Example 3.7. Given any differentiable function $f\colon \mathbb{R}^n \to \mathbb{R}$, by definition, for any $x \in \mathbb{R}^n$, the *total derivative* df_x of f at x is the linear form $df_x\colon \mathbb{R}^n \to \mathbb{R}$ defined so that for all $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$,

$$df_x(u) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(x) & \cdots & \frac{\partial f}{\partial x_n}(x) \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x)\, u_i.$$

Example 3.8. Let $\mathcal{C}([0,1])$ be the vector space of continuous functions $f\colon [0,1] \to \mathbb{R}$. The map $\mathcal{I}\colon \mathcal{C}([0,1]) \to \mathbb{R}$ given by

$$\mathcal{I}(f) = \int_0^1 f(x)dx \quad \text{for any } f \in \mathcal{C}([0,1])$$

is a linear form (integration).

Example 3.9. Consider the vector space $\mathrm{M}_n(\mathbb{R})$ of real $n \times n$ matrices. Let $\mathrm{tr}\colon \mathrm{M}_n(\mathbb{R}) \to \mathbb{R}$ be the function given by

$$\mathrm{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn},$$

called the *trace* of A. It is a linear form. Let $s\colon \mathrm{M}_n(\mathbb{R}) \to \mathbb{R}$ be the function given by

$$s(A) = \sum_{i,j=1}^{n} a_{ij},$$

where $A = (a_{ij})$. It is immediately verified that s is a linear form.

Given a vector space E and any basis $(u_i)_{i\in I}$ for E, we can associate to each u_i a linear form $u_i^* \in E^*$, and the u_i^* have some remarkable properties.

Definition 3.27. Given a vector space E and any basis $(u_i)_{i\in I}$ for E, by Proposition 3.18, for every $i \in I$, there is a unique linear form u_i^* such that

$$u_i^*(u_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$

for every $j \in I$. The linear form u_i^* is called the *coordinate form* of index i w.r.t. the basis $(u_i)_{i\in I}$.

Remark: Given an index set I, authors often define the so called “Kronecker symbol” $\delta_{i\,j}$ such that

$$\delta_{i\,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$

for all $i, j \in I$. Then, $u_i^*(u_j) = \delta_{i\,j}$.