We know from Section 35.6 that $K[X] \otimes_K E$ is a $K[X]$-module (obtained from the inclusion $K \subseteq K[X]$), which we will denote by $E[X]$. Since E is a vector space, $E[X]$ is a free $K[X]$-module, and if $(u_1, \ldots, u_n)$ is a basis of E, then $(1 \otimes u_1, \ldots, 1 \otimes u_n)$ is a basis of $E[X]$.

The free $K[X]$-module $E[X]$ is not as complicated as it looks. Over the basis $(1 \otimes u_1, \ldots, 1 \otimes u_n)$, every element $z \in E[X]$ can be written uniquely as

$$z = p_1(1 \otimes u_1) + \cdots + p_n(1 \otimes u_n) = p_1 \otimes u_1 + \cdots + p_n \otimes u_n,$$

where $p_1, \ldots, p_n$ are polynomials in $K[X]$. For notational simplicity, we may write

$$z = p_1 u_1 + \cdots + p_n u_n,$$

where $p_1, \ldots, p_n$ are viewed as coefficients in $K[X]$. With this notation, we see that $E[X]$ is isomorphic to $(K[X])^n$, which is easy to understand.

Observe that σ is $K[X]$-linear, because

$$\begin{aligned}
\sigma(q(p \otimes u)) &= \sigma((qp) \otimes u) \\
&= (qp) \cdot u \\
&= q(f)(p(f)(u)) \\
&= q \cdot (p(f)(u)) \\
&= q \cdot \sigma(p \otimes u).
\end{aligned}$$

Therefore, σ is a linear map of $K[X]$-modules, $\sigma\colon E[X] \to E_f$. Using our simplified notation, if $z = p_1 u_1 + \cdots + p_n u_n \in E[X]$, then

$$\sigma(z) = p_1(f)(u_1) + \cdots + p_n(f)(u_n),$$

which amounts to plugging f for X and evaluating. Similarly, f is a $K[X]$-linear map of E_f, because

$$\begin{aligned}
f(p \cdot u) &= f(p(f)(u)) \\
&= (fp(f))(u) \\
&= p(f)(f(u)) \\
&= p \cdot f(u),
\end{aligned}$$

where we used the fact that $fp(f) = p(f)f$ because $p(f)$ is a polynomial in f. By Proposition 35.40, the linear map $f\colon E \to E$ induces a $K[X]$-linear map $\overline{f}\colon E[X] \to E[X]$ such that

$$\overline{f}(p \otimes u) = p \otimes f(u).$$

Observe that we have

$$f(\sigma(p \otimes u)) = f(p(f)(u)) = p(f)(f(u))$$