Definition 37.36. Given two metric spaces, (E, d_E) and (F, d_F), a function, $f\colon E \to F$, is *uniformly continuous* if for every $\epsilon > 0$, there is some $\eta > 0$, such that, for all $a, b \in E$,

$$\text{if} \quad d_E(a, b) \leq \eta \quad \text{then} \quad d_F(f(a), f(b)) \leq \epsilon.$$

See Figures 37.42 and 37.43.

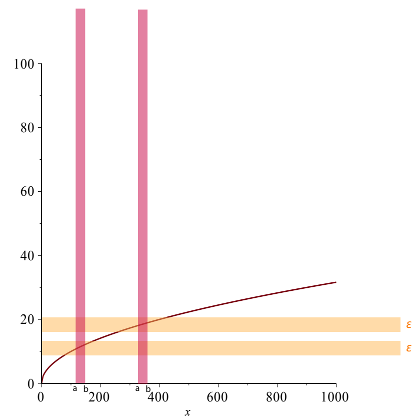

Figure 37.42: The real valued function $f(x) = \sqrt{x}$ is uniformly continuous over $(0, \infty)$. Fix ϵ. If the x values lie within the rose colored η strip, the y values always lie within the peach ϵ strip.

As we saw earlier, the metric on a metric space is uniformly continuous, and the norm on a normed metric space is uniformly continuous.

The *uniform continuity theorem* can be stated as follows:

Theorem 37.45. *Given two metric spaces, (E, d_E) and (F, d_F), if E is compact and if $f\colon E \to F$ is a continuous function, then f is uniformly continuous.*

Proof. Consider any $\epsilon > 0$ and let $(B_0(y, \epsilon/2))_{y \in F}$ be the open cover of F consisting of open balls of radius $\epsilon/2$. Since f is continuous, the family,

$$(f^{-1}(B_0(y, \epsilon/2)))_{y \in F},$$

is an open cover of E. Since, E is compact, by Lemma 37.44, there is a Lebesgue number, δ, such that for every open ball, $B_0(a, \eta)$, of radius $\eta \leq \delta$, then $B_0(a, \eta) \subseteq f^{-1}(B_0(y, \epsilon/2))$, for some $y \in F$. In particular, for any $a, b \in E$ such that $d_E(a, b) \leq \eta = \delta/2$, we have $a, b \in B_0(a, \delta)$ and thus, $a, b \in f^{-1}(B_0(y, \epsilon/2))$, which implies that $f(a), f(b) \in B_0(y, \epsilon/2)$. But then, $d_F(f(a), f(b)) \leq \epsilon$, as desired. $\square$

We now prove another lemma needed to obtain the characterization of compactness in metric spaces in terms of accumulation points.