Remark: The isomorphism (3) can be generalized to finite and even arbitrary direct sums $\bigoplus_{i\in I} E_i$ of vector spaces (where I is an arbitrary nonempty index set). We have an isomorphism

$$\left(\bigoplus_{i\in I} E_i\right) \otimes G \cong \bigoplus_{i\in I} (E_i \otimes G).$$

This isomorphism (with isomorphism (1)) can be used to give another proof of Proposition 33.12 (see Bertin [15], Chapter 4, Section 1) or Lang [109], Chapter XVI, Section 2).

Proposition 33.14. *Given any three vector spaces E, F, G, we have the canonical isomorphism*

$$\mathrm{Hom}(E, F; G) \cong \mathrm{Hom}(E, \mathrm{Hom}(F, G)).$$

Proof. Any bilinear map $f\colon E \times F \to G$ gives the linear map $\varphi(f) \in \mathrm{Hom}(E, \mathrm{Hom}(F, G))$, where $\varphi(f)(u)$ is the linear map in $\mathrm{Hom}(F, G)$ given by

$$\varphi(f)(u)(v) = f(u, v).$$

Conversely, given a linear map $g \in \mathrm{Hom}(E, \mathrm{Hom}(F, G))$, we get the bilinear map $\psi(g)$ given by

$$\psi(g)(u, v) = g(u)(v),$$

and it is clear that φ and ψ and mutual inverses. $\square$

Since by Proposition 33.7 there is a canonical isomorphism

$$\mathrm{Hom}(E \otimes F, G) \cong \mathrm{Hom}(E, F; G),$$

together with the isomorphism

$$\mathrm{Hom}(E, F; G) \cong \mathrm{Hom}(E, \mathrm{Hom}(F, G))$$

given by Proposition 33.14, we obtain the important corollary:

Proposition 33.15. *For any three vector spaces E, F, G, we have the canonical isomorphism*

$$\mathrm{Hom}(E \otimes F, G) \cong \mathrm{Hom}(E, \mathrm{Hom}(F, G)).$$

33.5 Duality for Tensor Products

In this section all vector spaces are assumed to have *finite dimension*, unless specified otherwise. Let us now see how tensor products behave under duality. For this, we define a pairing between $E_1^* \otimes \cdots \otimes E_n^*$ and $E_1 \otimes \cdots \otimes E_n$ as follows: For any fixed $(v_1^*, \ldots, v_n^*) \in E_1^* \times \cdots \times E_n^*$, we have the multilinear map

$$l_{v_1^*,\ldots,v_n^*}\colon (u_1, \ldots, u_n) \mapsto v_1^*(u_1) \cdots v_n^*(u_n)$$