of the cyclic subspace $E_i = Z(u_i; f)$, with $n_i = \deg(q_i)$, the matrix of the restriction of f to E_i is the *companion matrix* of $p_i(X)$, of the form

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & -a_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & -a_{n_i-2} \\ 0 & 0 & 0 & \cdots & 1 & -a_{n_i-1} \end{pmatrix}.$$

If we put all these bases together, we obtain a block matrix whose blocks are of the above form. Therefore, we proved the following result.

Theorem 36.6. *(Rational Canonical Form, First Version) Let $f\colon E \to E$ be an endomorphism on a K-vector space of dimension n. There exist n monic polynomials $q_1, \ldots, q_n \in K[X]$ such that*

$$q_1 \mid q_2 \mid \cdots \mid q_n,$$

with $q_1 = \cdots = q_{n-m} = 1$, and a basis of E such that the matrix M of f is a block matrix of the form

$$M = \begin{pmatrix} M_{n-m+1} & 0 & \cdots & 0 & 0 \\ 0 & M_{n-m+2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & M_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & M_n \end{pmatrix},$$

where each M_i is the companion matrix of q_i. The polynomials q_i satisfying the above conditions are unique, and q_n is the minimal polynomial of f.

Definition 36.2. A matrix M as in Theorem 36.6 is called a matrix in *rational form*. The polynomials $q_1, \ldots, q_n$ arising in Theorems 36.5 and 36.6 are called the *similarity invariants* (or *invariant factors*) of f.

Theorem 36.6 shows that every matrix is similar to a matrix in rational form. Such a matrix is unique.

Example 1 continued: We will calculate the rational canonical form for $f(x, y, z, w) = (x + w, y + z, y + z, x + w)$. The difficulty in finding the rational canonical form lies in determining the invariant factors q_1, q_2, q_3, q_4. As we will shortly discover, the invariant factors of f correspond to the invariant factors of $XI - M$. See Propositions 36.8 and 36.11. The invariant factors of $XI - M$ are found by converting $XI - M$ to Smith normal form. Section 36.5 describes an algorithmic procedure for computing the Smith normal form of a matrix. By applying the methodology of Section 36.5, we find that Smith normal form for