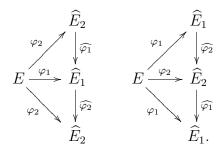
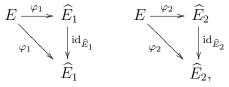
Consequently we have the following commutative diagrams:



However,  $\mathrm{id}_{\widehat{E}_1}$  and  $\mathrm{id}_{\widehat{E}_2}$  are uniformly continuous functions making the following diagrams commute



so by the uniqueness of extensions we must have

$$\widehat{\varphi_1}\circ\widehat{\varphi_2}=\mathrm{id}_{\widehat{E}_1}\quad\text{and}\quad \widehat{\varphi_2}\circ\widehat{\varphi_1}=\mathrm{id}_{\widehat{E}_2}.$$

This proves that  $\widehat{\varphi}_1$  and  $\widehat{\varphi}_2$  are mutual inverses. Now, since  $\varphi_2 = \widehat{\varphi}_2 \circ \varphi_1$ , we have

$$\widehat{\varphi_2}|\varphi_1(E)=\varphi_2\circ\varphi_1^{-1},$$

and since  $\varphi_1^{-1}$  and  $\varphi_2$  are isometries, so is  $\widehat{\varphi_2}|\varphi_1(E)$ . But we saw earlier that  $\widehat{\varphi_2}$  is the uniform continuous extension of  $\widehat{\varphi_2}|\varphi_1(E)$  and  $\varphi_1(E)$  is dense in  $\widehat{E}_1$ , so for any two elements  $\alpha, \beta \in \widehat{E}_1$ , if  $(a_n)$  and  $(b_n)$  are sequences in  $\varphi_1(E)$  converging to  $\alpha$  and  $\beta$ , we have

$$\widehat{d}_2((\widehat{\varphi}_2|\varphi_1(E))(a_n),((\widehat{\varphi}_2|\varphi_1(E))(b_n)) = \widehat{d}_1(a_n,b_n),$$

and by passing to the limit we get

$$\widehat{d}_2(\widehat{\varphi_2}(\alpha), \widehat{\varphi_2}(\beta)) = \widehat{d}_1(\alpha, \beta),$$

which shows that  $\widehat{\varphi_2}$  is an isometry (similarly,  $\widehat{\varphi_1}$  is an isometry).

## Remarks:

- 1. Except for Step 8 and Step 9, the proof of Theorem 37.53 is the proof given in Schwartz [149] (Chapter XI, Section 4, Theorem 1), and Kormogorov and Fomin [105] (Chapter 2, Section 7, Theorem 4).
- 2. The construction of  $\widehat{E}$  relies on the completeness of  $\mathbb{R}$ , and so it cannot be used to construct  $\mathbb{R}$  from  $\mathbb{Q}$ . However, this construction can be modified to yield a construction of  $\mathbb{R}$  from  $\mathbb{Q}$ .

We show in Section 37.12 that Theorem 37.53 yields a construction of the completion of a normed vector space.