(or contraction, if $\lambda < 1$). When $\lambda = 0$, $H_{a,0}(x) = a$ for all $x \in E$, and $H_{a,0}$ is the constant affine map sending every point to a. If we assume $\lambda \neq 1$, note that $H_{a,\lambda}$ is never the identity, and since a is a fixed point, $H_{a,\lambda}$ is never a translation.

We now consider the set $\widehat{E}$ of geometric transformations from E to E, consisting of the union of the (disjoint) sets of translations and dilatations of ratio $\lambda \neq 1$. We would like to give this set the structure of a vector space, in such a way that both E and $\overrightarrow{E}$ can be naturally embedded into $\widehat{E}$. In fact, it will turn out that barycenters show up quite naturally too!

In order to "add" two dilatations H_{a_1,λ_1} and H_{a_2,λ_2}, it turns out that it is more convenient to consider dilatations of the form $H_{a,1-\lambda}$, where $\lambda \neq 0$. To see this, let us see the effect of such a dilatation on a point $x \in E$: We have

$$H_{a,1-\lambda}(x) = a + (1-\lambda)\overrightarrow{ax} = a + \overrightarrow{ax} - \lambda\overrightarrow{ax} = x + \lambda\overrightarrow{xa}.$$

For simplicity of notation, let us denote $H_{a,1-\lambda}$ by $\langle a, \lambda\rangle$. Then, we have

$$\langle a, \lambda\rangle(x) = x + \lambda\overrightarrow{xa}.$$

Remarks:

(1) Note that $H_{a,1-\lambda}(x) = H_{x,\lambda}(a)$.

(2) Berger defines a map $h\colon E \to \overrightarrow{E}$ as a *vector field*. Thus, each $\langle a, \lambda\rangle$ can be viewed as the vector field $x \mapsto \lambda\overrightarrow{xa}$. Similarly, a translation u can be viewed as the constant vector field $x \mapsto u$. Thus, we could define $\widehat{E}$ as the (disjoint) union of these two vector fields. We prefer our view in terms of geometric transformations.

Then, since

$$\langle a_1, \lambda_1\rangle(x) = x + \lambda_1\overrightarrow{xa_1} \quad \text{and} \quad \langle a_2, \lambda_2\rangle(x) = x + \lambda_2\overrightarrow{xa_2},$$

if we want to define $\langle a_1, \lambda_1\rangle \mathbin{\widehat{+}} \langle a_2, \lambda_2\rangle$, we see that we have to distinguish between two cases:

(1) $\lambda_1 + \lambda_2 = 0$. In this case, since

$$\lambda_1\overrightarrow{xa_1} + \lambda_2\overrightarrow{xa_2} = \lambda_1\overrightarrow{xa_1} - \lambda_1\overrightarrow{xa_2} = \lambda_1\overrightarrow{a_2a_1},$$

we let

$$\langle a_1, \lambda_1\rangle \mathbin{\widehat{+}} \langle a_2, \lambda_2\rangle = \lambda_1\overrightarrow{a_2a_1},$$

where $\lambda_1\overrightarrow{a_2a_1}$ denotes the translation associated with the vector $\lambda_1\overrightarrow{a_2a_1}$.

(2) $\lambda_1 + \lambda_2 \neq 0$. In this case, the points a_1 and a_2 assigned the weights $\lambda_1/(\lambda_1 + \lambda_2)$ and $\lambda_2/(\lambda_1 + \lambda_2)$ have a barycenter

$$b = \frac{\lambda_1}{\lambda_1 + \lambda_2}a_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2}a_2,$$