which proves that $\mathrm{D}f_a$ is continuous at 0. By Proposition 37.56, $\mathrm{D}f_a$ is a continuous linear map. $\square$

Example 39.2. Consider the map, $f\colon \mathrm{M}_n(\mathbb{R}) \to \mathrm{M}_n(\mathbb{R})$, given by

$$f(A) = A^\top A - I,$$

where $\mathrm{M}_n(\mathbb{R})$ is equipped with any matrix norm, since they are all equivalent; for example, pick the Frobenius norm, $\|A\|_F = \sqrt{\mathrm{tr}(A^\top A)}$. We claim that

$$Df(A)(H) = A^\top H + H^\top A, \quad \text{for all } A \text{ and } H \text{ in } \mathrm{M}_n(\mathbb{R}).$$

We have

$$\begin{aligned} f(A+H) - f(A) - (A^\top H + H^\top A) &= (A+H)^\top (A+H) - I - (A^\top A - I) - A^\top H - H^\top A \\ &= A^\top A + A^\top H + H^\top A + H^\top H - A^\top A - A^\top H - H^\top A \\ &= H^\top H. \end{aligned}$$

It follows that

$$\epsilon(H) = \frac{f(A+H) - f(A) - (A^\top H + H^\top A)}{\|H\|} = \frac{H^\top H}{\|H\|},$$

and since our norm is the Frobenius norm,

$$\|\epsilon(H)\| = \left\| \frac{H^\top H}{\|H\|} \right\| \le \frac{\left\|H^\top\right\| \|H\|}{\|H\|} = \left\|H^\top\right\| = \|H\|,$$

so

$$\lim_{H \mapsto 0} \epsilon(H) = 0,$$

and we conclude that

$$Df(A)(H) = A^\top H + H^\top A.$$

If $\mathrm{D}f(a)$ exists for every $a \in A$, we get a map

$$\mathrm{D}f\colon A \to \mathcal{L}(\overrightarrow{E}; \overrightarrow{F}),$$

called the *derivative of f on A*, and also denoted by df. Recall that $\mathcal{L}(\overrightarrow{E}; \overrightarrow{F})$ denotes the vector space of all continuous maps from $\overrightarrow{E}$ to $\overrightarrow{F}$.

We now consider a number of standard results about derivatives.