for all $u \in E$, and since $f_\varphi(v_1+v_2)$ is the unique vector such that $\varphi(u, v_1+v_2) = \langle u, f_\varphi(v_1+v_2)\rangle$ for all $u \in E$, we must have

$$f_\varphi(v_1 + v_2) = f_\varphi(v_1) + f_\varphi(v_2).$$

For any $\lambda \in \mathbb{C}$ we have

$$\begin{aligned}
\varphi(u, \lambda v) &= \overline{\lambda}\varphi(u, v) && \varphi \text{ is sesquilinear}\\
&= \overline{\lambda}\langle u, f_\varphi(v)\rangle && \text{by definition of } f_\varphi\\
&= \langle u, \lambda f_\varphi(v)\rangle && \langle -, -\rangle \text{ is sesquilinear}
\end{aligned}$$

for all $u \in E$, and since $f_\varphi(\lambda v)$ is the unique vector such that $\varphi(u, \lambda v) = \langle u, f_\varphi(\lambda v)\rangle$ for all $u \in E$, we must have

$$f_\varphi(\lambda v) = \lambda f_\varphi(v).$$

Therefore f_φ is linear.

Then by definition of $\|\varphi\|$, we have

$$|\varphi_v(u)| = |\varphi(u, v)| \leq \|\varphi\| \, \|u\| \, \|v\| \,,$$

which shows that $\|\varphi_v\| \leq \|\varphi\| \, \|v\|$. Since $\|f_\varphi(v)\| = \|\varphi_v\|$, we have

$$\|f_\varphi(v)\| \leq \|\varphi\| \, \|v\| \,,$$

which shows that f_φ is continuous and that $\|f_\varphi\| \leq \|\varphi\|$. But by the Cauchy–Schwarz inequality we also have

$$|\varphi(u, v)| = |\langle u, f_\varphi(v)\rangle| \leq \|u\| \, \|f_\varphi(v)\| \leq \|u\| \, \|f_\varphi\| \, \|v\| \,,$$

so $\|\varphi\| \leq \|f_\varphi\|$, and thus

$$\|f_\varphi\| = \|\varphi\| \,.$$

If φ is Hermitian, $\varphi(v, u) = \overline{\varphi(u, v)}$, so

$$\langle f_\varphi(u), v\rangle = \overline{\langle v, f_\varphi(u)\rangle} = \overline{\varphi(v, u)} = \varphi(u, v) = \langle u, f_\varphi(v)\rangle,$$

which shows that f_φ is self-adjoint. $\square$

Proposition 48.11. *Given a Hilbert space E, for every continuous linear map $f \colon E \to E$, there is a unique continuous linear map $f^* \colon E \to E$, such that*

$$\langle f(u), v\rangle = \langle u, f^*(v)\rangle \quad \textit{for all } u, v \in E,$$

and we have $\|f^*\| = \|f\|$. The map f^* is called the adjoint of f.