The next two results generalize familiar results about derivatives to subdifferentials.

Proposition 51.23. *Let $f_1, \ldots, f_n$ be proper convex functions on $\mathbb{R}^n$, and let $f = f_1 + \cdots + f_n$. For $x \in \mathbb{R}^n$, we have*

$$\partial f(x) \supseteq \partial f_1(x) + \cdots + \partial f_n(x).$$

If $\bigcap_{i=1}^n \mathbf{relint}(\mathrm{dom}(f_i)) \neq \emptyset$, then

$$\partial f(x) = \partial f_1(x) + \cdots + \partial f_n(x).$$

Proposition 51.23 is proven in Rockafellar [138] (Theorem 23.8).

The next result can be viewed as a generalization of the chain rule.

Proposition 51.24. *Let f be the function given by $f(x) = h(Ax)$ for all $x \in \mathbb{R}^n$, where h is a proper convex function on $\mathbb{R}^m$ and A is an $m \times n$ matrix. Then*

$$\partial f(x) \supseteq A^\top(\partial h(Ax)) \quad \text{for all } x \in \mathbb{R}^n.$$

If the range of A contains a point of $\mathbf{relint}(\mathrm{dom}(h))$, then

$$\partial f(x) = A^\top(\partial h(Ax)).$$

Proposition 51.24 is proven in Rockafellar [138] (Theorem 23.9).

51.4 Additional Properties of Subdifferentials

In general, if $f\colon \mathbb{R}^n \to \mathbb{R}$ is a function (not necessarily convex) and f is differentiable at x, we expect that the gradient ∇f_x of f at x is normal to the level set $\{z \in \mathbb{R}^n \mid f(z) = f(x)\}$ at $f(x)$. An analogous result, as illustrated in Figure 51.22, holds for proper convex functions in terms of subdifferentials.

Proposition 51.25. *Let f be a proper convex function on $\mathbb{R}^n$, and let $x \in \mathbb{R}^n$ be a vector such that f is subdifferentiable at x but f does not achieve its minimum at x. Then the normal cone $N_C(x)$ at x to the sublevel set $C = \{z \in \mathbb{R}^n \mid f(z) \leq f(x)\}$ is the closure of the convex cone spanned by $\partial f(x)$.*

Proposition 51.25 is proven in Rockafellar [138] (Theorem 23.7).

The following result sharpens Proposition 51.8.

Proposition 51.26. *Let f be a closed proper convex function on $\mathbb{R}^n$, and let S be a nonempty closed and bounded subset of $\mathrm{int}(\mathrm{dom}(f))$. Then*

$$\partial f(S) = \bigcup_{x \in S} \partial f(x)$$