The problem of classifying sesquilinear forms up to equivalence is an important but very difficult problem. Solving this problem depends intimately on properties of the field K, and a complete answer is only known in a few cases. The problem is easily solved for $K = \mathbb{R}$, $K = \mathbb{C}$. It is also solved for finite fields and for $K = \mathbb{Q}$ (the rationals), but the solution is surprisingly involved!

It is hard to say anything interesting if φ_1 is degenerate and if the linear map f does not have adjoints. The next few propositions make use of natural conditions on φ_1 that yield a useful criterion for being a metric map.

Proposition 29.16. *With the same assumptions as in Definition 29.14 (which imply that φ_1 is nondegenerate), if $f\colon E_1 \to E_2$ is a bijective linear map, then we have*

$$\begin{aligned}&\varphi_1(x, y) = \varphi_2(f(x), f(y)) \quad \textit{for all } x, y \in E_1 \textit{ iff}\\ &f^{-1} = f^{*l} = f^{*r}.\end{aligned}$$

Proof. We have

$$\varphi_1(x, y) = \varphi_2(f(x), f(y))$$

iff

$$\varphi_1(x, y) = \varphi_2(f(x), f(y)) = \varphi_1(x, f^{*l}(f(y))$$

iff

$$\varphi_1(x, (\mathrm{id} - f^{*l} \circ f)(y)) = 0 \quad \text{for all } x \in E_1 \text{ and all } y \in E_2.$$

Since φ_1 is nondegenerate, we must have

$$f^{*l} \circ f = \mathrm{id},$$

which implies that $f^{-1} = f^{*l}$. Similarly,

$$\varphi_1(x, y) = \varphi_2(f(x), f(y))$$

iff

$$\varphi_1(x, y) = \varphi_2(f(x), f(y)) = \varphi_1(f^{*r}(f(x)), y)$$

iff

$$\varphi_1((\mathrm{id} - f^{*r} \circ f)(x), y) = 0 \quad \text{for all } x \in E_1 \text{ and all } y \in E_2.$$

Since φ_1 is nondegenerate, we must have

$$f^{*r} \circ f = \mathrm{id},$$

which implies that $f^{-1} = f^{*r}$. Therefore, $f^{-1} = f^{*l} = f^{*r}$. For the converse, do the computations in reverse. ∎

As a corollary, we get the following important proposition.