$a_i y_i$ can be expressed as a linear combination of $(u_1, \ldots, u_m)$. If we let $a = a_1 \ldots a_n$, then $a_1 \ldots a_n y_i \in Au_1 \oplus \cdots \oplus Au_m$ for $i = 1, \ldots, n$, which shows that

$$aM \subseteq Au_1 \oplus \cdots \oplus Au_m.$$

Now, $A$ is an integral domain, and since $a_i \neq 0$ for $i = 1, \ldots, n$, we have $a = a_1 \ldots a_n \neq 0$, and because $M$ is torsion-free, the map $x \mapsto ax$ is injective. It follows that $M$ is isomorphic to a submodule of the free module $Au_1 \oplus \cdots \oplus Au_m$. By Proposition 35.5, this submodule if free, and thus, $M$ is free. ☐

Although we will obtain this result as a corollary of the structure theorem for finitely generated modules over a PID, we are in the position to give a quick proof of the following theorem.

**Theorem 35.7.** *Let $M$ be a finitely generated module over a PID. Then $M/M_{\mathrm{tor}}$ is free, and there exit a free submodule $F$ of $M$ such that $M$ is the direct sum*

$$M = M_{\mathrm{tor}} \oplus F.$$

*The dimension of $F$ is uniquely determined.*

*Proof.* By Proposition 35.4 $M/M_{\mathrm{tor}}$ is torsion-free, and since $M$ is finitely generated, it is also finitely generated. By Proposition 35.6, $M/M_{\mathrm{tor}}$ is free. We have the quotient linear map $\pi\colon M \to M/M_{\mathrm{tor}}$, which is surjective, and $M/M_{\mathrm{tor}}$ is free, so by Proposition 35.2, there is a free module $F$ isomorphic to $M/M_{\mathrm{tor}}$ such that

$$M = \mathrm{Ker}\,(\pi) \oplus F = M_{\mathrm{tor}} \oplus F.$$

Since $F$ is isomorphic to $M/M_{\mathrm{tor}}$, the dimension of $F$ is uniquely determined. ☐

Theorem 35.7 reduces the study of finitely generated module over a PID to the study of finitely generated torsion modules. This is the path followed by Lang [109] (Chapter III, section 7).

## 35.2 Finite Presentations of Modules

Since modules are generally not free, it is natural to look for techniques for dealing with nonfree modules. The hint is that if $M$ is an $A$-module and if $(u_i)_{i\in I}$ is any set of generators for $M$, then we know that there is a surjective homomorphism $\varphi\colon A^{(I)} \to M$ from the free module $A^{(I)}$ generated by $I$ onto $M$. Furthermore $M$ is isomorphic to $A^{(I)}/\mathrm{Ker}\,(\varphi)$. Then, we can pick a set of generators $(v_j)_{j\in J}$ for $\mathrm{Ker}\,(\varphi)$, and again there is a surjective map $\psi\colon A^{(J)} \to \mathrm{Ker}\,(\varphi)$ from the free module $A^{(J)}$ generated by $J$ onto $\mathrm{Ker}\,(\varphi)$. The map $\psi$ can be viewed a linear map from $A^{(J)}$ to $A^{(I)}$, we have

$$\mathrm{Im}(\psi) = \mathrm{Ker}\,(\varphi),$$