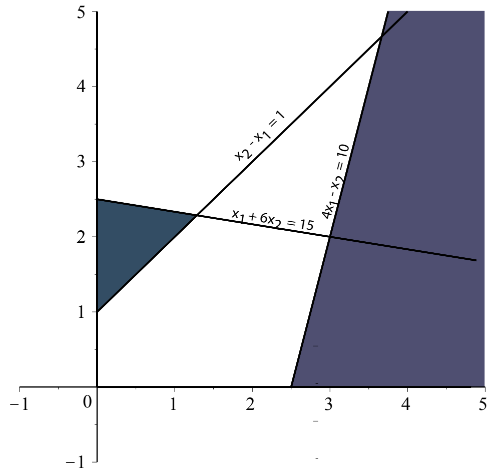

Figure 45.2: There is no $\mathcal{H}$-polyhedron associated with Example 45.2 since the blue and purple regions do not overlap.

Otherwise, we will prove shortly that if μ is the least upper bound of the set $\{cx \in \mathbb{R} \mid x \in \mathcal{P}(A, b)\}$, then there is some $p \in \mathcal{P}(A, b)$ such that

$$cp = \mu,$$

that is, the objective function $x \mapsto cx$ has a maximum value μ on $\mathcal{P}(A, b)$ which is achieved by some $p \in \mathcal{P}(A, b)$.

Definition 45.4. If the set $\{cx \in \mathbb{R} \mid x \in \mathcal{P}(A, b)\}$ is nonempty and bounded above, any point $p \in \mathcal{P}(A, b)$ such that $cp = \max\{cx \in \mathbb{R} \mid x \in \mathcal{P}(A, b)\}$ is called an *optimal solution* (or *optimum*) of (P). Optimal solutions are often denoted by an upper $*$; for example, p^*.

The linear program of Example 45.1 has a unique optimal solution $(3, 2)$, but observe that the linear program of Example 45.4 in which the objective function is $(1/6)x_1 + x_2$ has infinitely many optimal solutions; the maximum of the objective function is $15/6$ which occurs along the points of orange boundary line in Figure 45.1.

Example 45.4.

$$\begin{aligned}
\text{maximize} \quad & \frac{1}{6}x_1 + x_2 \\
\text{subject to} \quad & \\
& x_2 - x_1 \leq 1 \\
& x_1 + 6x_2 \leq 15 \\
& 4x_1 - x_2 \leq 10 \\
& x_1 \geq 0, \; x_2 \geq 0.
\end{aligned}$$