33.7 Symmetric Tensor Powers

Our goal is to come up with a notion of tensor product that will allow us to treat symmetric multilinear maps as linear maps. Note that we have to restrict ourselves to a *single* vector space E, rather then n vector spaces $E_1, \ldots, E_n$, so that symmetry makes sense.

Definition 33.15. A multilinear map $f \colon E^n \to F$ is *symmetric* iff

$$f(u_{\sigma(1)}, \ldots, u_{\sigma(n)}) = f(u_1, \ldots, u_n),$$

for all $u_i \in E$ and all permutations, $\sigma \colon \{1, \ldots, n\} \to \{1, \ldots, n\}$. The group of permutations on $\{1, \ldots, n\}$ (the *symmetric group*) is denoted $\mathfrak{S}_n$. The vector space of all symmetric multilinear maps $f \colon E^n \to F$ is denoted by $\mathrm{Sym}^n(E; F)$ or $\mathrm{Hom}_{\mathrm{symlin}}(E^n, F)$. Note that $\mathrm{Sym}^1(E; F) = \mathrm{Hom}(E, F)$.

We could proceed directly as in Theorem 33.6 and construct symmetric tensor products from scratch. However, since we already have the notion of a tensor product, there is a more economical method. First we define symmetric tensor powers.

Definition 33.16. An *n-th symmetric tensor power* of a vector space E, where $n \geq 1$, is a vector space S together with a symmetric multilinear map $\varphi \colon E^n \to S$ such that, for every vector space F and for every symmetric multilinear map $f \colon E^n \to F$, there is a unique linear map $f_\odot \colon S \to F$, with

$$f(u_1, \ldots, u_n) = f_\odot(\varphi(u_1, \ldots, u_n)),$$

for all $u_1, \ldots, u_n \in E$, or for short

$$f = f_\odot \circ \varphi.$$

Equivalently, there is a unique linear map $f_\odot$ such that the following diagram commutes.

$$\begin{tikzcd} E^n \arrow[r, "\varphi"] \arrow[dr, "f"'] & S \arrow[d, "f_\odot"] \\ & F \end{tikzcd}$$

The above property is called the *universal mapping property* of the symmetric tensor power (S, φ).

We next show that any two symmetric n-th tensor powers (S_1, φ_1) and (S_2, φ_2) for E are isomorphic.

Proposition 33.23. *Given any two symmetric n-th tensor powers (S_1, φ_1) and (S_2, φ_2) for E, there is an isomorphism $h \colon S_1 \to S_2$ such that*

$$\varphi_2 = h \circ \varphi_1.$$