Theorem 39.23. *(Taylor–Young) Given two normed affine spaces E and F, for any open subset $A \subseteq E$, for any function $f\colon A \to F$, for any $a \in A$, if $\mathrm{D}^k f$ exists in A for all k, $1 \leq k \leq m-1$, and if $\mathrm{D}^m f(a)$ exists, then we have:*

$$f(a+h) = f(a) + \frac{1}{1!}\mathrm{D}^1 f(a)(h) + \cdots + \frac{1}{m!}\mathrm{D}^m f(a)(h^m) + \|h\|^m \epsilon(h),$$

for any h such that $a + h \in A$, and where $\lim_{h \to 0,\, h \neq 0} \epsilon(h) = 0$.

The above version of Taylor's formula has applications to the study of relative maxima (or minima) of real-valued functions. It is also used to study the local properties of curves and surfaces.

The next version of Taylor's formula can be viewed as a generalization of Proposition 39.12. It is sometimes called the *Taylor formula with Lagrange remainder* or *generalized mean value theorem*.

Theorem 39.24. *(Generalized mean value theorem) Let E and F be two normed affine spaces, let A be an open subset of E, and let $f\colon A \to F$ be a function on A. Given any $a \in A$ and any $h \neq 0$ in $\overrightarrow{E}$, if the closed segment $[a, a+h]$ is contained in A, $\mathrm{D}^k f$ exists in A for all k, $1 \leq k \leq m$, $\mathrm{D}^{m+1} f(x)$ exists at every point x of the open segment $]a, a+h[$, and*

$$\max_{x \in (a, a+h)} \left\| \mathrm{D}^{m+1} f(x) \right\| \leq M,$$

for some $M \geq 0$, then

$$\left\| f(a+h) - f(a) - \Big(\frac{1}{1!}\mathrm{D}^1 f(a)(h) + \cdots + \frac{1}{m!}\mathrm{D}^m f(a)(h^m)\Big) \right\| \leq M \frac{\|h\|^{m+1}}{(m+1)!}.$$

As a corollary, if $L\colon \overrightarrow{E^{m+1}} \to \overrightarrow{F}$ is a continuous $(m+1)$-linear map, then

$$\left\| f(a+h) - f(a) - \Big(\frac{1}{1!}\mathrm{D}^1 f(a)(h) + \cdots + \frac{1}{m!}\mathrm{D}^m f(a)(h^m) + \frac{L(h^{m+1})}{(m+1)!}\Big) \right\| \leq M \frac{\|h\|^{m+1}}{(m+1)!},$$

where $M = \max_{x \in (a, a+h)} \|\mathrm{D}^{m+1} f(x) - L\|$.

The above theorem is sometimes stated under the slightly stronger assumption that f is a C^m-function on A. If $f\colon A \to \mathbb{R}$ is a real-valued function, Theorem 39.24 can be refined a little bit. This version is often called the *formula of Taylor–MacLaurin*.

Theorem 39.25. *(Taylor–MacLaurin) Let E be a normed affine space, let A be an open subset of E, and let $f\colon A \to \mathbb{R}$ be a real-valued function on A. Given any $a \in A$ and any $h \neq 0$ in $\overrightarrow{E}$, if the closed segment $[a, a+h]$ is contained in A, if $\mathrm{D}^k f$ exists in A for all k, $1 \leq k \leq m$, and $\mathrm{D}^{m+1} f(x)$ exists at every point x of the open segment $]a, a+h[$, then there is some $\theta \in \mathbb{R}$, with $0 < \theta < 1$, such that*

$$f(a+h) = f(a) + \frac{1}{1!}\mathrm{D}^1 f(a)(h) + \cdots + \frac{1}{m!}\mathrm{D}^m f(a)(h^m) + \frac{1}{(m+1)!}\mathrm{D}^{m+1} f(a + \theta h)(h^{m+1}).$$