*as in the following diagram:*

$$\begin{tikzcd} I \arrow[r, "\iota"] \arrow[dr, "f"'] & K^{(I)} \arrow[d, "\overline{f}"] \\ & F \end{tikzcd}$$

*Proof.* If such a linear map $\overline{f}\colon K^{(I)} \to F$ exists, since $f = \overline{f} \circ \iota$, we must have

$$f(i) = \overline{f}(\iota(i)) = \overline{f}(e_i),$$

for every $i \in I$. However, the family $(e_i)_{i \in I}$ is a basis of $K^{(I)}$, and $(f(i))_{i \in I}$ is a family of vectors in $F$, and by Proposition 3.18, there is a unique linear map $\overline{f}\colon K^{(I)} \to F$ such that $\overline{f}(e_i) = f(i)$ for every $i \in I$, which proves the existence and uniqueness of a linear map $\overline{f}$ such that $f = \overline{f} \circ \iota$. $\square$

The following simple proposition is also useful.

**Proposition 3.20.** *Given any two vector spaces $E$ and $F$, with $F$ nontrivial, given any family $(u_i)_{i \in I}$ of vectors in $E$, the following properties hold:*

*(1) The family $(u_i)_{i \in I}$ generates $E$ iff for every family of vectors $(v_i)_{i \in I}$ in $F$, there is at most one linear map $f\colon E \to F$ such that $f(u_i) = v_i$ for all $i \in I$.*

*(2) The family $(u_i)_{i \in I}$ is linearly independent iff for every family of vectors $(v_i)_{i \in I}$ in $F$, there is some linear map $f\colon E \to F$ such that $f(u_i) = v_i$ for all $i \in I$.*

*Proof.* (1) If there is any linear map $f\colon E \to F$ such that $f(u_i) = v_i$ for all $i \in I$, since $(u_i)_{i \in I}$ generates $E$, every vector $x \in E$ can be written as some linear combination

$$x = \sum_{i \in I} x_i u_i,$$

and by linearity, we must have

$$f(x) = \sum_{i \in I} x_i f(u_i) = \sum_{i \in I} x_i v_i.$$

This shows that $f$ is unique if it exists. Conversely, assume that $(u_i)_{i \in I}$ does not generate $E$. Since $F$ is nontrivial, there is some some vector $y \in F$ such that $y \neq 0$. Since $(u_i)_{i \in I}$ does not generate $E$, there is some vector $w \in E$ that is not in the subspace generated by $(u_i)_{i \in I}$. By Theorem 3.11, there is a linearly independent subfamily $(u_i)_{i \in I_0}$ of $(u_i)_{i \in I}$ generating the same subspace. Since by hypothesis, $w \in E$ is not in the subspace generated by $(u_i)_{i \in I_0}$, by Lemma 3.6 and by Theorem 3.11 again, there is a basis $(e_j)_{j \in I_0 \cup J}$ of $E$, such that $e_i = u_i$ for all $i \in I_0$, and $w = e_{j_0}$ for some $j_0 \in J$. Letting $(v_i)_{i \in I}$ be the family in $F$ such that $v_i = 0$ for all $i \in I$, defining $f\colon E \to F$ to be the constant linear map with value 0, we have a linear map such that $f(u_i) = 0$ for all $i \in I$. By Proposition 3.18, there is a unique linear