Since $u = p_H(u) + p_G(u)$, the Hermitian reflection $\rho_{w,\theta}$ is also expressed as

$$\rho_{w,\theta}(u) = u + (e^{i\theta} - 1)p_G(u),$$

or as

$$\rho_{w,\theta}(u) = u + (e^{i\theta} - 1)\,\frac{(u \cdot w)}{\|w\|^2}\,w.$$

Note that the case of a standard hyperplane reflection is obtained when $e^{i\theta} = -1$, i.e., $\theta = \pi$.

We leave as an easy exercise to check that $\rho_{w,\theta}$ is indeed an isometry, and that the inverse of $\rho_{w,\theta}$ is $\rho_{w,-\theta}$. If we pick an orthonormal basis $(e_1, \ldots, e_n)$ such that $(e_1, \ldots, e_{n-1})$ is an orthonormal basis of H, the matrix of $\rho_{w,\theta}$ is

$$\begin{pmatrix} I_{n-1} & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

We now come to the main surprise. Given any two distinct vectors u and v such that $\|u\| = \|v\|$, there isn't always a hyperplane reflection mapping u to v, but this can be done using two Hermitian reflections!

Proposition 28.1. *Let E be any nontrivial Hermitian space.*

(1) For any two vectors $u, v \in E$ such that $u \neq v$ and $\|u\| = \|v\|$, if $u \cdot v = e^{i\theta}|u \cdot v|$, then the (usual) reflection s about the hyperplane orthogonal to the vector $v - e^{-i\theta}u$ is such that $s(u) = e^{i\theta}v$.

(2) For any nonnull vector $v \in E$, for any unit complex number $e^{i\theta} \neq 1$, there is a Hermitian reflection $\rho_{v,\theta}$ such that

$$\rho_{v,\theta}(v) = e^{i\theta}v.$$

As a consequence, for u and v as in (1), we have $\rho_{v,-\theta} \circ s(u) = v$.

Proof. (1) Consider the (usual) reflection about the hyperplane orthogonal to $w = v - e^{-i\theta}u$. We have

$$s(u) = u - 2\,\frac{(u \cdot (v - e^{-i\theta}u))}{\|v - e^{-i\theta}u\|^2}\,(v - e^{-i\theta}u).$$

We need to compute

$$-2u \cdot (v - e^{-i\theta}u) \quad \text{and} \quad (v - e^{-i\theta}u) \cdot (v - e^{-i\theta}u).$$

Since $u \cdot v = e^{i\theta}|u \cdot v|$, we have

$$e^{-i\theta}u \cdot v = |u \cdot v| \quad \text{and} \quad e^{i\theta}v \cdot u = |u \cdot v|.$$

Using the above and the fact that $\|u\| = \|v\|$, we get

$$\begin{aligned} -2u \cdot (v - e^{-i\theta}u) &= 2e^{i\theta}\,\|u\|^2 - 2u \cdot v, \\ &= 2e^{i\theta}(\|u\|^2 - |u \cdot v|), \end{aligned}$$