This suggests defining the functions $U^i \in V_a$ by

$$U^i = \sum_{k=1}^{n} \mathbf{U}^i_k w_k.$$

Then, it immediate to check that

$$a(U^i, U^j) = (\mathbf{U}^i)^\top A \mathbf{U}^j = \delta_{ij},$$

which means that the functions $(U^1, \ldots, U^n)$ form an orthonormal basis of V_a for the inner product a. The functions $U^i \in V_a$ are called *modes* (or *modal vectors*).

As a final step, let us look again for a solution of our discrete weak formulation of the problem, this time expressing the unknown solution $u(x,t)$ over the modal basis $(U^1, \ldots, U^n)$, say

$$u = \sum_{j=1}^{n} \widetilde{u}_j(t) U^j,$$

where each $\widetilde{u}_j$ is a function of t. Because

$$u = \sum_{j=1}^{n} \widetilde{u}_j(t) U^j = \sum_{j=1}^{n} \widetilde{u}_j(t) \left(\sum_{k=1}^{n} \mathbf{U}^j_k w_k \right) = \sum_{k=1}^{n} \left(\sum_{j=1}^{n} \widetilde{u}_j(t) \mathbf{U}^j_k \right) w_k,$$

if we write $\mathbf{u} = (u_1, \ldots, u_n)$ with $u_k = \sum_{j=1}^{n} \widetilde{u}_j(t) \mathbf{U}^j_k$ for $k = 1, \ldots, n$, we see that

$$\mathbf{u} = \sum_{j=1}^{n} \widetilde{u}_j \mathbf{U}^j,$$

so using the fact that

$$K\mathbf{U}^j = \omega_j^2 A \mathbf{U}^j, \quad j = 1, \ldots, n,$$

the equation

$$A \frac{d^2 \mathbf{u}}{dt^2} + K\mathbf{u} = 0$$

yields

$$\sum_{j=1}^{n} [(\widetilde{u}_j)'' + \omega_j^2 \widetilde{u}_j] A \mathbf{U}^j = 0.$$

Since A is invertible and since $(\mathbf{U}^1, \ldots, \mathbf{U}^n)$ are linearly independent, the vectors $(A\mathbf{U}^1, \ldots, A\mathbf{U}^n)$ are linearly independent, and consequently we get the system of n ODEs'

$$(\widetilde{u}_j)'' + \omega_j^2 \widetilde{u}_j = 0, \quad 1 \leq j \leq n.$$

Each of these equation has a well-known solution of the form

$$\widetilde{u}_j = A_j \cos \omega_j t + B_j \sin \omega_j t.$$