We have $\text{dom}(f) = [0, +\infty)$, f is differentiable for all $x > 0$, but it is not subdifferentiable at $x = 0$. The only supporting hyperplane to $\mathbf{epi}(f)$ at $(0, 0)$ is the vertical line of equation $x = 0$ (the y-axis) as illustrated by Figure 51.18.

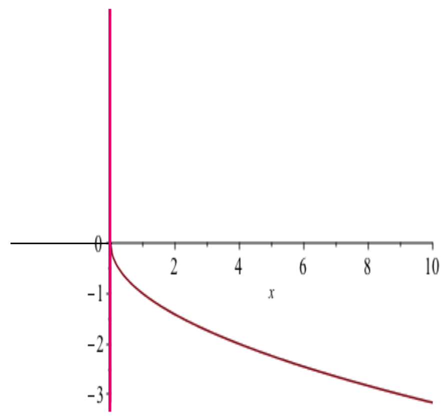

Figure 51.18: The graph of the partial function $f(x) = -\sqrt{x}$ and its red vertical supporting hyperplane at $x = 0$.

51.3 Basic Properties of Subgradients and Subdifferentials

A major tool to prove properties of subgradients is a variant of the notion of directional derivative.

Definition 51.15. Let $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$ be any function. For any $x \in \mathbb{R}^n$ such that $f(x)$ is finite ($f(x) \in \mathbb{R}$), for any $u \in \mathbb{R}^n$, the *one-sided directional derivative* $f'(x; u)$ is defined to be the limit

$$f'(x; u) = \lim_{\lambda \downarrow 0} \frac{f(x + \lambda u) - f(x)}{\lambda}$$

if it exists ($-\infty$ and $+\infty$ being allowed as limits). See Figure 51.19. The above notation for the limit means that we consider the limit when $\lambda > 0$ tends to 0.

Note that

$$\lim_{\lambda \uparrow 0} \frac{f(x + \lambda u) - f(x)}{\lambda}$$

denotes the one-sided limit when $\lambda < 0$ tends to zero, and that

$$-f'(x; -u) = \lim_{\lambda \uparrow 0} \frac{f(x + \lambda u) - f(x)}{\lambda},$$