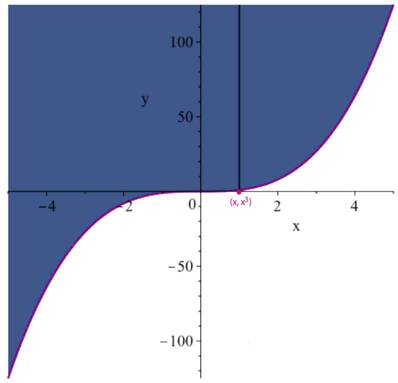

Figure 51.1: Let $f\colon \mathbb{R} \to \mathbb{R} \cup \{-\infty, +\infty\}$ be given by $f(x) = x^3$ for $x \in \mathbb{R}$. Its graph in $\mathbb{R}^2$ is the magenta curve, and its epigraph is the union of the magenta curve and blue region "above" this curve. For any point $x \in \mathbb{R}$, $\mathbf{epi}(f)$ contains the ray which starts at (x, x^3) and extends upward.

Definition 51.3. Given a nonempty subset S of $\mathbb{R}^n$, a (total) function $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$ is *convex on* S if its epigraph $\mathbf{epi}(f|S)$ is convex as a subset of $\mathbb{R}^{n+1}$. See Figure 51.2. The function f is *concave* on S if $-f$ is convex on S. The function f is *affine* on S if it is finite, convex, and concave. If $S = \mathbb{R}^n$, we simply that f is *convex* (resp. *concave*, resp. *affine*).

Definition 51.4. Given any function $f\colon \mathbb{R}^n \to \mathbb{R}\cup\{-\infty, +\infty\}$, the *effective domain* $\mathrm{dom}(f)$ of f is given by

$$\mathrm{dom}(f) = \{x \in \mathbb{R}^n \mid (\exists y \in \mathbb{R})((x, y) \in \mathbf{epi}(f))\} = \{x \in \mathbb{R}^n \mid f(x) < +\infty\}.$$

Observe that the effective domain of f contains the vectors $x \in \mathbb{R}^n$ such that $f(x) = -\infty$, but excludes the vectors $x \in \mathbb{R}^n$ such that $f(x) = +\infty$.

Example 51.1. The above fact is illustrated by the function $f\colon \mathbb{R} \to \mathbb{R}\cup\{-\infty, +\infty\}$ where

$$f(x) = \begin{cases} -x^2 & \text{if } x \geq 0 \\ +\infty & \text{if } x < 0. \end{cases}$$

The epigraph of this function is illustrated Figure 51.3. By definition $\mathrm{dom}(f) = [0, \infty)$.

If f is a convex function, since $\mathrm{dom}(f)$ is the image of $\mathbf{epi}(f)$ by a linear map (a projection), *it is convex*.

By definition, $\mathbf{epi}(f|S)$ is convex iff for any (x_1, y_1) and (x_2, y_2) with $x_1, x_2 \in S$ and $y_1, y_2 \in \mathbb{R}$ such that $f(x_1) \leq y_1$ and $f(x_2) \leq y_2$, for every λ such that $0 \leq \lambda \leq 1$, we have

$$(1 - \lambda)(x_1, y_1) + \lambda(x_2, y_2) = ((1 - \lambda)x_1 + \lambda x_2, (1 - \lambda)y_1 + \lambda y_2) \in \mathbf{epi}(f|S),$$