In fact, $\varphi_u^l = \varphi_u^r$, and because the inner product $\langle -, - \rangle$ is continuous, it is obvious that φ_v^r is continuous and linear, so that $\varphi_v^r \in E'$. To simplify notation, we write φ_v instead of φ_v^r.

Theorem 14.6 is generalized to Hilbert spaces as follows.

Proposition 48.9. *(Riesz representation theorem) Let E be a Hilbert space. Then the map $\flat \colon E \to E'$ defined such that*

$$\flat(v) = \varphi_v,$$

is semilinear, continuous, and bijective. Furthermore, for any continuous linear map $\psi \in E'$, if $u \in E$ is the unique vector such that

$$\psi(v) = \langle v, u \rangle \quad \text{for all } v \in E,$$

then we have $\|\psi\| = \|u\|$, where

$$\|\psi\| = \sup \left\{ \frac{|\psi(v)|}{\|v\|} \;\middle|\; v \in E,\ v \neq 0 \right\}.$$

Proof. The proof is basically identical to the proof of Theorem 14.6, except that a different argument is required for the surjectivity of $\flat \colon E \to E'$, since E may not be finite dimensional. For any nonnull linear operator $h \in E'$, the hyperplane $H = \operatorname{Ker} h = h^{-1}(0)$ is a closed subspace of E, and by Proposition 48.7, $H^{\perp}$ is a subspace of dimension one such that $E = H \oplus H^{\perp}$. Then picking any nonnull vector $w \in H^{\perp}$, observe that H is also the kernel of the linear operator φ_w, with

$$\varphi_w(u) = \langle u, w \rangle,$$

and thus, since any two nonzero linear forms defining the same hyperplane must be proportional, there is some nonzero scalar $\lambda \in \mathbb{C}$ such that $h = \lambda \varphi_w$. But then, $h = \varphi_{\overline{\lambda} w}$, proving that $\flat \colon E \to E'$ is surjective.

By the Cauchy–Schwarz inequality we have

$$|\psi(v)| = |\langle v, u \rangle| \leq \|v\| \, \|u\|,$$

so by definition of $\|\psi\|$ we get

$$\|\psi\| \leq \|u\|.$$

Obviously $\psi = 0$ iff $u = 0$ so assume $u \neq 0$. We have

$$\|u\|^2 = \langle u, u \rangle = \psi(u) \leq \|\psi\| \, \|u\|,$$

which yields $\|u\| \leq \|\psi\|$, and therefore $\|\psi\| = \|u\|$, as claimed. ∎

Proposition 48.9 is known as the *Riesz representation theorem or "Little Riesz Theorem."* It shows that the inner product on a Hilbert space induces a natural semilinear isomorphism between E and its dual E' (equivalently, a linear isomorphism between $\overline{E}$ and E'). This isomorphism is an isometry (it is preserves the norm).