if $\mu \neq 0$, *then*

$$\begin{aligned} \mu \cdot \langle a, \lambda \rangle &= \langle a, \lambda\mu \rangle, \\ 0 \cdot \langle a, \lambda \rangle &= 0; \end{aligned}$$

and

$$\lambda \cdot u = \lambda u.$$

Furthermore, the map $\omega \colon \widehat{E} \to \mathbb{R}$ *defined such that*

$$\begin{aligned} \omega(\langle a, \lambda \rangle) &= \lambda, \\ \omega(u) &= 0, \end{aligned}$$

is a linear form, $\omega^{-1}(0)$ *is a hyperplane isomorphic to* $\overrightarrow{E}$ *under the injective linear map* $i \colon \overrightarrow{E} \to \widehat{E}$ *such that* $i(u) = t_u$ *(the translation associated with* u*), and* $\omega^{-1}(1)$ *is an affine hyperplane isomorphic to* E *with direction* $i(\overrightarrow{E})$*, under the injective affine map* $j \colon E \to \widehat{E}$*, where* $j(a) = \langle a, 1 \rangle$ *for every* $a \in E$*. Finally, for every* $a \in E$*, we have*

$$\widehat{E} = i\big(\overrightarrow{E}\big) \oplus \mathbb{R}j(a).$$

Proof. The verification that $\widehat{E}$ is a vector space is straightforward. The linear map mapping a vector u to the translation defined by u is clearly an injection $i \colon \overrightarrow{E} \to \widehat{E}$ embedding $\overrightarrow{E}$ as an hyperplane in $\widehat{E}$. It is also clear that ω is a linear form. Note that

$$j(a + u) = \langle a + u, 1 \rangle = \langle a, 1 \rangle \mathbin{\widehat{+}} u,$$

where u stands for the translation associated with the vector u, and thus j is an affine injection with associated linear map i. Thus, $\omega^{-1}(1)$ is indeed an affine hyperplane isomorphic to E with direction $i\big(\overrightarrow{E}\big)$, under the map $j \colon E \to \widehat{E}$. Finally, from the definition of $\widehat{+}$, for every $a \in E$ and every $u \in \overrightarrow{E}$, since

$$i(u) \mathbin{\widehat{+}} \lambda \cdot j(a) = u \mathbin{\widehat{+}} \langle a, \lambda \rangle = \langle a + \lambda^{-1}u, \lambda \rangle,$$

when $\lambda \neq 0$, we get any arbitrary $v \in \widehat{E}$ by picking $\lambda = 0$ and $u = v$, and we get any arbitrary element $\langle b, \mu \rangle$, $\mu \neq 0$, by picking $\lambda = \mu$ and $u = \mu\overrightarrow{ab}$. Thus,

$$\widehat{E} = i\big(\overrightarrow{E}\big) + \mathbb{R}j(a),$$

and since $i\big(\overrightarrow{E}\big) \cap \mathbb{R}j(a) = \{0\}$, we have

$$\widehat{E} = i\big(\overrightarrow{E}\big) \oplus \mathbb{R}j(a),$$

for every $a \in E$. $\square$