and

$$M(f_A \circ (f_B \circ f_C)) = M(f_A)M(f_B \circ f_C) = A(BC),$$

and since composition of functions is associative, we have $(f_A \circ f_B) \circ f_C = f_A \circ (f_B \circ f_C)$, which implies that

$$(AB)C = A(BC).$$

(2) It is immediately verified that if $f_1, f_2 \in \mathrm{Hom}_K(E, F)$, $A, B \in \mathrm{M}_{m,n}(K)$, $(u_1, \ldots, u_n)$ is any basis of E, and $(v_1, \ldots, v_m)$ is any basis of F, then

$$\begin{aligned} M(f_1 + f_2) &= M(f_1) + M(f_2) \\ f_{A+B} &= f_A + f_B. \end{aligned}$$

Then we have

$$\begin{aligned} (A + B)C &= M(f_{A+B})M(f_C) \\ &= M(f_{A+B} \circ f_C) \\ &= M((f_A + f_B) \circ f_C)) \\ &= M((f_A \circ f_C) + (f_B \circ f_C)) \\ &= M(f_A \circ f_C) + M(f_B \circ f_C) \\ &= M(f_A)M(f_C) + M(f_B)M(f_C) \\ &= AC + BC. \end{aligned}$$

The equation $A(C + D) = AC + AD$ is proven in a similar fashion, and the last two equations are easily verified. We could also have verified all the identities by making matrix computations. $\square$

Note that Proposition 4.1 implies that the vector space $\mathrm{M}_n(K)$ of square matrices is a (noncommutative) ring with unit I_n. (It even shows that $\mathrm{M}_n(K)$ is an associative *algebra*.)

The following proposition states the main properties of the mapping $f \mapsto M(f)$ between $\mathrm{Hom}(E, F)$ and $\mathrm{M}_{m,n}$. In short, it is an isomorphism of vector spaces.

Proposition 4.2. *Given three vector spaces E, F, G, with respective bases $(u_1, \ldots, u_p)$, $(v_1, \ldots, v_n)$, and $(w_1, \ldots, w_m)$, the mapping $M\colon \mathrm{Hom}(E, F) \to \mathrm{M}_{n,p}$ that associates the matrix $M(g)$ to a linear map $g\colon E \to F$ satisfies the following properties for all $x \in E$, all $g, h\colon E \to F$, and all $f\colon F \to G$:*

$$\begin{aligned} M(g(x)) &= M(g)M(x) \\ M(g + h) &= M(g) + M(h) \\ M(\lambda g) &= \lambda M(g) \\ M(f \circ g) &= M(f)M(g), \end{aligned}$$