The notation $\dfrac{\partial f}{\partial x_j}(a)$ for a partial derivative, although customary and going back to Leibniz, is a "logical obscenity." Indeed, the variable x_j really has nothing to do with the formal definition. This is just another of these situations where tradition is just too hard to overthrow!

We now consider the situation where the normed affine space F is a finite direct sum $F = (F_1, b_1) \oplus \cdots \oplus (F_m, b_m)$.

Proposition 39.9. *Given normed affine spaces E and $F = (F_1, b_1) \oplus \cdots \oplus (F_m, b_m)$, given any open subset A of E, for any $a \in A$, for any function $f \colon A \to F$, letting $f = (f_1, \ldots, f_m)$, $\mathrm{D}f(a)$ exists iff every $\mathrm{D}f_i(a)$ exists, and*

$$\mathrm{D}f(a) = in_1 \circ \mathrm{D}f_1(a) + \cdots + in_m \circ \mathrm{D}f_m(a).$$

Proof. Observe that $f(a+h) - f(a) = (f(a+h) - b) - (f(a) - b)$, where $b = (b_1, \ldots, b_m)$, and thus, as far as dealing with derivatives, $\mathrm{D}f(a)$ is equal to $\mathrm{D}f_b(a)$, where $f_b \colon E \to \overrightarrow{F}$ is defined such that $f_b(x) = f(x) - b$, for every $x \in E$. Thus, we can work with the vector space $\overrightarrow{F}$ instead of the affine space F. The proposition is then a simple application of Theorem 39.6. $\square$

In the special case where F is a normed affine space of finite dimension m, for any frame $(b_0, (v_1, \ldots, v_m))$ of F, where $(v_1, \ldots, v_m)$ is a basis of $\overrightarrow{F}$, every point $x \in F$ can be expressed uniquely as

$$x = b_0 + x_1 v_1 + \cdots + x_m v_m,$$

where $(x_1, \ldots, x_m) \in K^m$, the coordinates of x in the frame $(b_0, (v_1, \ldots, v_m))$ (where $K = \mathbb{R}$ or $K = \mathbb{C}$). Thus, letting F_i be the standard normed affine space K with its natural structure, we note that F is isomorphic to the direct sum $F = (K, 0) \oplus \cdots \oplus (K, 0)$. Then, every function $f \colon E \to F$ is represented by m functions $(f_1, \ldots, f_m)$, where $f_i \colon E \to K$ (where $K = \mathbb{R}$ or $K = \mathbb{C}$), and

$$f(x) = b_0 + f_1(x) v_1 + \cdots + f_m(x) v_m,$$

for every $x \in E$. The following proposition is an immediate corollary of Proposition 39.9.

Proposition 39.10. *For any two normed affine spaces E and F, if F is of finite dimension m, for any frame $(b_0, (v_1, \ldots, v_m))$ of F, where $(v_1, \ldots, v_m)$ is a basis of $\overrightarrow{F}$, for every $a \in E$, a function $f \colon E \to F$ is differentiable at a iff each f_i is differentiable at a, and*

$$\mathrm{D}f(a)(u) = \mathrm{D}f_1(a)(u) v_1 + \cdots + \mathrm{D}f_m(a)(u) v_m,$$

for every $u \in \overrightarrow{E}$.