and since the matrix A_{R_i,S_j} represents $g_{ij}\colon F_j \to G_i$, the matrix B_{S_j,T_k} represents $f_{jk}\colon E_k \to F_j$, and the matrix C_{R_i,T_k} represents $h_{ik}\colon E_k \to G_i$, so $(*_5)$ implies the matrix equation

$$C_{ik} = \sum_{j=1}^{n} A_{ij}B_{jk}, \quad 1 \leq i \leq m,\ 1 \leq k \leq p, \tag{$*_6$}$$

establishing (when combined with Proposition 6.13) the fact that $[C] = [A][B]$, namely the product $C = AB$ of the matrices A and B can be performed by blocks, using the same product formula on matrices that is used on scalars.

We record the above fact in the following proposition.

Proposition 6.14. *Let M, N, P be any positive integers, and let $\{1, \ldots, M\} = R_1 \cup \cdots \cup R_m$, $\{1, \ldots, N\} = S_1 \cup \cdots \cup S_n$, and $\{1, \ldots, P\} = T_1 \cup \cdots \cup T_p$ be any partitions into nonempty subsets R_i, S_j, T_k, and write $r_i = |R_i|$, $s_j = |S_j|$ and $t_k = |T_k|$ $(1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq p)$. Let A be an $M \times N$ matrix, let $[A]$ be the corresponding $m \times n$ block matrix of $r_i \times s_j$ matrices A_{ij} $(1 \leq i \leq m, 1 \leq j \leq n)$, and let B be an $N \times P$ matrix and $[B]$ be the corresponding $n \times p$ block matrix of $s_j \times t_k$ matrices B_{jk} $(1 \leq j \leq n, 1 \leq k \leq p)$. Then the $M \times P$ matrix $C = AB$ corresponds to an $m \times p$ block matrix $[C]$ of $r_i \times t_k$ matrices C_{ik} $(1 \leq i \leq m, 1 \leq k \leq p)$, and we have*

$$[C] = [A][B],$$

which means that

$$C_{ik} = \sum_{j=1}^{n} A_{ij}B_{jk}, \quad 1 \leq i \leq m,\ 1 \leq k \leq p.$$

Remark: The product $A_{ij}B_{jk}$ of the blocks A_{ij} and B_{jk}, which are really the matrices A_{R_i,S_j} and B_{S_j,T_k}, can be computed using the matrices A'_{ij} and B'_{jk} (discussed after Example 6.3) that are indexed by the "canonical" index sets $\{1, \ldots, r_i\}$, $\{1, \ldots, s_j\}$ and $\{1, \ldots, t_k\}$. But after computing $A'_{ij}B'_{jk}$, we have to remember to insert it as a block in $[C]$ using the correct index sets R_i and T_k. This is easily achieved in `Matlab`.

Example 6.4. Consider the partition of the index set $R = \{1, 2, 3, 4, 5, 6\}$ given by $R_1 = \{1, 2\}$, $R_2 = \{3\}$, $R_3 = \{4, 5, 6\}$; of the index set $S = \{1, 2, 3\}$ given by $S_1 = \{1, 2\}$, $S_2 = \{3\}$; and of the index set $T = \{1, 2, 3, 4, 5, 6\}$ given by $T_1 = \{1\}$, $T_2 = \{2, 3\}$, $T_3 = \{4, 5, 6\}$. Let $[A]$ be the 3×2 block matrix

$$[A] = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{pmatrix} = \begin{pmatrix} \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} & \begin{bmatrix} \quad \\ \quad \end{bmatrix} \\ \begin{bmatrix} \quad & \quad \end{bmatrix} & \begin{bmatrix} \quad \end{bmatrix} \\ \begin{bmatrix} \quad & \quad \\ \quad & \quad \\ \quad & \quad \end{bmatrix} & \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix} \end{pmatrix}$$