from which we get

$$\begin{aligned}\|v-u\|^2 &= \|x+y+z\|^2 = \|x+z+y\|^2 \\ &= \|x+z\|^2 + \|y\|^2 + 2\Re\langle x,y\rangle + 2\Re\langle z,y\rangle \\ &\geq \|y\|^2 = \|p_X(v)-p_X(u)\|^2.\end{aligned}$$

However, $\|p_X(v)-p_X(u)\| \leq \|v-u\|$ obviously implies that p_X is continuous. $\square$

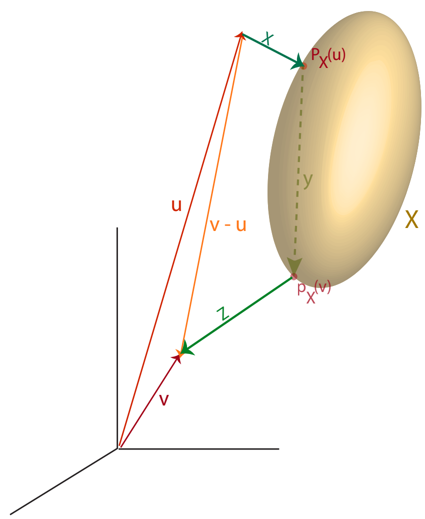

Figure 48.6: Let X be the solid gold ellipsoid. The vector $v-u$ is the sum of the three green vectors, each of which is determined by the appropriate projections.

We can now prove the following important proposition.

Proposition 48.7. *Let E be a Hilbert space.*

(1) For any closed subspace $V \subseteq E$, we have $E = V \oplus V^{\perp}$, and the map $p_V \colon E \to V$ is linear and continuous.

(2) For any $u \in E$, the projection $p_V(u)$ is the unique vector $w \in E$ such that

$$w \in V \quad \text{and} \quad \langle u-w, z\rangle = 0 \quad \text{for all } z \in V.$$

Proof. (1) First, we prove that $u - p_V(u) \in V^{\perp}$ for all $u \in E$. For any $v \in V$, since V is a subspace, $z = p_V(u) + \lambda v \in V$ for all $\lambda \in \mathbb{C}$, and since V is convex and nonempty (since it is a subspace), and closed by hypothesis, by Proposition 48.5(2), we have

$$\Re(\overline{\lambda}\langle u-p_V(u), v\rangle) = \Re(\langle u-p_V(u), \lambda v\rangle = \Re\langle u-p_V(u), z-p_V(u)\rangle \leq 0$$