However, by Proposition 35.14, we have isomorphisms

$$(K[X]/(q_iK[X])) \otimes_{K[X]} L[X] \approx L[X]/(q_iL[X]),$$

so we get

$$L[X] \otimes_{K[X]} E_f \approx L[X]/(q_1L[X]) \oplus \cdots \oplus L[X]/(q_nL[X]).$$

Since E_f is a $K[X]$-module, the $L[X]$ module $L[X] \otimes_{K[X]} E_f$ is the module obtained from E_f by the ring extension $K[X] \subseteq L[X]$. The L-module $E_{(L)} = L \otimes_K E$ becomes the $L[X]$-module $E_{(L)f_{(L)}}$ where

$$f_{(L)} = \mathrm{id}_L \otimes_K f.$$

We have the following proposition

Proposition 36.9. *For any field extension $K \subseteq L$, and any linear map $f\colon E \to E$ where E is a K-vector space, there is an isomorphism between the $L[X]$-modules $L[X] \otimes_{K[X]} E_f$ and $E_{(L)f_{(L)}}$.*

Proof. First we define the map $\alpha\colon L \times E \to L[X] \otimes_{K[X]} E_f$ by

$$\alpha(\lambda, u) = \lambda \otimes_{K[X]} u.$$

It is immediately verified that α is K-bilinear, so we obtain a K-linear map $\widetilde{\alpha}\colon L \otimes_K E \to L[X] \otimes_{K[X]} E_f$. Now $E_{(L)} = L \otimes_K E$ is a $L[X]$-module $(L \otimes_K E)_{f(L)}$, and let us denote this scalar multiplication by $\odot$. To describe $\odot$ it is enough to define how a monomial $aX^k \in L[X]$ acts on a generator $(\lambda \otimes_K u) \in L \otimes_K E$. We have

$$\begin{aligned} aX^k \odot (\lambda \otimes_K u) &= a(\mathrm{id}_L \otimes_K f)^k(\lambda \otimes_K u) \\ &= a(\lambda \otimes_K f^k(u)) \\ &= a\lambda \otimes_K f^k(u). \end{aligned}$$

We claim that $\widetilde{\alpha}$ is actually $L[X]$-linear. Indeed, we have

$$\begin{aligned} \widetilde{\alpha}(aX^k \odot (\lambda \otimes_K u)) &= \widetilde{\alpha}(a\lambda \otimes_K f^k(u)) \\ &= a\lambda \otimes_{K[X]} f^k(u), \end{aligned}$$

and by the definition of scalar multiplication in the $K[X]$-module E_f, we have $f^k(u) = X^k \cdot_f u$, so we have

$$\begin{aligned} \widetilde{\alpha}(aX^k \odot (\lambda \otimes_K u)) &= a\lambda \otimes_{K[X]} f^k(u) \\ &= a\lambda \otimes_{K[X]} X^k \cdot_f u \\ &= X^k \cdot (a\lambda \otimes_{K[X]} u) \\ &= aX^k \cdot (\lambda \otimes_{K[X]} u), \end{aligned}$$

which shows that $\widetilde{\alpha}$ is $L[X]$-linear.