If $v_1 = r_{1,1}e_1$, we let $h_1 = \mathrm{id}$. Otherwise, there is a unique hyperplane reflection h_1 such that

$$h_1(v_1) = r_{1,1}\, e_1,$$

defined such that

$$h_1(u) = u - 2\,\frac{(u \cdot w_1)}{\|w_1\|^2}\, w_1$$

for all $u \in E$, where

$$w_1 = r_{1,1}\, e_1 - v_1.$$

The map h_1 is the reflection about the hyperplane H_1 orthogonal to the vector $w_1 = r_{1,1}\, e_1 - v_1$. See Figure 13.4. Letting

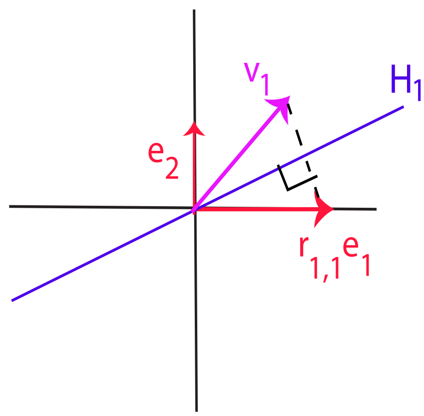

Figure 13.4: The construction of h_1 in Proposition 13.3.

$$r_1 = h_1(v_1) = r_{1,1}\, e_1,$$

it is obvious that r_1 belongs to the subspace spanned by e_1, and $r_{1,1} = \|v_1\|$ is nonnegative.

Next assume that we have found k linear maps $h_1, \ldots, h_k$, hyperplane reflections or the identity, where $1 \leq k \leq n-1$, such that if $(r_1, \ldots, r_k)$ are the vectors given by

$$r_j = h_k \circ \cdots \circ h_2 \circ h_1(v_j),$$

then every r_j is a linear combination of the vectors $(e_1, \ldots, e_j)$, $1 \leq j \leq k$. See Figure 13.5. The vectors $(e_1, \ldots, e_k)$ form a basis for the subspace denoted by U_k', the vectors $(e_{k+1}, \ldots, e_n)$ form a basis for the subspace denoted by U_k'', the subspaces U_k' and U_k'' are orthogonal, and $E = U_k' \oplus U_k''$. Let

$$u_{k+1} = h_k \circ \cdots \circ h_2 \circ h_1(v_{k+1}).$$

We can write

$$u_{k+1} = u_{k+1}' + u_{k+1}'',$$