The domain of f^* can be very small, even if the domain of f is big. For example, if $f\colon \mathbb{R} \to \mathbb{R}$ is the affine function given by $f(x) = ax + b$ (with $a, b \in \mathbb{R}$), then the function $x \mapsto yx - ax - b$ is unbounded above unless $y = a$, so

$$f^*(y) = \begin{cases} -b & \text{if } y = a \\ +\infty & \text{otherwise.} \end{cases}$$

The domain of f^* can also be bigger than the domain of f; see Example 50.8(3).

The conjugates of many functions that come up in optimization are derived in Boyd and Vandenberghe; see [29], Section 3.3. We mention a few that will be used in this chapter.

Example 50.8.

(1) *Negative logarithm*: $f(x) = -\log x$, with $\mathrm{dom}(f) = \{x \in \mathbb{R} \mid x > 0\}$. The function $x \mapsto yx + \log x$ is unbounded above if $y \geq 0$, and when $y < 0$, its maximum is obtained iff its derivative is zero, namely
$$y + \frac{1}{x} = 0.$$
Substituting for $x = -1/y$ in $yx + \log x$, we obtain $-1 + \log(-1/y) = -1 - \log(-y)$, so we have
$$f^*(y) = -\log(-y) - 1,$$
with $\mathrm{dom}(f^*) = \{y \in \mathbb{R} \mid y < 0\}$.

(2) *Exponential*: $f(x) = e^x$, with $\mathrm{dom}(f) = \mathbb{R}$. The function $x \mapsto yx - e^x$ is unbounded if $y < 0$. When $y > 0$, it reaches a maximum iff its derivative is zero, namely
$$y - e^x = 0.$$
Substituting for $x = \log y$ in $yx - e^x$, we obtain $y \log y - y$, so we have
$$f^*(y) = y \log y - y,$$
with $\mathrm{dom}(f^*) = \{y \in \mathbb{R} \mid y \geq 0\}$, with the convention that $0 \log 0 = 0$.

(3) *Negative Entropy*: $f(x) = x \log x$, with $\mathrm{dom}(f) = \{x \in \mathbb{R} \mid x \geq 0\}$, with the convention that $0 \log 0 = 0$. The function $x \mapsto yx - x \log x$ is bounded above for all $y > 0$, and it attains its maximum when its derivative is zero, namely
$$y - \log x - 1 = 0.$$
Substituting for $x = e^{y-1}$ in $yx - x \log x$, we obtain $ye^{y-1} - e^{y-1}(y - 1) = e^{y-1}$, which yields
$$f^*(y) = e^{y-1},$$
with $\mathrm{dom}(f^*) = \mathbb{R}$.