Definition 33.18. Symmetric tensors in $\mathrm{S}^n(E)$ are called *symmetric n-tensors*, and tensors of the form $u_1 \odot \cdots \odot u_n$, where $u_i \in E$, are called *simple (or decomposable) symmetric n-tensors*. Those symmetric n-tensors that are not simple are often called *compound symmetric n-tensors*.

Given linear map $f\colon E \to E'$, since the map $\iota'_\odot \circ (f \times f)$ is bilinear and symmetric, there is a unique linear map $f \odot f\colon \mathrm{S}^2(E) \to \mathrm{S}^2(E')$ making the following diagram commute.

$$\begin{tikzcd}
E^2 \arrow[r, "\iota_\odot"] \arrow[d, "f\times f"'] & \mathrm{S}^2(E) \arrow[d, "f\odot f"] \\
(E')^2 \arrow[r, "\iota'_\odot"'] & \mathrm{S}^2(E').
\end{tikzcd}$$

Observe that $f \odot g$ is determined by

$$(f \odot f)(u \odot v) = f(u) \odot f(v).$$

Proposition 33.27. *Given any two linear maps $f\colon E \to E'$ and $f'\colon E' \to E''$, we have*

$$(f' \circ f) \odot (f' \circ f) = (f' \odot f') \circ (f \odot f).$$

The generalization to the symmetric tensor product $f \odot \cdots \odot f$ of $n \geq 3$ copies of the linear map $f\colon E \to E'$ is immediate, and left to the reader.

33.8 Bases of Symmetric Powers

The vectors $u_1 \odot \cdots \odot u_m$ where $u_1, \ldots, u_m \in E$ generate $\mathrm{S}^m(E)$, but they are not linearly independent. We will prove a version of Proposition 33.12 for symmetric tensor powers using multisets.

Recall that a (finite) multiset over a set I is a function $M\colon I \to \mathbb{N}$, such that $M(i) \neq 0$ for finitely many $i \in I$. The set of all multisets over I is denoted as $\mathbb{N}^{(I)}$ and we let $\mathrm{dom}(M) = \{i \in I \mid M(i) \neq 0\}$, the finite set of elements in I that actually occur in M. The size of the multiset M is $|M| = \sum_{a\in A} M(a)$.

To explain the idea of the proof, consider the case when $m = 2$ and E has dimension 3. Given a basis (e_1, e_2, e_3) of E, we would like to prove that

$$e_1 \odot e_1, \quad e_1 \odot e_2, \quad e_1 \odot e_3, \quad e_2 \odot e_2, \quad e_2 \odot e_3, \quad e_3 \odot e_3$$

are linearly independent. To prove this, it suffices to show that for any vector space F, if $w_{11}, w_{12}, w_{13}, w_{22}, w_{23}, w_{33}$ are any vectors in F, then there is a symmetric bilinear map $h\colon E^2 \to F$ such that

$$h(e_i, e_j) = w_{ij}, \quad 1 \leq i \leq j \leq 3.$$