As the right hand side is an alternating map, we get a unique linear map $\bigwedge^n \eta\colon \bigwedge^n(E) \to E^{\otimes n}$ making the following diagram commute.

$$\begin{tikzcd} E^n \arrow[r, "\iota_\wedge"] \arrow[dr, "\eta"'] & \bigwedge^n(E) \arrow[d, "\bigwedge^n \eta"] \\ & E^{\otimes n}. \end{tikzcd}$$

Clearly, $\bigwedge^n \eta(\bigwedge^n(E))$ is the set of antisymmetrized tensors in $E^{\otimes n}$. If we consider the map $A = (\bigwedge^n \eta) \circ \pi\colon E^{\otimes n} \longrightarrow E^{\otimes n}$, it is easy to check that $A \circ A = A$. Therefore, A is a projection, and by linear algebra, we know that

$$E^{\otimes n} = A(E^{\otimes n}) \oplus \operatorname{Ker} A = \bigwedge^n \eta(\bigwedge^n(E)) \oplus \operatorname{Ker} A.$$

It turns out that $\operatorname{Ker} A = E^{\otimes n} \cap \mathfrak{I}_a = \operatorname{Ker}\, \pi$, where $\mathfrak{I}_a$ is the two-sided ideal of $T(E)$ generated by all tensors of the form $u \otimes u \in E^{\otimes 2}$ (for example, see Knapp [104], Appendix A). Therefore, $\bigwedge^n \eta$ is injective,

$$E^{\otimes n} = \bigwedge^n \eta(\bigwedge^n(E)) \oplus (E^{\otimes n} \cap \mathfrak{I}_a) = \bigwedge^n \eta(\bigwedge^n(E)) \oplus \operatorname{Ker} \pi,$$

and the exterior tensor power $\bigwedge^n(E)$ is naturally embedded into $E^{\otimes n}$.

34.5 Exterior Algebras

As in the case of symmetric tensors, we can pack together all the exterior powers $\bigwedge^n(V)$ into an algebra.

Definition 34.5. Gieven any vector space V, the vector space

$$\bigwedge(V) = \bigoplus_{m \geq 0} \bigwedge^m(V)$$

is called the *exterior algebra (or Grassmann algebra) of* V.

To make $\bigwedge(V)$ into an algebra, we mimic the procedure used for symmetric powers. If $\mathfrak{I}_a$ is the two-sided ideal generated by all tensors of the form $u \otimes u \in V^{\otimes 2}$, we set

$$\bigwedge^\bullet(V) = T(V)/\mathfrak{I}_a.$$

Then $\bigwedge^\bullet(V)$ automatically inherits a multiplication operation, called *wedge product*, and since $T(V)$ is graded, that is

$$T(V) = \bigoplus_{m \geq 0} V^{\otimes m},$$