Chapter 40

Extrema of Real-Valued Functions

This chapter deals with extrema of real-valued functions. In most optimization problems, we need to find necessary conditions for a function $J\colon \Omega \to \mathbb{R}$ to have a local extremum with respect to a subset U of Ω (where Ω is open). This can be done in two cases:

(1) The set U is defined by a set of equations,

$$U = \{x \in \Omega \mid \varphi_i(x) = 0,\ 1 \leq i \leq m\},$$

where the functions $\varphi_i\colon \Omega \to \mathbb{R}$ are continuous (and usually differentiable).

(2) The set U is defined by a set of inequalities,

$$U = \{x \in \Omega \mid \varphi_i(x) \leq 0,\ 1 \leq i \leq m\},$$

where the functions $\varphi_i\colon \Omega \to \mathbb{R}$ are continuous (and usually differentiable).

In (1), the equations $\varphi_i(x) = 0$ are called *equality constraints*, and in (2), the inequalities $\varphi_i(x) \leq 0$ are called *inequality constraints*. The case of equality constraints is much easier to deal with and is treated in this chapter.

If the functions φ_i are convex and Ω is convex, then U is convex. This is a very important case that we discuss later. In particular, if the functions φ_i are affine, then the equality constraints can be written as $Ax = b$, and the inequality constraints as $Ax \leq b$, for some $m \times n$ matrix A and some vector $b \in \mathbb{R}^m$. We will also discuss the case of affine constraints later.

In the case of equality constraints, a necessary condition for a local extremum with respect to U can be given in terms of *Lagrange multipliers*. In the case of inequality constraints, there is also a necessary condition for a local extremum with respect to U in terms of generalized Lagrange multipliers and the *Karush–Kuhn–Tucker* conditions. This will be discussed in Chapter 50.