is called the *golden ratio*. Show that the eigenvalues of A are φ and $-\varphi^{-1}$.

(3) Prove that A is diagonalized as

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} \varphi & -\varphi^{-1} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \varphi & 0 \\ 0 & -\varphi^{-1} \end{pmatrix} \begin{pmatrix} 1 & \varphi^{-1} \\ -1 & \varphi \end{pmatrix}.$$

Prove that

$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} \varphi & -\varphi^{-1} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \varphi^n \\ -(-\varphi^{-1})^n \end{pmatrix},$$

and thus

$$F_n = \frac{1}{\sqrt{5}}(\varphi^n - (-\varphi^{-1})^n) = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right], \quad n \geq 0.$$

Problem 15.8. Let A be an $n \times n$ matrix. For any subset I of $\{1, \ldots, n\}$, let $A_{I,I}$ be the matrix obtained from A by first selecting the columns whose indices belong to I, and then the rows whose indices also belong to I. Prove that

$$\tau_k(A) = \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ |I|=k}} \det(A_{I,I}).$$

Problem 15.9. (1) Consider the matrix

$$A = \begin{pmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{pmatrix}.$$

Prove that the characteristic polynomial $\chi_A(z) = \det(zI - A)$ of A is given by

$$\chi_A(z) = z^3 + a_1 z^2 + a_2 z + a_3.$$

(2) Consider the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & -a_4 \\ 1 & 0 & 0 & -a_3 \\ 0 & 1 & 0 & -a_2 \\ 0 & 0 & 1 & -a_1 \end{pmatrix}.$$

Prove that the characteristic polynomial $\chi_A(z) = \det(zI - A)$ of A is given by

$$\chi_A(z) = z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4.$$