**Example 39.6.** Let $E = \mathbb{R}^2$, $F = G = \mathbb{R}$, $\Omega = \mathbb{R}^2 \times \mathbb{R} \cong \mathbb{R}^3$, $f\colon \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ given by

$$f((x_1, x_2), x_3) = x_1^2 + x_2^2 + x_3^2 - 1,$$

$a = \big(\sqrt{3}/(2\sqrt{2}), \sqrt{3}/(2\sqrt{2})\big)$, $b = 1/2$, and $c = 0$. The set of vectors $(x_1, x_2, x_3) \in \mathbb{R}^2$ such that

$$f((x_1, x_2), x_3) = x_1^2 + x_2^2 + x_3^2 - 1 = 0$$

is the unit sphere in $\mathbb{R}^3$. The vector $(a, b)$ belongs to the unit sphere since $\|a\|_2^2 + b^2 - 1 = 0$. The function $g\colon \mathbb{R}^2 \to \mathbb{R}$ given by

$$g(x_1, x_2) = \sqrt{1 - x_1^2 - x_2^2}$$

satisfies the equation

$$f(x_1, x_2, g(x_1, x_2)) = 0$$

all for $(x_1, x_2)$ in the open disk $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1\}$, and $g(a) = b$. Observe that if we had picked $b = -1/2$, then we would need to consider the function

$$g(x_1, x_2) = -\sqrt{1 - x_1^2 - x_2^2}.$$

We now state a very general version of the implicit function theorem. The proof of this theorem is fairly involved and uses a fixed-point theorem for contracting mappings in complete metric spaces; it is given in Schwartz [151]. Other proofs can be found in Lang [111] and Cartan [34].

**Theorem 39.14.** *Let $E, F$, and $G$, be normed affine spaces, let $\Omega$ be an open subset of $E \times F$, let $f\colon \Omega \to G$ be a function defined on $\Omega$, let $(a, b) \in \Omega$, let $c \in G$, and assume that $f(a, b) = c$. If the following assumptions hold*

*(1) The function $f\colon \Omega \to G$ is continuous on $\Omega$;*

*(2) $F$ is a complete normed affine space (and so is $G$);*

*(3) $\dfrac{\partial f}{\partial y}(x, y)$ exists for every $(x, y) \in \Omega$, and $\dfrac{\partial f}{\partial y}\colon \Omega \to \mathcal{L}(\overrightarrow{F}; \overrightarrow{G})$ is continuous;*

*(4) $\dfrac{\partial f}{\partial y}(a, b)$ is a bijection of $\mathcal{L}(\overrightarrow{F}; \overrightarrow{G})$, and $\Big(\dfrac{\partial f}{\partial y}(a, b)\Big)^{-1} \in \mathcal{L}(\overrightarrow{G}; \overrightarrow{F})$;*

*then the following properties hold:*

*(a) There exist some open subset $A \subseteq E$ containing $a$ and some open subset $B \subseteq F$ containing $b$, such that $A \times B \subseteq \Omega$, and for every $x \in A$, the equation $f(x, y) = c$ has a single solution $y = g(x)$, and thus, there is a unique function $g\colon A \to B$ such that $f(x, g(x)) = c$, for all $x \in A$;*