(4) Let the total order of the basis (E_{ij}) extending the partial ordering defined in (2) be given by

$$(i,j) < (h,k) \quad \text{iff} \quad \begin{cases} i = h \text{ and } j > k \\ \text{or } i < h. \end{cases}$$

Let R be the $n \times n$ permutation matrix given by

$$R = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Observe that $R^{-1} = R$. Prove that for any $n \geq 1$, the matrix of L_A is given by $A \otimes I_n$, and the matrix of R_A is given by $I_n \otimes RA^\top R$ (over the basis (E_{ij}) ordered as specified above), where $\otimes$ is the *Kronecker product* (also called *tensor product*) of matrices defined in Definition 5.4.

Hint. Figure out what are $R_B(E_{ij}) = E_{ij}B$ and $L_B(E_{ij}) = BE_{ij}$.

(5) Prove that if A is upper triangular, then the matrices representing L_A and R_A are also upper triangular.

Note that if instead of the ordering

$$E_{1n}, E_{1n-1}, \dots, E_{11}, E_{2n}, \dots, E_{21}, \dots, E_{nn}, \dots, E_{n1},$$

that I proposed you use the standard lexicographic ordering

$$E_{11}, E_{12}, \dots, E_{1n}, E_{21}, \dots, E_{2n}, \dots, E_{n1}, \dots, E_{nn},$$

then the matrix representing L_A is still $A \otimes I_n$, but the matrix representing R_A is $I_n \otimes A^\top$. In this case, if A is upper-triangular, then the matrix of R_A is *lower triangular*. This is the motivation for using the first basis (avoid upper becoming lower).