Similarly, we say that $J$ has a *local maximum* (or *relative maximum*) at the point $u \in U$ *with respect to* $U$ if there is some open subset $W \subseteq \Omega$ containing $u$ such that

$$J(u) \geq J(w) \quad \text{for all } w \in U \cap W.$$

In either case, we say that $J$ has a *local extremum* at $u$ *with respect to* $U$.

In order to find necessary conditions for a function $J\colon \Omega \to \mathbb{R}$ to have a local extremum with respect to a subset $U$ of $\Omega$ (where $\Omega$ is open), we need to somehow incorporate the definition of $U$ into these conditions. This can be done in two cases:

(1) The set $U$ is defined by a set of equations,

$$U = \{x \in \Omega \mid \varphi_i(x) = 0,\ 1 \leq i \leq m\},$$

where the functions $\varphi_i\colon \Omega \to \mathbb{R}$ are continuous (and usually differentiable).

(2) The set $U$ is defined by a set of inequalities,

$$U = \{x \in \Omega \mid \varphi_i(x) \leq 0,\ 1 \leq i \leq m\},$$

where the functions $\varphi_i\colon \Omega \to \mathbb{R}$ are continuous (and usually differentiable).

In (1), the equations $\varphi_i(x) = 0$ are called *equality constraints*, and in (2), the inequalities $\varphi_i(x) \leq 0$ are called *inequality constraints*.

An inequality constraint of the form $\varphi_i(x) \geq 0$ is equivalent to the inequality constraint $-\varphi_x(x) \leq 0$. An equality constraint $\varphi_i(x) = 0$ is equivalent to the conjunction of the two inequality constraints $\varphi_i(x) \leq 0$ and $-\varphi_i(x) \leq 0$, so the case of inequality constraints subsumes the case of equality constraints. However, the case of equality constraints is easier to deal with, and in this chapter we will restrict our attention to this case.

If the functions $\varphi_i$ are convex and $\Omega$ is convex, then $U$ is convex. This is a very important case that we will discuss later. In particular, if the functions $\varphi_i$ are affine, then the equality constraints can be written as $Ax = b$, and the inequality constraints as $Ax \leq b$, for some $m \times n$ matrix $A$ and some vector $b \in \mathbb{R}^m$. We will also discuss the case of affine constraints later.

In the case of equality constraints, a necessary condition for a local extremum with respect to $U$ can be given in terms of *Lagrange multipliers*. In the case of inequality constraints, there is also a necessary condition for a local extremum with respect to $U$ in terms of generalized Lagrange multipliers and the *Karush–Kuhn–Tucker* conditions. This will be discussed in Chapter 50.

We begin by considering the case where $\Omega \subseteq E_1 \times E_2$ is an open subset of a product of normed vector spaces and where $U$ is the zero locus of some continuous function $\varphi\colon \Omega \to E_2$, which means that

$$U = \{(u_1, u_2) \in \Omega \mid \varphi(u_1, u_2) = 0\}.$$