In particular, this is the case if

$$U = \mathbb{R}^n_+ = \{v \in \mathbb{R}^n \mid v \geq 0\}$$

and if

$$J(v) = \frac{1}{2}\langle Av, a\rangle - \langle b, v\rangle$$

is an elliptic quadratic functional on $\mathbb{R}^n$. Then the vector $u_{k+1} = (u_1^{k+1}, \ldots, u_n^{k+1})$ is given in terms of $u_k = (u_1^k, \ldots, u_n^k)$ by

$$u_i^{k+1} = \max\{u_i^k - \rho_k(Au_k - b)_i, 0\}, \quad 1 \leq i \leq n.$$

## 49.12 Penalty Methods for Constrained Optimization

In the case where $V = \mathbb{R}^n$, another method to deal with constrained optimization is to incorporate the domain $U$ into the objective function $J$ by adding a penalty function.

**Definition 49.11.** Given a nonempty closed convex subset $U$ of $\mathbb{R}^n$, a function $\psi\colon \mathbb{R}^n \to \mathbb{R}$ is called a *penalty function* for $U$ if $\psi$ is convex and continuous and if the following conditions hold:

$$\psi(v) \geq 0 \quad \text{for all} \quad v \in \mathbb{R}^n, \quad \text{and} \quad \psi(v) = 0 \quad \text{iff} \quad v \in U.$$

The following proposition shows that the use of penalty functions reduces a constrained optimization problem to a sequence of unconstrained optimization problems.

**Proposition 49.19.** *Let $J\colon \mathbb{R}^n \to \mathbb{R}$ be a continuous, coercive, strictly convex function, $U$ be a nonempty, convex, closed subset of $\mathbb{R}^n$, $\psi\colon \mathbb{R}^n \to \mathbb{R}$ be a penalty function for $U$, and let $J_\epsilon\colon \mathbb{R}^n \to \mathbb{R}$ be the penalized objective function given by*

$$J_\epsilon(v) = J(v) + \frac{1}{\epsilon}\,\psi(v) \quad \text{for all } v \in \mathbb{R}^n.$$

*Then for every $\epsilon > 0$, there exists a unique element $u_\epsilon \in \mathbb{R}^n$ such that*

$$J_\epsilon(u_\epsilon) = \inf_{v \in \mathbb{R}^n} J_\epsilon(v).$$

*Furthermore, if $u \in U$ is the unique minimizer of $J$ over $U$, so that $J(u) = \inf_{v \in U} J(v)$, then*

$$\lim_{\epsilon \mapsto 0} u_\epsilon = u.$$

*Proof.* Observe that since $J$ is coercive, since $\psi(v) \geq 0$ for all $v \in \mathbb{R}^n$, and $J_\epsilon = J + (1/\epsilon)\psi$, we have $J_\epsilon(v) \geq J(v)$ for all $v \in \mathbb{R}^n$, so $J_\epsilon$ is also coercive. Since $J$ is strictly convex and $(1/\epsilon)\psi$ is convex, it is immediately checked that $J_\epsilon = J + (1/\epsilon)\psi$ is also strictly convex. Then by Proposition 49.1 (and the fact that $J$ and $J_\epsilon$ are strictly convex), $J$ has a unique minimizer $u \in U$, and $J_\epsilon$ has a unique minimizer $u_\epsilon \in \mathbb{R}^n$.