What is this decomposition for $a = -1$?

(2) Recall that a rotation matrix R (a member of the group $\mathbf{SO}(2)$) is a matrix of the form

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Prove that if $\theta \neq k\pi$ (with $k \in \mathbb{Z}$), any rotation matrix can be written as a product

$$R = ULU,$$

where U is upper triangular and L is lower triangular of the form

$$U = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}.$$

Therefore, every plane rotation (except a flip about the origin when $\theta = \pi$) can be written as the composition of three shear transformations!

Problem 8.11. (1) Recall that $E_{i,d}$ is the diagonal matrix

$$E_{i,d} = \mathrm{diag}(1, \ldots, 1, d, 1, \ldots, 1),$$

whose diagonal entries are all $+1$, except the (i,i)th entry which is equal to d.

Given any $n \times n$ matrix A, for any pair (i,j) of distinct row indices ($1 \leq i, j \leq n$), prove that there exist two elementary matrices $E_1(i,j)$ and $E_2(i,j)$ of the form $E_{k,\ell;\beta}$, such that

$$E_{j,-1}E_1(i,j)E_2(i,j)E_1(i,j)A = P(i,j)A,$$

the matrix obtained from the matrix A by permuting row i and row j. Equivalently, we have

$$E_1(i,j)E_2(i,j)E_1(i,j)A = E_{j,-1}P(i,j)A,$$

the matrix obtained from A by permuting row i and row j and multiplying row j by -1.

Prove that for every $i = 2, \ldots, n$, there exist four elementary matrices $E_3(i,d), E_4(i,d), E_5(i,d), E_6(i,d)$ of the form $E_{k,\ell;\beta}$, such that

$$E_6(i,d)E_5(i,d)E_4(i,d)E_3(i,d)E_{n,d} = E_{i,d}.$$

What happens when $d = -1$, that is, what kind of simplifications occur?

Prove that all permutation matrices can be written as products of elementary operations of the form $E_{k,\ell;\beta}$ and the operation $E_{n,-1}$.

(2) Prove that for every invertible $n \times n$ matrix A, there is a matrix S such that

$$SA = \begin{pmatrix} I_{n-1} & 0 \\ 0 & d \end{pmatrix} = E_{n,d},$$