respect to some projective frame $(a_1, \ldots, a_{n+2})$), then the equation of the unique hyperplane H containing $P_1, \ldots, P_n$ is given by the equation

$$\begin{vmatrix} x_1 & x_2 & \cdots & x_n & x_{n+1} \\ u_1^1 & u_2^1 & \cdots & u_n^1 & u_{n+1}^1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ u_1^{n-1} & u_2^{n-1} & \cdots & u_n^{n-1} & u_{n+1}^{n-1} \\ u_1^n & u_2^n & \cdots & u_n^n & u_{n+1}^n \end{vmatrix} = 0.$$

We also have the following proposition giving another characterization of projective frames.

Proposition 26.3. *A family $(a_i)_{1 \leq i \leq n+2}$ of $n+2$ points is a projective frame of $\mathbf{P}(E)$ iff for every i, $1 \leq i \leq n+2$, the subfamily $(a_j)_{j \neq i}$ is projectively independent.*

Proof. We leave as an (easy) exercise the fact that if $(a_i)_{1 \leq i \leq n+2}$ is a projective frame, then each subfamily $(a_j)_{j \neq i}$ is projectively independent. Conversely, pick some $u_i \in E - \{0\}$ such that $a_i = p(u_i)$, $1 \leq i \leq n+2$. Since $(a_j)_{j \neq n+2}$ is projectively independent, $(u_1, \ldots, u_{n+1})$ is a basis of E. Thus, we must have

$$u_{n+2} = \lambda_1 u_1 + \cdots + \lambda_{n+1} u_{n+1},$$

for some $\lambda_i \in K$. However, since for every i, $1 \leq i \leq n+1$, the family $(a_j)_{j \neq i}$ is projectively independent, we must have $\lambda_i \neq 0$, and thus $(\lambda_1 u_1, \ldots, \lambda_{n+1} u_{n+1})$ is also a basis of E, and since

$$u_{n+2} = \lambda_1 u_1 + \cdots + \lambda_{n+1} u_{n+1},$$

it induces the projective frame $(a_i)_{1 \leq i \leq n+2}$. $\square$

Figure 26.9 shows a projective frame (a, b, c, d) in a projective plane. With respect to this projective frame, the points a, b, c, d have homogeneous coordinates $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and $(1, 1, 1)$. Let a' be the intersection of $\langle d, a \rangle$ and $\langle b, c \rangle$, b' be the intersection of $\langle d, b \rangle$ and $\langle a, c \rangle$, and c' be the intersection of $\langle d, c \rangle$ and $\langle a, b \rangle$. Then the points a', b', c' have homogeneous coordinates $(0, 1, 1)$, $(1, 0, 1)$, and $(1, 1, 0)$. The diagram formed by the line segments $\langle a, c' \rangle$, $\langle a, b' \rangle$, $\langle b, b' \rangle$, $\langle c, c' \rangle$, $\langle a, d \rangle$, and $\langle b, c \rangle$ is sometimes called a *Möbius net*; see Hilbert and Cohn-Vossen [92] (Chapter III, §15, page 96).

Recall that the equation of a line (a hyperplane in a projective plane) in terms of homogeneous coordinates with respect to the projective frame (a, b, c, d) is given by a homogeneous equation of the form

$$\alpha x + \beta y + \gamma z = 0,$$

where α, β, γ are not all zero. It is easily verified that the equations of the lines $\langle a, b \rangle$, $\langle a, c \rangle$, $\langle b, c \rangle$, are $z = 0$, $y = 0$, and $x = 0$, and the equations of the lines $\langle a, d \rangle$, $\langle b, d \rangle$, and $\langle c, d \rangle$,