Definition 34.13. For any basis $(e_1, \ldots, e_n)$ of E, if we let $M = \{1, \ldots, n\}$, $e = e_1 \wedge \cdots \wedge e_n$, and $e^* = e_1^* \wedge \cdots \wedge e_n^*$, define $\gamma\colon \bigwedge^p E \to \bigwedge^{n-p} E^*$ and $\delta\colon \bigwedge^p E^* \to \bigwedge^{n-p} E$ as

$$\gamma(u) = u \lrcorner e^* \quad \text{and} \quad \delta(v^*) = e \llcorner v^*,$$

for all $u \in \bigwedge^p E$ and all $v^* \in \bigwedge^p E^*$.

Proposition 34.23. *The linear maps $\gamma\colon \bigwedge^p E \to \bigwedge^{n-p} E^*$ and $\delta\colon \bigwedge^p E^* \to \bigwedge^{n-p} E$ are isomorphims, and $\gamma^{-1} = \delta$. The isomorphisms γ and δ map decomposable vectors to decomposable vectors. Furthermore, if $z \in \bigwedge^p E$ is decomposable, say $z = u_1 \wedge \cdots \wedge u_p$ for some $u_i \in E$, then $\gamma(z) = v_1^* \wedge \cdots \wedge v_{n-p}^*$ for some $v_j^* \in E^*$, and $v_j^*(u_i) = 0$ for all i, j. A similar property holds for $v^* \in \bigwedge^p E^*$ and $\delta(v^*)$. If $(e_1', \ldots, e_n')$ is any other basis of E and $\gamma'\colon \bigwedge^p E \to \bigwedge^{n-p} E^*$ and $\delta'\colon \bigwedge^p E^* \to \bigwedge^{n-p} E$ are the corresponding isomorphisms, then $\gamma' = \lambda\gamma$ and $\delta' = \lambda^{-1}\delta$ for some nonzero $\lambda \in K$.*

Proof. Using Propositions 34.18 and 34.21, for any subset $J \subseteq \{1, \ldots, n\} = M$ such that $|J| = p$, we have

$$\gamma(e_J) = e_J \lrcorner e^* = \rho_{M-J,J} e^*_{M-J} \quad \text{and} \quad \delta(e^*_{M-J}) = e \llcorner e^*_{M-J} = \rho_{M-J,J} e_J.$$

Thus,

$$\delta \circ \gamma(e_J) = \rho_{M-J,J}\rho_{M-J,J} e_J = e_J,$$

since $\rho_{M-J,J} = \pm 1$. A similar result holds for $\gamma \circ \delta$. This implies that

$$\delta \circ \gamma = \mathrm{id} \quad \text{and} \quad \gamma \circ \delta = \mathrm{id}.$$

Thus, γ and δ are inverse isomorphisms.

If $z \in \bigwedge^p E$ is decomposable, then $z = u_1 \wedge \cdots \wedge u_p$ where $u_1, \ldots, u_p$ are linearly independent since $z \neq 0$, and we can pick a basis of E of the form $(u_1, \ldots, u_n)$. Then the above formulae show that

$$\gamma(z) = \pm u^*_{p+1} \wedge \cdots \wedge u^*_n.$$

Since $(u_1^*, \ldots, u_n^*)$ is the dual basis of $(u_1, \ldots, u_n)$, we have $u_i^*(u_j) = \delta_{ij}$, If $(e_1', \ldots, e_n')$ is any other basis of E, because $\bigwedge^n E$ has dimension 1, we have

$$e_1' \wedge \cdots \wedge e_n' = \lambda e_1 \wedge \cdots \wedge e_n$$

for some nonzero $\lambda \in K$, and the rest is trivial. $\square$

Applying Proposition 34.23 to the case where $p = n - 1$, the isomorphism $\gamma\colon \bigwedge^{n-1} E \to \bigwedge^1 E^*$ maps indecomposable vectors in $\bigwedge^{n-1} E$ to indecomposable vectors in $\bigwedge^1 E^* = E^*$. But every vector in E^* is decomposable, so every vector in $\bigwedge^{n-1} E$ is decomposable.

Corollary 34.24. *If E is a finite-dimensional vector space, then every vector in $\bigwedge^{n-1} E$ is decomposable.*