where $y \in (\mathbb{R}^*)^m$. Since $c \leq 0$, observe that $y = 0_m^\top$ is a feasible solution of the dual.

If a basic solution u of $(P2)$ is found such that $u \geq 0$, then $cu = yb$ for $y = c_K A_K^{-1}$, and we have found an optimal solution u for $(P2)$ and y for (D). The dual simplex method makes progress by attempting to make negative components of u zero and by decreasing the objective function of the dual program.

The dual simplex method starts with a basic solution (u, K) of $Ax = b$ which is not feasible but for which $y = c_K A_K^{-1}$ is dual feasible. In many cases the original linear program is specified by a set of inequalities $Ax \leq b$ with some $b_i < 0$, so by adding slack variables it is easy to find such basic solution u, and if in addition $c \leq 0$, then because the cost associated with slack variables is 0, we see that $y = 0$ is a feasible solution of the dual.

Given a basic solution (u, K) of $Ax = b$ (feasible or not), $y = c_K A_K^{-1}$ is dual feasible iff $c_K A_K^{-1} A \geq c$, and since $c_K A_K^{-1} A_K = c_K$, the inequality $c_K A_K^{-1} A \geq c$ is equivalent to $c_K A_K^{-1} A_N \geq c_N$, that is,

$$c_N - c_K A_K^{-1} A_N \leq 0, \tag{$*_1$}$$

where $N = \{1, \ldots, n\} - K$. Equation $(*_1)$ is equivalent to

$$c_j - c_K \gamma_K^j \leq 0 \quad \text{for all } j \in N, \tag{$*_2$}$$

where $\gamma_K^j = A_K^{-1} A^j$. Recall that the notation $\overline{c}_j$ is used to denote $c_j - c_K \gamma_K^j$, which is called the *reduced cost* of the variable x_j.

As in the simplex algorithm we need to decide which column A^k leaves the basis K and which column A^j enters the new basis K^+, in such a way that $y^+ = c_{K^+} A_{K^+}^{-1}$ is a feasible solution of (D), that is, $c_{N^+} - c_{K^+} A_{K^+}^{-1} A_{N^+} \leq 0$, where $N^+ = \{1, \ldots, n\} - K^+$. We use Proposition 46.2 to decide wich column k^- should leave the basis.

Suppose (u, K) is a solution of $Ax = b$ for which $y = c_K A_K^{-1}$ is dual feasible.

Case (A). If $u \geq 0$, then u is an optimal solution of $(P2)$.

Case (B). There is some $k \in K$ such that $u_k < 0$. In this case pick some $k^- \in K$ such that $u_{k^-} < 0$ (according to some pivot rule).

Case (B1). Suppose that $\gamma_{k^-}^j \geq 0$ for all $j \notin K$ (in fact, for all j, since $\gamma_{k^-}^j \in \{0, 1\}$ for all $j \in K$). If so, we we claim that $(P2)$ is not feasible.

Indeed, let v be some basic feasible solution. We have $v \geq 0$ and $Av = b$, that is,

$$\sum_{j=1}^{n} v_j A^j = b,$$

so by multiplying both sides by A_K^{-1} and using the fact that by definition $\gamma_K^j = A_K^{-1} A^j$, we obtain

$$\sum_{j=1}^{n} v_j \gamma_K^j = A_K^{-1} b = u_K.$$