$\sqrt{\pi}$, and 1 by $\sqrt{2\pi}$; we won't because it looks messy!), the fact that a function $f \in \mathcal{C}^0[-\pi, \pi]$ can be written as a Fourier series as

$$f(x) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

does not mean that $(\sin px)_{p\geq 1} \cup (\cos qx)_{q\geq 0}$ is a basis of this vector space of functions, because in general, the families (a_k) and (b_k) **do not** have finite support! In order for this infinite linear combination to make sense, it is necessary to prove that the partial sums

$$a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

of the series converge to a limit when n goes to infinity. This requires a topology on the space.

A very important property of Euclidean spaces of finite dimension is that the inner product induces a canonical bijection (i.e., independent of the choice of bases) between the vector space E and its dual E^*. The reason is that an inner product $\cdot\colon E \times E \to \mathbb{R}$ defines a nondegenerate pairing, as defined in Definition 11.4. Indeed, if $u \cdot v = 0$ for all $v \in E$ then $u = 0$, and similarly if $u \cdot v = 0$ for all $u \in E$ then $v = 0$ (since an inner product is positive definite and symmetric). By Proposition 11.6, there is a canonical isomorphism between E and E^*. We feel that the reader will appreciate if we exhibit this mapping explicitly and reprove that it is an isomorphism.

The mapping from E to E^* is defined as follows.

Definition 12.3. For any vector $u \in E$, let $\varphi_u\colon E \to \mathbb{R}$ be the map defined such that

$$\varphi_u(v) = u \cdot v, \quad \text{for all } v \in E.$$

Since the inner product is bilinear, the map φ_u is a linear form in E^*. Thus, we have a map $\flat\colon E \to E^*$, defined such that

$$\flat(u) = \varphi_u.$$

Theorem 12.6. *Given a Euclidean space E, the map $\flat\colon E \to E^*$ defined such that*

$$\flat(u) = \varphi_u$$

is linear and injective. When E is also of finite dimension, the map $\flat\colon E \to E^*$ is a canonical isomorphism.