*(d)* $\mathrm{Ker}\, f = (0)$.

*Proof.* Obviously, (a) implies (b).

If $f$ is surjective, then $\mathrm{Im}\, f = F$, and so $\dim(\mathrm{Im}\, f) = n$. By Theorem 6.16,

$$\dim(E) = \dim(\mathrm{Ker}\, f) + \dim(\mathrm{Im}\, f),$$

and since $\dim(E) = n$ and $\dim(\mathrm{Im}\, f) = n$, we get $\dim(\mathrm{Ker}\, f) = 0$, which means that $\mathrm{Ker}\, f = (0)$, and so $f$ is injective (see Proposition 3.17). This proves that (b) implies (c).

If $f$ is injective, then by Proposition 3.17, $\mathrm{Ker}\, f = (0)$, so (c) implies (d).

Finally, assume that $\mathrm{Ker}\, f = (0)$, so that $\dim(\mathrm{Ker}\, f) = 0$ and $f$ is injective (by Proposition 3.17). By Theorem 6.16,

$$\dim(E) = \dim(\mathrm{Ker}\, f) + \dim(\mathrm{Im}\, f),$$

and since $\dim(\mathrm{Ker}\, f) = 0$, we get

$$\dim(\mathrm{Im}\, f) = \dim(E) = \dim(F),$$

which proves that $f$ is also surjective, and thus bijective. This proves that (d) implies (a) and concludes the proof. ☐

One should be warned that Proposition 6.19 fails in infinite dimension.

Here are a few applications of Proposition 6.19. Let $A$ be an $n \times n$ matrix and assume that $A$ some right inverse $B$, which means that $B$ is an $n \times n$ matrix such that

$$AB = I.$$

The linear map associated with $A$ is surjective, since for every $u \in \mathbb{R}^n$, we have $A(Bu) = u$. By Proposition 6.19, this map is bijective so $B$ is actually the inverse of $A$; in particular $BA = I$.

Similarly, assume that $A$ has a left inverse $B$, so that

$$BA = I.$$

This time the linear map associated with $A$ is injective, because if $Au = 0$, then $BAu = B0 = 0$, and since $BA = I$ we get $u = 0$. Again, by Proposition 6.19, this map is bijective so $B$ is actually the inverse of $A$; in particular $AB = I$.

Now assume that the linear system $Ax = b$ has some solution for every $b$. Then the linear map associated with $A$ is surjective and by Proposition 6.19, $A$ is invertible.

Finally assume that the linear system $Ax = b$ has at most one solution for every $b$. Then the linear map associated with $A$ is injective and by Proposition 6.19, $A$ is invertible.

The following Proposition will also be useful.