Proposition 56.3. *If $\nu < (m-1)/m$, then $p_f < \lfloor m/2 \rfloor$ and $q_f < \lfloor m/2 \rfloor$.*

Proof. By Proposition 56.2, $\max\{2p_f/m, 2q_f/m\} \leq \nu$. If m is even, say $m = 2k$, then

$$2p_f/m = 2p_f/(2k) \leq \nu < (m-1)/m = (2k-1)/2k,$$

so $2p_f < 2k - 1$, which implies $p_f < k = \lfloor m/2 \rfloor$. A similar argument shows that $q_f < k = \lfloor m/2 \rfloor$.

If m is odd, say $m = 2k+1$, then

$$2p_f/m = 2p_f/(2k+1) \leq \nu < (m-1)/m = 2k/(2k+1),$$

so $2p_f < 2k$, which implies $p_f < k = \lfloor m/2 \rfloor$. A similar argument shows that $q_f < k = \lfloor m/2 \rfloor$. ∎

Since $p_{sf} \leq p_f$ and $q_{sf} \leq q_f$, we also have $p_{sf} < \lfloor m/2 \rfloor$ and $q_{sf} < \lfloor m/2 \rfloor$. This implies that $\{1, \ldots, m\} - (E_\lambda \cup E_\mu)$ contains at least two elements and there are constraints corresponding to at least two $i \notin (E_\lambda \cup E_\mu)$ (in which case $\xi_i = \xi'_i = 0$), of the form

$$\begin{aligned} w^\top x_i + b - y_i &\leq \epsilon & \qquad & i \notin (E_\lambda \cup E_\mu) \\ -w^\top x_i - b + y_i &\leq \epsilon & & i \notin (E_\lambda \cup E_\mu). \end{aligned}$$

If $w^\top x_i + b - y_i = \epsilon$ for some $i \notin (E_\lambda \cup E_\mu)$ and $-w^\top x_j - b + y_j = \epsilon$ for some $j \notin (E_\lambda \cup E_\mu)$ with $i \neq j$, then we have a blue support vector and a red support vector. Otherwise, we show how to modify b and ϵ to obtain an optimal solution with a blue support vector and a red support vector.

Proposition 56.4. *For every optimal solution $(w, b, \epsilon, \xi, \xi')$ with $w \neq 0$ and $\epsilon > 0$, if*

$$\nu < (m-1)/m$$

and if either no x_i is a blue support vector or no x_i is a red support vector, then there is another optimal solution (for the same w) with some i_0 such that $\xi_{i_0} = 0$ and $w^\top x_{i_0} + b - y_{i_0} = \epsilon$, and there is some j_0 such that $\xi'_{j_0} = 0$ and $-w^\top x_{j_0} - b + y_{j_0} = \epsilon$; in other words, some x_{i_0} is a blue support vector and some x_{j_0} is a red support vector (with $i_0 \neq j_0$). If all points (x_i, y_i) that are not errors lie on one of the margin hyperplanes, then there is an optimal solution for which $\epsilon = 0$.

Proof. By Proposition 56.3 if $\nu < (m-1)/m$, then $p_f < \lfloor m/2 \rfloor$ and $q_f < \lfloor m/2 \rfloor$, so the following constraints hold:

$$\begin{aligned} w^\top x_i + b - y_i &= \epsilon + \xi_i & \qquad & \xi_i > 0 & \qquad & i \in E_\lambda \\ -w^\top x_j - b + y_j &= \epsilon + \xi'_j & & \xi'_j > 0 & & j \in E_\mu \\ w^\top x_i + b - y_i &\leq \epsilon & & & & i \notin (E_\lambda \cup E_\mu) \\ -w^\top x_i - b + y_i &\leq \epsilon & & & & i \notin (E_\lambda \cup E_\mu), \end{aligned}$$