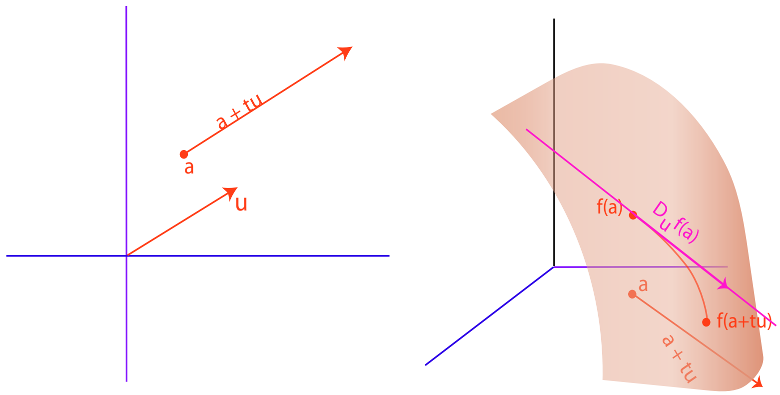

Figure 39.1: Let $f\colon \mathbb{R}^2 \to \mathbb{R}$. The graph of f is the peach surface in $\mathbb{R}^3$, and $t \mapsto f(a+tu)$ is the embedded orange curve connecting $f(a)$ to $f(a+tu)$. Then $\mathrm{D}_u f(a)$ is the slope of the pink tangent line in the direction of u.

Since the map $t \mapsto a + tu$ is continuous, and since $A - \{a\}$ is open, the inverse image U of $A - \{a\}$ under the above map is open, and the definition of the limit in Definition 39.2 makes sense.

Remark: Since the notion of limit is purely topological, the existence and value of a directional derivative is independent of the choice of norms in E and F, as long as they are equivalent norms.

The directional derivative is sometimes called the *Gâteaux derivative*.

In the special case where $E = \mathbb{R}$ and $F = \mathbb{R}$, and we let $u = 1$ (i.e., the real number 1, viewed as a vector), it is immediately verified that $\mathrm{D}_1 f(a) = f'(a)$, in the sense of Definition 39.1. When $E = \mathbb{R}$ (or $E = \mathbb{C}$) and F is any normed vector space, the derivative $\mathrm{D}_1 f(a)$, also denoted by $f'(a)$, provides a suitable generalization of the notion of derivative.

However, when E has dimension ≥ 2, directional derivatives present a serious problem, which is that their definition is not sufficiently uniform. Indeed, there is no reason to believe that the directional derivatives w.r.t. all nonnull vectors u share something in common. As a consequence, a function can have all directional derivatives at a, and yet not be continuous at a. Two functions may have all directional derivatives in some open sets, and yet their composition may not.

Example 39.1. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function given by

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^4 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

The graph of $f(x,y)$ is illustrated in Figure 39.2.