and since (α_m) is a Cauchy sequence, so is $(\varphi(a_n))$, and as φ is an isometry, the sequence (a_n) is a Cauchy sequence in E. Let $\alpha \in \widehat{E}$ be the equivalence class of (a_n). Since

$$\widehat{d}(\alpha, \varphi(a_n)) = \lim_{m \mapsto \infty} d(a_m, a_n)$$

and (a_n) is a Cauchy sequence, we deduce that the sequence $(\varphi(a_n))$ converges to α, and since $d(\alpha_n, \varphi(a_n)) \leq 1/n$ for all $n > 0$, the sequence (α_n) also converges to α.

Step 8. Let us prove the extension property. Let F be any complete metric space and let $f\colon E \to F$ be any uniformly continuous function. The function $\varphi\colon E \to \widehat{E}$ is an isometry and a bijection between E and its image $\varphi(E)$, so its inverse $\varphi^{-1}\colon \varphi(E) \to E$ is also an isometry, and thus is uniformly continuous. If we let $g = f \circ \varphi^{-1}$, then $g\colon \varphi(E) \to F$ is a uniformly continuous function, and $\varphi(E)$ is dense in $\widehat{E}$, so by Theorem 37.52 there is a unique uniformly continuous function $\widehat{f}\colon \widehat{E} \to F$ extending $g = f \circ \varphi^{-1}$; see the diagram below:

$$\begin{tikzcd}
E \arrow[drr, "f"'] & \varphi(E) \arrow[l, "\varphi^{-1}"'] \arrow[dr, "g"] & \subseteq & \widehat{E} \arrow[dl, "\widehat{f}"] \\
& & F &
\end{tikzcd}$$

This means that

$$\widehat{f}|\varphi(E) = f \circ \varphi^{-1},$$

which implies that

$$(\widehat{f}|\varphi(E)) \circ \varphi = f,$$

that is, $f = \widehat{f} \circ \varphi$, as illustrated in the diagram below:

$$\begin{tikzcd}
E \arrow[r, "\varphi"] \arrow[dr, "f"'] & \widehat{E} \arrow[d, "\widehat{f}"] \\
& F
\end{tikzcd}$$

If $h\colon \widehat{E} \to F$ is any other uniformly continuous function such that $f = h \circ \varphi$, then $g = f \circ \varphi^{-1} = h|\varphi(E)$, so h is a uniformly continuous function extending g, and by Theorem 37.52, we have have $h = \widehat{f}$, so $\widehat{f}$ is indeed unique.

Step 9. Uniqueness of the completion $(\widehat{E}, \widehat{d})$ up to a bijective isometry.

Let $(\widehat{E_1}, \widehat{d_1})$ and $(\widehat{E_2}, \widehat{d_2})$ be any two completions of (E, d). Then we have two uniformly continuous isometries $\varphi_1\colon E \to \widehat{E_1}$ and $\varphi_2\colon E \to \widehat{E_2}$, so by the unique extension property, there exist unique uniformly continuous maps $\widehat{\varphi_2}\colon \widehat{E_1} \to \widehat{E_2}$ and $\widehat{\varphi_1}\colon \widehat{E_2} \to \widehat{E_1}$ such that the following diagrams commute:

$$\begin{tikzcd}
E \arrow[r, "\varphi_1"] \arrow[dr, "\varphi_2"'] & \widehat{E_1} \arrow[d, "\widehat{\varphi_2}"] \\
& \widehat{E_2}
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[r, "\varphi_2"] \arrow[dr, "\varphi_1"'] & \widehat{E_2} \arrow[d, "\widehat{\varphi_1}"] \\
& \widehat{E_1}.
\end{tikzcd}$$