Note that u'_{k+1} is obtained by subtracting from e_{k+1} the projection of e_{k+1} itself onto the orthonormal vectors $u_1, \ldots, u_k$ that have already been computed. Then u'_{k+1} is normalized.

Example 12.9. For a specific example of this procedure, let $E = \mathbb{R}^3$ with the standard Euclidean norm. Take the basis

$$e_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad e_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Then

$$u_1 = 1/\sqrt{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

and

$$u'_2 = e_2 - (e_2 \cdot u_1)u_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - 2/3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1/3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

This implies that

$$u_2 = 1/\sqrt{6} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix},$$

and that

$$u'_3 = e_3 - (e_3 \cdot u_1)u_1 - (e_3 \cdot u_2)u_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - 2/3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 1/6 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 1/2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

To complete the orthonormal basis, normalize u'_3 to obtain

$$u_3 = 1/\sqrt{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

An illustration of this example is provided by Figure 12.4.

Remarks:

(1) The QR-decomposition can now be obtained very easily, but we postpone this until Section 12.8.

(2) The proof of Proposition 12.10 also works for a countably infinite basis for E, producing a countably infinite orthonormal basis.