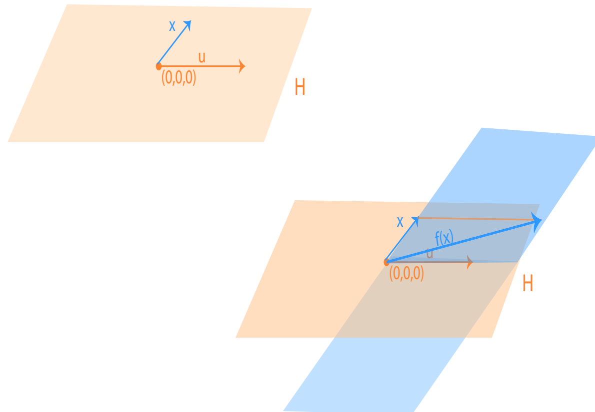

Figure 26.35: A transvection $\tau_{\varphi,u}$ of the xy-plane in direction $u = (0, 1, 0)$, where $\varphi(x, y, z) = z$. Every vector x not in the xy-plane determines a light-blue plane through x and u. The image $f(x)$ stays in the light-blue hyperplane since it is "stretched" in the u direction by a factor of $\varphi(x, y, z)$.

Proposition 26.23, which we repeat here for the convenience of the reader, characterizes the linear isomorphisms $f \neq \mathrm{id}$ that leave every point in the hyperplane H fixed.

Proposition 26.23. *Let $f : E \to E$ be a bijective linear map of a finite-dimensional vector space E and assume that $f \neq \mathrm{id}$ and that $f(x) = x$ for all $x \in H$, where H is some hyperplane in E. If $\det(f) = 1$, then f is a transvection of hyperplane H; otherwise, f is a dilatation of hyperplane H. In either case, the vector u is uniquely defined up to a nonzero scalar.*

Proof. Only the last part was not proved in Proposition 8.23, Since f is bijective and the identity on H, the linear map $f - \mathrm{id}$ has kernel exactly H. Since H is a hyperplane in E, the image of $f - \mathrm{id}$ has dimension 1, and since u belong to this image, it is uniquely defined up to a nonzero scalar. ∎

The proof of Proposition 8.23 shows that if $\dim(E) = n + 1$ and if f is a dilatation of hyperplane H, direction $D = Ku$, and scale α, then 1 is an eigenvalue of f with multiplicity n and $\alpha \neq 0, 1$ is an eigenvalue of f with multiplicity 1; the vector u is an eigenvector for α, and f is diagonalizable. If f is a transvection of hyperplane H and direction u, then 1 is the only eigenvalue of f, and it has multiplicity n; the vector u is an eigenvector for 1, and f is not diagonalizable.

A homology is the projective version of the type of maps involved in Proposition 26.23.

Definition 26.11. For any vector space E and any hyperplane H in E, a homography $h : \mathbb{P}(E) \to \mathbb{P}(E)$ is a *homology of axis (or base)* $\mathbb{P}(H)$ if $h(P) = P$ for all $P \in \mathbb{P}(H)$. In other words, the restriction of h to $\mathbb{P}(H)$ is the identity. More explicitly, if $h = \mathbb{P}(f)$ for some linear isomorphism $f : E \to E$, we have $\mathbb{P}(f)(P) = P$ for all points $P = [u] \in \mathbb{P}(H)$.