Using $(*_2)$, we also have

$$\|f_m(u)\| \leq \|f_m\| \, \|u\| \leq (\|f_n\| + \epsilon) \, \|u\| \quad \text{for all } m, n \geq N,$$

that is,

$$\|f_m(u)\| \leq (\|f_n\| + \epsilon) \, \|u\| \quad \text{for all } m, n \geq N. \tag{$*_3$}$$

Hold $n \geq N$ fixed and let m tend to $+\infty$ in $(*_3)$. Since the norm is continuous, we get

$$\|f(u)\| \leq (\|f_n\| + \epsilon) \, \|u\| \, ,$$

which shows that f is continuous.

Step 4. The function f is the limit of (f_n) for the operator norm.

Recall $(*_1)$:

$$\|f_m(u) - f_n(u)\| \leq \epsilon \, \|u\| \quad \text{for all } m, n \geq N. \tag{$*_1$}$$

Hold $n \geq N$ fixed but this time let m tend to $+\infty$ in $(*_1)$. By continuity of the norm we get

$$\|f(u) - f_n(u)\| = \|(f - f_n)(u)\| \leq \epsilon \, \|u\| \, .$$

By definition of the operator norm,

$$\|f - f_n\| = \sup\{\|(f - f_n)(u)\| \mid \|u\| = 1\} \leq \epsilon \quad \text{for all } n \geq N,$$

which proves that f_n converges to f for the operator norm. $\square$

As a special case of Theorem 37.62, if we let $F = \mathbb{R}$ (or $F = \mathbb{C}$ in the case of complex vector spaces) we see that $E' = \mathcal{L}(E; \mathbb{R})$ (or $E' = \mathcal{L}(E; \mathbb{C})$) is complete (since $\mathbb{R}$ and $\mathbb{C}$ are complete). The space E' of continuous linear forms on E is called the *dual* of E. It is a subspace of the *algebraic dual* E^* of E which consists of *all* linear forms on E, not necessarily continuous.

It can also be shown that if E, F and G are normed vector spaces, and if G is a Banach space, then $\mathcal{L}_2(E, F; G)$ is a Banach space. The proof is essentially identical.

37.12 Completion of a Normed Vector Space

An easy corollary of Theorem 37.53 and Theorem 37.52 is that every normed vector space can be embedded in a complete normed vector space, that is, a Banach space.

Theorem 37.63. *If $(E, \| \, \|)$ is a normed vector space, then its completion $(\widehat{E}, \widehat{d})$ as a metric space (where E is given the metric $d(x, y) = \|x - y\|$) can be given a unique vector space structure extending the vector space structure on E, and a norm $\| \, \|_{\widehat{E}}$, so that $(\widehat{E}, \| \, \|_{\widehat{E}})$ is a Banach space, and the metric $\widehat{d}$ is associated with the norm $\| \, \|_{\widehat{E}}$. Furthermore, the isometry $\varphi \colon E \to \widehat{E}$ is a linear isometry.*