For example, if $\varphi(\overline{1} \otimes \overline{s}) = 0$, because $\varphi(\overline{1} \otimes \overline{s}) = s \pmod{\mathfrak{a}_1 + \mathfrak{a}_2}$, we have $s \in \mathfrak{a}_1 + \mathfrak{a}_2$, so we can write $s = a + b$ with $a \in \mathfrak{a}_1$ and $b \in \mathfrak{a}_2$. Then

$$\begin{aligned}
\overline{1} \otimes \overline{s} &= \overline{1} \otimes \overline{a+b} \\
&= \overline{1} \otimes (\overline{a} + \overline{b}) \\
&= \overline{1} \otimes \overline{a} + \overline{1} \otimes \overline{b} \\
&= \overline{a} \otimes \overline{1} + \overline{1} \otimes \overline{b} \\
&= 0 + 0 = 0,
\end{aligned}$$

since $a \in \mathfrak{a}_1$ and $b \in \mathfrak{a}_2$, which proves injectivity. ∎

Recall that the exterior algebra of an A-module M is defined by

$$\bigwedge M = \bigoplus_{k \geq 0} \bigwedge^k (M).$$

Proposition 35.27. *If A is a commutative ring, then for any n modules M_i, there is an isomorphism*

$$\bigwedge \Big(\bigoplus_{i=1}^n M_i\Big) \approx \bigotimes_{i=1}^n \bigwedge M_i.$$

A proof can be found in Bourbaki [25] (Chapter III, Section 7, No 7, Proposition 10).

Proposition 35.28. *Let A be a commutative ring and let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ be n ideals of A. If the module M is the direct sum of n cyclic modules*

$$M = A/\mathfrak{a}_1 \oplus \cdots \oplus A/\mathfrak{a}_n,$$

then for every $p > 0$, the exterior power $\bigwedge^p M$ is isomorphic to the direct sum of the modules $A/\mathfrak{a}_H$, where H ranges over all subsets $H \subseteq \{1, \ldots, n\}$ with p elements, and with

$$\mathfrak{a}_H = \sum_{h \in H} \mathfrak{a}_h.$$

Proof. If u_i is the image of 1 in $A/\mathfrak{a}_i$, then $A/\mathfrak{a}_i$ is equal to Au_i. By Proposition 35.27, we have

$$\bigwedge M \approx \bigotimes_{i=1}^n \bigwedge (Au_i).$$

We also have

$$\bigwedge (Au_i) = \bigoplus_{k \geq 0} \bigwedge^k (Au_i) \approx A \oplus Au_i,$$