$(a_n)_{n\geq 1}$ of elements $a_n \in E$ is a *Cauchy sequence* iff for every $\epsilon > 0$, there is some $N \geq 1$ such that

$$d(a_m, a_n) < \epsilon \quad \text{for all} \quad m, n \geq N.$$

We say that E is *complete* iff every Cauchy sequence converges to a limit (which is unique, since a metric space is Hausdorff).

Every finite dimensional vector space over $\mathbb{R}$ *or* $\mathbb{C}$ *is complete.* For example, one can show by induction that given any basis $(e_1, \ldots, e_n)$ of E, the linear map $h\colon \mathbb{C}^n \to E$ defined such that

$$h((z_1, \ldots, z_n)) = z_1 e_1 + \cdots + z_n e_n$$

is a homeomorphism (using the *sup*-norm on $\mathbb{C}^n$). One can also use the fact that any two norms on a finite dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$ are equivalent (see Chapter 9, or Lang [112], Dixmier [51], Schwartz [150]).

However, if E has infinite dimension, it may not be complete. When a Hermitian space is complete, a number of the properties that hold for finite dimensional Hermitian spaces also hold for infinite dimensional spaces. For example, any closed subspace has an orthogonal complement, and in particular, a finite dimensional subspace has an orthogonal complement. Hermitian spaces that are also complete play an important role in analysis. Since they were first studied by Hilbert, they are called Hilbert spaces.

Definition 48.1. A (complex) Hermitian space $\langle E, \varphi\rangle$ which is a complete normed vector space under the norm $\| \, \|$ induced by φ is called a *Hilbert space*. A real Euclidean space $\langle E, \varphi\rangle$ which is complete under the norm $\| \, \|$ induced by φ is called a *real Hilbert space*.

All the results in this section hold for complex Hilbert spaces as well as for real Hilbert spaces. We state all results for the complex case only, since they also apply to the real case, and since the proofs in the complex case need a little more care.

Example 48.1. The space ℓ^2 of all countably infinite sequences $x = (x_i)_{i\in\mathbb{N}}$ of complex numbers such that $\sum_{i=0}^{\infty} |x_i|^2 < \infty$ is a Hilbert space. It will be shown later that the map $\varphi\colon \ell^2 \times \ell^2 \to \mathbb{C}$ defined such that

$$\varphi\left((x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}\right) = \sum_{i=0}^{\infty} x_i \overline{y_i}$$

is well defined, and that ℓ^2 is a Hilbert space under φ. In fact, we will prove a more general result (Proposition A.3).

Example 48.2. The set $\mathcal{C}^{\infty}[a, b]$ of smooth functions $f\colon [a, b] \to \mathbb{C}$ is a Hermitian space under the Hermitian form

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}dx,$$

but it is not a Hilbert space because it is not complete. It is possible to construct its completion $\mathrm{L}^2([a, b])$, which turns out to be the space of Lebesgue integrable functions on $[a, b]$.