$\lambda \in K$, then

$$h([u]) = [f(u)] = [\lambda u] = [u]$$

since $\lambda \neq 0$ because f is an isomorphism, which means that the point $[u] \in \mathbf{P}(E)$ is a fixed pointh of h. In other words, *eigenvectors of f induce fixed points of $h = \mathbb{P}(f)$.*

Consequently, it makes sense to try to classify homographies in terms of their fixed points. Of course this depends on the field K. If K is algebraically closed, for instance $K = \mathbb{C}$, then all the eigenvalues of f belong to K, and we can use the Jordan form of a matrix representing f. If $K = \mathbb{R}$, which is of particular interest to us, then we can use the real Jordan form, and we can obtain a compete classification for $E = \mathbb{R}^2$ and $E = \mathbb{R}^3$. We will also see that special kinds of homographies that leave every point of some projective hyperplane $\mathbf{P}(H)$ fixed, called *homologies*, play a special role.

We begin with the classification of the homographies of the real projective line $\mathbb{RP}^1$. Since a homography h of $\mathbb{RP}^1$ is represented by a real invertible 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and since A either 0, 1, or 2, real eigenvalues, the homography h has 0, 1, or 2 fixed points.

Definition 26.10. A homography of the real projective line $\mathbb{RP}^1$ not equal to the identity is *elliptic* if is has no fixed point, *parabolic* if it has a single fixed point, or *hyperbolic* if it has two fixed points.

(1) *Elliptic homographies.* In this case, $(a + d)^2 - 4(ad - bc) < 0$, so A has two distinct complex conjugate eigenvalues $\alpha \pm i\beta$, and in $\mathbb{C}^2$, they correspond to two complex eigenvectors $w_1 = u + iv$ and $w_2 = u - iv$, with $u, v \in \mathbb{R}^2$. Since

$$f(w_1) = (\alpha - i\beta)w_1$$

we obtain

$$f(u) + if(v) = \alpha u + \beta v + i(-\beta u + \alpha v),$$

which shows that in the basis (u, v), the homography h is represented by the matrix

$$\Gamma = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}.$$

If we let $\theta \in (0, 2\pi)$ be the angle given by

$$\cos\theta = \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}}$$
$$\sin\theta = \frac{\beta}{\sqrt{\alpha^2 + \beta^2}}$$