(a) Either the constraints $\varphi_i$ are affine for all $i \in I(u)$, or

(b) There is some nonzero vector $w \in V$ such that the following conditions hold for all $i \in I(u)$:

  (i) $(\varphi_i')_u(w) \leq 0$.

  (ii) If $\varphi_i$ is not affine, then $(\varphi_i')_u(w) < 0$.

Condition (b)(ii) implies that $u$ is not a critical point of $\varphi_i$ for every $i \in I(u)$, so there is no singularity at $u$ in the zero locus of $\varphi_i$. Intuitively, if the constraints are qualified at $u$ then the boundary of $U$ near $u$ behaves "nicely."

The boundary points illustrated in Figure 50.6 are qualified. Observe that $U = \{x \in \mathbb{R}^2 \mid \varphi_1(x, y) = y^2 - x \leq 0,\ \varphi_2(x, y) = x^2 - y \leq 0\}$. For $u = (1, 1)$, $I(u) = \{1, 2\}$, $(\varphi_1')_{(1,1)} = (-1\ \ 2)$, $(\varphi_2')_{(1,1)} = (2\ \ -1)$, and $w = (-1, -1)$ ensures that $(\varphi_1')_{(1,1)}$ and $(\varphi_1')_{(1,1)}$ satisfy Condition (b) of Definition 50.5. For $u = (1/4, 1/2)$, $I(u) = \{1\}$, $(\varphi_1')_{(1,1)} = (-1\ \ 1)$, and $w = (1, 0)$ will satisfy Condition (b).

In Example 50.1, the constraint $\varphi_2(u_1, u_2) = 0$ is not qualified at the origin because $(\varphi_2')_{(0,0)} = (0, 0)$; in fact, the origin is a self-intersection. In the example below, the origin is also a singular point, but for a different reason.

**Example 50.2.** Consider the region $U \subseteq \mathbb{R}^2$ determined by the two curves given by

$$\begin{aligned}\varphi_1(u_1, u_2) &= u_2 - \max(0, u_1^3)\\ \varphi_2(u_1, u_2) &= u_1^4 - u_2.\end{aligned}$$

We have $I(0, 0) = \{1, 2\}$, and since $(\varphi_1)'_{(0,0)}(w_1, w_2) = (0\ 1)\binom{w_1}{w_2} = w_2$ and $(\varphi_2')_{(0,0)}(w_1, w_2) = (0\ \ -1)\binom{w_1}{w_2} = -w_2$, we have $C^*(0) = \{(u_1, u_2) \in \mathbb{R}^2 \mid u_2 = 0\}$, but the constraints are not qualified at $(0, 0)$ since it is impossible to have simultaneously $(\varphi_1')_{(0,0)}(w_1, w_2) < 0$ and $(\varphi_2')_{(0,0)}(w_1, w_2) < 0$, so in fact $C(0) = \{(u_1, u_2) \in \mathbb{R}^2 \mid u_1 \geq 0,\ u_2 = 0\}$ is strictly contained in $C^*(0)$; see Figure 50.8.

**Proposition 50.2.** *Let $u$ be any point of the set*

$$U = \{x \in \Omega \mid \varphi_i(x) \leq 0,\ 1 \leq i \leq m\},$$

*where $\Omega$ is an open subset of the normed vector space $V$, and assume that the functions $\varphi_i$ are differentiable at $u$ (in fact, we only this for $i \in I(u)$). Then the following facts hold:*

*(1) The cone $C(u)$ of feasible directions at $u$ is contained in the convex cone $C^*(u)$; that is,*

$$C(u) \subseteq C^*(u) = \{v \in V \mid (\varphi_i')_u(v) \leq 0,\ \ i \in I(u)\}.$$