Observe that $[A]$ viewed as a 3×3 scalar matrix is definitely different from

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

In practice, $S = \{1, \ldots, M\}$ and $T = \{1, \ldots, N\}$, so there are bijections $\alpha_i \colon \{1, \ldots, s_i\} \to S_i$ and $\beta_j \colon \{1, \ldots, t_j\} \to T_j$, $1 \leq i \leq m$, $1 \leq j \leq n$. Each $s_i \times t_j$ matrix A_{S_i,T_j} is considered as a submatrix of A, this is why the rows are indexed by S_i and the columns are indexed by T_j, but this matrix can also be viewed as an $s_i \times t_j$ matrix $A'_{ij} = \big((a'_{ij})_{st}\big)$ by itself, with the rows indexed by $\{1, \ldots, s_i\}$ and the columns indexed by $\{1, \ldots, t_j\}$, with

$$(a'_{ij})_{st} = a_{\alpha(s)\beta(t)}, \quad 1 \leq s \leq s_i,\ 1 \leq t \leq t_j.$$

Symbolic systems like `Matlab` have commands to construct such matrices. But be careful that to put a matrix such as A'_{ij} back into A at the correct row and column locations requires viewing this matrix as A_{S_i,T_j}. Symbolic systems like `Matlab` also have commands to assign row vectors and column vectors to specific rows or columns of a matrix. Technically, to be completely rigorous, the matrices A_{S_i,T_j} and A'_{ij} are different, even though they contain the same entries. The reason they are different is that in A_{S_i,T_j} the entries are indexed by the index sets S_i and T_j, but in A'_{ij} they are indexed by the index sets $\{1, \ldots, s_i\}$ and $\{1, \ldots, t_j\}$. This depends whether we view A_{S_i,T_j} as a submatrix of A or as a matrix on its own.

In most cases, the partitions $S = S_1 \cup \cdots \cup S_m$ and $T = T_1 \cup \cdots \cup T_n$ are chosen so that

$$\begin{aligned} S_i &= \{s \mid s_1 + \cdots + s_{i-1} + 1 \leq s \leq s_1 + \cdots + s_i\} \\ T_j &= \{t \mid t_1 + \cdots + t_{i-1} + 1 \leq t \leq t_1 + \cdots + t_j\}, \end{aligned}$$

with $s_i = |S_i| \geq 1$, $t_j = |T_j| \geq 1$, $1 \leq i \leq m, 1 \leq j \leq n$. For $i = 1$, we have $S_1 = \{1, \ldots, s_1\}$ and $T_1 = \{1, \ldots, t_1\}$. *This means that we partition into consecutive subsets of consecutive integers and we preserve the order of the bases*. In this case, $[A]$ can be viewed as A. But the results about block multiplication hold in the general case.

Finally we tackle block multiplication. But first we observe that the computation made in Section 4.2 can be immediately adapted to matrices indexed by arbitrary finite index sets I, J, K, not necessary of the form $\{1, \ldots, p\}$, $\{1, \ldots, n\}$, $\{1, \ldots, m\}$. We need this to deal with products of matrices occurring as blocks in other matrices, since such matrices are of the form A_{S_i,T_j}, *etc.*

We can prove immediately the following result generalizing Equation (4) proven in Section 4.2 (also see the fourth equation of Proposition 4.2).

Proposition 6.13. *Let I, J, K be any nonempty finite index sets. If the $I \times J$ matrix $A = (a_{ij})_{(i,j) \in I \times J}$ represents the linear map $g \colon F \to G$ with respect to the basis $(v_j)_{j \in J}$ of F and the basis $(w_i)_{i \in I}$ of G and if the $J \times K$ matrix $B = (b_{jk})_{(j,k) \in J \times K}$ represents the linear map $f \colon E \to F$ with respect to the basis $(u_k)_{k \in K}$ of E and the basis $(v_j)_{j \in J}$ of F, then the*