4. the first k columns of C match the first k columns of I_m.

We prove this claim by induction on k.

For the base case $k = 1$, we already know that $u_1 = v_1 = \ell_1^m$. We also have

$$c_1 = C\ell_1^m = Cv_1 = u_1 = \ell_1^m.$$

If $v_j = \lambda \ell_1^m$ for some $\lambda \in \mathbb{R}$, then

$$u_j = U\ell_j^n = CV\ell_j^n = Cv_j = \lambda C\ell_1^m = \lambda c_1 = \lambda \ell_1^m = v_j.$$

A similar argument using C^{-1} shows that if $u_j = \lambda \ell_1^m$, then $v_j = u_j$. Therefore, all the columns of U and V proportional to ℓ_1^m match, which establishes the base case. Observe that if ℓ_2^m appears in U, then it must appear in both U and V for the same index, and if not then $n_1 = n$ and $U = V$.

Next us now prove the induction step. If $n_k = n$, then $U = V$ and we are done. If $k = r$, then C is a block matrix of the form

$$C = \begin{pmatrix} I_r & B \\ 0_{m-r,r} & C \end{pmatrix}$$

and since the last $m - r$ rows of both U and V are zero rows, C acts as the identity on the first r rows, and so $U = V$. Otherwise $k < r$, $n_k < n$, and ℓ_{k+1}^m appears in both U and V, in which case, by (2) and (3) of the induction hypothesis, it appears in both U and V for the same index, say j_{k+1}. Thus, $u_{j_{k+1}} = v_{j_{k+1}} = \ell_{k+1}^m$. It follows that

$$c_{k+1} = C\ell_{k+1}^m = Cv_{j_{k+1}} = u_{j_{k+1}} = \ell_{k+1}^m,$$

so the first $k+1$ columns of C match the first $k+1$ columns of I_m.

Consider any subsequent column v_j (with $j > j_{k+1}$) whose elements beyond the $(k+1)$th all vanish. Then v_j is a linear combination of columns of V to the left of v_j, so

$$u_j = Cv_j = v_j.$$

because the first $k+1$ columns of C match the first $k+1$ column of I_m. Similarly, any subsequent column u_j (with $j > j_{k+1}$) whose elements beyond the $(k+1)$th all vanish is equal to v_j. Therefore, all the subsequent columns in U and V (of index $> j_{k+1}$) whose elements beyond the $(k+1)$th all vanish also match, which completes the induction hypothesis. $\square$

Remark: Observe that $C = E_p \cdots E_1 F_1^{-1} \cdots F_q^{-1}$ is *not* necessarily the identity matrix I_m. However, $C = I_m$ if $r = m$ (A has row rank m).

The reduction to row echelon form also provides a method to describe the set of solutions of a linear system of the form $Ax = b$.