Let $H$ be any hyperplane in $E$, and pick some (nonzero) vector $v \in E$ such that $v \notin H$, so that

$$E = H \oplus Kv.$$

Assume that $f \colon E \to E$ is a linear isomorphism such that $f(u) = u$ for all $u \in H$, and that $f$ is not the identity. We have

$$f(v) = h + \alpha v, \quad \text{for some } h \in H \text{ and some } \alpha \in K,$$

with $\alpha \neq 0$, because otherwise we would have $f(v) = h = f(h)$ since $h \in H$, contradicting the injectivity of $f$ ($v \neq h$ since $v \notin H$). For any $x \in E$, if we write

$$x = y + tv, \quad \text{for some } y \in H \text{ and some } t \in K,$$

then

$$f(x) = f(y) + f(tv) = y + tf(v) = y + th + t\alpha v,$$

and since $\alpha x = \alpha y + t\alpha v$, we get

$$\begin{aligned} f(x) - \alpha x &= (1 - \alpha)y + th \\ f(x) - x &= t(h + (\alpha - 1)v). \end{aligned}$$

Observe that if $E$ is finite-dimensional, by picking a basis of $E$ consisting of $v$ and basis vectors of $H$, then the matrix of $f$ is a lower triangular matrix whose diagonal entries are all 1 except the first entry which is equal to $\alpha$. Therefore, $\det(f) = \alpha$.

*Case 1*. $\alpha \neq 1$.

We have $f(x) = \alpha x$ iff $(1 - \alpha)y + th = 0$ iff

$$y = \frac{t}{\alpha - 1}h.$$

Then if we let $w = h + (\alpha - 1)v$, for $y = (t/(\alpha - 1))h$, we have

$$x = y + tv = \frac{t}{\alpha - 1}h + tv = \frac{t}{\alpha - 1}(h + (\alpha - 1)v) = \frac{t}{\alpha - 1}w,$$

which shows that $f(x) = \alpha x$ iff $x \in Kw$. Note that $w \notin H$, since $\alpha \neq 1$ and $v \notin H$. Therefore,

$$E = H \oplus Kw,$$

and $f$ is the identity on $H$ and a magnification by $\alpha$ on the line $D = Kw$.

**Definition 8.8.** Given a vector space $E$, for any hyperplane $H$ in $E$, any nonzero vector $u \in E$ such that $u \notin H$, and any scalar $\alpha \neq 0, 1$, a linear map $f$ such that $f(x) = x$ for all $x \in H$ and $f(x) = \alpha x$ for every $x \in D = Ku$ is called a *dilatation of hyperplane $H$, direction $D$, and scale factor $\alpha$.*