and since $f_j = 0$ for $j = r+1, \ldots, n$,

$$\langle v_i, f_j \rangle = 0 \quad 1 \leq i \leq n,\ r+1 \leq j \leq n.$$

If V is the matrix whose columns are $v_1, \ldots, v_n$, then V is orthogonal and the above equations prove that

$$V^\top AU = D,$$

which yields $A = VDU^\top$, as required.

The equation $A = VDU^\top$ implies that

$$A^\top A = UD^2U^\top, \quad AA^\top = VD^2V^\top,$$

which shows that $A^\top A$ and $AA^\top$ have the same eigenvalues, that the columns of U are eigenvectors of $A^\top A$, and that the columns of V are eigenvectors of $AA^\top$. $\square$

Example 22.4. Here is a simple example of how to use the proof of Theorem 22.5 to obtain an SVD decomposition. Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. Then $A^\top = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $A^\top A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, and $AA^\top = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$. A simple calculation shows that the eigenvalues of $A^\top A$ are 2 and 0, and for the eigenvalue 2, a unit eigenvector is $\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$, while a unit eigenvector for the eigenvalue 0 is $\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$. Observe that the singular values are $\sigma_1 = \sqrt{2}$ and $\sigma_2 = 0$. Furthermore, $U = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} = U^\top$. To determine V, the proof of Theorem 22.5 tells us to first calculate

$$AU = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix},$$

and then set

$$v_1 = (1/\sqrt{2}) \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Once v_1 is determined, since $\sigma_2 = 0$, we have the freedom to choose v_2 such that (v_1, v_2) forms an orthonormal basis for $\mathbb{R}^2$. Naturally, we chose $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and set $V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The columns of V are unit eigenvectors of $AA^\top$, but finding V by computing unit eigenvectors of $AA^\top$ does not guarantee that these vectors are consistent with U so that $A = V\Sigma U^\top$. Thus one has to use AU instead. We leave it to the reader to check that

$$A = V \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix} U^\top.$$

Theorem 22.5 suggests the following definition.