which implies that

$$L(u) = \frac{f(a+tu) - f(a)}{t} - \frac{|t|}{t}\epsilon(tu)\|u\|,$$

and since $\lim_{t\mapsto 0}\epsilon(tu) = 0$, we deduce that

$$L(u) = \mathrm{D}f(a)(u) = \mathrm{D}_u f(a).$$

Because

$$f(a+h) = f(a) + L(h) + \epsilon(h)\|h\|$$

for all h such that $\|h\|$ is small enough, L is continuous, and $\lim_{h\mapsto 0}\epsilon(h)\|h\| = 0$, we have $\lim_{h\mapsto 0} f(a+h) = f(a)$, that is, f is continuous at a. $\square$

When E is of finite dimension, every linear map is continuous (see Proposition 9.8 or Theorem 37.58), and this assumption is then redundant.

It is important to note that the derivative $\mathrm{D}f(a)$ of f at a is a continuous linear map from the *vector space* $\overrightarrow{E}$ to the *vector space* $\overrightarrow{F}$, and not a function from the affine space E to the affine space F.

Although this may not be immediately obvious, the reason for requiring the linear map $\mathrm{D}f_a$ to be continuous is to ensure that if a function f is differentiable at a, then it is continuous at a. This is certainly a desirable property of a differentiable function. In finite dimension this holds, but in infinite dimension this is not the case. The following proposition shows that if $\mathrm{D}f_a$ exists at a and if f is continuous at a, then $\mathrm{D}f_a$ must be a continuous map. So if a function is differentiable at a, then it is continuous iff the linear map $\mathrm{D}f_a$ is continuous. We chose to include the second condition rather that the first in the definition of a differentiable function.

Proposition 39.2. *Let E and F be two normed affine spaces, let A be a nonempty open subset of E, and let $f\colon A \to F$ be any function. For any $a \in A$, if $\mathrm{D}f_a$ is defined, then f is continuous at a iff $\mathrm{D}f_a$ is a continuous linear map.*

Proof. Proposition 39.1 shows that if $\mathrm{D}f_a$ is defined and continuous then f is continuous at a. Conversely, assume that $\mathrm{D}f_a$ exists and that f is continuous at a. Since f is continuous at a and since $\mathrm{D}f_a$ exists, for any $\eta > 0$ there is some ρ with $0 < \rho < 1$ such that if $\|h\| \leq \rho$ then

$$\|f(a+h) - f(a)\| \leq \frac{\eta}{2},$$

and

$$\|f(a+h) - f(a) - \mathrm{D}_a(h)\| \leq \frac{\eta}{2}\,\|h\| \leq \frac{\eta}{2},$$

so we have

$$\begin{aligned}\|\mathrm{D}_a(h)\| &= \|\mathrm{D}_a(h) - (f(a+h) - f(a)) + f(a+h) - f(a)\| \\ &\leq \|f(a+h) - f(a) - \mathrm{D}_a(h)\| + \|f(a+h) - f(a)\| \\ &\leq \frac{\eta}{2} + \frac{\eta}{2} = \eta,\end{aligned}$$