so we get

$$(I+B)^{-1} = I - B(I+B)^{-1},$$

which yields

$$\left\|(I+B)^{-1}\right\| \leq 1 + \|B\| \left\|(I+B)^{-1}\right\|,$$

and finally,

$$\left\|(I+B)^{-1}\right\| \leq \frac{1}{1-\|B\|}.$$

(2) If $I+B$ is singular, then -1 is an eigenvalue of B, and by Proposition 9.6, we get $\rho(B) \leq \|B\|$, which implies $1 \leq \rho(B) \leq \|B\|$. $\square$

The second inequality is a result is that is needed to deal with the convergence of sequences of powers of matrices.

Proposition 9.12. *For every matrix $A \in \mathrm{M}_n(\mathbb{C})$ and for every $\epsilon > 0$, there is some subordinate matrix norm $\| \; \|$ such that*

$$\|A\| \leq \rho(A) + \epsilon.$$

Proof. By Theorem 15.5, there exists some invertible matrix U and some upper triangular matrix T such that

$$A = UTU^{-1},$$

and say that

$$T = \begin{pmatrix} \lambda_1 & t_{12} & t_{13} & \cdots & t_{1n} \\ 0 & \lambda_2 & t_{23} & \cdots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_{n-1} & t_{n-1\,n} \\ 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix},$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A. For every $\delta \neq 0$, define the diagonal matrix

$$D_\delta = \mathrm{diag}(1, \delta, \delta^2, \ldots, \delta^{n-1}),$$

and consider the matrix

$$(UD_\delta)^{-1}A(UD_\delta) = D_\delta^{-1}TD_\delta = \begin{pmatrix} \lambda_1 & \delta t_{12} & \delta^2 t_{13} & \cdots & \delta^{n-1} t_{1n} \\ 0 & \lambda_2 & \delta t_{23} & \cdots & \delta^{n-2} t_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_{n-1} & \delta t_{n-1\,n} \\ 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix}.$$

Now define the function $\| \; \| \colon \mathrm{M}_n(\mathbb{C}) \to \mathbb{R}$ by

$$\|B\| = \left\|(UD_\delta)^{-1}B(UD_\delta)\right\|_\infty,$$