Problem 16.10. Let i, j, and k, be the unit vectors of coordinates $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ in $\mathbb{R}^3$.

(1) Describe geometrically the rotations defined by the following quaternions:

$$p = (0, i), \quad q = (0, j).$$

Prove that the interpolant $Z(\lambda) = p(p^{-1}q)^{\lambda}$ is given by

$$Z(\lambda) = (0,\, \cos(\lambda\pi/2)i + \sin(\lambda\pi/2)j)\,.$$

Describe geometrically what this rotation is.

(2) Repeat Question (1) with the rotations defined by the quaternions

$$p = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}i\right), \quad q = (0, j).$$

Prove that the interpolant $Z(\lambda)$ is given by

$$Z(\lambda) = \left(\frac{1}{2}\cos(\lambda\pi/2),\, \frac{\sqrt{3}}{2}\cos(\lambda\pi/2)i + \sin(\lambda\pi/2)j\right).$$

Describe geometrically what this rotation is.

(3) Repeat Question (1) with the rotations defined by the quaternions

$$p = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}i\right), \quad q = \left(0, \frac{1}{\sqrt{2}}(i+j)\right).$$

Prove that the interpolant $Z(\lambda)$ is given by

$$Z(\lambda) = \left(\frac{1}{\sqrt{2}}\cos(\lambda\pi/3) - \frac{1}{\sqrt{6}}\sin(\lambda\pi/3),\right.$$
$$\left.(1/\sqrt{2}\cos(\lambda\pi/3) + 1/\sqrt{6}\sin(\lambda\pi/3))i + \frac{2}{\sqrt{6}}\sin(\lambda\pi/3)j\right).$$

Problem 16.11. Prove that

$$w \times (u \times v) = (w \cdot v)u - (u \cdot w)v.$$

Conclude that

$$u \times (u \times v) = (u \cdot v)u - (u \cdot u)v.$$