and we say that our minimization problem has no solution, or that it is unbounded (below). For example, if $V = \Omega = \mathbb{R}$, $U = \{x \in \mathbb{R} \mid x \leq 0\}$, and $J(x) = x$, then the function $J(x)$ is not bounded below and $\inf_{v \in U} J(v) = -\infty$.

The issue is that J^* may not belong to $\{J(u) \mid u \in U\}$, that is, it may not be achieved by some element $u \in U$, and solving the above problem consists in finding some $u \in U$ that achieves the value J^* in the sense that $J(u) = J^*$. If no such $u \in U$ exists, again we say that our minimization problem has no solution.

The minimization problem

$$\begin{aligned} &\text{find} \quad u \\ &\text{such that} \;\; u \in U \;\text{ and }\; J(u) = \inf_{v \in U} J(v) \end{aligned}$$

is often presented in the following more informal way:

$$\begin{aligned} &\text{minimize} \quad J(v) \\ &\text{subject to} \;\; v \in U. \end{aligned} \qquad \textbf{(Problem M)}$$

A vector $u \in U$ such that $J(u) = \inf_{v \in U} J(v)$ is often called a *minimizer* of J over U. Some authors denote the set of minimizers of J over U by $\arg\min_{v \in U} J(v)$ and write

$$u \in \arg\min_{v \in U} J(v)$$

to express that u is such a minimizer. When such a minimizer is unique, by abuse of notation, this unique minimizer u is denoted by

$$u = \arg\min_{v \in U} J(v).$$

We prefer not to use this notation, although it seems to have invaded the literature.

If we need to maximize rather than minimize a function, then we try to find some $u \in U$ such that

$$J(u) = \sup_{v \in U} J(v).$$

Here $\sup_{v \in U} J(v)$ is the least upper bound of the set $\{J(u) \mid u \in U\}$. Some authors denote the set of *maximizers* of J over U by $\arg\max_{v \in U} J(v)$.

Remark: Some authors define an *extended real-valued function* as a function $f \colon \Omega \to \mathbb{R}$ which is allowed to take the value $-\infty$ or even $+\infty$ for some of its arguments. Although this may be convenient to deal with situations where we need to consider $\inf_{v \in U} J(v)$ or $\sup_{v \in U} J(v)$, such "functions" are really partial functions and we prefer not to use the notion of extended real-valued function.