Remarks: As with the previous proposition, the assumption of finite dimensionality is crucial. The proof provides an *a priori* bound on the error $\|u_k - u\|$.

If J is a an elliptic quadratic functional

$$J(v) = \frac{1}{2}\langle Av, v\rangle - \langle b, v\rangle,$$

we can use the orthogonality of the descent directions ∇J_{u_k} and $\nabla J_{u_{k+1}}$ to compute ρ_k. Indeed, we have $\nabla J_v = Av - b$, so

$$0 = \langle \nabla J_{u_{k+1}}, \nabla J_{u_k}\rangle = \langle A(u_k - \rho_k(Au_k - b)) - b, Au_k - b\rangle,$$

which yields

$$\rho_k = \frac{\|w_k\|^2}{\langle Aw_k, w_k\rangle}, \quad \text{with} \quad w_k = Au_k - b = \nabla J_{u_k}.$$

Consequently, a step of the iteration method takes the following form:

(1) Compute the vector
$$w_k = Au_k - b.$$

(2) Compute the scalar
$$\rho_k = \frac{\|w_k\|^2}{\langle Aw_k, w_k\rangle}.$$

(3) Compute the next vector u_{k+1} by
$$u_{k+1} = u_k - \rho_k w_k.$$

This method is of particular interest when the computation of Aw for a given vector w is cheap, which is the case if A is sparse.

Example 49.1. For a particular illustration of this method, we turn to the example provided by Shewchuk, with $A = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}$ and $b = \begin{pmatrix} 2 \\ -8 \end{pmatrix}$, namely

$$\begin{aligned} J(x, y) &= \frac{1}{2}\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 2 & -8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\ &= \frac{3}{2}x^2 + 2xy + 3y^2 - 2x + 8y. \end{aligned}$$

This quadratic ellipsoid, which is illustrated in Figure 49.2, has a unique minimum at $(2, -2)$. In order to find this minimum via the gradient descent with optimal step size