Another version of Proposition 43.3 using the Schur complement of A instead of the Schur complement of C also holds. The proof uses the factorization of M using the Schur complement of A (see Section 43.1).

Proposition 43.4. *For any symmetric matrix M of the form*

$$M = \begin{pmatrix} A & B \\ B^\top & C \end{pmatrix},$$

if A is invertible then the following properties hold:

(1) $M \succ 0$ iff $A \succ 0$ and $C - B^\top A^{-1} B \succ 0$.

(2) If $A \succ 0$, then $M \succeq 0$ iff $C - B^\top A^{-1} B \succeq 0$.

Here is an illustration of Proposition 43.4(2). Consider the nonlinear quadratic constraint

$$(Ax + b)^\top (Ax + b) \leq c^\top x + d,$$

were $A \in \mathrm{M}_n(\mathbb{R}), x, b, c \in \mathbb{R}^n$ and $d \in \mathbb{R}$. Since obviously $I = I_n$ is invertible and $I \succ 0$, we have

$$\begin{pmatrix} I & Ax + b \\ (Ax + b)^\top & c^\top x + d \end{pmatrix} \succeq 0$$

iff $c^\top x + d - (Ax + b)^\top (Ax + b) \succeq 0$ iff $(Ax + b)^\top (Ax + b) \leq c^\top x + d$, since the matrix (a scalar) $c^\top x + d - (Ax + b)^\top (Ax + b)$ is the Schur complement of I in the above matrix.

The trick of using Schur complements to convert nonlinear inequality constraints into linear constraints on symmetric matrices involving the semidefinite ordering $\succeq$ is used extensively to convert nonlinear problems into semidefinite programs; see Boyd and Vandenberghe [29].

When C is singular (or A is singular), it is still possible to characterize when a symmetric matrix M as above is positive semidefinite, but this requires using a version of the Schur complement involving the pseudo-inverse of C, namely $A - BC^+ B^\top$ (or the Schur complement, $C - B^\top A^+ B$, of A). We use the criterion of Proposition 42.5, which tells us when a quadratic function of the form $\frac{1}{2} x^\top P x - x^\top b$ has a minimum and what this optimum value is (where P is a symmetric matrix).

43.3 Symmetric Positive Semidefinite Matrices and Schur Complements

We now return to our original problem, characterizing when a symmetric matrix

$$M = \begin{pmatrix} A & B \\ B^\top & C \end{pmatrix}$$