Proof. Consider the set $\mathcal{E}$ of all Cauchy sequences (x_n) in E, and define the relation $\sim$ on $\mathcal{E}$ as follows:

$$(x_n) \sim (y_n) \quad \text{iff} \quad \lim_{n \mapsto \infty} d(x_n, y_n) = 0.$$

It is easy to check that $\sim$ is an equivalence relation on $\mathcal{E}$, and let $\widehat{E} = \mathcal{E}/\sim$ be the quotient set, that is, the set of equivalence classes modulo $\sim$. Our goal is to show that we can endow $\widehat{E}$ with a distance that makes it into a complete metric space satisfying the conditions of the theorem. We proceed in several steps.

Step 1. First, let us construct the function $\varphi\colon E \to \widehat{E}$. For every $a \in E$, we have the constant sequence (a_n) such that $a_n = a$ for all $n \geq 0$, which is obviously a Cauchy sequence. Let $\varphi(a) \in \widehat{E}$ be the equivalence class $[(a_n)]$ of the constant sequence (a_n) with $a_n = a$ for all n. By definition of $\sim$, the equivalence class $\varphi(a)$ is also the equivalence class of all sequences converging to a. The map $a \mapsto \varphi(a)$ is injective because a metric space is Hausdorff, so if $a \neq b$, then a sequence converging to a does not converge to b. After having defined a distance on $\widehat{E}$, we will check that φ is an isometry.

Step 2. Let us now define a distance on $\widehat{E}$. Let $\alpha = [(a_n)]$ and $\beta = [(b_n)]$ be two equivalence classes of Cauchy sequences in E. The triangle inequality implies that

$$d(a_m, b_m) \leq d(a_m, a_n) + d(a_n, b_n) + d(b_n, b_m) = d(a_n, b_n) + d(a_m, a_n) + d(b_m, b_n)$$

and

$$d(a_n, b_n) \leq d(a_n, a_m) + d(a_m, b_m) + d(b_m, b_n) = d(a_m, b_m) + d(a_m, a_n) + d(b_m, b_n),$$

which implies that

$$|d(a_m, b_m) - d(a_n, b_n)| \leq d(a_m, a_n) + d(b_m, b_n).$$

Since (a_n) and (b_n) are Cauchy sequences, it follows that $(d(a_n, b_n))$ is a Cauchy sequence of nonnegative reals. Since $\mathbb{R}$ is complete, the sequence $(d(a_n, b_n))$ has a limit, which we denote by $\widehat{d}(\alpha, \beta)$; that is, we set

$$\widehat{d}(\alpha, \beta) = \lim_{n \mapsto \infty} d(a_n, b_n), \quad \alpha = [(a_n)],\ \beta = [(b_n)].$$

Step 3. Let us check that $\widehat{d}(\alpha, \beta)$ does not depend on the Cauchy sequences (a_n) and (b_n) chosen in the equivalence classes α and β.

If $(a_n) \sim (a'_n)$ and $(b_n) \sim (b'_n)$, then $\lim_{n \mapsto \infty} d(a_n, a'_n) = 0$ and $\lim_{n \mapsto \infty} d(b_n, b'_n) = 0$, and since

$$d(a'_n, b'_n) \leq d(a'_n, a_n) + d(a_n, b_n) + d(b_n, b'_n) = d(a_n, b_n) + d(a_n, a'_n) + d(b_n, b'_n)$$

and

$$d(a_n, b_n) \leq d(a_n, a'_n) + d(a'_n, b'_n) + d(b'_n, b_n) = d(a'_n, b'_n) + d(a_n, a'_n) + d(b_n, b'_n)$$