which shows that $v_i^* = f_i^* \circ h$ for $i = 1, \ldots, m$. This means that h has a unique expression in terms of linear forms as in $(*)$. Define the map α from $(E^*)^m$ to $\mathrm{Hom}(E, F)$ by

$$\alpha(v_1^*, \ldots, v_m^*)(u) = \sum_{j=1}^{m} v_j^*(u) f_j \quad \text{for all } u \in E.$$

This map is linear. For any $h \in \mathrm{Hom}(E, F)$, we showed earlier that the expression of h in $(*)$ is unique, thus α is an isomorphism. Similarly, $E^* \otimes F$ is isomorphic to $(E^*)^m$. Any tensor $\omega \in E^* \otimes F$ can be written as a linear combination

$$\sum_{k=1}^{p} u_k^* \otimes y_k$$

for some $u_k^* \in E^*$ and some $y_k \in F$, and since $(f_1, \ldots, f_m)$ is a basis of F, each y_k can be written as a linear combination of $(f_1, \ldots, f_m)$, so ω can be expressed as

$$\omega = \sum_{i=1}^{m} v_i^* \otimes f_i, \tag{$\dagger$}$$

for some linear forms $v_i^* \in E^*$ which are linear combinations of the u_k^*. If we pick a basis $(w_i^*)_{i \in I}$ for E^*, then we know that the family $(w_i^* \otimes f_j)_{i \in I, 1 \leq j \leq m}$ is a basis of $E^* \otimes F$, and this implies that the v_i^* in $(\dagger)$ are unique. Define the linear map β from $(E^*)^m$ to $E^* \otimes F$ by

$$\beta(v_1^*, \ldots, v_m^*) = \sum_{i=1}^{m} v_i^* \otimes f_i.$$

Since every tensor $\omega \in E^* \otimes F$ can be written in a unique way as in $(\dagger)$, this map is an isomorphism. $\square$

Note that in Proposition 33.17, we have an isomorphism if either E or F has finite dimension. The following proposition allows us to view a multilinear as a tensor product.

Proposition 33.18. *If the $E_1, \ldots E_n$ are finite-dimensional vector spaces and F is any vector space, then we have the canonical isomorphism*

$$\mathrm{Hom}(E_1, \ldots, E_n; F) \cong E_1^* \otimes \cdots \otimes E_n^* \otimes F.$$

Proof. In view of the canonical isomorphism

$$\mathrm{Hom}(E_1, \ldots, E_n; F) \cong \mathrm{Hom}(E_1 \otimes \cdots \otimes E_n, F)$$

given by Proposition 33.7 and the canonical isomorphism $(E_1 \otimes \cdots \otimes E_n)^* \cong E_1^* \otimes \cdots \otimes E_n^*$ given by Proposition 33.16, if the E_i's are finite-dimensional, then Proposition 33.17 yields the canonical isomorphism

$$\mathrm{Hom}(E_1, \ldots, E_n; F) \cong E_1^* \otimes \cdots \otimes E_n^* \otimes F,$$

as claimed. $\square$