Proof. Since A is symmetric, each $A(1:k, 1:k)$ is also symmetric. If $w \in \mathbb{R}^k$, with $1 \leq k \leq n$, we let $x \in \mathbb{R}^n$ be the vector with $x_i = w_i$ for $i = 1, \ldots, k$ and $x_i = 0$ for $i = k+1, \ldots, n$. Now since A is symmetric positive definite, we have $x^\top A x > 0$ for all $x \in \mathbb{R}^n$ with $x \neq 0$. This holds in particular for all vectors x obtained from nonzero vectors $w \in \mathbb{R}^k$ as defined earlier, and clearly

$$x^\top A x = w^\top A(1:k, 1:k)\, w,$$

which implies that $A(1:k, 1:k)$ is symmetric positive definite. Thus, by Fact 1 above, $A(1:k, 1:k)$ is also invertible. $\square$

Proposition 8.9 also holds for a complex Hermitian positive definite matrix. Proposition 8.9 can be strengthened as follows: *A real (resp. complex) matrix A is symmetric (resp. Hermitian) positive definite iff* $\det(A(1:k, 1:k)) > 0$ *for* $k = 1, \ldots, n$.

The above fact is known as *Sylvester's criterion*. We will prove it after establishing the Cholesky factorization.

Let A be an $n \times n$ real symmetric positive definite matrix and write

$$A = \begin{pmatrix} a_{1\,1} & W^\top \\ W & C \end{pmatrix},$$

where C is an $(n-1) \times (n-1)$ symmetric matrix and W is an $(n-1) \times 1$ matrix. Since A is symmetric positive definite, $a_{1\,1} > 0$, and we can compute $\alpha = \sqrt{a_{1\,1}}$. The trick is that we can factor A uniquely as

$$A = \begin{pmatrix} a_{1\,1} & W^\top \\ W & C \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ W/\alpha & I \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & C - WW^\top/a_{1\,1} \end{pmatrix} \begin{pmatrix} \alpha & W^\top/\alpha \\ 0 & I \end{pmatrix},$$

i.e., as $A = B_1 A_1 B_1^\top$, where B_1 is lower-triangular with positive diagonal entries. Thus, B_1 is invertible, and by Fact (3) above, A_1 is also symmetric positive definite.

Remark: The matrix $C - WW^\top/a_{1\,1}$ is known as the *Schur complement* of the 1×1 matrix (a_{11}) in A.

Theorem 8.10. *(Cholesky factorization) Let A be a real symmetric positive definite matrix. Then there is some real lower-triangular matrix B so that $A = BB^\top$. Furthermore, B can be chosen so that its diagonal elements are strictly positive, in which case B is unique.*

Proof. We proceed by induction on the dimension n of A. For $n = 1$, we must have $a_{1\,1} > 0$, and if we let $\alpha = \sqrt{a_{1\,1}}$ and $B = (\alpha)$, the theorem holds trivially. If $n \geq 2$, as we explained above, again we must have $a_{1\,1} > 0$, and we can write

$$A = \begin{pmatrix} a_{1\,1} & W^\top \\ W & C \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ W/\alpha & I \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & C - WW^\top/a_{1\,1} \end{pmatrix} \begin{pmatrix} \alpha & W^\top/\alpha \\ 0 & I \end{pmatrix} = B_1 A_1 B_1^\top,$$