with $\alpha = a + ib$, $\beta = c + id$, and $a^2 + b^2 + c^2 + d^2 = 1$ $(a, b, c, d \in \mathbb{R})$, to find the matrix representing the rotation r_q we need to compute

$$q(x\sigma_3 + y\sigma_2 + z\sigma_1)q^* = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \begin{pmatrix} x & z - iy \\ z + iy & -x \end{pmatrix} \begin{pmatrix} \overline{\alpha} & -\beta \\ \overline{\beta} & \alpha \end{pmatrix}.$$

First, we have

$$\begin{pmatrix} x & z - iy \\ z + iy & -x \end{pmatrix} \begin{pmatrix} \overline{\alpha} & -\beta \\ \overline{\beta} & \alpha \end{pmatrix} = \begin{pmatrix} x\overline{\alpha} + z\overline{\beta} - iy\overline{\beta} & -x\beta + z\alpha - iy\alpha \\ z\overline{\alpha} + iy\overline{\alpha} - x\overline{\beta} & -z\beta - iy\beta - x\alpha \end{pmatrix}.$$

Next, we have

$$\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \begin{pmatrix} x\overline{\alpha} + z\overline{\beta} - iy\overline{\beta} & -x\beta + z\alpha - iy\alpha \\ z\overline{\alpha} + iy\overline{\alpha} - x\overline{\beta} & -z\beta - iy\beta - x\alpha \end{pmatrix} =$$

$$\begin{pmatrix} (\alpha\overline{\alpha} - \beta\overline{\beta})x + i(\overline{\alpha}\beta - \alpha\overline{\beta})y + (\alpha\overline{\beta} + \overline{\alpha}\beta)z & -2\alpha\beta x - i(\alpha^2 + \beta^2)y + (\alpha^2 - \beta^2)z \\ -2\overline{\alpha}\overline{\beta}x + i(\overline{\alpha}^2 + \overline{\beta}^2)y + (\overline{\alpha}^2 - \overline{\beta}^2)z & -(\alpha\overline{\alpha} - \beta\overline{\beta})x - i(\overline{\alpha}\beta - \alpha\overline{\beta})y - (\alpha\overline{\beta} + \overline{\alpha}\beta)z \end{pmatrix}$$

Since $\alpha = a + ib$ and $\beta = c + id$, with $a, b, c, d \in \mathbb{R}$, we have

$$\begin{aligned} \alpha\overline{\alpha} - \beta\overline{\beta} &= a^2 + b^2 - c^2 - d^2 \\ i(\overline{\alpha}\beta - \alpha\overline{\beta}) &= 2(bc - ad) \\ \alpha\overline{\beta} + \overline{\alpha}\beta &= 2(ac + bd) \\ -\alpha\beta &= -ac + bd - i(ad + bc) \\ -i(\alpha^2 + \beta^2) &= 2(ab + cd) - i(a^2 - b^2 + c^2 - d^2) \\ \alpha^2 - \beta^2 &= a^2 - b^2 - c^2 + d^2 + i2(ab - cd). \end{aligned}$$

Using the above, we get

$$(\alpha\overline{\alpha} - \beta\overline{\beta})x + i(\overline{\alpha}\beta - \alpha\overline{\beta})y + (\alpha\overline{\beta} + \overline{\alpha}\beta)z = (a^2 + b^2 - c^2 - d^2)x + 2(bc - ad)y + 2(ac + bd)z,$$

and

$$\begin{aligned} -2\alpha\beta x - i(\alpha^2 + \beta^2)y + (\alpha^2 - \beta^2)z = 2(-ac + bd)x + 2(ab + cd)y + (a^2 - b^2 - c^2 + d^2)z \\ - i[2(ad + bc)x + (a^2 - b^2 + c^2 - d^2)y + 2(-ab + cd)z]. \end{aligned}$$

If we write

$$q(x\sigma_3 + y\sigma_2 + z\sigma_1)q^* = \begin{pmatrix} x' & z' - iy' \\ z' + iy' & -x' \end{pmatrix},$$

we obtain

$$\begin{aligned} x' &= (a^2 + b^2 - c^2 - d^2)x + 2(bc - ad)y + 2(ac + bd)z \\ y' &= 2(ad + bc)x + (a^2 - b^2 + c^2 - d^2)y + 2(-ab + cd)z \\ z' &= 2(-ac + bd)x + 2(ab + cd)y + (a^2 - b^2 - c^2 + d^2)z. \end{aligned}$$

In summary, we proved the following result.