is a unit eigenvector of A associated with λ_1. If λ_1 is real, then

$$v = \lim_{k \mapsto \infty} x^k$$

is a unit eigenvector of A associated with λ_1. Actually some condition on x^0 is needed: x^0 must have a nonzero component in the eigenspace E associated with λ_1 (in any direct sum of $\mathbb{C}^m$ in which E is a summand).

The eigenvalue λ_1 is found as follows. If λ_1 is complex, and if $v_j \neq 0$ is any nonzero coordinate of v, then

$$\lambda_1 = \lim_{k \mapsto \infty} \frac{(Ax^k)_j}{x_j^k}.$$

If λ_1 is real, then we can define the sequence $(\lambda^{(k)})$ by

$$\lambda^{(k+1)} = (x^{k+1})^* A x^{k+1}, \quad k \geq 0,$$

and we have

$$\lambda_1 = \lim_{k \mapsto \infty} \lambda^{(k)}.$$

Indeed, in this case, since $v = \lim_{k \mapsto \infty} x^k$ and v is a unit eigenvector for λ_1, we have

$$\lim_{k \mapsto \infty} \lambda^{(k)} = \lim_{k \mapsto \infty} (x^{k+1})^* A x^{k+1} = v^* A v = \lambda_1 v^* v = \lambda_1.$$

Note that since x^{k+1} is a unit vector, $(x^{k+1})^* A x^{k+1}$ is a Rayleigh ratio.

If A is a Hermitian matrix, then the eigenvalues are real and we can say more about the rate of convergence, which is not great (only linear). For details, see Trefethen and Bau [176] (Lecture 27).

If $\lambda_1 = 0$, then there is some power $\ell < m$ such that $Ax^\ell = 0$.

The *inverse iteration method* is designed to find an eigenvector associated with an eigenvalue λ of A for which we know a good approximation μ.

Pick some initial unit vector x^0 and compute the following sequences (w^k) and (x^k), where w^{k+1} is the solution of the system

$$(A - \mu I) w^{k+1} = x^k \quad \text{equivalently} \quad w^{k+1} = (A - \mu I)^{-1} x^k, \quad k \geq 0,$$

and

$$x^{k+1} = \frac{w^{k+1}}{\|w^{k+1}\|}, \quad k \geq 0.$$

The following result is proven in Ciarlet [41] (Theorem 6.4.1).