where $M(x)$ is the column vector associated with the vector x and $M(g(x))$ is the column vector associated with $g(x)$, as explained in Definition 4.1.

Thus, $M\colon \mathrm{Hom}(E,F) \to \mathrm{M}_{n,p}$ is an isomorphism of vector spaces, and when $p = n$ and the basis $(v_1,\ldots,v_n)$ is identical to the basis $(u_1,\ldots,u_p)$, $M\colon \mathrm{Hom}(E,E) \to \mathrm{M}_n$ is an isomorphism of rings.

Proof. That $M(g(x)) = M(g)M(x)$ was shown by Definition 4.2 or equivalently by Formula (1). The identities $M(g+h) = M(g)+M(h)$ and $M(\lambda g) = \lambda M(g)$ are straightforward, and $M(f\circ g) = M(f)M(g)$ follows from Identity (4) and the definition of matrix multiplication. The mapping $M\colon \mathrm{Hom}(E,F) \to \mathrm{M}_{n,p}$ is clearly injective, and since every matrix defines a linear map (see Proposition 4.1), it is also surjective, and thus bijective. In view of the above identities, it is an isomorphism (and similarly for $M\colon \mathrm{Hom}(E,E) \to \mathrm{M}_n$, where Proposition 4.1 is used to show that M_n is a ring). $\square$

In view of Proposition 4.2, it seems preferable to represent vectors from a vector space of finite dimension as column vectors rather than row vectors. *Thus, from now on, we will denote vectors of $\mathbb{R}^n$ (or more generally, of K^n) as column vectors.*

We explained in Section 3.9 that if the space E is finite-dimensional and has a finite basis $(u_1,\ldots,u_n)$, then a linear form $f^*\colon E \to K$ is represented by the *row vector* of coefficients

$$\begin{pmatrix} f^*(u_1) & \cdots & f^*(u_n) \end{pmatrix}, \tag{1}$$

over the bases $(u_1,\ldots,u_n)$ and 1 (in K), and that over the dual basis $(u_1^*,\ldots,u_n^*)$ of E^*, the linear form f^* is represented by the same coefficients, but as the *column vector*

$$\begin{pmatrix} f^*(u_1) \\ \vdots \\ f^*(u_n) \end{pmatrix}, \tag{2}$$

which is the transpose of the row vector in (1).

This is a special case of a more general phenomenon. A linear map $f\colon E \to F$ induces a map $f^\top\colon F^* \to E^*$ called the *transpose* of f (note that $f^\top$ maps F^* to E^*, *not* E^* to F^*), and if $(u_1\ldots,u_n)$ is a basis of E, $(v_1\ldots,v_m)$ is a basis of F, and if f is represented by the $m\times n$ matrix A over these bases, then over the dual bases $(v_1^*,\ldots,v_m^*)$ and $(u_1^*,\ldots,u_n^*)$, the linear map $f^\top$ is represented by $A^\top$, the transpose of the matrix A.

This is because over the basis $(v_1,\ldots,v_m)$, a linear form $\varphi \in F^*$ is represented by the row vector

$$\lambda = \begin{pmatrix} \varphi(v_1) & \cdots & \varphi(v_m) \end{pmatrix},$$

and we define $f^\top(\varphi)$ as the linear form represented by the row vector

$$\lambda A$$