that is,

$$\mu_i \leq \lambda_{n-m+i}, \quad i = 1, \ldots, m.$$

Therefore,

$$\lambda_i \leq \mu_i \leq \lambda_{n-m+i}, \quad i = 1, \ldots, m,$$

as desired.

(b) If $\lambda_i = \mu_i$, then

$$\lambda_i = \frac{(Rv)^\top ARv}{(Rv)^\top Rv} = \frac{v^\top Bv}{v^\top v} = \mu_i,$$

so v must be an eigenvector for B and Rv must be an eigenvector for A, both for the eigenvalue $\lambda_i = \mu_i$. $\square$

Proposition 17.25 immediately implies the *Poincaré separation theorem*. It can be used in situations, such as in quantum mechanics, where one has information about the inner products $u_i^\top Au_j$.

Proposition 17.26. *(Poincaré separation theorem) Let A be a $n \times n$ symmetric (or Hermitian) matrix, let m be some integer with $1 \leq m \leq n$, and let $(u_1, \ldots, u_m)$ be m orthonormal vectors. Let $B = (u_i^\top Au_j)$ (an $m \times m$ matrix), let $\lambda_1(A) \leq \ldots \leq \lambda_n(A)$ be the eigenvalues of A and $\lambda_1(B) \leq \ldots \leq \lambda_m(B)$ be the eigenvalues of B; then we have*

$$\lambda_k(A) \leq \lambda_k(B) \leq \lambda_{k+n-m}(A), \quad k = 1, \ldots, m.$$

Observe that Proposition 17.25 implies that

$$\lambda_1 + \cdots + \lambda_m \leq \operatorname{tr}(R^\top AR) \leq \lambda_{n-m+1} + \cdots + \lambda_n.$$

If P_1 is the the $n \times (n-1)$ matrix obtained from the identity matrix by dropping its last column, we have $P_1^\top P_1 = I$, and the matrix $B = P_1^\top AP_1$ is the matrix obtained from A by deleting its last row and its last column. In this case the interlacing result is

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \mu_{n-2} \leq \lambda_{n-1} \leq \mu_{n-1} \leq \lambda_n,$$

a genuine interlacing. We obtain similar results with the matrix P_{n-m} obtained by dropping the last $n-m$ columns of the identity matrix and setting $B = P_{n-m}^\top AP_{n-m}$ (B is the $m \times m$ matrix obtained from A by deleting its last $n-m$ rows and columns). In this case we have the following interlacing inequalities known as *Cauchy interlacing theorem*:

$$\lambda_k \leq \mu_k \leq \lambda_{k+n-m}, \quad k = 1, \ldots, m. \qquad (*)$$