and then the minimum u_λ of Problem (P) is given by

$$u_\lambda = A^{-1}(b - C^\top \lambda).$$

If C has rank $< m$, then we can find $\lambda \geq 0$ by finding a feasible solution of the linear program whose set of constraints is given by

$$-CA^{-1}C^\top \mu + CA^{-1}b - d = 0,$$

using the standard method of adding nonnegative slack variables $\xi_1, \ldots, \xi_m$ and maximizing $-(\xi_1 + \cdots + \xi_m)$.

50.9 Handling Equality Constraints Explicitly

Sometimes it is desirable to handle equality constraints explicitly (for instance, this is what Boyd and Vandenberghe do, see [29]). The only difference is that the Lagrange multipliers associated with *equality constraints* are *not required* to be nonnegative, as we now show.

Consider the *Optimization Problem* (P')

$$\begin{aligned} &\text{minimize} && J(v) \\ &\text{subject to} && \varphi_i(v) \leq 0, \quad i = 1, \ldots, m \\ &&& \psi_j(v) = 0, \quad j = 1, \ldots, p. \end{aligned}$$

We treat each equality constraint $\psi_j(u) = 0$ as the conjunction of the inequalities $\psi_j(u) \leq 0$ and $-\psi_j(u) \leq 0$, and we associate Lagrange multipliers $\lambda \in \mathbb{R}^m_+$, and $\nu^+, \nu^- \in \mathbb{R}^p_+$. Assuming that the constraints are qualified, by Theorem 50.5, the KKT conditions are

$$J'_u + \sum_{i=1}^m \lambda_i (\varphi'_i)_u + \sum_{j=1}^p \nu^+_j (\psi'_j)_u - \sum_{j=1}^p \nu^-_j (\psi'_j)_u = 0,$$

and

$$\sum_{i=1}^m \lambda_i \varphi_i(u) + \sum_{j=1}^p \nu^+_j \psi_j(u) - \sum_{j=1}^p \nu^-_j \psi_j(u) = 0,$$

with $\lambda \geq 0, \nu^+ \geq 0, \nu^- \geq 0$. Since $\psi_j(u) = 0$ for $j = 1, \ldots, p$, these equations can be rewritten as

$$J'_u + \sum_{i=1}^m \lambda_i (\varphi'_i)_u + \sum_{j=1}^p (\nu^+_j - \nu^-_j)(\psi'_j)_u = 0,$$

and

$$\sum_{i=1}^m \lambda_i \varphi_i(u) = 0$$