of *algebraic geometry*, a beautiful but formidable subject. For a taste of algebraic geometry, see Lang [109] or Dummit and Foote [54].

The duality theorem (Theorem 11.4) shows that the situation is much simpler if we restrict our attention to linear subspaces; in this case

$$U = \mathcal{I}(\mathcal{Z}(U)) \quad \text{and} \quad V = \mathcal{Z}(\mathcal{I}(V)).$$

Proposition 11.3. *We have $V \subseteq V^{00}$ for every subspace V of E, and $U \subseteq U^{00}$ for every subspace U of E^*.*

Proof. Indeed, for any $v \in V$, to show that $v \in V^{00}$ we need to prove that $u^*(v) = 0$ for all $u^* \in V^0$. However, V^0 consists of all linear forms u^* such that $u^*(y) = 0$ for *all* $y \in V$; in particular, for a fixed $v \in V$, we have $u^*(v) = 0$ for all $u^* \in V^0$, as required.

Similarly, for any $u^* \in U$, to show that $u^* \in U^{00}$ we need to prove that $u^*(v) = 0$ for all $v \in U^0$. However, U^0 consists of all vectors v such that $f^*(v) = 0$ for *all* $f^* \in U$; in particular, for a fixed $u^* \in U$, we have $u^*(v) = 0$ for all $v \in U^0$, as required. $\square$

We will see shortly that in finite dimension, we have $V = V^{00}$ and $U = U^{00}$.

However, even though $V = V^{00}$ is always true, when E is of infinite dimension, it is not always true that $U = U^{00}$.

Given a vector space E and a subspace U of E, by Theorem 3.7, every basis $(u_i)_{i\in I}$ of U can be extended to a basis $(u_j)_{j\in I\cup J}$ of E, where $I \cap J = \emptyset$.

11.3 The Duality Theorem and Some Consequences

We have the following important theorem adapted from E. Artin [6] (Chapter 1).

Theorem 11.4. *(Duality theorem) Let E be a vector space. The following properties hold:*

(a) For every basis $(u_i)_{i\in I}$ of E, the family $(u_i^*)_{i\in I}$ of coordinate forms is linearly independent.

(b) For every subspace V of E, we have $V^{00} = V$.

(c) For every subspace V of finite codimension m of E, for every subspace W of E such that $E = V \oplus W$ (where W is of finite dimension m), for every basis $(u_i)_{i\in I}$ of E such that $(u_1, \ldots, u_m)$ is a basis of W, the family $(u_1^*, \ldots, u_m^*)$ is a basis of the orthogonal V^0 of V in E^*, so that

$$\dim(V^0) = \operatorname{codim}(V).$$

Furthermore, we have $V^{00} = V$.