which is strictly positive, since $\lambda_i > 0$ for $i = 1, \ldots, n$, and $x_i^2 > 0$ for some i, since $x \neq 0$.

Conversely, assume that

$$\langle f(x), x\rangle > 0$$

for all $x \neq 0$. Then for $x = e_i$, we get

$$\langle f(e_i), e_i\rangle = \langle \lambda_i e_i, e_i\rangle = \lambda_i,$$

and thus $\lambda_i > 0$ for all $i = 1, \ldots, n$.

(2) As in (1), we have

$$\langle f(x), x\rangle = \sum_{i=1}^{n} \lambda_i x_i^2,$$

and since $\lambda_i \geq 0$ for $i = 1, \ldots, n$ because f is positive semidefinite, we have $\langle f(x), x\rangle \geq 0$, as claimed. The converse is as in (1) except that we get only $\lambda_i \geq 0$ since $\langle f(e_i), e_i\rangle \geq 0$. $\square$

Some special notation is customary (especially in the field of convex optimization) to express that a symmetric matrix is positive definite or positive semidefinite.

Definition 42.2. Given any $n \times n$ symmetric matrix A we write $A \succeq 0$ if A is positive semidefinite and we write $A \succ 0$ if A is positive definite.

Remark: It should be noted that we can define the relation

$$A \succeq B$$

between any two $n \times n$ matrices (symmetric or not) iff $A - B$ is symmetric positive semidefinite. It is easy to check that this relation is actually a partial order on matrices, called the *positive semidefinite cone ordering*; for details, see Boyd and Vandenberghe [29], Section 2.4.

If A is symmetric positive definite, it is easily checked that A^{-1} is also symmetric positive definite. Also, if C is a symmetric positive definite $m \times m$ matrix and A is an $m \times n$ matrix of rank n (and so $m \geq n$ and the map $x \mapsto Ax$ is injective), then $A^\top CA$ is symmetric positive definite.

We can now prove that

$$Q(x) = \frac{1}{2} x^\top Ax - x^\top b$$

has a global minimum when A is symmetric positive definite.

Proposition 42.2. *Given a quadratic function*

$$Q(x) = \frac{1}{2} x^\top Ax - x^\top b,$$

if A is symmetric positive definite, then $Q(x)$ has a unique global minimum for the solution $x_0 = A^{-1}b$ of the linear system $Ax = b$. The minimum value of $Q(x)$ is

$$Q(A^{-1}b) = -\frac{1}{2} b^\top A^{-1} b.$$