Proof. (1) Consider the binary relation $\simeq$ on $A \times (A - \{0\})$ defined as follows:

$$(a, b) \simeq (a', b') \quad \text{iff} \quad ab' = a'b.$$

It is easily seen that $\simeq$ is an equivalence relation. Note that the fact that A is an integral domain is used to prove transitivity. The equivalence class of (a, b) is denoted by a/b. Clearly, $(0, b) \simeq (0, 1)$ for all $b \in A$, and we denote the class of $(0, 1)$ also by 0. The equivalence class $a/1$ of $(a, 1)$ is also denoted by a. We define addition and multiplication on $A \times (A - \{0\})$ as follows:

$$\begin{aligned}(a, b) + (a', b') &= (ab' + a'b, bb'),\\ (a, b) \cdot (a', b') &= (aa', bb').\end{aligned}$$

It is easily verified that $\simeq$ is congruential w.r.t. $+$ and $\cdot$, which means that $+$ and $\cdot$ are well-defined on equivalence classes modulo $\simeq$. When $a, b \neq 0$, the inverse of a/b is b/a, and it is easily verified that F is a field. The map $i \colon A \to F$ defined such that $i(a) = a/1$ is an injection of A into F and clearly

$$\frac{a}{b} = i(a)i(b)^{-1}.$$

(2) Given an injective ring homomorphism $h \colon A \to K$ into a field K,

$$\frac{a}{b} = \frac{a'}{b'} \quad \text{iff} \quad ab' = a'b,$$

which implies that

$$h(a)h(b') = h(a')h(b),$$

and since h is injective and $b, b' \neq 0$, we get

$$h(a)h(b)^{-1} = h(a')h(b')^{-1}.$$

Thus, there is a map $\widehat{h} \colon F \to K$ such that

$$\widehat{h}(a/b) = \widehat{h}(i(a)i(b)^{-1}) = h(a)h(b)^{-1}$$

for all $a, b \in A$, $b \neq 0$, and it is easily checked that $\widehat{h}$ is a field homomorphism. The map $\widehat{h}$ is clearly unique.

(3) The uniqueness of F up to isomorphism follows from (2), and is left as an exercise. $\square$

The field F given by Proposition 32.7 is called the *fraction field of* A, and it is denoted by $\mathrm{Frac}(A)$.

In particular, given an integral domain A, since $A[X_1, \ldots, X_n]$ is also an integral domain, we can form the fraction field of the polynomial ring $A[X_1, \ldots, X_n]$, denoted by $F(X_1, \ldots, X_n)$, where $F = \mathrm{Frac}(A)$ is the fraction field of A. It is also called the field