where $\alpha = \sqrt{a_{1\,1}}$, the matrix B_1 is invertible and

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & C - WW^\top/a_{1\,1} \end{pmatrix}$$

is symmetric positive definite. However, this implies that $C - WW^\top/a_{1\,1}$ is also symmetric positive definite (consider $x^\top A_1 x$ for every $x \in \mathbb{R}^n$ with $x \neq 0$ and $x_1 = 0$). Thus, we can apply the induction hypothesis to $C - WW^\top/a_{1\,1}$ (which is an $(n-1) \times (n-1)$ matrix), and we find a unique lower-triangular matrix L with positive diagonal entries so that

$$C - WW^\top/a_{1\,1} = LL^\top.$$

But then we get

$$\begin{aligned} A &= \begin{pmatrix} \alpha & 0 \\ W/\alpha & I \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & C - WW^\top/a_{1\,1} \end{pmatrix} \begin{pmatrix} \alpha & W^\top/\alpha \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} \alpha & 0 \\ W/\alpha & I \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & LL^\top \end{pmatrix} \begin{pmatrix} \alpha & W^\top/\alpha \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} \alpha & 0 \\ W/\alpha & I \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & L \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & L^\top \end{pmatrix} \begin{pmatrix} \alpha & W^\top/\alpha \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} \alpha & 0 \\ W/\alpha & L \end{pmatrix} \begin{pmatrix} \alpha & W^\top/\alpha \\ 0 & L^\top \end{pmatrix}. \end{aligned}$$

Therefore, if we let

$$B = \begin{pmatrix} \alpha & 0 \\ W/\alpha & L \end{pmatrix},$$

we have a unique lower-triangular matrix with positive diagonal entries and $A = BB^\top$. $\square$

Remark: The uniqueness of the Cholesky decomposition can also be established using the uniqueness of an LU-decomposition. Indeed, if $A = B_1 B_1^\top = B_2 B_2^\top$ where B_1 and B_2 are lower triangular with positive diagonal entries, if we let Δ_1 (resp. Δ_2) be the diagonal matrix consisting of the diagonal entries of B_1 (resp. B_2) so that $(\Delta_k)_{ii} = (B_k)_{ii}$ for $k = 1, 2$, then we have two LU-decompositions

$$A = (B_1 \Delta_1^{-1})(\Delta_1 B_1^\top) = (B_2 \Delta_2^{-1})(\Delta_2 B_2^\top)$$

with $B_1 \Delta_1^{-1}, B_2 \Delta_2^{-1}$ unit lower triangular, and $\Delta_1 B_1^\top, \Delta_2 B_2^\top$ upper triangular. By uniquenes of LU-factorization (Theorem 8.5(1)), we have

$$B_1 \Delta_1^{-1} = B_2 \Delta_2^{-1}, \quad \Delta_1 B_1^\top = \Delta_2 B_2^\top,$$

and the second equation yields

$$B_1 \Delta_1 = B_2 \Delta_2. \tag{$*$}$$