in $\overrightarrow{E}$. Let $u \in \overrightarrow{E} - \overrightarrow{H}$ be any nonnull vector such that $A = a_1 + \mathbb{R}u$. Since A is not parallel to H, we have $\overrightarrow{E} = \overrightarrow{H} \oplus \mathbb{R}u$, and thus we can define the linear map $p\colon \overrightarrow{E} \to \mathbb{R}u$, the projection on $\mathbb{R}u$ parallel to $\overrightarrow{H}$. This linear map induces an affine map $f\colon E \to A$, by defining f such that

$$f(b_1 + w) = a_1 + p(w),$$

for all $w \in \overrightarrow{E}$. Clearly, $f(b_1) = a_1$, and since H_1, H_2, H_3 all have direction $\overrightarrow{H}$, we also have $f(b_2) = a_2$ and $f(b_3) = a_3$. Since f is affine, it preserves ratios, and thus

$$\frac{\overrightarrow{a_1a_3}}{\overrightarrow{a_1a_2}} = \frac{\overrightarrow{b_1b_3}}{\overrightarrow{b_1b_2}}.$$

The converse is immediate. $\square$

We also have the following simple proposition, whose proof is left as an easy exercise.

Proposition 24.11. *Given any affine space E, given any two distinct points $a, b \in E$, and for any affine dilatation f different from the identity, if $a' = f(a)$, $D = \langle a, b\rangle$ is the line passing through a and b, and D' is the line parallel to D and passing through a', the following are equivalent:*

(i) $b' = f(b)$;

(ii) If f is a translation, then b' is the intersection of D' with the line parallel to $\langle a, a'\rangle$ passing through b;

If f is a dilatation of center c, then $b' = D' \cap \langle c, b\rangle$.

The first case is the parallelogram law, and the second case follows easily from Thales' theorem. For an illustration, see Figure 24.22.

We are now ready to prove two classical results of affine geometry, Pappus's theorem and Desargues's theorem. Actually, these results are theorems of projective geometry, and we are stating affine versions of these important results. There are stronger versions that are best proved using projective geometry.

Proposition 24.12. *Given any affine plane E, any two distinct lines D and D', then for any distinct points a, b, c on D and a', b', c' on D', if a, b, c, a', b', c' are distinct from the intersection of D and D' (if D and D' intersect) and if the lines $\langle a, b'\rangle$ and $\langle a', b\rangle$ are parallel, and the lines $\langle b, c'\rangle$ and $\langle b', c\rangle$ are parallel, then the lines $\langle a, c'\rangle$ and $\langle a', c\rangle$ are parallel.*