Therefore, the solutions of the original least-squares problem are precisely the solutions of the the so-called *normal equations*

$$A^\top Ax = A^\top b,$$

discovered by Gauss and Legendre around 1800. We also proved that the normal equations always have a solution.

Computationally, it is best not to solve the normal equations directly, and instead, to use methods such as the QR-decomposition (applied to A) or the SVD-decomposition (in the form of the pseudo-inverse). We will come back to this point later on.

Here is another important corollary of Proposition 48.7.

Corollary 48.8. *For any continuous nonnull linear map $h\colon E \to \mathbb{C}$, the null space*

$$H = \mathrm{Ker}\, h = \{u \in E \mid h(u) = 0\} = h^{-1}(0)$$

is a closed hyperplane H, and thus, $H^\perp$ is a subspace of dimension one such that $E = H \oplus H^\perp$.

The above suggests defining the dual space of E as the set of all continuous maps $h\colon E \to \mathbb{C}$.

Remark: If $h\colon E \to \mathbb{C}$ is a linear map which is **not** continuous, then it can be shown that the hyperplane $H = \mathrm{Ker}\, h$ is dense in E! Thus, $H^\perp$ is reduced to the trivial subspace $\{0\}$. This goes against our intuition of what a hyperplane in $\mathbb{R}^n$ (or $\mathbb{C}^n$) is, and warns us not to trust our "physical" intuition too much when dealing with infinite dimensions. As a consequence, the map $\flat\colon E \to E^*$ introduced in Section 14.2 (see just after Definition 48.4 below) is not surjective, since the linear forms of the form $u \mapsto \langle u, v\rangle$ (for some fixed vector $v \in E$) are continuous (the inner product is continuous).

48.2 Duality and the Riesz Representation Theorem

We now show that by redefining the dual space of a Hilbert space as the set of continuous linear forms on E we recover Theorem 14.6.

Definition 48.4. Given a Hilbert space E, we define the *dual space E' of E* as the vector space of all continuous linear forms $h\colon E \to \mathbb{C}$. Maps in E' are also called *bounded linear operators, bounded linear functionals, or simply operators or functionals*.

As in Section 14.2, for all $u, v \in E$, we define the maps $\varphi^l_u\colon E \to \mathbb{C}$ and $\varphi^r_v\colon E \to \mathbb{C}$ such that

$$\varphi^l_u(v) = \overline{\langle u, v\rangle},$$

and

$$\varphi^r_v(u) = \langle u, v\rangle\,.$$