Theorem 36.21. *If M is an $m \times n$ matrix over a PID A, then there exist some invertible $n \times n$ matrix P and some invertible $m \times m$ matrix Q, where P and Q are products of elementary matrices and matrices of the form*

$$\begin{pmatrix} x & y & 0 & 0 & \cdots & 0 \\ s & t & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 \end{pmatrix}$$

with $xt - ys = 1$, and a $m \times n$ matrix D of the form

$$D = \begin{pmatrix} \alpha_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \alpha_r & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}$$

for some nonzero $\alpha_i \in A$, such that

(1) $\alpha_1 \mid \alpha_2 \mid \cdots \mid \alpha_r$, and

(2) $M = QDP^{-1}$.

Proof sketch. In Step 2a, if a_{11} does not divide a_{k1}, then first permute row 2 and row k (if $k \neq 2$). Then, if we write $a = a_{11}$ and $b = a_{k1}$, if d is a gcd of a and b and if x, y, s, t are determined as explained above, multiply on the left by the matrix

$$\begin{pmatrix} x & y & 0 & 0 & \cdots & 0 \\ s & t & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 1 \end{pmatrix}$$

to obtain a matrix of the form

$$\begin{pmatrix} d & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$$