(2) We have

$$U_i \cap \left(\sum_{j=1, j\neq i}^{p} U_j \right) = (0), \quad i = 1, \ldots, p.$$

(3) We have

$$U_i \cap \left(\sum_{j=1}^{i-1} U_j \right) = (0), \quad i = 2, \ldots, p.$$

Because of the isomorphism

$$U_1 \times \cdots \times U_p \approx U_1 \oplus \cdots \oplus U_p,$$

we have

Proposition 6.7. *If E is any vector space, for any (finite-dimensional) subspaces $U_1, \ldots,$ U_p of E, we have*

$$\dim(U_1 \oplus \cdots \oplus U_p) = \dim(U_1) + \cdots + \dim(U_p).$$

If E is a direct sum

$$E = U_1 \oplus \cdots \oplus U_p,$$

since every $u \in E$ can be written in a unique way as

$$u = u_1 + \cdots + u_p$$

with $u_i \in U_i$ for $i = 1 \ldots, p$, we can define the maps $\pi_i \colon E \to U_i$, called *projections*, by

$$\pi_i(u) = \pi_i(u_1 + \cdots + u_p) = u_i.$$

It is easy to check that these maps are linear and satisfy the following properties:

$$\pi_j \circ \pi_i = \begin{cases} \pi_i & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$
$$\pi_1 + \cdots + \pi_p = \mathrm{id}_E.$$

For example, in the case of the direct sum

$$\mathrm{M}_n = \mathbf{S}(n) \oplus \mathbf{Skew}(n),$$

the projection onto $\mathbf{S}(n)$ is given by

$$\pi_1(A) = H(A) = \frac{A + A^\top}{2},$$