Proposition 35.2. *Let $f\colon E \to F$ be a surjective linear map between two A-modules with F a free module. Given any basis $(v_1, \ldots, v_r)$ of F, for any r vectors $u_1, \ldots, u_r \in E$ such that $f(u_i) = v_i$ for $i = 1, \ldots, r$, the vectors $(u_1, \ldots, u_r)$ are linearly independent and the module E is the direct sum*

$$E = \mathrm{Ker}\,(f) \oplus U,$$

where U is the free submodule of E spanned by the basis $(u_1, \ldots, u_r)$.

Proof. Pick any $w \in E$, write $f(w)$ over the basis $(v_1, \ldots, v_r)$ as $f(w) = a_1 v_1 + \cdots + a_r v_r$, and let $u = a_1 u_1 + \cdots + a_r u_r$. Observe that

$$\begin{aligned} f(w - u) &= f(w) - f(u) \\ &= a_1 v_1 + \cdots + a_r v_r - (a_1 f(u_1) + \cdots + a_r f(u_r)) \\ &= a_1 v_1 + \cdots + a_r v_r - (a_1 v_1 + \cdots + a_r v_r) \\ &= 0. \end{aligned}$$

Therefore, $h = w - u \in \mathrm{Ker}\,(f)$, and since $w = h + u$ with $h \in \mathrm{Ker}\,(f)$ and $u \in U$, we have $E = \mathrm{Ker}\,(f) + U$.

If $u = a_1 u_1 + \cdots + a_r u_r \in U$ also belongs to $\mathrm{Ker}\,(f)$, then

$$0 = f(u) = f(a_1 u_1 + \cdots + a_r u_r) = a_1 v_1 + \cdots + a_r v_r,$$

and since $(v_1, \ldots, v_r)$ is a basis, $a_i = 0$ for $i = 1, \ldots, r$, which shows that $\mathrm{Ker}\,(f) \cap U = (0)$. Therefore, we have a direct sum

$$E = \mathrm{Ker}\,(f) \oplus U.$$

Finally, if

$$a_1 u_1 + \cdots + a_r u_r = 0,$$

the above reasoning shows that $a_i = 0$ for $i = 1, \ldots, r$, so $(u_1, \ldots, u_r)$ are linearly independent. Therefore, the module U is a free module. $\square$

One should be aware that if we have a direct sum of modules

$$U = U_1 \oplus \cdots \oplus U_m,$$

every vector $u \in U$ can be written is a unique way as

$$u = u_1 + \cdots + u_m,$$

with $u_i \in U_i$ but, unlike the case of vector spaces, this does not imply that any m nonzero vectors $(u_1, \ldots, u_m)$ are linearly independent. For example, we have the direct sum

$$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$