which shows that $z = \psi(y)$ for some $y \in E[X]$.

Finally, we prove that ψ is injective as follows. We have

$$\begin{aligned}\psi(z) &= \psi\Big(\sum_k X^k \otimes u_k\Big)\\ &= (X1 - \overline{f})\Big(\sum_k X^k \otimes u_k\Big)\\ &= \sum_k X^{k+1} \otimes (u_k - f(u_{k+1})),\end{aligned}$$

where (u_k) is a family of finite support of $u_k \in E$. If $\psi(z) = 0$, then

$$\sum_k X^{k+1} \otimes (u_k - f(u_{k+1})) = 0,$$

and because the X^k form a basis of $A[X]$, we must have

$$u_k - f(u_{k+1}) = 0, \quad \text{for all } k.$$

Since (u_k) has finite support, there is a largest k, say $m+1$ so that $u_{m+1} = 0$, and then from

$$u_k = f(u_{k+1}),$$

we deduce that $u_k = 0$ for all k. Therefore, $z = 0$, and ψ is injective. $\square$

Remark: The exact sequence of Theorem 36.3 yields a *presentation* of M_f.

Since $A[X]$ is a free A-module, $A[X] \otimes_A M$ is a free A-module, but $A[X] \otimes_A M$ is generally not a free $A[X]$-module. However, if M is a free module, then $M[X]$ is a free $A[X]$-module, since if $(u_i)_{i \in I}$ is a basis for M, then $(1 \otimes u_i)_{i \in I}$ is a basis for $M[X]$. This allows us to define the characterisctic polynomial $\chi_f(X)$ of an endomorphism of a free module M as

$$\chi_f(X) = \det(X1 - \overline{f}).$$

Note that to have a correct definition, we need to define the determinant of a linear map allowing the indeterminate X as a scalar, and this is what the definition of $M[X]$ achieves (among other things). Theorem 36.3 can be used to give a short proof of the Cayley-Hamilton Theorem, see Bourbaki [25] (Chapter III, Section 8, Proposition 20). Proposition 7.10 is still the crucial ingredient of the proof.