Proposition 12.17. *(Hadamard) For any real $n \times n$ matrix $A = (a_{ij})$, we have*

$$|\det(A)| \leq \prod_{i=1}^{n}\Big(\sum_{j=1}^{n} a_{ij}^2\Big)^{1/2} \quad \textit{and} \quad |\det(A)| \leq \prod_{j=1}^{n}\Big(\sum_{i=1}^{n} a_{ij}^2\Big)^{1/2}.$$

Moreover, equality holds iff either A has orthogonal rows in the left inequality or orthogonal columns in the right inequality.

Proof. If $\det(A) = 0$, then the inequality is trivial. In addition, if the righthand side is also 0, then either some column or some row is zero. If $\det(A) \neq 0$, then we can factor A as $A = QR$, with Q is orthogonal and $R = (r_{ij})$ upper triangular with positive diagonal entries. Then since Q is orthogonal $\det(Q) = \pm 1$, so

$$|\det(A)| = |\det(Q)|\,|\det(R)| = \prod_{j=1} r_{jj}.$$

Now as Q is orthogonal, it preserves the Euclidean norm, so

$$\sum_{i=1}^{n} a_{ij}^2 = \left\|A^j\right\|_2^2 = \left\|QR^j\right\|_2^2 = \left\|R^j\right\|_2^2 = \sum_{i=1}^{n} r_{ij}^2 \geq r_{jj}^2,$$

which implies that

$$|\det(A)| = \prod_{j=1}^{n} r_{jj} \leq \prod_{j=1}^{n} \left\|R^j\right\|_2 = \prod_{j=1}^{n}\Big(\sum_{i=1}^{n} a_{ij}^2\Big)^{1/2}.$$

The other inequality is obtained by replacing A by $A^\top$. Finally, if $\det(A) \neq 0$ and equality holds, then we must have

$$r_{jj} = \left\|A^j\right\|_2, \quad 1 \leq j \leq n,$$

which can only occur if A has orthogonal columns. ☐

Another version of Hadamard's inequality applies to symmetric positive semidefinite matrices.

Proposition 12.18. *(Hadamard) For any real $n \times n$ matrix $A = (a_{ij})$, if A is symmetric positive semidefinite, then we have*

$$\det(A) \leq \prod_{i=1}^{n} a_{ii}.$$

Moreover, if A is positive definite, then equality holds iff A is a diagonal matrix.