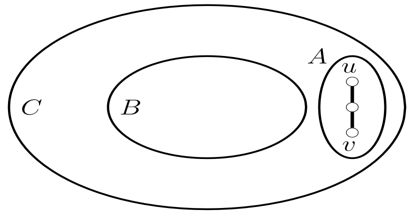

Figure 48.1: Inequality of Proposition 48.3.

Proof. Since A is convex, $\frac{1}{2}(u+v) \in A$ if $u, v \in A$, and thus, $\|\frac{1}{2}(u+v)\| \geq d$. From the parallelogram equality written in the form

$$\left\|\frac{1}{2}(u+v)\right\|^2 + \left\|\frac{1}{2}(u-v)\right\|^2 = \frac{1}{2}\left(\|u\|^2 + \|v\|^2\right),$$

since $\delta < d$, we get

$$\left\|\frac{1}{2}(u-v)\right\|^2 = \frac{1}{2}\left(\|u\|^2 + \|v\|^2\right) - \left\|\frac{1}{2}(u+v)\right\|^2 \leq (d+\delta)^2 - d^2 = 2d\delta + \delta^2 \leq 3d\delta,$$

from which

$$\|v-u\| \leq \sqrt{12d\delta}.$$

$\square$

Definition 48.2. If X is a nonempty subset of a metric space (E, d), for any $a \in E$, recall that we define the *distance* $d(a, X)$ *of* a *to* X as

$$d(a, X) = \inf_{b \in X} d(a, b).$$

Also, the *diameter* $\delta(X)$ *of* X is defined by

$$\delta(X) = \sup\{d(a, b) \mid a, b \in X\}.$$

It is possible that $\delta(X) = \infty$.

We leave the following standard two facts as an exercise (see Dixmier [51]):

Proposition 48.4. *Let E be a metric space.*

(1) *For every subset $X \subseteq E$, $\delta(X) = \delta(\overline{X})$.*

(2) *If E is a complete metric space, for every sequence (F_n) of closed nonempty subsets of E such that $F_{n+1} \subseteq F_n$, if $\lim_{n\to\infty} \delta(F_n) = 0$, then $\bigcap_{n=1}^{\infty} F_n$ consists of a single point.*

We are now ready to prove the crucial projection lemma.