and by substituting in the second equation we have

$$u_2 = v_3 + v_4 - v_5 = v_3 + u_1 - v_5 - v_5 = u_1 + v_3 - 2v_5.$$

From the above equation we get

$$v_3 = -u_1 + u_2 + 2v_5,$$

and so

$$u_3 = v_1 + v_2 + v_3 = v_1 + v_2 - u_1 + u_2 + 2v_5.$$

Finally, we get

$$v_1 = u_1 - u_2 + u_3 - v_2 - 2v_5$$

Therefore we have

$$\begin{aligned} v_1 &= u_1 - u_2 + u_3 - v_2 - 2v_5 \\ v_3 &= -u_1 + u_2 + 2v_5 \\ v_4 &= u_1 - v_5, \end{aligned}$$

which shows that $(u_1, u_2, u_3, v_2, v_5)$ spans the same subspace as $(v_1, v_2, v_3, v_4, v_5)$. The vectors (v_1, v_3, v_4) have been replaced by (u_1, u_2, u_3), and the vectors left over are (v_2, v_5). We can rename them (v_4, v_5).

For the sake of completeness, here is a more formal statement of the replacement lemma (and its proof).

Proposition 3.10. *(Replacement lemma, version 2) Given a vector space E, let $(u_i)_{i\in I}$ be any finite linearly independent family in E, where $|I| = m$, and let $(v_j)_{j\in J}$ be any finite family such that every u_i is a linear combination of $(v_j)_{j\in J}$, where $|J| = n$. Then there exists a set L and an injection $\rho\colon L \to J$ (a relabeling function) such that $L \cap I = \emptyset$, $|L| = n - m$, and the families $(u_i)_{i\in I} \cup (v_{\rho(l)})_{l\in L}$ and $(v_j)_{j\in J}$ generate the same subspace of E. In particular, $m \leq n$.*

Proof. We proceed by induction on $|I| = m$. When $m = 0$, the family $(u_i)_{i\in I}$ is empty, and the proposition holds trivially with $L = J$ (ρ is the identity). Assume $|I| = m+1$. Consider the linearly independent family $(u_i)_{i\in(I-\{p\})}$, where p is any member of I. By the induction hypothesis, there exists a set L and an injection $\rho\colon L \to J$ such that $L \cap (I - \{p\}) = \emptyset$, $|L| = n-m$, and the families $(u_i)_{i\in(I-\{p\})} \cup (v_{\rho(l)})_{l\in L}$ and $(v_j)_{j\in J}$ generate the same subspace of E. If $p \in L$, we can replace L by $(L - \{p\}) \cup \{p'\}$ where p' does not belong to $I \cup L$, and replace ρ by the injection ρ' which agrees with ρ on $L - \{p\}$ and such that $\rho'(p') = \rho(p)$. Thus, we can always assume that $L \cap I = \emptyset$. Since u_p is a linear combination of $(v_j)_{j\in J}$ and the families $(u_i)_{i\in(I-\{p\})} \cup (v_{\rho(l)})_{l\in L}$ and $(v_j)_{j\in J}$ generate the same subspace of E, u_p is a linear combination of $(u_i)_{i\in(I-\{p\})} \cup (v_{\rho(l)})_{l\in L}$. Let

$$u_p = \sum_{i\in(I-\{p\})} \lambda_i u_i + \sum_{l\in L} \lambda_l v_{\rho(l)}. \tag{1}$$