But $(*_1)$ and $(*_4)$ are exactly the KKT equations, and by Theorem 51.41, we conclude that $\widetilde{x}, \widetilde{z}, \widetilde{\lambda}$ are optimal solutions.

Step 9. Prove (A1). This is the most tedious step of the proof. We begin by adding up (A2) and (A3), and then perform quite a bit or rewriting and manipulation. The complete derivation can be found in Boyd et al. [28]. □

Remarks:

(1) In view of Theorem 51.42, we could replace Assumption (3) by the slightly stronger assumptions that the optimum value of our program is finite and that the constraints are qualified. Since the constraints are affine, this means that there is some feasible solution in $\mathbf{relint}(\mathrm{dom}(f)) \cap \mathbf{relint}(\mathrm{dom}(g))$. These assumptions are more practical than Assumption (3).

(2) Actually, Assumption (3) implies Assumption (2). Indeed, we know from Theorem 51.41 that the existence of a saddle point implies that our program has a finite optimal solution. However, if either $A^\top A$ or $B^\top B$ is not invertible, then Program (P) may not have a finite optimal solution, as shown by the following counterexample.

Example 52.5. Let

$$f(x, y) = x, \quad g(z) = 0, \quad y - z = 0.$$

Then

$$L_\rho(x, y, z, \lambda) = x + \lambda(y - z) + (\rho/2)(y - z)^2,$$

but minimizing over (x, y) with z held constant yields $-\infty$, which implies that the above program has no finite optimal solution. See Figure 52.4.

The problem is that

$$A = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 \end{pmatrix},$$

but

$$A^\top A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

is not invertible.

(3) Proving (A1), (A2), (A3), and the convergence of (r^k) to 0 and of (p^k) to p^* does not require Assumption (2). The proof, using the ingeneous Inequality (A1) (and (B)) is the proof given in Boyd et al. [28]. We were also able to prove that (λ^k), (Ax^k) and (Bz^k) converge without Assumption (2), but to prove that (x^k), (y^k), and (λ^k) converge to optimal solutions, we had to use Assumption (2).