37.7 Sequential Compactness

For a general topological Hausdorff space E, the definition of compactness relies on the existence of finite cover. However, when E has a countable basis or is a metric space, we may define the notion of compactness in terms of sequences. To understand how this is done, we need to first define accumulation points.

Definition 37.35. Given a topological Hausdorff space, E, given any sequence, (x_n), of points in E, a point, $l \in E$, is an *accumulation point (or cluster point)* of the sequence (x_n) if every open set, U, containing l contains x_n for infinitely many n. See Figure 37.38.

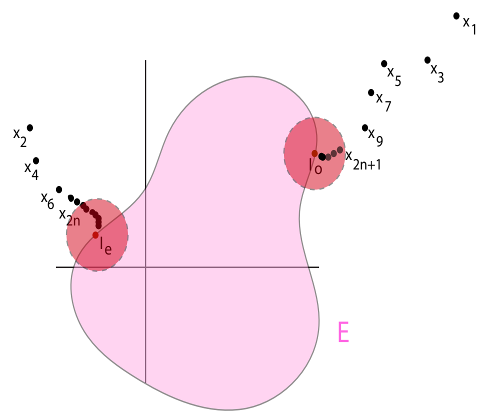

Figure 37.38: The space E is the closed, bounded pink subset of $\mathbb{R}^2$. The sequence (x_n) has two accumulation points, one for the subsequence (x_{2n+1}) and one for (x_{2n}).

Clearly, if l is a limit of the sequence, (x_n), then it is an accumulation point, since every open set, U, containing a contains all x_n except for finitely many n.

For second-countable spaces we are able to give another characterization of accumulation points.

Proposition 37.42. *Given a second-countable topological Hausdorff space, E, a point, l, is an accumulation point of the sequence, (x_n), iff l is the limit of some subsequence, (x_{n_k}), of (x_n).*

Proof. Clearly, if l is the limit of some subsequence (x_{n_k}) of (x_n), it is an accumulation point of (x_n).

Conversely, let $(U_k)_{k\geq 0}$ be the sequence of open sets containing l, where each U_k belongs to a countable basis of E, and let $V_k = U_1 \cap \cdots \cap U_k$. For every $k \geq 1$, we can find some $n_k > n_{k-1}$ such that $x_{n_k} \in V_k$, since l is an accumulation point of (x_n). Now, since every open set containing l contains some U_{k_0} and since $x_{n_k} \in U_{k_0}$ for all $k \geq 0$, the sequence (x_{n_k}) has limit l. $\square$