Remark: It is easy to show that the conditions $A\,A^\top = I_n$, $A^\top A = I_n$, and $A^{-1} = A^\top$, are equivalent.

Given any two orthonormal bases $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$, if P is the change of basis matrix from $(u_1, \ldots, u_n)$ to $(v_1, \ldots, v_n)$, since the columns of P are the coordinates of the vectors v_j with respect to the basis $(u_1, \ldots, u_n)$, if $v_{j_1} = \sum_{i_1=1}^n p_{i_1 j_1} u_{i_1}$ and $v_{j_2} = \sum_{i_2=1}^n p_{i_2 j_2} u_{i_2}$, since $(u_1, \ldots, u_n)$ is orthonormal,

$$v_{j_1} \cdot v_{j_2} = \sum_{i_1=1}^n \sum_{i_2=1}^n p_{i_1 j_1} p_{i_2 j_2} (u_{i_1} \cdot u_{i_2}) = \sum_{i=1}^n p_{i j_1} p_{i j_2},$$

and since $(v_1, \ldots, v_n)$ is orthonormal, $v_{j_1} \cdot v_{j_2} = \delta_{j_1\, j_2}$, so the columns of P are orthonormal, and by Proposition 12.14 (2), the matrix P is orthogonal.

The proof of Proposition 12.12 (3) also shows that if f is an isometry, then the image of an orthonormal basis $(u_1, \ldots, u_n)$ is an orthonormal basis. Students often ask why ortho*gon*al matrices are not called ortho*norm*al matrices, since their columns (and rows) are orthonormal bases! I have no good answer, but isometries do preserve orthogonality, and orthogonal matrices correspond to isometries.

Recall that the determinant $\det(f)$ of a linear map $f \colon E \to E$ is independent of the choice of a basis in E. Also, for every matrix $A \in \mathrm{M}_n(\mathbb{R})$, we have $\det(A) = \det(A^\top)$, and for any two $n \times n$ matrices A and B, we have $\det(AB) = \det(A)\det(B)$. Then if f is an isometry, and A is its matrix with respect to any orthonormal basis, $A\,A^\top = A^\top A = I_n$ implies that $\det(A)^2 = 1$, that is, either $\det(A) = 1$, or $\det(A) = -1$. It is also clear that the isometries of a Euclidean space of dimension n form a group, and that the isometries of determinant $+1$ form a subgroup. This leads to the following definition.

Definition 12.7. Given a Euclidean space E of dimension n, the set of isometries $f \colon E \to E$ forms a subgroup of $\mathbf{GL}(E)$ denoted by $\mathbf{O}(E)$, or $\mathbf{O}(n)$ when $E = \mathbb{R}^n$, called the *orthogonal group (of E)*. For every isometry f, we have $\det(f) = \pm 1$, where $\det(f)$ denotes the determinant of f. The isometries such that $\det(f) = 1$ are called *rotations, or proper isometries, or proper orthogonal transformations*, and they form a subgroup of the special linear group $\mathbf{SL}(E)$ (and of $\mathbf{O}(E)$), denoted by $\mathbf{SO}(E)$, or $\mathbf{SO}(n)$ when $E = \mathbb{R}^n$, called the *special orthogonal group (of E)*. The isometries such that $\det(f) = -1$ are called *improper isometries, or improper orthogonal transformations, or flip transformations*.

12.7 The Rodrigues Formula

When $n = 3$ and A is a skew symmetric matrix, it is possible to work out an explicit formula for e^A. For any 3×3 real skew symmetric matrix

$$A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix},$$