Example 50.5. Consider the Linear Program (P)

$$\begin{aligned} &\text{minimize} \quad c^\top v \\ &\text{subject to} \quad Av \leq b, \;\; v \geq 0, \end{aligned}$$

where A is an $m \times n$ matrix. The constraints $v \geq 0$ are rewritten as $-v_i \leq 0$, so we introduce Lagrange multipliers $\mu \in \mathbb{R}^m_+$ and $\nu \in \mathbb{R}^n_+$, and we have the Lagrangian

$$\begin{aligned} L(v, \mu, \nu) &= c^\top v + \mu^\top (Av - b) - \nu^\top v \\ &= -b^\top \mu + (c + A^\top \mu - \nu)^\top v. \end{aligned}$$

The linear function $v \mapsto (c + A^\top \mu - \nu)^\top v$ is unbounded below unless $c + A^\top \mu - \nu = 0$, so the dual function $G(\mu, \nu) = \inf_{v \in \mathbb{R}^n} L(v, \mu, \nu)$ is given for all $\mu \geq 0$ and $\nu \geq 0$ by

$$G(\mu, \nu) = \begin{cases} -b^\top \mu & \text{if } A^\top \mu - \nu + c = 0, \\ -\infty & \text{otherwise.} \end{cases}$$

The domain of G is a proper subset of $\mathbb{R}^m_+ \times \mathbb{R}^n_+$.

Observe that the value $G(\mu, \nu)$ of the function G, when it is defined, is independent of the second argument ν. Since we are interested in maximizing G, this suggests introducing the function $\widehat{G}$ of the single argument μ given by

$$\widehat{G}(\mu) = -b^\top \mu,$$

which is defined for all $\mu \in \mathbb{R}^m_+$.

Of course, $\sup_{\mu \in \mathbb{R}^m_+} \widehat{G}(\mu)$ and $\sup_{(\mu,\nu) \in \mathbb{R}^m_+ \times \mathbb{R}^n_+} G(\mu, \nu)$ are generally different, but note that $\widehat{G}(\mu) = G(\mu, \nu)$ iff there is some $\nu \in \mathbb{R}^n_+$ such that $A^\top \mu - \nu + c = 0$ iff $A^\top \mu + c \geq 0$. Therefore, finding $\sup_{(\mu,\nu) \in \mathbb{R}^m_+ \times \mathbb{R}^n_+} G(\mu, \nu)$ is equivalent to the constrained Problem (D_1)

$$\begin{aligned} &\text{maximize} \quad -b^\top \mu \\ &\text{subject to} \quad A^\top \mu \geq -c, \;\; \mu \geq 0. \end{aligned}$$

The above problem is the dual of the Linear Program (P).

In summary, the dual function G of a primary Problem (P) often contains hidden inequality constraints that define its domain, and sometimes it is possible to make these domain constraints $\psi_1(\mu) \leq 0, \ldots, \psi_p(\mu) \leq 0$ explicit, to define a new function $\widehat{G}$ that depends only on $q < m$ of the variables μ_i and is defined for all values $\mu_i \geq 0$ of these variables, and to replace the Maximization Problem (D), find $\sup_{\mu \in \mathbb{R}^m_+} G(\mu)$, by the constrained Problem (D_1)

$$\begin{aligned} &\text{maximize} \quad \widehat{G}(\mu) \\ &\text{subject to} \quad \psi_i(\mu) \leq 0, \quad i = 1, \ldots, p. \end{aligned}$$

Problem (D_1) is different from the Dual Program (D), but it is equivalent to (D) as a maximization problem.