establishing the induction step. It follows that for any polynomial $p(X) = \sum_{k=0}^{n} a_k X^k$, we have

$$\begin{aligned} g(p(X) \cdot_f u) &= g\Big(\sum_{k=0}^{n} a_k f^k(u)\Big) \\ &= \sum_{k=0}^{n} a_k g \circ f^k(u) \\ &= \sum_{k=0}^{n} a_k f'^k \circ g(u) \\ &= \Big(\sum_{k=0}^{n} a_k f'^k\Big)(g(u)) \\ &= p(X) \cdot_{f'} g(u), \end{aligned}$$

so, g is indeed $K[X]$-linear. $\square$

Definition 36.1. We say that the linear maps $f \colon E \to E$ and $f' \colon E' \to E'$ are *similar* iff there is an isomorphism $g \colon E \to E'$ such that

$$f' = g \circ f \circ g^{-1},$$

or equivalently,

$$g \circ f = f' \circ g.$$

Then, Proposition 36.1 shows the following fact:

Proposition 36.2. *With notation of Proposition 36.1, two linear maps f and f' are similar iff g is an isomorphism between E_f and $E'_{f'}$.*

Later on, we will see that the isomorphism of finitely generated torsion modules can be characterized in terms of invariant factors, and this will be translated into a characterization of similarity of linear maps in terms of so-called similarity invariants. If f and f' are represented by matrices A and A' over bases of E and E', then f and f' are similar iff the matrices A and A' are similar (there is an invertible matrix P such that $A' = PAP^{-1}$). Similar matrices (and endomorphisms) have the same characteristic polynomial.

It turns out that there is a useful relationship between E_f and the module $K[X] \otimes_K E$. Observe that the map $\cdot \colon K[X] \times E \to E$ given by

$$p \cdot u = p(f)(u)$$

is K-bilinear, so it yields a K-linear map $\sigma \colon K[X] \otimes_K E \to E$ such that

$$\sigma(p \otimes u) = p \cdot u = p(f)(u).$$