Chapter 21

Spectral Graph Drawing

21.1 Graph Drawing and Energy Minimization

Let $G = (V, E)$ be some undirected graph. It is often desirable to draw a graph, usually in the plane but possibly in 3D, and it turns out that the graph Laplacian can be used to design surprisingly good methods. Say $|V| = m$. The idea is to assign a point $\rho(v_i)$ in $\mathbb{R}^n$ to the vertex $v_i \in V$, for every $v_i \in V$, and to draw a line segment between the points $\rho(v_i)$ and $\rho(v_j)$ iff there is an edge $\{v_i, v_j\}$.

Definition 21.1. Let $G = (V, E)$ be some undirected graph with m vertices. A *graph drawing* is a function $\rho \colon V \to \mathbb{R}^n$, for some $n \geq 1$. The *matrix of a graph drawing ρ (in $\mathbb{R}^n$)* is a $m \times n$ matrix R whose ith row consists of the row vector $\rho(v_i)$ corresponding to the point representing v_i in $\mathbb{R}^n$.

For a graph drawing to be useful we want $n \leq m$; in fact n should be much smaller than m, typically $n = 2$ or $n = 3$.

Definition 21.2. A graph drawing is *balanced* iff the sum of the entries of every column of the matrix of the graph drawing is zero, that is,

$$\mathbf{1}^\top R = 0.$$

If a graph drawing is not balanced, it can be made balanced by a suitable translation. We may also assume that the columns of R are linearly independent, since any basis of the column space also determines the drawing. Thus, from now on, we may assume that $n \leq m$.

Remark: A graph drawing $\rho \colon V \to \mathbb{R}^n$ is not required to be injective, which may result in degenerate drawings where distinct vertices are drawn as the same point. For this reason, we prefer not to use the terminology *graph embedding*, which is often used in the literature. This is because in differential geometry, an embedding always refers to an injective map. The term *graph immersion* would be more appropriate.