which is just the set of all polynomials in $\mathbb{Z}[X]$ whose constant term is of the form $2c$ for some $c \in \mathbb{Z}$. The ideal (X) is indeed properly contained in the ideal $(X, 2)$. If $P(X)Q(X) \in (X, 2)$, let a be the constant term in $P(X)$ and let b be the constant term in $Q(X)$. Since $P(X)Q(X) \in (X, 2)$, we must have $ab = 2c$ for some $c \in \mathbb{Z}$, and since 2 is prime, either a is divisible by 2 or b is divisible by 2. It follows that either $P(X) \in (X, 2)$ or $Q(X) \in (X, 2)$, which shows that $(X, 2)$ is a prime ideal.

Definition 30.6. An integral domain in which every ideal is a principal ideal is called a *principal ring or principal ideal domain*, for short, a *PID*.

The ring $\mathbb{Z}$ is a PID. This is a consequence of the existence of a (Euclidean) division algorithm. As we shall see next, when K is a field, the ring $K[X]$ is also a principal ring.

However, when $n \geq 2$, the ring $K[X_1, \ldots, X_n]$ is not principal. For example, in the ring $K[X, Y]$, the ideal (X, Y) generated by X and Y is not principal. First, since (X, Y) is the set of all polynomials of the form $Xq_1 + Yq_2$, where $q_1, q_2 \in K[X, Y]$, except when $Xq_1 + Yq_2 = 0$, we have $\deg(Xq_1 + Yq_2) \geq 1$. Thus, $1 \notin (X, Y)$. Now if there was some $p \in K[X, Y]$ such that $(X, Y) = (p)$, since $1 \notin (X, Y)$, we must have $\deg(p) \geq 1$. But we would also have $X = pq_1$ and $Y = pq_2$, for some $q_1, q_2 \in K[X, Y]$. Since $\deg(X) = \deg(Y) = 1$, this is impossible.

Even though $K[X, Y]$ is not a principal ring, a suitable version of unique factorization in terms of irreducible factors holds. The ring $K[X, Y]$ (and more generally $K[X_1, \ldots, X_n]$) is what is called a *unique factorization domain*, for short, UFD, or a *factorial ring*.

From this point until Definition 30.11, we consider polynomials in one variable over a field K.

Remark: Although we already proved part (1) of Proposition 30.10 in a more general situation above, we reprove it in the special case of polynomials. This may offend the purists, but most readers will probably not mind.

Proposition 30.10. *Let K be a field. The following properties hold:*

(1) For any two nonzero polynomials $f, g \in K[X]$, $(f) = (g)$ iff there is some $\lambda \neq 0$ in K such that $g = \lambda f$.

(2) For every nonnull ideal $\mathfrak{I}$ in $K[X]$, there is a unique monic polynomial $f \in K[X]$ such that $\mathfrak{I} = (f)$.

Proof. (1) If $(f) = (g)$, there are some nonzero polynomials $q_1, q_2 \in K[X]$ such that $g = fq_1$ and $f = gq_2$. Thus, we have $f = fq_1q_2$, which implies $f(1 - q_1q_2) = 0$. Since K is a field, by Proposition 30.1, $K[X]$ has no zero divisor, and since we assumed $f \neq 0$, we must have $q_1q_2 = 1$. However, if either q_1 or q_2 is not a constant, by Proposition 30.1 again, $\deg(q_1q_2) = \deg(q_1) + \deg(q_2) \geq 1$, contradicting $q_1q_2 = 1$, since $\deg(1) = 0$. Thus, both $q_1, q_2 \in K - \{0\}$, and (1) holds with $\lambda = q_1$. In the other direction, it is obvious that $g = \lambda f$ implies that $(f) = (g)$.