Proof. (1) Since J is a C^1-function, by Taylor's formula with integral remainder in the case $m = 0$ (Theorem 39.26), we get

$$\begin{aligned}
J(v) - J(u) &= \int_0^1 dJ_{u+t(v-u)}(v-u)dt \\
&= \int_0^1 \langle \nabla J_{u+t(v-u)}, v-u\rangle dt \\
&= \langle \nabla J_u, v-u\rangle + \int_0^1 \langle \nabla J_{u+t(v-u)} - \nabla J_u, v-u\rangle dt \\
&= \langle \nabla J_u, v-u\rangle + \int_0^1 \frac{\langle \nabla J_{u+t(v-u)} - \nabla J_u, t(v-u)\rangle}{t} dt \\
&\geq \langle \nabla J_u, v-u\rangle + \int_0^1 \alpha t \left\|v-u\right\|^2 dt \qquad \text{since } J \text{ is elliptic} \\
&= \langle \nabla J_u, v-u\rangle + \frac{\alpha}{2}\left\|v-u\right\|^2 .
\end{aligned}$$

Using the inequality

$$J(v) - J(u) \geq \langle \nabla J_u, v-u\rangle + \frac{\alpha}{2}\left\|v-u\right\|^2 \quad \text{for all } u, v \in V,$$

by Proposition 40.11(2), since

$$J(v) > J(u) + \langle \nabla J_u, v-u\rangle \quad \text{for all } u, v \in V,\ v \neq u,$$

the function J is strictly convex. It is coercive because (using Cauchy–Schwarz)

$$\begin{aligned}
J(v) &\geq J(0) + \langle \nabla J_0, v\rangle + \frac{\alpha}{2}\left\|v\right\|^2 \\
&\geq J(0) - \left\|\nabla J_0\right\|\left\|v\right\| + \frac{\alpha}{2}\left\|v\right\|^2 ,
\end{aligned}$$

and the term $(-\left\|\nabla J_0\right\| + \frac{\alpha}{2}\left\|v\right\|)\left\|v\right\|$ goes to $+\infty$ when $\left\|v\right\|$ tends to $+\infty$.

(2) Since by (1) the functional J is coercive, by Theorem 49.2, Problem (P) has a solution. Since J is strictly convex, by Theorem 40.13(2), it has a unique minimum.

(3) These are just the conditions of Theorem 40.13(3, 4).

(4) If J is twice differentiable, we showed in Section 39.6 that we have

$$\mathrm{D}^2 J_u(w,w) = \mathrm{D}_w(\mathrm{D}J)(u) = \lim_{\theta \mapsto 0} \frac{\mathrm{D}J_{u+\theta w}(w) - \mathrm{D}J_u(w)}{\theta},$$

and since

$$\begin{aligned}
\mathrm{D}^2 J_u(w,w) &= \langle \nabla^2 J_u(w), w\rangle \\
\mathrm{D}J_{u+\theta w}(w) &= \langle \nabla J_{u+\theta w}, w\rangle \\
\mathrm{D}J_u(w) &= \langle \nabla J_u, w\rangle,
\end{aligned}$$