined as the intersection of a finite number s of closed half-spaces C_i. An example of an $\mathcal{H}$-polyhedron is shown in Figure 44.6. An $\mathcal{H}$-*polytope* is a bounded $\mathcal{H}$-polyhedron, which means that there is a closed ball $B_r(x)$ of center x and radius $r > 0$ such that $\mathcal{P} \subseteq B_r(x)$. An example of a $\mathcal{H}$-polytope is shown in Figure 44.5.

By convention, we agree that $\mathbb{R}^n$ itself is an $\mathcal{H}$-polyhedron.

Remark: The $\mathcal{H}$-polyhedra of Definition 44.10 are always convex. For this reason, as in the case of cones we use the simpler terminology $\mathcal{H}$-polyhedron instead of convex $\mathcal{H}$-polyhedron. In algebraic topology, there are more general polyhedra that are not convex.

It can be shown that an $\mathcal{H}$-polytope $\mathcal{P}$ is equal to the convex hull of finitely many points (the extreme points of $\mathcal{P}$). This is a nontrivial result whose proof takes a significant amount of work; see Gallier [73] and Ziegler [195].