or

$$J(h)(a) = \begin{pmatrix} \frac{\partial g_1}{\partial y_1}(b) & \frac{\partial g_1}{\partial y_2}(b) & \dots & \frac{\partial g_1}{\partial y_n}(b) \\ \frac{\partial g_2}{\partial y_1}(b) & \frac{\partial g_2}{\partial y_2}(b) & \dots & \frac{\partial g_2}{\partial y_n}(b) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial y_1}(b) & \frac{\partial g_m}{\partial y_2}(b) & \dots & \frac{\partial g_m}{\partial y_n}(b) \end{pmatrix} \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \frac{\partial f_1}{\partial x_2}(a) & \dots & \frac{\partial f_1}{\partial x_p}(a) \\ \frac{\partial f_2}{\partial x_1}(a) & \frac{\partial f_2}{\partial x_2}(a) & \dots & \frac{\partial f_2}{\partial x_p}(a) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(a) & \frac{\partial f_n}{\partial x_2}(a) & \dots & \frac{\partial f_n}{\partial x_p}(a) \end{pmatrix}.$$

Thus, we have the familiar formula

$$\frac{\partial h_i}{\partial x_j}(a) = \sum_{k=1}^{k=n} \frac{\partial g_i}{\partial y_k}(b) \frac{\partial f_k}{\partial x_j}(a).$$

Given two normed affine spaces E and F of finite dimension, given an open subset A of E, if a function $f\colon A \to F$ is differentiable at $a \in A$, then its Jacobian matrix is well defined.

One should be warned that the converse is false. There are functions such that all the partial derivatives exist at some $a \in A$, but yet, the function is not differentiable at a, and not even continuous at a. For example, consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$, defined such that $f(0,0) = 0$, and

$$f(x,y) = \frac{x^2 y}{x^4 + y^2} \qquad \text{if } (x,y) \neq (0,0).$$

For any $u \neq 0$, letting $u = \begin{pmatrix} h \\ k \end{pmatrix}$, we have

$$\frac{f(0 + tu) - f(0)}{t} = \frac{h^2 k}{t^2 h^4 + k^2},$$

so that

$$\mathrm{D}_u f(0,0) = \begin{cases} \frac{h^2}{k} & \text{if } k \neq 0 \\ 0 & \text{if } k = 0. \end{cases}$$

Thus, $\mathrm{D}_u f(0,0)$ exists for all $u \neq 0$. On the other hand, if $\mathrm{D}f(0,0)$ existed, it would be a linear map $\mathrm{D}f(0,0)\colon \mathbb{R}^2 \to \mathbb{R}$ represented by a row matrix $(\alpha \;\; \beta)$, and we would have $\mathrm{D}_u f(0,0) = \mathrm{D}f(0,0)(u) = \alpha h + \beta k$, but the explicit formula for $\mathrm{D}_u f(0,0)$ is not linear. As a matter of fact, the function f is not continuous at $(0,0)$. For example, on the parabola $y = x^2$, $f(x,y) = \frac{1}{2}$, and when we approach the origin on this parabola, the limit is $\frac{1}{2}$, when in fact, $f(0,0) = 0$.

However, there are sufficient conditions on the partial derivatives for $\mathrm{D}f(a)$ to exist, namely, continuity of the partial derivatives.