and the map $x \mapsto \|x\|_p$ is not convex.

For $p = 0$, for any $x \in \mathbb{R}^n$, we have

$$\|x\|_0 = |\{i \in \{1, \ldots, n\} \mid x_i \neq 0\}|,$$

the number of nonzero components of x. The map $x \mapsto \|x\|_0$ is not a norm this time because Axiom (N2) fails. For example,

$$\|(1,0)\|_0 = \|(10,0)\|_0 = 1 \neq 10 = 10 \, \|(1,0)\|_0 \, .$$

The map $x \mapsto \|x\|_0$ is also not convex. For example,

$$\|(1/2)(2,2)\|_0 = \|(1,1)\|_0 = 2,$$

and

$$\|(2,0)\|_0 = \|(0,2)\|_0 = 1,$$

but

$$\|(1/2)(2,2)\|_0 = 2 > 1 = (1/2)\, \|(2,0)\|_0 + (1/2)\, \|(0,2)\|_0 \, .$$

Nevertheless, the "zero-norm" $x \mapsto \|x\|_0$ is used in machine learning as a regularizing term which encourages sparsity, namely increases the number of zero components of the vector x.

The following proposition is easy to show.

Proposition 9.3. *The following inequalities hold for all $x \in \mathbb{R}^n$ (or $x \in \mathbb{C}^n$):*

$$\begin{aligned}
\|x\|_\infty &\leq \|x\|_1 \leq n\|x\|_\infty, \\
\|x\|_\infty &\leq \|x\|_2 \leq \sqrt{n}\|x\|_\infty, \\
\|x\|_2 &\leq \|x\|_1 \leq \sqrt{n}\|x\|_2.
\end{aligned}$$

Proposition 9.3 is actually a special case of a very important result: *in a finite-dimensional vector space, any two norms are equivalent.*

Definition 9.2. Given any (real or complex) vector space E, two norms $\| \, \|_a$ and $\| \, \|_b$ are *equivalent* iff there exists some positive reals $C_1, C_2 > 0$, such that

$$\|u\|_a \leq C_1 \, \|u\|_b \quad \text{and} \quad \|u\|_b \leq C_2 \, \|u\|_a \, , \text{ for all } u \in E.$$

There is an illuminating interpretation of Definition 9.2 in terms of open balls. For any radius $\rho > 0$ and any $x \in E$, consider the open a-ball of center x and radius ρ (with respect the norm $\| \, \|_a$),

$$B_a(x, \rho) = \{z \in E \mid \|z - x\|_a < \rho\}.$$