Proposition 30.13. *(Euclid's proposition) Let K be a field and let $f, g, h \in K[X]$ be any nonzero polynomials. If f divides gh and f is relatively prime to g, then f divides h.*

Proof. From Proposition 30.11, f and g are relatively prime iff there exist some polynomials $u, v \in K[X]$ such that

$$uf + vg = 1.$$

Then, we have

$$ufh + vgh = h,$$

and since f divides gh, it divides both ufh and vgh, and so, f divides h. $\square$

Proposition 30.14. *Let K be a field and let $f, g_1, \ldots, g_m \in K[X]$ be some nonzero polynomials. If f and g_i are relatively prime for all i, $1 \leq i \leq m$, then f and $g_1 \cdots g_m$ are relatively prime.*

Proof. We proceed by induction on m. The case $m = 1$ is trivial. Let $h = g_2 \cdots g_m$. By the induction hypothesis, f and h are relatively prime. Let d be a gcd of f and $g_1 h$. We claim that d is relatively prime to g_1. Otherwise, d and g_1 would have some nonconstant gcd d_1 which would divide both f and g_1, contradicting the fact that f and g_1 are relatively prime. Now, by Proposition 30.13, since d divides $g_1 h$ and d and g_1 are relatively prime, d divides $h = g_2 \cdots g_m$. But then, d is a divisor of f and h, and since f and h are relatively prime, d must be a constant, and f and $g_1 \cdots g_m$ are relatively prime. $\square$

Definition 30.7 is generalized to any finite number of polynomials as follows.

Definition 30.8. Given any nonzero polynomials $f_1, \ldots, f_n \in K[X]$, where $n \geq 2$, a polynomial $d \in K[X]$ is a *greatest common divisor of* $f_1, \ldots, f_n$ (for short, a *gcd of* $f_1, \ldots, f_n$) if d divides each f_i and whenever $h \in K[X]$ divides each f_i, then h divides d. We say that $f_1, \ldots, f_n$ are *relatively prime* if 1 is a gcd of $f_1, \ldots, f_n$.

It is easily shown that Proposition 30.11 can be generalized to any finite number of polynomials, and similarly for its relevant corollaries. The details are left as an exercise.

Proposition 30.15. *Let K be a field and let $f_1, \ldots, f_n \in K[X]$ be any $n \geq 2$ nonzero polynomials. For every polynomial $d \in K[X]$, the following properties are equivalent:*

(1) The polynomial d is a gcd of $f_1, \ldots, f_n$.

(2) The polynomial d divides each f_i and there exist $u_1, \ldots, u_n \in K[X]$ such that

$$d = u_1 f_1 + \cdots + u_n f_n.$$

(3) The ideals (f_i), and (d) satisfy the equation

$$(d) = (f_1) + \cdots + (f_n).$$