31.4 The Primary Decomposition Theorem

If $f\colon E \to E$ is a linear map and $\lambda \in K$ is an eigenvalue of f, recall that the eigenspace E_λ associated with λ is the kernel of the linear map $\lambda \mathrm{id} - f$. If all the eigenvalues $\lambda_1 \ldots, \lambda_k$ of f are in K, it may happen that

$$E = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k},$$

but in general there are not enough eigenvectors to span E. What if we generalize the notion of eigenvector and look for (nonzero) vectors u such that

$$(\lambda \mathrm{id} - f)^r(u) = 0, \quad \text{for some } r \geq 1?$$

It turns out that if the minimal polynomial of f is of the form

$$m = (X - \lambda_1)^{r_1} \cdots (X - \lambda_k)^{r_k},$$

then $r = r_i$ does the job for λ_i; that is, if we let

$$W_i = \mathrm{Ker}\,(\lambda_i \mathrm{id} - f)^{r_i},$$

then

$$E = W_1 \oplus \cdots \oplus W_k.$$

This result is very nice but seems to require that the eigenvalues of f all belong to K. Actually, it is a special case of a more general result involving the factorization of the minimal polynomial m into its irreducible monic factors (see Theorem 30.17),

$$m = p_1^{r_1} \cdots p_k^{r_k},$$

where the p_i are distinct irreducible monic polynomials over K.

Theorem 31.10. *(Primary Decomposition Theorem) Let $f\colon E \to E$ be a linear map on the finite-dimensional vector space E over the field K. Write the minimal polynomial m of f as*

$$m = p_1^{r_1} \cdots p_k^{r_k},$$

where the p_i are distinct irreducible monic polynomials over K, and the r_i are positive integers. Let

$$W_i = \mathrm{Ker}\,(p_i^{r_i}(f)), \quad i = 1, \ldots, k.$$

Then

(a) $E = W_1 \oplus \cdots \oplus W_k$.

(b) Each W_i is invariant under f.

(c) The minimal polynomial of the restriction $f \mid W_i$ of f to W_i is $p_i^{r_i}$.