Lemma 7.4 can be reformulated nicely as follows.

Proposition 7.8. *Let $f\colon E \times \ldots \times E \to F$ be an n-linear alternating map. Let $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ be two families of n vectors, such that*

$$\begin{aligned} v_1 &= a_{1\,1} u_1 + \cdots + a_{1\,n} u_n, \\ &\cdots \\ v_n &= a_{n\,1} u_1 + \cdots + a_{n\,n} u_n. \end{aligned}$$

Equivalently, letting

$$A = \begin{pmatrix} a_{1\,1} & a_{1\,2} & \ldots & a_{1\,n} \\ a_{2\,1} & a_{2\,2} & \ldots & a_{2\,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n\,1} & a_{n\,2} & \ldots & a_{n\,n} \end{pmatrix},$$

assume that we have

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = A \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}.$$

Then,

$$f(v_1, \ldots, v_n) = \det(A) f(u_1, \ldots, u_n).$$

Proof. The only difference with Lemma 7.4 is that here we are using $A^\top$ instead of A. Thus, by Lemma 7.4 and Corollary 7.7, we get the desired result. □

As a consequence, we get the very useful property that the determinant of a product of matrices is the product of the determinants of these matrices.

Proposition 7.9. *For any two $n \times n$-matrices A and B, we have* $\det(AB) = \det(A)\det(B)$.

Proof. We use Proposition 7.8 as follows: let $(e_1, \ldots, e_n)$ be the standard basis of K^n, and let

$$\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = AB \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix}.$$

Then we get

$$\det(w_1, \ldots, w_n) = \det(AB)\det(e_1, \ldots, e_n) = \det(AB),$$