Furthermore, observe that φ is Hermitian iff $G = G^*$, and φ is positive definite iff the matrix G is positive definite, that is,

$$(Gx)^\top \overline{x} = x^* G x > 0 \quad \text{for all } x \in \mathbb{C}^n,\ x \neq 0.$$

Definition 14.5. The matrix G associated with a Hermitian product is called the *Gram matrix* of the Hermitian product with respect to the basis $(e_1, \ldots, e_n)$.

Conversely, if A is a Hermitian positive definite $n \times n$ matrix, it is easy to check that the Hermitian form

$$\langle x, y\rangle = y^* A x$$

is positive definite. If we make a change of basis from the basis $(e_1, \ldots, e_n)$ to the basis $(f_1, \ldots, f_n)$, and if the change of basis matrix is P (where the jth column of P consists of the coordinates of f_j over the basis $(e_1, \ldots, e_n)$), then with respect to coordinates x' and y' over the basis $(f_1, \ldots, f_n)$, we have

$$y^* G x = (y')^* P^* G P x',$$

so the matrix of our inner product over the basis $(f_1, \ldots, f_n)$ is $P^* G P$. We summarize these facts in the following proposition.

Proposition 14.2. *Let E be a finite-dimensional vector space, and let $(e_1, \ldots, e_n)$ be a basis of E.*

1. *For any Hermitian inner product $\langle -, -\rangle$ on E, if $G = (g_{ij})$ with $g_{ij} = \langle e_j, e_i\rangle$ is the Gram matrix of the Hermitian product $\langle -, -\rangle$ w.r.t. the basis $(e_1, \ldots, e_n)$, then G is Hermitian positive definite.*

2. *For any change of basis matrix P, the Gram matrix of $\langle -, -\rangle$ with respect to the new basis is $P^* G P$.*

3. *If A is any $n \times n$ Hermitian positive definite matrix, then*

$$\langle x, y\rangle = y^* A x$$

is a Hermitian product on E.

We will see later that a Hermitian matrix is positive definite iff its eigenvalues are all positive.

The following result reminiscent of the first polarization identity of Proposition 14.1 can be used to prove that two linear maps are identical.

Proposition 14.3. *Given any Hermitian space E with Hermitian product $\langle -, -\rangle$, for any linear map $f\colon E \to E$, if $\langle f(x), x\rangle = 0$ for all $x \in E$, then $f = 0$.*