50.8 Weak and Strong Duality

Another important property of the dual function G is that it provides a *lower bound* on the value of the objective function J. Indeed, we have

$$G(\mu) \leq L(u, \mu) \leq J(u) \quad \text{for all } u \in U \text{ and all } \mu \in \mathbb{R}^m_+, \tag{$\dagger$}$$

since $\mu \geq 0$ and $\varphi_i(u) \leq 0$ for $i = 1, \ldots, m$, so

$$G(\mu) = \inf_{v \in \Omega} L(v, \mu) \leq L(u, \mu) = J(u) + \sum_{i=1}^m \mu_i \varphi_i(u) \leq J(u).$$

If the Primal Problem (P) has a minimum denoted p^* and the Dual Problem (D) has a maximum denoted d^*, then the above inequality implies that

$$d^* \leq p^* \tag{$\dagger_w$}$$

known as *weak duality*. Equivalently, for every optimal solution λ^* of the dual problem and every optimal solution u^* of the primal problem, we have

$$G(\lambda^*) \leq J(u^*). \tag{$\dagger_{w'}$}$$

In particular, if $p^* = -\infty$, which means that the primal problem is unbounded below, then the dual problem is unfeasible. Conversely, if $d^* = +\infty$, which means that the dual problem is unbounded above, then the primal problem is unfeasible.

Definition 50.10. The difference $p^* - d^* \geq 0$ is called the *optimal duality gap*. If the duality gap is zero, that is, $p^* = d^*$, then we say that *strong duality* holds.

Even when the duality gap is strictly positive, the inequality $(\dagger_w)$ can be helpful to find a lower bound on the optimal value of a primal problem that is difficult to solve, since the dual problem is *always* convex.

If the primal problem and the dual problem are feasible and if the optimal values p^* and d^* are finite and $p^* = d^*$ (no duality gap), then the complementary slackness conditions hold for the inequality constraints.

Proposition 50.16. *(Complementary Slackness) Given the Minimization Problem* (P)

$$\begin{aligned} &\text{minimize} && J(v) \\ &\text{subject to} && \varphi_i(v) \leq 0, \quad i = 1, \ldots, m, \end{aligned}$$

and its Dual Problem (D)

$$\begin{aligned} &\text{maximize} && G(\mu) \\ &\text{subject to} && \mu \in \mathbb{R}^m_+, \end{aligned}$$