8.5 $PA = LU$ Factorization

The following easy proposition shows that, in principle, A can be premultiplied by some permutation matrix P, so that PA can be converted to upper-triangular form without using any pivoting. Permutations are discussed in some detail in Section 30.3, but for now we just need this definition. For the precise connection between the notion of permutation (as discussed in Section 30.3) and permutation matrices, see Problem 8.16.

Definition 8.3. A *permutation matrix* is a square matrix that has a single 1 in every row and every column and zeros everywhere else.

It is shown in Section 30.3 that every permutation matrix is a product of transposition matrices (the $P(i,k)$s), and that P is invertible with inverse $P^{\top}$.

Proposition 8.4. *Let A be an invertible $n \times n$-matrix. There is some permutation matrix P so that $(PA)(1:k, 1:k)$ is invertible for $k = 1, \ldots, n$.*

Proof. The case $n = 1$ is trivial, and so is the case $n = 2$ (we swap the rows if necessary). If $n \geq 3$, we proceed by induction. Since A is invertible, its columns are linearly independent; in particular, its first $n-1$ columns are also linearly independent. Delete the last column of A. Since the remaining $n-1$ columns are linearly independent, there are also $n-1$ linearly independent rows in the corresponding $n \times (n-1)$ matrix. Thus, there is a permutation of these n rows so that the $(n-1) \times (n-1)$ matrix consisting of the first $n-1$ rows is invertible. But then there is a corresponding permutation matrix P_1, so that the first $n-1$ rows and columns of $P_1 A$ form an invertible matrix A'. Applying the induction hypothesis to the $(n-1) \times (n-1)$ matrix A', we see that there some permutation matrix P_2 (leaving the nth row fixed), so that $(P_2 P_1 A)(1:k, 1:k)$ is invertible, for $k = 1, \ldots, n-1$. Since A is invertible in the first place and P_1 and P_2 are invertible, $P_1 P_2 A$ is also invertible, and we are done. $\square$

Remark: One can also prove Proposition 8.4 using a clever reordering of the Gaussian elimination steps suggested by Trefethen and Bau [176] (Lecture 21). Indeed, we know that if A is invertible, then there are permutation matrices P_i and products of elementary matrices E_i, so that

$$A_n = E_{n-1} P_{n-1} \cdots E_2 P_2 E_1 P_1 A,$$

where $U = A_n$ is upper-triangular. For example, when $n = 4$, we have $E_3 P_3 E_2 P_2 E_1 P_1 A = U$. We can define new matrices E_1', E_2', E_3' which are still products of elementary matrices so that we have

$$E_3' E_2' E_1' P_3 P_2 P_1 A = U.$$

Indeed, if we let $E_3' = E_3$, $E_2' = P_3 E_2 P_3^{-1}$, and $E_1' = P_3 P_2 E_1 P_2^{-1} P_3^{-1}$, we easily verify that each E_k' is a product of elementary matrices and that

$$E_3' E_2' E_1' P_3 P_2 P_1 = E_3 (P_3 E_2 P_3^{-1})(P_3 P_2 E_1 P_2^{-1} P_3^{-1}) P_3 P_2 P_1 = E_3 P_3 E_2 P_2 E_1 P_1.$$