linear form $y \in (\mathbb{R}^m)^*$ such that $yA \geq 0_n^\top$ and $yb < 0$.

We will use the version of Farkas lemma obtained by taking a contrapositive, namely: *if $yA \geq 0_n^\top$ implies $yb \geq 0$ for all linear forms $y \in (\mathbb{R}^m)^*$, then the linear system $Ax = b$ has some solution $x \geq 0$.*

Actually, it is more convenient to use a version of Farkas lemma applying to a Euclidean vector space (with an inner product denoted $\langle -, - \rangle$). This version also applies to an infinite dimensional real Hilbert space; see Theorem 48.12. Recall that in a Euclidean space V the inner product induces an isomorphism between V and V', the space of continuous linear forms on V. In our case, we need the isomorphism $\sharp$ from V' to V defined such that for every linear form $\omega \in V'$, the vector $\omega^\sharp \in V$ is uniquely defined by the equation

$$\omega(v) = \langle v, \omega^\sharp \rangle \quad \text{for all } v \in V.$$

In $\mathbb{R}^n$, the isomorphism between $\mathbb{R}^n$ and $(\mathbb{R}^n)^*$ amounts to *transposition*: if $y \in (\mathbb{R}^n)^*$ is a linear form and $v \in \mathbb{R}^n$ is a vector, then

$$yv = v^\top y^\top.$$

The version of the Farkas–Minskowski lemma in term of an inner product is as follows.

Proposition 50.4. *(Farkas–Minkowski) Let V be a Euclidean space of finite dimension with inner product $\langle -, - \rangle$ (more generally, a Hilbert space). For any finite family $(a_1, \ldots, a_m)$ of m vectors $a_i \in V$ and any vector $b \in V$, for any $v \in V$,*

$$\textit{if } \langle a_i, v \rangle \geq 0 \textit{ for } i = 1, \ldots, m \textit{ implies that } \langle b, v \rangle \geq 0,$$

then there exist $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ such that

$$\lambda_i \geq 0 \textit{ for } i = 1, \ldots, m, \textit{ and } b = \sum_{i=1}^m \lambda_i a_i,$$

that is, b belong to the polyhedral cone $\mathrm{cone}(a_1, \ldots, a_m)$.

Proposition 50.4 is the special case of Theorem 48.12 which holds for real Hilbert spaces.

We can now prove the following theorem.

Theorem 50.5. *Let $\varphi_i \colon \Omega \to \mathbb{R}$ be m constraints defined on some open subset Ω of a finite-dimensional Euclidean vector space V (more generally, a real Hilbert space V), let $J \colon \Omega \to \mathbb{R}$ be some function, and let U be given by*

$$U = \{x \in \Omega \mid \varphi_i(x) \leq 0,\ 1 \leq i \leq m\}.$$

For any $u \in U$, let

$$I(u) = \{i \in \{1, \ldots, m\} \mid \varphi_i(u) = 0\},$$