Problem 4.5. Consider the $n \times n$ matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & 0 & \cdots & 0 & -a_{n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & -a_2 \\ 0 & 0 & 0 & \cdots & 1 & -a_1 \end{pmatrix},$$

with $a_n \neq 0$.

(1) Find a matrix P such that

$$A^\top = P^{-1}AP.$$

What happens when $a_n = 0$?

Hint. First, try $n = 3, 4, 5$. Such a matrix must have zeros above the "antidiagonal," and identical entries p_{ij} for all $i, j \geq 0$ such that $i + j = n + k$, where $k = 1, \ldots, n$.

(2) Prove that if $a_n = 1$ and if $a_1, \ldots, a_{n-1}$ are integers, then P can be chosen so that the entries in P^{-1} are also integers.

Problem 4.6. For any matrix $A \in \mathrm{M}_n(\mathbb{C})$, let R_A and L_A be the maps from $\mathrm{M}_n(\mathbb{C})$ to itself defined so that

$$L_A(B) = AB, \quad R_A(B) = BA, \quad \text{for all } B \in \mathrm{M}_n(\mathbb{C}).$$

(1) Check that L_A and R_A are linear, and that L_A and R_B commute for all A, B.

Let $\mathrm{ad_A} \colon \mathrm{M}_n(\mathbb{C}) \to \mathrm{M}_n(\mathbb{C})$ be the linear map given by

$$\mathrm{ad}_A(B) = L_A(B) - R_A(B) = AB - BA = [A, B], \quad \text{for all } B \in \mathrm{M}_n(\mathbb{C}).$$

Note that $[A, B]$ is the Lie bracket.

(2) Prove that if A is invertible, then L_A and R_A are invertible; in fact, $(L_A)^{-1} = L_{A^{-1}}$ and $(R_A)^{-1} = R_{A^{-1}}$. Prove that if $A = PBP^{-1}$ for some invertible matrix P, then

$$L_A = L_P \circ L_B \circ L_P^{-1}, \quad R_A = R_P^{-1} \circ R_B \circ R_P.$$

(3) Recall that the n^2 matrices E_{ij} defined such that all entries in E_{ij} are zero except the (i, j)th entry, which is equal to 1, form a basis of the vector space $\mathrm{M}_n(\mathbb{C})$. Consider the partial ordering of the E_{ij} defined such that for $i = 1, \ldots, n$, if $n \geq j > k \geq 1$, then then E_{ij} precedes E_{ik}, and for $j = 1, \ldots, n$, if $1 \leq i < h \leq n$, then E_{ij} precedes E_{hj}.

Draw the Hasse diagram of the partial order defined above when $n = 3$.

There are total orderings extending this partial ordering. How would you find them algorithmically? Check that the following is such a total order:

$$(1, 3), \ (1, 2), \ (1, 1), \ (2, 3), \ (2, 2), \ (2, 1), \ (3, 3), \ (3, 2), \ (3, 1).$$