

and each zero by the column vector

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Now the first half of W_n , that is the matrix consisting of the first 2^{n-1} columns of W_n , can be obtained from W_{n-1} by forming the $2^n \times 2^{n-1}$ matrix obtained by replacing each 1 by the column vector

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

each -1 by the column vector

$$\begin{pmatrix} -1 \\ -1 \end{pmatrix},$$

and each zero by the column vector

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For $n = 3$, the first half of W_3 is the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \\ 1 & -1 & 0 & -1 \end{pmatrix}$$

which is indeed obtained from

$$W_2 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{pmatrix}$$

using the process that we just described.

These matrix manipulations can be described conveniently using a product operation on matrices known as the Kronecker product.

Definition 5.4. Given a $m \times n$ matrix $A = (a_{ij})$ and a $p \times q$ matrix $B = (b_{ij})$, the *Kronecker product* (or *tensor product*) $A \otimes B$ of A and B is the $mp \times nq$ matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}.$$