else if $n \geq m$, then we let

$$D = \begin{pmatrix} \sigma_1 & & \dots & & 0 & \dots & 0 \\ & \sigma_2 & \dots & & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 & \vdots & 0 \\ & & \dots & \sigma_m & 0 & \dots & 0 \end{pmatrix}.$$

In either case, the above equations prove that

$$V^\top A U = D,$$

which yields $A = VDU^\top$, as required.

The equation $A = VDU^\top$ implies that

$$A^\top A = UD^\top DU^\top = U\mathrm{diag}(\sigma_1^2, \dots, \sigma_r^2, \underbrace{0, \dots, 0}_{n-r})U^\top$$

and

$$AA^\top = VDD^\top V^\top = V\mathrm{diag}(\sigma_1^2, \dots, \sigma_r^2, \underbrace{0, \dots, 0}_{m-r})V^\top,$$

which shows that $A^\top A$ and $AA^\top$ have the same nonzero eigenvalues, that the columns of U are eigenvectors of $A^\top A$, and that the columns of V are eigenvectors of $AA^\top$. ☐

A triple (U, D, V) such that $A = V D U^\top$ is called a *singular value decomposition (SVD) of* A. If $D = \mathrm{diag}(\sigma_1, \dots, \sigma_p)$ (with $p = \min(m, n)$), it is customary to assume that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p$.

Example 22.7. Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$. Then $A^\top = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ $A^\top A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, and $AA^\top = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. The reader should verify that $A^\top A = U\Sigma^2 U^\top$ where $\Sigma^2 = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ and $U = U^\top = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$. Since $AU = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$, set $v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, and complete an orthonormal basis for $\mathbb{R}^3$ by assigning $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Thus $V = I_3$, and the reader should verify that $A = VDU^\top$, where $D = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$.