Since d_k is a descent direction, we must have $\langle \nabla J_{u_k}, d_k \rangle < 0$, so for t small enough the condition $J(u_k + td_k) \leq J(u_k) + \alpha t \langle \nabla J_{u_k}, d_k \rangle$ will hold and the search will stop. It can be shown that the exit inequality $J(u_k + td_k) \leq J(u_k) + \alpha t \langle \nabla J_{u_k}, d_k \rangle$ holds for all $t \in (0, t_0]$, for some $t_0 > 0$. Thus the backtracking line search stops with a step length ρ_k that satisfies $\rho_k = 1$ or $\rho_k \in (\beta t_0, t_0]$. Care has to be exercised so that $u_k + \rho_k d_k \in \mathrm{dom}(J)$. For more details, see Boyd and Vandenberghe [29] (Section 9.2).

We now consider one of the simplest methods for choosing the directions of descent in the case where $V = \mathbb{R}^n$, which is to pick the directions of the coordinate axes in a cyclic fashion. Such a method is called the *method of relaxation*.

If we write

$$u_k = (u_1^k, u_2^k, \ldots, u_n^k),$$

then the components u_i^{k+1} of u_{k+1} are computed in terms of u_k by solving from top down the following system of equations:

$$\begin{aligned} J(\mathbf{u_1^{k+1}}, u_2^k, u_3^k, \ldots, u_n^k) &= \inf_{\lambda \in \mathbb{R}} J(\lambda, u_2^k, u_3^k, \ldots, u_n^k) \\ J(u_1^{k+1}, \mathbf{u_2^{k+1}}, u_3^k, \ldots, u_n^k) &= \inf_{\lambda \in \mathbb{R}} J(u_1^{k+1}, \lambda, u_3^k, \ldots, u_n^k) \\ &\vdots \\ J(u_1^{k+1}, \ldots, u_{n-1}^{k+1}, \mathbf{u_n^{k+1}}) &= \inf_{\lambda \in \mathbb{R}} J(u_1^{k+1}, \ldots, u_{n-1}^{k+1}, \lambda). \end{aligned}$$

Another and more informative way to write the above system is to define the vectors $u_{k;i}$ by

$$\begin{aligned} u_{k;0} &= (u_1^k, u_2^k, \ldots, u_n^k) \\ u_{k;1} &= (u_1^{k+1}, u_2^k, \ldots, u_n^k) \\ &\vdots \\ u_{k;i} &= (u_1^{k+1}, \ldots, u_i^{k+1}, u_{i+1}^k, \ldots, u_n^k) \\ &\vdots \\ u_{k;n} &= (u_1^{k+1}, u_2^{k+1}, \ldots, u_n^{k+1}). \end{aligned}$$

Note that $u_{k;0} = u_k$ and $u_{k;n} = u_{k+1}$. Then our minimization problem can be written as

$$\begin{aligned} J(u_{k;1}) &= \inf_{\lambda \in \mathbb{R}} J(u_{k;0} + \lambda e_1) \\ &\vdots \\ J(u_{k;i}) &= \inf_{\lambda \in \mathbb{R}} J(u_{k;i-1} + \lambda e_i) \\ &\vdots \\ J(u_{k;n}) &= \inf_{\lambda \in \mathbb{R}} J(u_{k;n-1} + \lambda e_n), \end{aligned}$$