*Proof.* The first statement is a direct consequence of Theorem 29.4. If $K = \mathbb{C}$, then every $\lambda_i$ has a square root $\mu_i$, and if replace $e_i$ by $e_i/\mu_i$, we obtained the desired form.

If $K = \mathbb{R}$, then there are two cases:

1. If $\lambda_i > 0$, let $\mu_i$ be a positive square root of $\lambda_i$ and replace $e_i$ by $e_i/\mu_i$.

2. If $\lambda_i < 0$, et $\mu_i$ be a positive square root of $-\lambda_i$ and replace $e_i$ by $e_i/\mu_i$.

□

In the nondegenerate case, the matrices corresponding to the complex and the real case are, $I_n, -I_n$, and $I_{p,q}$. Observe that the second statement of Proposition 29.6 holds in any field in which every element has a square root. In the case $K = \mathbb{R}$, we can show that$(p, q)$ only depends on $\varphi$.

**Definition 29.7.** Let $\varphi\colon E \times E \to \mathbb{R}$ be any symmetric real bilinear form. For any subspace $U$ of $E$, we say that $\varphi$ is *positive definite on* $U$ iff $\varphi(u, u) > 0$ for all nonzero $u \in U$, and we say that $\varphi$ is *negative definite on* $U$ iff $\varphi(u, u) < 0$ for all nonzero $u \in U$. Then, let

$$r = \max\{\dim(U) \mid U \subseteq E,\ \varphi \text{ is positive definite on } U\}$$

and let

$$s = \max\{\dim(U) \mid U \subseteq E,\ \varphi \text{ is negative definite on } U\}$$

**Proposition 29.7.** *(Sylvester's inertia law) Given any symmetric bilinear form $\varphi\colon E \times E \to \mathbb{R}$ with $\dim(E) = n$, for any basis $(e_1, \ldots, e_n)$ of $E$ such that*

$$\Phi\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{p} x_i^2 - \sum_{i=p+1}^{p+q} x_i^2,$$

*with $0 \leq p, q$ and $p + q \leq n$, the integers $p, q$ depend only on $\varphi$; in fact, $p = r$ and $q = s$, with $r$ and $s$ as defined above.*

*Proof.* If we let $U$ be the subspace spanned by $(e_1, \ldots, e_p)$, then $\varphi$ is positive definite on $U$, so $r \geq p$. Similarly, if we let $V$ be the subspace spanned by $(e_{p+1}, \ldots, e_{p+q})$, then $\varphi$ is negative definite on $V$, so $s \geq q$.

Next, if $W_1$ is any subspace of maximum dimension such that $\varphi$ is positive definite on $W_1$, and if we let $V'$ be the subspace spanned by $(e_{p+1}, \ldots, e_n)$, then $\varphi(u, u) \leq 0$ on $V'$, so $W_1 \cap V' = (0)$, which implies that $\dim(W_1) + \dim(V') \leq n$, and thus, $r + n - p \leq n$; that is, $r \leq p$. Similarly, if $W_2$ is any subspace of maximum dimension such that $\varphi$ is negative definite on $W_2$, and if we let $U'$ be the subspace spanned by $(e_1, \ldots, e_p, e_{p+q+1}, \ldots, e_n)$, then $\varphi(u, u) \geq 0$ on $U'$, so $W_2 \cap U' = (0)$, which implies that $s + n - q \leq n$; that is, $s \leq q$. Therefore, $p = r$ and $q = s$, as claimed □

These last two results can be generalized to ordered fields. For example, see Snapper and Troyer [162], Artin [6], and Bourbaki [24].