and Theorem 11.12 shows that

$$\mathrm{rk}(f) = \mathrm{rk}(f^\top).$$

It is instructive to translate these relations in terms of matrices (actually, certain linear algebra books make a big deal about this!). If $\dim(E) = n$ and $\dim(F) = m$, given any basis $(u_1, \ldots, u_n)$ of E and a basis $(v_1, \ldots, v_m)$ of F, we know that f is represented by an $m \times n$ matrix $A = (a_{i\,j})$, where the jth column of A is equal to $f(u_j)$ over the basis $(v_1, \ldots, v_m)$. Furthermore, the transpose map $f^\top$ is represented by the $n \times m$ matrix $A^\top$ (with respect to the dual bases). Consequently, the four fundamental spaces

$$\mathrm{Im}\, f, \; \mathrm{Im}\, f^\top, \; \mathrm{Ker}\, f, \; \mathrm{Ker}\, f^\top$$

correspond to

(1) The *column space* of A, denoted by $\mathrm{Im}\, A$ or $\mathcal{R}(A)$; this is the subspace of $\mathbb{R}^m$ spanned by the columns of A, which corresponds to the image $\mathrm{Im}\, f$ of f.

(2) The *kernel* or *nullspace* of A, denoted by $\mathrm{Ker}\, A$ or $\mathcal{N}(A)$; this is the subspace of $\mathbb{R}^n$ consisting of all vectors $x \in \mathbb{R}^n$ such that $Ax = 0$.

(3) The *row space* of A, denoted by $\mathrm{Im}\, A^\top$ or $\mathcal{R}(A^\top)$; this is the subspace of $\mathbb{R}^n$ spanned by the rows of A, or equivalently, spanned by the columns of $A^\top$, which corresponds to the image $\mathrm{Im}\, f^\top$ of $f^\top$.

(4) The *left kernel* or *left nullspace* of A denoted by $\mathrm{Ker}\, A^\top$ or $\mathcal{N}(A^\top)$; this is the kernel (nullspace) of $A^\top$, the subspace of $\mathbb{R}^m$ consisting of all vectors $y \in \mathbb{R}^m$ such that $A^\top y = 0$, or equivalently, $y^\top A = 0$.

Recall that the dimension r of $\mathrm{Im}\, f$, which is also equal to the dimension of the column space $\mathrm{Im}\, A = \mathcal{R}(A)$, is the *rank* of A (and f). Then, some our previous results can be reformulated as follows:

1. The column space $\mathcal{R}(A)$ of A has dimension r.
2. The nullspace $\mathcal{N}(A)$ of A has dimension $n - r$.
3. The row space $\mathcal{R}(A^\top)$ has dimension r.
4. The left nullspace $\mathcal{N}(A^\top)$ of A has dimension $m - r$.

The above statements constitute what Strang calls the *Fundamental Theorem of Linear Algebra, Part I* (see Strang [170]).

The two statements

$$\begin{aligned} \mathrm{Ker}\, f &= (\mathrm{Im}\, f^\top)^0 \\ \mathrm{Ker}\, f^\top &= (\mathrm{Im}\, f)^0 \end{aligned}$$

translate to