given by

$$(v_1^*, \ldots, v_n^*, u_1, \ldots, u_n) \mapsto \sum_{\sigma \in \mathfrak{S}_n} v_{\sigma(1)}^*(u_1) \cdots v_{\sigma(n)}^*(u_n).$$

Note that the expression on the right-hand side is "almost" the determinant $\det(v_j^*(u_i))$, except that the sign $\mathrm{sgn}(\sigma)$ is missing (where $\mathrm{sgn}(\sigma)$ is the signature of the permutation σ; that is, the parity of the number of transpositions into which σ can be factored). Such an expression is called a *permanent*.

It can be verified that this expression is symmetric w.r.t. the u_i's and also w.r.t. the v_j^*. For any fixed $(v_1^*, \ldots, v_n^*) \in (E^*)^n$, we get a symmetric multilinear map

$$l_{v_1^*, \ldots, v_n^*} \colon (u_1, \ldots, u_n) \mapsto \sum_{\sigma \in \mathfrak{S}_n} v_{\sigma(1)}^*(u_1) \cdots v_{\sigma(n)}^*(u_n)$$

from E^n to K. The map $l_{v_1^*, \ldots, v_n^*}$ extends uniquely to a linear map $L_{v_1^*, \ldots, v_n^*} \colon \mathrm{S}^n(E) \to K$ making the following diagram commute:

$$\begin{tikzcd} E^n \arrow[r, "\iota_\odot"] \arrow[dr, "l_{v_1^*, \ldots, v_n^*}"'] & \mathrm{S}^n(E) \arrow[d, "L_{v_1^*, \ldots, v_n^*}"] \\ & K. \end{tikzcd}$$

We also have the symmetric multilinear map

$$(v_1^*, \ldots, v_n^*) \mapsto L_{v_1^*, \ldots, v_n^*}$$

from $(E^*)^n$ to $\mathrm{Hom}(\mathrm{S}^n(E), K)$, which extends to a linear map L from $\mathrm{S}^n(E^*)$ to $\mathrm{Hom}(\mathrm{S}^n(E), K)$ making the following diagram commute:

$$\begin{tikzcd} (E^*)^n \arrow[r, "\iota_{\odot^*}"] \arrow[dr] & \mathrm{S}^n(E^*) \arrow[d, "L"] \\ & \mathrm{Hom}(\mathrm{S}^n(E), K). \end{tikzcd}$$

However, in view of the isomorphism

$$\mathrm{Hom}(U \otimes V, W) \cong \mathrm{Hom}(U, \mathrm{Hom}(V, W)),$$

with $U = \mathrm{S}^n(E^*)$, $V = \mathrm{S}^n(E)$ and $W = K$, we can view L as a linear map

$$L \colon \mathrm{S}^n(E^*) \otimes \mathrm{S}^n(E) \longrightarrow K,$$

which by Proposition 33.8 corresponds to a bilinear map

$$\langle -, - \rangle \colon \mathrm{S}^n(E^*) \times \mathrm{S}^n(E) \longrightarrow K. \tag{$*$}$$