with $u \in U$ and $v \in U^\perp$. If we let $p = p_U(b) \in U$, then for any point $y \in U$, the vectors $\overrightarrow{py} = y - p \in U$ and $\overrightarrow{bp} = p - b \in U^\perp$ are orthogonal, which implies that

$$\|\overrightarrow{by}\|_2^2 = \|\overrightarrow{bp}\|_2^2 + \|\overrightarrow{py}\|_2^2,$$

where $\overrightarrow{by} = y - b$. Thus, p is indeed the unique point in U that minimizes the distance from b to any point in U. See Figure 23.2.

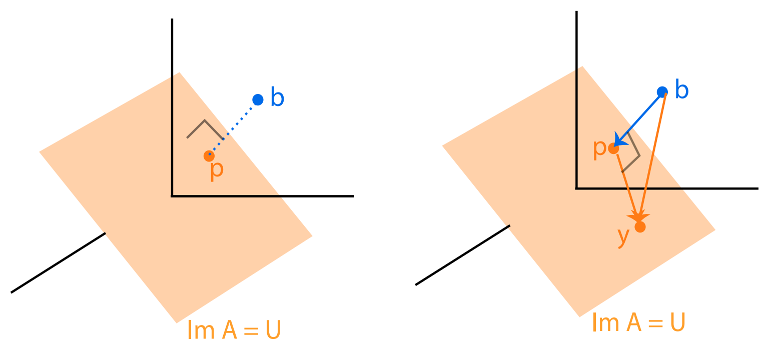

Figure 23.2: Given a 3×2 matrix A, $U = \operatorname{Im} A$ is the peach plane in $\mathbb{R}^3$ and p is the orthogonal projection of b onto U. Furthermore, given $y \in U$, the points b, y, and p are the vertices of a right triangle.

Thus the problem has been reduced to proving that there is a unique x^+ of minimum norm such that $Ax^+ = p$, with $p = p_U(b) \in U$, the orthogonal projection of b onto U. We use the fact that

$$\mathbb{R}^n = \operatorname{Ker} A \oplus (\operatorname{Ker} A)^\perp.$$

Consequently, every $x \in \mathbb{R}^n$ can be written uniquely as $x = u + v$, where $u \in \operatorname{Ker} A$ and $v \in (\operatorname{Ker} A)^\perp$, and since u and v are orthogonal,

$$\|x\|_2^2 = \|u\|_2^2 + \|v\|_2^2.$$

Furthermore, since $u \in \operatorname{Ker} A$, we have $Au = 0$, and thus $Ax = p$ iff $Av = p$, which shows that the solutions of $Ax = p$ for which x has minimum norm must belong to $(\operatorname{Ker} A)^\perp$. However, the restriction of A to $(\operatorname{Ker} A)^\perp$ is injective. This is because if $Av_1 = Av_2$, where $v_1, v_2 \in (\operatorname{Ker} A)^\perp$, then $A(v_2 - v_1) = 0$, which implies $v_2 - v_1 \in \operatorname{Ker} A$, and since $v_1, v_2 \in (\operatorname{Ker} A)^\perp$, we also have $v_2 - v_1 \in (\operatorname{Ker} A)^\perp$, and consequently, $v_2 - v_1 = 0$. This shows that there is a unique x^+ of minimum norm such that $Ax^+ = p$, and that x^+ must belong to $(\operatorname{Ker} A)^\perp$. By our previous reasoning, x^+ is the unique vector of minimum norm minimizing $\|Ax - b\|_2^2$. $\square$