Definition 37.23. Given a topological space, $(E, \mathcal{O})$, we say that two points, $a, b \in E$, are *connected* if there is some connected subset, A, of E such that $a \in A$ and $b \in A$.

It is immediately verified that the relation "a and b are connected in E" is an equivalence relation. Only transitivity is not obvious, but it follows immediately as a special case of Lemma 37.19. Thus, the above equivalence relation defines a partition of E into nonempty disjoint *connected components*. The following proposition is easily proved using Lemma 37.19 and Lemma 37.20:

Proposition 37.21. *Given any topological space, E, for any $a \in E$, the connected component containing a is the largest connected set containing a. The connected components of E are closed.*

The notion of a locally connected space is also useful.

Definition 37.24. A topological space, $(E, \mathcal{O})$, is *locally connected* if for every $a \in E$, for every neighborhood, V, of a, there is a connected neighborhood, U, of a such that $U \subseteq V$. See Figure 37.24.

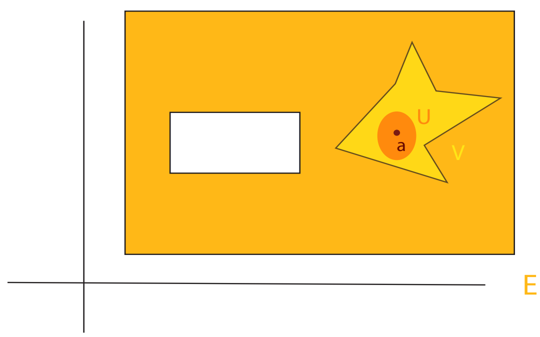

Figure 37.24: The topological space E, which is homeomorphic to an annulus, is locally connected since each point is surrounded by a small disk contained in E.

As we shall see in a moment, it would be equivalent to require that E has a basis of connected open sets.

There are connected spaces that are not locally connected and there are locally connected spaces that are not connected. The two properties are independent. For example, the subspace of $\mathbb{R}^2$ $S = \{(x, \sin(1/x)), \mid x > 0\} \cup \{(0, y) \mid -1 \leq y \leq 1\}$ is connected but not locally connected. See Figure 37.25. The subspace S of $\mathbb{R}$ consisting $[0, 1] \cup [2, 3]$ is locally connected but not connected.