Remark: If E and F are nontrivial, it can be shown that $\|\mathrm{app}\| = 1$. It can also be shown that composition

$$\circ\colon \mathcal{L}(E;F) \times \mathcal{L}(F;G) \to \mathcal{L}(E;G),$$

is bilinear and continuous.

The above propositions and definition generalize to arbitrary n-multilinear maps, with $n \geq 2$. Proposition 37.59 extends in the obvious way to any n-multilinear map $f\colon E_1 \times \cdots \times E_n \to F$, but condition (3) becomes:

There is a constant $k \geq 0$ such that,

$$\|f(u_1, \ldots, u_n)\| \leq k\|u_1\| \cdots \|u_n\|, \text{ for all } u_1 \in E_1, \ldots, u_n \in E_n.$$

Definition 37.42 also extends easily to

$$\begin{aligned} \|f\| &= \inf \{k \geq 0 \mid \|f(x_1, \ldots, x_n)\| \leq k\|x_1\| \cdots \|x_n\|, \text{ for all } x_i \in E_i, 1 \leq i \leq n\} \\ &= \sup \{\|f(x_1, \ldots, x_n)\| \mid \|x_n\|, \ldots, \|x_n\| \leq 1\}. \end{aligned}$$

Proposition 37.60 is also easily extended, and we get an isomorphism between continuous n-multilinear maps in $\mathcal{L}_n(E_1, \ldots, E_n; F)$, and continuous linear maps in

$$\mathcal{L}(E_1; \mathcal{L}(E_2; \ldots; \mathcal{L}(E_n; F)))$$

An obvious extension of Proposition 37.61 also holds.

Definition 37.43. A normed vector space $(E, \| \|)$ over $\mathbb{R}$ (or $\mathbb{C}$) which is a complete metric space for the distance $d(u, v) = \|v - u\|$, is called a *Banach space*.

The following theorem is a key result of the theory of Banach spaces worth proving.

Theorem 37.62. *If E and F are normed vector spaces, and if F is a Banach space, then $\mathcal{L}(E;F)$ is a Banach space (with the operator norm).*

Proof. Let $(f)_{n\geq 1}$ be a Cauchy sequence of continuous linear maps $f_n\colon E \to F$. We proceed in several steps.

Step 1. Define the pointwise limit $f\colon E \to F$ of the sequence $(f_n)_{n\geq 1}$.

Since $(f)_{n\geq 1}$ is a Cauchy sequence, for every $\epsilon > 0$, there is some $N > 0$ such that $\|f_m - f_n\| < \epsilon$ for all $m, n \geq N$. Since $\| \|$ is the operator norm, we deduce that for any $u \in E$, we have

$$\|f_m(u) - f_n(u)\| = \|(f_m - f_n)(u)\| \leq \|f_m - f_n\| \, \|u\| \leq \epsilon \, \|u\| \quad \text{for all } m, n \geq N,$$

that is,

$$\|f_m(u) - f_n(u)\| \leq \epsilon \, \|u\| \quad \text{for all } m, n \geq N. \tag{$*_1$}$$