In general we proceed as follows. For any $x = x_1 + \cdots + x_n$ with $x_j \in E_j$, if $y = f(x)$, since $F = F_1 \oplus \cdots \oplus F_m$, the vector $y \in F$ has a unique decomposition $y = y_1 + \cdots + y_m$ with $y_i \in F_i$, and since $f_{ij} \colon E_j \to F_i$, we have $\sum_{j=1}^n f_{ij}(x_j) \in F_i$, so $\sum_{j=1}^n f_{ij}(x_j) \in F_i$ is the ith component of $f(x)$ over the direct sum $F = F_1 \oplus \cdots \oplus F_m$; equivalently

$$pr_i^F(f(x)) = \sum_{j=1}^n f_{ij}(x_j), \quad 1 \leq i \leq m.$$

Consequently, we have

$$y_i = \sum_{j=1}^n f_{ij}(x_j), \quad 1 \leq i \leq m. \tag{$\dagger_2$}$$

This time we are summing over the index j, which eventually corresponds to multiplying the ith row of the matrix representing f by the n-tuple $(x_1, \ldots, x_n)$; see Definition 6.7.

All this suggests a generalization of the matrix notation Ax, where A is a matrix of scalars and x is a column vector of scalars, namely to write

$$\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} f_{1\,1} & \ldots & f_{1\,n} \\ \vdots & \ddots & \vdots \\ f_{m\,1} & \ldots & f_{m\,n} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \tag{$\dagger_3$}$$

which is an abbreviation for the m equations

$$y_i = \sum_{j=1}^n f_{ij}(x_j), \quad i = 1, \ldots, m.$$

The interpretation of the multiplication of an $m \times n$ matrix of linear maps f_{ij} by a column vector of n vectors $x_j \in E_j$ is to apply each f_{ij} to x_j to obtain $f_{ij}(x_j)$ and to sum over the index j to obtain the ith output vector. This is the generalization of multiplying the scalar a_{ij} by the scalar x_j. Also note that the jth column of the matrix (f_{ij}) consists of the maps $(f_{1j}, \ldots, f_{mj})$ such that $(f_{1j}(x_j), \ldots, f_{mj}(x_j))$ are the components of $f(x_j) = f_j(x_j)$ over the direct sum $F = F_1 \oplus \cdots \oplus F_m$.

In the special case in which each E_j and each F_i is one-dimensional, this is equivalent to choosing a basis $(u_1, \ldots, u_n)$ in E so that E_j is the one-dimensional subspace $E_j = Ku_j$, and a basis $(v_1, \ldots, v_m)$ in F_j so that F_i is the one-dimensional subspace $F_i = Kv_i$. In this case every vector $x \in E$ is expressed as $x = x_1u_1 + \cdots + x_nu_n$, where the $x_i \in K$ are scalars and similarly every vector $y \in F$ is expressed as $y = y_1v_1 + \cdots + y_mv_m$, where the $y_i \in K$ are scalars. Each linear map $f_{ij} \colon E_j \to F_i$ is a map between the one-dimensional spaces Ku_j and Kv_i, so it is of the form $f_{ij}(x_ju_j) = a_{ij}x_jv_i$, with $x_j \in K$, and so the matrix (f_{ij}) of linear maps f_{ij} is in one-to-one correspondence with the matrix (a_{ij}) of scalars in K, and Equation ($\dagger_3$) where the x_j and y_i are vectors is equivalent to the same familar equation where the x_j and y_i are the scalar coordinates of x and y over the respective bases.