We claim that

$$x = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

works. Indeed, using the above congruences, for $i = 2, \ldots, n$, we get

$$x \equiv x_1 y_1 + x_i \pmod{\mathfrak{a}_i}, \tag{$*$}$$

but since $\mathfrak{a}_2 \cdots \mathfrak{a}_n \subseteq \mathfrak{a}_i$ for $i = 2, \ldots, n$ and $y_1 \equiv 0 \pmod{\mathfrak{a}_2 \cdots \mathfrak{a}_n}$, we have

$$x_1 y_1 \equiv 0 \pmod{\mathfrak{a}_i}, \quad i = 2, \ldots, n$$

and equation $(*)$ reduces to

$$x \equiv x_i \pmod{\mathfrak{a}_i}, \quad i = 2, \ldots, n.$$

For $i = 1$, we get

$$x \equiv x_1 \pmod{\mathfrak{a}_1},$$

therefore

$$x \equiv x_i \pmod{\mathfrak{a}_i}, \quad i = 1, \ldots, n.$$

proving surjectivity. $\square$

The classical version of the Chinese Remainder Theorem is the case where $A = \mathbb{Z}$ and where the ideals $\mathfrak{a}_i$ are defined by n pairwise relatively prime integers $m_1, \ldots, m_n$. By the Bezout identity, since m_i and m_j are relatively prime whenever $i \neq j$, there exist some $u_i, u_j \in \mathbb{Z}$ such that $u_i m_i + u_j m_j = 1$, and so $m_i\mathbb{Z} + m_j\mathbb{Z} = \mathbb{Z}$. In this case, we get an isomorphism

$$\mathbb{Z}/(m_1 \cdots m_n)\mathbb{Z} \approx \prod_{i=1}^{n} \mathbb{Z}/m_i\mathbb{Z}.$$

In particular, if m is an integer greater than 1 and

$$m = \prod_i p_i^{r_i}$$

is its factorization into prime factors, then

$$\mathbb{Z}/m\mathbb{Z} \approx \prod_i \mathbb{Z}/p_i^{r_i}\mathbb{Z}.$$

In the previous situation where the integers $m_1, \ldots, m_n$ are pairwise relatively prime, if we write $m = m_1 \cdots m_n$ and $m_i' = m/m_i$ for $i = 1 \ldots, n$, then m_i and m_i' are relatively prime, and so m_i' has an inverse modulo m_i. If t_i is such an inverse, so that

$$m_i' t_i \equiv 1 \pmod{m_i},$$