We take a closer look at the projectivities of the projective line $\mathbb{P}^1_K$, since they play a role in the "change of parameters" for projective curves. A projectivity $f\colon \mathbb{P}^1_K \to \mathbb{P}^1_K$ is induced by some bijective linear map $g\colon K^2 \to K^2$ given by some invertible matrix

$$M(g) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with $ad - bc \neq 0$. Since the projective line $\mathbb{P}^1_K$ is isomorphic to $K \cup \{\infty\}$, it is easily verified that f is defined as follows:

$$c \neq 0 \begin{cases} z \mapsto \dfrac{az+b}{cz+d} & \text{if } z \neq -\dfrac{d}{c}, \\ -\dfrac{d}{c} \mapsto \infty, \\ \infty \mapsto \dfrac{a}{c}; \end{cases} \qquad c = 0 \begin{cases} z \mapsto \dfrac{az+b}{d}, \\ \infty \mapsto \infty. \end{cases}$$

From Section 26.4, we know that the points not at infinity are repesented by vectors of the form $(z, 1)$ where $z \in K$ and that ∞ is represented by $(1, 0)$. First, assume $c \neq 0$. Since

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} az+b \\ cz+d \end{pmatrix},$$

if $cz + d \neq 0$, that is, $z \neq -d/c$, then

$$(az+b, cz+d) \sim \left(\frac{az+b}{cz+d}, 1 \right),$$

so z is is mapped to $(az+d)/cz+d)$. If $cz + d = 0$, then

$$(az+d, 0) \sim (1, 0) = \infty,$$

so $-d/c$ is mapped to ∞. We also have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix},$$

and since $c \neq 0$ we have

$$(a, c) \sim (a/c, 1),$$

so ∞ is mapped to a/c. The case where $c = 0$ is handled similarly.

If $K = \mathbb{R}$ or $K = \mathbb{C}$, note that a/c is the limit of $(az + b)/(cz + d)$, as z approaches infinity, and the limit of $(az + b)/(cz + d)$ as z approaches $-d/c$ is ∞ (when $c \neq 0$).

Projections between hyperplanes form an important example of projectivities.