Proof. Since we already know that $(\bigwedge^n(E^*))\otimes F$ and $\mathrm{Alt}^n(E;F)$ are isomorphic, it is enough to show that μ_F maps some basis of $(\bigwedge^n(E^*))\otimes F$ to linearly independent elements. Pick some bases $(e_1,\ldots,e_p)$ in E and $(f_j)_{j\in J}$ in F. Then we know that the vectors $e_I^*\otimes f_j$, where $I\subseteq\{1,\ldots,p\}$ and $|I|=n$, form a basis of $(\bigwedge^n(E^*))\otimes F$. If we have a linear dependence

$$\sum_{I,j}\lambda_{I,j}\mu_F(e_I^*\otimes f_j)=0,$$

applying the above combination to each $(e_{i_1},\ldots,e_{i_n})$ ($I=\{i_1,\ldots,i_n\}$, $i_1<\cdots<i_n$), we get the linear combination

$$\sum_j\lambda_{I,j}f_j=0,$$

and by linear independence of the f_j's, we get $\lambda_{I,j}=0$ for all I and all j. Therefore, the $\mu_F(e_I^*\otimes f_j)$ are linearly independent, and we are done. The second part of the proposition is checked using a simple computation. □

The following proposition will be useful in dealing with vector-valued differential forms.

Proposition 34.34. *If $(e_1,\ldots,e_p)$ is any basis of E, then every element $\omega\in(\bigwedge^n(E^*))\otimes F$ can be written in a unique way as*

$$\omega=\sum_I e_I^*\otimes f_I,\qquad f_I\in F,$$

where the e_I^* are defined as in Section 34.2.

Proof. Since, by Proposition 34.7, the e_I^* form a basis of $\bigwedge^n(E^*)$, elements of the form $e_I^*\otimes f$ span $(\bigwedge^n(E^*))\otimes F$. Now if we apply $\mu_F(\omega)$ to $(e_{i_1},\ldots,e_{i_n})$, where $I=\{i_1,\ldots,i_n\}\subseteq\{1,\ldots,p\}$, we get

$$\mu_F(\omega)(e_{i_1},\ldots,e_{i_n})=\mu_F(e_I^*\otimes f_I)(e_{i_1},\ldots,e_{i_n})=f_I.$$

Therefore, the f_I are uniquely determined by f. □

Proposition 34.34 can also be formulated in terms of alternating multilinear maps, a fact that will be useful to deal with differential forms.

Corollary 34.35. *Define the product $\cdot\colon \mathrm{Alt}^n(E;\mathbb{R})\times F\to\mathrm{Alt}^n(E;F)$ as follows: For all $\omega\in\mathrm{Alt}^n(E;\mathbb{R})$ and all $f\in F$,*

$$(\omega\cdot f)(u_1,\ldots,u_n)=\omega(u_1,\ldots,u_n)f,$$

for all $u_1,\ldots,u_n\in E$. Then for every $\omega\in(\bigwedge^n(E^*))\otimes F$ of the form

$$\omega=u_1^*\wedge\cdots\wedge u_n^*\otimes f,$$

we have

$$\mu_F(u_1^*\wedge\cdots\wedge u_n^*\otimes f)=\mu_F(u_1^*\wedge\cdots\wedge u_n^*)\cdot f.$$