If U is a subspace of a space E, recall that the *codimension* of U is the dimension of E/U, which is also equal to the dimension of any subspace V such that E is a direct sum of U and V ($E = U \oplus V$).

Proposition 29.12 implies the following useful fact.

Proposition 29.13. *Let $\varphi\colon E \times F \to K$ be any nondegenerate sesquilinear form. A subspace U of E has finite dimension iff $U^\perp$ has finite codimension in F. If $\dim(U)$ is finite, then $\mathrm{codim}(U^\perp) = \dim(U)$, and $U^{\perp\perp} = U$.*

Proof. Since φ is nondegenerate $E^\perp = \{0\}$ and $F^\perp = \{0\}$, so Proposition 29.12 applied to the restriction of φ to $U \times F$ implies that a subspace U of E has finite dimension iff $U^\perp$ has finite codimension in F, and that if $\dim(U)$ is finite, then $\mathrm{codim}(U^\perp) = \dim(U)$. Since $U^\perp$ and $U^{\perp\perp}$ are orthogonal, and since $\mathrm{codim}(U^\perp)$ is finite, $\dim(U^{\perp\perp})$ is finite and we have $\dim(U^{\perp\perp}) = \mathrm{codim}(U^{\perp\perp\perp}) = \mathrm{codim}(U^\perp) = \dim(U)$. Since $U \subseteq U^{\perp\perp}$, we must have $U = U^{\perp\perp}$. ∎

Proposition 29.14. *Let $\varphi\colon E \times F \to K$ be any sesquilinear form. Given any two subspaces U and V of E, we have*

$$(U + V)^\perp = U^\perp \cap V^\perp.$$

Furthermore, if φ is nondegenerate and if U and V are finite-dimensional, then

$$(U \cap V)^\perp = U^\perp + V^\perp.$$

Proof. If $w \in (U + V)^\perp$, then $\varphi(u + v, w) = 0$ for all $u \in U$ and all $v \in V$. In particular, with $v = 0$, we have $\varphi(u, w) = 0$ for all $u \in U$, and with $u = 0$, we have $\varphi(v, w) = 0$ for all $v \in V$, so $w \in U^\perp \cap V^\perp$. Conversely, if $w \in U^\perp \cap V^\perp$, then $\varphi(u, w) = 0$ for all $u \in U$ and $\varphi(v, w) = 0$ for all $v \in V$. By bilinearity, $\varphi(u + v, w) = \varphi(u, w) + \varphi(v, w) = 0$, which shows that $w \in (U + V)^\perp$. Therefore, the first identity holds.

Now, assume that φ is nondegenerate and that U and V are finite-dimensional, and let $W = U^\perp + V^\perp$. Using the equation that we just established and the fact that U and V are finite-dimensional, by Proposition 29.13, we get

$$W^\perp = U^{\perp\perp} \cap V^{\perp\perp} = U \cap V.$$

We can apply Proposition 29.12 to the restriction of φ to $U \times W$ (since $U^\perp \subseteq W$ and $W^\perp \subseteq U$), and we get

$$\dim(U/W^\perp) = \dim(U/(U \cap V)) = \dim(W/U^\perp).$$

If T is a supplement of $U^\perp$ in W so that $W = U^\perp \oplus T$ and if S is a supplement of W in E so that $E = W \oplus S$, then $\mathrm{codim}(W) = \dim(S)$, $\dim(T) = \dim(W/U^\perp)$, and we have the direct sum

$$E = U^\perp \oplus T \oplus S$$