Problem 49.7. If P is a symmetric positive definite matrix, prove that $\|z\|_P = (z^\top P z)^{1/2} = \left\|P^{1/2} z\right\|_2$ is a norm. Prove that the normalized steepest descent direction is

$$d_{\text{nsd,k}} = -(\nabla J_{u_k}^\top P^{-1} \nabla J_{u_k})^{-1/2} P^{-1} \nabla J_{u_k},$$

the dual norm is $\|z\|^D = \left\|P^{-1/2} z\right\|_2$, and the steepest descent direction with respect to $\| \, \|_P$ is given by

$$d_{\text{sd},k} = -P^{-1} \nabla J_{u_k}.$$

Problem 49.8. If $\| \, \|$ is the ℓ^1-norm, then show that $d_{\text{nsd,k}}$ is determined as follows: let i be any index for which $\|\nabla J_{u_k}\|_\infty = |(\nabla J_{u_k})_i|$. Then

$$d_{\text{nsd,k}} = -\text{sign}\left(\frac{\partial J}{\partial x_i}(u_k)\right) e_i,$$

where e_i is the ith canonical basis vector, and

$$d_{\text{sd,k}} = -\frac{\partial J}{\partial x_i}(u_k) e_i.$$

Problem 49.9. (From Boyd and Vandenberghe [29], Problem 9.12). If $\nabla^2 f(x)$ is singular (or very ill-conditioned), the Newton step $d_{\text{nt}} = -(\nabla^2 J(x))^{-1} \nabla J_x$ is not well defined. Instead we can define a search direction d_{tr} as the solution of the problem

$$\begin{aligned} &\text{minimize} && (1/2)\langle Hv, v\rangle + \langle g, v\rangle \\ &\text{subject to} && \|v\|_2 \leq \gamma, \end{aligned}$$

where $H = \nabla^2 f_x$, $g = \nabla f_x$, and γ is some positive constant. The idea is to use a *trust region*, which is the closed ball $\{v \mid \|v\|_2 \leq \gamma\}$. The point $x + d_{\text{tr}}$ minimizes the second-order approximation of f at x, subject to the constraint that

$$\|x + d_{\text{tr}} - x\|_2 \leq \gamma.$$

The parameter γ, called the *trust parameter*, reflects our confidence in the second-order approximation.

Prove that d_{tr} minimizes

$$\frac{1}{2}\langle Hv, v\rangle + \langle g, v\rangle + \widehat{\beta} \, \|v\|_2^2,$$

for some $\widehat{\beta}$.

Problem 49.10. (From Boyd and Vandenberghe [29], Problem 9.9). Prove that the Newton decrement $\lambda(x)$ is given by

$$\lambda(x) = \sup_{v \neq 0} -\frac{\langle \nabla J_x, v\rangle}{(\langle \nabla^2 J_x v, v\rangle)^{1/2}}.$$