we deduce that the subsequence $(\widetilde{R}_\ell)$ also converges to some matrix $\widetilde{R}$, which is also upper triangular with positive diagonal entries. By passing to the limit (using the subsequences), we get $\widetilde{R} = (\widetilde{Q})^*$, that is,

$$I = \widetilde{Q}\widetilde{R}.$$

By the uniqueness of a QR-decomposition (when the diagonal entries of R are positive), we get

$$\widetilde{Q} = \widetilde{R} = I.$$

Since the above reasoning applies to any subsequences of $(\widetilde{Q}_k)$ and $(\widetilde{R}_k)$, by the uniqueness of limits, we conclude that the "full" sequences $(\widetilde{Q}_k)$ and $(\widetilde{R}_k)$ converge:

$$\lim_{k\mapsto\infty} \widetilde{Q}_k = I, \quad \lim_{k\mapsto\infty} \widetilde{R}_k = I.$$

Since by $(*_4)$,

$$A^k = QR(\Lambda^k L \Lambda^{-k})\Lambda^k U,$$

by $(*_5)$,

$$R(\Lambda^k L \Lambda^{-k}) = (I + RF_k R^{-1})R,$$

and by $(*_6)$

$$I + RF_k R^{-1} = \widetilde{Q}_k \widetilde{R}_k,$$

we proved that

$$A^k = (Q\widetilde{Q}_k)(\widetilde{R}_k R \Lambda^k U). \tag{$*_7$}$$

Observe that $Q\widetilde{Q}_k$ is a unitary matrix, and $\widetilde{R}_k R\Lambda^k U$ is an upper triangular matrix, as a product of upper triangular matrices. However, some entries in Λ may be negative, so we can't claim that $\widetilde{R}_k R\Lambda^k U$ has positive diagonal entries. Nevertheless, we have another QR-decomposition of A^k,

$$A^k = (Q\widetilde{Q}_k)(\widetilde{R}_k R \Lambda^k U) = P_k \mathcal{R}_k.$$

It is easy to prove that there is diagonal matrix D_k with $|(D_k)_{ii}| = 1$ for $i = 1, \ldots, n$, such that

$$P_k = Q\widetilde{Q}_k D_k. \tag{$*_8$}$$

The existence of D_k is consequence of the following fact: If an invertible matrix B has two QR factorizations $B = Q_1R_1 = Q_2R_2$, then there is a diagonal matrix D with unit entries such that $Q_2 = DQ_1$.

The expression for P_k in $(*_8)$ is that which we were seeking.

Step 3. Asymptotic behavior of the matrices $A_{k+1} = P_k^* A P_k$.

Since $A = P\Lambda P^{-1} = QR\Lambda R^{-1}Q^{-1}$ and by $(*_8)$, $P_k = Q\widetilde{Q}_k D_k$, we get

$$A_{k+1} = D_k^*(\widetilde{Q}_k)^* Q^* QR\Lambda R^{-1} Q^{-1} Q\widetilde{Q}_k D_k = D_k^*(\widetilde{Q}_k)^* R\Lambda R^{-1}\widetilde{Q}_k D_k. \tag{$*_9$}$$