(1) Given any two permutations $\pi_1, \pi_2 \colon [n] \to [n]$, the permutation matrix $P_{\pi_2 \circ \pi_1}$ representing the composition of π_1 and π_2 is equal to the product $P_{\pi_2} P_{\pi_1}$ of the permutation matrices P_{π_1} and P_{π_2} representing π_1 and π_2; that is,

$$P_{\pi_2 \circ \pi_1} = P_{\pi_2} P_{\pi_1}.$$

(2) The matrix $P_{\pi_1^{-1}}$ representing the inverse of the permutation π_1 is the inverse $P_{\pi_1}^{-1}$ of the matrix P_{π_1} representing the permutation π_1; that is,

$$P_{\pi_1^{-1}} = P_{\pi_1}^{-1}.$$

Furthermore,

$$P_{\pi_1}^{-1} = (P_{\pi_1})^{\top}.$$

(3) Prove that if P is the matrix associated with a transposition, then $\det(P) = -1$.

(4) Prove that if P is a permutation matrix, then $\det(P) = \pm 1$.

(5) Use permutation matrices to give another proof of the fact that the parity of the number of transpositions used to express a permutation π depends only on π.