Proof. For any vectors $u, v \in E$, we define the function $T\colon \mathbb{R} \to \mathbb{R}$ such that

$$T(\lambda) = \Phi(u + \lambda v),$$

for all $\lambda \in \mathbb{R}$. Using bilinearity and symmetry, we have

$$\begin{aligned}
\Phi(u + \lambda v) &= \varphi(u + \lambda v,\, u + \lambda v) \\
&= \varphi(u,\, u + \lambda v) + \lambda\varphi(v,\, u + \lambda v) \\
&= \varphi(u,\, u) + 2\lambda\varphi(u,\, v) + \lambda^2\varphi(v,\, v) \\
&= \Phi(u) + 2\lambda\varphi(u,\, v) + \lambda^2\Phi(v).
\end{aligned}$$

Since φ is positive definite, Φ is nonnegative, and thus $T(\lambda) \geq 0$ for all $\lambda \in \mathbb{R}$. If $\Phi(v) = 0$, then $v = 0$, and we also have $\varphi(u,\, v) = 0$. In this case, the Cauchy–Schwarz inequality is trivial, and $v = 0$ and u are linearly dependent.

Now assume $\Phi(v) > 0$. Since $T(\lambda) \geq 0$, the quadratic equation

$$\lambda^2\Phi(v) + 2\lambda\varphi(u,\, v) + \Phi(u) = 0$$

cannot have distinct real roots, which means that its discriminant

$$\Delta = 4(\varphi(u,\, v)^2 - \Phi(u)\Phi(v))$$

is null or negative, which is precisely the Cauchy–Schwarz inequality

$$\varphi(u, v)^2 \leq \Phi(u)\Phi(v).$$

Let us now consider the case where we have the equality

$$\varphi(u, v)^2 = \Phi(u)\Phi(v).$$

There are two cases. If $\Phi(v) = 0$, then $v = 0$ and u and v are linearly dependent. If $\Phi(v) \neq 0$, then the above quadratic equation has a double root λ_0, and we have $\Phi(u + \lambda_0 v) = 0$. Since φ is positive definite, $\Phi(u + \lambda_0 v) = 0$ implies that $u + \lambda_0 v = 0$, which shows that u and v are linearly dependent. Conversely, it is easy to check that we have equality when u and v are linearly dependent.

The Minkowski inequality

$$\sqrt{\Phi(u + v)} \leq \sqrt{\Phi(u)} + \sqrt{\Phi(v)}$$

is equivalent to

$$\Phi(u + v) \leq \Phi(u) + \Phi(v) + 2\sqrt{\Phi(u)\Phi(v)}.$$

However, we have shown that

$$2\varphi(u,\, v) = \Phi(u + v) - \Phi(u) - \Phi(v),$$