Similarly, we have the commutative diagram

$$\begin{tikzcd} E_1 \times \cdots \times E_n \arrow[r, "\varphi_2"] \arrow[dr, "\varphi_2"'] & T_2 \arrow[d, "\mathrm{id}"] \\ & T_2, \end{tikzcd}$$

and we must have

$$(\varphi_2)_{\otimes} \circ (\varphi_1)_{\otimes} = \mathrm{id}.$$

This shows that $(\varphi_1)_{\otimes}$ and $(\varphi_2)_{\otimes}$ are inverse linear maps, and thus, $(\varphi_2)_{\otimes}\colon T_1 \to T_2$ is an isomorphism between T_1 and T_2. $\square$

Now that we have shown that tensor products are unique up to isomorphism, we give a construction that produces them. Tensor products are obtained from free vector spaces by a quotient process, so let us begin by describing the construction of the free vector space generated by a set.

For simplicity assume that our set I is finite, say

$$I = \{\heartsuit, \diamondsuit, \spadesuit, \clubsuit\}.$$

The construction works for any field K (and in fact for any commutative ring A, in which case we obtain the free A-module generated by I). Assume that $K = \mathbb{R}$. The *free vector space generated by* I is the set of all formal linear combinations of the form

$$a\heartsuit + b\diamondsuit + c\spadesuit + d\clubsuit,$$

with $a, b, c, d \in \mathbb{R}$. It is assumed that the order of the terms does not matter. For example,

$$2\heartsuit - 5\diamondsuit + 3\spadesuit = -5\diamondsuit + 2\heartsuit + 3\spadesuit.$$

Addition and multiplication by a scalar are are defined as follows:

$$\begin{aligned}(a_1\heartsuit + b_1\diamondsuit + c_1\spadesuit + d_1\clubsuit) &+ (a_2\heartsuit + b_2\diamondsuit + c_2\spadesuit + d_2\clubsuit) \\ &= (a_1 + a_2)\heartsuit + (b_1 + b_2)\diamondsuit + (c_1 + c_2)\spadesuit + (d_1 + d_2)\clubsuit,\end{aligned}$$

and

$$\alpha \cdot (a\heartsuit + b\diamondsuit + c\spadesuit + d\clubsuit) = \alpha a\heartsuit + \alpha b\diamondsuit + \alpha c\spadesuit + \alpha d\clubsuit,$$

for all $a, b, c, d, \alpha \in \mathbb{R}$. With these operations, it is immediately verified that we obtain a vector space denoted $\mathbb{R}^{(I)}$. The set I can be viewed as embedded in $\mathbb{R}^{(I)}$ by the injection ι given by

$$\iota(\heartsuit) = 1\heartsuit, \quad \iota(\diamondsuit) = 1\diamondsuit, \quad \iota(\spadesuit) = 1\spadesuit, \quad \iota(\clubsuit) = 1\clubsuit.$$

Thus, $\mathbb{R}^{(I)}$ can be viewed as the vector space with the special basis $I = \{\heartsuit, \diamondsuit, \spadesuit, \clubsuit\}$. In our case, $\mathbb{R}^{(I)}$ is isomorophic to $\mathbb{R}^4$.