for any $s \in S_i$ and any $j \in T_j$, so in fact $X = [A]$ and $X(i,j) = A_{S_i,T_j}$. But remember that we abbreviate $X(i,j)$ as X_{ij}, so the (i,j)th entry in the block matrix $[A]$ of A associated with the partitions $S = S_1 \cup \cdots \cup S_m$ and $T = T_1 \cup \cdots \cup T_n$ should be denoted by $[A]_{ij}$. To minimize notation we will use the simpler notation A_{ij}. Schematically we represent the block matrix $[A]$ as

$$[A] = \begin{pmatrix} A_{S_1,T_1} & \cdots & A_{S_1,T_n} \\ \vdots & \ddots & \vdots \\ A_{S_m,T_1} & \cdots & A_{S_m,T_n} \end{pmatrix} \quad \text{or simply as} \quad [A] = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{pmatrix}.$$

In the simplified notation we lose the information about the index sets of the blocks.

Remark: It is easy to check that the set of $m \times n$ block matrices induced by two partitions $S = S_1 \cup \cdots \cup S_m$ and $T = T_1 \cup \cdots \cup T_n$ is a vector space. In fact, it is isomorphic to the direct sum

$$\bigoplus_{(i,j)\in[m]\times[n]} \mathrm{M}_{S_i,T_j}(K).$$

Addition and rescaling are performed blockwise.

Example 6.2. Let $S = \{1,2,3,4,5,6\}$, with $S_1 = \{1,2\}$, $S_2 = \{3\}$, $S_3 = \{4,5,6\}$, and $T = \{1,2,3,4,5\}$, with $T_1 = \{1,2\}$, $T_2 = \{3,4\}$, $T_3 = \{5\}$, and Then $s_1 = 2$, $s_2 = 1$, $s_3 = 3$ and $t_1 = 2, t_2 = 2, t_3 = 1$. The original matrix is a 6×5 matrix $A = (a_{ij})$. Schematically we obtain a 3×3 matrix of nine blocks. where A_{11}, A_{12}, A_{13} are respectively 2×2, 2×2 and 2×1, A_{21}, A_{22}, A_{23} are respectively 1×2, 1×2 and 1×1, and A_{31}, A_{32}, A_{33} are respectively 3×2, 3×2 and 3×1, as illustrated below.

$$[A] = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \begin{pmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} & \begin{bmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{bmatrix} & \begin{bmatrix} a_{15} \\ a_{25} \end{bmatrix} \\ \begin{bmatrix} a_{31} & a_{32} \end{bmatrix} & \begin{bmatrix} a_{33} & a_{34} \end{bmatrix} & \begin{bmatrix} a_{35} \end{bmatrix} \\ \begin{bmatrix} a_{41} & a_{42} \\ a_{51} & a_{52} \\ a_{61} & a_{62} \end{bmatrix} & \begin{bmatrix} a_{43} & a_{44} \\ a_{53} & a_{54} \\ a_{63} & a_{64} \end{bmatrix} & \begin{bmatrix} a_{45} \\ a_{55} \\ a_{65} \end{bmatrix} \end{pmatrix}.$$

Technically, the blocks are obtained from A in terms of the subsets S_i, T_j. For example,

$$A_{12} = A_{\{1,2\},\{3,4\}} = \begin{bmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{bmatrix}.$$

Example 6.3. Let $S = \{1,2,3\}$, with $S_1 = \{1,3\}$, $S_2 = \{2\}$, and $T = \{1,2,3\}$, with $T_1 = \{1,3\}$, $T_2 = \{2\}$. Then $s_1 = 2$, $s_2 = 1$, and $t_1 = 2, t_2 = 1$. The block 2×2 matrix $[A]$ associated with above partitions is

$$[A] = \begin{pmatrix} A_{\{1,3\},\{1,3\}} & A_{\{1,3\},\{2\}} \\ A_{\{2\},\{1,3\}} & A_{\{2\},\{2\}} \end{pmatrix} = \begin{pmatrix} \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} & \begin{bmatrix} a_{12} \\ a_{32} \end{bmatrix} \\ \begin{bmatrix} a_{21} & a_{23} \end{bmatrix} & \begin{bmatrix} a_{22} \end{bmatrix} \end{pmatrix}.$$