This pairing is given explicitly on generators by

$$\langle v_1^* \odot \cdots \odot v_n^*, u_1, \ldots, u_n \rangle = \sum_{\sigma \in \mathfrak{S}_n} v_{\sigma(1)}^*(u_1) \cdots v_{\sigma(n)}^*(u_n).$$

Now this pairing in nondegenerate. This can be shown using bases.[2] If $(e_1, \ldots, e_m)$ is a basis of E, then for every basis element $(e_{i_1}^*)^{\odot n_1} \odot \cdots \odot (e_{i_k}^*)^{\odot n_k}$ of $\mathrm{S}^n(E^*)$, with $n_1 + \cdots + n_k = n$, we have

$$\langle (e_{i_1}^*)^{\odot n_1} \odot \cdots \odot (e_{i_k}^*)^{\odot n_k}, e_{i_1}^{\odot n_1} \odot \cdots \odot e_{i_k}^{\odot n_k} \rangle = n_1! \cdots n_k!,$$

and

$$\langle (e_{i_1}^*)^{\odot n_1} \odot \cdots \odot (e_{i_k}^*)^{\odot n_k}, e_{j_1} \odot \cdots \odot e_{j_n} \rangle = 0$$

if $(j_1 \ldots, j_n) \neq (\underbrace{i_1, \ldots, i_1}_{n_1}, \ldots, \underbrace{i_k, \ldots, i_k}_{n_k})$.

If the field K has characteristic zero, then $n_1! \cdots n_k! \neq 0$. We leave the details as an exercise to the reader. Therefore we get a canonical isomorphism

$$(\mathrm{S}^n(E))^* \cong \mathrm{S}^n(E^*).$$

The following proposition summarizes the duality properties of symmetric powers.

Proposition 33.30. *Assume the field K has characteristic zero. We have the canonical isomorphisms*

$$(\mathrm{S}^n(E))^* \cong \mathrm{S}^n(E^*)$$

and

$$\mathrm{S}^n(E^*) \cong \mathrm{Sym}^n(E; K) = \mathrm{Hom}_{\mathrm{symlin}}(E^n, K),$$

which allows us to interpret symmetric tensors over E^* as symmetric multilinear maps.

Proof. The isomorphism

$$\mu \colon \mathrm{S}^n(E^*) \cong \mathrm{Sym}^n(E; K)$$

follows from the isomorphisms $(\mathrm{S}^n(E))^* \cong \mathrm{S}^n(E^*)$ and $(\mathrm{S}^n(E))^* \cong \mathrm{Sym}^n(E; K)$ given by Proposition 33.26. ∎

Remarks:

1. The isomorphism $\mu \colon \mathrm{S}^n(E^*) \cong \mathrm{Sym}^n(E; K)$ discussed above can be described explicitly as the linear extension of the map given by

$$\mu(v_1^* \odot \cdots \odot v_n^*)(u_1, \ldots, u_n) = \sum_{\sigma \in \mathfrak{S}_n} v_{\sigma(1)}^*(u_1) \cdots v_{\sigma(n)}^*(u_n).$$

[2] This is where the assumption that we are in finite dimension and that the field has characteristic zero are used.