Proposition 8.11. *Let A be any $n \times n$ real symmetric matrix. The following conditions are equivalent:*

(a) A is positive definite.

(b) All principal minors of A are positive; that is: $\det(A(1:k, 1:k)) > 0$ *for* $k = 1, \ldots, n$ *(Sylvester's criterion).*

(c) A has an LU-factorization and all pivots are positive.

(d) A has an $LDL^\top$-factorization and all pivots in D are positive.

Proof. By Proposition 8.9, if A is symmetric positive definite, then each matrix $A(1:k, 1:k)$ is symmetric positive definite for $k = 1, \ldots, n$. By the Cholesky decomposition, $A(1:k, 1:k) = Q^\top Q$ for some invertible matrix Q, so $\det(A(1:k, 1:k)) = \det(Q)^2 > 0$. This shows that (a) implies (b).

If $\det(A(1:k, 1:k)) > 0$ for $k = 1, \ldots, n$, then each $A(1:k, 1:k)$ is invertible. By Proposition 8.2, the matrix A has an LU-factorization, and since the pivots π_k are given by

$$\pi_k = \begin{cases} a_{11} = \det(A(1:1, 1:1)) & \text{if } k = 1 \\ \dfrac{\det(A(1:k, 1:k))}{\det(A(1:k-1, 1:k-1))} & \text{if } k = 2, \ldots, n, \end{cases}$$

we see that $\pi_k > 0$ for $k = 1, \ldots, n$. Thus (b) implies (c).

Assume A has an LU-factorization and that the pivots are all positive. Since A is symmetric, this implies that A has a factorization of the form

$$A = LDL^\top,$$

with L lower-triangular with 1s on its diagonal, and where D is a diagonal matrix with positive entries on the diagonal (the pivots). This shows that (c) implies (d).

Given a factorization $A = LDL^\top$ with all pivots in D positive, if we form the diagonal matrix

$$\sqrt{D} = \mathrm{diag}(\sqrt{\pi_1}, \ldots, \sqrt{\pi_n})$$

and if we let $B = L\sqrt{D}$, then we have

$$A = BB^\top,$$

with B lower-triangular and invertible. By the remark before Proposition 8.11, A is positive definite. Hence, (d) implies (a). $\square$

Criterion (c) yields a simple computational test to check whether a symmetric matrix is positive definite. There is one more criterion for a symmetric matrix to be positive definite: