Definition 14.3. Given a sesquilinear form $\varphi\colon E \times E \to \mathbb{C}$, the function $\Phi\colon E \to \mathbb{C}$ defined such that $\Phi(u) = \varphi(u, u)$ for all $u \in E$ is called the *quadratic form* associated with φ.

The standard example of a Hermitian form on $\mathbb{C}^n$ is the map φ defined such that

$$\varphi((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = x_1\overline{y_1} + x_2\overline{y_2} + \cdots + x_n\overline{y_n}.$$

This map is also positive definite, but before dealing with these issues, we show the following useful proposition.

Proposition 14.1. *Given a complex vector space E, the following properties hold:*

(1) A sesquilinear form $\varphi\colon E \times E \to \mathbb{C}$ is a Hermitian form iff $\varphi(u, u) \in \mathbb{R}$ for all $u \in E$.

(2) If $\varphi\colon E \times E \to \mathbb{C}$ is a sesquilinear form, then

$$\begin{aligned} 4\varphi(u, v) &= \varphi(u + v, u + v) - \varphi(u - v, u - v) \\ &\quad + i\varphi(u + iv, u + iv) - i\varphi(u - iv, u - iv), \end{aligned}$$

and

$$2\varphi(u, v) = (1 + i)(\varphi(u, u) + \varphi(v, v)) - \varphi(u - v, u - v) - i\varphi(u - iv, u - iv).$$

These are called **polarization identities**.

Proof. (1) If φ is a Hermitian form, then

$$\varphi(v, u) = \overline{\varphi(u, v)}$$

implies that

$$\varphi(u, u) = \overline{\varphi(u, u)},$$

and thus $\varphi(u, u) \in \mathbb{R}$. If φ is sesquilinear and $\varphi(u, u) \in \mathbb{R}$ for all $u \in E$, then

$$\varphi(u + v, u + v) = \varphi(u, u) + \varphi(u, v) + \varphi(v, u) + \varphi(v, v),$$

which proves that

$$\varphi(u, v) + \varphi(v, u) = \alpha,$$

where α is real, and changing u to iu, we have

$$i(\varphi(u, v) - \varphi(v, u)) = \beta,$$

where β is real, and thus

$$\varphi(u, v) = \frac{\alpha - i\beta}{2} \quad \text{and} \quad \varphi(v, u) = \frac{\alpha + i\beta}{2},$$

proving that φ is Hermitian.

(2) These identities are verified by expanding the right-hand side, and we leave them as an exercise. ∎