where Q is an orthogonal matrix and D is a diagonal matrix,

$$D = \mathrm{diag}(d_1, \ldots, d_n),$$

with $d_i > 0$, for $i = 1, \ldots, n$. If we define the matrices $B^{1/2}$ and $B^{-1/2}$ by

$$B^{1/2} = Q\, \mathrm{diag}\left(\sqrt{d_1}, \ldots, \sqrt{d_n}\right) Q^\top$$

and

$$B^{-1/2} = Q\, \mathrm{diag}\left(1/\sqrt{d_1}, \ldots, 1/\sqrt{d_n}\right) Q^\top,$$

it is clear that these matrices are symmetric, that $B^{-1/2}BB^{-1/2} = I$, and that $B^{1/2}$ and $B^{-1/2}$ are mutual inverses. Then if we make the change of variable

$$x = B^{-1/2}y,$$

the equation $x^\top Bx = 1$ becomes $y^\top y = 1$, and the optimization problem

$$\begin{array}{ll} \text{minimize} & x^\top Ax \\ \text{subject to} & x^\top Bx = 1,\ x \in \mathbb{R}^n, \end{array}$$

is equivalent to the problem

$$\begin{array}{ll} \text{minimize} & y^\top B^{-1/2}AB^{-1/2}y \\ \text{subject to} & y^\top y = 1,\ y \in \mathbb{R}^n, \end{array}$$

where $y = B^{1/2}x$ and $B^{-1/2}AB^{-1/2}$ are symmetric.

The complex version of our basic optimization problem in which A is a Hermitian matrix also arises in computer vision. Namely, given an $n \times n$ complex Hermitian matrix A,

$$\begin{array}{ll} \text{maximize} & x^*Ax \\ \text{subject to} & x^*x = 1,\ x \in \mathbb{C}^n. \end{array}$$

Again by Proposition 23.10, the maximum value of x^*Ax on the unit sphere is equal to the largest eigenvalue λ_1 of the matrix A, and it is achieved for any unit eigenvector u_1 associated with λ_1.

Remark: It is worth pointing out that if A is a *skew-Hermitian* matrix, that is, if $A^* = -A$, then x^*Ax is *pure imaginary or zero*.

Indeed, since $z = x^*Ax$ is a scalar, we have $z^* = \overline{z}$ (the conjugate of z), so we have

$$\overline{x^*Ax} = (x^*Ax)^* = x^*A^*x = -x^*Ax,$$

so $\overline{x^*Ax} + x^*Ax = 2\mathrm{Re}(x^*Ax) = 0$, which means that x^*Ax is pure imaginary or zero.