Since the rref of $A^\top$ is

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix},$$

the above system is equivalent to

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} u_1 + u_2 \\ u_3 + u_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where the free variables are associated with u_2 and u_4. Thus to determine a basis for the kernel of $A^\top$, we set $u_2 = 1, u_4 = 0$ and $u_2 = 0, u_4 = 1$ and obtain a basis for V^0 as

$$\begin{pmatrix} 1 & -1 & 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 1 & -1 \end{pmatrix}.$$

Problem 2 . Let us now consider the problem of finding a basis of the hyperplane H in $\mathbb{R}^n$ defined by the equation

$$c_1x_1 + \cdots + c_nx_n = 0.$$

More precisely, if $u^*(x_1, \ldots, x_n)$ is the linear form in $(\mathbb{R}^n)^*$ given by $u^*(x_1, \ldots, x_n) = c_1x_1 + \cdots + c_nx_n$, then the hyperplane H is the kernel of u^*. Of course we assume that some c_j is nonzero, in which case the linear form u^* spans a one-dimensional subspace U of $(\mathbb{R}^n)^*$, and $U^0 = H$ has dimension $n - 1$.

Since u^* is not the linear form which is identically zero, there is a smallest positive index $j \leq n$ such that $c_j \neq 0$, so our linear form is really $u^*(x_1, \ldots, x_n) = c_jx_j + \cdots + c_nx_n$. We claim that the following $n - 1$ vectors (in $\mathbb{R}^n$) form a basis of H:

$$\begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ j-1 \\ j \\ j+1 \\ j+2 \\ \vdots \\ n \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} 1 & 2 & \ldots & j-1 & j & j+1 & \ldots & n-1 \end{array} \\
\begin{pmatrix}
1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & -c_{j+1}/c_j & -c_{j+2}/c_j & \ldots & -c_n/c_j \\
0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1
\end{pmatrix}
\end{array}.$$

Observe that the $(n-1) \times (n-1)$ matrix obtained by deleting row j is the identity matrix, so the columns of the above matrix are linearly independent. A simple calculation also shows that the linear form $u^*(x_1, \ldots, x_n) = c_jx_j + \cdots + c_nx_n$ vanishes on every column of the above