Thus, $\mathcal{L}$ is an ideal in A (this can also be proved directly). Since A is noetherian, $\mathcal{L}$ is finitely generated, and let $\{a_1, \ldots, a_n\}$ be a set of generators of $\mathcal{L}$. Let $f_1(X), \ldots, f_n(X)$ be polynomials in $\mathfrak{A}$ having respectively $a_1, \ldots, a_n$ as highest degree term coefficients. These polynomials generate an ideal $\mathfrak{B}$. Let q be the maximum of the degrees of the $f_i(X)$'s. Now, pick any polynomial $g(X) \in \mathfrak{A}$ of degree $d \geq q$, and let aX^d be its term of highest degree. Since $a \in \mathcal{L}$, we have

$$a = \lambda_1 a_1 + \cdots + \lambda_n a_n,$$

for some $\lambda_i \in A$. Consider the polynomial

$$g_1(X) = \sum_{i=1}^{n} \lambda_i f_i(X) X^{d-d_i},$$

where d_i is the degree of $f_i(X)$. Now, $g(X) - g_1(X)$ is a polynomial in $\mathfrak{A}$ of degree at most $d-1$. By repeating this procedure, we get a sequence of polynomials $g_i(X)$ in $\mathfrak{B}$, having strictly decreasing degrees, and such that the polynomial

$$g(X) - (g_1(X) + \cdots + g_i(X))$$

is of degree at most $d - i$. This polynomial must be of degree at most $q - 1$ as soon as $i = d - q + 1$. Thus, we proved that every polynomial in $\mathfrak{A}$ of degree $d \geq q$ belongs to $\mathfrak{B}$.

It remains to take care of the polynomials in $\mathfrak{A}$ of degree at most $q - 1$. Since A is noetherian, each ideal $L_i(\mathfrak{A})$ is finitely generated, and let $\{a_{i1}, \ldots, a_{in_i}\}$ be a set of generators for $L_i(\mathfrak{A})$ (for $i = 0, \ldots, q-1$). Let $f_{ij}(X)$ be a polynomial in $\mathfrak{A}$ having $a_{ij}X^i$ as its highest degree term. Given any polynomial $g(X) \in \mathfrak{A}$ of degree $d \leq q - 1$, if we denote its term of highest degree by aX^d, then, as in the previous argument, we can write

$$a = \lambda_1 a_{d1} + \cdots + \lambda_{n_d} a_{dn_d},$$

and we define

$$g_1(X) = \sum_{i=1}^{n_d} \lambda_i f_{di}(X) X^{d-d_i},$$

where d_i is the degree of $f_{di}(X)$. Then, $g(X) - g_1(X)$ is a polynomial in $\mathfrak{A}$ of degree at most $d-1$, and by repeating this procedure at most q times, we get an element of $\mathfrak{A}$ of degree 0, and the latter is a linear combination of the f_{0i}'s. This proves that every polynomial in $\mathfrak{A}$ of degree at most $q - 1$ is a combination of the polynomials $f_{ij}(X)$, for $0 \leq i \leq q - 1$ and $1 \leq j \leq n_i$. Therefore, $\mathfrak{A}$ is generated by the $f_k(X)$'s and the $f_{ij}(X)$'s, a finite number of polynomials. $\square$

Remark: Only a small part of Lemma 32.19 was used in the above proof, namely, the fact that $L_i(\mathfrak{A})$ is an ideal. A shorter proof of Theorem 32.21 making full use of Lemma 32.19 can be given as follows: