and since $U + V$ is a subspace of $\mathbb{R}^5$, $\dim(U + V) \leq 5$, which implies

$$6 \leq 5 + \dim(U \cap V),$$

that is $1 \leq \dim(U \cap V)$.

As another consequence of Proposition 6.17, if U and V are two hyperplanes in a vector space of dimension n, so that $\dim(U) = n - 1$ and $\dim(V) = n - 1$, the reader should show that

$$\dim(U \cap V) \geq n - 2,$$

and so, if $U \neq V$, then

$$\dim(U \cap V) = n - 2.$$

Here is a characterization of direct sums that follows directly from Theorem 6.16.

Proposition 6.18. *If $U_1, \ldots, U_p$ are any subspaces of a finite dimensional vector space E, then*

$$\dim(U_1 + \cdots + U_p) \leq \dim(U_1) + \cdots + \dim(U_p),$$

and

$$\dim(U_1 + \cdots + U_p) = \dim(U_1) + \cdots + \dim(U_p)$$

iff the U_is form a direct sum $U_1 \oplus \cdots \oplus U_p$.

Proof. If we apply Theorem 6.16 to the linear map

$$a \colon U_1 \times \cdots \times U_p \to U_1 + \cdots + U_p$$

given by $a(u_1, \ldots, u_p) = u_1 + \cdots + u_p$, we get

$$\begin{aligned} \dim(U_1 + \cdots + U_p) &= \dim(U_1 \times \cdots \times U_p) - \dim(\mathrm{Ker}\, a) \\ &= \dim(U_1) + \cdots + \dim(U_p) - \dim(\mathrm{Ker}\, a), \end{aligned}$$

so the inequality follows. Since a is injective iff $\mathrm{Ker}\, a = (0)$, the U_is form a direct sum iff the second equation holds. $\square$

Another important corollary of Theorem 6.16 is the following result:

Proposition 6.19. *Let E and F be two vector spaces with the same finite dimension $\dim(E) = \dim(F) = n$. For every linear map $f \colon E \to F$, the following properties are equivalent:*

(a) f is bijective.

(b) f is surjective.

(c) f is injective.