Recall from Definition 30.10 that a *Euclidean ring* is an integral domain A such that there exists a function $\sigma\colon A \to \mathbb{N}$ with the following property: For all $a, b \in A$ with $b \neq 0$, there are some $q, r \in A$ such that

$$a = bq + r \quad \text{and} \quad \sigma(r) < \sigma(b).$$

Note that the pair (q, r) is not necessarily unique.

We make use of the elementary row and column operations $P(i,k)$, $E_{i,j;\beta}$, and $E_{i,\lambda}$ described in Chapter 8, where we require the scalar λ used in $E_{i,\lambda}$ to be a unit.

Theorem 36.18. *If M is an $m \times n$ matrix over a Euclidean ring A, then there exist some invertible $n \times n$ matrix P and some invertible $m \times m$ matrix Q, where P and Q are products of elementary matrices, and a $m \times n$ matrix D of the form*

$$D = \begin{pmatrix} \alpha_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \alpha_r & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}$$

for some nonzero $\alpha_i \in A$, such that

(1) $\alpha_1 \mid \alpha_2 \mid \cdots \mid \alpha_r$, and

(2) $M = QDP^{-1}$.

Proof. We follow Jacobson's proof [98] (Chapter 3, Theorem 3.8). We proceed by induction on $m + n$.

If $m = n = 1$, let $P = (1)$ and $Q = (1)$.

For the induction step, if $M = 0$, let $P = I_n$ and $Q = I_m$. If $M \neq 0$, the stategy is to apply a sequence of elementary transformations that converts M to a matrix of the form

$$M' = \begin{pmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & Y & \\ 0 & & & \end{pmatrix}$$

where Y is a $(m-1) \times (n-1)$-matrix such that α_1 divides every entry in Y. Then, we proceed by induction on Y. To find M', we perform the following steps.

Step 1. Pick some nonzero entry a_{ij} in M such that $\sigma(a_{ij})$ is minimal. Then permute column j and column 1, and permute row i and row 1, to bring this entry in position $(1,1)$. We denote this new matrix again by M.