since $au_i \wedge bu_i = 0$, and it follows that

$$\bigwedge^p M \approx \bigoplus_{\substack{H \subseteq \{1,\ldots,n\} \\ H = \{k_1,\ldots,k_p\}}} (Au_{k_1}) \otimes \cdots \otimes (Au_{k_p}).$$

However, by Proposition 35.26, we have

$$(Au_{k_1}) \otimes \cdots \otimes (Au_{k_p}) = A/\mathfrak{a}_{k_1} \otimes \cdots \otimes A/\mathfrak{a}_{k_p} \approx A/(\mathfrak{a}_{k_1} + \cdots + \mathfrak{a}_{k_p}) = A/\mathfrak{a}_H.$$

Therefore,

$$\bigwedge^p M \approx \bigoplus_{\substack{H \subseteq \{1,\ldots,n\} \\ |H| = p}} A/\mathfrak{a}_H,$$

as claimed. $\square$

Example 35.1 continued: Recall that M is the $\mathbb{Z}$-module generated by $\{e_1, e_2, e_3, e_4\}$ subject to $6e_3 = 0$, $2e_2 = 0$. Then

$$\begin{aligned}
\bigwedge^1 M &= \mathrm{span}\{e_1, e_2, e_3, e_4\} \\
\bigwedge^2 M &= \mathrm{span}\{e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4\} \\
\bigwedge^3 M &= \mathrm{span}\{e_1 \wedge e_2 \wedge e_3, e_1 \wedge e_2 \wedge e_4, e_1 \wedge e_3 \wedge e_4, e_2 \wedge e_3 \wedge e_4\} \\
\bigwedge^3 M &= \mathrm{span}\{e_1 \wedge e_2 \wedge e_3 \wedge e_4\}.
\end{aligned}$$

Since $6e_3 = 0$, each element of $\{e_1 \wedge e_3, e_2 \wedge e_3, e_1 \wedge e_2 \wedge e_3\}$ is annihilated by $6\mathbb{Z} = (6)$. Since $2e_4 = 0$, each element of $\{e_1 \wedge e_4, e_2 \wedge e_4, e_3 \wedge e_4, e_1 \wedge e_2 \wedge e_4, e_1 \wedge e_3 \wedge e_4, e_2 \wedge e_3 \wedge e_4, e_1 \wedge e_2 \wedge e_3 \wedge e_4\}$ is annihilated by $2\mathbb{Z} = (2)$. We have shown that

$$M \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/(6) \oplus \mathbb{Z}/(2),$$

where $\mathfrak{a}_1 = (0) = \mathfrak{a}_2$, $\mathfrak{a}_3 = (6)$,and $\mathfrak{a}_4 = (2)$. Then Proposition 35.28 implies that

$$\begin{aligned}
\bigwedge^1 M &\cong \mathbb{Z}/\mathfrak{a}_1 \oplus \mathbb{Z}/\mathfrak{a}_2 \oplus \mathbb{Z}/\mathfrak{a}_3 \oplus \mathbb{Z}/\mathfrak{a}_4 = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/(6) \oplus \mathbb{Z}/(2) \\
\bigwedge^2 M &\cong \mathbb{Z}/(\mathfrak{a}_1 + \mathfrak{a}_2) \oplus \mathbb{Z}/(\mathfrak{a}_1 + \mathfrak{a}_3) \oplus \mathbb{Z}/(\mathfrak{a}_1 + \mathfrak{a}_4) \oplus \mathbb{Z}/(\mathfrak{a}_2 + \mathfrak{a}_3) \oplus \mathbb{Z}/(\mathfrak{a}_2 + \mathfrak{a}_3) \\
&\qquad \oplus \mathbb{Z}/(\mathfrak{a}_3 + \mathfrak{a}_4) = \mathbb{Z} \oplus \mathbb{Z}/(6) \oplus \mathbb{Z}/(2) \oplus \mathbb{Z}/(6) \oplus \mathbb{Z}/(2) \oplus \mathbb{Z}/(2) \\
\bigwedge^3 M &\cong \mathbb{Z}/(\mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{a}_3) \oplus \mathbb{Z}/(\mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{a}_4) \oplus \mathbb{Z}/(\mathfrak{a}_1 + \mathfrak{a}_3 + \mathfrak{a}_4) \oplus \mathbb{Z}/(\mathfrak{a}_1 + \mathfrak{a}_3 + \mathfrak{a}_4) \\
&= \mathbb{Z}/(6) \oplus \mathbb{Z}/(2) \oplus \mathbb{Z}/(2) \oplus \mathbb{Z}(2) \\
\bigwedge^4 M &\cong \mathbb{Z}/(\mathfrak{a}_1 + \mathfrak{a}_2 + \mathfrak{a}_3 + \mathfrak{a}_4) = \mathbb{Z}/(2).
\end{aligned}$$