where $\sigma_1, \ldots, \sigma_r$ are the singular values of A, i.e. the (positive) square roots of the nonzero eigenvalues of $A^\top A$ and $A\,A^\top$, and $\sigma_{r+1} = \ldots = \sigma_p = 0$, where $p = \min(m, n)$. The columns of U are eigenvectors of $A^\top A$, and the columns of V are eigenvectors of $A\,A^\top$.

Proof. As in the proof of Theorem 22.5, since $A^\top A$ is symmetric positive semidefinite, there exists an $n \times n$ orthogonal matrix U such that

$$A^\top A = U\Sigma^2 U^\top,$$

with $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_r, 0, \ldots, 0)$, where $\sigma_1^2, \ldots, \sigma_r^2$ are the nonzero eigenvalues of $A^\top A$, and where r is the rank of A. Observe that $r \leq \min\{m, n\}$, and AU is an $m \times n$ matrix. It follows that

$$U^\top A^\top AU = (AU)^\top AU = \Sigma^2,$$

and if we let $f_j \in \mathbb{R}^m$ be the jth column of AU for $j = 1, \ldots, n$, then we have

$$\langle f_i, f_j\rangle = \sigma_i^2 \delta_{ij}, \quad 1 \leq i, j \leq r$$

and

$$f_j = 0, \quad r + 1 \leq j \leq n.$$

If we define $(v_1, \ldots, v_r)$ by

$$v_j = \sigma_j^{-1} f_j, \quad 1 \leq j \leq r,$$

then we have

$$\langle v_i, v_j\rangle = \delta_{ij}, \quad 1 \leq i, j \leq r,$$

so complete $(v_1, \ldots, v_r)$ into an orthonormal basis $(v_1, \ldots, v_r, v_{r+1}, \ldots, v_m)$ (for example, using Gram–Schmidt).

Now since $f_j = \sigma_j v_j$ for $j = 1\ldots, r$, we have

$$\langle v_i, f_j\rangle = \sigma_j \langle v_i, v_j\rangle = \sigma_j \delta_{i,j}, \quad 1 \leq i \leq m,\ 1 \leq j \leq r$$

and since $f_j = 0$ for $j = r + 1, \ldots, n$, we have

$$\langle v_i, f_j\rangle = 0 \quad 1 \leq i \leq m,\ r + 1 \leq j \leq n.$$

If V is the matrix whose columns are $v_1, \ldots, v_m$, then V is an $m \times m$ orthogonal matrix and if $m \geq n$, we let

$$D = \begin{pmatrix} \Sigma \\ 0_{m-n} \end{pmatrix} = \begin{pmatrix} \sigma_1 & & \ldots & \\ & \sigma_2 & \ldots & \\ \vdots & \vdots & \ddots & \vdots \\ & & \ldots & \sigma_n \\ 0 & \vdots & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \vdots & \ldots & 0 \end{pmatrix},$$