One needs to verify that the map Ad_q is an invertible linear map from $\mathfrak{su}(2)$ to itself, and that Ad is a group homomorphism, which is easy to do.

In order to associate a rotation ρ_q (in $\mathbf{SO}(3)$) to q, we need to embed $\mathbb{R}^3$ into $\mathbb{H}$ as the pure quaternions, by

$$\psi(x, y, z) = \begin{pmatrix} ix & y + iz \\ -y + iz & -ix \end{pmatrix}, \quad (x, y, z) \in \mathbb{R}^3.$$

Then q defines the map ρ_q (on $\mathbb{R}^3$) given by

$$\rho_q(x, y, z) = \psi^{-1}(q\psi(x, y, z)q^*).$$

Therefore, modulo the isomorphism ψ, the linear map ρ_q is the linear isomorphism Ad_q. In fact, it turns out that ρ_q is a rotation (and so is Ad_q), which we will prove shortly. So, the representation of rotations in $\mathbf{SO}(3)$ by unit quaternions is just the adjoint representation of $\mathbf{SU}(2)$; its image is a subgroup of $\mathbf{GL}(\mathfrak{su}(2))$ isomorphic to $\mathbf{SO}(3)$.

Technically, it is a bit simpler to embed $\mathbb{R}^3$ in the (real) vector spaces of Hermitian matrices with zero trace,

$$\left\{ \begin{pmatrix} x & z - iy \\ z + iy & -x \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\}.$$

Since the matrix $\psi(x, y, z)$ is skew-Hermitian, the matrix $-i\psi(x, y, z)$ is Hermitian, and we have

$$-i\psi(x, y, z) = \begin{pmatrix} x & z - iy \\ z + iy & -x \end{pmatrix} = x\sigma_3 + y\sigma_2 + z\sigma_1,$$

where $\sigma_1, \sigma_2, \sigma_3$ are the *Pauli spin matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Matrices of the form $x\sigma_3 + y\sigma_2 + z\sigma_1$ are Hermitian matrices with zero trace.

It is easy to see that every 2×2 Hermitian matrix with zero trace must be of this form. (observe that $(i\sigma_1, i\sigma_2, i\sigma_3)$ forms a basis of $\mathfrak{su}(2)$. Also, $\mathbf{i} = i\sigma_3$, $\mathbf{j} = i\sigma_2$, $\mathbf{k} = i\sigma_1$.)

Now, if $A = x\sigma_3 + y\sigma_2 + z\sigma_1$ is a Hermitian 2×2 matrix with zero trace, we have

$$(qAq^*)^* = qA^*q^* = qAq^*,$$

so qAq^* is also Hermitian, and

$$\mathrm{tr}(qAq^*) = \mathrm{tr}(Aq^*q) = \mathrm{tr}(A),$$

and qAq^* also has zero trace. Therefore, the map $A \mapsto qAq^*$ preserves the Hermitian matrices with zero trace. We also have

$$\det(x\sigma_3 + y\sigma_2 + z\sigma_1) = \det \begin{pmatrix} x & z - iy \\ z + iy & -x \end{pmatrix} = -(x^2 + y^2 + z^2),$$