This holds for $k = 1$, since $A(1:1, 1:1) = (a_{1\,1})$, so $a_{1\,1} \neq 0$. Assume that no pivoting was necessary for the first $k-1$ steps ($2 \leq k \leq n-1$). In this case, we have

$$E_{k-1} \cdots E_2 E_1 A = A_k,$$

where $L = E_{k-1} \cdots E_2 E_1$ is a unit lower-triangular matrix and $A_k(1:k, 1:k)$ is upper-triangular, so that $LA = A_k$ can be written as

$$\begin{pmatrix} L_1 & 0 \\ L_3 & L_4 \end{pmatrix} \begin{pmatrix} A(1:k, 1:k) & A_2 \\ A_3 & A_4 \end{pmatrix} = \begin{pmatrix} U_1 & B_2 \\ 0 & B_4 \end{pmatrix},$$

where L_1 is unit lower-triangular and U_1 is upper-triangular. (Once again $A(1:k, 1:k)$, L_1, and U_1 are $k \times k$ matrices; A_2 and B_2 are $k \times (n-k)$ matrices; A_3 and L_3 are $(n-k) \times k$ matrices; A_4, L_4, and B_4 are $(n-k) \times (n-k)$ matrices.) But then,

$$L_1 A(1:k, 1:k)) = U_1,$$

where L_1 is invertible (in fact, $\det(L_1) = 1$), and since by hypothesis $A(1:k, 1:k)$ is invertible, U_1 is also invertible, which implies that $(U_1)_{kk} \neq 0$, since U_1 is upper-triangular. Therefore, no pivoting is needed in Step k, establishing the induction step. Since $\det(L_1) = 1$, we also have

$$\det(U_1) = \det(L_1 A(1:k, 1:k)) = \det(L_1)\det(A(1:k, 1:k)) = \det(A(1:k, 1:k)),$$

and since U_1 is upper-triangular and has the pivots $\pi_1, \ldots, \pi_k$ on its diagonal, we get

$$\det(A(1:k, 1:k)) = \pi_1 \pi_2 \cdots \pi_k, \quad k = 1, \ldots, n,$$

as claimed. $\square$

Remark: The use of determinants in the first part of the proof of Proposition 8.2 can be avoided if we use the fact that a triangular matrix is invertible iff all its diagonal entries are nonzero.

Corollary 8.3. *(LU-Factorization) Let A be an invertible $n \times n$-matrix. If every matrix $A(1:k, 1:k)$ is invertible for $k = 1, \ldots, n$, then Gaussian elimination requires no pivoting and yields an LU-factorization $A = LU$.*

Proof. We proved in Proposition 8.2 that in this case Gaussian elimination requires no pivoting. Then since every elementary matrix $E_{i,k;\beta}$ is lower-triangular (since we always arrange that the pivot π_k occurs above the rows that it operates on), since $E_{i,k;\beta}^{-1} = E_{i,k;-\beta}$ and the $E_k s$ are products of $E_{i,k;\beta_{i,k}} s$, from

$$E_{n-1} \cdots E_2 E_1 A = U,$$

where U is an upper-triangular matrix, we get

$$A = LU,$$

where $L = E_1^{-1} E_2^{-1} \cdots E_{n-1}^{-1}$ is a lower-triangular matrix. Furthermore, as the diagonal entries of each $E_{i,k;\beta}$ are 1, the diagonal entries of each E_k are also 1. $\square$