Consequently, $B^\sigma(B^\sigma)^\top$ is independent of the orientation of the underlying graph of G and $L = D - W$ is symmetric and positive semidefinite; that is, the eigenvalues of $L = D - W$ are real and nonnegative.

Another way to prove that L is positive semidefinite is to evaluate the quadratic form $x^\top Lx$.

Proposition 20.4. *For any $m \times m$ symmetric matrix $W = (w_{ij})$, if we let $L = D - W$ where D is the degree matrix associated with W (that is, $d_i = \sum_{j=1}^m w_{ij}$), then we have*

$$x^\top Lx = \frac{1}{2}\sum_{i,j=1}^m w_{i\,j}(x_i - x_j)^2 \quad \text{for all } x \in \mathbb{R}^m.$$

Consequently, $x^\top Lx$ does not depend on the diagonal entries in W, and if $w_{i\,j} \geq 0$ for all $i, j \in \{1, \ldots, m\}$, then L is positive semidefinite.

Proof. We have

$$\begin{aligned}
x^\top Lx &= x^\top Dx - x^\top Wx \\
&= \sum_{i=1}^m d_i x_i^2 - \sum_{i,j=1}^m w_{i\,j}x_i x_j \\
&= \frac{1}{2}\left(\sum_{i=1}^m d_i x_i^2 - 2\sum_{i,j=1}^m w_{i\,j}x_i x_j + \sum_{i=1}^m d_i x_i^2\right) \\
&= \frac{1}{2}\sum_{i,j=1}^m w_{i\,j}(x_i - x_j)^2.
\end{aligned}$$

Obviously, the quantity on the right-hand side does not depend on the diagonal entries in W, and if $w_{i\,j} \geq 0$ for all i, j, then this quantity is nonnegative. ☐

Proposition 20.4 immediately implies the following facts: For any weighted graph $G = (V, W)$,

1. The eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m$ of L are real and nonnegative, and there is an orthonormal basis of eigenvectors of L.

2. The smallest eigenvalue λ_1 of L is equal to 0, and $\mathbf{1}$ is a corresponding eigenvector.

It turns out that the dimension of the nullspace of L (the eigenspace of 0) is equal to the number of connected components of the underlying graph of G.