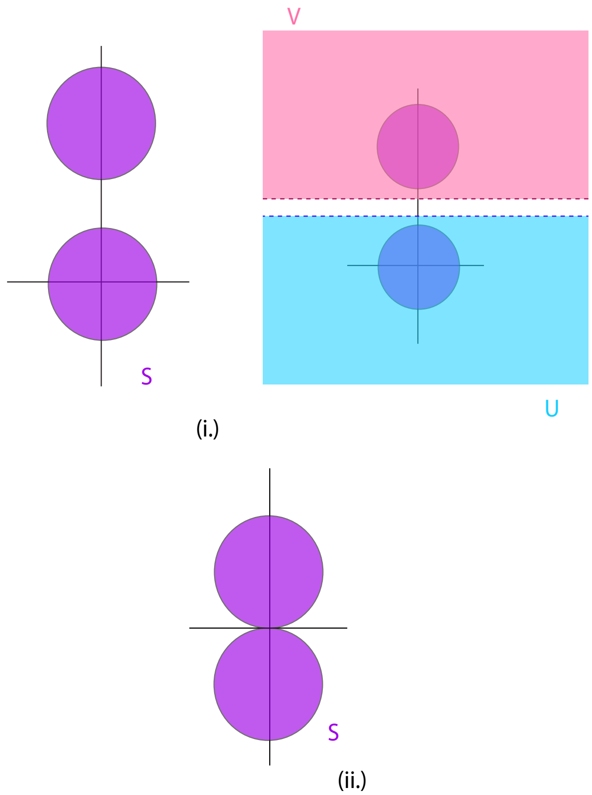

Figure 37.22: Figure (i) shows that the union of two disjoint disks in $\mathbb{R}^2$ is a disconnected set since each circle can be separated by open half regions. Figure (ii) is an example of a connected subset of $\mathbb{R}^2$ since the two disks can not separated by open sets.

Conversely, we show that an interval, I, must be connected. Let A be any nonempty subset of I which is both open and closed in I. We show that $I = A$. Fix any $x \in A$ and consider the set, R_x, of all y such that $[x, y] \subseteq A$. If the set R_x is unbounded, then $R_x = [x, +\infty)$. Otherwise, if this set is bounded, let b be its least upper bound. We claim that b is the right boundary of the interval I. Because A is closed in I, unless I is open on the right and b is its right boundary, we must have $b \in A$. In the first case, $A \cap [x, b) = I \cap [x, b) = [x, b)$. In the second case, because A is also open in I, unless b is the right boundary of the interval I (closed on the right), there is some open set $(b - \eta, b + \eta)$ contained in A, which implies that $[x, b + \eta/2] \subseteq A$, contradicting the fact that b is the least upper bound of the set R_x. Thus, b must be the right boundary of the interval I (closed on the right). A similar argument applies to the set, L_y, of all x such that $[x, y] \subseteq A$ and either L_y is unbounded, or its greatest lower bound a is the left boundary of I (open or closed on the left). In all cases, we showed that $A = I$, and the interval must be connected. $\square$

Intuitively, if a space is not connected, it is possible to define a continuous function which