Consequently, the relative error in the result $\|\Delta x\| / \|x\|$ is bounded in terms of the relative error $\|\Delta b\| / \|b\|$ in the data as follows:

$$\frac{\|\Delta x\|}{\|x\|} \leq \left(\|A\| \left\|A^{-1}\right\| \right) \frac{\|\Delta b\|}{\|b\|}.$$

Now let us assume that A is perturbed to $A+\Delta A$, and let us analyze the change between the exact solutions of the two systems

$$\begin{aligned} Ax &= b \\ (A+\Delta A)(x+\Delta x) &= b. \end{aligned}$$

The second equation yields $Ax + A\Delta x + \Delta A(x+\Delta x) = b$, and by subtracting the first equation we get

$$\Delta x = -A^{-1}\Delta A(x+\Delta x).$$

It follows that

$$\|\Delta x\| \leq \left\|A^{-1}\right\| \|\Delta A\| \|x+\Delta x\|,$$

which can be rewritten as

$$\frac{\|\Delta x\|}{\|x+\Delta x\|} \leq \left(\|A\| \left\|A^{-1}\right\| \right) \frac{\|\Delta A\|}{\|A\|}.$$

Observe that the above reasoning is valid even if the matrix $A+\Delta A$ is singular, as long as $x+\Delta x$ is a solution of the second system. Furthermore, if $\|\Delta A\|$ is small enough, it is not unreasonable to expect that the ratio $\|\Delta x\| / \|x+\Delta x\|$ is close to $\|\Delta x\| / \|x\|$. This will be made more precise later.

In summary, for each of the two perturbations, we see that the relative error in the result is bounded by the relative error in the data, *multiplied the number* $\|A\| \|A^{-1}\|$. In fact, this factor turns out to be optimal and this suggests the following definition:

Definition 9.10. For any subordinate matrix norm $\| \; \|$, for any invertible matrix A, the number

$$\mathrm{cond}(A) = \|A\| \left\|A^{-1}\right\|$$

is called the *condition number* of A relative to $\| \; \|$.

The condition number $\mathrm{cond}(A)$ measures the sensitivity of the linear system $Ax = b$ to variations in the data b and A; a feature referred to as the *condition* of the system. Thus, when we says that a linear system is *ill-conditioned*, we mean that the condition number of its matrix is large. We can sharpen the preceding analysis as follows:

Proposition 9.13. *Let A be an invertible matrix and let x and $x+\Delta x$ be the solutions of the linear systems*

$$\begin{aligned} Ax &= b \\ A(x+\Delta x) &= b+\Delta b. \end{aligned}$$