The product of the 1×2 block matrix $[A]$ and the 2×2 block matrix $[B]$ is the 1×2 block matrix $[C]$ given by

$$\begin{aligned}
[C] = [A][B] &= \begin{pmatrix} \begin{bmatrix} a_{11} & a_{13} \end{bmatrix} & \begin{bmatrix} a_{12} \end{bmatrix} \end{pmatrix} \begin{pmatrix} \begin{bmatrix} b_{12} \\ b_{32} \end{bmatrix} & \begin{bmatrix} b_{11} \\ b_{31} \end{bmatrix} \\ \begin{bmatrix} b_{22} \end{bmatrix} & \begin{bmatrix} b_{21} \end{bmatrix} \end{pmatrix} \\
&= \begin{pmatrix} \begin{bmatrix} a_{11} & a_{13} \end{bmatrix} \begin{bmatrix} b_{12} \\ b_{32} \end{bmatrix} + \begin{bmatrix} a_{12} \end{bmatrix} \begin{bmatrix} b_{22} \end{bmatrix} & \begin{bmatrix} a_{11} & a_{13} \end{bmatrix} \begin{bmatrix} b_{11} \\ b_{31} \end{bmatrix} + \begin{bmatrix} a_{12} \end{bmatrix} \begin{bmatrix} b_{21} \end{bmatrix} \end{pmatrix} \\
&= \begin{pmatrix} \begin{bmatrix} a_{11}b_{12} + a_{13}b_{32} + a_{12}b_{22} \end{bmatrix} & \begin{bmatrix} a_{11}b_{11} + a_{13}b_{31} + a_{12}b_{21} \end{bmatrix} \end{pmatrix} \\
&= \begin{pmatrix} \begin{bmatrix} a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \end{bmatrix} & \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} \end{bmatrix} \end{pmatrix}.
\end{aligned}$$

The block matrix $[C]$ is obtained from the 1×2 matrix $C = AB$ using the partitions of $R = \{1\}$ given by $R_1 = \{1\}$ and of $T = \{1, 2\}$ given by $T_1 = \{2\}$, $T_2 = \{1\}$, so

$$[C] = \begin{pmatrix} C_{\{1\},\{2\}} & C_{\{1\},\{1\}} \end{pmatrix},$$

which means that $[C]$ is obtained from C by permuting its two columns. Since

$$\begin{aligned}
C = AB &= \begin{pmatrix} a_{11} & a_{12} & a_{13} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} \\
&= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \end{pmatrix},
\end{aligned}$$

we have confirmed that $[C]$ is correct.

Example 6.6. Matrix block multiplication is a very effective method to prove that if an upper-triangular matrix A is invertible, then its inverse is also upper-triangular. We proceed by induction on the dimensiopn n of A. If $n = 1$, then $A = (a)$, where a is a scalar, so A is invertible iff $a \neq 0$, and $A^{-1} = (a^{-1})$, which is trivially upper-triangular. For the induction step we can write an $(n+1) \times (n+1)$ upper triangular matrix A in block form as

$$A = \begin{pmatrix} T & U \\ 0_{1,n} & \alpha \end{pmatrix},$$

where T is an $n \times n$ upper triangular matrix, U is an $n \times 1$ matrix and $\alpha \in \mathbb{R}$. Assume that A is invertible and let B be its inverse, written in block form as

$$B = \begin{pmatrix} C & V \\ W & \beta \end{pmatrix},$$

where C is an $n \times n$ matrix, V is an $n \times 1$ matrix, W is a $1 \times n$ matrix, and $\beta \in \mathbb{R}$. Since B is the inverse of A, we have $AB = I_{n+1}$, which yields

$$\begin{pmatrix} T & U \\ 0_{1,n-1} & \alpha \end{pmatrix} \begin{pmatrix} C & V \\ W & \beta \end{pmatrix} = \begin{pmatrix} I_n & 0_{n,1} \\ 0_{1,n} & 1 \end{pmatrix}.$$