The following is an example of an 8×8 matrix consisting of three diagonal unreduced Hessenberg blocks:

$$H = \begin{pmatrix} \star & \star & \star & * & * & * & * & * \\ \mathbf{h_{21}} & \star & \star & * & * & * & * & * \\ \mathbf{0} & \mathbf{h_{32}} & \star & * & * & * & * & * \\ 0 & 0 & 0 & \star & \star & \star & * & * \\ 0 & 0 & 0 & \mathbf{h_{54}} & \star & \star & * & * \\ 0 & 0 & 0 & \mathbf{0} & \mathbf{h_{65}} & \star & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \star & \star \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{h_{87}} & \star \end{pmatrix}.$$

An interesting and important property of unreduced Hessenberg matrices is the following.

Proposition 18.3. *Let H be an $n \times n$ complex or real unreduced Hessenberg matrix. Then every eigenvalue of H is geometrically simple, that is,* $\dim(E_\lambda) = 1$ *for every eigenvalue λ, where E_λ is the eigenspace associated with λ. Furthermore, if H is diagonalizable, then every eigenvalue is simple, that is, H has n distinct eigenvalues.*

Proof. We follow Serre's proof [156] (Proposition 3.26). Let λ be any eigenvalue of H, let $M = \lambda I_n - H$, and let N be the $(n-1) \times (n-1)$ matrix obtained from M by deleting its first row and its last column. Since H is upper Hessenberg, N is a diagonal matrix with entries $-h_{i+1i} \neq 0$, $i = 1, \ldots, n-1$. Thus N is invertible and has rank $n-1$. But a matrix has rank greater than or equal to the rank of any of its submatrices, so $\mathrm{rank}(M) = n-1$, since M is singular. By the rank-nullity theorem, $\mathrm{rank}(\mathrm{Ker}\, N) = 1$, that is, $\dim(E_\lambda) = 1$, as claimed.

If H is diagonalizable, then the sum of the dimensions of the eigenspaces is equal to n, which implies that the eigenvalues of H are distinct. $\square$

As we said earlier, a case where Theorem 18.1 applies is the case where A is a symmetric (or Hermitian) positive definite matrix. This follows from two facts.

The first fact is that if A is Hermitian (or symmetric in the real case), then it is easy to show that the Hessenberg matrix similar to A is a Hermitian (or symmetric in real case) *tridiagonal matrix*. The conversion method is also more efficient. Here is an example of a symmetric tridiagonal matrix consisting of three unreduced blocks:

$$H = \begin{pmatrix} \alpha_1 & \beta_1 & \mathbf{0} & 0 & 0 & 0 & 0 & 0 \\ \beta_1 & \alpha_2 & \beta_2 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{0} & \beta_2 & \alpha_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_4 & \beta_4 & \mathbf{0} & 0 & 0 \\ 0 & 0 & 0 & \beta_4 & \alpha_5 & \beta_5 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{0} & \beta_5 & \alpha_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \alpha_7 & \beta_7 \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta_7 & \alpha_8 \end{pmatrix}.$$