Since by $(*_1)$ and $(*_4)$ we have $\nabla J_{u_{\ell+1}} = \nabla J_{u_\ell} + A\Delta_\ell = \nabla J_{u_\ell} - \rho_\ell A d_\ell$, the gradient $\nabla J_{u_{\ell+1}}$ can be computed iteratively:

$$\begin{aligned} \nabla J_0 &= Au_0 - b \\ \nabla J_{u_{\ell+1}} &= \nabla J_{u_\ell} - \rho_\ell A d_\ell. \end{aligned}$$

Since by Proposition 49.17 we have

$$d_k = \nabla J_{u_k} + \frac{\left\|\nabla J_{u_k}\right\|^2}{\left\|\nabla J_{u_{k-1}}\right\|^2}\, d_{k-1}$$

and since d_{k-1} is a linear combination of the gradients ∇J_{u_i} for $i = 0, \ldots, k-1$, which are all orthogonal to ∇J_{u_k}, we have

$$\rho_k = \frac{\langle \nabla J_{u_k}, d_k\rangle}{\langle A d_k, d_k\rangle} = \frac{\left\|\nabla J_{u_k}\right\|^2}{\langle A d_k, d_k\rangle}.$$

It is customary to introduce the term r_k defined as

$$r_k = \nabla J_{u_k} = Au_k - b \tag{$*_7$}$$

and to call it the *residual*. Then the conjugate gradient method consists of the following steps. We intitialize the method starting from any vector u_0 and set

$$d_0 = r_0 = Au_0 - b.$$

The main iteration step is $(k \geq 0)$:

$$(*_8) \qquad \begin{cases} \rho_k = \dfrac{\left\|r_k\right\|^2}{\langle A d_k, d_k\rangle} \\ u_{k+1} = u_k - \rho_k d_k \\ r_{k+1} = r_k - \rho_k A d_k \\ \beta_{k+1} = \dfrac{\left\|r_{k+1}\right\|^2}{\left\|r_k\right\|^2} \\ d_{k+1} = r_{k+1} + \beta_{k+1}\, d_k. \end{cases}$$

Beware that some authors define the residual r_k as $r_k = b - Au_k$ and the descent direction d_k as $-d_k$. In this case, the second equation becomes

$$u_{k+1} = u_k + \rho_k d_k.$$

Since $d_0 = r_0$, the equations

$$\begin{aligned} r_{k+1} &= r_k - \rho_k A d_k \\ d_{k+1} &= r_{k+1} + \beta_{k+1} d_k \end{aligned}$$