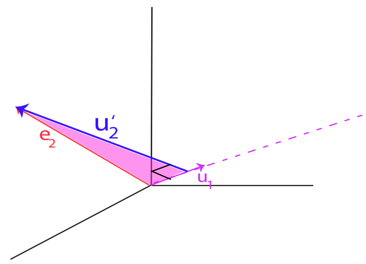

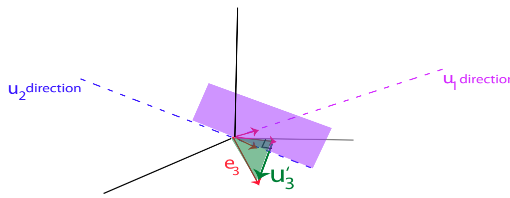

Figure 12.4: The top figure shows the construction of the blue u_2' as perpendicular to the orthogonal projection of e_2 onto u_1, while the bottom figure shows the construction of the green u_3' as normal to the plane determined by u_1 and u_2.

It should also be said that the Gram–Schmidt orthonormalization procedure that we have presented is not very stable numerically, and instead, one should use the *modified Gram–Schmidt method*. To compute u'_{k+1}, instead of projecting e_{k+1} onto $u_1, \ldots, u_k$ in a single step, it is better to perform k projections. We compute $u_1^{k+1}, u_2^{k+1}, \ldots, u_k^{k+1}$ as follows:

$$\begin{aligned} u_1^{k+1} &= e_{k+1} - (e_{k+1} \cdot u_1)\, u_1, \\ u_{i+1}^{k+1} &= u_i^{k+1} - (u_i^{k+1} \cdot u_{i+1})\, u_{i+1}, \end{aligned}$$

where $1 \leq i \leq k-1$. It is easily shown that $u'_{k+1} = u_k^{k+1}$.

Example 12.10. Let us apply the modified Gram–Schmidt method to the (e_1, e_2, e_3) basis of Example 12.9. The only change is the computation of u'_3. For the modified Gram–Schmidt procedure, we first calculate

$$u_1^3 = e_3 - (e_3 \cdot u_1)u_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - 2/3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1/3 \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.$$

Then

$$u_2^3 = u_1^3 - (u_1^3 \cdot u_2)u_2 = 1/3 \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} + 1/6 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 1/2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$