Thus the problem of finding the eigenvalues of a symmetric (or Hermitian) matrix reduces to the problem of finding the eigenvalues of a symmetric (resp. Hermitian) tridiagonal matrix, and this can be done much more efficiently.

The second fact is that if H is an upper Hessenberg matrix and if it is diagonalizable, then there is an invertible matrix P such that $H = P\Lambda P^{-1}$ with Λ a diagonal matrix consisting of the eigenvalues of H, such that P^{-1} has an LU-decomposition; see Serre [156] (Theorem 13.3).

As a consequence, since any symmetric (or Hermitian) tridiagonal matrix is a block diagonal matrix of unreduced symmetric (resp. Hermitian) tridiagonal matrices, by Proposition 18.3, we see that the QR algorithm applied to a tridiagonal matrix which is symmetric (or Hermitian) positive definite converges to a diagonal matrix consisting of its eigenvalues. Let us record this important fact.

Theorem 18.4. *Let H be a symmetric (or Hermitian) positive definite tridiagonal matrix. If H is unreduced, then the QR algorithm converges to a diagonal matrix consisting of the eigenvalues of H.*

Since every symmetric (or Hermitian) positive definite matrix is similar to a tridiagonal symmetric (resp. Hermitian) positive definite matrix, we deduce that we have a method for finding the eigenvalues of a symmetric (resp. Hermitian) positive definite matrix (more accurately, to find approximations as good as we want for these eigenvalues).

If A is a symmetric (or Hermitian) matrix, since its eigenvalues are real, for some $\mu > 0$ large enough (pick $\mu > \rho(A)$), $A + \mu I$ is symmetric (resp. Hermitan) positive definite, so we can apply the QR algorithm to an upper Hessenberg matrix similar to $A + \mu I$ to find its eigenvalues, and then the eigenvalues of A are obtained by subtracting μ.

The problem of finding the eigenvalues of a symmetric matrix is discussed extensively in Parlett [135], one of the best references on this topic.

The upper Hessenberg form also yields a way to handle singular matrices. First, checking the proof of Proposition 14.21 that an $n \times n$ complex matrix A (possibly singular) can be factored as $A = QR$ where Q is a unitary matrix which is a product of Householder reflections and R is upper triangular, it is easy to see that if A is upper Hessenberg, then Q is also upper Hessenberg. If H is an unreduced upper Hessenberg matrix, since Q is upper Hessenberg and R is upper triangular, we have $h_{i+1i} = q_{i+1i}r_{ii}$ for $i = 1\ldots, n-1$, and since H is unreduced, $r_{ii} \neq 0$ for $i = 1, \ldots, n-1$. Consequently H is singular iff $r_{nn} = 0$. Then the matrix RQ is a matrix whose last row consists of zero's thus we can deflate the problem by considering the $(n-1) \times (n-1)$ unreduced Hessenberg matrix obtained by deleting the last row and the last column. After finitely many steps (not larger that the multiplicity of the eigenvalue 0), there remains an invertible unreduced Hessenberg matrix. As an alternative, see Serre [156] (Chapter 13, Section 13.3.2).

As is, the QR algorithm, although very simple, is quite inefficient for several reasons. In the next section, we indicate how to make the method more efficient. This involves a lot of work and we only discuss the main ideas at a high level.