(1) Prove that $T = P^\top AP$ is a symmetric tridiagonal $(2n) \times (2n)$ matrix with zero main diagonal of the form

$$T = \begin{pmatrix} 0 & a_1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ a_1 & 0 & b_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & b_1 & 0 & a_2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & 0 & b_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & 0 & b_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & b_{n-1} & 0 & a_n \\ 0 & 0 & 0 & \cdots & 0 & 0 & a_n & 0 \end{pmatrix}.$$

(2) Prove that if x_i is a unit eigenvector for an eigenvalue λ_i of T, then $\lambda_i = \pm\sigma_i$ where σ_i is a singular value of B, and that

$$Px_i = \frac{1}{\sqrt{2}} \begin{pmatrix} u_i \\ \pm v_i \end{pmatrix},$$

where the u_i are unit eigenvectors of $B^\top B$ and the v_i are unit eigenvectors of $BB^\top$.

Problem 22.4. Find the SVD of the matrix

$$A = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}.$$

Problem 22.5. Let $u, v \in \mathbb{R}^n$ be two nonzero vectors, and let $A = uv^\top$ be the corresponding rank 1 matrix. Prove that the nonzero singular value of A is $\|u\|_2 \|v\|_2$.

Problem 22.6. Let A be a $n \times n$ real matrix. Prove that if $\sigma_1, \ldots, \sigma_n$ are the singular values of A, then $\sigma_1^3, \ldots, \sigma_n^3$ are the singular values of $AA^\top A$.

Problem 22.7. Let A be a real $n \times n$ matrix.

(1) Prove that the largest singular value σ_1 of A is given by

$$\sigma_1 = \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2},$$

and that this supremum is achieved at $x = u_1$, the first column in U in an SVD $A = V\Sigma U^\top$.

(2) Extend the above result to real $m \times n$ matrices.

Problem 22.8. Let A be a real $m \times n$ matrix. Prove that if B is any submatrix of A (by keeping $M \leq m$ rows and $N \leq n$ columns of A), then $(\sigma_1)_B \leq (\sigma_1)_A$ (where $(\sigma_1)_A$ is the largest singular value of A and similarly for $(\sigma_1)_B$).