Prove that
$$\det(B) = (-1)^n(n-2)2^{n-1}.$$

Problem 7.8. Given a field K (say $K = \mathbb{R}$ or $K = \mathbb{C}$), given any two polynomials $p(X), q(X) \in K[X]$, we says that $q(X)$ *divides* $p(X)$ (and that $p(X)$ *is a multiple of* $q(X)$) iff there is some polynomial $s(X) \in K[X]$ such that
$$p(X) = q(X)s(X).$$
In this case we say that $q(X)$ *is a factor of* $p(X)$, and if $q(X)$ has degree at least one, we say that $q(X)$ *is a nontrivial factor of* $p(X)$.

Let $f(X)$ and $g(X)$ be two polynomials in $K[X]$ with
$$f(X) = a_0X^m + a_1X^{m-1} + \cdots + a_m$$
of degree $m \geq 1$ and
$$g(X) = b_0X^n + b_1X^{n-1} + \cdots + b_n$$
of degree $n \geq 1$ (with $a_0, b_0 \neq 0$).

You will need the following result which you need not prove:

Two polynomials $f(X)$ and $g(X)$ with $\deg(f) = m \geq 1$ and $\deg(g) = n \geq 1$ have some common nontrivial factor iff there exist two nonzero polynomials $p(X)$ and $q(X)$ such that
$$fp = gq,$$
with $\deg(p) \leq n-1$ and $\deg(q) \leq m-1$.

(1) Let $\mathcal{P}_m$ denote the vector space of all polynomials in $K[X]$ of degree at most $m-1$, and let $T\colon \mathcal{P}_n \times \mathcal{P}_m \to \mathcal{P}_{m+n}$ be the map given by
$$T(p,q) = fp + gq, \quad p \in \mathcal{P}_n,\ q \in \mathcal{P}_m,$$
where f and g are some fixed polynomials of degree $m \geq 1$ and $n \geq 1$.

Prove that the map T is linear.

(2) Prove that T is not injective iff f and g have a common nontrivial factor.

(3) Prove that f and g have a nontrivial common factor iff $R(f,g) = 0$, where $R(f,g)$ is the determinant given by
$$R(f,g) = \begin{vmatrix}
a_0 & a_1 & \cdots & \cdots & a_m & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & a_0 & a_1 & \cdots & \cdots & a_m & 0 & \cdots & \cdots & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & a_0 & a_1 & \cdots & \cdots & a_m \\
b_0 & b_1 & \cdots & \cdots & \cdots & \cdots & \cdots & b_n & 0 & \cdots & 0 \\
0 & b_0 & b_1 & \cdots & \cdots & \cdots & \cdots & \cdots & b_n & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & b_0 & b_1 & \cdots & \cdots & \cdots & \cdots & \cdots & b_n
\end{vmatrix}.$$