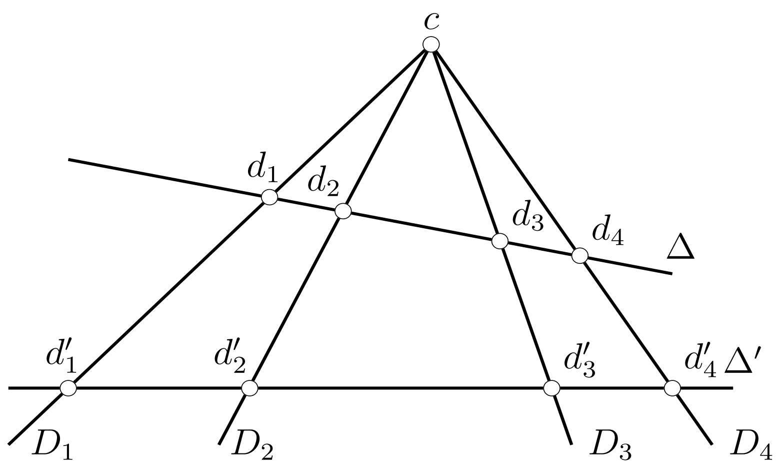

Figure 26.11: A projection of center c between two lines Δ and Δ'.

We now turn to the issue of determining when two linear maps f, g determine the same projective map, i.e., when $\mathbf{P}(f) = \mathbf{P}(g)$. The following proposition gives us a complete answer.

Proposition 26.4. *Given two nontrivial vector spaces E and F, for any two linear maps $f\colon E \to F$ and $g\colon E \to F$, we have $\mathbf{P}(f) = \mathbf{P}(g)$ iff there is some scalar $\lambda \in K - \{0\}$ such that $g = \lambda f$.*

Proof. If $g = \lambda f$, it is clear that $\mathbf{P}(f) = \mathbf{P}(g)$. Conversely, in order to have $\mathbf{P}(f) = \mathbf{P}(g)$, we must have $\operatorname{Ker} f = \operatorname{Ker} g$. If $\operatorname{Ker} f = \operatorname{Ker} g = E$, then f and g are both the null map, and this case is trivial. If $E - \operatorname{Ker} f \neq \emptyset$, by taking a basis of $\operatorname{Im} f$ and some inverse image of this basis, we obtain a basis B of a subspace G of E such that $E = \operatorname{Ker} f \oplus G$. If $\dim(G) = 1$, the restriction of any linear map $f\colon E \to F$ to G is determined by some nonzero vector $u \in E$ and some scalar $\lambda \in K$, and the proposition is obvious. Thus, assume that $\dim(G) \geq 2$. For any two distinct basis vectors $u, v \in B$, since $\mathbf{P}(f) = \mathbf{P}(g)$, there must be some nonzero scalars $\lambda(u)$, $\lambda(v)$, and $\lambda(u + v)$ such that

$$g(u) = \lambda(u)f(u), \quad g(v) = \lambda(v)f(v), \quad g(u + v) = \lambda(u + v)f(u + v).$$

Since f and g are linear, we get

$$g(u) + g(v) = \lambda(u)f(u) + \lambda(v)f(v) = \lambda(u + v)(f(u) + f(v)),$$

that is,

$$(\lambda(u + v) - \lambda(u))f(u) + (\lambda(u + v) - \lambda(v))f(v) = 0.$$

Since f is injective on G and $u, v \in B \subseteq G$ are linearly independent, $f(u)$ and $f(v)$ are also linearly independent, and thus we have

$$\lambda(u + v) = \lambda(u) = \lambda(v).$$