every m, the 2^m Walsh functions are pairwise orthogonal. The countable set of Walsh functions $\mathrm{Wal}(k,t)$ for all $m \geq 0$ and all k such that $0 \leq k \leq 2^m - 1$ can be ordered in such a way that it is an orthogonal Hilbert basis of the Hilbert space $\mathrm{L}^2([0,1))$; see Seberry, Wysocki and Wysocki [154].

The Sylvester–Hadamard matrix H_{2^m} plays a role in various algorithms for dimension reduction and low-rank matrix approximation. There is a type of structured dimension-reduction map known as the *subsampled randomized Hadamard transform*, for short SRHT; see Tropp [177] and Halko, Martinsson and Tropp [86]. For $\ell \ll n = 2^m$, an *SRHT matrix* is an $\ell \times n$ matrix of the form

$$\Phi = \sqrt{\frac{n}{\ell}} RHD,$$

where

1. D is a random $n \times n$ diagonal matrix whose entries are independent random signs.

2. $H = n^{-1/2} H_n$, a normalized Sylvester–Hadamard matrix of dimension n.

3. R is a random $\ell \times n$ matrix that restricts an n-dimensional vector to ℓ coordinates, chosen uniformly at random.

It is explained in Tropp [177] that for any input x such that $\|x\|_2 = 1$, the probability that $|(HDx)_i| \geq \sqrt{n^{-1}\log(n)}$ for any i is quite small. Thus HD has the effect of "flattening" the input x. The main result about the SRHT is that it preserves the geometry of an entire subspace of vectors; see Tropp [177] (Theorem 1.3).

5.7 Summary

The main concepts and results of this chapter are listed below:

- Haar basis vectors and a glimpse at *Haar wavelets*.

- *Kronecker product* (or *tensor product*) of matrices.

- Hadamard and Sylvester–Hadamard matrices.

- Walsh functions.