Proof. Pick any norm on $\mathbb{C}^n$ (or $\mathbb{R}^n$) and let $\|\|$ be the corresponding operator norm on $\mathrm{M}_n(\mathbb{C})$. Since $\mathrm{M}_n(\mathbb{C})$ has dimension n^2, it is complete. By Proposition 9.18, it suffices to show that the series of nonnegative reals $\sum_{k=0}^{n} \left\| \frac{A^k}{k!} \right\|$ converges. Since $\| \|$ is an operator norm, this a matrix norm, so we have

$$\sum_{k=0}^{n} \left\| \frac{A^k}{k!} \right\| \leq \sum_{k=0}^{n} \frac{\|A\|^k}{k!} \leq e^{\|A\|}.$$

Thus, the nondecreasing sequence of positive real numbers $\sum_{k=0}^{n} \left\| \frac{A^k}{k!} \right\|$ is bounded by $e^{\|A\|}$, and by a fundamental property of $\mathbb{R}$, it has a least upper bound which is its limit. $\square$

Definition 9.16. Let E be a finite-dimensional real or complex normed vector space. For any $n \times n$ matrix A, the limit of the series

$$\sum_{k=0}^{\infty} \frac{A^k}{k!}$$

is the *exponential of* A and is denoted e^A.

A basic property of the exponential $x \mapsto e^x$ with $x \in \mathbb{C}$ is

$$e^{x+y} = e^x e^y, \quad \text{for all } x, y \in \mathbb{C}.$$

As a consequence, e^x is always invertible and $(e^x)^{-1} = e^{-x}$. For matrices, because matrix multiplication is not commutative, in general,

$$e^{A+B} = e^A e^B$$

fails! This result is salvaged as follows.

Proposition 9.21. *For any two $n \times n$ complex matrices A and B, if A and B commute, that is, $AB = BA$, then*

$$e^{A+B} = e^A e^B.$$

A proof of Proposition 9.21 can be found in Gallier [72].

Since A and $-A$ commute, as a corollary of Proposition 9.21, we see that e^A is always invertible and that

$$(e^A)^{-1} = e^{-A}.$$

It is also easy to see that

$$(e^A)^{\top} = e^{A^{\top}}.$$