If $K = \{0, 1\}$, since the only nonzero scalar is 1, it is immediate that $g(y) = \overrightarrow{f}(y)$, and we are done. Otherwise, for $\nu \neq 0, 1$, we get $\lambda(y) = \mu$ for all $y \in \overrightarrow{G}$. Then equation

$$\lambda(y)w + \lambda(y)\overrightarrow{f}(y) = \mu w + g(y)$$

yields $g = \mu\overrightarrow{f}$ on G, and since g vanishes on $\mathrm{Ker}\,\overrightarrow{f}$ we get $g = \mu\overrightarrow{f}$ on $\overrightarrow{E}$ and the restriction of $\widetilde{f} = \mathbf{P}(g)$ to $\mathbf{P}(\overrightarrow{E})$ is equal to $\mathbf{P}(\overrightarrow{f})$. But now, by Proposition 25.6 and since $\widehat{F_H}$ is isomorphic to F, the linear map $\widehat{f}$ is completely determined by

$$\widehat{f}(u \mathbin{\widehat{+}} \lambda a) = \lambda f(a) + \overrightarrow{f}(u) = \lambda w + \overrightarrow{f}(u),$$

and g is completely determined by

$$g(u \mathbin{\widehat{+}} \lambda a) = \lambda g(a) + g(u) = \lambda\mu w + \mu\overrightarrow{f}(u).$$

Thus, we have $g = \mu\widehat{f}$.

Otherwise, if $\dim(\overrightarrow{G}) \geq 2$, then for any two distinct basis vectors u and v in B,

$$\begin{aligned} \lambda(u)w + \lambda(u)\overrightarrow{f}(u) &= \mu w + g(u), \\ \lambda(v)w + \lambda(v)\overrightarrow{f}(v) &= \mu w + g(v), \end{aligned}$$

and

$$\lambda(u+v)w + \lambda(u+v)\overrightarrow{f}(u+v) = \mu w + g(u+v),$$

and by linearity, we get

$$(\lambda(u+v) - \lambda(u) - \lambda(v) + \mu)w + (\lambda(u+v) - \lambda(u))\overrightarrow{f}(u) + (\lambda(u+v) - \lambda(v))\overrightarrow{f}(v) = 0.$$

Since $F = Kw \oplus H$, $\overrightarrow{f}\colon \overrightarrow{E} \to H$, and $\overrightarrow{f}(u)$ and $\overrightarrow{f}(v)$ are linearly independent (because $\overrightarrow{f}$ in injective on $\overrightarrow{G}$), we must have

$$\lambda(u+v) = \lambda(u) = \lambda(v) = \mu,$$

which implies that $g = \mu\overrightarrow{f}$ on $\overrightarrow{E}$, and the restriction of $\widetilde{f} = \mathbf{P}(g)$ to $\mathbf{P}(\overrightarrow{E})$ is equal to $\mathbf{P}(\overrightarrow{f})$. As in the previous case, g is completely determined by

$$g(u \mathbin{\widehat{+}} \lambda a) = \lambda g(a) + g(u) = \lambda\mu w + \mu\overrightarrow{f}(u).$$

Again, we have $g = \mu\widehat{f}$, and thus $\widetilde{f}$ is unique. $\square$