There is a nice geometric interpretation of harmonic divisions in terms of quadrangles (or complete quadrilaterals). Consider the quadrangle (projective frame) (a, b, c, d) in a projective plane, and let a' be the intersection of $\langle d, a\rangle$ and $\langle b, c\rangle$, b' be the intersection of $\langle d, b\rangle$ and $\langle a, c\rangle$, and c' be the intersection of $\langle d, c\rangle$ and $\langle a, b\rangle$. If we let g be the intersection of $\langle a, b\rangle$ and $\langle a', b'\rangle$, then it is an interesting exercise to show that (a, b, g, c') is a harmonic division. One way to prove this is to pick (a, c, b, d) as a projective frame and to compute the coordinates of a', b', c', and g. Then because $\langle a, c\rangle$ is the line at infinity, $[a, b, g, c'] = [\infty, b, g, c']$, which is computed using the above formula. Another way is to send some well chosen points to infinity; see Berger [11] (Chapter 6, Section 6.4).

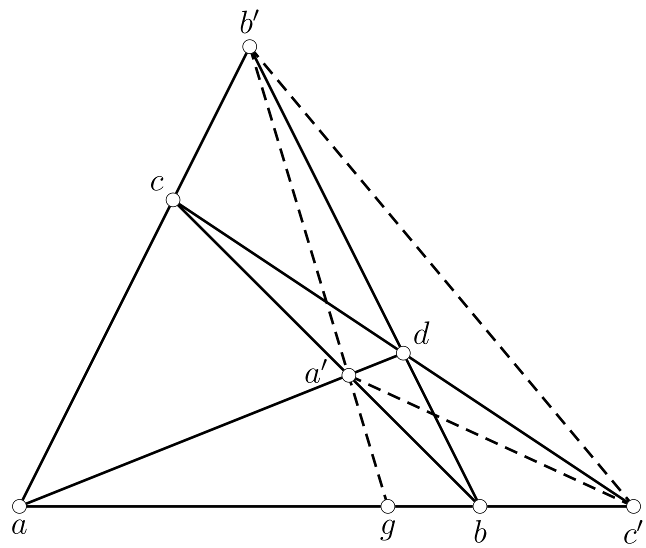

Figure 26.23: A quadrangle, and harmonic divisions.

In fact, it can be shown that the following quadruples of lines induce harmonic divisions: $(\langle c, a\rangle, \langle b', a'\rangle, \langle d, b\rangle, \langle b', c'\rangle)$ on $\langle a, b\rangle$, $(\langle b, a\rangle, \langle c', a'\rangle, \langle d, c\rangle, \langle c', b'\rangle)$ on $\langle a, c\rangle$, and $(\langle b, c\rangle, \langle a', c'\rangle, \langle a, d\rangle, \langle a', b'\rangle)$ on $\langle c, d\rangle$; see Figure 26.23. For more on harmonic divisions, the interested reader should consult any text on projective geometry (for example, Berger [11, 12], Samuel [142], Sidler [161], Tisseron [175], or Pedoe [136]).

26.11 Fixed Points of Homographies and Homologies; Homographies of $\mathbb{RP}^1$ and $\mathbb{RP}^2$

Let $\mathbb{P}(E)$ be a projective space where E is a vector space over some field K, and let $h\colon \mathbb{P}(E) \to \mathbb{P}(E)$ be homography (or projectivity) of $\mathbb{P}(E)$ where h is given by the linear isomorphism $f\colon E \to E$ so that $h = \mathbb{P}(f)$. Observe that if $u \in E$ is an eigenvector of f for some eigenvalue