associated with $h = g \circ f$ (with respect to the decomposition of E and G as direct sums) is given by

$$C = AB,$$

with

$$h_{ik} = \sum_{j=1}^{n} g_{ij} \circ f_{jk}, \quad 1 \leq i \leq m,\ 1 \leq k \leq p.$$

We will use Proposition 6.12 to justify the rule for the block multiplication of matrices. The difficulty is mostly notational. Again suppose that E and F are expressed as direct sums

$$E = \bigoplus_{j=1}^{n} E_j, \quad F = \bigoplus_{i=1}^{m} F_i,$$

and let $f \colon E \to F$ be a linear map. Furthermore, suppose that E has a finite basis $(u_t)_{t \in T}$, where T is the disjoint union $T = T_1 \cup \cdots \cup T_n$ of nonempty subsets T_j so that $(u_t)_{t \in T_j}$ is a basis of E_j, and similarly F has a finite basis $(v_s)_{s \in S}$, where S is the disjoint union $S = S_1 \cup \cdots \cup S_m$ of nonempty subsets S_i so that $(v_s)_{s \in S_i}$ is a basis of F_i. Let $M = |S|$, $N = |T|$, $s_i = |S_i|$, and let $t_j = |T_j|$. Since s_i is the number of elements in the basis $(v_s)_{s \in S_i}$ of F_i and $F = F_1 \oplus \cdots \oplus F_m$, we have $M = \dim(F) = s_1 + \cdots + s_m$. Similarly, since t_j is the number of elements in the basis $(u_t)_{t \in T_j}$ of E_j and $E = E_1 \oplus \cdots \oplus E_n$, we have $N = \dim(E) = t_1 + \cdots + t_n$.

Let $A = (a_{st})_{(s,t) \in S \times T}$ be the (ordinary) $M \times N$ matrix of scalars (in K) representing f with respect to the basis $(u_t)_{t \in T}$ of E and the basis $(v_s)_{s \in S}$ of F with $M = r_1 + \cdots + r_m$ and $N = s_1 + \cdots + s_n$, which means that for any $t \in T$, the tth column of A consists of the components a_{st} of $f(u_t)$ over the basis $(v_s)_{s \in S}$, as in the beginning of Section 4.1.

For any i and any j such that $1 \leq i \leq m$ and $1 \leq j \leq n$, we can form the $s_i \times t_j$ matrix A_{S_i,T_j} obtained by deleting all rows in A of index $s \notin S_i$ and all columns in A of index $t \notin T_j$. The matrix A_{S_i,T_j} is the indexed family $(a_{st})_{(s,t) \in S_i \times T_j}$, as explained at the beginning of Section 4.1.

Observe that the matrix A_{S_i,T_j} is actually the matrix representing the linear map $f_{ij} \colon E_j \to F_i$ of Definition 6.7 with respect to the basis $(u_t)_{t \in T_j}$ of E_j and the basis $(v_s)_{s \in S_i}$ of F_i, in the sense that for any $t \in T_j$, the tth column of A_{S_i,T_j} consists of the components a_{st} of $f_{ij}(u_t)$ over the basis $(v_s)_{s \in S_i}$.

Definition 6.8. Given an $M \times N$ matrix A (with entries in K), we define the $m \times n$ matrix $(A_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ whose entry A_{ij} is the matrix $A_{ij} = A_{S_i,T_j}$, $1 \leq i \leq m$, $1 \leq j \leq n$, and we call it the *block matrix of A associated with the partitions $S = S_1 \cup \cdots \cup S_m$ and $T = T_1 \cup \cdots \cup T_n$*. The matrix A_{S_i,T_j} is an $s_i \times t_j$ matrix called the (i,j)th *block* of this block matrix.

Here we run into a notational dilemma which does not seem to be addressed in the literature. Horn and Johnson [95] (Section 0.7) define partitioned matrices as we do, but they do not propose a notation for block matrices.