we get

$$\|g(v)\| = \|v\|$$

for all $v \in E$. In other words, g preserves both the distance and the norm.

To prove that g preserves the inner product, we use the simple fact that

$$2u \cdot v = \|u\|^2 + \|v\|^2 - \|u - v\|^2$$

for all $u, v \in E$. Then since g preserves distance and norm, we have

$$\begin{aligned} 2g(u) \cdot g(v) &= \|g(u)\|^2 + \|g(v)\|^2 - \|g(u) - g(v)\|^2 \\ &= \|u\|^2 + \|v\|^2 - \|u - v\|^2 \\ &= 2u \cdot v, \end{aligned}$$

and thus $g(u) \cdot g(v) = u \cdot v$, for all $u, v \in E$, which is (3). In particular, if $f(0) = 0$, by letting $\tau = 0$, we have $g = f$, and f preserves the scalar product, i.e., (3) holds.

Now assume that (3) holds. Since E is of finite dimension, we can pick an orthonormal basis $(e_1, \ldots, e_n)$ for E. Since f preserves inner products, $(f(e_1), \ldots, f(e_n))$ is also orthonormal, and since F also has dimension n, it is a basis of F. Then note that since $(e_1, \ldots, e_n)$ and $(f(e_1), \ldots, f(e_n))$ are orthonormal bases, for any $u \in E$ we have

$$u = \sum_{i=1}^{n} (u \cdot e_i) e_i = \sum_{i=1}^{n} u_i e_i$$

and

$$f(u) = \sum_{i=1}^{n} (f(u) \cdot f(e_i)) f(e_i),$$

and since f preserves inner products, this shows that

$$f(u) = \sum_{i=1}^{n} (f(u) \cdot f(e_i)) f(e_i) = \sum_{i=1}^{n} (u \cdot e_i) f(e_i) = \sum_{i=1}^{n} u_i f(e_i),$$

which proves that f is linear. Obviously, f preserves the Euclidean norm, and (3) implies (1).

Finally, if $f(u) = f(v)$, then by linearity $f(v - u) = 0$, so that $\|f(v - u)\| = 0$, and since f preserves norms, we must have $\|v - u\| = 0$, and thus $u = v$. Thus, f is injective, and since E and F have the same finite dimension, f is bijective. $\square$

Remarks:

(i) The dimension assumption is needed only to prove that (3) implies (1) when f is not known to be linear, and to prove that f is surjective, but the proof shows that (1) implies that f is injective.