that is, $f(\ldots, x_i, x_{i+1}, \ldots) = -f(\ldots, x_{i+1}, x_i, \ldots)$.

(2) If $x_i = x_j$ and i and j are not adjacent, we can interchange x_i and x_{i+1}, and then x_i and x_{i+2}, etc, until x_i and x_j become adjacent. By (1),

$$f(\ldots, x_i, \ldots, x_j, \ldots) = \epsilon f(\ldots, x_i, x_j, \ldots),$$

where $\epsilon = +1$ or -1, but $f(\ldots, x_i, x_j, \ldots) = 0$, since $x_i = x_j$, and (2) holds.

(3) follows from (2) as in (1). (4) is an immediate consequence of (2). $\square$

Proposition 7.3 will now be used to show a fundamental property of alternating multilinear maps. First we need to extend the matrix notation a little bit. Let E be a vector space over K. Given an $n \times n$ matrix $A = (a_{i\,j})$ over K, we can define a map $L(A)\colon E^n \to E^n$ as follows:

$$\begin{aligned} L(A)_1(u) &= a_{1\,1}u_1 + \cdots + a_{1\,n}u_n, \\ &\cdots \\ L(A)_n(u) &= a_{n\,1}u_1 + \cdots + a_{n\,n}u_n, \end{aligned}$$

for all $u_1, \ldots, u_n \in E$ and with $u = (u_1, \ldots, u_n)$. It is immediately verified that $L(A)$ is linear. Then given two $n \times n$ matrices $A = (a_{i\,j})$ and $B = (b_{i\,j})$, by repeating the calculations establishing the product of matrices (just before Definition 3.12), we can show that

$$L(AB) = L(A) \circ L(B).$$

It is then convenient to use the matrix notation to describe the effect of the linear map $L(A)$, as

$$\begin{pmatrix} L(A)_1(u) \\ L(A)_2(u) \\ \vdots \\ L(A)_n(u) \end{pmatrix} = \begin{pmatrix} a_{1\,1} & a_{1\,2} & \ldots & a_{1\,n} \\ a_{2\,1} & a_{2\,2} & \ldots & a_{2\,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n\,1} & a_{n\,2} & \ldots & a_{n\,n} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}.$$

Lemma 7.4. *Let $f\colon E \times \ldots \times E \to F$ be an n-linear alternating map. Let $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ be two families of n vectors, such that,*

$$\begin{aligned} v_1 &= a_{1\,1}u_1 + \cdots + a_{n\,1}u_n, \\ &\cdots \\ v_n &= a_{1\,n}u_1 + \cdots + a_{n\,n}u_n. \end{aligned}$$

Equivalently, letting

$$A = \begin{pmatrix} a_{1\,1} & a_{1\,2} & \ldots & a_{1\,n} \\ a_{2\,1} & a_{2\,2} & \ldots & a_{2\,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n\,1} & a_{n\,2} & \ldots & a_{n\,n} \end{pmatrix},$$