Example 52.8. When the function f is simple enough, the proximity operator can be computed analytically. This is the case in particular when $f = I_C$, the indicator function of a nonempty closed convex set C. In this case, it is easy to see that

$$x^+ = \arg\min_x \left(I_C(x) + (\rho/2)\,\|x - v\|_2^2\right) = \Pi_C(v),$$

the orthogonal projection of v onto C. In the special case where $C = \mathbb{R}^n_+$ (the first orthant), then

$$x^+ = (v)_+,$$

the vector obtained by setting the negative components of v to zero.

Example 52.9. A second case where simplifications arise is the case where f is a convex quadratic functional of the form

$$f(x) = \frac{1}{2}x^\top Px + q^\top x + r,$$

where P is an $n \times n$ symmetric positive semidefinite matrix, $q \in \mathbb{R}^n$ and $r \in \mathbb{R}$. In this case the gradient of the map

$$x \mapsto f(x) + (\rho/2)\,\|Ax - v\|_2^2 = \frac{1}{2}x^\top Px + q^\top x + r + \frac{\rho}{2}x^\top(A^\top A)x - \rho x^\top A^\top v + \frac{\rho}{2}v^\top v$$

is given by

$$(P + \rho A^\top A)x + q - \rho A^\top v,$$

and since A has rank n, the matrix $A^\top A$ is symmetric positive definite, so we get

$$x^+ = (P + \rho A^\top A))^{-1}(\rho A^\top v - q).$$

Methods from numerical linear algebra can be used so compute x^+ fairly efficiently; see Boyd et al. [28] (Section 4).

Example 52.10. A third case where simplifications arise is the variation of the previous case where f is a convex quadratic functional of the form

$$f(x) = \frac{1}{2}x^\top Px + q^\top x + r,$$

except that f is constrained by equality constraints $Cx = b$, as in Section 50.4, which means that $\mathrm{dom}(f) = \{x \in \mathbb{R}^n \mid Cx = b\}$, and $A = I$. The x-minimization step consists in minimizing the function

$$J(x) = \frac{1}{2}x^\top Px + q^\top x + r + \frac{\rho}{2}x^\top x - \rho x^\top v + \frac{\rho}{2}v^\top v$$

subject to the constraint

$$Cx = b,$$