- Distance of a point to a subset, diameter.
- Projection onto a closed and convex subset.
- Orthogonal complement of a closed subspace.
- Dual of a Hilbert space.
- Bounded linear operator (or functional).
- Riesz representation theorem.
- Adjoint of a continuous linear map.
- Farkas–Minkowski lemma.

48.5 Problems

Problem 48.1. Let V be a Hilbert space. Prove that a subspace W of V is dense in V if and only if there is no nonzero vector orthogonal to W.

Problem 48.2. Prove that the adjoint satisfies the following properties:

$$\begin{aligned}(f+g)^* &= f^* + g^*\\ (\lambda f)^* &= \overline{\lambda} f^*\\ (f \circ g)^* &= g^* \circ f^*\\ f^{**} &= f.\end{aligned}$$

Problem 48.3. Prove that $\|f^* \circ f\| = \|f\|^2$.

Problem 48.4. Let V be a (real) Hilbert space and let C be a nonempty closed convex subset of V. Define the map $h\colon V \to \mathbb{R} \cup \{+\infty\}$ by

$$h(u) = \sup_{v \in C} \langle u, v \rangle.$$

Prove that

$$C = \bigcap_{u \in V} \{v \in V \mid \langle u, v\rangle \leq h(u)\} = \bigcap_{u \in \Lambda_C} \{v \in V \mid \langle u, v\rangle \leq h(u)\},$$

where $\Lambda_C = \{u \in V \mid h(u) \neq +\infty\}$.

Describe Λ_C when C is also a subspace of V.

Problem 48.5. Let A be a real $m \times n$ matrix, and let (u_k) be a sequence of vectors $u_k \in \mathbb{R}^n$ such that $u_k \geq 0$. Prove that if the sequence (Au_k) converges, then there is some $u \in \mathbb{R}^n$ such that

$$Au = \lim_{k \mapsto \infty} Au_k \quad \text{and} \quad u \geq 0.$$