22.3 Polar Form for Square Matrices

A notion closely related to the SVD is the polar form of a matrix.

Definition 22.4. A pair (R, S) such that $A = RS$ with R orthogonal and S symmetric positive semidefinite is called a *polar decomposition of A*.

Theorem 22.5 implies that for every real $n \times n$ matrix A, there is some orthogonal matrix R and some positive semidefinite symmetric matrix S such that

$$A = RS.$$

This is easy to show and we will prove it below. Furthermore, R, S are unique if A is invertible, but this is harder to prove; see Problem 22.9.

For example, the matrix

$$A = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

is both orthogonal and symmetric, and $A = RS$ with $R = A$ and $S = I$, which implies that some of the eigenvalues of A are negative.

Remark: In the complex case, the polar decomposition states that for every complex $n \times n$ matrix A, there is some unitary matrix U and some positive semidefinite Hermitian matrix H such that

$$A = UH.$$

It is easy to go from the polar form to the SVD, and conversely.

Given an SVD $A = VDU^\top$, let $R = VU^\top$ and $S = UDU^\top$. It is clear that R is orthogonal and that S is positive semidefinite symmetric, and

$$RS = VU^\top UDU^\top = VDU^\top = A.$$

Example 22.5. Recall from Example 22.4 that $A = VDU^\top$ where $V = I_2$ and

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, \qquad D = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix}.$$

Set $R = VU^\top = U$ and

$$S = UDU^\top = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Since $S = \frac{1}{\sqrt{2}}A^\top A$, S has eigenvalues $\sqrt{2}$ and 0. We leave it to the reader to check that $A = RS$.