Then prove the desired result by writing the power series for e^A and regrouping terms so that the power series for $\cos\theta$ and $\sin\theta$ show up. In particular

$$\begin{aligned} e^A &= I_3 + \sum_{p\geq 1}\frac{A^p}{p!} = I_3 + \sum_{p\geq 0}\frac{A^{2p+1}}{(2p+1)!} + \sum_{p\geq 1}\frac{A^{2p}}{(2p)!} \\ &= I_3 + \sum_{p\geq 0}\frac{(-1)^p\theta^{2p}}{(2p+1)!}A + \sum_{p\geq 1}\frac{(-1)^{p-1}\theta^{2(p-1)}}{(2p)!}A^2 \\ &= I_3 + \frac{A}{\theta}\sum_{p\geq 0}\frac{(-1)^p\theta^{2p+1}}{(2p+1)!} - \frac{A^2}{\theta^2}\sum_{p\geq 1}\frac{(-1)^p\theta^{2p}}{(2p)!} \\ &= I_3 + \frac{\sin\theta}{\theta}A - \frac{A^2}{\theta^2}\sum_{p\geq 0}\frac{(-1)^p\theta^{2p}}{(2p)!} + \frac{A^2}{\theta^2} \\ &= I_3 + \frac{\sin\theta}{\theta}A + \frac{(1-\cos\theta)}{\theta^2}A^2, \end{aligned}$$

as claimed. □

The above formulae are the well-known formulae expressing a rotation of axis specified by the vector (a, b, c) and angle θ.

The Rodrigues formula can used to show that the exponential map $\exp\colon \mathfrak{so}(3) \to \mathbf{SO}(3)$ is surjective.

Given any rotation matrix $R \in \mathbf{SO}(3)$, we have the following cases:

(1) The case $R = I$ is trivial.

(2) If $R \neq I$ and $\mathrm{tr}(R) \neq -1$, then

$$\exp^{-1}(R) = \left\{ \frac{\theta}{2\sin\theta}(R - R^T) \;\middle|\; 1 + 2\cos\theta = \mathrm{tr}(R) \right\}.$$

(Recall that $\mathrm{tr}(R) = r_{11} + r_{22} + r_{33}$, the *trace* of the matrix R).

Then there is a unique skew-symmetric B with corresponding θ satisfying $0 < \theta < \pi$ such that $e^B = R$.

(3) If $R \neq I$ and $\mathrm{tr}(R) = -1$, then R is a rotation by the angle π and things are more complicated, but a matrix B can be found. We leave this part as a good exercise: see Problem 17.8.

The computation of a logarithm of a rotation in $\mathbf{SO}(3)$ as sketched above has applications in kinematics, robotics, and motion interpolation.

As an immediate corollary of the Gram–Schmidt orthonormalization procedure, we obtain the QR-decomposition for invertible matrices.