Example 51.2. Here is an example of an improper convex function $f\colon \mathbb{R} \to \mathbb{R} \cup \{-\infty, +\infty\}$:

$$f(x) = \begin{cases} -\infty & \text{if } |x| < 1 \\ 0 & \text{if } |x| = 1 \\ +\infty & \text{if } |x| > 1 \end{cases}$$

Observe that $\mathrm{dom}(f) = [-1, 1]$, and that $\mathbf{epi}(f)$ is not closed. See Figure 51.4.

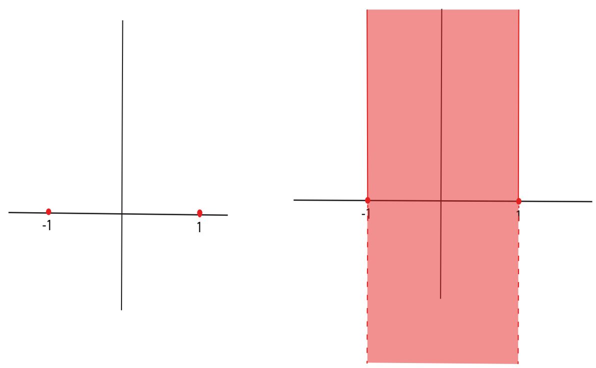

Figure 51.4: The improper convex function of Example 51.2 and its epigraph depicted as a rose colored region of $\mathbb{R}^2$.

Functions whose epigraph are closed tend to have better properties. To characterize such functions we introduce sublevel sets.

Definition 51.6. Given a function $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$, for any $\alpha \in \mathbb{R} \cup \{-\infty, +\infty\}$, the *sublevel sets* $\mathrm{sublev}_\alpha(f)$ and $\mathrm{sublev}_{<\alpha}(f)$ are the sets

$$\mathrm{sublev}_\alpha(f) = \{x \in \mathbb{R}^n \mid f(x) \leq \alpha\} \quad \text{and} \quad \mathrm{sublev}_{<\alpha}(f) = \{x \in \mathbb{R}^n \mid f(x) < \alpha\}.$$

For the improper convex function of Example 51.2, we have

$\mathrm{sublev}_{-\infty}(f) = (-1, 1)$ while $\mathrm{sublev}_{<-\infty}(f) = \emptyset$.

$\mathrm{sublev}_\alpha(f) = (-1, 1) = \mathrm{sublev}_{<\alpha}(f)$ whenever $-\infty < \alpha < 0$.

$\mathrm{sublev}_0(f) = [-1, 1]$ while $\mathrm{sublev}_{<0}(f) = (-1, 1)$.

$\mathrm{sublev}_\alpha(f) = [-1, 1] = \mathrm{sublev}_{<\alpha}(f)$ whenever $0 < \alpha < +\infty$.

$\mathrm{sublev}_{+\infty}(f) = \mathbb{R}$ while $\mathrm{sublev}_{<+\infty}(f) = [-1, 1]$.

A useful corollary of Proposition 51.1 is the following result whose (easy) proof can be found in Rockafellar [138] (Theorem 4.6).