Observe that the Laplacian $L_{\mathrm{sym}} = D^{-1/2}LD^{-1/2}$ is a symmetric matrix (because L and $D^{-1/2}$ are symmetric) and that

$$L_{\mathrm{rw}} = D^{-1/2}L_{\mathrm{sym}}D^{1/2}.$$

The reason for the notation L_{rw} is that this matrix is closely related to a random walk on the graph G.

Example 20.1. As an example, the matrices L_{sym} and L_{rw} associated with the graph G_1 are

$$L_{\mathrm{sym}} = \begin{pmatrix} 1.0000 & -0.3536 & -0.4082 & 0 & 0 \\ -0.3536 & 1.0000 & -0.2887 & -0.2887 & -0.3536 \\ -0.4082 & -0.2887 & 1.0000 & -0.3333 & 0 \\ 0 & -0.2887 & -0.3333 & 1.0000 & -0.4082 \\ 0 & -0.3536 & 0 & -0.4082 & 1.0000 \end{pmatrix}$$

and

$$L_{\mathrm{rw}} = \begin{pmatrix} 1.0000 & -0.5000 & -0.5000 & 0 & 0 \\ -0.2500 & 1.0000 & -0.2500 & -0.2500 & -0.2500 \\ -0.3333 & -0.3333 & 1.0000 & -0.3333 & 0 \\ 0 & -0.3333 & -0.3333 & 1.0000 & -0.3333 \\ 0 & -0.5000 & 0 & -0.5000 & 1.0000 \end{pmatrix}.$$

Since the unnormalized Laplacian L can be written as $L = BB^{\top}$, where B is the incidence matrix of any oriented graph obtained from the underlying graph of $G = (V, W)$, if we let

$$B_{\mathrm{sym}} = D^{-1/2}B,$$

we get

$$L_{\mathrm{sym}} = B_{\mathrm{sym}}B_{\mathrm{sym}}^{\top}.$$

In particular, for any singular decomposition $B_{\mathrm{sym}} = U\Sigma V^{\top}$ of B_{sym} (with U an $m \times m$ orthogonal matrix, Σ a "diagonal" $m \times n$ matrix of singular values, and V an $n \times n$ orthogonal matrix), the eigenvalues of L_{sym} are the squares of the top m singular values of B_{sym}, and the vectors in U are orthonormal eigenvectors of L_{sym} with respect to these eigenvalues (the squares of the top m diagonal entries of Σ). Computing the SVD of B_{sym} generally yields more accurate results than diagonalizing L_{sym}, especially when L_{sym} has eigenvalues with high multiplicity.

There are simple relationships between the eigenvalues and the eigenvectors of L_{sym}, and L_{rw}. There is also a simple relationship with the generalized eigenvalue problem $Lx = \lambda Dx$.

Proposition 20.6. *Let $G = (V, W)$ be a weighted graph without isolated vertices. The graph Laplacians, L, L_{sym}, and L_{rw} satisfy the following properties:*