for every j, $1 \leq j \leq p$; in matrix form, we have

$$\begin{array}{c} \\ w_1 \\ w_2 \\ \vdots \\ w_m \end{array} \begin{array}{c} \begin{array}{cccc} f(v_1) & f(v_2) & \dots & f(v_n) \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \end{array}$$

and

$$\begin{array}{c} \\ v_1 \\ v_2 \\ \vdots \\ v_n \end{array} \begin{array}{c} \begin{array}{cccc} g(u_1) & g(u_2) & \dots & g(u_p) \end{array} \\ \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{pmatrix} \end{array}.$$

By previous considerations, for every

$$x = x_1 u_1 + \cdots + x_p u_p,$$

letting $g(x) = y = y_1 v_1 + \cdots + y_n v_n$, we have

$$y_k = \sum_{j=1}^{p} b_{k\,j} x_j \tag{2}$$

for all k, $1 \leq k \leq n$, and for every

$$y = y_1 v_1 + \cdots + y_n v_n,$$

letting $f(y) = z = z_1 w_1 + \cdots + z_m w_m$, we have

$$z_i = \sum_{k=1}^{n} a_{i\,k} y_k \tag{3}$$

for all i, $1 \leq i \leq m$. Then if $y = g(x)$ and $z = f(y)$, we have $z = f(g(x))$, and in view of (2) and (3), we have

$$\begin{aligned} z_i &= \sum_{k=1}^{n} a_{i\,k} (\sum_{j=1}^{p} b_{k\,j} x_j) \\ &= \sum_{k=1}^{n} \sum_{j=1}^{p} a_{i\,k} b_{k\,j} x_j \\ &= \sum_{j=1}^{p} \sum_{k=1}^{n} a_{i\,k} b_{k\,j} x_j \\ &= \sum_{j=1}^{p} (\sum_{k=1}^{n} a_{i\,k} b_{k\,j}) x_j. \end{aligned}$$