Definition 37.29. A family $(F_i)_{i\in I}$ of sets has the *finite intersection property* if $\bigcap_{j\in J} F_j \neq \emptyset$ for every finite subset J of I.

Proposition 37.24. *A topological Hausdorff space E is compact iff for every family $(F_i)_{i\in I}$ of closed sets having the finite intersection property, then $\bigcap_{i\in I} F_i \neq \emptyset$.*

Proof. If E is compact and $(F_i)_{i\in I}$ is a family of closed sets having the finite intersection property, then $\bigcap_{i\in I} F_i$ cannot be empty, since otherwise we would have $\bigcap_{j\in J} F_j = \emptyset$ for some finite subset J of I, a contradiction. The converse is equally obvious. $\square$

Another useful consequence of compactness is as follows. For any family $(F_i)_{i\in I}$ of closed sets such that $F_{i+1} \subseteq F_i$ for all $i \in I$, if $\bigcap_{i\in I} F_i = \emptyset$, then $F_i = \emptyset$ for some $i \in I$. Indeed, there must be some finite subset J of I such that $\bigcap_{j\in J} F_j = \emptyset$, and since $F_{i+1} \subseteq F_i$ for all $i \in I$, we must have $F_j = \emptyset$ for the smallest F_j in $(F_j)_{j\in J}$. Using this fact, we note that $\mathbb{R}$ is *not* compact. Indeed, the family of closed sets, $([n, +\infty))_{n\geq 0}$, is decreasing and has an empty intersection.

It is immediately verified that every finite union of compact subsets is compact. Similarly, every finite union of relatively compact subsets is relatively compact (use the fact that $\overline{A \cup B} = \overline{A} \cap \overline{B}$).

Given a metric space, if we define a *bounded subset* to be a subset that can be enclosed in some closed ball (of finite radius), then any nonbounded subset of a metric space is not compact. However, a closed interval $[a, b]$ of the real line is compact.

Proposition 37.25. *Every closed interval, $[a, b]$, of the real line is compact.*

Proof. We proceed by contradiction. Let $(U_i)_{i\in I}$ be any open cover of $[a, b]$ and assume that there is no finite open subcover. Let $c = (a + b)/2$. If both $[a, c]$ and $[c, b]$ had some finite open subcover, so would $[a, b]$, and thus, either $[a, c]$ does not have any finite subcover, or $[c, b]$ does not have any finite open subcover. Let $[a_1, b_1]$ be such a bad subinterval. The same argument applies and we split $[a_1, b_1]$ into two equal subintervals, one of which must be bad. Thus, having defined $[a_n, b_n]$ of length $(b - a)/2^n$ as an interval having no finite open subcover, splitting $[a_n, b_n]$ into two equal intervals, we know that at least one of the two has no finite open subcover and we denote such a bad interval by $[a_{n+1}, b_{n+1}]$. See Figure 37.29. The sequence (a_n) is nondecreasing and bounded from above by b, and thus, by a fundamental property of the real line, it converges to its least upper bound, α. Similarly, the sequence (b_n) is nonincreasing and bounded from below by a and thus, it converges to its greatest lowest bound, β. Since $[a_n, b_n]$ has length $(b-a)/2^n$, we must have $\alpha = \beta$. However, the common limit $\alpha = \beta$ of the sequences (a_n) and (b_n) must belong to some open set, U_i, of the open cover and since U_i is open, it must contain some interval $[c, d]$ containing α. Then, because α is the common limit of the sequences (a_n) and (b_n), there is some N such that the intervals $[a_n, b_n]$ are all contained in the interval $[c, d]$ for all $n \geq N$, which contradicts the fact that none of the intervals $[a_n, b_n]$ has a finite open subcover. Thus, $[a, b]$ is indeed compact. $\square$