Another useful criterion for a square matrix to be invertible is stated next.

Proposition 3.15. *A square matrix $A \in \mathrm{M}_n(K)$ is invertible iff for any $x \in K^n$, the equation $Ax = 0$ implies that $x = 0$.*

Proof. If A is invertible and if $Ax = 0$, then by multiplying both sides of the equation $x = 0$ by A^{-1}, we get

$$A^{-1}Ax = I_n x = x = A^{-1}0 = 0.$$

Conversely, for any $x = (x_1, \ldots, x_n) \in K^n$, since

$$Ax = x_1 A^1 + \cdots + x_n A^n,$$

the condition $Ax = 0$ implies $x = 0$ is equivalent to the linear independence of the columns $(A^1, \ldots, A^n)$ of A. By Proposition 3.14, the matrix A is invertible. ☐

It is immediately verified that the set $\mathrm{M}_{m,n}(K)$ of $m \times n$ matrices is a *vector space* under addition of matrices and multiplication of a matrix by a scalar.

Definition 3.17. The $m \times n$-matrices $E_{ij} = (e_{h\,k})$, are defined such that $e_{i\,j} = 1$, and $e_{h\,k} = 0$, if $h \neq i$ or $k \neq j$; in other words, the (i, j)-entry is equal to 1 and all other entries are 0.

Here are the E_{ij} matrices for $m = 2$ and $n = 3$:

$$E_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$
$$E_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad E_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad E_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is clear that every matrix $A = (a_{i\,j}) \in \mathrm{M}_{m,n}(K)$ can be written in a unique way as

$$A = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{i\,j} E_{ij}.$$

Thus, the family $(E_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ is a basis of the vector space $\mathrm{M}_{m,n}(K)$, which has dimension mn.

Remark: Definition 3.12 and Definition 3.13 also make perfect sense when K is a (commutative) ring rather than a field. In this more general setting, the framework of vector spaces is too narrow, but we can consider structures over a commutative ring A satisfying all the axioms of Definition 3.1. Such structures are called *modules*. The theory of modules is (much) more complicated than that of vector spaces. For example, modules do not always have a basis, and other properties holding for vector spaces usually fail for modules. When a module has a basis, it is called a *free module*. For example, when A is a commutative