Chapter 42

Quadratic Optimization Problems

In this chapter we consider two classes of quadratic optimization problems that appear frequently in engineering and in computer science (especially in computer vision):

1. Minimizing

$$Q(x) = \frac{1}{2}x^\top A x - x^\top b$$

over all $x \in \mathbb{R}^n$, or subject to linear or affine constraints.

2. Minimizing

$$Q(x) = \frac{1}{2}x^\top A x - x^\top b$$

over the unit sphere.

In both cases, A is a symmetric matrix. We also seek necessary and sufficient conditions for Q to have a global minimum.

42.1 Quadratic Optimization: The Positive Definite Case

Many problems in physics and engineering can be stated as the minimization of some energy function, with or without constraints. Indeed, it is a fundamental principle of mechanics that nature acts so as to minimize energy. Furthermore, if a physical system is in a stable state of equilibrium, then the energy in that state should be minimal. For example, a small ball placed on top of a sphere is in an unstable equilibrium position. A small motion causes the ball to roll down. On the other hand, a ball placed inside and at the bottom of a sphere is in a stable equilibrium position because the potential energy is minimal.

The simplest kind of energy function is a quadratic function. Such functions can be conveniently defined in the form

$$Q(x) = x^\top A x - x^\top b,$$

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