which implies that

$$\dim(T) = \operatorname{codim}(U^{\perp}) - \dim(S) = \operatorname{codim}(U^{\perp}) - \operatorname{codim}(W)$$

so

$$\dim(U/(U \cap V)) = \dim(W/U^{\perp}) = \operatorname{codim}(U^{\perp}) - \operatorname{codim}(W),$$

and since $\operatorname{codim}(U^{\perp}) = \dim(U)$, we deduce that

$$\dim(U \cap V) = \operatorname{codim}(W).$$

However, by Proposition 29.13, we have $\dim(U \cap V) = \operatorname{codim}((U \cap V)^{\perp})$, so $\operatorname{codim}(W) = \operatorname{codim}((U \cap V)^{\perp})$, and since $W \subseteq W^{\perp\perp} = (U \cap V)^{\perp}$, we must have $W = (U \cap V)^{\perp}$, as claimed. $\square$

In view of Proposition 29.12, we can make the following definition.

Definition 29.13. Let $\varphi \colon E \times F \to K$ be any sesquilinear form. If $E/F^{\perp}$ and $F/E^{\perp}$ are finite-dimensional, then their common dimension is called the *rank* of the form φ. If $E/F^{\perp}$ and $F/E^{\perp}$ have infinite dimension, we say that φ has infinite rank.

Not surprisingly, the rank of φ is related to the ranks of l_{φ} and r_{φ}.

Proposition 29.15. *Let $\varphi \colon E \times F \to K$ be any sesquilinear form. If φ has finite rank r, then l_{φ} and r_{φ} have the same rank, which is equal to r.*

Proof. Because for every $u \in E$,

$$l_{\varphi}(u)(y) = \overline{\varphi(u, y)} \quad \text{for all } y \in F,$$

and for every $v \in F$,

$$r_{\varphi}(v)(x) = \varphi(x, v) \quad \text{for all } x \in E,$$

it is clear that the kernel of $l_{\varphi} \colon \overline{E} \to F^*$ is equal to $F^{\perp}$ and that, the kernel of $r_{\varphi} \colon \overline{F} \to E^*$ is equal to $E^{\perp}$. Therefore, $\operatorname{rank}(l_{\varphi}) = \dim(\operatorname{Im} l_{\varphi}) = \dim(E/F^{\perp}) = r$, and similarly $\operatorname{rank}(r_{\varphi}) = \dim(F/E^{\perp}) = r$. $\square$

Remark: If the sesquilinear form φ is represented by the matrix $n \times m$ matrix M with respect to the bases $(e_1, \ldots, e_m)$ in E and $(f_1, \ldots, f_n)$ in F, it can be shown that the matrix representing l_{φ} with respect to the bases $(e_1, \ldots, e_m)$ and $(f_1^*, \ldots, f_n^*)$ is $\overline{M}$, and that the matrix representing r_{φ} with respect to the bases $(f_1, \ldots, f_n)$ and $(e_1^*, \ldots, e_m^*)$ is $M^{\top}$. It follows that the rank of φ is equal to the rank of M.