the partial derivatives above all vanish for $x = y = 0$, so at a local extremum we should also have

$$\frac{\partial J}{\partial x}(0,0,z) = 0, \quad \frac{\partial J}{\partial y}(0,0,z) = 0, \quad \frac{\partial J}{\partial z}(0,0,z) = 0,$$

but this is absurd since

$$\frac{\partial J}{\partial x}(x,y,z) = 1, \quad \frac{\partial J}{\partial y}(x,y,z) = 1, \quad \frac{\partial J}{\partial z}(x,y,z) = 2z.$$

The reader should enjoy finding the reason for the flaw in the argument.

One should also keep in mind that Theorem 40.2 gives only a necessary condition. The (u, λ) may *not* correspond to local extrema! Thus, it is always necessary to analyze the local behavior of J near a critical point u. This is generally difficult, but in the case where J is affine or quadratic and the constraints are affine or quadratic, this is possible (although not always easy).

Example 40.2. Let us apply the above method to the following example in which $E_1 = \mathbb{R}$, $E_2 = \mathbb{R}$, $\Omega = \mathbb{R}^2$, and

$$\begin{aligned} J(x_1, x_2) &= -x_2 \\ \varphi(x_1, x_2) &= x_1^2 + x_2^2 - 1. \end{aligned}$$

Observe that

$$U = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1\}$$

is the unit circle, and since

$$\nabla\varphi(x_1, x_2) = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix},$$

it is clear that $\nabla\varphi(x_1, x_2) \neq 0$ for every point $= (x_1, x_2)$ on the unit circle. If we form the Lagrangian

$$L(x_1, x_2, \lambda) = -x_2 + \lambda(x_1^2 + x_2^2 - 1),$$

Theorem 40.2 says that a necessary condition for J to have a constrained local extremum is that $\nabla L(x_1, x_2, \lambda) = 0$, so the following equations must hold:

$$\begin{aligned} 2\lambda x_1 &= 0 \\ -1 + 2\lambda x_2 &= 0 \\ x_1^2 + x_2^2 &= 1. \end{aligned}$$

The second equation implies that $\lambda \neq 0$, and then the first yields $x_1 = 0$, so the third yields $x_2 = \pm 1$, and we get two solutions:

$$\begin{aligned} \lambda &= \frac{1}{2}, & (x_1, x_2) &= (0, 1) \\ \lambda &= -\frac{1}{2}, & (x_1', x_2') &= (0, -1). \end{aligned}$$