(c) If U is any subspace supplementary to $\mathrm{rad}(E)$*, so that*

$$E = \mathrm{rad}(E) \oplus U,$$

then U is nondegenerate, and $\mathrm{rad}(E)$ *and U are orthogonal.*

Proof. (a) If U and V are orthogonal, then $U \subseteq V^\perp$ and $V \subseteq U^\perp$. We get

$$\begin{aligned} \mathrm{rad}(U+V) &= (U+V) \cap (U+V)^\perp \\ &= (U+V) \cap U^\perp \cap V^\perp \\ &= U \cap U^\perp \cap V^\perp + V \cap U^\perp \cap V^\perp \\ &= U \cap U^\perp + V \cap V^\perp \\ &= \mathrm{rad}(U) + \mathrm{rad}(V). \end{aligned}$$

(b) By definition, $\mathrm{rad}(E) = E^\perp$, and obviously $E = E^{\perp\perp}$, so we get

$$\mathrm{rad}(\mathrm{rad}(E)) = E^\perp \cap E^{\perp\perp} = E^\perp \cap E = E^\perp = \mathrm{rad}(E).$$

(c) If $E = \mathrm{rad}(E) \oplus U$, by definition of $\mathrm{rad}(E)$, the subspaces $\mathrm{rad}(E)$ and U are orthogonal. From (a) and (b), we get

$$\mathrm{rad}(E) = \mathrm{rad}(E) + \mathrm{rad}(U).$$

Since $\mathrm{rad}(U) = U \cap U^\perp \subseteq U$ and since $\mathrm{rad}(E) \oplus U$ is a direct sum, we have a direct sum

$$\mathrm{rad}(E) = \mathrm{rad}(E) \oplus \mathrm{rad}(U),$$

which implies that $\mathrm{rad}(U) = (0)$; that is, U is nondegenerate. $\square$

Proposition 29.19(c) shows that the restriction of φ to any supplement U of $\mathrm{rad}(E)$ is nondegenerate and φ is zero on $\mathrm{rad}(U)$, so we may restrict our attention to nondegenerate forms.

The following is also a key result.

Proposition 29.20. *Given an ϵ-Hermitian form $\varphi\colon E \times E \to K$ on E, if U is a finite-dimensional nondegenerate subspace of E, then $E = U \oplus U^\perp$.*

Proof. By hypothesis, the restriction φ_U of φ to U is nondegenerate, so the semilinear map $r_{\varphi_U}\colon U \to U^*$ is injective. Since U is finite-dimensional, r_{φ_U} is actually bijective, so for every $v \in E$, if we consider the linear form in U^* given by $u \mapsto \varphi(u, v)$ $(u \in U)$, there is a unique $v_0 \in U$ such that

$$\varphi(u, v_0) = \varphi(u, v) \quad \text{for all } u \in U;$$

that is, $\varphi(u, v - v_0) = 0$ for all $u \in U$, so $v - v_0 \in U^\perp$. It follows that $v = v_0 + v - v_0$, with $v_0 \in U$ and $v_0 - v \in U^\perp$, and since U is nondegenerate $U \cap U^\perp = (0)$, and $E = U \oplus U^\perp$. $\square$

As a corollary of Proposition 29.20, we get the following result.