so the image of AA^+ is indeed the range of A. It is also clear that $\mathrm{Ker}(A) \subseteq \mathrm{Ker}(A^+A)$, and since $AA^+A = A$, we also have $\mathrm{Ker}(A^+A) \subseteq \mathrm{Ker}(A)$, and so

$$\mathrm{Ker}(A^+A) = \mathrm{Ker}(A).$$

Since A^+A is symmetric, $\mathrm{range}(A^+A) = \mathrm{range}((A^+A)^\top) = \mathrm{Ker}(A^+A)^\perp = \mathrm{Ker}(A)^\perp$, as claimed. $\square$

Proposition 23.5. *The set* $\mathrm{range}(A) = \mathrm{range}(AA^+)$ *consists of all vectors* $y \in \mathbb{R}^m$ *such that*

$$V^\top y = \begin{pmatrix} z \\ 0 \end{pmatrix},$$

with $z \in \mathbb{R}^r$.

Proof. Indeed, if $y = Ax$, then

$$V^\top y = V^\top Ax = V^\top V\Sigma U^\top x = \Sigma U^\top x = \begin{pmatrix} \Sigma_r & 0 \\ 0 & 0_{m-r} \end{pmatrix} U^\top x = \begin{pmatrix} z \\ 0 \end{pmatrix},$$

where Σ_r is the $r \times r$ diagonal matrix $\mathrm{diag}(\sigma_1, \ldots, \sigma_r)$. Conversely, if $V^\top y = \left(\begin{smallmatrix} z \\ 0 \end{smallmatrix}\right)$, then $y = V\left(\begin{smallmatrix} z \\ 0 \end{smallmatrix}\right)$, and

$$\begin{aligned} AA^+y &= V\begin{pmatrix} I_r & 0 \\ 0 & 0_{m-r} \end{pmatrix} V^\top y \\ &= V\begin{pmatrix} I_r & 0 \\ 0 & 0_{m-r} \end{pmatrix} V^\top V\begin{pmatrix} z \\ 0 \end{pmatrix} \\ &= V\begin{pmatrix} I_r & 0 \\ 0 & 0_{m-r} \end{pmatrix}\begin{pmatrix} z \\ 0 \end{pmatrix} \\ &= V\begin{pmatrix} z \\ 0 \end{pmatrix} = y, \end{aligned}$$

which shows that y belongs to the range of A. $\square$

Similarly, we have the following result.

Proposition 23.6. *The set* $\mathrm{range}(A^+A) = \mathrm{Ker}(A)^\perp$ *consists of all vectors* $y \in \mathbb{R}^n$ *such that*

$$U^\top y = \begin{pmatrix} z \\ 0 \end{pmatrix},$$

with $z \in \mathbb{R}^r$.