(4) Prove that the eigenvalues of $(A \otimes I_n) + (I_n \otimes B)$ are the scalars $\lambda_i + \mu_j$, for $i, j = 1, \ldots, n$. Deduce that the eigenvalues of $(A \otimes I_n) + (I_n \otimes A^\top)$ are $\lambda_i + \lambda_j$, for $i, j = 1, \ldots, n$. Thus $(A \otimes I_n) + (I_n \otimes A^\top)$ is invertible iff $\lambda_i + \lambda_j \neq 0$, for $i, j = 1, \ldots, n$. In particular, prove that if A is symmetric positive definite, then so is $(A \otimes I_n) + (I_n \otimes A^\top)$.

Hint. Use (3).

(5) Prove that if A and B are symmetric and $(A \otimes I_n) + (I_n \otimes A^\top)$ is invertible, then the unique solution X of the equation $AX + XA = B$ is symmetric.

(6) Starting with a symmetric positive definite matrix X_0, the general step of Newton's method is

$$X_{k+1} = X_k - (f'_{X_k})^{-1}(X_k^2 - C) = X_k - (g'_{X_k})^{-1}(X_k^2 - C),$$

and since g'_{X_k} is linear, this is equivalent to

$$X_{k+1} = X_k - (g'_{X_k})^{-1}(X_k^2) + (g'_{X_k})^{-1}(C).$$

But since X_k is SPD, $(g'_{X_k})^{-1}(X_k^2)$ is the unique solution of

$$X_k Y + Y X_k = X_k^2$$

whose solution is obviously $Y = (1/2)X_k$. Therefore the Newton step is

$$X_{k+1} = X_k - (g'_{X_k})^{-1}(X_k^2) + (g'_{X_k})^{-1}(C) = X_k - \frac{1}{2}X_k + (g'_{X_k})^{-1}(C) = \frac{1}{2}X_k + (g'_{X_k})^{-1}(C),$$

so we have

$$X_{k+1} = \frac{1}{2}X_k + (g'_{X_k})^{-1}(C) = (g'_{X_k})^{-1}(X_k^2 + C).$$

Prove that if X_k and C are symmetric positive definite, then $(g'_{X_k})^{-1}(C)$ is symmetric positive definite, and if C is symmetric positive semidefinite, then $(g'_{X_k})^{-1}(C)$ is symmetric positive semidefinite.

Hint. By (5) the unique solution Z of the equation $X_k Z + Z X_k = C$ (where C is symmetric) is symmetric so it can be diagonalized as $Z = QDQ^\top$ with Q orthogonal and D a real diagonal matrix. Prove that

$$Q^\top X_k Q D + D Q^\top X_k Q = Q^\top C Q,$$

and solve the system using the diagonal elements.

Deduce that if X_k and C are SPD, then X_{k+1} is SPD.

Since $C = P\Sigma P^\top$ is SPD, it has an SPD square root (in fact unique) $C^{1/2} = P\Sigma^{1/2}P^\top$. Prove that

$$X_{k+1} - C^{1/2} = (g'_{X_k})^{-1}(X_k - C^{1/2})^2.$$

Prove that

$$\left\|(g'_{X_k})^{-1}\right\|_2 = \frac{1}{2\left\|X_k\right\|_2}.$$