Thus, we could describe the vector $\mathrm{D}^2 f(a)(u, v)$ in terms of an $mn \times mn$-matrix consisting of m diagonal blocks, which are the above Hessians, and the row matrix $(U^\top, \ldots, U^\top)$ (m times) and the column matrix consisting of m copies of V.

We now indicate briefly how higher-order derivatives are defined. Let $m \geq 2$. Given a function $f\colon A \to F$ as before, for any $a \in A$, if the derivatives $\mathrm{D}^i f$ exist on A for all i, $1 \leq i \leq m-1$, by induction, $\mathrm{D}^{m-1} f$ can be considered to be a continuous function $\mathrm{D}^{m-1} f\colon A \to \mathcal{L}_{m-1}(\overrightarrow{E^{m-1}}; \overrightarrow{F})$.

Definition 39.16. Define $\mathrm{D}^m f(a)$, the *m-th derivative of f at a*, as

$$\mathrm{D}^m f(a) = \mathrm{D}(\mathrm{D}^{m-1} f)(a).$$

Then $\mathrm{D}^m f(a)$ *can be identified with a continuous m-multilinear map in* $\mathcal{L}_m(\overrightarrow{E^m}; \overrightarrow{F})$. We can then show (as we did before), that if $\mathrm{D}^m f(a)$ is defined, then

$$\mathrm{D}^m f(a)(u_1, \ldots, u_m) = \mathrm{D}_{u_1} \ldots \mathrm{D}_{u_m} f(a).$$

Definition 39.17. When E if of finite dimension n and $(a_0, (e_1, \ldots, e_n))$ is a frame for E, if $\mathrm{D}^m f(a)$ exists, for every $j_1, \ldots, j_m \in \{1, \ldots, n\}$, we denote $\mathrm{D}_{e_{j_m}} \ldots \mathrm{D}_{e_{j_1}} f(a)$ by

$$\frac{\partial^m f}{\partial x_{j_1} \ldots \partial x_{j_m}}(a).$$

Example 39.12. Going back to the function f of Example 39.10 given by $f(A) = \log\det(A)$, using the formula for the derivative of the inversion map, the chain rule and the product rule, we can show that

$$\mathrm{D}^m f(A)(X_1, \ldots, X_m) = (-1)^{m-1} \sum_{\sigma \in \mathfrak{S}_{m-1}} \mathrm{tr}(A^{-1} X_1 A^{-1} X_{\sigma(1)+1} A^{-1} X_{\sigma(2)+1} \cdots A^{-1} X_{\sigma(m-1)+1})$$

for any $m \geq 1$, where $A \in \mathbf{GL}^+(n, \mathbb{R})$ and $X_1, \ldots X_m$ are any $n \times n$ real matrices.

Given a m-multilinear map $f \in \mathcal{L}_m(\overrightarrow{E^m}; \overrightarrow{F})$, recall that f is *symmetric* if

$$f(u_{\pi(1)}, \ldots, u_{\pi(m)}) = f(u_1, \ldots, u_m),$$

for all $u_1, \ldots, u_m \in \overrightarrow{E}$, and all permutations π on $\{1, \ldots, m\}$. Then, the following generalization of Schwarz's lemma holds.

Proposition 39.21. *Given two normed affine spaces E and F, given any open subset A of E, given any $f\colon A \to F$, for every $a \in A$, for every $m \geq 1$, if $\mathrm{D}^m f(a)$ exists, then $\mathrm{D}^m f(a) \in \mathcal{L}_m(\overrightarrow{E^m}; \overrightarrow{F})$ is a continuous symmetric m-multilinear map. As a corollary, if E is of finite dimension n, and $(a_0, (e_1, \ldots, e_n))$ is a frame for E, we have*

$$\frac{\partial^m f}{\partial x_{j_1} \ldots \partial x_{j_m}}(a) = \frac{\partial^m f}{\partial x_{\pi(j_1)} \ldots \partial x_{\pi(j_m)}}(a),$$

for every $j_1, \ldots, j_m \in \{1, \ldots, n\}$, and for every permutation π on $\{1, \ldots, m\}$.