(2) We have

$$V^\top AU = \begin{pmatrix} E & 0 \\ 0 & H \end{pmatrix}, \quad V^\top BU = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix},$$

where E is diagonal, so deduce that

1. $D = \mathrm{diag}(\sigma_1, \ldots, \sigma_k)$.
2. The singular values of H must be the smallest $n - k$ singular values of A.
3. The minimum of $\|A - B\|_F$ must be $\|H\|_F = (\sigma_{k+1}^2 + \cdots + \sigma_r^2)^{1/2}$.

Problem 23.5. Prove that the closest rank 1 approximation (in $\| \ \|_2$) of the matrix

$$A = \begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix}$$

is

$$A_1 = \frac{3}{2} \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}.$$

Show that the Eckart–Young theorem fails for the operator norm $\| \ \|_\infty$ by finding a rank 1 matrix B such that $\|A - B\|_\infty < \|A - A_1\|_\infty$.

Problem 23.6. Find a closest rank 1 approximation (in $\| \ \|_2$) for the matrices

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Problem 23.7. Find a closest rank 1 approximation (in $\| \ \|_2$) for the matrix

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Problem 23.8. Let S be a real symmetric positive definite matrix and let $S = U\Sigma U^\top$ be a diagonalization of S. Prove that the closest rank 1 matrix (in the L^2-norm) to S is $u_1\sigma_1 u_1^\top$, where u_1 is the first column of U.