and thus, $\mathrm{Im}(\psi) \subseteq \mathrm{Ker}\,(\sigma)$. It remains to prove that $\mathrm{Ker}\,(\sigma) \subseteq \mathrm{Im}(\psi)$.

Since the monomials X^k form a basis of $A[X]$, by Proposition 35.13 (with the roles of M and N exchanged), every $z \in E[X] = A[X] \otimes_A E$ has a unique expression as

$$z = \sum_k X^k \otimes u_k,$$

for a family (u_k) of finite support of $u_k \in E$. If $z \in \mathrm{Ker}\,(\sigma)$, then

$$0 = \sigma(z) = \sum_k f^k(u_k),$$

which allows us to write

$$\begin{aligned}
z &= \sum_k X^k \otimes u_k - 1 \otimes 0 \\
&= \sum_k X^k \otimes u_k - 1 \otimes \left(\sum_k f^k(u_k) \right) \\
&= \sum_k (X^k \otimes u_k - 1 \otimes f^k(u_k)) \\
&= \sum_k (X^k(1 \otimes u_k) - \overline{f}^k(1 \otimes u_k)) \\
&= \sum_k (X^k 1 - \overline{f}^k)(1 \otimes u_k).
\end{aligned}$$

Now, $X1$ and $\overline{f}$ commute, since

$$\begin{aligned}
(X1 \circ \overline{f})(p \otimes u) &= (X1)(p \otimes f(u)) \\
&= (Xp) \otimes f(u)
\end{aligned}$$

and

$$\begin{aligned}
(\overline{f} \circ X1)(p \otimes u) &= \overline{f}((Xp) \otimes u) \\
&= (Xp) \otimes f(u),
\end{aligned}$$

so we can write

$$X^k 1 - \overline{f}^k = (X1 - \overline{f}) \left(\sum_{j=0}^{k-1} (X1)^j \overline{f}^{k-j-1} \right),$$

and

$$z = (X1 - \overline{f}) \left(\sum_k \left(\sum_{j=0}^{k-1} (X1)^j \overline{f}^{k-j-1} \right) (1 \otimes u_k) \right),$$