Definition 3.1. Given a field K (with addition $+$ and multiplication $*$), a *vector space over* K (or K*-vector space*) is a set E (of vectors) together with two operations $+\colon E \times E \to E$ (called *vector addition*),[1] and $\cdot\colon K \times E \to E$ (called *scalar multiplication*) satisfying the following conditions for all $\alpha, \beta \in K$ and all $u, v \in E$;

(V0) E is an abelian group w.r.t. $+$, with identity element 0;[2]

(V1) $\alpha \cdot (u + v) = (\alpha \cdot u) + (\alpha \cdot v)$;

(V2) $(\alpha + \beta) \cdot u = (\alpha \cdot u) + (\beta \cdot u)$;

(V3) $(\alpha * \beta) \cdot u = \alpha \cdot (\beta \cdot u)$;

(V4) $1 \cdot u = u$.

In (V3), $*$ denotes multiplication in the field K.

Given $\alpha \in K$ and $v \in E$, the element $\alpha \cdot v$ is also denoted by αv. The field K is often called the field of scalars.

Unless specified otherwise or unless we are dealing with several different fields, in the rest of this chapter, we assume that all K-vector spaces are defined with respect to a fixed field K. Thus, we will refer to a K-vector space simply as a vector space. In most cases, the field K will be the field $\mathbb{R}$ of reals.

From (V0), a vector space always contains the null vector 0, and thus is nonempty. From (V1), we get $\alpha \cdot 0 = 0$, and $\alpha \cdot (-v) = -(\alpha \cdot v)$. From (V2), we get $0 \cdot v = 0$, and $(-\alpha) \cdot v = -(\alpha \cdot v)$.

Another important consequence of the axioms is the following fact:

Proposition 3.1. *For any $u \in E$ and any $\lambda \in K$, if $\lambda \neq 0$ and $\lambda \cdot u = 0$, then $u = 0$.*

Proof. Indeed, since $\lambda \neq 0$, it has a multiplicative inverse λ^{-1}, so from $\lambda \cdot u = 0$, we get

$$\lambda^{-1} \cdot (\lambda \cdot u) = \lambda^{-1} \cdot 0.$$

However, we just observed that $\lambda^{-1} \cdot 0 = 0$, and from (V3) and (V4), we have

$$\lambda^{-1} \cdot (\lambda \cdot u) = (\lambda^{-1}\lambda) \cdot u = 1 \cdot u = u,$$

and we deduce that $u = 0$. ∎

[1] The symbol $+$ is overloaded, since it denotes both addition in the field K and addition of vectors in E. It is usually clear from the context which $+$ is intended.

[2] The symbol 0 is also overloaded, since it represents both the zero in K (a scalar) and the identity element of E (the zero vector). Confusion rarely arises, but one may prefer using $\mathbf{0}$ for the zero vector.