from $E_1 \times \cdots \times E_n$ to K. The map $l_{v_1^*,\ldots,v_n^*}$ extends uniquely to a linear map $L_{v_1^*,\ldots,v_n^*} : E_1 \otimes \cdots \otimes E_n \longrightarrow K$ making the following diagram commute.

$$\begin{tikzcd} E_1 \times \cdots \times E_n \arrow[r, "\iota_\otimes"] \arrow[dr, "l_{v_1^*,\ldots,v_n^*}"'] & E_1 \otimes \cdots \otimes E_n \arrow[d, "L_{v_1^*,\ldots,v_n^*}"] \\ & K \end{tikzcd}$$

We also have the multilinear map

$$(v_1^*, \ldots, v_n^*) \mapsto L_{v_1^*,\ldots,v_n^*}$$

from $E_1^* \times \cdots \times E_n^*$ to $\mathrm{Hom}(E_1 \otimes \cdots \otimes E_n, K)$, which extends to a unique linear map L from $E_1^* \otimes \cdots \otimes E_n^*$ to $\mathrm{Hom}(E_1 \otimes \cdots \otimes E_n, K)$ making the following diagram commute.

$$\begin{tikzcd} E_1^* \times \cdots \times E_n^* \arrow[r, "\iota_\otimes"] \arrow[dr, "L_{v_1^*,\ldots,v_n^*}"'] & E_1^* \otimes \cdots \otimes E_n^* \arrow[d, "L"] \\ & \mathrm{Hom}(E_1 \otimes \cdots \otimes E_n; K) \end{tikzcd}$$

However, in view of the isomorphism

$$\mathrm{Hom}(U \otimes V, W) \cong \mathrm{Hom}(U, \mathrm{Hom}(V, W))$$

given by Proposition 33.15, with $U = E_1^* \otimes \cdots \otimes E_n^*$, $V = E_1 \otimes \cdots \otimes E_n$ and $W = K$, we can view L as a linear map

$$L \colon (E_1^* \otimes \cdots \otimes E_n^*) \otimes (E_1 \otimes \cdots \otimes E_n) \to K,$$

which corresponds to a bilinear map

$$\langle -, - \rangle \colon (E_1^* \otimes \cdots \otimes E_n^*) \times (E_1 \otimes \cdots \otimes E_n) \longrightarrow K, \tag{$\dagger\dagger$}$$

via the isomorphism $(U \otimes V)^* \cong \mathrm{Hom}(U, V; K)$ given by Proposition 33.8. This pairing is given explicitly on generators by

$$\langle v_1^* \otimes \cdots \otimes v_n^*, u_1 \ldots, u_n \rangle = v_1^*(u_1) \cdots v_n^*(u_n).$$

This pairing is nondegenerate, as proved below.

Proof. If $(e_1^1, \ldots, e_{m_1}^1), \ldots, (e_1^n, \ldots, e_{m_n}^n)$ are bases for $E_1, \ldots, E_n$, then for every basis element $(e_{i_1}^1)^* \otimes \cdots \otimes (e_{i_n}^n)^*$ of $E_1^* \otimes \cdots \otimes E_n^*$, and any basis element $e_{j_1}^1 \otimes \cdots \otimes e_{j_n}^n$ of $E_1 \otimes \cdots \otimes E_n$, we have

$$\langle (e_{i_1}^1)^* \otimes \cdots \otimes (e_{i_n}^n)^*, e_{j_1}^1 \otimes \cdots \otimes e_{j_n}^n \rangle = \delta_{i_1\, j_1} \cdots \delta_{i_n\, j_n},$$

where $\delta_{i\, j}$ is *Kronecker delta*, defined such that $\delta_{i\, j} = 1$ if $i = j$, and 0 otherwise. Given any $\alpha \in E_1^* \otimes \cdots \otimes E_n^*$, assume that $\langle \alpha, \beta \rangle = 0$ for all $\beta \in E_1 \otimes \cdots \otimes E_n$. The vector α is a finite