47.6 The Primal-Dual Algorithm

Let $(P2)$ be a linear program in standard form

$$\begin{aligned} &\text{maximize} \quad cx \\ &\text{subject to} \quad Ax = b \ \text{and} \ x \geq 0, \end{aligned}$$

where A is an $m \times n$ matrix of rank m, and (D) be its dual given by

$$\begin{aligned} &\text{minimize} \quad yb \\ &\text{subject to} \quad yA \geq c, \end{aligned}$$

where $y \in (\mathbb{R}^m)^*$.

First we may assume that $b \geq 0$ by changing every equation $\sum_{j=1}^n a_{ij} x_j = b_i$ with $b_i < 0$ to $\sum_{j=1}^n -a_{ij} x_j = -b_i$. If we happen to have some feasible solution y of the dual program (D), we know from Theorem 47.13 that a feasible solution x of $(P2)$ is an optimal solution iff the equations in $(*_P)$ hold. If we denote by J the subset of $\{1, \ldots, n\}$ for which the equalities

$$yA^j = c_j$$

hold, then by Theorem 47.13 a feasible solution x of $(P2)$ is an optimal solution iff

$$x_j = 0 \quad \text{for all } j \notin J.$$

Let $|J| = p$ and $N = \{1, \ldots, n\} - J$. The above suggests looking for $x \in \mathbb{R}^n$ such that

$$\begin{aligned} \sum_{j \in J} x_j A^j &= b \\ x_j &\geq 0 \quad \text{for all } j \in J \\ x_j &= 0 \quad \text{for all } j \notin J, \end{aligned}$$

or equivalently

$$A_J x_J = b, \quad x_J \geq 0, \tag{$*_1$}$$

and

$$x_N = 0_{n-p}.$$

To search for such an x, we just need to look for a feasible x_J, and for this we can use the *Restricted Primal* linear program (RP) defined as follows:

$$\begin{aligned} &\text{maximize} \quad -(\xi_1 + \cdots + \xi_m) \\ &\text{subject to} \quad \begin{pmatrix} A_J & I_m \end{pmatrix} \begin{pmatrix} x_J \\ \xi \end{pmatrix} = b \ \text{and} \ x, \xi \geq 0. \end{aligned}$$