We have the following simple but important proposition.

Proposition 40.3. *There exists some $\mu = (\mu_1, \ldots, \mu_m)$ and some $u \in U$ such that*

$$dJ(u) + \mu_1 d\varphi_1(u) + \cdots + \mu_m d\varphi_m(u) = 0$$

if and only if

$$dL(u, \mu) = 0,$$

or equivalently

$$\nabla L(u, \mu) = 0;$$

that is, iff (u, μ) is a critical point of the Lagrangian L.

Proof. Indeed $dL(u, \mu) = 0$ is equivalent to

$$\begin{aligned} \frac{\partial L}{\partial v}(u, \mu) &= 0 \\ \frac{\partial L}{\partial \lambda_1}(u, \mu) &= 0 \\ &\vdots \\ \frac{\partial L}{\partial \lambda_m}(u, \mu) &= 0, \end{aligned}$$

and since

$$\frac{\partial L}{\partial v}(u, \mu) = dJ(u) + \mu_1 d\varphi_1(u) + \cdots + \mu_m d\varphi_m(u)$$

and

$$\frac{\partial L}{\partial \lambda_i}(u, \mu) = \varphi_i(u),$$

we get

$$dJ(u) + \mu_1 d\varphi_1(u) + \cdots + \mu_m d\varphi_m(u) = 0$$

and

$$\varphi_1(u) = \cdots = \varphi_m(u) = 0,$$

that is, $u \in U$. The converse is proven in a similar fashion (essentially by reversing the argument). $\square$

If we write out explicitly the condition

$$dJ(u) + \mu_1 d\varphi_1(u) + \cdots + \mu_m d\varphi_m(u) = 0,$$