and the projection onto $\mathbf{Skew}(n)$ is given by

$$\pi_2(A) = S(A) = \frac{A - A^\top}{2}.$$

Clearly, $H(A)+S(A) = A$, $H(H(A)) = H(A)$, $S(S(A)) = S(A)$, and $H(S(A)) = S(H(A)) = 0$.

A function f such that $f \circ f = f$ is said to be *idempotent*. Thus, the projections π_i are idempotent. Conversely, the following proposition can be shown:

Proposition 6.8. *Let E be a vector space. For any $p \geq 2$ linear maps $f_i \colon E \to E$, if*

$$f_j \circ f_i = \begin{cases} f_i & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$
$$f_1 + \cdots + f_p = \mathrm{id}_E,$$

then if we let $U_i = f_i(E)$, we have a direct sum

$$E = U_1 \oplus \cdots \oplus U_p.$$

We also have the following proposition characterizing idempotent linear maps whose proof is also left as an exercise.

Proposition 6.9. *For every vector space E, if $f \colon E \to E$ is an idempotent linear map, i.e., $f \circ f = f$, then we have a direct sum*

$$E = \mathrm{Ker}\, f \oplus \mathrm{Im}\, f,$$

so that f is the projection onto its image $\mathrm{Im}\, f$.

We now give the definition of a direct sum for any arbitrary nonempty index set I. First, let us recall the notion of the product of a family $(E_i)_{i\in I}$. Given a family of sets $(E_i)_{i\in I}$, its product $\prod_{i\in I} E_i$, is the set of all functions $f \colon I \to \bigcup_{i\in I} E_i$, such that, $f(i) \in E_i$, for every $i \in I$. It is one of the many versions of the axiom of choice, that, if $E_i \neq \emptyset$ for every $i \in I$, then $\prod_{i\in I} E_i \neq \emptyset$. A member $f \in \prod_{i\in I} E_i$, is often denoted as $(f_i)_{i\in I}$. For every $i \in I$, we have the *projection* $\pi_i \colon \prod_{i\in I} E_i \to E_i$, defined such that, $\pi_i((f_i)_{i\in I}) = f_i$. We now define direct sums.

Definition 6.5. Let I be any nonempty set, and let $(E_i)_{i\in I}$ be a family of vector spaces. The *(external) direct sum* $\coprod_{i\in I} E_i$ (or *coproduct*) of the family $(E_i)_{i\in I}$ is defined as follows:

$\coprod_{i\in I} E_i$ consists of all $f \in \prod_{i\in I} E_i$, which have finite support, and addition and multiplication by a scalar are defined as follows:

$$(f_i)_{i\in I} + (g_i)_{i\in I} = (f_i + g_i)_{i\in I},$$
$$\lambda(f_i)_{i\in I} = (\lambda f_i)_{i\in I}.$$

We also have *injection maps* $in_i \colon E_i \to \coprod_{i\in I} E_i$, defined such that, $in_i(x) = (f_i)_{i\in I}$, where $f_i = x$, and $f_j = 0$, for all $j \in (I - \{i\})$.