The following proposition is an obvious generalization of Proposition 6.1.

Proposition 6.10. *Let I be any nonempty set, let $(E_i)_{i\in I}$ be a family of vector spaces, and let G be any vector space. The direct sum $\coprod_{i\in I} E_i$ is a vector space, and for every family $(h_i)_{i\in I}$ of linear maps $h_i\colon E_i \to G$, there is a unique linear map*

$$\Big(\sum_{i\in I} h_i\Big)\colon \coprod_{i\in I} E_i \to G,$$

such that, $(\sum_{i\in I} h_i) \circ in_i = h_i$, for every $i \in I$.

Remarks:

(1) One might wonder why the direct sum $\coprod_{i\in I} E_i$ consists of familes of finite support instead of arbitrary families; in other words, why didn't we define the direct sum of the family $(E_i)_{i\in I}$ as $\prod_{i\in I} E_i$? The product space $\prod_{i\in I} E_i$ with addition and scalar multiplication defined as above is also a vector space but the problem is that any linear map $\widehat{h}\colon \prod_{i\in I} E_i \to G$ such that $\widehat{h} \circ in_i = h_i$ for all $\in I$ must be given by

$$\widehat{h}((u_i)_{\in I}) = \sum_{i\in I} h_i(u_i),$$

and if I is infinite, the sum on the right-hand side is infinite, and thus undefined! If I is finite then $\prod_{i\in I} E_i$ and $\coprod_{i\in I} E_i$ are isomorphic.

(2) When $E_i = E$, for all $i \in I$, we denote $\coprod_{i\in I} E_i$ by $E^{(I)}$. In particular, when $E_i = K$, for all $i \in I$, we find the vector space $K^{(I)}$ of Definition 3.11.

We also have the following basic proposition about injective or surjective linear maps.

Proposition 6.11. *Let E and F be vector spaces, and let $f\colon E \to F$ be a linear map. If $f\colon E \to F$ is injective, then there is a surjective linear map $r\colon F \to E$ called a retraction, such that $r \circ f = \mathrm{id}_E$. See Figure 6.2. If $f\colon E \to F$ is surjective, then there is an injective linear map $s\colon F \to E$ called a section, such that $f \circ s = \mathrm{id}_F$. See Figure 6.3.*

Proof. Let $(u_i)_{i\in I}$ be a basis of E. Since $f\colon E \to F$ is an injective linear map, by Proposition 3.18, $(f(u_i))_{i\in I}$ is linearly independent in F. By Theorem 3.7, there is a basis $(v_j)_{j\in J}$ of F, where $I \subseteq J$, and where $v_i = f(u_i)$, for all $i \in I$. By Proposition 3.18, a linear map $r\colon F \to E$ can be defined such that $r(v_i) = u_i$, for all $i \in I$, and $r(v_j) = w$ for all $j \in (J - I)$, where w is any given vector in E, say $w = 0$. Since $r(f(u_i)) = u_i$ for all $i \in I$, by Proposition 3.18, we have $r \circ f = \mathrm{id}_E$.

Now, assume that $f\colon E \to F$ is surjective. Let $(v_j)_{j\in J}$ be a basis of F. Since $f\colon E \to F$ is surjective, for every $v_j \in F$, there is some $u_j \in E$ such that $f(u_j) = v_j$. Since $(v_j)_{j\in J}$ is a basis of F, by Proposition 3.18, there is a unique linear map $s\colon F \to E$ such that $s(v_j) = u_j$. Also, since $f(s(v_j)) = v_j$, by Proposition 3.18 (again), we must have $f \circ s = \mathrm{id}_F$. $\square$

The converse of Proposition 6.11 is obvious.