Proposition 29.31. *Any two ϵ-Hermitian neutral forms satisfying property (T) defined on spaces of the same dimension are equivalent.*

The following proposition shows that every subspace U of E can be embedded into a nondegenerate subspace. It is needed to prove a version of the Witt extension theorem (Theorem 29.48).

Proposition 29.32. *Let φ be an ϵ-Hermitian form on E which is nondegenerate and satisfies property (T). For any subspace U of E of finite dimension, if we write*

$$U = V \overset{\perp}{\oplus} W,$$

for some orthogonal complement W of $V = \mathrm{rad}(U)$, and if we let $r = \dim(\mathrm{rad}(U))$, then there exists a totally isotropic subspace V' of dimension r such that $V \cap V' = (0)$, and $(V \oplus V') \overset{\perp}{\oplus} W = V' \oplus U$ is nondegenerate. Furthermore, any isometry f from U into another space (E', φ') where φ' is an ϵ-Hermitian form satisfying the same assumptions as φ can be extended to an isometry on $(V \oplus V') \overset{\perp}{\oplus} W$.

Proof. Since W is nondegenerate, $W^\perp$ is also nondegenerate, and $V \subseteq W^\perp$. Therefore, we can apply Theorem 29.30 to the restriction of φ to $W^\perp$ and to V to obtain the required V'. We know that $V \oplus V'$ is nondegenerate and orthogonal to W, which is also nondegenerate, so $(V \oplus V') \overset{\perp}{\oplus} W = V' \oplus U$ is nondegenerate.

We leave the second statement about extending f as an exercise (use the fact that $f(U) = f(V) \overset{\perp}{\oplus} f(W)$, where $V_1 = f(V)$ is totally isotropic of dimension r, to find another totally isotropic susbpace V_1' of dimension r such that $V_1 \cap V_1' = (0)$ and $V_1 \oplus V_1'$ is orthogonal to $f(W)$). ∎

The subspace $(V \oplus V') \overset{\perp}{\oplus} W = V' \oplus U$ is often called a *nondegenerate completion* of U. The subspace $V \oplus V'$ is called an *Artinian space*. Proposition 29.29 show that $V \oplus V'$ has a basis $(u_1, v_1, \ldots, u_r, v_r)$ consisting of vectors $u_i \in V$ and $v_j \in V'$ such that $\varphi(u_i, u_j) = \delta_{ij}$. The subspace spanned by (u_i, v_i) is an Artinian plane, so $V \oplus V'$ is the orthogonal direct sum of r Artinian planes. Such a space is often denoted by Ar_{2r}.

In order to obtain the stronger version of the Witt decomposition when φ has some nonzero isotropic vector and W is anisotropic we now sharpen Proposition 29.29

Theorem 29.33. *Let φ be an ϵ-Hermitian form on E which is nondegenerate and satisfies property (T). Let U_1 and U_2 be two totally isotropic maximal subspaces of E, with U_1 or U_2 of finite dimension ≥ 1. Write $U = U_1 \cap U_2$, let S_1 be a supplement of U in U_1 and S_2 be a supplement of U in U_2 (so that $U_1 = U \oplus S_1$, $U_2 = U \oplus S_2$), and let $S = S_1 + S_2$. Then, there exist two subspaces W and D of E such that:*

(a) The subspaces S, $U + W$, and D are nondegenerate and pairwise orthogonal.