Proposition 51.15 is proven in Rockafellar [138] (Theorem 23.1). The proof is not difficult but not very informative.

Remark: As a convex function of u, it can be shown that the effective domain of the function $u \mapsto f'(x; u)$ is the convex cone generated by $\mathrm{dom}(f) - x$.

We will now state without proof some of the most important properties of subgradients and subdifferentials. Complete details can be found in Rockafellar [138] (Part V, Section 23).

In order to state the next proposition, we need the following definition.

Definition 51.16. For any convex set C in $\mathbb{R}^n$, the *support function* $\delta^*(-|C)$ of C is defined by

$$\delta^*(x|C) = \sup_{y \in C} \langle x, y \rangle, \quad x \in \mathbb{R}^n.$$

According to Definition 50.11, the conjugate of the indicator function I_C of a convex set C is given by

$$I_C^*(x) = \sup_{y \in \mathbb{R}^n} (\langle x, y \rangle - I_C(y)) = \sup_{y \in C} \langle x, y \rangle = \delta^*(x|C).$$

Thus $\delta^*(-|C) = I_C^*$, *the conjugate of the indicator function* I_C.

The following proposition relates directional derivatives at x and the subdifferential at x.

Proposition 51.16. *Let $f \colon \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$ be a convex function. For any $x \in \mathbb{R}^n$, if $f(x)$ is finite, then a vector $u \in \mathbb{R}^n$ is a subgradient to f at x if and only if*

$$f'(x; y) \geq \langle y, u \rangle \quad \text{for all } y \in \mathbb{R}^n.$$

Furthermore, the closure of the convex function $y \mapsto f'(x; y)$ is the support function of the closed convex set $\partial f(x)$, the subdifferential of f at x:

$$\mathrm{cl}(f'(x; -)) = \delta^*(-|\partial f(x)).$$

Sketch of proof. Proposition 51.16 is proven in Rockafellar [138] (Theorem 23.2). We prove the inequality. If we write $z = x + \lambda y$ with $\lambda > 0$, then the subgradient inequality implies

$$f(x + \lambda u) \geq f(x) + \langle z - x, u \rangle = f(x) + \lambda \langle y, u \rangle,$$

so we get

$$\frac{f(x + \lambda y) - f(x)}{\lambda} \geq \langle y, u \rangle.$$

Since the expression on the left tends to $f'(x; y)$ as $\lambda > 0$ tends to zero, we obtain the desired inequality. The second part follows from Corollary 13.2.1 in Rockafellar [138]. $\square$