In Section 54.7 we investigate conditions on ν that ensure that some point u_{i_0} and some point v_{j_0} is a support vector. Theorem 54.3 shows that for every optimal solution $(w, b, \eta, \epsilon, \xi)$ of Problem $(\mathrm{SVM}_{s2'})$ with $w \neq 0$ and $\eta > 0$, if

$$\max\{2p_f/(p+q), 2q_f/(p+q)\} < \nu < \min\{2p/(p+q), 2q/(p+q)\},$$

then some u_{i_0} and some v_{j_0} is a support vector. Under the same conditions on ν Proposition 54.4 shows that η and b can always be determined in terms of (λ, μ) using a single support vector.

(3) **Soft margin ν-SVM Problem** (SVM_{s3}). This is the variation of Problem $(\mathrm{SVM}_{s2'})$ obtained by adding the term $(1/2)b^2$ to the objective function. The result is that in minimizing the Lagrangian to find the dual function G, not just w but also b is determined. We also suppress the constraint $\eta \geq 0$ which turns out to be redundant. If $\nu > (p_f + q_f)/(p+q)$, then η is also determined. The fact that b and η are determined by the dual seems to be an advantage of Problem (SVM_{s3}).

The optimization problem is

$$\begin{aligned}
&\text{minimize} \quad \frac{1}{2}w^\top w + \frac{1}{2}b^2 + (p+q)K_s\left(-\nu\eta + \frac{1}{p+q}\begin{pmatrix}\epsilon^\top & \xi^\top\end{pmatrix}\mathbf{1}_{p+q}\right)\\
&\text{subject to}\\
&\qquad w^\top u_i - b \geq \eta - \epsilon_i, \quad \epsilon_i \geq 0 \qquad i = 1, \ldots, p\\
&\qquad -w^\top v_j + b \geq \eta - \xi_j, \quad \xi_j \geq 0 \qquad j = 1, \ldots, q.
\end{aligned}$$

Theoretically it is convenient to assume that $K_s = 1/(p+q)$. Otherwise, ν needs to be replaced by $(p+q)K_s\nu$ in all the formulae below.

It is shown in Section 54.13 that the dual is given by

Dual of the Soft margin ν-SVM Problem (SVM_{s3}):

$$\begin{aligned}
&\text{minimize} \quad \frac{1}{2}\begin{pmatrix}\lambda^\top & \mu^\top\end{pmatrix}\left(X^\top X + \begin{pmatrix}\mathbf{1}_p\mathbf{1}_p^\top & -\mathbf{1}_p\mathbf{1}_q^\top\\ -\mathbf{1}_q\mathbf{1}_p^\top & \mathbf{1}_q\mathbf{1}_q^\top\end{pmatrix}\right)\begin{pmatrix}\lambda\\ \mu\end{pmatrix}\\
&\text{subject to}\\
&\qquad \sum_{i=1}^p \lambda_i + \sum_{j=1}^q \mu_j = \nu\\
&\qquad 0 \leq \lambda_i \leq K_s, \quad i = 1, \ldots, p\\
&\qquad 0 \leq \mu_j \leq K_s, \quad j = 1, \ldots, q.
\end{aligned}$$