If φ is symmetric, then the group $\mathbf{Isom}(\varphi)$ is denoted $\mathbf{O}(\varphi)$ and called the *orthogonal group* of φ. If φ is alternating, then the group $\mathbf{Isom}(\varphi)$ is denoted $\mathbf{Sp}(\varphi)$ and called the *symplectic group* of φ. If φ is ϵ-Hermitian, then the group $\mathbf{Isom}(\varphi)$ is denoted $\mathbf{U}_\epsilon(\varphi)$ and called the *ϵ-unitary group* of φ. When $\epsilon = 1$, we drop ϵ and just say *unitary group*.

If $(e_1, \ldots, e_n)$ is a basis of E, φ is the represented by the $n \times n$ matrix M, and f is represented by the $n \times n$ matrix A, since $A^{-1} = A^{*l} = A^{*r} = M^{-1}A^*M$, then we find that $f \in \mathbf{Isom}(\varphi)$ iff

$$A^*MA = M,$$

and A^{-1} is given by $A^{-1} = M^{-1}A^*M$.

More specifically, we define the following groups, using the matrices $I_{p,q}$, $J_{m,m}$ and $A_{m,m}$ defined at the end of Section 29.1.

(1) $K = \mathbb{R}$. We have

$$\begin{aligned}
\mathbf{O}(n) &= \{A \in \mathrm{M}_n(\mathbb{R}) \mid A^\top A = I_n\} \\
\mathbf{O}(p,q) &= \{A \in \mathrm{M}_{p+q}(\mathbb{R}) \mid A^\top I_{p,q} A = I_{p,q}\} \\
\mathbf{Sp}(2n,\mathbb{R}) &= \{A \in \mathrm{M}_{2n}(\mathbb{R}) \mid A^\top J_{n,n} A = J_{n,n}\} \\
\mathbf{SO}(n) &= \{A \in \mathrm{M}_n(\mathbb{R}) \mid A^\top A = I_n,\ \det(A) = 1\} \\
\mathbf{SO}(p,q) &= \{A \in \mathrm{M}_{p+q}(\mathbb{R}) \mid A^\top I_{p,q} A = I_{p,q},\ \det(A) = 1\}.
\end{aligned}$$

The group $\mathbf{O}(n)$ is the *orthogonal group*, $\mathbf{Sp}(2n,\mathbb{R})$ is the *real symplectic group*, and $\mathbf{SO}(n)$ is the *special orthogonal group*. We can define the group

$$\{A \in \mathrm{M}_{2n}(\mathbb{R}) \mid A^\top A_{n,n} A = A_{n,n}\},$$

but it is isomorphic to $\mathbf{O}(n,n)$.

(2) $K = \mathbb{C}$. We have

$$\begin{aligned}
\mathbf{U}(n) &= \{A \in \mathrm{M}_n(\mathbb{C}) \mid A^* A = I_n\} \\
\mathbf{U}(p,q) &= \{A \in \mathrm{M}_{p+q}(\mathbb{C}) \mid A^* I_{p,q} A = I_{p,q}\} \\
\mathbf{Sp}(2n,\mathbb{C}) &= \{A \in \mathrm{M}_{2n}(\mathbb{C}) \mid A^\top J_{n,n} A = J_{n,n}\} \\
\mathbf{SU}(n) &= \{A \in \mathrm{M}_n(\mathbb{C}) \mid A^* A = I_n,\ \det(A) = 1\} \\
\mathbf{SU}(p,q) &= \{A \in \mathrm{M}_{p+q}(\mathbb{C}) \mid A^* I_{p,q} A = I_{p,q},\ \det(A) = 1\}.
\end{aligned}$$

The group $\mathbf{U}(n)$ is the *unitary group*, $\mathbf{Sp}(2n,\mathbb{C})$ is the *complex symplectic group*, and $\mathbf{SU}(n)$ is the *special unitary group*.

It can be shown that if $A \in \mathbf{Sp}(2n,\mathbb{R})$ or if $A \in \mathbf{Sp}(2n,\mathbb{C})$, then $\det(A) = 1$.