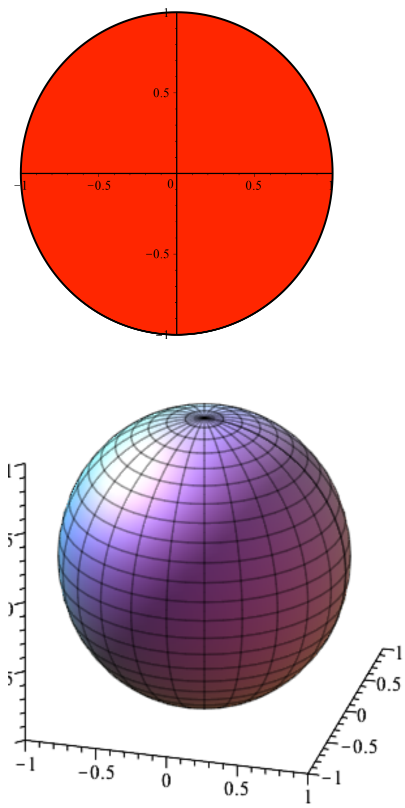

Figure 9.2: The top figure is $\{x \in \mathbb{R}^2 \mid \|x\|_2 \leq 1\}$, while the bottom figure is $\{x \in \mathbb{R}^3 \mid \|x\|_2 \leq 1\}$.

Proposition 9.1. *If $E = \mathbb{C}^n$ or $E = \mathbb{R}^n$, for every real number $p \geq 1$, the ℓ^p-norm is indeed a norm.*

Proof. The cases $p = 1$ and $p = \infty$ are easy and left to the reader. If $p > 1$, then let $q > 1$ such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

We will make use of the following fact: for all $\alpha, \beta \in \mathbb{R}$, if $\alpha, \beta \geq 0$, then

$$\alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}. \tag{$*$}$$

To prove the above inequality, we use the fact that the exponential function $t \mapsto e^t$ satisfies the following convexity inequality:

$$e^{\theta x + (1-\theta)y} \leq \theta e^x + (1-\theta)e^y,$$