In most cases, U is defined as the set of solutions of a finite sets of *constraints*, either equality constraints $\varphi_i(v) = 0$, or inequality constraints $\varphi_i(v) \leq 0$, where the $\varphi_i \colon \Omega \to \mathbb{R}$ are some given functions. The function J is often called the *functional* of the optimization problem. This is a slightly odd terminology, but it is justified if V is a function space.

The following questions arise naturally:

(1) Results concerning the *existence and uniqueness* of a solution for Problem (M). In the next section we state sufficient conditions either on the domain U or on the function J that ensure the existence of a solution.

(2) The *characterization* of the possible solutions of Problem M. These are conditions for any element $u \in U$ to be a solution of the problem. Such conditions usually involve the derivative dJ_u of J, and possibly the derivatives of the functions φ_i defining U. Some of these conditions become sufficient when the functions φ_i are convex,

(3) The effective construction of *algorithms*, typically iterative algorithms that construct a sequence $(u_k)_{k\geq 1}$ of elements of U whose limit is a solution $u \in U$ of our problem. It is then necessary to understand when and how quickly such sequences converge. Gradient descent methods fall under this category. As a general rule, unconstrained problems (for which $U = \Omega = V$) are (much) easier to deal with than constrained problems (where $U \neq V$).

The material of this chapter is heavily inspired by Ciarlet [41]. *In this chapter it is assumed that V is a real vector space with an inner product $\langle -, - \rangle$.* If V is infinite dimensional, then we assume that it is a real Hilbert space (it is complete). As usual, we write $\|u\| = \langle u, u \rangle^{1/2}$ for the norm associated with the inner product $\langle -, - \rangle$. The reader may want to review Section 48.1, especially the projection lemma and the Riesz representation theorem.

As a matter of terminology, if U is defined by inequality and equality constraints as

$$U = \{v \in \Omega \mid \varphi_i(v) \leq 0,\ i = 1, \ldots, m,\ \psi_j(v) = 0,\ j = 1, \ldots, p\},$$

if J and all the functions φ_i and ψ_j are affine, the problem is said to be *linear* (or a *linear program*), and otherwise *nonlinear*. If J is of the form

$$J(v) = \langle Av, v \rangle - \langle b, v \rangle$$

where A is a nonzero symmetric positive semidefinite matrix and the constraints are affine, the problem is called a *quadratic programming problem*. If the inner product $\langle -, - \rangle$ is the standard Euclidean inner product, J is also expressed as

$$J(v) = v^\top Av - b^\top v.$$