then the matrix representing φ becomes

$$\begin{pmatrix} 0 & I_r & 0 \\ -I_r & 0 & 0 \\ 0 & 0 & 0_{n-2r} \end{pmatrix}.$$

This particularly simple matrix is often preferable, especially when dealing with the matrices (symplectic matrices) representing the isometries of φ (in which case $n = 2r$).

As a warm up for Proposition 29.29 of the next section, we prove an analog of Proposition 29.23 in the case of a symmetric bilinear form.

Proposition 29.26. *Let $\varphi\colon E \times E \to K$ be a nondegenerate symmetric bilinear form with K a field of characteristic different from 2. For any nonzero isotropic vector u, there is another nonzero isotropic vector v such that $\varphi(u, v) = 2$, and u and v are linearly independent. In the basis $(u, v/2)$, the restriction of φ to the plane spanned by u and $v/2$ is of the form*

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Proof. Since φ is nondegenerate, there is some nonzero vector z such that (rescaling z if necessary) $\varphi(u, z) = 1$. If

$$v = 2z - \varphi(z, z)u,$$

then since $\varphi(u, u) = 0$ and $\varphi(u, z) = 1$, note that

$$\varphi(u, v) = \varphi(u, 2z - \varphi(z, z)u) = 2\varphi(u, z) - \varphi(z, z)\varphi(u, u) = 2,$$

and

$$\begin{aligned} \varphi(v, v) &= \varphi(2z - \varphi(z, z)u, 2z - \varphi(z, z)u) \\ &= 4\varphi(z, z) - 4\varphi(z, z)\varphi(u, z) + \varphi(z, z)^2\varphi(u, u) \\ &= 4\varphi(z, z) - 4\varphi(z, z) = 0. \end{aligned}$$

If u and z were linearly dependent, as $u, z \neq 0$, we could write $z = \mu u$ for some $\mu \neq 0$, but then, we would have

$$\varphi(u, z) = \varphi(u, \mu u) = \mu\varphi(u, u) = 0,$$

contradicting the fact that $\varphi(u, z) \neq 0$. Then u and $v = 2z - \varphi(z, z)u$ are also linearly independent, since otherwise z could be expressed as a multiple of u. The rest is obvious. $\square$

Proposition 29.26 yields a plane spanned by two vectors u_1, v_1 such that $\varphi(u_1, u_1) = \varphi(v_1, v_1) = 0$ and $\varphi(u_1, v_1) = 1$. Such a plane is called an *Artinian plane*. Proposition 29.26 also shows that nonzero isotropic vectors come in pair.

Proposition 29.26 has the following corollary which has applications in number theory; see Serre [157], Chapter IV.