(2) Since $\langle -, - \rangle$ is continuous, for every $\epsilon > 0$, there are some $\eta_1 > 0$ and $\eta_2 > 0$, such that

$$|\langle x, y \rangle| < \epsilon$$

whenever $\|x\| < \eta_1$ and $\|y\| < \eta_2$. Since $v = \sum_{k \in K} \lambda_k u_k$ and $w = \sum_{k \in K} \mu_k u_k$, there is some finite subset I_1 of K such that

$$\Big\| v - \sum_{j \in J} \lambda_j u_j \Big\| < \eta_1$$

for every finite subset J of K such that $I_1 \subseteq J$, and there is some finite subset I_2 of K such that

$$\Big\| w - \sum_{j \in J} \mu_j u_j \Big\| < \eta_2$$

for every finite subset J of K such that $I_2 \subseteq J$. Letting $I = I_1 \cup I_2$, we get

$$\left| \left\langle v - \sum_{i \in I} \lambda_i u_i, w - \sum_{i \in I} \mu_i u_i \right\rangle \right| < \epsilon.$$

Furthermore,

$$\begin{aligned}
\langle v, w \rangle &= \left\langle v - \sum_{i \in I} \lambda_i u_i + \sum_{i \in I} \lambda_i u_i, w - \sum_{i \in I} \mu_i u_i + \sum_{i \in I} \mu_i u_i \right\rangle \\
&= \left\langle v - \sum_{i \in I} \lambda_i u_i, w - \sum_{i \in I} \mu_i u_i \right\rangle + \sum_{i \in I} \lambda_i \overline{\mu_i},
\end{aligned}$$

since the u_i are orthogonal to $v - \sum_{i \in I} \lambda_i u_i$ and $w - \sum_{i \in I} \mu_i u_i$ for all $i \in I$. This proves that for every $\epsilon > 0$, there is some finite subset I of K such that

$$\left| \langle v, w \rangle - \sum_{i \in I} \lambda_i \overline{\mu_i} \right| < \epsilon.$$

We already know from Proposition A.3 that $(\lambda_k \overline{\mu_k})_{k \in K}$ is summable, and since $\epsilon > 0$ is arbitrary we get

$$\langle v, w \rangle = \sum_{k \in K} \lambda_k \overline{\mu_k}.$$

$\square$

The next proposition states properties characterizing Hilbert bases (total orthogonal families).

Proposition A.5. *Let E be a Hilbert space, and let $(u_k)_{k \in K}$ be an orthogonal family in E. The following properties are equivalent:*