

Let $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$ be the matrices

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

then \mathbb{H} is the set of all matrices of the form

$$X = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}, \quad a, b, c, d \in \mathbb{R}.$$

Indeed, every matrix in \mathbb{H} is of the form

$$X = \begin{pmatrix} a + ib & c + id \\ -(c - id) & a - ib \end{pmatrix}, \quad a, b, c, d \in \mathbb{R}.$$

It is easy (but a bit tedious) to verify that the quaternions $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfy the famous identities discovered by Hamilton:

$$\begin{aligned} \mathbf{i}^2 &= \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -\mathbf{1}, \\ \mathbf{ij} &= -\mathbf{ji} = \mathbf{k}, \\ \mathbf{jk} &= -\mathbf{kj} = \mathbf{i}, \\ \mathbf{ki} &= -\mathbf{ik} = \mathbf{j}. \end{aligned}$$

Thus, the quaternions are a generalization of the complex numbers, but there are three square roots of $-\mathbf{1}$ and multiplication is not commutative.

Given any two quaternions $X = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ and $Y = a'\mathbf{1} + b'\mathbf{i} + c'\mathbf{j} + d'\mathbf{k}$, Hamilton's famous formula

$$\begin{aligned} XY &= (aa' - bb' - cc' - dd')\mathbf{1} + (ab' + ba' + cd' - dc')\mathbf{i} \\ &\quad + (ac' + ca' + db' - bd')\mathbf{j} + (ad' + da' + bc' - cb')\mathbf{k} \end{aligned}$$

looks mysterious, but it is simply the result of multiplying the two matrices

$$X = \begin{pmatrix} a + ib & c + id \\ -(c - id) & a - ib \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} a' + ib' & c' + id' \\ -(c' - id') & a' - ib' \end{pmatrix}.$$

It is worth noting that this formula was discovered independently by Olinde Rodrigues in 1840, a few years before Hamilton (Veblen and Young [184]). However, Rodrigues was working with a different formalism, homogeneous transformations, and he did not discover the quaternions.

If

$$X = \begin{pmatrix} a + ib & c + id \\ -(c - id) & a - ib \end{pmatrix}, \quad a, b, c, d \in \mathbb{R},$$

it is immediately verified that

$$XX^* = X^*X = (a^2 + b^2 + c^2 + d^2)\mathbf{1}.$$