The above kernel is called the *intersection kernel*. If we assume that μ is normalized so that $\mu(D) = 1$, then we also have the *union complement kernel*:

$$\kappa_2(A_1, A_2) = \mu(\overline{A_1} \cap \overline{A_2}) = 1 - \mu(A_1 \cup A_2).$$

The sum κ_3 of the kernels κ_1 and κ_2 is the *agreement kernel*:

$$\kappa_s(A_1, A_2) = 1 - \mu(A_1 - A_2) - \mu(A_2 - A_1).$$

Many other kinds of kernels can be designed, in particular, graph kernels. For comprehensive presentations of kernels, see Schölkopf and Smola [145] and Shawe–Taylor and Christianini [159].

Kernel functions have the following important property.

Proposition 53.1. *Let X be any nonempty set, let H be any (complex) Hilbert space, let $\varphi\colon X \to H$ be any function, and let $\kappa\colon X \times X \to \mathbb{C}$ be the kernel given by*

$$\kappa(x, y) = \langle \varphi(x), \varphi(y) \rangle, \qquad x, y \in X.$$

For any finite subset $S = \{x_1, \ldots, x_p\}$ of X, if K_S is the $p \times p$ matrix

$$K_S = (\kappa(x_j, x_i))_{1 \leq i,j \leq p} = (\langle \varphi(x_j), \varphi(x_i) \rangle)_{1 \leq i,j \leq p},$$

then we have

$$u^* K_S\, u \geq 0, \qquad \text{for all } u \in \mathbb{C}^p.$$

Proof. We have

$$\begin{aligned} u^* K_S\, u = u^\top K_S^\top\, \overline{u} &= \sum_{i,j=1}^p \kappa(x_i, x_j) u_i \overline{u_j} \\ &= \sum_{i,j=1}^p \langle \varphi(x), \varphi(y) \rangle u_i \overline{u_j} \\ &= \left\langle \sum_{i=1}^p u_i \varphi(x_i), \sum_{j=1}^p u_j \varphi(x_j) \right\rangle = \left\| \sum_{i=1}^p u_i \varphi(x_i) \right\|^2 \geq 0, \end{aligned}$$

as claimed. ∎