by

$$\omega \wedge_\Phi \eta = (\alpha \otimes f) \wedge_\Phi (\beta \otimes g) = (\alpha \wedge \beta) \otimes \Phi(f, g).$$

As in Section 34.5 (following H. Cartan [35]), we can also define a multiplication

$$\wedge_\Phi \colon \mathrm{Alt}^m(E; F) \times \mathrm{Alt}^n(E; G) \longrightarrow \mathrm{Alt}^{m+n}(E; H)$$

directly on alternating multilinear maps as follows: For $f \in \mathrm{Alt}^m(E; F)$ and $g \in \mathrm{Alt}^n(E; G)$,

$$(f \wedge_\Phi g)(u_1, \ldots, u_{m+n}) = \sum_{\sigma \in \mathrm{shuffle}(m,n)} \mathrm{sgn}(\sigma)\, \Phi\Big(f(u_{\sigma(1)}, \ldots, u_{\sigma(m)}), g(u_{\sigma(m+1)}, \ldots, u_{\sigma(m+n)})\Big),$$

where $\mathrm{shuffle}(m, n)$ consists of all (m, n)-"shuffles;" that is, permutations σ of $\{1, \ldots m+n\}$ such that $\sigma(1) < \cdots < \sigma(m)$ and $\sigma(m+1) < \cdots < \sigma(m+n)$.

A special case of interest is the case where $F = G = H$ is a Lie algebra and $\Phi(a, b) = [a, b]$ is the Lie bracket of F. In this case, using a basis $(f_1, \ldots, f_r)$ of F, if we write $\omega = \sum_i \alpha_i \otimes f_i$ and $\eta = \sum_j \beta_j \otimes f_j$, we have

$$\omega \wedge_\Phi \eta = [\omega, \eta] = \sum_{i,j} \alpha_i \wedge \beta_j \otimes [f_i, f_j].$$

It is customary to denote $\omega \wedge_\Phi \eta$ by $[\omega, \eta]$ (unfortunately, the bracket notation is overloaded). Consequently,

$$[\eta, \omega] = (-1)^{mn+1}[\omega, \eta].$$

In general not much can be said about $\wedge_\Phi$, unless Φ has some additional properties. In particular, $\wedge_\Phi$ is generally not associative.

We now use vector-valued alternating forms to generalize both the μ map of Proposition 34.14 and generalize Proposition 33.17 by defining the map

$$\mu_F \colon \left(\bigwedge^n (E^*)\right) \otimes F \longrightarrow \mathrm{Alt}^n(E; F)$$

on generators by

$$\mu_F((v_1^* \wedge \cdots \wedge v_n^*) \otimes f)(u_1, \ldots, u_n) = (\det(v_j^*(u_i))f,$$

with $v_1^*, \ldots, v_n^* \in E^*$, $u_1, \ldots, u_n \in E$, and $f \in F$.

Proposition 34.33. *The map*

$$\mu_F \colon \left(\bigwedge^n (E^*)\right) \otimes F \longrightarrow \mathrm{Alt}^n(E; F)$$

defined as above is a canonical isomorphism for every $n \geq 0$. Furthermore, given any three vector spaces, F, G, H, and any bilinear map $\Phi \colon F \times G \to H$, for all $\omega \in (\bigwedge^n(E^*)) \otimes F$ and all $\eta \in (\bigwedge^n(E^*)) \otimes G$,

$$\mu_H(\omega \wedge_\Phi \eta) = \mu_F(\omega) \wedge_\Phi \mu_G(\eta).$$