We also define the map $\beta\colon L[X] \times E_f \to (L \otimes_K E)_{f_{(L)}}$ by

$$\beta(q(X), u) = q(X) \odot (1 \otimes_K u).$$

Using a computation similar to the computation that we just performed, we can check that β is $K[X]$-bilinear so we obtain a map $\widetilde{\beta}\colon L[X] \otimes_{K[X]} E_f \to (L \otimes_K E)_{f_{(L)}}$. To finish the proof, it suffices to prove that $\widetilde{\alpha} \circ \widetilde{\beta}$ and $\widetilde{\beta} \circ \widetilde{\alpha}$ are the identity on generators. We have

$$\widetilde{\alpha} \circ \widetilde{\beta}(q(X) \otimes_{K[X]} u) = \widetilde{\alpha}(q(X) \odot (1 \otimes_K u)) = q(X) \cdot (1 \otimes_{K[X]} u)) = q(X) \otimes_{K[X]} u,$$

and

$$\widetilde{\beta} \circ \widetilde{\alpha}(\lambda \otimes_K u) = \widetilde{\beta}(\lambda \otimes_{K[X]} u) = \lambda \odot (1 \otimes_K u) = \lambda \otimes_K u,$$

which finishes the proof. $\square$

By Proposition 36.9,

$$E_{(L)f_{(L)}} \approx L[X] \otimes_{K[X]} E_f \approx L[X]/(q_1 L[X]) \oplus \cdots \oplus L[X]/(q_n L[X]),$$

which shows that $(q_1, \ldots, q_n)$ are the similarity invariants of $f_{(L)}$. $\square$

Proposition 36.8 justifies the terminology "invariant" in similarity invariants. Indeed, under a field extension $K \subseteq L$, the similarity invariants of $f_{(L)}$ remain the same. This is not true of the elementary divisors, which depend on the field; indeed, an irreducible polynomial $p \in K[X]$ may split over $L[X]$. Since q_n is the minimal polynomial of f, the above reasoning also shows that the minimal polynomial of $f_{(L)}$ remains the same under a field extension.

Proposition 36.8 has the following corollary.

Proposition 36.10. *Let K be a field and let $L \supseteq K$ be a field extension of K. For any two square matrices A and B over K, if there is an invertible matrix Q over L such that $B = QAQ^{-1}$, then there is an invertible matrix P over K such that $B = PAP^{-1}$.*

Recall from Theorem 36.3 that the sequence of $K[X]$-linear maps

$$0 \longrightarrow E[X] \xrightarrow{\psi} E[X] \xrightarrow{\sigma} E_f \longrightarrow 0$$

is exact, and as a consequence, E_f is isomorphic to the quotient of $E[X]$ by $\mathrm{Im}(X1 - \overline{f})$. Furthermore, because E is a vector space, $E[X]$ is a free module with basis $(1 \otimes u_1, \ldots, 1 \otimes u_n)$, where $(u_1, \ldots, u_n)$ is a basis of E, and since ψ is injective, the module $\mathrm{Im}(X1 - \overline{f})$ has rank n. By Theorem 35.31, we have an isomorphism

$$E_f \approx K[X]/(q_1 K[X]) \oplus \cdots \oplus K[X]/(q_n K[X]),$$

and by Proposition 35.32, $E[X]/\mathrm{Im}(X1 - \overline{f})$ is isomorphic to a direct sum

$$E[X]/\mathrm{Im}(X1 - \overline{f}) \approx K[X]/(p_1 K[X]) \oplus \cdots \oplus K[X]/(p_n K[X]),$$