Proof. Obviously, (1) implies (2). The proof that that (2) implies (1) is similar to the proof of Proposition 27.7, but uses Proposition 14.14 instead of Proposition 12.12. The details are left as an exercise. □

Inspection of the proof shows immediately that Proposition 27.8 holds for Hermitian spaces. For the sake of completeness, we restate the Proposition in the complex case.

Proposition 28.11. *Let E be any complex affine space of finite dimension For every affine map $f\colon E \to E$, let $Fix(f) = \{a \in E \mid f(a) = a\}$ be the set of fixed points of f. The following properties hold:*

(1) If f has some fixed point a, so that $Fix(f) \neq \emptyset$, then $Fix(f)$ is an affine subspace of E such that

$$Fix(f) = a + E(1, \overrightarrow{f}) = a + \mathrm{Ker}\,(\overrightarrow{f} - \mathrm{id}),$$

where $E(1, \overrightarrow{f})$ is the eigenspace of the linear map $\overrightarrow{f}$ for the eigenvalue 1.

(2) The affine map f has a unique fixed point iff $E(1, \overrightarrow{f}) = \mathrm{Ker}\,(\overrightarrow{f} - \mathrm{id}) = \{0\}$.

Affine orthogonal symmetries are defined just as in the Euclidean case, and Proposition 27.9 also applies to complex affine spaces.

Proposition 28.12. *Given any affine complex space E, if $f\colon E \to E$ and $g\colon E \to E$ are affine orthogonal symmetries about parallel affine subspaces F_1 and F_2, then $g \circ f$ is a translation defined by the vector $2\overrightarrow{ab}$, where $\overrightarrow{ab}$ is any vector perpendicular to the common direction $\overrightarrow{F}$ of F_1 and F_2 such that $\left\|\overrightarrow{ab}\right\|$ is the distance between F_1 and F_2, with $a \in F_1$ and $b \in F_2$. Conversely, every translation by a vector τ is obtained as the composition of two affine orthogonal symmetries about parallel affine subspaces F_1 and F_2 whose common direction is orthogonal to $\tau = \overrightarrow{ab}$, for some $a \in F_1$ and some $b \in F_2$ such that the distance betwen F_1 and F_2 is $\left\|\overrightarrow{ab}\right\|/2$.*

It is easy to check that the proof of Proposition 27.10 also holds in the Hermitian case.

Proposition 28.13. *Let E be a Hermitian affine space of finite dimension n. For every affine isometry $f\colon E \to E$, there is a unique affine isometry $g\colon E \to E$ and a unique translation $t = t_\tau$, with $\overrightarrow{f}(\tau) = \tau$ (i.e., $\tau \in \mathrm{Ker}\,(\overrightarrow{f} - \mathrm{id})$), such that the set $Fix(g) = \{a \in E, \mid g(a) = a\}$ of fixed points of g is a nonempty affine subspace of E of direction*

$$\overrightarrow{G} = \mathrm{Ker}\,(\overrightarrow{f} - \mathrm{id}) = E(1, \overrightarrow{f}),$$

and such that

$$f = t \circ g \quad \text{and} \quad t \circ g = g \circ t.$$

Furthermore, we have the following additional properties: