Assuming again that E is a Hermitian space, observe that Proposition 17.1 also holds. We deduce the following corollary.

Proposition 17.2. *Given a Hermitian space E, for any normal linear map $f\colon E \to E$, we have* $\mathrm{Ker}\,(f) \cap \mathrm{Im}(f) = (0)$.

Proof. Assume $v \in \mathrm{Ker}\,(f) \cap \mathrm{Im}(f)$, which means that $v = f(u)$ for some $u \in E$, and $f(v) = 0$. By Proposition 17.1, $\mathrm{Ker}\,(f) = \mathrm{Ker}\,(f^*)$, so $f(v) = 0$ implies that $f^*(v) = 0$. Consequently,

$$\begin{aligned}
0 &= \langle f^*(v), u\rangle \\
&= \langle v, f(u)\rangle \\
&= \langle v, v\rangle,
\end{aligned}$$

and thus, $v = 0$. $\square$

We also have the following crucial proposition relating the eigenvalues of f and f^*.

Proposition 17.3. *Given a Hermitian space E, for any normal linear map $f\colon E \to E$, a vector u is an eigenvector of f for the eigenvalue λ (in $\mathbb{C}$) iff u is an eigenvector of f^* for the eigenvalue $\overline{\lambda}$.*

Proof. First it is immediately verified that the adjoint of $f - \lambda\,\mathrm{id}$ is $f^* - \overline{\lambda}\,\mathrm{id}$. Furthermore, $f - \lambda\,\mathrm{id}$ is normal. Indeed,

$$\begin{aligned}
(f - \lambda\,\mathrm{id}) \circ (f - \lambda\,\mathrm{id})^* &= (f - \lambda\,\mathrm{id}) \circ (f^* - \overline{\lambda}\,\mathrm{id}), \\
&= f \circ f^* - \overline{\lambda} f - \lambda f^* + \lambda\overline{\lambda}\,\mathrm{id}, \\
&= f^* \circ f - \lambda f^* - \overline{\lambda} f + \overline{\lambda}\lambda\,\mathrm{id}, \\
&= (f^* - \overline{\lambda}\,\mathrm{id}) \circ (f - \lambda\,\mathrm{id}), \\
&= (f - \lambda\,\mathrm{id})^* \circ (f - \lambda\,\mathrm{id}).
\end{aligned}$$

Applying Proposition 17.1 to $f - \lambda\,\mathrm{id}$, for every nonnull vector u, we see that

$$(f - \lambda\,\mathrm{id})(u) = 0 \quad \text{iff} \quad (f^* - \overline{\lambda}\,\mathrm{id})(u) = 0,$$

which is exactly the statement of the proposition. $\square$

The next proposition shows a very important property of normal linear maps: eigenvectors corresponding to distinct eigenvalues are orthogonal.

Proposition 17.4. *Given a Hermitian space E, for any normal linear map $f\colon E \to E$, if u and v are eigenvectors of f associated with the eigenvalues λ and μ (in $\mathbb{C}$) where $\lambda \neq \mu$, then $\langle u, v\rangle = 0$.*