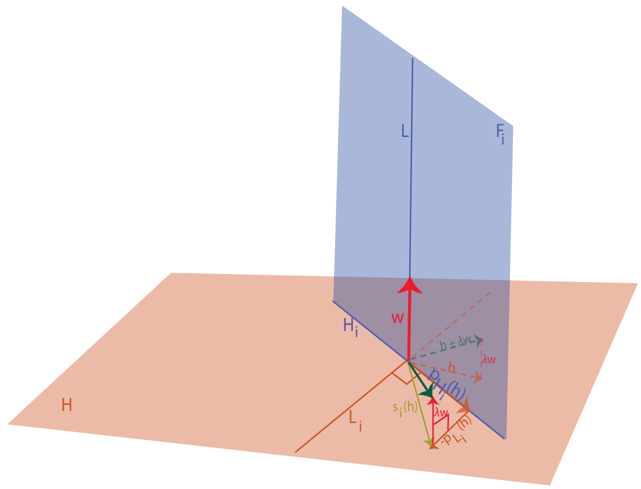

Figure 27.1: An illustration of how to extend the reflection s_i of Case 1 in Theorem 27.1 to E. The result of this extended reflection is the bold green vector.

Since $s^2 = \mathrm{id}$, we cannot have $s \circ f = \mathrm{id}$, since this would imply that $f = s$, where s is the identity on H, contradicting the fact that f is not the identity on any vector. Thus, we are back to Case 1. Thus, there are $k \leq n-1$ hyperplane reflections such that $s \circ f = s_k \circ \cdots \circ s_1$, from which we get

$$f = s \circ s_k \circ \cdots \circ s_1,$$

with at most $k+1 \leq n$ reflections. $\square$

Remarks:

(1) A slightly different proof can be given. Either f is the identity, or there is some nonnull vector u such that $f(u) \neq u$. In the second case, proceed as in the second part of the proof, to get back to the case where f admits 1 as an eigenvalue.

(2) Theorem 27.1 still holds if the inner product on E is replaced by a nondegenerate symmetric bilinear form φ, but the proof is a lot harder; see Section 29.9.

(3) The proof of Theorem 27.1 shows more than stated. If 1 is an eigenvalue of f, for any eigenvector w associated with 1 (i.e., $f(w) = w$, $w \neq 0$), then f is the composition of $k \leq n-1$ reflections about hyperplanes F_i such that $F_i = H_i \oplus L$, where L is the line $\mathbb{R}w$ and the H_i are subspaces of dimension $n-2$ all orthogonal to L (the H_i are hyperplanes in H). This situation is illustrated in Figure 27.3.

    If 1 is not an eigenvalue of f, then f is the composition of $k \leq n$ reflections about hyperplanes $H, F_1, \ldots, F_{k-1}$, such that $F_i = H_i \oplus L$, where L is a line intersecting H, and the H_i are subspaces of dimension $n-2$ all orthogonal to L (the H_i are hyperplanes in $L^{\perp}$). This situation is illustrated in Figure 27.4.