Also observe that

$$X^* = \begin{pmatrix} a - ib & -(c+id) \\ c - id & a + ib \end{pmatrix} = a\mathbf{1} - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}.$$

This implies that if $X \neq 0$, then X is invertible and its inverse is given by

$$X^{-1} = (a^2 + b^2 + c^2 + d^2)^{-1} X^*.$$

As a consequence, it can be verified that $\mathbb{H}$ is a skew field (a noncommutative field). It is also a real vector space of dimension 4 with basis $(\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k})$; thus as a vector space, $\mathbb{H}$ is isomorphic to $\mathbb{R}^4$.

Definition 16.2. A concise notation for the quaternion X defined by $\alpha = a + ib$ and $\beta = c + id$ is

$$X = [a, (b, c, d)].$$

We call a the *scalar part* of X and (b, c, d) the *vector part* of X. With this notation, $X^* = [a, -(b, c, d)]$, which is often denoted by $\overline{X}$. The quaternion $\overline{X}$ is called the *conjugate* of X. If q is a unit quaternion, then $\overline{q}$ is the multiplicative inverse of q.

16.2 Representation of Rotations in $\mathbf{SO}(3)$ by Quaternions in $\mathbf{SU}(2)$

The key to representation of rotations in $\mathbf{SO}(3)$ by unit quaternions is a certain group homomorphism called the *adjoint representation of* $\mathbf{SU}(2)$. To define this mapping, first we define the real vector space $\mathfrak{su}(2)$ of skew Hermitian matrices.

Definition 16.3. The (real) vector space $\mathfrak{su}(2)$ of 2×2 *skew Hermitian matrices with zero trace* is given by

$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} ix & y + iz \\ -y + iz & -ix \end{pmatrix} \;\middle|\; (x, y, z) \in \mathbb{R}^3 \right\}.$$

Observe that for every matrix $A \in \mathfrak{su}(2)$, we have $A^* = -A$, that is, A is skew Hermitian, and that $\mathrm{tr}(A) = 0$.

Definition 16.4. The *adjoint representation* of the group $\mathbf{SU}(2)$ is the group homomorphism $\mathrm{Ad}\colon \mathbf{SU}(2) \to \mathbf{GL}(\mathfrak{su}(2))$ defined such that for every $q \in \mathbf{SU}(2)$, with

$$q = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \in \mathbf{SU}(2),$$

we have

$$\mathrm{Ad}_q(A) = qAq^*, \quad A \in \mathfrak{su}(2),$$

where q^* is the inverse of q (since $\mathbf{SU}(2)$ is a unitary group) and is given by

$$q^* = \begin{pmatrix} \overline{\alpha} & -\beta \\ \overline{\beta} & \alpha \end{pmatrix}.$$