Proof. Part (1) follows from Theorem 29.33. By Proposition 29.30, we obtain a totally isotropic subspace U' of dimension r such that $U \cap U' = (0)$. By applying Theorem 29.33 to $U_1 = U$ and $U_2 = U'$, we get $U = W = (0)$, which proves (2). Part (3) is an immediate consequence of (2). $\square$

As a consequence of Theorem 29.34, we make the following definition.

Definition 29.19. Let E be a vector space of finite dimension n, and let φ be an ϵ-Hermitian form on E which is nondegenerate and satisfies property (T). The *index* (or *Witt index*) ν of φ, is the common dimension of all totally isotropic maximal subspaces of E. We have $2\nu \leq n$.

Neutral forms only exist if n is even, in which case, $\nu = n/2$. Forms of index $\nu = 0$ have no nonzero isotropic vectors. When $K = \mathbb{R}$, this is satisfied by positive definite or negative definite symmetric forms. When $K = \mathbb{C}$, this is satisfied by positive definite or negative definite Hermitian forms. The vector space of a neutral Hermitian form ($\epsilon = +1$) is an Artinian space, and the vector space of a neutral alternating form is a hyperbolic space.

If the field K is algebraically closed, we can describe all nondegenerate quadratic forms.

Proposition 29.35. *If K is algebraically closed and E has dimension n, then for every nondegenerate quadratic form Φ, there is a basis $(e_1, \ldots, e_n)$ such that Φ is given by*

$$\Phi\left(\sum_{i-1}^{n} x_i e_i\right) = \begin{cases} \sum_{i=1}^{m} x_i x_{m+i} & \text{if } n = 2m \\ \sum_{i=1}^{m} x_i x_{m+i} + x_{2m+1}^2 & \text{if } n = 2m+1. \end{cases}$$

Proof. We work with the polar form φ of Φ. Let U_1 and U_2 be some totally isotropic subspaces such that $U_1 \cap U_2 = (0)$ given by Theorem 29.34, and let q be their common dimension. Then, $W = U = (0)$. Since we can pick bases $(e_1, \ldots e_q)$ in U_1 and $(e_{q+1}, \ldots, e_{2q})$ in U_2 such that $\varphi(e_i, e_{i+q}) = 0$, for $i, j = 1, \ldots, q$, it suffices to proves that $\dim(D) \leq 1$. If $x, y \in D$ with $x \neq 0$, from the identity

$$\Phi(y - \lambda x) = \Phi(y) - \lambda\varphi(x, y) + \lambda^2 \Phi(x)$$

and the fact that $\Phi(x) \neq 0$ since $x \in D$ and $x \neq 0$, we see that the equation $\Phi(y - \lambda y) = 0$ has at least one solution. Since $\Phi(z) \neq 0$ for every nonzero $z \in D$, we get $y = \lambda x$, and thus $\dim(D) \leq 1$, as claimed. $\square$

Proposition 29.35 shows that for every nondegenerate quadratic form Φ over an algebraically closed field, if $\dim(E) = 2m$ or $\dim(E) = 2m + 1$ with $m \geq 1$, then Φ has some nonzero isotropic vector.