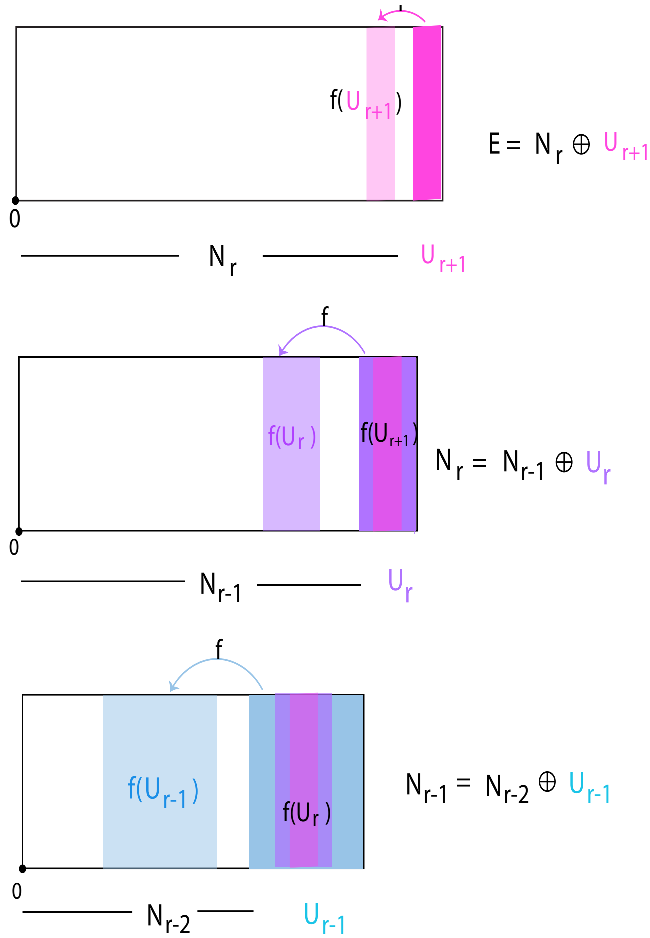

Figure 31.2: A schematic illustration of $N_i = N_{i-1} \oplus U_i$ with $f(U_i) \subseteq U_{i-1}$ for $i = r+1, r, r-1$.

Proof. We proceed inductively, by defining the sequence $U_{r+1}, U_r, \ldots, U_1$. We pick U_{r+1} to be any supplement of N_r in $N_{r+1} = E$, so that

$$E = N_{r+1} = N_r \oplus U_{r+1}.$$

Since $f^{r+1} = 0$ and $N_r = \mathrm{Ker}\,(f^r)$, we have $f(U_{r+1}) \subseteq N_r$, and by Proposition 31.14, as $U_{r+1} \cap N_r = (0)$, we have $f(U_{r+1}) \cap N_{r-1} = (0)$. As a consequence, we can pick a supplement U_r of N_{r-1} in N_r so that $f(U_{r+1}) \subseteq U_r$. We have

$$N_r = N_{r-1} \oplus U_r \quad \text{and} \quad f(U_{r+1}) \subseteq U_r.$$

By Proposition 31.14, f is an injection from U_{r+1} to U_r. Assume inductively that $U_{r+1}, \ldots, U_i$ have been defined for $i \geq 2$ and that they satisfy (1) and (2). Since

$$N_i = N_{i-1} \oplus U_i,$$