This proves that $f(y) - y$ is isotropic for any nonzero isotropic vector y. Since by hypothesis $f(u) - u$ is isotropic for every nonisotropic vector u, we proved that $f(u) - u$ is isotropic for *every* $u \in E$. If we let $W = \mathrm{Im}(f - \mathrm{id})$, then every vector in W is isotropic, and thus W is totally isotropic (recall that we assumed that $\mathrm{char}(K) \neq 2$, so φ is determined by Φ). For any $u \in E$ and any $v \in W^{\perp}$, since W is totally isotropic, we have

$$\varphi(f(u) - u, f(v) - v) = 0,$$

and since $f(u) - u \in W$ and $v \in W^{\perp}$, we have $\varphi(f(u) - u, v) = 0$, and so

$$\begin{aligned} 0 &= \varphi(f(u) - u, f(v) - v) \\ &= \varphi(f(u), f(v)) - \varphi(u, f(v)) - \varphi(f(u) - u, v) \\ &= \varphi(u, v) - \varphi(u, f(v)) \\ &= \varphi(u, v - f(v)), \end{aligned}$$

for all $u \in E$. Since φ is nonsingular, this means that $f(v) = v$, for all $v \in W^{\perp}$. However, by hypothesis, no nonisotropic vector is left fixed, which implies that $W^{\perp}$ is also totally isotropic. In summary, we proved that $W \subseteq W^{\perp}$ and $W^{\perp} \subseteq W^{\perp\perp} = W$, that is,

$$W = W^{\perp}.$$

Since, $\dim(W) + \dim(W^{\perp}) = n$, we conclude that W is a totally isotropic subspace of E such that

$$\dim(W) = n/2.$$

By Proposition 29.29, the space E is an Artinian space of dimension $n = 2m$. Since $W = W^{\perp}$ and $f(W) = W$, by Proposition 29.42, the isometry f is a rotation. $\square$

Remarks:

1. Another way to finish the proof of Proposition 29.43 is to prove that if f is an isometry, then

   $$\mathrm{Ker}\,(f - \mathrm{id}) = (\mathrm{Im}(f - \mathrm{id}))^{\perp}.$$

   After having proved that $W = \mathrm{Im}(f - \mathrm{id})$ is totally isotropic, we get

   $$\mathrm{Ker}\,(f - \mathrm{id}) = \mathrm{Im}(f - \mathrm{id}),$$

   which implies that $(f - \mathrm{id})^2 = 0$. From this, we deduce that $\det(f) = 1$. For details, see Jacobson [98] (Chapter 6, Section 6).

2. If $f = \tau_{H_k} \circ \cdots \circ \tau_{H_1}$, where the H_i are hyperplanes, then it can be shown that

   $$\dim(H_1 \cap H_2 \cap \cdots \cap H_s) \geq n - s.$$

   Now, since each H_i is left fixed by τ_{H_i}, we see that every vector in $H_1 \cap \cdots \cap H_s$ is left fixed by f. In particular, if $s < n$, then f has some nonzero fixed point. As a consequence, an isometry without fixed points requires n hyperplane reflections.