Problem 9.16. Recall that the Hilbert matrix is given by

$$H_{ij}^{(n)} = \left(\frac{1}{i+j-1}\right).$$

(1) Prove that

$$\det(H^{(n)}) = \frac{(1!2!\cdots(n-1)!)^4}{1!2!\cdots(2n-1)!},$$

thus the reciprocal of an integer.

Hint. Use Problem 7.13.

(2) Amazingly, the entries of the inverse of $H^{(n)}$ are integers. Prove that $(H^{(n)})^{-1} = (\alpha_{ij})$, with

$$\alpha_{ij} = (-1)^{i+j}(i+j-1)\binom{n+i-1}{n-j}\binom{n+j-1}{n-i}\binom{i+j-2}{i-1}^2.$$