Proposition 29.17. *If $\varphi\colon E \times E \to K$ is a sesquilinear map, and if l_φ and r_φ are bijective, for every bijective linear map $f\colon E \to E$, then we have*

$$\begin{aligned}&\varphi(f(x), f(y)) = \varphi(x, y) \quad \text{for all } x, y \in E \text{ iff}\\ &f^{-1} = f^{*_l} = f^{*_r}.\end{aligned}$$

We also have the following facts.

Proposition 29.18. *(1) If $\varphi\colon E \times E \to K$ is a sesquilinear map and if l_φ is injective, then for every linear map $f\colon E \to E$, if*

$$\varphi(f(x), f(y)) = \varphi(x, y) \quad \text{for all } x, y \in E, \tag{$*$}$$

then f is injective.

(2) If E is finite-dimensional and if φ is nondegenerate, then the linear maps $f\colon E \to E$ satisfying $(*)$ form a group. The inverse of f is given by $f^{-1} = f^*$.

Proof. (1) If $f(x) = 0$, then

$$\varphi(x, y) = \varphi(f(x), f(y)) = \varphi(0, f(y)) = 0 \quad \text{for all } y \in E.$$

Since l_φ is injective, we must have $x = 0$, and thus f is injective.

(2) If E is finite-dimensional, since a linear map satisfying $(*)$ is injective, it is a bijection. By Proposition 29.17, we have $f^{-1} = f^*$. We also have

$$\varphi(f(x), f(y)) = \varphi((f^* \circ f)(x), y) = \varphi(x, y) = \varphi((f \circ f^*)(x), y) = \varphi(f^*(x), f^*(y)),$$

which shows that f^* satisfies $(*)$. If $\varphi(f(x), f(y)) = \varphi(x, y)$ for all $x, y \in E$ and $\varphi(g(x), g(y)) = \varphi(x, y)$ for all $x, y \in E$, then we have

$$\varphi((g \circ f)(x), (g \circ f)(y)) = \varphi(f(x), f(y)) = \varphi(x, y) \quad \text{for all } x, y \in E.$$

Obviously, the identity map id_E satisfies $(*)$. Therefore, the set of linear maps satisfying $(*)$ is a group. ∎

The above considerations motivate the following definition.

Definition 29.16. Let $\varphi\colon E \times E \to K$ be a sesquilinear map, and assume that E is finite-dimensional and that φ is nondegenerate. A linear map $f\colon E \to E$ is an *isometry* of E (with respect to φ) iff

$$\varphi(f(x), f(y)) = \varphi(x, y) \quad \text{for all } x, y \in E.$$

The set of all isometries of E is a group denoted by $\mathbf{Isom}(\varphi)$.