we have

$$\|u - p_X(u)\|^2 - \|u - v\|^2 = (\|u - p_X(u)\| - \|u - v\|)(\|u - p_X(u)\| + \|u - v\|) \le 0,$$

and since Equation (†) holds for all λ such that $0 < \lambda \le 1$, if $\|u - p_X(u)\|^2 - \|u - v\|^2 < 0$, then for $\lambda > 0$ small enough we have

$$\frac{1}{2\lambda}\left(\|u - p_X(u)\|^2 - \|u - v\|^2\right) + \frac{\lambda}{2}\|z - p_X(u)\|^2 < 0,$$

and if $\|u - p_X(u)\|^2 - \|u - v\|^2 = 0$, then the limit of $\frac{\lambda}{2}\|z - p_X(u)\|^2$ as $\lambda > 0$ goes to zero is zero, so in all cases, by (†), we have

$$\Re \langle u - p_X(u), z - p_X(u)\rangle \le 0.$$

Conversely, assume that $w \in X$ satisfies the condition

$$\Re \langle u - w, z - w\rangle \le 0$$

for all $z \in X$. For all $z \in X$, we have

$$\|u - z\|^2 = \|u - w\|^2 + \|z - w\|^2 - 2\Re \langle u - w, z - w\rangle \ge \|u - w\|^2,$$

which implies that $\|u - w\| = d(u, X) = d$, and from (1), that $w = p_X(u)$.

(3) If X is a subspace of E and $w \in X$, when z ranges over X the vector $z - w$ also ranges over the whole of X so Condition $(*)$ is equivalent to

$$w \in X \quad \text{and} \quad \Re\langle u - w, z\rangle \le 0 \quad \text{for all } z \in X. \tag{$*_1$}$$

Since X is a subspace, if $z \in X$, then $-z \in X$, which implies that $(*_1)$ is equivalent to

$$w \in X \quad \text{and} \quad \Re\langle u - w, z\rangle = 0 \quad \text{for all } z \in X. \tag{$*_2$}$$

Finally, since X is a subspace, if $z \in X$,then $iz \in X$, and this implies that

$$0 = \Re\langle u - w, iz\rangle = -i\Im\langle u - w, z\rangle,$$

so $\Im\langle u - w, z\rangle = 0$, but since we also have $\Re\langle u - w, z\rangle = 0$, we see that $(*_2)$ is equivalent to

$$w \in X \quad \text{and} \quad \langle u - w, z\rangle = 0 \quad \text{for all } z \in X, \tag{$**$}$$

as claimed. $\square$

Definition 48.3. The vector $p_X(u)$ is called the *projection of u onto X*, and the map $p_X \colon E \to X$ is called the *projection of E onto X*.