

where  $R$  is upper triangular and every  $H_i$  is either a Householder matrix or the identity. However,  $h_i \circ h_i = \text{id}$  for all  $i$ ,  $1 \leq i \leq n$ , and so

$$v_j = h_1 \circ h_2 \circ \cdots \circ h_n(r_j)$$

for all  $j$ ,  $1 \leq j \leq n$ . But  $\rho = h_1 \circ h_2 \circ \cdots \circ h_n$  is an isometry represented by the orthogonal matrix  $Q = H_1 H_2 \cdots H_n$ . It is clear that  $A = QR$ , where  $R$  is upper triangular. As we noted in Proposition 13.3, the diagonal entries of  $R$  can be chosen to be nonnegative.  $\square$

### Remarks:

(1) Letting

$$A_{k+1} = H_k \cdots H_2 H_1 A,$$

with  $A_1 = A$ ,  $1 \leq k \leq n$ , the proof of Proposition 13.3 can be interpreted in terms of the computation of the sequence of matrices  $A_1, \dots, A_{n+1} = R$ . The matrix  $A_{k+1}$  has the shape

$$A_{k+1} = \begin{pmatrix} \times & \times & \times & u_1^{k+1} & \times & \times & \times & \times \\ 0 & \times & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \times & u_k^{k+1} & \times & \times & \times & \times \\ 0 & 0 & 0 & u_{k+1}^{k+1} & \times & \times & \times & \times \\ 0 & 0 & 0 & u_{k+2}^{k+1} & \times & \times & \times & \times \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & u_{n-1}^{k+1} & \times & \times & \times & \times \\ 0 & 0 & 0 & u_n^{k+1} & \times & \times & \times & \times \end{pmatrix},$$

where the  $(k+1)$ th column of the matrix is the vector

$$u_{k+1} = h_k \circ \cdots \circ h_2 \circ h_1(v_{k+1}),$$

and thus

$$u'_{k+1} = (u_1^{k+1}, \dots, u_k^{k+1})$$

and

$$u''_{k+1} = (u_{k+1}^{k+1}, u_{k+2}^{k+1}, \dots, u_n^{k+1}).$$

If the last  $n - k - 1$  entries in column  $k+1$  are all zero, there is nothing to do, and we let  $H_{k+1} = I$ . Otherwise, we kill these  $n - k - 1$  entries by multiplying  $A_{k+1}$  on the left by the Householder matrix  $H_{k+1}$  sending

$$(0, \dots, 0, u_{k+1}^{k+1}, \dots, u_n^{k+1}) \quad \text{to} \quad (0, \dots, 0, r_{k+1,k+1}, 0, \dots, 0),$$

where  $r_{k+1,k+1} = \|(u_{k+1}^{k+1}, \dots, u_n^{k+1})\|$ .