and

$$\sigma(\overline{f}(p \otimes u)) = \sigma(p \otimes f(u)) = p(f)(f(u)),$$

so we get

$$\sigma \circ \overline{f} = f \circ \sigma. \tag{$*$}$$

Using our simplified notation,

$$\overline{f}(p_1 u_1 + \cdots + p_n u_n) = p_1 f(u_1) + \cdots + p_n f(u_n).$$

Define the $K[X]$-linear map $\psi \colon E[X] \to E[X]$ by

$$\psi(p \otimes u) = (Xp) \otimes u - p \otimes f(u).$$

Observe that $\psi = X1_{E[X]} - \overline{f}$, which we abbreviate as $X1 - \overline{f}$. Using our simplified notation

$$\psi(p_1 u_1 + \cdots + p_n u_n) = Xp_1 u_1 + \cdots + Xp_n u_n - (p_1 f(u_1) + \cdots + p_n f(u_n)).$$

It should be noted that everything we did in Section 36.1 applies to modules over a commutative ring A, except for the statements that assume that $A[X]$ is a PID. So, if M is an A-module, we can define the $A[X]$-modules M_f and $M[X] = A[X] \otimes_A M$, except that M_f is generally not a torsion module, and all the results showed above hold. Then, we have the following remarkable result.

Theorem 36.3. *(The Characteristic Sequence) Let A be a ring and let E be an A-module. The following sequence of $A[X]$-linear maps is exact:*

$$0 \longrightarrow E[X] \xrightarrow{\psi} E[X] \xrightarrow{\sigma} E_f \longrightarrow 0.$$

This means that ψ is injective, σ is surjective, and that $\mathrm{Im}(\psi) = \mathrm{Ker}\,(\sigma)$. As a consequence, E_f is isomorphic to the quotient of $E[X]$ by $\mathrm{Im}(X1 - \overline{f})$.

Proof. Because $\sigma(1 \otimes u) = u$ for all $u \in E$, the map σ is surjective. We have

$$\begin{aligned} \sigma(X(p \otimes u)) &= X \cdot \sigma(p \otimes u) \\ &= f(\sigma(p \otimes u)), \end{aligned}$$

which shows that

$$\sigma \circ X1 = f \circ \sigma = \sigma \circ \overline{f},$$

using $(*)$. This implies that

$$\begin{aligned} \sigma \circ \psi &= \sigma \circ (X1 - \overline{f}) \\ &= \sigma \circ X1 - \sigma \circ \overline{f} \\ &= \sigma \circ \overline{f} - \sigma \circ \overline{f} = 0, \end{aligned}$$