and we say that a vector $y \in \mathbb{R}^n$ is *orthogonal* (or *perpendicular*) to H iff y is orthogonal to H_0. Let h be the intersection of H with the line through the origin and perpendicular to H. Prove that the coordinates of h are given by

$$\frac{c}{a_1^2 + \cdots + a_n^2}(a_1, \ldots, a_n).$$

(2) For any point $p \in H$, prove that $\|h\| \leq \|p\|$. Thus, it is natural to define the *distance* $d(O, H)$ from the origin O to the hyperplane H as $d(O, H) = \|h\|$. Prove that

$$d(O, H) = \frac{|c|}{(a_1^2 + \cdots + a_n^2)^{\frac{1}{2}}}.$$

(3) Let S be a finite set of $n \geq 3$ points in the plane ($\mathbb{R}^2$). Prove that if for every pair of distinct points $p_i, p_j \in S$, there is a third point $p_k \in S$ (distinct from p_i and p_j) such that p_i, p_j, p_k belong to the same (affine) line, then all points in S belong to a common (affine) line.

Hint. Proceed by contradiction and use a minimality argument. This is either ∞-hard or relatively easy, depending how you proceed!

Problem 12.14. (The space of closed polygons in $\mathbb{R}^2$, after Hausmann and Knutson)

An *open polygon* P in the plane is a sequence $P = (v_1, \ldots, v_{n+1})$ of points $v_i \in \mathbb{R}^2$ called *vertices* (with $n \geq 1$). A *closed polygon*, for short a *polygon*, is an open polygon $P = (v_1, \ldots, v_{n+1})$ such that $v_{n+1} = v_1$. The sequence of *edge vectors* $(e_1, \ldots, e_n)$ associated with the open (or closed) polygon $P = (v_1, \ldots, v_{n+1})$ is defined by

$$e_i = v_{i+1} - v_i, \quad i = 1, \ldots, n.$$

Thus, a closed or open polygon is also defined by a pair $(v_1, (e_1, \ldots, e_n))$, with the vertices given by

$$v_{i+1} = v_i + e_i, \quad i = 1, \ldots, n.$$

Observe that a polygon $(v_1, (e_1, \ldots, e_n))$ is closed iff

$$e_1 + \cdots + e_n = 0.$$

Since every polygon $(v_1, (e_1, \ldots, e_n))$ can be translated by $-v_1$, so that $v_1 = (0, 0)$, we may assume that our polygons are specified by a sequence of edge vectors.

Recall that the plane $\mathbb{R}^2$ is isomorphic to $\mathbb{C}$, via the isomorphism

$$(x, y) \mapsto x + iy.$$

We will represent each edge vector e_k by the square of a complex number $w_k = a_k + ib_k$. Thus, every sequence of complex numbers $(w_1, \ldots, w_n)$ defines a polygon (namely, $(w_1^2, \ldots, w_n^2)$).