Definition 37.14. We say that a family $\mathcal{B}$ of subsets of E is a *basis for the topology* $\mathcal{O}$, if $\mathcal{B}$ is a subset of $\mathcal{O}$, and if every open set U in $\mathcal{O}$ can be obtained as some union (possibly infinite) of sets in $\mathcal{B}$ (agreeing that the empty union is the empty set).

For example, given any metric space (E, d), $\mathcal{B} = \{B_0(a, \rho) \mid a \in E, \rho > 0\}$. In particular, if $d = \| \|_2$, the open intervals form a basis for $\mathbb{R}$, while the open disks form a basis for $\mathbb{R}^2$. The open rectangles also form a basis for $\mathbb{R}^2$ with the standard topology. See Figure 37.13.

It is immediately verified that if a family $\mathcal{B} = (U_i)_{i \in I}$ is a basis for the topology of $(E, \mathcal{O})$, then $E = \bigcup_{i \in I} U_i$, and the intersection of any two sets $U_i, U_j \in \mathcal{B}$ is the union of some sets in the family $\mathcal{B}$ (again, agreeing that the empty union is the empty set). Conversely, a family $\mathcal{B}$ with these properties is the basis of the topology obtained by forming arbitrary unions of sets in $\mathcal{B}$.

Definition 37.15. A *subbasis for* $\mathcal{O}$ is a family $\mathcal{S}$ of subsets of E, such that the family $\mathcal{B}$ of all finite intersections of sets in $\mathcal{S}$ (including E itself, in case of the empty intersection) is a basis of $\mathcal{O}$. See Figure 37.13.

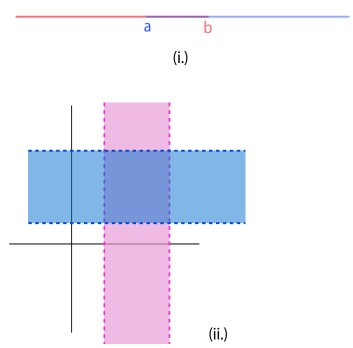

Figure 37.13: Figure $(i.)$ shows that the set of infinite open intervals forms a subbasis for $\mathbb{R}$. Figure $(ii.)$ shows that the infinite open strips form a subbasis for $\mathbb{R}^2$.

The following proposition gives useful criteria for determining whether a family of open subsets is a basis of a topological space.

Proposition 37.8. *Given a topological space* $(E, \mathcal{O})$ *and a family* $\mathcal{B}$ *of open subsets in* $\mathcal{O}$ *the following properties hold:*

(1) The family $\mathcal{B}$ *is a basis for the topology* $\mathcal{O}$ *iff for every open set* $U \in \mathcal{O}$ *and every* $x \in U$*, there is some* $B \in \mathcal{B}$ *such that* $x \in B$ *and* $B \subseteq U$*. See Figure 37.14.*

(2) The family $\mathcal{B}$ *is a basis for the topology* $\mathcal{O}$ *iff*

(a) For every $x \in E$*, there is some* $B \in \mathcal{B}$ *such that* $x \in B$*.*