These conditions are obviously analogous, and we can make this analogy more precise as follows. If $p_U \colon V \to U$ is the projection map onto U, we have the following chain of equivalences:

$$\begin{aligned}
& u \in U \quad \text{and} \quad J(u) = \inf_{v \in U} J(v) \quad \text{iff} \\
& u \in U \quad \text{and} \quad \langle \nabla J_u, v - u \rangle \geq 0 \quad \text{for every } v \in U, \text{ iff} \\
& u \in U \quad \text{and} \quad \langle u - (u - \rho \nabla J_u), v - u \rangle \geq 0 \quad \text{for every } v \in U \text{ and every } \rho > 0, \text{ iff} \\
& u = p_U(u - \rho \nabla J_u) \quad \text{for every } \rho > 0.
\end{aligned}$$

In other words, for every $\rho > 0$, $u \in V$ is a *fixed-point* of the function $g \colon V \to U$ given by

$$g(v) = p_U(v - \rho \nabla J_v).$$

The above suggests finding u by the method of successive approximations for finding the fixed-point of a contracting mapping, namely given any initial $u_0 \in V$, to define the sequence $(u_k)_{k \geq 0}$ such that

$$u_{k+1} = p_U(u_k - \rho_k \nabla J_{u_k}),$$

where the parameter $\rho_k > 0$ is chosen at each step. This method is called the *projected-gradient method with variable stepsize parameter*. Observe that if $U = V$, then this is just the gradient method with variable stepsize. We have the following result about the convergence of this method.

Proposition 49.18. *Let $J \colon V \to \mathbb{R}$ be a continuously differentiable functional defined on a Hilbert space V, and let U be nonempty, convex, closed subset of V. Suppose there exists two constants $\alpha > 0$ and $M > 0$ such that*

$$\langle \nabla J_v - \nabla J_u, v - u \rangle \geq \alpha \left\| v - u \right\|^2 \quad \textit{for all } u, v \in V,$$

and

$$\left\| \nabla J_v - \nabla J_u \right\| \leq M \left\| v - u \right\| \quad \textit{for all } u, v \in V.$$

If there exists two real nunbers $a, b \in \mathbb{R}$ such that

$$0 < a \leq \rho_k \leq b \leq \frac{2\alpha}{M^2} \quad \textit{for all } k \geq 0,$$

then the projected-gradient method with variable stepsize parameter converges. Furthermore, there is some constant $\beta > 0$ (depending on α, M, a, b) such that

$$\beta < 1 \quad \textit{and} \quad \left\| u_k - u \right\| \leq \beta^k \left\| u_0 - u \right\|,$$

where $u \in M$ is the unique minimum of J.