Definition 33.7. The linear map $f \otimes g\colon E \otimes F \to E' \otimes F'$ given by Proposition 33.9 is called the *tensor product* of $f\colon E \to E'$ and $g\colon F \to F'$.

Another way to define $f \otimes g$ proceeds as follows. Given two linear maps $f\colon E \to E'$ and $g\colon F \to F'$, the map $f \times g$ is the linear map from $E \times F$ to $E' \times F'$ given by

$$(f \times g)(u, v) = (f(u), g(v)), \quad \text{for all } u \in E \text{ and all } v \in F.$$

Then the map h in the proof of Proposition 33.9 is given by $h = \iota'_{\otimes} \circ (f \times g)$, and $f \otimes g$ is the unique linear map making the following diagram commute.

$$\begin{tikzcd} E \times F \arrow[r, "\iota_{\otimes}"] \arrow[d, "f\times g"'] & E \otimes F \arrow[d, "f\otimes g"] \\ E' \times F' \arrow[r, "\iota'_{\otimes}"'] & E' \otimes F' \end{tikzcd}$$

Remark: The notation $f \otimes g$ is potentially ambiguous, because $\mathrm{Hom}(E, F)$ and $\mathrm{Hom}(E', F')$ are vector spaces, so we can form the tensor product $\mathrm{Hom}(E, F) \otimes \mathrm{Hom}(E', F')$ which contains elements also denoted $f \otimes g$. To avoid confusion, the first kind of tensor product of linear maps defined in Proposition 33.9 (which yields a linear map in $\mathrm{Hom}(E \otimes F, E' \otimes F')$) can be denoted by $T(f, g)$. If we denote the tensor product $E \otimes F$ by $T(E, F)$, this notation makes it clearer that T is a bifunctor. If E, E' and F, F' are finite dimensional, by picking bases it is not hard to show that the map induced by $f \otimes g \mapsto T(f, g)$ is an isomorphism

$$\mathrm{Hom}(E, F) \otimes \mathrm{Hom}(E', F') \cong \mathrm{Hom}(E \otimes F, E' \otimes F').$$

Proposition 33.10. *Suppose we have linear maps $f\colon E \to E'$, $g\colon F \to F'$, $f'\colon E' \to E''$ and $g'\colon F' \to F''$. Then the following identity holds:*

$$(f' \circ f) \otimes (g' \circ g) = (f' \otimes g') \circ (f \otimes g). \tag{$*$}$$

Proof. We have the commutative diagram

$$\begin{tikzcd} E \times F \arrow[r, "\iota_{\otimes}"] \arrow[d, "f\times g"'] & E \otimes F \arrow[d, "f\otimes g"] \\ E' \times F' \arrow[r, "\iota'_{\otimes}"] \arrow[d, "f'\times g'"'] & E' \otimes F' \arrow[d, "f'\otimes g'"] \\ E'' \times F'' \arrow[r, "\iota''_{\otimes}"'] & E'' \otimes F'', \end{tikzcd}$$

and thus the commutative diagram.

$$\begin{tikzcd} E \times F \arrow[r, "\iota_{\otimes}"] \arrow[d, "(f'\times g')\circ(f\times g)"'] & E \otimes F \arrow[d, "(f'\otimes g')\circ(f\otimes g)"] \\ E'' \times F'' \arrow[r, "\iota''_{\otimes}"'] & E'' \otimes F'' \end{tikzcd}$$