48.3 Farkas–Minkowski Lemma in Hilbert Spaces

In this section $(V, \langle -, - \rangle)$ is assumed to be a *real* Hilbert space. The projection lemma can be used to show an interesting version of the Farkas–Minkowski lemma in a Hilbert space.

Given a finite sequence of vectors $(a_1, \ldots, a_m)$ with $a_i \in V$, let C be the polyhedral cone

$$C = \mathrm{cone}(a_1, \ldots, a_m) = \left\{ \sum_{i=1}^{m} \lambda_i a_i \mid \lambda_i \geq 0,\ i = 1, \ldots, m \right\}.$$

For any vector $b \in V$, the Farkas–Minkowski lemma gives a criterion for checking whether $b \in C$.

In Proposition 44.2 we proved that every polyhedral cone $\mathrm{cone}(a_1, \ldots, a_m)$ with $a_i \in \mathbb{R}^n$ is closed. Close examination of the proof shows that it goes through if $a_i \in V$ where V is any vector space possibly of infinite dimension, because the important fact is that the number m of these vectors is finite, not their dimension.

Theorem 48.12. *(Farkas–Minkowski Lemma in Hilbert Spaces) Let $(V, \langle -, - \rangle)$ be a real Hilbert space. For any finite sequence of vectors $(a_1, \ldots, a_m)$ with $a_i \in V$, if C is the polyhedral cone $C = \mathrm{cone}(a_1, \ldots, a_m)$, for any vector $b \in V$, we have $b \notin C$ iff there is a vector $u \in V$ such that*

$$\langle a_i, u \rangle \geq 0 \quad i = 1, \ldots, m, \quad \textit{and} \quad \langle b, u \rangle < 0.$$

Equivalently, $b \in C$ iff for all $u \in V$,

$$\textit{if } \langle a_i, u \rangle \geq 0 \quad i = 1, \ldots, m, \quad \textit{then} \quad \langle b, u \rangle \geq 0.$$

Proof. We follow Ciarlet [41] (Chapter 9, Theorem 9.1.1). We already established in Proposition 44.2 that the polyhedral cone $C = \mathrm{cone}(a_1, \ldots, a_m)$ is closed. Next we claim the following:

Claim: If C is a nonempty, closed, convex subset of a Hilbert space V, and $b \in V$ is any vector such that $b \notin C$, then there exist some $u \in V$ and infinitely many scalars $\alpha \in \mathbb{R}$ such that

$$\begin{aligned} \langle v, u \rangle &> \alpha \quad \text{for every } v \in C \\ \langle b, u \rangle &< \alpha. \end{aligned}$$

We use the projection lemma (Proposition 48.5) which says that since $b \notin C$ there is some unique $c = p_C(b) \in C$ such that

$$\begin{aligned} \|b - c\| = \inf_{v \in C} \|b - v\| > 0 \\ \langle b - c, v - c \rangle \leq 0 \quad \text{for all } v \in C, \end{aligned}$$