and so f is represented by the following matrix known as the *companion matrix* of $q(X)$:

$$U = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & -a_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & -a_{n-2} \\ 0 & 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix}.$$

It is an easy exercise to prove that the characteristic polynomial $\chi_U(X)$ of U gives back $q(X)$:

$$\chi_U(X) = q(X).$$

We will need the following proposition to characterize when two linear maps are similar.

Proposition 36.1. *Let $f\colon E \to E$ and $f'\colon E' \to E'$ be two linear maps over the vector spaces E and E'. A linear map $g\colon E \to E'$ can be viewed as a linear map between the $K[X]$-modules E_f and $E_{f'}$ iff*

$$g \circ f = f' \circ g.$$

Proof. First, suppose g is $K[X]$-linear. Then, we have

$$g(p \cdot_f u) = p \cdot_{f'} g(u)$$

for all $p \in K[X]$ and all $u \in E$, so for $p = X$ we get

$$g(p \cdot_f u) = g(X \cdot_f u) = g(f(u))$$

and

$$p \cdot_{f'} g(u) = X \cdot_{f'} g(u) = f'(g(u)),$$

which means that $g \circ f = f' \circ g$.

Conversely, if $g \circ f = f' \circ g$, we prove by induction that

$$g \circ f^n = f'^n \circ g, \quad \text{for all } n \geq 1.$$

Indeed, we have

$$\begin{aligned} g \circ f^{n+1} &= g \circ f^n \circ f \\ &= f'^n \circ g \circ f \\ &= f'^n \circ f' \circ g \\ &= f'^{n+1} \circ g, \end{aligned}$$