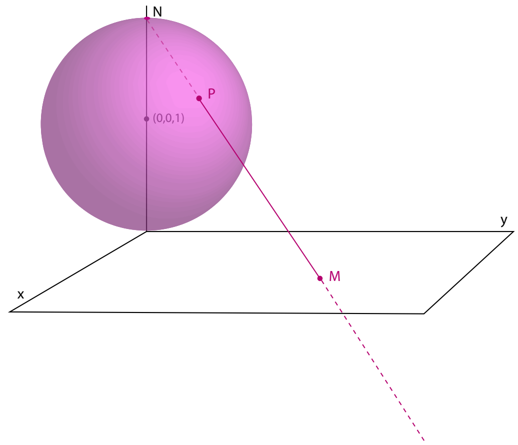

Figure 37.35: The stereographic projections of $x^2 + y^2 + (z - 1)^2 = 1$ onto the xy-plane.

according to stereographic projection. See Figure 37.36. A simpler example takes a line and gets a circle as its compactification. The Alexandroff compactification is a generalization of these simple constructions.

Definition 37.32. Let $(E, \mathcal{O})$ be a locally compact space. Let ω be any point not in E, and let $E_\omega = E \cup \{\omega\}$. Define the family, $\mathcal{O}_\omega$, as follows:

$$\mathcal{O}_\omega = \mathcal{O} \cup \{(E - K) \cup \{\omega\} \mid K \text{ compact in } E\}.$$

The pair, $(E_\omega, \mathcal{O}_\omega)$, is called the *Alexandroff compactification (or one point compactification) of* $(E, \mathcal{O})$. See Figure 37.37.

The following theorem shows that $(E_\omega, \mathcal{O}_\omega)$ is indeed a topological space, and that it is compact.

Theorem 37.36. *Let E be a locally compact topological space. The Alexandroff compactification, E_ω, of E is a compact space such that E is a subspace of E_ω and if E is not compact, then $\overline{E} = E_\omega$.*

Proof. The verification that $\mathcal{O}_\omega$ is a family of open sets is not difficult but a bit tedious. Details can be found in Munkres [131] or Schwartz [150]. Let us show that E_ω is compact. For every open cover, $(U_i)_{i \in I}$, of E_ω, since ω must be covered, there is some U_{i_0} of the form

$$U_{i_0} = (E - K_0) \cup \{\omega\}$$

where K_0 is compact in E. Consider the family, $(V_i)_{i \in I}$, defined as follows:

$$\begin{aligned} V_i &= U_i \quad \text{if} \quad U_i \in \mathcal{O}, \\ V_i &= E - K \quad \text{if} \quad U_i = (E - K) \cup \{\omega\}, \end{aligned}$$