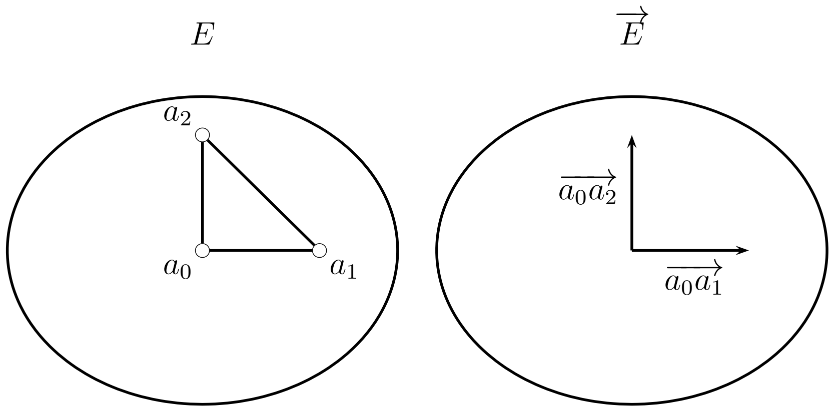

Figure 24.14: Affine independence and linear independence

Definition 24.4 is reasonable, because by Proposition 24.4, the independence of the family $(\overrightarrow{a_ia_j})_{j\in(I-\{i\})}$ does not depend on the choice of a_i. A crucial property of linearly independent vectors $(u_1, \ldots, u_m)$ is that if a vector v is a linear combination

$$v = \sum_{i=1}^{m} \lambda_i u_i$$

of the u_i, then the λ_i are unique. A similar result holds for affinely independent points.

Proposition 24.5. *Given an affine space $\langle E, \overrightarrow{E}, +\rangle$, let $(a_0, \ldots, a_m)$ be a family of $m+1$ points in E. Let $x \in E$, and assume that $x = \sum_{i=0}^{m} \lambda_i a_i$, where $\sum_{i=0}^{m} \lambda_i = 1$. Then, the family $(\lambda_0, \ldots, \lambda_m)$ such that $x = \sum_{i=0}^{m} \lambda_i a_i$ is unique iff the family $(\overrightarrow{a_0a_1}, \ldots, \overrightarrow{a_0a_m})$ is linearly independent.*

Proof. The proof is straightforward and is omitted. It is also given in Gallier [70]. □

Proposition 24.5 suggests the notion of affine frame. Affine frames are the affine analogues of bases in vector spaces. Let $\langle E, \overrightarrow{E}, +\rangle$ be a nonempty affine space, and let $(a_0, \ldots, a_m)$ be a family of $m+1$ points in E. The family $(a_0, \ldots, a_m)$ determines the family of m vectors $(\overrightarrow{a_0a_1}, \ldots, \overrightarrow{a_0a_m})$ in $\overrightarrow{E}$. Conversely, given a point a_0 in E and a family of m vectors $(u_1, \ldots, u_m)$ in $\overrightarrow{E}$, we obtain the family of $m+1$ points $(a_0, \ldots, a_m)$ in E, where $a_i = a_0 + u_i$, $1 \leq i \leq m$.

Thus, for any $m \geq 1$, it is equivalent to consider a family of $m+1$ points $(a_0, \ldots, a_m)$ in E, and a pair $(a_0, (u_1, \ldots, u_m))$, where the u_i are vectors in $\overrightarrow{E}$. Figure 24.14 illustrates the notion of affine independence.