and the matrix

$$\Delta A = \beta I$$

sastisfy the equations

$$\begin{aligned}
Ax &= b \\
(A + \Delta A)(x + \Delta x) &= b \\
\|\Delta x\| &= |\beta| \left\|A^{-1}w\right\| = \|\Delta A\| \left\|A^{-1}\right\| \|x + \Delta x\| .
\end{aligned}$$

Finally we can pick β so that $-\beta$ is not equal to any of the eigenvalues of A, so that $A + \Delta A = A + \beta I$ is invertible and b is is nonzero.

If $\|\Delta A\| < 1/\left\|A^{-1}\right\|$, then

$$\left\|A^{-1}\Delta A\right\| \leq \left\|A^{-1}\right\| \|\Delta A\| < 1,$$

so by Proposition 9.11, the matrix $I + A^{-1}\Delta A$ is invertible and

$$\left\|(I + A^{-1}\Delta A)^{-1}\right\| \leq \frac{1}{1 - \|A^{-1}\Delta A\|} \leq \frac{1}{1 - \|A^{-1}\| \|\Delta A\|}.$$

Recall that we proved earlier that

$$\Delta x = -A^{-1}\Delta A(x + \Delta x),$$

and by adding x to both sides and moving the right-hand side to the left-hand side yields

$$(I + A^{-1}\Delta A)(x + \Delta x) = x,$$

and thus

$$x + \Delta x = (I + A^{-1}\Delta A)^{-1}x,$$

which yields

$$\begin{aligned}
\Delta x &= ((I + A^{-1}\Delta A)^{-1} - I)x = (I + A^{-1}\Delta A)^{-1}(I - (I + A^{-1}\Delta A))x \\
&= -(I + A^{-1}\Delta A)^{-1}A^{-1}(\Delta A)x.
\end{aligned}$$

From this and

$$\left\|(I + A^{-1}\Delta A)^{-1}\right\| \leq \frac{1}{1 - \|A^{-1}\| \|\Delta A\|},$$

we get

$$\|\Delta x\| \leq \frac{\|A^{-1}\| \|\Delta A\|}{1 - \|A^{-1}\| \|\Delta A\|} \|x\| ,$$

which can be written as

$$\frac{\|\Delta x\|}{\|x\|} \leq \mathrm{cond}(A)\frac{\|\Delta A\|}{\|A\|} \left(\frac{1}{1 - \|A^{-1}\| \|\Delta A\|}\right),$$

which is the kind of inequality that we were seeking. $\square$