Observe that the Problem (SVM_{h1}) has an optimal solution $\delta > 0$ iff the two subsets are linearly separable. We used the constraint $\|w\| \leq 1$ rather than $\|w\| = 1$ because the former is qualified, whereas the latter is not. But if (w, b, δ) is an optimal solution, then $\|w\| = 1$, as shown in the following proposition.

Proposition 50.12. *If (w, b, δ) is an optimal solution of Problem (SVM_{h1}), so in particular $\delta > 0$, then we must have $\|w\| = 1$.*

Proof. First, if $w = 0$, then we get the two inequalities

$$-b \geq \delta, \quad b \geq \delta,$$

which imply that $b \leq -\delta$ and $b \geq \delta$ for some positive δ, which is impossible. But then, if $w \neq 0$ and $\|w\| < 1$, by dividing both sides of the inequalities by $\|w\| < 1$ we would obtain the better solution $(w/\|w\|, b/\|w\|, \delta/\|w\|)$, since $\|w\| < 1$ implies that $\delta/\|w\| > \delta$. $\square$

We now prove that if the two subsets are linearly separable, then Problem (SVM_{h1}) has a unique optimal solution.

Theorem 50.13. *If two disjoint subsets of p blue points $\{u_i\}_{i=1}^p$ and q red points $\{v_j\}_{j=1}^q$ are linearly separable, then Problem (SVM_{h1}) has a unique optimal solution consisting of a hyperplane of equation $w^\top x - b = 0$ separating the two subsets with maximum margin δ. Furthermore, if we define $c_1(w)$ and $c_2(w)$ by*

$$\begin{aligned} c_1(w) &= \min_{1 \leq i \leq p} w^\top u_i \\ c_2(w) &= \max_{1 \leq j \leq q} w^\top v_j, \end{aligned}$$

then w is the unique maximum of the function

$$\rho(w) = \frac{c_1(w) - c_2(w)}{2}$$

over the convex subset U of $\mathbb{R}^n$ given by the inequalities

$$\begin{aligned} w^\top u_i - b &\geq \delta \qquad i = 1, \ldots, p \\ -w^\top v_j + b &\geq \delta \qquad j = 1, \ldots, q \\ \|w\| &\leq 1, \end{aligned}$$

and

$$b = \frac{c_1(w) + c_2(w)}{2}.$$