

# Chapter 3

## Vector Spaces, Bases, Linear Maps

### 3.1 Motivations: Linear Combinations, Linear Independence and Rank

In linear optimization problems, we often encounter systems of linear equations. For example, consider the problem of solving the following system of three linear equations in the three variables  $x_1, x_2, x_3 \in \mathbb{R}$ :

$$\begin{aligned}x_1 + 2x_2 - x_3 &= 1 \\2x_1 + x_2 + x_3 &= 2 \\x_1 - 2x_2 - 2x_3 &= 3.\end{aligned}$$

One way to approach this problem is introduce the “vectors”  $u, v, w$ , and  $b$ , given by

$$u = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad v = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \quad w = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

and to write our linear system as

$$x_1 u + x_2 v + x_3 w = b.$$

In the above equation, we used implicitly the fact that a vector  $z$  can be multiplied by a scalar  $\lambda \in \mathbb{R}$ , where

$$\lambda z = \lambda \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} \lambda z_1 \\ \lambda z_2 \\ \lambda z_3 \end{pmatrix},$$

and two vectors  $y$  and  $z$  can be added, where

$$y + z = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} y_1 + z_1 \\ y_2 + z_2 \\ y_3 + z_3 \end{pmatrix}.$$