We give a brief introduction to these topics. As a reward, we provide several criteria for testing whether a system of inequalities

$$Ax \leq b, \ x \geq 0$$

has a solution or not in terms of versions of the *Farkas lemma* (see Proposition 50.3 and Proposition 47.4). Then we give a complete proof of the strong duality theorem for linear programming (see Theorem 47.7). We also discuss the complementary slackness conditions and show that they can be exploited to design an algorithm for solving a linear program that uses both the primal problem and its dual. This algorithm known as the *primal dual algorithm*, although not used much nowadays, has been the source of inspiration for a whole class of approximation algorithms also known as primal dual algorithms.

We hope that this chapter and the next three will be a motivation for learning more about linear programming, convex optimization, but also convex geometry. The "bible" in convex optimization is Boyd and Vandenberghe [29], and one of the best sources for convex geometry is Ziegler [195]. This is a rather advanced text, so the reader may want to begin with Gallier [73].

44.2 Affine Subsets, Convex Sets, Affine Hyperplanes, Half-Spaces

We view $\mathbb{R}^n$ as consisting of *column vectors* ($n \times 1$ matrices). As usual, row vectors represent *linear forms*, that is linear maps $\varphi\colon \mathbb{R}^n \to \mathbb{R}$, in the sense that the row vector y (a $1 \times n$ matrix) represents the linear form φ if $\varphi(x) = yx$ for all $x \in \mathbb{R}^n$. We denote the space of linear forms (row vectors) by $(\mathbb{R}^n)^*$.

Recall that a *linear combination* of vectors in $\mathbb{R}^n$ is an expression

$$\lambda_1 x_1 + \cdots + \lambda_m x_m$$

where $x_1, \ldots, x_m \in \mathbb{R}^n$ and where $\lambda_1, \ldots, \lambda_m$ are *arbitrary* scalars in $\mathbb{R}$. Given a sequence of vectors $S = (x_1, \ldots, x_m)$ with $x_i \in \mathbb{R}^n$, the set of all linear combinations of the vectors in S is the smallest (linear) subspace containing S called the *linear span* of S, and denoted $\mathrm{span}(S)$. A *linear subspace* of $\mathbb{R}^n$ is any nonempty subset of $\mathbb{R}^n$ closed under linear combinations.

Definition 44.1. An *affine combination* of vectors in $\mathbb{R}^n$ is an expression

$$\lambda_1 x_1 + \cdots + \lambda_m x_m$$

where $x_1, \ldots, x_m \in \mathbb{R}^n$ and where $\lambda_1, \ldots, \lambda_m$ are scalars in $\mathbb{R}$ *satisfying the condition*

$$\lambda_1 + \cdots + \lambda_m = 1.$$

Given a sequence of vectors $S = (x_1, \ldots, x_m)$ with $x_i \in \mathbb{R}^n$, the set of all affine combinations of the vectors in S is the smallest affine subspace containing S called the *affine hull* of S and denoted $\mathrm{aff}(S)$.