Chapter 39

Differential Calculus

39.1 Directional Derivatives, Total Derivatives

This chapter contains a review of basic notions of differential calculus. First, we review the definition of the derivative of a function $f\colon \mathbb{R} \to \mathbb{R}$. Next, we define directional derivatives and the total derivative of a function $f\colon E \to F$ between normed affine spaces. Basic properties of derivatives are shown, including the chain rule. We show how derivatives are represented by Jacobian matrices. The mean value theorem is stated, as well as the implicit function theorem and the inverse function theorem. Diffeomorphisms and local diffeomorphisms are defined. Tangent spaces are defined. Higher-order derivatives are defined, as well as the Hessian. Schwarz's lemma (about the commutativity of partials) is stated. Several versions of Taylor's formula are stated, and a famous formula due to Faà di Bruno's is given.

We first review the notion of the derivative of a real-valued function whose domain is an open subset of $\mathbb{R}$.

Let $f\colon A \to \mathbb{R}$, where A is a nonempty open subset of $\mathbb{R}$, and consider any $a \in A$. The main idea behind the concept of the derivative of f at a, denoted by $f'(a)$, is that locally around a (that is, in some small open set $U \subseteq A$ containing a), the function f is approximated linearly[1] by the map

$$x \mapsto f(a) + f'(a)(x - a).$$

As pointed out by Dieudonné in the early 1960s, it is an "unfortunate accident" that if V is vector space of dimension one, then there is a bijection between the space V^* of linear forms defined on V and the field of scalars. As a consequence, the derivative of a real-valued function f defined on an open subset A of the reals can be defined as the scalar $f'(a)$ (for any $a \in A$). But as soon as f is a function of several arguments, the scalar interpretation of the derivative breaks down.

[1]Actually, the approximation is affine, but everybody commits this abuse of language.