12.8 QR-Decomposition for Invertible Matrices

Now that we have the definition of an orthogonal matrix, we can explain how the Gram–Schmidt orthonormalization procedure immediately yields the QR-decomposition for matrices.

Definition 12.8. Given any real $n \times n$ matrix A, a *QR-decomposition* of A is any pair of $n \times n$ matrices (Q, R), where Q is an orthogonal matrix and R is an upper triangular matrix such that $A = QR$.

Note that if A is not invertible, then some diagonal entry in R must be zero.

Proposition 12.16. *Given any real $n \times n$ matrix A, if A is invertible, then there is an orthogonal matrix Q and an upper triangular matrix R with positive diagonal entries such that $A = QR$.*

Proof. We can view the columns of A as vectors $A^1, \ldots, A^n$ in $\mathbb{E}^n$. If A is invertible, then they are linearly independent, and we can apply Proposition 12.10 to produce an orthonormal basis using the Gram–Schmidt orthonormalization procedure. Recall that we construct vectors Q^k and Q'^k as follows:

$$Q'^1 = A^1, \qquad Q^1 = \frac{Q'^1}{\|Q'^1\|},$$

and for the inductive step

$$Q'^{k+1} = A^{k+1} - \sum_{i=1}^{k} (A^{k+1} \cdot Q^i)\, Q^i, \qquad Q^{k+1} = \frac{Q'^{k+1}}{\|Q'^{k+1}\|},$$

where $1 \leq k \leq n-1$. If we express the vectors A^k in terms of the Q^i and Q'^i, we get the triangular system

$$\begin{aligned}
A^1 &= \|Q'^1\| Q^1, \\
&\ \vdots \\
A^j &= (A^j \cdot Q^1)\, Q^1 + \cdots + (A^j \cdot Q^i)\, Q^i + \cdots + (A^j \cdot Q^{j-1})\, Q^{j-1} + \|Q'^j\| Q^j, \\
&\ \vdots \\
A^n &= (A^n \cdot Q^1)\, Q^1 + \cdots + (A^n \cdot Q^{n-1})\, Q^{n-1} + \|Q'^n\| Q^n.
\end{aligned}$$

Letting $r_{k\,k} = \|Q'^k\|$, and $r_{i\,j} = A^j \cdot Q^i$ (the reversal of i and j on the right-hand side *is* intentional!), where $1 \leq k \leq n$, $2 \leq j \leq n$, and $1 \leq i \leq j-1$, and letting $q_{i\,j}$ be the ith component of Q^j, we note that $a_{i\,j}$, the ith component of A^j, is given by

$$a_{i\,j} = r_{1\,j} q_{i\,1} + \cdots + r_{i\,j} q_{i\,i} + \cdots + r_{j\,j} q_{i\,j} = q_{i\,1} r_{1\,j} + \cdots + q_{i\,i} r_{i\,j} + \cdots + q_{i\,j} r_{j\,j}.$$

If we let $Q = (q_{i\,j})$, the matrix whose columns are the components of the Q^j, and $R = (r_{i\,j})$, the above equations show that $A = QR$, where R is upper triangular. The diagonal entries $r_{k\,k} = \|Q'^k\| = A^k \cdot Q^k$ are indeed positive. $\square$