Proof. If $\mathcal{B} = (B_n)$ is a countable basis for the topology of E, then for any set S obtained by picking some point s_n in B_n, since every nonempty open subset U of E is the union of some of the B_n, the intersection $U \cap S$ is nonempty, and so S is dense in E.

Conversely, assume that there is a countable subset $S = (s_n)$ of E which is dense in E. We claim that the countable family $\mathcal{B}$ of open balls $B_0(s_n, 1/m)$ ($m \in \mathbb{N}, m > 0$) is a basis for the topology of E. For every $x \in E$ and every $r > 0$, there is some $m > 0$ such that $1/m < r/2$, and some n such that $s_n \in B_0(x, 1/m)$. It follows that $x \in B_0(s_n, 1/m)$. For all $y \in B_0(s_n, 1/m)$, we have

$$d(x, y) \leq d(x, s_n) + d(s_n, y) \leq 2/m < r,$$

thus $B_0(s_n, 1/m) \subseteq B_0(x, r)$, which by Proposition 37.8(a) implies that $\mathcal{B}$ is a basis for the topology of E. $\square$

Proposition 37.38. *If E is a compact metric space, then E is separable.*

Proof. For every $n > 0$, the family of open balls of radius $1/n$ forms an open cover of E, and since E is compact, there is a finite subset A_n of E such that $E = \bigcup_{a_i \in A_n} B_0(a_i, 1/n)$. It is easy to see that this is equivalent to the condition $d(x, A_n) < 1/n$ for all $x \in E$. Let $A = \bigcup_{n \geq 1} A_n$. Then A is countable, and for evey $x \in E$, we have

$$d(x, A) \leq d(x, A_n) < \frac{1}{n}, \quad \text{for all } n \geq 1,$$

which implies that $d(x, A) = 0$; that is, A is dense in E. $\square$

The following theorem due to Uryshon gives a very useful sufficient condition for a topological space to be metrizable.

Theorem 37.39. *(Urysohn metrization theorem) If a topological space E is regular and second-countable, then it is metrizable.*

The proof of Theorem 37.39 can be found in Munkres [131] (Chapter 4, Theorem 34.1). As a corollary of Theorem 37.39, every (second-countable) manifold, and thus every Lie group, is metrizable.

The following technical result shows that a locally compact metrizable space which is also separable can be expressed as the union of a countable monotonic sequence of compact subsets. This gives us a method for generalizing various properties of compact metric spaces to locally compact metric spaces of the above kind.

Proposition 37.40. *Let E be a locally compact metric space. The following properties are equivalent:*

(1) There is a sequence $(U_n)_{n\geq 0}$ of open subsets such that for all $n \in \mathbb{N}$, $U_n \subseteq U_{n+1}$, $\overline{U_n}$ is compact, $\overline{U_n} \subseteq U_{n+1}$, and $E = \bigcup_{n\geq 0} U_n = \bigcup_{n\geq 0} \overline{U_n}$.