(1) If $g_1 N = g_1' N$ and $g_2 N = g_2' N$, then $g_1 g_2 N = g_1' g_2' N$, and

(2) If $g_1 N = g_2 N$, then $g_1^{-1} N = g_2^{-1} N$.

As a consequence, we can define a group structure on the set $G/\sim$ of equivalence classes modulo $\sim$, by setting

$$(g_1 N)(g_2 N) = (g_1 g_2)N.$$

Definition 2.11. Let G be a group and N be a normal subgroup of G. The group obtained by defining the multiplication of (left) cosets by

$$(g_1 N)(g_2 N) = (g_1 g_2)N, \quad g_1, g_2 \in G$$

is denoted G/N, and called the *quotient of* G *by* N. The equivalence class gN of an element $g \in G$ is also denoted $\overline{g}$ (or $[g]$). The map $\pi\colon G \to G/N$ given by

$$\pi(g) = \overline{g} = gN$$

is a group homomorphism called the *canonical projection*.

Since the kernel of a homomorphism is a normal subgroup, we obtain the following very useful result.

Proposition 2.12. *Given a homomorphism of groups $\varphi\colon G \to G'$, the groups $G/\mathrm{Ker}\ \varphi$ and $\mathrm{Im}\ \varphi = \varphi(G)$ are isomorphic.*

Proof. Since φ is surjective onto its image, we may assume that φ is surjective, so that $G' = \mathrm{Im}\ \varphi$. We define a map $\overline{\varphi}\colon G/\mathrm{Ker}\ \varphi \to G'$ as follows:

$$\overline{\varphi}(\overline{g}) = \varphi(g), \quad g \in G.$$

We need to check that the definition of this map does not depend on the representative chosen in the coset $\overline{g} = g\,\mathrm{Ker}\ \varphi$, and that it is a homomorphism. If g' is another element in the coset $g\,\mathrm{Ker}\ \varphi$, which means that $g' = gh$ for some $h \in \mathrm{Ker}\,\varphi$, then

$$\varphi(g') = \varphi(gh) = \varphi(g)\varphi(h) = \varphi(g)e' = \varphi(g),$$

since $\varphi(h) = e'$ as $h \in \mathrm{Ker}\ \varphi$. This shows that

$$\overline{\varphi}(\overline{g'}) = \varphi(g') = \varphi(g) = \overline{\varphi}(\overline{g}),$$

so the map $\overline{\varphi}$ is well defined. It is a homomorphism because

$$\begin{aligned}
\overline{\varphi}(\overline{g}\,\overline{g'}) &= \overline{\varphi}(\overline{gg'}) \\
&= \varphi(gg') \\
&= \varphi(g)\varphi(g') \\
&= \overline{\varphi}(\overline{g})\overline{\varphi}(\overline{g'}).
\end{aligned}$$

The map $\overline{\varphi}$ is injective because $\overline{\varphi}(\overline{g}) = e'$ iff $\varphi(g) = e'$ iff $g \in \mathrm{Ker}\ \varphi$, iff $\overline{g} = \overline{e}$. The map $\overline{\varphi}$ is surjective because φ is surjective. Therefore $\overline{\varphi}$ is a bijective homomorphism, and thus an isomorphism, as claimed. $\square$