Chapter 28

Isometries of Hermitian Spaces

28.1 The Cartan–Dieudonné Theorem, Hermitian Case

The Cartan-Dieudonné theorem can be generalized (Theorem 28.2), but this requires allowing new types of hyperplane reflections that we call Hermitian reflections. After doing so, every isometry in $\mathbf{U}(n)$ can always be written as a composition of at most n Hermitian reflections (for $n \geq 2$). Better yet, every rotation in $\mathbf{SU}(n)$ can be expressed as the composition of at most $2n-2$ (standard) hyperplane reflections! This implies that every unitary transformation in $\mathbf{U}(n)$ is the composition of at most $2n-1$ isometries, with at most one Hermitian reflection, the other isometries being (standard) hyperplane reflections. The crucial Proposition 13.2 is false as is, and needs to be amended. The QR-decomposition of arbitrary complex matrices in terms of Householder matrices can also be generalized, using a trick.

In order to generalize the Cartan–Dieudonné theorem and the QR-decomposition in terms of Householder transformations, we need to introduce new kinds of hyperplane reflections. This is not really surprising, since in the Hermitian case, there are improper isometries whose determinant can be any unit complex number. Hyperplane reflections are generalized as follows.

Definition 28.1. Let E be a Hermitian space of finite dimension. For any hyperplane H, for any nonnull vector w orthogonal to H, so that $E = H \oplus G$, where $G = \mathbb{C}w$, a *Hermitian reflection about H of angle θ* is a linear map of the form $\rho_{H,\theta} \colon E \to E$, defined such that

$$\rho_{H,\theta}(u) = p_H(u) + e^{i\theta} p_G(u),$$

for any unit complex number $e^{i\theta} \neq 1$ (i.e. $\theta \neq k2\pi$). For any nonzero vector $w \in E$, we denote by $\rho_{w,\theta}$ the Hermitian reflection given by $\rho_{H,\theta}$, where H is the hyperplane orthogonal to w.