for any $\lambda \in \mathbb{R}$ (or $\lambda \in \mathbb{C}$). Since $f(x)$ is defined by

$$f(x) = \lim_{n \mapsto \infty} f_0(x_n)$$

independently of the sequence (x_n) converging to x, and similarly for $f(y)$ and $f(x+y)$, since f_0 is linear, we have

$$\begin{aligned} f(x+y) &= \lim_{n \mapsto \infty} f_0(x_n + y_n) \\ &= \lim_{n \mapsto \infty} (f_0(x_n) + f_0(y_n)) \\ &= \lim_{n \mapsto \infty} f_0(x_n) + \lim_{n \mapsto \infty} f_0(y_n) \\ &= f(x) + f(y). \end{aligned}$$

Similarly,

$$\begin{aligned} f(\lambda x) &= \lim_{n \mapsto \infty} f_0(\lambda x_n) \\ &= \lim_{n \mapsto \infty} \lambda f_0(x_n) \\ &= \lambda \lim_{n \mapsto \infty} f_0(x_n) \\ &= \lambda f(x). \end{aligned}$$

Therefore, f is linear. Since the norm is continuous, we have

$$\|f(x)\| = \left\| \lim_{n \mapsto \infty} f_0(x_n) \right\| = \lim_{n \mapsto \infty} \|f_0(x_n)\|,$$

and since f_0 is continuous

$$\|f_0(x_n)\| \leq \|f_0\| \, \|x_n\| \quad \text{for all } n \geq 1,$$

so we get

$$\lim_{n \mapsto \infty} \|f_0(x_n)\| \leq \lim_{n \mapsto \infty} \|f_0\| \, \|x_n\| \quad \text{for all } n \geq 1,$$

that is,

$$\|f(x)\| \leq \|f_0\| \, \|x\|.$$

Since

$$\|f\| = \sup_{\|x\|=1,\, x \in E} \|f(x)\|,$$

we deduce that $\|f\| \leq \|f_0\|$. But since $E_0 \subseteq E$ and f agrees with f_0 on E_0, we also have

$$\|f_0\| = \sup_{\|x\|=1,\, x \in E_0} \|f_0(x)\| = \sup_{\|x\|=1,\, x \in E_0} \|f(x)\| \leq \sup_{\|x\|=1,\, x \in E} \|f(x)\| = \|f\|,$$

and thus $\|f\| = \|f_0\|$. $\square$

Finally, we consider normed affine spaces.