with $e_0 = u_0 - \widetilde{u} = v_0 - \widetilde{u}$. Again, by Proposition 10.2, for every $\epsilon > 0$, there is some natural number $N(\epsilon)$ such that if $k \geq N(\epsilon)$, then

$$\sup_{\|e_0\|=1} \|B_1^k e_0\|^{1/k} \leq \rho(B_1) + \epsilon.$$

Furthermore, for all $k \geq N(\epsilon)$, there exists a vector $e_0 = e_0(k)$ (for some suitable choice of u_0) such that

$$\|e_0\| = 1 \quad \text{and} \quad \|B_2^k e_0\|^{1/k} = \|B_2^k\|^{1/k} \geq \rho(B_2),$$

which implies Statement (2). $\square$

In light of the above, we see that when we investigate new iterative methods, we have to deal with the following two problems:

1. Given an iterative method with matrix B, determine whether the method is convergent. This involves determining whether $\rho(B) < 1$, or equivalently whether there is a subordinate matrix norm such that $\|B\| < 1$. By Proposition 9.11, this implies that $I - B$ is invertible (since $\|-B\| = \|B\|$, Proposition 9.11 applies).

2. Given two convergent iterative methods, compare them. The iterative method which is faster is that whose matrix has the smaller spectral radius.

We now discuss three iterative methods for solving linear systems:

1. Jacobi's method

2. Gauss–Seidel's method

3. The relaxation method.

10.3 Description of the Methods of Jacobi, Gauss–Seidel, and Relaxation

The methods described in this section are instances of the following scheme: Given a linear system $Ax = b$, with A invertible, suppose we can write A in the form

$$A = M - N,$$

with M invertible, and "easy to invert," which means that M is close to being a diagonal or a triangular matrix (perhaps by blocks). Then $Au = b$ is equivalent to

$$Mu = Nu + b,$$

that is,

$$u = M^{-1}Nu + M^{-1}b.$$