alternating multilinear maps. As in the case of general tensors, the isomorphisms provided by Propositions 34.5, 33.17, and 34.10, namely

$$\begin{aligned} \mathrm{Alt}^n(E;F) &\cong \mathrm{Hom}\Big(\bigwedge^n(E), F\Big) \\ \mathrm{Hom}\Big(\bigwedge^n(E), F\Big) &\cong \Big(\bigwedge^n(E)\Big)^* \otimes F \\ \Big(\bigwedge^n(E)\Big)^* &\cong \bigwedge^n(E^*) \end{aligned}$$

yield a canonical isomorphism

$$\mathrm{Alt}^n(E;F) \cong \Big(\bigwedge^n(E^*)\Big) \otimes F$$

which we record as a corollary.

Corollary 34.32. *For any finite-dimensional vecgtor space E and any vector space F, we have a canonical isomorphism*

$$\mathrm{Alt}^n(E;F) \cong \Big(\bigwedge^n(E^*)\Big) \otimes F.$$

Note that F may have infinite dimension. This isomorphism allows us to view the tensors in $\bigwedge^n(E^*) \otimes F$ as *vector-valued alternating forms*, a point of view that is useful in differential geometry. If $(f_1, \ldots, f_r)$ is a basis of F, every tensor $\omega \in \bigwedge^n(E^*) \otimes F$ can be written as some linear combination

$$\omega = \sum_{i=1}^r \alpha_i \otimes f_i,$$

with $\alpha_i \in \bigwedge^n(E^*)$.We also let

$$\bigwedge(E;F) = \bigoplus_{n=0} \Big(\bigwedge^n(E^*)\Big) \otimes F = \Big(\bigwedge(E)\Big) \otimes F.$$

Given three vector spaces, F, G, H, if we have some bilinear map $\Phi\colon F \times G \to H$, then we can define a multiplication operation

$$\wedge_\Phi \colon \bigwedge(E;F) \times \bigwedge(E;G) \to \bigwedge(E;H)$$

as follows: For every pair (m,n), we define the multiplication

$$\wedge_\Phi \colon \bigg(\Big(\bigwedge^m(E^*)\Big) \otimes F\bigg) \times \bigg(\Big(\bigwedge^n(E^*)\Big) \otimes G\bigg) \longrightarrow \Big(\bigwedge^{m+n}(E^*)\Big) \otimes H$$