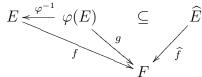
and since (α_m) is a Cauchy sequence, so is $(\varphi(a_n))$, and as φ is an isometry, the sequence (a_n) is a Cauchy sequence in E. Let $\alpha \in \widehat{E}$ be the equivalence class of (a_n) . Since

$$\widehat{d}(\alpha, \varphi(a_n)) = \lim_{m \to \infty} d(a_m, a_n)$$

and (a_n) is a Cauchy sequence, we deduce that the sequence $(\varphi(a_n))$ converges to α , and since $d(\alpha_n, \varphi(a_n)) \leq 1/n$ for all n > 0, the sequence (α_n) also converges to α .

Step 8. Let us prove the extension property. Let F be any complete metric space and let $f \colon E \to F$ be any uniformly continuous function. The function $\varphi \colon E \to \widehat{E}$ is an isometry and a bijection between E and its image $\varphi(E)$, so its inverse $\varphi^{-1} \colon \varphi(E) \to E$ is also an isometry, and thus is uniformly continuous. If we let $g = f \circ \varphi^{-1}$, then $g \colon \varphi(E) \to F$ is a uniformly continuous function, and $\varphi(E)$ is dense in \widehat{E} , so by Theorem 37.52 there is a unique uniformly continuous function $\widehat{f} \colon \widehat{E} \to F$ extending $g = f \circ \varphi^{-1}$; see the diagram below:



This means that

$$\widehat{f}|\varphi(E) = f \circ \varphi^{-1},$$

which implies that

$$(\widehat{f}|\varphi(E))\circ\varphi=f,$$

that is, $f = \hat{f} \circ \varphi$, as illustrated in the diagram below:



If $h: \widehat{E} \to F$ is any other uniformly continuous function such that $f = h \circ \varphi$, then $g = f \circ \varphi^{-1} = h | \varphi(E)$, so h is a uniformly continuous function extending g, and by Theorem 37.52, we have $h = \widehat{f}$, so \widehat{f} is indeed unique.

Step 9. Uniqueness of the completion $(\widehat{E},\widehat{d})$ up to a bijective isometry.

Let $(\widehat{E}_1, \widehat{d}_1)$ and $(\widehat{E}_2, \widehat{d}_2)$ be any two completions of (E, d). Then we have two uniformly continuous isometries $\varphi_1 \colon E \to \widehat{E}_1$ and $\varphi_2 \colon E \to \widehat{E}_2$, so by the unique extension property, there exist unique uniformly continuous maps $\widehat{\varphi_2} \colon \widehat{E}_1 \to \widehat{E}_2$ and $\widehat{\varphi_1} \colon \widehat{E}_2 \to \widehat{E}_1$ such that the following diagrams commute:

