polynomials used in approximation theory and in physics arise by a suitable choice of the weight function W. Besides the previous two examples, the *Hermite polynomials* correspond to $W(x) = e^{-x^2}$, the *Laguerre polynomials* to $W(x) = e^{-x}$, and the *Jacobi polynomials* to $W(x) = (1-x)^\alpha(1+x)^\beta$, with $\alpha, \beta > -1$. Comprehensive treatments of orthogonal polynomials can be found in Lebedev [114], Sansone [144], and Andrews, Askey and Roy [3].

We can also prove the following proposition regarding orthogonal spaces.

Proposition 12.11. *Given any nontrivial Euclidean space E of finite dimension $n \geq 1$, for any subspace F of dimension k, the orthogonal complement $F^\perp$ of F has dimension $n-k$, and $E = F \oplus F^\perp$. Furthermore, we have $F^{\perp\perp} = F$.*

Proof. From Proposition 12.9, the subspace F has some orthonormal basis $(u_1, \ldots, u_k)$. This linearly independent family $(u_1, \ldots, u_k)$ can be extended to a basis $(u_1, \ldots, u_k, v_{k+1}, \ldots, v_n)$, and by Proposition 12.10, it can be converted to an orthonormal basis $(u_1, \ldots, u_n)$, which contains $(u_1, \ldots, u_k)$ as an orthonormal basis of F. Now any vector $w = w_1u_1 + \cdots + w_nu_n \in E$ is orthogonal to F iff $w \cdot u_i = 0$, for every i, where $1 \leq i \leq k$, iff $w_i = 0$ for every i, where $1 \leq i \leq k$. Clearly, this shows that $(u_{k+1}, \ldots, u_n)$ is a basis of $F^\perp$, and thus $E = F \oplus F^\perp$, and $F^\perp$ has dimension $n-k$. Similarly, any vector $w = w_1u_1 + \cdots + w_nu_n \in E$ is orthogonal to $F^\perp$ iff $w \cdot u_i = 0$, for every i, where $k+1 \leq i \leq n$, iff $w_i = 0$ for every i, where $k+1 \leq i \leq n$. Thus, $(u_1, \ldots, u_k)$ is a basis of $F^{\perp\perp}$, and $F^{\perp\perp} = F$. $\square$

12.5 Linear Isometries (Orthogonal Transformations)

In this section we consider linear maps between Euclidean spaces that preserve the Euclidean norm. These transformations, sometimes called *rigid motions*, play an important role in geometry.

Definition 12.5. Given any two nontrivial Euclidean spaces E and F of the same finite dimension n, a function $f \colon E \to F$ is *an orthogonal transformation, or a linear isometry*, if it is linear and

$$\|f(u)\| = \|u\|, \quad \text{for all } u \in E.$$

Remarks:

(1) A linear isometry is often defined as a linear map such that

$$\|f(v) - f(u)\| = \|v - u\|,$$

for all $u, v \in E$. Since the map f is linear, the two definitions are equivalent. The second definition just focuses on preserving the distance between vectors.

(2) Sometimes, a linear map satisfying the condition of Definition 12.5 is called a *metric map*, and a linear isometry is defined as a *bijective* metric map.