We claim that there is some open b-ball $B_b(x, r)$ of radius $r > 0$ and center x,

$$B_b(x, r) = \{z \in E \mid \|z - x\|_n < r\},$$

such that

$$B_b(x, r) \subseteq B_a(x, \rho).$$

Indeed, if we pick $r = \rho/C_1$, for any $z \in E$, if $\|z - x\|_b < \rho/C_1$, then

$$\|z - x\|_a \leq C_1 \|z - x\|_b < C_1(\rho/C_1) = \rho,$$

which means that

$$B_b(x, \rho/C_1) \subseteq B_a(x, \rho).$$

Similarly, for any radius $\rho > 0$ and any $x \in E$, we have

$$B_a(x, \rho/C_2) \subseteq B_b(x, \rho).$$

Now given a normed vector space $(E, \| \|)$, a subset U of E is said to be *open* (with respect to the norm $\| \|$) if either $U = \emptyset$ or if for every $x \in U$, there is some open ball $B(x, \rho)$ (for some $\rho > 0$) such that $B(x, \rho) \subseteq U$.

The collection $\mathcal{U}$ of open sets defined by the norm $\| \|$ is called the *topology on E induced by the norm $\| \|$*. What we showed above regarding the containments of open a-balls and open b-balls immediately implies that *two equivalent norms induce the same topology on E*. This is the reason why the notion of equivalent norms is important.

Given any norm $\| \|$ on a vector space of dimension n, for any basis $(e_1, \ldots, e_n)$ of E, observe that for any vector $x = x_1e_1 + \cdots + x_ne_n$, we have

$$\|x\| = \|x_1e_1 + \cdots + x_ne_n\| \leq |x_1| \|e_1\| + \cdots + |x_n| \|e_n\| \leq C(|x_1| + \cdots + |x_n|) = C \|x\|_1,$$

with $C = \max_{1 \leq i \leq n} \|e_i\|$ and with the norm $\|x\|_1$ defined as

$$\|x\|_1 = \|x_1e_1 + \cdots + x_ne_n\| = |x_1| + \cdots + |x_n|.$$

The above implies that

$$| \|u\| - \|v\| | \leq \|u - v\| \leq C \|u - v\|_1,$$

and this implies the following corollary.

Corollary 9.4. *For any norm $u \mapsto \|u\|$ on a finite-dimensional (complex or real) vector space E, the map $u \mapsto \|u\|$ is continuous with respect to the norm $\| \|_1$.*