as claimed. This formula shows that the gradient ∇f_u of f at u is given by

$$\nabla f_u = 2Au.$$

As a first corollary we obtain the gradient of a function of the form

$$f(x) = \frac{1}{2} x^\top A x - b^\top x,$$

where A is a symmetric $n \times n$ matrix and b is some vector $b \in \mathbb{R}^n$. Since the derivative of a linear function is itself, we obtain

$$df_u(h) = u^\top A h - b^\top h,$$

and the gradient of f is given by

$$\nabla f_u = Au - b.$$

As a second corollary we obtain the gradient of the function

$$f(x) = \|Ax - b\|_2^2 = (Ax - b)^\top (Ax - b) = (x^\top A^\top - b^\top)(Ax - b)$$

which is the function to minimize in a least squares problem, where A is an $m \times n$ matrix. We have

$$f(x) = x^\top A^\top A x - x^\top A^\top b - b^\top A x + b^\top b = x^\top A^\top A x - 2b^\top A x + b^\top b,$$

and since the derivative of a constant function is 0 and the derivative of a linear function is itself, we get

$$df_u(h) = 2u^\top A^\top A h - 2b^\top A h.$$

Consequently, the gradient of f is given by

$$\nabla f_u = 2A^\top A u - 2A^\top b.$$

When E, F, and G have finite dimensions, and $(a_0, (u_1, \ldots, u_p))$ is an affine frame for E, $(b_0, (v_1, \ldots, v_n))$ is an affine frame for F, and $(c_0, (w_1, \ldots, w_m))$ is an affine frame for G, if A is an open subset of E, B is an open subset of F, for any functions $f\colon A \to F$ and $g\colon B \to G$, such that $f(A) \subseteq B$, for any $a \in A$, letting $b = f(a)$, and $h = g \circ f$, if $\mathrm{D}f(a)$ exists and $\mathrm{D}g(b)$ exists, by Theorem 39.6, the Jacobian matrix $J(h)(a) = J(g \circ f)(a)$ w.r.t. the bases $(u_1, \ldots, u_p)$ and $(w_1, \ldots, w_m)$ is the product of the Jacobian matrices $J(g)(b)$ w.r.t. the bases $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_m)$, and $J(f)(a)$ w.r.t. the bases $(u_1, \ldots, u_p)$ and $(v_1, \ldots, v_n)$:

$$J(h)(a) = \begin{pmatrix} \partial_1 g_1(b) & \partial_2 g_1(b) & \ldots & \partial_n g_1(b) \\ \partial_1 g_2(b) & \partial_2 g_2(b) & \ldots & \partial_n g_2(b) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_1 g_m(b) & \partial_2 g_m(b) & \ldots & \partial_n g_m(b) \end{pmatrix} \begin{pmatrix} \partial_1 f_1(a) & \partial_2 f_1(a) & \ldots & \partial_p f_1(a) \\ \partial_1 f_2(a) & \partial_2 f_2(a) & \ldots & \partial_p f_2(a) \\ \vdots & \vdots & \ddots & \vdots \\ \partial_1 f_n(a) & \partial_2 f_n(a) & \ldots & \partial_p f_n(a) \end{pmatrix}$$