34.9 The Grassmann-Plücker's Equations and Grassmannian Manifolds ⊛

We follow an argument adapted from Bourbaki [25] (Chapter III, §11, Section 13).

Let E be a vector space of dimensions n, let $(e_1, \ldots, e_n)$ be a basis of E, and let $(e_1^*, \ldots, e_n^*)$ be its dual basis. Our objective is to determine whether a nonzero vector $z \in \bigwedge^p E$ is decomposable. By Proposition 34.27, the vector z is decomposable iff $(u^* \lrcorner z) \wedge z = 0$ for all $u^* \in \bigwedge^{p-1} E^*$. We can let u^* range over a basis of $\bigwedge^{p-1} E^*$, and then the conditions are

$$(e_H^* \lrcorner z) \wedge z = 0$$

for all $H \subseteq \{1, \ldots, n\}$, with $|H| = p-1$. Since $(e_H^* \lrcorner z) \wedge z \in \bigwedge^{p+1} E$, this is equivalent to

$$\langle e_J^*, (e_H^* \lrcorner z) \wedge z \rangle = 0$$

for all $H, J \subseteq \{1, \ldots, n\}$, with $|H| = p-1$ and $|J| = p+1$. Then, for all $I, I' \subseteq \{1, \ldots, n\}$ with $|I| = |I'| = p$, Formulae (2) and (4) of Proposition 34.18 show that

$$\langle e_J^*, (e_H^* \lrcorner e_I) \wedge e_{I'} \rangle = 0,$$

unless there is some $i \in \{1, \ldots, n\}$ such that

$$I - H = \{i\}, \quad J - I' = \{i\}.$$

In this case, $I = H \cup \{i\}$ and $I' = J - \{i\}$, and using Formulae (2) and (4) of Proposition 34.18, we have

$$\langle e_J^*, (e_H^* \lrcorner e_{H\cup\{i\}}) \wedge e_{J-\{i\}} \rangle = \langle e_J^*, \rho_{\{i\},H} e_i \wedge e_{J-\{i\}} \rangle = \langle e_J^*, \rho_{\{i\},H} \rho_{\{i\},J-\{i\}} e_J \rangle = \rho_{\{i\},H} \rho_{\{i\},J-\{i\}}.$$

If we let

$$\epsilon_{i,J,H} = \rho_{\{i\},H} \rho_{\{i\},J-\{i\}},$$

we have $\epsilon_{i,J,H} = +1$ if the parity of the number of $j \in J$ such that $j < i$ is the same as the parity of the number of $h \in H$ such that $h < i$, and $\epsilon_{i,J,H} = -1$ otherwise.

Finally we obtain the following criterion in terms of quadratic equations (*Plücker's equations*) for the decomposability of an alternating tensor.

Proposition 34.29. *(Grassmann-Plücker's Equations) For $z = \sum_I \lambda_I e_I \in \bigwedge^p E$, the conditions for $z \neq 0$ to be decomposable are*

$$\sum_{i \in J-H} \epsilon_{i,J,H} \lambda_{H\cup\{i\}} \lambda_{J-\{i\}} = 0,$$

with $\epsilon_{i,J,H} = \rho_{\{i\},H} \rho_{\{i\},J-\{i\}}$, for all $H, J \subseteq \{1, \ldots, n\}$ such that $|H| = p-1$, $|J| = p+1$, and all $i \in J - H$.