Chapter 46

The Simplex Algorithm

46.1 The Idea Behind the Simplex Algorithm

The simplex algorithm, due to Dantzig, applies to a linear program (P) in standard form, where the constraints are given by $Ax = b$ and $x \geq 0$, with A a $m \times n$ matrix of rank m, and with an objective function $x \mapsto cx$. This algorithm either reports that (P) has no feasible solution, or that (P) is unbounded, or yields an optimal solution. Geometrically, the algorithm climbs from vertex to vertex in the polyhedron $\mathcal{P}(A, b)$, trying to improve the value of the objective function. Since vertices correspond to basic feasible solutions, the simplex algorithm actually works with basic feasible solutions.

Recall that a basic feasible solution x is a feasible solution for which there is a subset $K \subseteq \{1, \ldots, n\}$ of size m such that the matrix A_K consisting of the columns of A whose indices belong to K are linearly independent, and that $x_j = 0$ for all $j \notin K$. We also let $J_>(x)$ be the set of indices

$$J_>(x) = \{j \in \{1, \ldots, n\} \mid x_j > 0\},$$

so for a basic feasible solution x associated with K, we have $J_>(x) \subseteq K$. In fact, by Proposition 45.2, a feasible solution x is a basic feasible solution iff the columns of $A_{J_>(x)}$ are linearly independent.

If $J_>(x)$ had cardinality m for all basic feasible solutions x, then the simplex algorithm would make progress at every step, in the sense that it would strictly increase the value of the objective function. Unfortunately, it is possible that $|J_>(x)| < m$ for certain basic feasible solutions, and in this case a step of the simplex algorithm may not increase the value of the objective function. Worse, in rare cases, it is possible that the algorithm enters an infinite loop. This phenomenon called *cycling* can be detected, but in this case the algorithm fails to give a conclusive answer.

Fortunately, there are ways of preventing the simplex algorithm from cycling (for example, Bland's rule discussed later), although proving that these rules work correctly is quite involved.