The requirement that the restriction of $\widetilde{f} = \mathbf{P}(g)$ to $\mathbf{P}\big(\overrightarrow{E}\big)$ be equal to $\mathbf{P}\big(\overrightarrow{f}\big)$ is necessary for the uniqueness of $\widetilde{f}$. The problem comes up when f is a constant map. Indeed, if f is the constant map defined such that $f(a) = [w]$ for some fixed vector $w \in F$, it can be shown that any linear map $g\colon \widehat{E} \to F$ defined such that $g(a) = \mu w$ and $g(u) = \varphi(u)w$ for all $u \in \overrightarrow{E}$, for some $\mu \neq 0$, and some linear form $\varphi\colon \overrightarrow{E} \to F$ satisfies $f = \mathbf{P}(g) \circ i$.

Proposition 26.17 shows that $\langle \widetilde{E}, \mathbf{P}\big(\overrightarrow{E}\big), i\rangle$ is the projective completion of the affine space E.

The projective completion $\widetilde{E}$ of an affine space E is a very handy place in which to do geometry in, mainly because the following facts can be easily established.

There is a bijection between affine subspaces of E and projective subspaces of $\widetilde{E}$ not contained in $\mathbf{P}\big(\overrightarrow{E}\big)$. Two affine subspaces of E are parallel iff the corresponding projective subspaces of $\widetilde{E}$ have the same intersection with the hyperplane at infinity $\mathbf{P}\big(\overrightarrow{E}\big)$. There is also a bijection between affine maps from E to F and projective maps from $\widetilde{E}$ to $\widetilde{F}$ mapping the hyperplane at infinity $\mathbf{P}\big(\overrightarrow{E}\big)$ into the hyperplane at infinity $\mathbf{P}\big(\overrightarrow{F}\big)$. In the projective plane, two distinct lines intersect in a single point (possibly at infinity, when the lines are parallel). In the projective space, two distinct planes intersect in a single line (possibly at infinity, when the planes are parallel). In the projective space, a plane and a line not contained in that plane intersect in a single point (possibly at infinity, when the plane and the line are parallel).

26.9 Making Good Use of Hyperplanes at Infinity

Given a vector space E and a hyperplane H in E, we have already observed that the projective spaces $\widetilde{E_H}$ and $\mathbf{P}(E)$ are isomorphic. Thus, $\mathbf{P}(H)$ can be viewed as the hyperplane at infinity in $\mathbf{P}(E)$, and the considerations applying to the projective completion of an affine space apply to the affine patch E_H on $\mathbf{P}(E)$. This fact yields a powerful and elegant method for proving theorems in projective geometry. The general schema is to choose some projective hyperplane $\mathbf{P}(H)$ in $\mathbf{P}(E)$, view it as the "hyperplane at infinity," then prove an affine version of the desired result in the affine patch E_H (the complement of $\mathbf{P}(H)$ in $\mathbf{P}(E)$, which has an affine structure), and then transfer this result back to the projective space $\mathbf{P}(E)$. This technique is often called "sending objects to infinity." We refer the reader to geometry textbooks for a comprehensive development of these ideas (for example, Berger [11, 12], Samuel [142], Sidler [161], Tisseron [175], or Pedoe [136]), but we cannot resist presenting the projective versions of the theorems of Pappus and Desargues. Indeed, the method of sending points to infinity provides some strikingly elegant proofs. We begin with Pappus's theorem, illustrated in Figure 26.20.

Proposition 26.18. *(Pappus) Given any projective plane $\mathbf{P}(E)$ and any two distinct lines D and D', for any distinct points a, b, c, a', b', c', with a, b, c on D and a', b', c' on D', if*