Proposition 16.5. *The matrix representing r_q is*

$$R_q = \begin{pmatrix} a^2+b^2-c^2-d^2 & 2bc-2ad & 2ac+2bd \\ 2bc+2ad & a^2-b^2+c^2-d^2 & -2ab+2cd \\ -2ac+2bd & 2ab+2cd & a^2-b^2-c^2+d^2 \end{pmatrix}.$$

Since $a^2+b^2+c^2+d^2=1$, this matrix can also be written as

$$R_q = \begin{pmatrix} 2a^2+2b^2-1 & 2bc-2ad & 2ac+2bd \\ 2bc+2ad & 2a^2+2c^2-1 & -2ab+2cd \\ -2ac+2bd & 2ab+2cd & 2a^2+2d^2-1 \end{pmatrix}.$$

The above is the rotation matrix in Euler form induced by the quaternion q, which is the matrix corresponding to ρ_q. This is because

$$\varphi = -i\psi, \quad \varphi^{-1} = i\psi^{-1},$$

so

$$r_q(x,y,z) = \varphi^{-1}(q\varphi(x,y,z)q^*) = i\psi^{-1}(q(-i\psi(x,y,z))q^*) = \psi^{-1}(q\psi(x,y,z)q^*) = \rho_q(x,y,z),$$

and so $r_q = \rho_q$.

We showed that every unit quaternion $q \in \mathbf{SU}(2)$ induces a rotation $r_q \in \mathbf{SO}(3)$, but it is not obvious that every rotation can be represented by a quaternion. This can shown in various ways.

One way to is use the fact that every rotation in $\mathbf{SO}(3)$ is the composition of two reflections, and that every reflection σ of $\mathbb{R}^3$ can be represented by a quaternion q, in the sense that

$$\sigma(x,y,z) = -\varphi^{-1}(q\varphi(x,y,z)q^*).$$

Note the presence of the negative sign. This is the method used in Gallier [72] (Chapter 9).

16.4 An Algorithm to Find a Quaternion Representing a Rotation

Theorem 16.6. *The homomorphism $r\colon \mathbf{SU}(2) \to \mathbf{SO}(3)$ is surjective.*

Here is an algorithmic method to find a unit quaternion q representing a rotation matrix R, which provides a proof of Theorem 16.6.

Let

$$q = \begin{pmatrix} a+ib & c+id \\ -(c-id) & a-ib \end{pmatrix}, \quad a^2+b^2+c^2+d^2=1,\ a,b,c,d \in \mathbb{R}.$$