Therefore, f^{*l} is linear. We call it the *left adjoint* of f.

Now, for any fixed $u \in E_2$, we can consider the linear form in E_1^* given by

$$x \mapsto \overline{\varphi_2(u, f(x))} \quad x \in E_1.$$

Since $l_{\varphi_1} \colon \overline{E_1} \to E_1^*$ is bijective, there is a unique $y \in E_1$ so that

$$\overline{\varphi_2(u, f(x))} = \overline{\varphi_1(y, x)}, \quad \text{for all } x \in E_1.$$

If we denote this unique $y \in E_1$ by $f^{*r}(u)$, then we have

$$\varphi_2(u, f(x)) = \varphi_1(f^{*r}(u), x), \quad \text{for all } x \in E_1, \text{ and all } u \in E_2.$$

Thus, we get a function $f^{*r} \colon E_2 \to E_1$. As in the previous situation, it easy to check that f^{*r} is linear. We call it the *right adjoint* of f. In summary, we make the following definition.

Definition 29.14. Let E_1 and E_2 be two K-vector spaces, and let $\varphi_1 \colon E_1 \times E_1 \to K$ and $\varphi_2 \colon E_2 \times E_2 \to K$ be two sesquilinear forms. Assume that l_{φ_1} and r_{φ_1} are bijective, so that φ_1 is nondegnerate. For every linear map $f \colon E_1 \to E_2$, there exist unique linear maps $f^{*l} \colon E_2 \to E_1$ and $f^{*r} \colon E_2 \to E_1$, such that

$$\begin{aligned} \varphi_2(f(x), u) &= \varphi_1(x, f^{*l}(u)), \quad \text{for all } x \in E_1, \text{ and all } u \in E_2 \\ \varphi_2(u, f(x)) &= \varphi_1(f^{*r}(u), x), \quad \text{for all } x \in E_1, \text{ and all } u \in E_2. \end{aligned}$$

The map f^{*l} is called the *left adjoint* of f, and the map f^{*r} is called the *right adjoint* of f.

If E_1 and E_2 are finite-dimensional with bases $(e_1, \ldots, e_m)$ and $(f_1, \ldots, f_n)$, then we can work out the matrices A^{*l} and A^{*r} corresponding to the left adjoint f^{*l} and the right adjoint f^{*r} of f. Assumine that f is represented by the $n \times m$ matrix A, φ_1 is represented by the $m \times m$ matrix M_1, and φ_2 is represented by the $n \times n$ matrix M_2. Since

$$\begin{aligned} \varphi_1(x, f^{*l}(u)) &= (A^{*l}u)^* M_1 x = u^*(A^{*l})^* M_1 x \\ \varphi_2(f(x), u) &= u^* M_2 A x \end{aligned}$$

we find that $(A^{*l})^* M_1 = M_2 A$, that is $(A^{*l})^* = M_2 A M_1^{-1}$, and similarly

$$\begin{aligned} \varphi_1(f^{*r}(u), x) &= x^* M_1 A^{*r} u \\ \varphi_2(u, f(x)) &= (Ax)^* M_2 u = x^* A^* M_2 u, \end{aligned}$$

we have $M_1 A^{*r} = A^* M_2$, that is $A^{*r} = (M_1)^{-1} A^* M_2$. Thus, we obtain

$$\begin{aligned} A^{*l} &= (M_1^*)^{-1} A^* M_2^* \\ A^{*r} &= (M_1)^{-1} A^* M_2. \end{aligned}$$