(1) For every $u \in E$, if $\varphi(u, v) = 0$ for all $v \in F$, then $u = 0$, and

(2) For every $v \in F$, if $\varphi(u, v) = 0$ for all $u \in E$, then $v = 0$.

Proposition 29.1 translates into the following proposition. The proof is left as an exercise.

Proposition 29.9. *Given a sesquilinear map $\varphi\colon E \times F \to K$, the following properties hold:*

(a) The map l_φ is injective iff Property (1) of Definition 29.10 holds.

(b) The map r_φ is injective iff Property (2) of Definition 29.10 holds.

(c) The sesquilinear form φ is nondegenerate and iff l_φ and r_φ are injective.

(d) If the sesquillinear form φ is nondegenerate and if E and F have finite dimensions, then $\dim(E) = \dim(F)$, and $l_\varphi\colon \overline{E} \to F^*$ and $r_\varphi\colon \overline{F} \to E^*$ are linear isomorphisms.

Propositions 29.2 and 29.3 also generalize to sesquilinear forms. We also have the following version of Theorem 29.4, whose proof is left as an exercise.

Theorem 29.10. *Given any sesquilinear form $\varphi\colon E \times E \to K$ with $\dim(E) = n$, if φ is Hermitian and K does not have characteristic 2, then there is a basis $(e_1, \ldots, e_n)$ of E such that $\varphi(e_i, e_j) = 0$, for all $i \neq j$.*

As in Section 29.1, if E and F are finite-dimensional vector spaces and if $(e_1, \ldots, e_m)$ is a basis of E and $(f_1, \ldots, f_n)$ is a basis of F then the sesquilinearity of φ yields

$$\varphi\left(\sum_{i=1}^{m} x_i e_i, \sum_{j=1}^{n} y_j f_j\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} x_i \varphi(e_i, f_j) \overline{y}_j.$$

This shows that φ is completely determined by the $n \times m$ matrix $M = (m_{ij})$ with $m_{ij} = \varphi(e_j, f_i)$, and in matrix form, we have

$$\varphi(x, y) = x^\top M^\top \overline{y} = y^* M x,$$

where x and $\overline{y}$ are the column vectors associated with $(x_1, \ldots, x_m) \in K^m$ and $(\overline{y}_1, \ldots, \overline{y}_n) \in K^n$, and $y^* = \overline{y}^\top$. As earlier, we are committing the slight abuse of notation of letting x denote both the vector $x = \sum_{i=1}^n x_i e_i$ and the column vector associated with $(x_1, \ldots, x_n)$ (and similarly for y).

Definition 29.11. If $(e_1, \ldots, e_m)$ is a basis of E and $(f_1, \ldots, f_n)$ is a basis of F, for any sesquilinear form $\varphi\colon E \times F \to K$, the $n \times m$ matrix $M = (m_{ij})$ given by $m_{ij} = \varphi(e_j, f_i)$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$ is called the *matrix of φ with respect to the bases* $(e_1, \ldots, e_m)$ *and* $(f_1, \ldots, f_n)$.