Figure 44.5: An icosahedron is an example of an $\mathcal{H}$-polytope.

which is tangent to the x-axis at the origin. Then the cone(D) consists of the open upper half-plane *plus* the origin $(0, 0)$, but this set is not closed.

Proposition 44.2. *Every polyhedral cone C is closed.*

Proof. This is proven by showing that

1. Every primitive cone is closed, where a *primitive cone* is a polyhedral cone spanned by linearly independent vectors.

2. A polyhedral cone C is the union of finitely many primitive cones.

Assume that $(a_1, \ldots, a_m)$ are linearly independent vectors in $\mathbb{R}^n$, and consider any sequence $(x^{(k)})_{k\geq 0}$

$$x^{(k)} = \sum_{i=1}^{m} \lambda_i^{(k)} a_i$$

of vectors in the primitive cone cone($\{a_1, \ldots, a_m\}$), which means that $\lambda_j^{(k)} \geq 0$ for $i = 1, \ldots, m$ and all $k \geq 0$. The vectors $x^{(k)}$ belong to the subspace U spanned by $(a_1, \ldots, a_m)$, and U is closed. Assume that the sequence $(x^{(k)})_{k\geq 0}$ converges to a limit $x \in \mathbb{R}^n$. Since U is closed and $x^{(k)} \in U$ for all $k \geq 0$, we have $x \in U$. If we write $x = x_1 a_1 + \cdots + x_m a_m$, we would like to prove that $x_i \geq 0$ for $i = 1, \ldots, m$. The sequence the $(x^{(k)})_{k\geq 0}$ converges to x iff

$$\lim_{k \mapsto \infty} \left\| x^{(k)} - x \right\| = 0,$$

iff

$$\lim_{k \mapsto \infty} \left(\sum_{i=1}^{m} |\lambda_i^{(k)} - x_i|^2 \right)^{1/2} = 0$$

iff

$$\lim_{k \mapsto \infty} \lambda_i^{(k)} = x_i, \quad i = 1, \ldots, m.$$