8.12 Uniqueness of RREF Form

For the sake of completeness, we prove that the reduced row echelon form of a matrix is unique. The neat proof given below is borrowed and adapted from W. Kahan.

Proposition 8.19. *Let A be any $m \times n$ matrix. If U and V are two reduced row echelon matrices obtained from A by applying two sequences of elementary row operations $E_1, \ldots, E_p$ and $F_1, \ldots, F_q$, so that*

$$U = E_p \cdots E_1 A \quad \textit{and} \quad V = F_q \cdots F_1 A,$$

then $U = V$. In other words, the reduced row echelon form of any matrix is unique.

Proof. Let

$$C = E_p \cdots E_1 F_1^{-1} \cdots F_q^{-1}$$

so that

$$U = CV \quad \text{and} \quad V = C^{-1} U.$$

Recall from Proposition 8.13 that U and V have the same row rank r, and since U and V are in *rref*, this is the number of nonzero rows in both U and V. We prove by induction on n that $U = V$ (and that the first r columns of C are the first r columns in I_m). If $r = 0$ then $A = U = V = 0$ and the result is trivial. We now assume that $r \geq 1$.

Let ℓ_j^n denote the jth column of the identity matrix I_n, and let $u_j = U\ell_j^n$, $v_j = V\ell_j^n$, $c_j = C\ell_j^m$, and $a_j = A\ell_j^n$, be the jth column of U, V, C, and A respectively.

First I claim that $u_j = 0$ iff $v_j = 0$ iff $a_j = 0$.

Indeed, if $v_j = 0$, then (because $U = CV$) $u_j = Cv_j = 0$, and if $u_j = 0$, then $v_j = C^{-1}u_j = 0$. Since $U = E_p \cdots E_1 A$, we also get $a_j = 0$ iff $u_j = 0$.

Therefore, we may simplify our task by striking out columns of zeros from U, V, and A, since they will have corresponding indices. We still use n to denote the number of columns of A. Observe that because U and V are reduced row echelon matrices with no zero columns, we must have $u_1 = v_1 = \ell_1^m$.

Claim. If U and V are reduced row echelon matrices without zero columns such that $U = CV$, for all $k \geq 1$, if $k \leq m$, then ℓ_k^m occurs in U iff ℓ_k^m occurs in V, and if ℓ_k^m does occur in U, then

1. ℓ_k^m occurs for the same column index j_k in both U and V;
2. the first j_k columns of U and V match;
3. the subsequent columns in U and V (of column index $> j_k$) whose coordinates of index $k+1$ through m are all equal to 0 also match. Let n_k be the rightmost index of such a column, with $n_k = j_k$ if there is none.