(b) *There exist some elements* $e_1, \ldots, e_n$ *of* A *such that*

$$\begin{aligned} e_i^2 &= e_i \\ e_i e_j &= 0, \quad i \neq j \\ e_1 + \cdots + e_n &= 1_A, \end{aligned}$$

and $\mathfrak{b}_i = (1_A - e_i)A$, *for* $i, j = 1, \ldots, n$.

(c) *We have* $\mathfrak{b}_i + \mathfrak{b}_j = A$ *for all* $i \neq j$, *and* $\mathfrak{b}_1 \cdots \mathfrak{b}_n = (0)$.

(d) *We have* $\mathfrak{b}_i + \mathfrak{b}_j = A$ *for all* $i \neq j$, *and* $\mathfrak{b}_1 \cap \cdots \cap \mathfrak{b}_n = (0)$.

Proof. Assume (a). Since we have an isomorphism $A \approx A/\mathfrak{b}_1 \times \cdots \times A/\mathfrak{b}_n$, we may identify A with $A/\mathfrak{b}_1 \times \cdots \times A/\mathfrak{b}_n$, and $\mathfrak{b}_\mathfrak{i}$ with $\mathrm{Ker}\,(pr_i)$. Then, $e_1, \ldots, e_n$ are the elements defined just before Definition 32.3. As noted, $\mathfrak{b}_i = \mathrm{Ker}\,(pr_i) = (1_A - e_i)A$. This proves (b).

Assume (b). Since $\mathfrak{b}_i = (1_A - e_i)A$ and A is a ring with unit 1_A, we have $1_A - e_i \in \mathfrak{b}_i$ for $i = 1, \ldots, n$. For all $i \neq j$, we also have $e_i(1_A - e_j) = e_i - e_i e_j = e_i$, so (because $\mathfrak{b}_j$ is an ideal), $e_i \in \mathfrak{b}_j$, and thus, $1_A = 1_A - e_i + e_i \in \mathfrak{b}_\mathfrak{i} + \mathfrak{b}_j$, which shows that $\mathfrak{b}_i + \mathfrak{b}_j = A$ for all $i \neq j$. Furthermore, for any $x_i \in A$, with $1 \leq i \leq n$, we have

$$\begin{aligned} \prod_{i=1}^n x_i(1_A - e_i) &= \left(\prod_{i=1}^n x_i\right) \prod_{i=1}^n (1_A - e_i) \\ &= \left(\prod_{i=1}^n x_i\right)(1_A - \sum_{i=1}^n e_i) \\ &= 0, \end{aligned}$$

which proves that $\mathfrak{b}_1 \cdots \mathfrak{b}_n = (0)$. Thus, (c) holds.

The equivalence of (c) and (d) follows from the proof of Theorem 32.15.

The fact that (c) implies (a) is an immediate consequence of Theorem 32.15. ☐

Here is example of Theorem 32.16. Take the commutative ring of residue classes mod 30, namely

$$A := \mathbb{Z}/30\mathbb{Z} = \{\bar{i}\}_{i=0}^{29}.$$

Let

$$\begin{aligned} \mathfrak{b}_1 &= 2\mathbb{Z}/30\mathbb{Z} := \{2\bar{i}\}_{i=0}^{14} \\ \mathfrak{b}_2 &= 3\mathbb{Z}/30\mathbb{Z} := \{3\bar{i}\}_{i=0}^{9} \\ \mathfrak{b}_3 &= 5\mathbb{Z}/30\mathbb{Z} := \{5\bar{i}\}_{i=0}^{5}. \end{aligned}$$