Chapter 50

Introduction to Nonlinear Optimization

This chapter contains the most important results of nonlinear optimization theory.

In Chapter 40 we investigated the problem of determining when a function $J\colon \Omega \to \mathbb{R}$ defined on some open subset Ω of a normed vector space E has a local extremum in a subset U of Ω defined by equational constraints, namely

$$U = \{x \in \Omega \mid \varphi_i(x) = 0,\ 1 \leq i \leq m\},$$

where the functions $\varphi_i\colon \Omega \to \mathbb{R}$ are continuous (and usually differentiable). Theorem 40.2 gave a necessary condition in terms of the Lagrange multipliers. In Section 40.3 we assumed that U was a convex subset of Ω; then Theorem 40.9 gave us a necessary condition for the function $J\colon \Omega \to \mathbb{R}$ to have a local minimum at u with respect to U if dJ_u exists, namely

$$dJ_u(v - u) \geq 0 \quad \text{for all } v \in U.$$

Our first goal is to find a necessary criterion for a function $J\colon \Omega \to \mathbb{R}$ to have a minimum on a subset U, even is this subset is *not* convex. This can be done by introducing a notion of "tangent cone" at a point $u \in U$. We define the cone of feasible directions and then state a necessary condition for a function to have local minimum on a set U that is not necessarily convex in terms of the cone of feasible directions. The cone of feasible directions is not always convex, but it is if the constraints are inequality constraints. An inequality constraint $\varphi(u) \leq 0$ is said to be *active* if $\varphi(u) = 0$. One can also define the notion of *qualified constraint*. Theorem 50.5 gives necessary conditions for a function J to have a minimum on a subset U defined by qualified inequality constraints in terms of the Karush–Kuhn–Tucker conditions (for short KKT conditions), which involve nonnegative Lagrange multipliers. The proof relies on a version of the Farkas–Minkowski lemma. Some of the KTT conditions assert that $\lambda_i\varphi_i(u) = 0$, where $\lambda_i \geq 0$ is the Lagrange multiplier associated with the constraint $\varphi_i \leq 0$. To some extent, this implies that active constaints are more important than inactive constraints, since if $\varphi_i(u) < 0$ is an inactive constraint, then $\lambda_i = 0$. In general,