and $x_n \in U_l$ for only finitely many n. Thus, if (x_n) does not have any accumulation point, the family, $(U_l)_{l\in E}$, is an open cover of E and since E is compact, it has some finite open subcover, $(U_l)_{l\in J}$, where J is a finite subset of E. But every U_l with $l \in J$ is such that $x_n \in U_l$ for only finitely many n, and since J is finite, $x_n \in \bigcup_{l\in J} U_l$ for only finitely many n, which contradicts the fact that $(U_l)_{l\in J}$ is an open cover of E, and thus contains all the x_n. Thus, (x_n) has some accumulation point. See Figure 37.40. $\square$

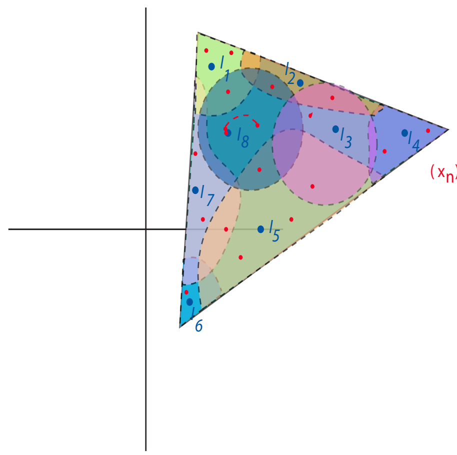

Figure 37.40: The space E the closed triangular region of $\mathbb{R}^2$. Given a sequence (x_n) of red points in E, if the sequence has no accumulation points, then each l_i for $1 \leq i \leq 8$, is not an accumulation point. But as implied by the illustration, l_8 actually is an accumulation point of (x_n).

Remarks:

1. By combining Propositions 37.42 and 37.43, we have observe that a second-countable Hausdorff space E is compact iff every sequence (x_n) has a convergent subsequence (x_{n_k}). In other words, we say a second-countable Hausdorff space E is compact iff it is *sequentially compact*.

2. It should be noted that the proof showing that if E is compact, then every sequence has some accumulation point, holds for any arbitrary compact space (the proof does not use a countable basis for the topology). The converse also holds for metric spaces. We will prove this converse since it is a major property of metric spaces.

Given a metric space in which every sequence has some accumulation point, we first prove the existence of a *Lebesgue number*.