For example, if $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 0 & 1 \end{pmatrix}$, then A has rank 2 and since $m \geq n$, $A^+ = (A^\top A)^{-1} A^\top$ where

$$A^+ = \begin{pmatrix} 5 & 8 \\ 8 & 14 \end{pmatrix}^{-1} A^\top = \begin{pmatrix} 7/3 & -4/3 \\ 4/3 & 5/6 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} -1/3 & 2/3 & -4/3 \\ 1/3 & -1/6 & 5/6 \end{pmatrix}.$$

If $A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & -1 \end{pmatrix}$, since A has rank 2 and $n \geq m$, then $A^+ = A^\top (AA^\top)^{-1}$ where

$$A^+ = A^\top \begin{pmatrix} 14 & 5 \\ 5 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 3/17 & -5/17 \\ -5/17 & 14/17 \end{pmatrix} = \begin{pmatrix} 3/17 & -5/17 \\ 1/17 & 4/17 \\ 4/17 & -1/17 \\ 5/17 & -14/17 \end{pmatrix}.$$

Let $A = V\Sigma U^\top$ be an SVD for any $m \times n$ matrix A. It is easy to check that both AA^+ and A^+A are symmetric matrices. In fact,

$$AA^+ = V\Sigma U^\top U\Sigma^+ V^\top = V\Sigma\Sigma^+ V^\top = V \begin{pmatrix} I_r & 0 \\ 0 & 0_{m-r} \end{pmatrix} V^\top$$

and

$$A^+A = U\Sigma^+ V^\top V\Sigma U^\top = U\Sigma^+\Sigma U^\top = U \begin{pmatrix} I_r & 0 \\ 0 & 0_{n-r} \end{pmatrix} U^\top.$$

From the above expressions we immediately deduce that

$$\begin{aligned} AA^+A &= A, \\ A^+AA^+ &= A^+, \end{aligned}$$

and that

$$\begin{aligned} (AA^+)^2 &= AA^+, \\ (A^+A)^2 &= A^+A, \end{aligned}$$

so both AA^+ and A^+A are orthogonal projections (since they are both symmetric).

Proposition 23.4. *The matrix AA^+ is the orthogonal projection onto the range of A and A^+A is the orthogonal projection onto* $\mathrm{Ker}(A)^\perp = \mathrm{Im}(A^\top)$*, the range of $A^\top$.*

Proof. Obviously, we have $\mathrm{range}(AA^+) \subseteq \mathrm{range}(A)$, and for any $y = Ax \in \mathrm{range}(A)$, since $AA^+A = A$, we have

$$AA^+y = AA^+Ax = Ax = y,$$