and

$$(v - e^{-i\theta}u) \cdot (v - e^{-i\theta}u) = \|v\|^2 + \|u\|^2 - e^{-i\theta}u \cdot v - e^{i\theta}v \cdot u,$$
$$= 2(\|u\|^2 - |u \cdot v|),$$

and thus,

$$-2\,\frac{(u \cdot (v - e^{-i\theta}u))}{\|(v - e^{-i\theta}u)\|^2}\,(v - e^{-i\theta}u) = e^{i\theta}(v - e^{-i\theta}u).$$

But then,

$$s(u) = u + e^{i\theta}(v - e^{-i\theta}u) = u + e^{i\theta}v - u = e^{i\theta}v,$$

and $s(u) = e^{i\theta}v$, as claimed.

(2) This part is easier. Consider the Hermitian reflection

$$\rho_{v,\theta}(u) = u + (e^{i\theta} - 1)\,\frac{(u \cdot v)}{\|v\|^2}\,v.$$

We have

$$\begin{aligned}\rho_{v,\theta}(v) &= v + (e^{i\theta} - 1)\,\frac{(v \cdot v)}{\|v\|^2}\,v,\\ &= v + (e^{i\theta} - 1)v,\\ &= e^{i\theta}v.\end{aligned}$$

Thus, $\rho_{v,\theta}(v) = e^{i\theta}v$. Since $\rho_{v,\theta}$ is linear, changing the argument v to $e^{i\theta}v$, we get

$$\rho_{v,-\theta}(e^{i\theta}v) = v,$$

and thus, $\rho_{v,-\theta} \circ s(u) = v$. ☐

Remarks:

(1) If we use the vector $v + e^{-i\theta}u$ instead of $v - e^{-i\theta}u$, we get $s(u) = -e^{i\theta}v$.

(2) Certain authors, such as Kincaid and Cheney [102] and Ciarlet [41], use the vector $u + e^{i\theta}v$ instead of our vector $v + e^{-i\theta}u$. The effect of this choice is that they also get $s(u) = -e^{i\theta}v$.

(3) If $v = \|u\|\,e_1$, where e_1 is a basis vector, $u \cdot e_1 = a_1$, where a_1 is just the coefficient of u over the basis vector e_1. Then, since $u \cdot e_1 = e^{i\theta}|a_1|$, the choice of the plus sign in the vector $\|u\|\,e_1 + e^{-i\theta}u$ has the effect that the coefficient of this vector over e_1 is $\|u\| + |a_1|$, and no cancellations takes place, which is preferable for numerical stability (we need to divide by the square norm of this vector).