Suppose that A is an $n \times n$ matrix and that we are trying to solve the linear system

$$Ax = b,$$

with $b \in \mathbb{R}^n$. Suppose we can find an $n \times n$ matrix B such that

$$BA^i = e_i, \quad i = 1, \ldots, n,$$

with $e_i = (0, \ldots, 0, 1, 0 \ldots, 0)$, where the only nonzero entry is 1 in the ith slot. If we form the $n \times n$ matrix

$$I_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

called the *identity matrix*, whose ith column is e_i, then the above is equivalent to

$$BA = I_n.$$

If $Ax = b$, then multiplying both sides on the left by B, we get

$$B(Ax) = Bb.$$

But is is easy to see that $B(Ax) = (BA)x = I_n x = x$, so we must have

$$x = Bb.$$

We can verify that $x = Bb$ is indeed a solution, because it can be shown that

$$A(Bb) = (AB)b = I_n b = b.$$

What is not obvious is that $BA = I_n$ implies $AB = I_n$, but this is indeed provable. The matrix B is usually denoted A^{-1} and called the *inverse* of A. It can be shown that it is the unique matrix such that

$$AA^{-1} = A^{-1}A = I_n.$$

If a square matrix A has an inverse, then we say that it is *invertible* or *nonsingular*, otherwise we say that it is *singular*. We will show later that a square matrix is invertible iff its columns are linearly independent iff its determinant is nonzero.

In summary, if A is a square invertible matrix, then the linear system $Ax = b$ has the unique solution $x = A^{-1}b$. In practice, this is not a good way to solve a linear system because computing A^{-1} is too expensive. A practical method for solving a linear system is Gaussian elimination, discussed in Chapter 8. Other practical methods for solving a linear system