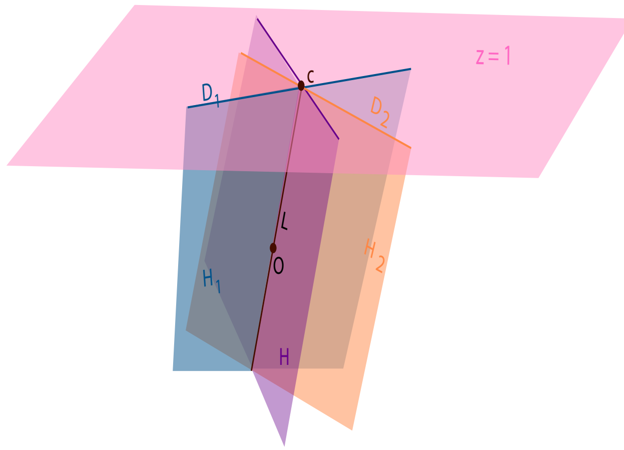

Figure 26.6: A pencil of lines through c in the hyperplane model of $\mathbb{RP}^2$

$\mathbf{P}(E)$, i.e., the set of all hyperplanes containing some given projective subspace $W = p(V)$ of dimension $n-2$. For $n = 3$, a pencil of planes in $\mathbb{RP}^3 = \mathbf{P}(\mathbb{R}^4)$ is the set of all planes (in $\mathbb{RP}^3$) containing some given line W. Other examples of unusual projective spaces and pencils will be given in Section 26.4.

Next, we define the projective analogues of bases (or frames) and linear maps.

26.4 Projective Frames

As all good notions in projective geometry, the concept of a projective frame turns out to be uniquely defined up to a scalar.

Definition 26.3. Given a nontrivial vector space E of dimension $n+1$, a family $(a_i)_{1\leq i\leq n+2}$ of $n+2$ points of the projective space $\mathbf{P}(E)$ is a *projective frame (or basis) of* $\mathbf{P}(E)$ if there exists some basis $(e_1, \ldots, e_{n+1})$ of E such that $a_i = p(e_i)$ for $1 \leq i \leq n+1$, and $a_{n+2} = p(e_1 + \cdots + e_{n+1})$. Any basis with the above property is said to be *associated with the projective frame* $(a_i)_{1\leq i\leq n+2}$.

The justification of Definition 26.3 is given by the following proposition.

Proposition 26.2. *If $(a_i)_{1\leq i\leq n+2}$ is a projective frame of $\mathbf{P}(E)$, for any two bases $(u_1, \ldots, u_{n+1})$, $(v_1, \ldots, v_{n+1})$ of E such that $a_i = p(u_i) = p(v_i)$ for $1 \leq i \leq n+1$, and $a_{n+2} = p(u_1 + \cdots + u_{n+1}) = p(v_1 + \cdots + v_{n+1})$, there is a nonzero scalar $\lambda \in K$ such that $v_i = \lambda u_i$, for all i, $1 \leq i \leq n+1$.*

Proof. Since $p(u_i) = p(v_i)$ for $1 \leq i \leq n+1$, there exist some nonzero scalars $\lambda_i \in K$ such that $v_i = \lambda_i u_i$ for all i, $1 \leq i \leq n+1$. Since we must have

$$p(u_1 + \cdots + u_{n+1}) = p(v_1 + \cdots + v_{n+1}),$$