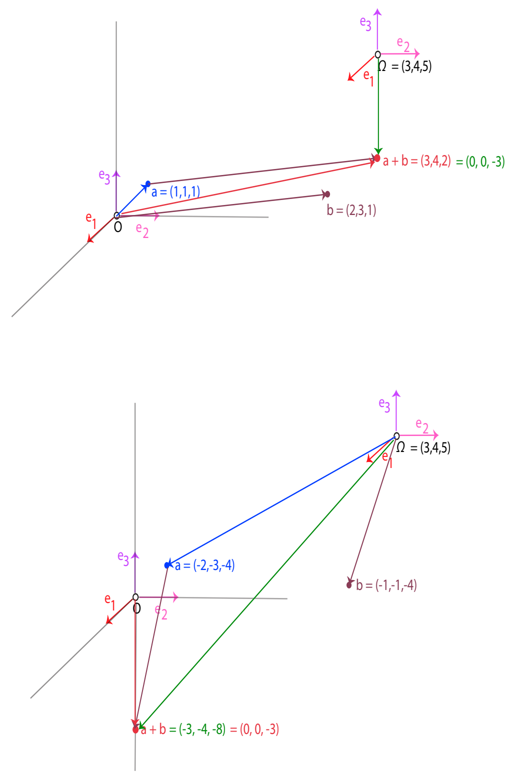

Figure 24.3: The top figure shows the location of the "point" sum $a+b$ with respect to the frame $(O,(e_1,e_2,e_3))$, while the bottom figure shows the location of the "point" sum $a+b$ with respect to the frame $(\Omega,(e_1,e_2,e_3))$.

A clean way to handle the problem of frame invariance and to deal with points in a more intrinsic manner is to make a clearer distinction between points and vectors. We duplicate $\mathbb{R}^3$ into two copies, the first copy corresponding to points, where we forget the vector space structure, and the second copy corresponding to free vectors, where the vector space structure is important. Furthermore, we make explicit the important fact that the vector space $\mathbb{R}^3$ acts on the set of points $\mathbb{R}^3$: Given any **point** $a=(a_1,a_2,a_3)$ and any **vector** $v=(v_1,v_2,v_3)$, we obtain the **point**

$$a+v=(a_1+v_1,a_2+v_2,a_3+v_3),$$

which can be thought of as the result of translating a to b using the vector v. We can imagine that v is placed such that its origin coincides with a and that its tip coincides with b. This action $+\colon \mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}^3$ satisfies some crucial properties. For example,

$$\begin{aligned} a+0 &= a,\\ (a+u)+v &= a+(u+v), \end{aligned}$$