of a row of vectors $\begin{pmatrix} u_1 & \cdots & u_n \end{pmatrix}$ by the jth column of P as the linear combination

$$\sum_{i=1}^{n} p_{ij} u_i.$$

Such a definition is needed since scalar multiplication of a vector by a scalar is only defined if the scalar is on the left of the vector, but in the matrix expression $(*)$ above, the vectors are on the left of the scalars!

Even though matrices are indispensable since they are *the* major tool in applications of linear algebra, one should not lose track of the fact that

linear maps are more fundamental because they are intrinsic objects that do not depend on the choice of bases. Consequently, we advise the reader to try to think in terms of linear maps rather than reduce everything to matrices.

In our experience, this is particularly effective when it comes to proving results about linear maps and matrices, where proofs involving linear maps are often more "conceptual." These proofs are usually more general because they do not depend on the fact that the dimension is finite. Also, instead of thinking of a matrix decomposition as a purely algebraic operation, it is often illuminating to view it as a *geometric decomposition*. This is the case of the SVD, which in geometric terms says that every linear map can be factored as a rotation, followed by a rescaling along orthogonal axes and then another rotation.

After all,

a matrix is a representation of a linear map,

and most decompositions of a matrix reflect the fact that with a *suitable choice of a basis (or bases)*, the linear map is a represented by a matrix having a special shape. The problem is then to find such bases.

Still, for the beginner, matrices have a certain irresistible appeal, and we confess that it takes a certain amount of practice to reach the point where it becomes more natural to deal with linear maps. We still recommend it! For example, try to translate a result stated in terms of matrices into a result stated in terms of linear maps. Whenever we tried this exercise, we learned something.

Also, always try to keep in mind that

linear maps are geometric in nature; they act on space.