We claim that $x + y \notin U$. Otherwise, since $y = x + y - x$, with $x + y, x \in U$ and since $y \in f(U)$, we would have $y \in U \cap f(U) = H$, a contradiction. Similarly, $x + y \notin f(U)$. It follows that

$$U + f(U) = U \oplus K(x + y) = f(U) \oplus K(x + y).$$

Now, pick W to be any supplement of $U + f(U)$ in $D^{\perp}$ so that $D^{\perp} = (U + f(U)) \oplus W$, and let

$$V = K(x + y) + W.$$

Then, since $x \in U, y \in f(U)$, $W \subseteq D^{\perp}$, and $U + f(U) \subseteq D^{\perp}$, we have $V \subseteq D^{\perp}$. We also have

$$U \oplus V = U \oplus K(x + y) \oplus W = (U + f(U)) \oplus W = D^{\perp}$$

and

$$f(U) \oplus V = f(U) \oplus K(x + y) \oplus W = (U + f(U)) \oplus W = D^{\perp},$$

so as we showed as a consequence of hypothesis (V), f can be extended to an isometry of the hyperplane $D^{\perp} = U \oplus V$, and D is still the line $\{f(w) - w \mid w \in U \oplus V\}$. $\square$

The argument in the proof of Case (b) shows that we are reduced to the situation where $U = D^{\perp}$ is a hyperplane in E and f is an isometry of U. If we pick any $v \notin U$, then $E = U \oplus Kv$, so suppose we can find some $v_1 \in E$ such that

$$\begin{aligned}\varphi(f(u), v_1) &= \varphi(u, v) \quad \text{for all } u \in U \\ \varphi(v_1, v_1) &= \varphi(v, v).\end{aligned}$$

The first condition is condition $(*)$ of Proposition 29.44, and the second condition asserts that the map $\lambda v \mapsto \lambda v_2$ from the line Kv to the line Kv_1 is a metric map. Then, by Proposition 29.44, we can extend f to a metric map g of $U + Kv = E$ such that $g(v) = v_1$.

To find v_1, let us prove that for every $v \in E$, there is some $v' \in E$ such that

$$\varphi(f(u), v') = \varphi(u, v) \quad \text{for all } u \in U. \tag{$\dagger$}$$

This is because the linear form $u \mapsto \varphi(f^{-1}(u), v)$ $(u \in U)$ is the restriction of a linear form $\psi \in E^*$, and since φ is nondegenerate, there is some (unique) $v' \in E$, such that

$$\psi(x) = \varphi(x, v') \quad \text{for all } x \in E,$$

which implies that

$$\varphi(u, v') = \varphi(f^{-1}(u), v) \quad \text{for all } u \in U,$$

and since f is an automorphism of U, that $(\dagger)$ holds. Furthermore, observe that formula $(\dagger)$ still holds if we add to v' any vector y in D, since $f(U) = U = D^{\perp}$. Therefore, for any $v_1 = v' + y$ with $y \in D$, if we extend f to a linear map of E by setting $g(v) = v_1$, then by $(\dagger)$ we have

$$\varphi(g(u), g(v)) = \varphi(u, v) \quad \text{for all } u \in U.$$