where τ and the τ_i are hyperplane reflections, with $k \geq 2n-3$, and we get a total of $2n-2$ hyperplane reflections.

Case 3. $f(u) \neq u$ and $f(u) \neq -u$.

Note that $f(u)-u$ and $f(u)+u$ are orthogonal, since

$$\begin{aligned}\varphi(f(u)-u, f(u)+u) &= \varphi(f(u), f(u)) + \varphi(f(u), u) - \varphi(u, f(u)) - \varphi(u, u) \\ &= \varphi(u, u) - \varphi(u, u) = 0.\end{aligned}$$

We also have

$$\begin{aligned}\varphi(u, u) &= \varphi((f(u)+u-(f(u)-u))/2, (f(u)+u-(f(u)-u))/2) \\ &= \frac{1}{4}\varphi(f(u)+u, f(u)+u) + \frac{1}{4}\varphi(f(u)-u, f(u)-u),\end{aligned}$$

so $f(u)+u$ and $f(u)-u$ cannot be both isotropic, since u is not isotropic.

If $f(u)-u$ is not isotropic, then the reflection $\tau_{f(u)-u}$ is such that

$$\tau_{f(u)-u}(u) = f(u),$$

and since $\tau_{f(u)-u}^2 = \mathrm{id}$, if $g = \tau_{f(u)-u} \circ f$, then $g(u) = u$, and we are back to case (1). We obtain

$$f = \tau_{f(u)-u} \circ \tau_k \circ \cdots \circ \tau_1$$

where $\tau_{f(u)-u}$ and the τ_i are hyperplane reflections, with $k \geq 2n-3$, and we get a total of $2n-2$ hyperplane reflections.

If $f(u)+u$ is not isotropic, then the reflection $\tau_{f(u)+u}$ is such that

$$\tau_{f(u)+u}(u) = -f(u),$$

and since $\tau_{f(u)+u}^2 = \mathrm{id}$, if $g = \tau_{f(u)+u} \circ f$, then $g(u) = -u$, and we are back to case (2). We obtain

$$f = \tau_{f(u)-u} \circ \tau \circ \tau_k \circ \cdots \circ \tau_1$$

where $\tau, \tau_{f(u)-u}$ and the τ_i are hyperplane reflections, with $k \geq 2n-3$, and we get a total of $2n-1$ hyperplane reflections. This proves the induction step. $\square$

The bound $2n-1$ is not optimal. The strong version of the Cartan–Dieudonné theorem says that at most n reflections are needed, but the proof is harder. Here is a neat proof due to E. Artin (see [6], Chapter III, Section 4).

Case 1 remains unchanged. Case 2 is slightly different: $f(u)-u \neq 0$ is not isotropic. Since $\varphi(f(u)+u, f(u)-u) = 0$, as in the first subcase of Case (3), $g = \tau_{f(u)-u} \circ f$ is such that $g(u) = u$ and we are back to Case 1. This only costs one more reflection.

The new (bad) case is: