(5) As in the proof of Proposition 53.5 (adapted to the real case) there is a matrix R such that

$$B = RR^\top,$$

so

$$\kappa(x, y) = x^\top By = x^\top RR^\top y = (R^\top x)^\top R^\top y = \langle R^\top x, R^\top y\rangle,$$

so κ is the kernel function given by the feature map $\varphi(x) = R^\top x$ from $\mathbb{R}^n$ to itself, and by Proposition 53.1, it is a symmetric positive definite kernel. $\square$

Proposition 53.7. *Let $\kappa_1 \colon X \times X \to \mathbb{C}$ be a positive definite kernel, and let $p(z)$ be a polynomial with nonnegative coefficients. Then the following functions κ defined below are also positive definite kernels.*

(1) $\kappa(x, y) = p(\kappa_1(x, y))$.

(2) $\kappa(x, y) = e^{\kappa_1(x,y)}$.

(3) If X is real Hilbert space with inner product $\langle -, -\rangle_X$ and corresponding norm $\| \|_X$,

$$\kappa(x, y) = e^{-\frac{\|x-y\|_X^2}{2\sigma^2}}$$

for any $\sigma > 0$.

Proof. (1) If $p(z) = a_m z^m + \cdots + a_1 z + a_0$, then

$$p(\kappa_1(x, y)) = a_m \kappa_1(x, y)^m + \cdots + a_1 \kappa_1(x, y) + a_0.$$

Since $a_k \geq 0$ for $k = 0, \ldots, m$, by Proposition 53.5 and Proposition 53.6(2), each function $a_k \kappa_i(x, y)^k$ with $1 \leq k \leq m$ is a positive definite kernel, by Proposition 53.6(3) with $f(x) = \sqrt{a_0}$, the constant function a_0 is a positive definite kernel, and by Proposition 53.6(1), $p(\kappa_1(x, y))$ is a positive definite kernel.

(2) We have

$$e^{\kappa_1(x,y)} = \sum_{k=0}^{\infty} \frac{\kappa_1(x, y)^k}{k!}.$$

By (1), the partial sums

$$\sum_{k=0}^{m} \frac{\kappa_1(x, y)^k}{k!}$$

are positive definite kernels, and since $e^{\kappa_1(x,y)}$ is the (uniform) pointwise limit of positive definite kernels, it is also a positive definite kernel.

(3) By Proposition 53.6(2), since the map $(x, y) \mapsto \langle x, y\rangle_X$ is obviously a positive definite kernel (the feature map is the identity) and since $\sigma \neq 0$, the function $(x, y) \mapsto \langle x, y\rangle_X/\sigma^2$ is a positive definite kernel (by Proposition 53.6(2)), so by (2),

$$\kappa_1(x, y) = e^{\frac{\langle x,y\rangle_X}{\sigma^2}}$$