(3) Consider the $n \times n$ matrix (called a *companion matrix*)

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & 0 & \cdots & 0 & -a_{n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & -a_2 \\ 0 & 0 & 0 & \cdots & 1 & -a_1 \end{pmatrix}.$$

Prove that the characteristic polynomial $\chi_A(z) = \det(zI - A)$ of A is given by

$$\chi_A(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n.$$

Hint. Use induction.

Explain why finding the roots of a polynomial (with real or complex coefficients) and finding the eigenvalues of a (real or complex) matrix are equivalent problems, in the sense that if we have a method for solving one of these problems, then we have a method to solve the other.

Problem 15.10. Let A be a complex $n \times n$ matrix. Prove that if A is invertible and if the eigenvalues of A are $(\lambda_1, \ldots, \lambda_n)$, then the eigenvalues of A^{-1} are $(\lambda_1^{-1}, \ldots, \lambda_n^{-1})$. Prove that if u is an eigenvector of A for λ_i, then u is an eigenvector of A^{-1} for λ_i^{-1}.

Problem 15.11. Prove that every complex matrix is the limit of a sequence of diagonalizable matrices that have distinct eigenvalues.

Problem 15.12. Consider the following tridiagonal $n \times n$ matrices

$$A = \begin{pmatrix} 2 & -1 & 0 & & & \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & 2 & -1 \\ & & 0 & -1 & 2 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 & 0 & & \\ 1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & 1 \\ & & 0 & 1 & 0 \end{pmatrix}.$$

Observe that $A = 2I - S$ and show that the eigenvalues of A are $\lambda_k = 2 - \mu_k$, where the μ_k are the eigenvalues of S.

(2) Using Problem 10.6, prove that the eigenvalues of the matrix A are given by

$$\lambda_k = 4\sin^2\left(\frac{k\pi}{2(n+1)}\right), \quad k = 1, \ldots, n.$$

Show that A is symmetric positive definite.

(3) Find the condition number of A with respect to the 2-norm.

(4) Show that an eigenvector $(y_1^{(k)}, \ldots, y_n^{(k)})$ associated wih the eigenvalue λ_k is given by

$$y_j^{(k)} = \sin\left(\frac{kj\pi}{n+1}\right), \quad j = 1, \ldots, n.$$