and by definition of weak convergence

$$\lim_{\ell \mapsto \infty} \langle \nabla J_v, z_\ell \rangle = \langle \nabla J_v, v \rangle,$$

so $\lim_{\ell \mapsto \infty} \langle \nabla J_v, z_\ell - v \rangle = 0$, and by definition of lim inf we get

$$J(v) \leq \liminf_{\ell \mapsto \infty} J(z_\ell)$$

for every sequence $(z_\ell)_{\ell \geq 0}$ converging weakly to some element $v \in V$.

Step 4. The weak limit $u \in U$ of the subsequence $(w_\ell)_{\ell \geq 0}$ extracted from the minimizing sequence $(u_k)_{k \geq 0}$ satisfies the equation

$$J(u) = \inf_{v \in U} J(v).$$

By Step (1) and Step (2) the subsequence $(w_\ell)_{\ell \geq 0}$ of the sequence $(u_k)_{k \geq 0}$ converges weakly to some element $u \in U$, so by Step (3) we have

$$J(u) \leq \liminf_{\ell \mapsto \infty} J(w_\ell).$$

On the other hand, by definition of $(w_\ell)_{\ell \geq 0}$ as a subsequence of $(u_k)_{k \geq 0}$, since the sequence $(J(u_k))_{k \geq 0}$ converges to $J(v)$, we have

$$J(u) \leq \liminf_{\ell \mapsto \infty} J(w_\ell) = \lim_{k \mapsto \infty} J(u_k) = \inf_{v \in U} J(v),$$

which proves that $u \in U$ achieves the minimum of J on U. $\square$

Remark: Theorem 49.2 still holds if we only assume that J is convex and continuous. It also holds in a reflexive Banach space, of which Hilbert spaces are a special case; see Brezis [31], Corollary 3.23.

Theorem 49.2 is a rather general theorem whose proof is quite involved. For functions J of a certain type, we can obtain existence and uniqueness results that are easier to prove. This is true in particular for quadratic functionals.

49.3 Minima of Quadratic Functionals

Definition 49.4. Let V be a real Hilbert space. A function $J\colon V \to \mathbb{R}$ is called a *quadratic functional* if it is of the form

$$J(v) = \frac{1}{2} a(v, v) - h(v),$$

where $a\colon V \times V \to \mathbb{R}$ is a bilinear form which is symmetric and continuous, and $h\colon V \to \mathbb{R}$ is a continuous linear form.