Observe that the matrix P_π has a single 1 on every row and every column, all other entries being zero, and that if we multiply any 4×4 matrix A by P_π on the left, then the rows of A are permuted according to the permutation π; that is, the $\pi(i)$th row of $P_\pi A$ is the ith row of A. For example,

$$P_\pi A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = \begin{pmatrix} a_{41} & a_{42} & a_{43} & a_{44} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}.$$

Equivalently, the ith row of $P_\pi A$ is the $\pi^{-1}(i)$th row of A. In order for the matrix P_π to move the ith row of A to the $\pi(i)$th row, the $\pi(i)$th row of P_π must have a 1 in column i and zeros everywhere else; this means that the ith column of P_π contains the basis vector $e_{\pi(i)}$, the vector that has a 1 in position $\pi(i)$ and zeros everywhere else.

This is the general situation and it leads to the following definition.

Definition 8.10. Given any permutation $\pi\colon [n] \to [n]$, the *permutation matrix* $P_\pi = (p_{ij})$ representing π is the matrix given by

$$p_{ij} = \begin{cases} 1 & \text{if } i = \pi(j) \\ 0 & \text{if } i \neq \pi(j); \end{cases}$$

equivalently, the jth column of P_π is the basis vector $e_{\pi(j)}$. A *permutation matrix* P is any matrix of the form P_π (where P is an $n \times n$ matrix, and $\pi\colon [n] \to [n]$ is a permutation, for some $n \geq 1$).

Remark: There is a confusing point about the notation for permutation matrices. A permutation matrix P acts on a matrix A by multiplication on the left by permuting the rows of A. As we said before, this means that the $\pi(i)$th row of $P_\pi A$ is the ith row of A, or equivalently that the ith row of $P_\pi A$ is the $\pi^{-1}(i)$th row of A. But then observe that the row index of the entries of the ith row of PA is $\pi^{-1}(i)$, and not $\pi(i)$! See the following example:

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = \begin{pmatrix} a_{41} & a_{42} & a_{43} & a_{44} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix},$$

where

$$\begin{aligned} \pi^{-1}(1) &= 4 \\ \pi^{-1}(2) &= 3 \\ \pi^{-1}(3) &= 1 \\ \pi^{-1}(4) &= 2. \end{aligned}$$

Prove the following results