Problem 15.5. Find the eigenvalues of the matrices

$$A = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}, \quad C = A + B = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}.$$

Check that the eigenvalues of $A + B$ are not equal to the sums of eigenvalues of A plus eigenvalues of B.

Problem 15.6. Let A be a real symmetric $n \times n$ matrix and B be a real symmetric positive definite $n \times n$ matrix. We would like to solve the *generalized eigenvalue problem*: find $\lambda \in \mathbb{R}$ and $u \neq 0$ such that

$$Au = \lambda Bu. \tag{$*$}$$

(1) Use the Cholesky decomposition $B = CC^{\top}$ to show that λ and u are solutions of the generalized eigenvalue problem $(*)$ iff λ and v are solutions of the (ordinary) eigenvalue problem

$$C^{-1}A(C^{\top})^{-1}v = \lambda v, \quad \text{with } v = C^{\top}u.$$

Check that $C^{-1}A(C^{\top})^{-1}$ is symmetric.

(2) Prove that if $Au_1 = \lambda_1 Bu_1$, $Au_2 = \lambda_2 Bu_2$, with $u_1 \neq 0$, $u_2 \neq 0$ and $\lambda_1 \neq \lambda_2$, then $u_1^{\top} Bu_2 = 0$.

(3) Prove that $B^{-1}A$ and $C^{-1}A(C^{\top})^{-1}$ have the same eigenvalues.

Problem 15.7. The sequence of *Fibonacci numbers*, $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$, is given by the recurrence

$$F_{n+2} = F_{n+1} + F_n,$$

with $F_0 = 0$ and $F_1 = 1$. In matrix form, we can write

$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix}, \quad n \geq 1, \quad \begin{pmatrix} F_1 \\ F_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(1) Show that

$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(2) Prove that the eigenvalues of the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

are

$$\lambda = \frac{1 \pm \sqrt{5}}{2}.$$

The number

$$\varphi = \frac{1 + \sqrt{5}}{2}$$