Proof. From
$$\langle f^{\top}(\varphi), u\rangle = \langle \varphi, f(u)\rangle$$
for all $\varphi \in F^*$ and all $u \in E$, we see that if $u \in \mathrm{Ker}\,(f)$, then $\langle f^{\top}(\varphi), u\rangle = \langle \varphi, 0\rangle = 0$, which means that $f^{\top}(\varphi) \in (\mathrm{Ker}\,(f))^0$, and thus, $\mathrm{Im}\, f^{\top} \subseteq (\mathrm{Ker}\,(f))^0$. For the converse, since $\dim(E)$ is finite, we have

$$\dim((\mathrm{Ker}\,(f))^0) = \dim(E) - \dim(\mathrm{Ker}\,(f)) = \dim(\mathrm{Im}\, f),$$

but we just proved that $\dim(\mathrm{Im}\, f^{\top}) = \dim(\mathrm{Im}\, f)$, so we get

$$\dim((\mathrm{Ker}\,(f))^0) = \dim(\mathrm{Im}\, f^{\top}),$$

and since $\mathrm{Im}\, f^{\top} \subseteq (\mathrm{Ker}\,(f))^0$, we obtain

$$\mathrm{Im}\, f^{\top} = (\mathrm{Ker}\,(f))^0,$$

as claimed. $\square$

Remarks:

1. By the duality theorem, since $(\mathrm{Ker}\,(f))^{00} = \mathrm{Ker}\,(f)$, the above equation yields another proof of the fact that
$$\mathrm{Ker}\,(f) = (\mathrm{Im}\, f^{\top})^0,$$
when E is finite-dimensional.

2. The equation
$$\mathrm{Im}\, f^{\top} = (\mathrm{Ker}\,(f))^0$$
is actually valid even if when E if infinite-dimensional, but we will not prove this here.

11.8 The Four Fundamental Subspaces

Given a linear map $f\colon E \to F$ (where E and F are finite-dimensional), Proposition 11.11 revealed that the four spaces

$$\mathrm{Im}\, f,\ \mathrm{Im}\, f^{\top},\ \mathrm{Ker}\, f,\ \mathrm{Ker}\, f^{\top}$$

play a special role. They are often called the *fundamental subspaces* associated with f. These spaces are related in an intimate manner, since Proposition 11.11 shows that

$$\begin{aligned} \mathrm{Ker}\, f &= (\mathrm{Im}\, f^{\top})^0 \\ \mathrm{Ker}\, f^{\top} &= (\mathrm{Im}\, f)^0, \end{aligned}$$