where

$$\nu = |\{(h,l) \ \mid \ (h,l) \in H \times L,\, h > l\}|.$$

Observe that when $H \cap L = \emptyset$, $|H| = p$ and $|L| = q$, the number ν is the number of inversions of the sequence

$$(h_1,\, \cdots,\, h_p,\, \ell_1,\, \cdots,\, \ell_q),$$

where an inversion is a pair (h_i, ℓ_j) such that $h_i > \ell_j$.

Unless $p + q = n$, the function whose graph is given by

$$\begin{pmatrix} 1 & \cdots & p & p+1 & \cdots & p+q \\ h_1 & \cdots & h_p & \ell_1 & \cdots & \ell_q \end{pmatrix}$$

is **not** a permutation of $\{1, \ldots, n\}$. We can view ν as a slight generalization of the notion of the number of inversions of a permutation.

Proposition 34.18. *For any basis $(e_1, \ldots, e_n)$ of E the following properties hold:*

(1) If $H \cap L = \emptyset$, $|H| = p$, and $|L| = q$, then

$$\rho_{H,L}\rho_{L,H} = (-1)^{\nu}(-1)^{pq-\nu} = (-1)^{pq}.$$

(2) For $H, L \subseteq \{1, \ldots, m\}$ listed in increasing order, we have

$$e_H \wedge e_L = \rho_{H,L} e_{H \cup L}.$$

Similarly,

$$e_H^* \wedge e_L^* = \rho_{H,L} e_{H \cup L}^*.$$

(3) For the left hook

$$\lrcorner \colon \bigwedge^p E \times \bigwedge^{p+q} E^* \longrightarrow \bigwedge^q E^*,$$

we have

$$\begin{aligned} &e_H \lrcorner\, e_L^* = 0 \quad \textit{if } H \not\subseteq L \\ &e_H \lrcorner\, e_L^* = \rho_{L-H,H} e_{L-H}^* \quad \textit{if } H \subseteq L. \end{aligned}$$

(4) For the left hook

$$\lrcorner \colon \bigwedge^p E^* \times \bigwedge^{p+q} E \longrightarrow \bigwedge^q E,$$

we have

$$\begin{aligned} &e_H^* \lrcorner\, e_L = 0 \quad \textit{if } H \not\subseteq L \\ &e_H^* \lrcorner\, e_L = \rho_{L-H,H} e_{L-H} \quad \textit{if } H \subseteq L. \end{aligned}$$