If $B$ is a skew symmetric matrix, since $\mathrm{tr}(B) = 0$, we deduce that $\det(e^B) = e^0 = 1$. This allows us to obtain the following result. Recall that the (real) vector space of skew symmetric matrices is denoted by $\mathfrak{so}(n)$.

**Proposition 15.15.** *For every skew symmetric matrix $B \in \mathfrak{so}(n)$, we have $e^B \in \mathbf{SO}(n)$, that is, $e^B$ is a rotation.*

*Proof.* By Proposition 9.23, $e^B$ is an orthogonal matrix. Since $\mathrm{tr}(B) = 0$, we deduce that $\det(e^B) = e^0 = 1$. Therefore, $e^B \in \mathbf{SO}(n)$. $\square$

Proposition 15.15 shows that the map $B \mapsto e^B$ is a map $\exp\colon \mathfrak{so}(n) \to \mathbf{SO}(n)$. It is not injective, but it can be shown (using one of the spectral theorems) that it is surjective.

If $B$ is a (real) symmetric matrix, then

$$(e^B)^\top = e^{B^\top} = e^B,$$

so $e^B$ is also symmetric. Since the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $B$ are real, by Proposition 15.13, since the eigenvalues of $e^B$ are $e^{\lambda_1}, \ldots, e^{\lambda_n}$ and the $\lambda_i$ are real, we have $e^{\lambda_i} > 0$ for $i = 1, \ldots, n$, which implies that $e^B$ is symmetric positive definite. In fact, it can be shown that for every symmetric positive definite matrix $A$, there is a *unique* symmetric matrix $B$ such that $A = e^B$; see Gallier [72].

## 15.6 Summary

The main concepts and results of this chapter are listed below:

- *Diagonal matrix*.
- *Eigenvalues, eigenvectors*; the *eigenspace associated* with an eigenvalue.
- *Characteristic polynomial*.
- *Trace*.
- *Algebraic and geometric multiplicity*.
- Eigenspaces associated with distinct eigenvalues form a direct sum (Proposition 15.3).
- Reduction of a matrix to an upper-triangular matrix.
- *Schur decomposition*.
- The *Gershgorin's discs* can be used to locate the eigenvalues of a complex matrix; see Theorems 15.9 and 15.10.
- The conditioning of eigenvalue problems.
- Eigenvalues of the matrix exponential. The formula $\det(e^A) = e^{\mathrm{tr}(A)}$.