and since $\mathrm{D}^2 f(a)(u) = \mathrm{D}_u(\mathrm{D}f)(a)$, we get

$$\mathrm{D}^2 f(a)(u)(v) = \mathrm{D}_u(\mathrm{D}_v f)(a).$$

Thus, when $\mathrm{D}^2 f(a)$ exists, $\mathrm{D}_u(\mathrm{D}_v f)(a)$ exists, and

$$\mathrm{D}^2 f(a)(u)(v) = \mathrm{D}_u(\mathrm{D}_v f)(a),$$

for all $u, v \in \overrightarrow{E}$. $\square$

Definition 39.13. We denote $\mathrm{D}_u(\mathrm{D}_v f)(a)$ by $\mathrm{D}^2_{u,v} f(a)$ (or $\mathrm{D}_u \mathrm{D}_v f(a)$).

Recall from Proposition 37.60, that the map from $\mathcal{L}_2(\overrightarrow{E}, \overrightarrow{E}; \overrightarrow{F})$ to $\mathcal{L}(\overrightarrow{E}; \mathcal{L}(\overrightarrow{E}; \overrightarrow{F}))$ defined such that $g \mapsto \varphi$ iff for every $g \in \mathcal{L}_2(\overrightarrow{E}, \overrightarrow{E}; \overrightarrow{F})$,

$$\varphi(u)(v) = g(u, v),$$

is an isomorphism of vector spaces. Thus, we will consider $\mathrm{D}^2 f(a) \in \mathcal{L}(\overrightarrow{E}; \mathcal{L}(\overrightarrow{E}; \overrightarrow{F}))$ as a continuous bilinear map in $\mathcal{L}_2(\overrightarrow{E}, \overrightarrow{E}; \overrightarrow{F})$, and we will write $\mathrm{D}^2 f(a)(u, v)$, instead of $\mathrm{D}^2 f(a)(u)(v)$.

Then, the above discussion can be summarized by saying that when $\mathrm{D}^2 f(a)$ is defined, we have

$$\mathrm{D}^2 f(a)(u, v) = \mathrm{D}_u \mathrm{D}_v f(a).$$

Definition 39.14. When E has finite dimension and $(a_0, (e_1, \ldots, e_n))$ is a frame for E, we denote $\mathrm{D}_{e_j} \mathrm{D}_{e_i} f(a)$ by $\dfrac{\partial^2 f}{\partial x_i \partial x_j}(a)$, when $i \neq j$, and we denote $\mathrm{D}_{e_i} \mathrm{D}_{e_i} f(a)$ by $\dfrac{\partial^2 f}{\partial x_i^2}(a)$.

The following important lemma attributed to Schwarz can be shown, using Proposition 39.12. Given a bilinear map $f\colon \overrightarrow{E} \times \overrightarrow{E} \to \overrightarrow{F}$, recall that f is *symmetric*, if

$$f(u, v) = f(v, u),$$

for all $u, v \in \overrightarrow{E}$.

Proposition 39.20. *(Schwarz's lemma) Given two normed affine spaces E and F, given any open subset A of E, given any $f\colon A \to F$, for every $a \in A$, if $\mathrm{D}^2 f(a)$ exists, then $\mathrm{D}^2 f(a) \in \mathcal{L}_2(\overrightarrow{E}, \overrightarrow{E}; \overrightarrow{F})$ is a continuous symmetric bilinear map. As a corollary, if E is of finite dimension n, and $(a_0, (e_1, \ldots, e_n))$ is a frame for E, we have*

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(a) = \frac{\partial^2 f}{\partial x_j \partial x_i}(a).$$

Remark: There is a variation of the above result which does not assume the existence of $\mathrm{D}^2 f(a)$, but instead assumes that $\mathrm{D}_u \mathrm{D}_v f$ and $\mathrm{D}_v \mathrm{D}_u f$ exist on an open subset containing a and are continuous at a, and concludes that $\mathrm{D}_u \mathrm{D}_v f(a) = \mathrm{D}_v \mathrm{D}_u f(a)$. This is just a different result which does not imply Proposition 39.20, and is not a consequence of Proposition 39.20.