

Note that tensors in $T^{r,0}(V)$ are just our “old tensors” in $V^{\otimes r}$. We make $T^{\bullet,\bullet}(V)$ into an algebra by defining multiplication operations

$$T^{r_1,s_1}(V) \times T^{r_2,s_2}(V) \longrightarrow T^{r_1+r_2,s_1+s_2}(V)$$

in the usual way, namely: For $u = u_1 \otimes \cdots \otimes u_{r_1} \otimes u_1^* \otimes \cdots \otimes u_{s_1}^*$ and $v = v_1 \otimes \cdots \otimes v_{r_2} \otimes v_1^* \otimes \cdots \otimes v_{s_2}^*$, let

$$u \otimes v = u_1 \otimes \cdots \otimes u_{r_1} \otimes v_1 \otimes \cdots \otimes v_{r_2} \otimes u_1^* \otimes \cdots \otimes u_{s_1}^* \otimes v_1^* \otimes \cdots \otimes v_{s_2}^*.$$

Denote by $\text{Hom}(V^r, (V^*)^s; W)$ the vector space of all multilinear maps from $V^r \times (V^*)^s$ to W . Then we have the universal mapping property which asserts that there is a canonical isomorphism

$$\text{Hom}(T^{r,s}(V), W) \cong \text{Hom}(V^r, (V^*)^s; W).$$

In particular,

$$(T^{r,s}(V))^* \cong \text{Hom}(V^r, (V^*)^s; K).$$

For finite dimensional vector spaces, the duality of Section 33.5 is also easily extended to the tensor spaces $T^{r,s}(V)$. We define the pairing

$$T^{r,s}(V^*) \times T^{r,s}(V) \longrightarrow K$$

as follows: if

$$v^* = v_1^* \otimes \cdots \otimes v_r^* \otimes u_{r+1} \otimes \cdots \otimes u_{r+s} \in T^{r,s}(V^*)$$

and

$$u = u_1 \otimes \cdots \otimes u_r \otimes v_{r+1}^* \otimes \cdots \otimes v_{r+s}^* \in T^{r,s}(V),$$

then

$$(v^*, u) = v_1^*(u_1) \cdots v_{r+s}^*(u_{r+s}).$$

This is a nondegenerate pairing, and thus we get a canonical isomorphism

$$(T^{r,s}(V))^* \cong T^{r,s}(V^*).$$

Consequently, we get a canonical isomorphism

$$T^{r,s}(V^*) \cong \text{Hom}(V^r, (V^*)^s; K).$$

We summarize these results in the following proposition.

Proposition 33.21. *Let V be a vector space and let*

$$T^{r,s}(V) = V^{\otimes r} \otimes (V^*)^{\otimes s} = \underbrace{V \otimes \cdots \otimes V}_r \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_s.$$

We have the canonical isomorphisms

$$(T^{r,s}(V))^* \cong T^{r,s}(V^*),$$

and

$$T^{r,s}(V^*) \cong \text{Hom}(V^r, (V^*)^s; K).$$