An improper transformation is either a reflection about a plane or the product of three reflections, or equivalently the product of a reflection about a plane with a rotation, and we noted in the discussion following Theorem 27.1 that the axis of rotation is orthogonal to the plane of the reflection. Thus, an improper transformation is represented by a matrix of the form

$$S = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

When $n \geq 3$, the group of rotations $\mathbf{SO}(n)$ is not only generated by hyperplane reflections, but also by flips (about subspaces of dimension $n-2$). We will also see, in Section 27.2, that every proper affine rigid motion can be expressed as the composition of at most n flips, which is perhaps even more surprising! The proof of these results uses the following key lemma.

Proposition 27.4. *Given any Euclidean space E of dimension $n \geq 3$, for any two reflections h_1 and h_2 about some hyperplanes H_1 and H_2, there exist two flips f_1 and f_2 such that $h_2 \circ h_1 = f_2 \circ f_1$.*

Proof. If $h_1 = h_2$, it is obvious that

$$h_1 \circ h_2 = h_1 \circ h_1 = \mathrm{id} = f_1 \circ f_1$$

for any flip f_1. If $h_1 \neq h_2$, then $H_1 \cap H_2 = F$, where $\dim(F) = n-2$ (by the Grassmann relation). We can pick an orthonormal basis $(e_1, \ldots, e_n)$ of E such that $(e_1, \ldots, e_{n-2})$ is an orthonormal basis of F. We can also extend $(e_1, \ldots, e_{n-2})$ to an orthonormal basis $(e_1, \ldots, e_{n-2}, u_1, v_1)$ of E, where $(e_1, \ldots, e_{n-2}, u_1)$ is an orthonormal basis of H_1, in which case

$$\begin{aligned} e_{n-1} &= \cos\theta_1\, u_1 + \sin\theta_1\, v_1, \\ e_n &= \sin\theta_1\, u_1 - \cos\theta_1\, v_1, \end{aligned}$$

for some $\theta_1 \in [0, 2\pi]$. See Figure 27.6

Since h_1 is the identity on H_1 and v_1 is orthogonal to H_1, it follows that $h_1(u_1) = u_1$, $h_1(v_1) = -v_1$, and we get

$$\begin{aligned} h_1(e_{n-1}) &= \cos\theta_1\, u_1 - \sin\theta_1\, v_1, \\ h_1(e_n) &= \sin\theta_1\, u_1 + \cos\theta_1\, v_1. \end{aligned}$$

After some simple calculations, we get

$$\begin{aligned} h_1(e_{n-1}) &= \cos 2\theta_1\, e_{n-1} + \sin 2\theta_1\, e_n, \\ h_1(e_n) &= \sin 2\theta_1\, e_{n-1} - \cos 2\theta_1\, e_n. \end{aligned}$$