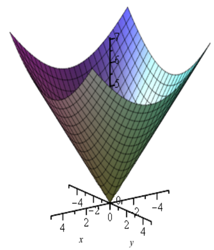

Figure 1

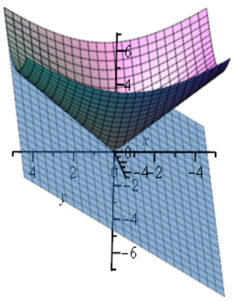

Figure 2

Figure 51.14: Figure (1) shows the graph in $\mathbb{R}^3$ of $f(x,y) = \|(x,y)\|_2 = \sqrt{x^2+y^2}$. Figure (2) shows the supporting hyperplane with normal $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -1)$, where $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \in \partial f(0)$.

iff

$$\langle z, y\rangle \leq \langle x, y\rangle \quad \text{for all } z \geq 0.$$

In particular, for $z = 0$ we get $\langle x, y\rangle \geq 0$, and for $z = 2x \geq 0$, we have $\langle x, y\rangle \leq 0$, so $\langle x, y\rangle = 0$. As a consequence, $y \in N_C(x)$ iff $\langle x, y\rangle = 0$ and

$$\langle z, y\rangle \leq 0 \quad \text{for all } z \geq 0.$$

For $z = e_j \geq 0$, we get $y_j \leq 0$. Conversely, if $y \leq 0$ and $\langle x, y\rangle = 0$, since $x \geq 0$, we get $\langle z, y\rangle \leq 0$ for all $z \geq 0$, and so

$$\partial f(x) = \{y = (y_1, \ldots, y_n) \in \mathbb{R}^n \mid y \leq 0,\ \langle x, y\rangle = 0\}.$$

But for $x \geq 0$ and $y \leq 0$ we have $\langle x, y\rangle = \sum_{j=1}^n x_j y_j = 0$ iff $x_j y_j = 0$ for $j = 1, \ldots, n$, thus we see that $y \in \partial f(x)$ iff we have

$$x_j \geq 0,\ y_j \leq 0,\ x_j y_j = 0, \quad 1 \leq j \leq n,$$

which are complementary slackness conditions.

Supporting hyperplanes to the epigraph of a proper convex function f can be used to prove a property which plays a key role in optimization theory. The proof uses a classical result of convex geometry, namely the Minkowski supporting hyperplane theorem.