A fundamental property of $\mathbb{R}$ is that *it is complete*. It follows immediately that $\mathbb{C}$ is also complete. If E is a finite-dimensional real or complex vector space, since any two norms are equivalent, we can pick the ℓ^∞ norm, and then by picking a basis in E, a sequence (u_n) of vectors in E converges iff the n sequences of coordinates (u_n^i) $(1 \leq i \leq n)$ converge, so *any finite-dimensional real or complex vector space is a Banach space*.

Let us now consider the convergence of series.

Definition 9.14. Given a normed vector space $(E, \| \|)$, a *series* is an infinite sum $\sum_{k=0}^\infty u_k$ of elements $u_k \in E$. We denote by S_n the partial sum of the first $n+1$ elements,

$$S_n = \sum_{k=0}^n u_k.$$

Definition 9.15. We say that the series $\sum_{k=0}^\infty u_k$ *converges* to the limit $v \in E$ if the sequence (S_n) converges to v, i.e., given any $\epsilon > 0$, there exists a positive integer N such that for all $n \geq N$,

$$\|S_n - v\| < \epsilon.$$

In this case, we say that the series is *convergent*. We say that the series $\sum_{k=0}^\infty u_k$ *converges absolutely* if the series of norms $\sum_{k=0}^\infty \|u_k\|$ is convergent.

If the series $\sum_{k=0}^\infty u_k$ converges to v, since for all m, n with $m > n$ we have

$$\sum_{k=0}^m u_k - S_n = \sum_{k=0}^m u_k - \sum_{k=0}^n u_k = \sum_{k=n+1}^m u_k,$$

if we let m go to infinity (with n fixed), we see that the series $\sum_{k=n+1}^\infty u_k$ converges and that

$$v - S_n = \sum_{k=n+1}^\infty u_k.$$

There are series that are convergent but not absolutely convergent; for example, the series

$$\sum_{k=1}^\infty \frac{(-1)^{k-1}}{k}$$

converges to $\ln 2$, but $\sum_{k=1}^\infty \frac{1}{k}$ does not converge (this sum is infinite).

If E is complete, the converse is an enormously useful result.

Proposition 9.18. *Assume $(E, \| \|)$ is a complete normed vector space. If a series $\sum_{k=0}^\infty u_k$ is absolutely convergent, then it is convergent.*