and

$$d(x_0, y_0) \leq d(x_0, x) + d(x, y) + d(y, y_0) = d(x, y) + d(x_0, x) + d(y_0, y).$$

Consequently,

$$|d(x, y) - d(x_0, y_0)| \leq d(x_0, x) + d(y_0, y),$$

which proves that d is continuous at (x_0, y_0). In fact this shows that d is uniformly continuous; see Definition 37.36.

Given any nonempty subset A of E, by Proposition 37.2, the map $x \mapsto d(x, A)$ is continuous (in fact, uniformy continuous).

Similarly, for a normed vector space $(E, \| \, \|)$, the norm $\| \, \| \colon E \to \mathbb{R}$ is (uniformly) continuous.

Given a function $f \colon E_1 \times \cdots \times E_n \to F$, we can fix $n - 1$ of the arguments, say $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n$, and view f as a function of the remaining argument,

$$x_i \mapsto f(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_n),$$

where $x_i \in E_i$. If f is continuous, it is clear that each f_i is continuous.

One should be careful that the converse is false! For example, consider the function $f \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, defined such that,

$$f(x, y) = \frac{xy}{x^2 + y^2} \quad \text{if } (x, y) \neq (0, 0), \quad \text{and} \quad f(0, 0) = 0.$$

The function f is continuous on $\mathbb{R} \times \mathbb{R} - \{(0, 0)\}$, but on the line $y = mx$, with $m \neq 0$, we have $f(x, y) = \frac{m}{1+m^2} \neq 0$, and thus, on this line, $f(x, y)$ does not approach 0 when (x, y) approaches $(0, 0)$. See Figure 37.18.

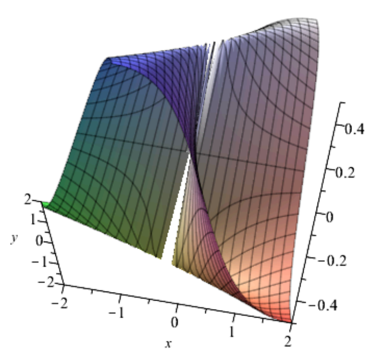

Figure 37.18: The graph of $f(x, y) = \frac{xy}{x^2+y^2}$ for $(x, y) \neq (0, 0)$. The bottom of this graph, which shows the approach along the line $y = -x$, does not have a z value of 0.

The following proposition is useful for showing that real-valued functions are continuous.