Example 39.13. Going back to the function f of Example 39.10 given by $f(A) = \log\det(A)$, we know from Example 39.12 that

$$\mathrm{D}^m f(A)(X_1, \ldots, X_m) = (-1)^{m-1} \sum_{\sigma \in \mathfrak{S}_{m-1}} \mathrm{tr}(A^{-1}X_1A^{-1}X_{\sigma(1)+1} \cdots A^{-1}X_{\sigma(m-1)+1}) \qquad (*)$$

for all $m \geq 1$, with $A \in \mathbf{GL}^+(n, \mathbb{R})$. If we make the stronger assumption that A is symmetric positive definite, then for any other symmetric positive definite matrix B, since the symmetric positive definite matrices form a convex set, the matrices $A + \theta(B - A) = (1-\theta)A + \theta B$ are also symmetric positive definite for $\theta \in [0, 1]$. Theorem 39.25 applies with $H = B - A$ (a symmetric matrix), and using $(*)$, we obtain

$$\begin{aligned}\log\det(A + H) = \log\det(A) + \mathrm{tr}\Bigg(&A^{-1}H - \frac{1}{2}(A^{-1}H)^2 + \cdots + \frac{(-1)^{m-1}}{m}(A^{-1}H)^m \\ &+ \frac{(-1)^m}{m+1}((A + \theta H)^{-1}H)^{m+1}\Bigg),\end{aligned}$$

for some θ such that $0 < \theta < 1$. In particular, if $A = I$, for any symmetric matrix H such that $I + H$ is symmetric positive definite, we obtain

$$\begin{aligned}\log\det(I + H) = \mathrm{tr}\Bigg(&H - \frac{1}{2}H^2 + \cdots + \frac{(-1)^{m-1}}{m}H^m \\ &+ \frac{(-1)^m}{m+1}((I + \theta H)^{-1}H)^{m+1}\Bigg),\end{aligned}$$

for some θ such that $0 < \theta < 1$. In the special case when $n = 1$, we have $I = 1$, H is a real such that $1 + H > 0$ and the trace function is the identity, so we recognize the partial sum of the series for $x \mapsto \log(1 + x)$,

$$\begin{aligned}\log(1 + H) = H &- \frac{1}{2}H^2 + \cdots + \frac{(-1)^{m-1}}{m}H^m \\ &+ \frac{(-1)^m}{m+1}(1 + \theta H)^{-(m+1)}H^{m+1}.\end{aligned}$$

We also mention for "mathematical culture," a version with integral remainder, in the case of a real-valued function. This is usually called *Taylor's formula with integral remainder*.

Theorem 39.26. *(Taylor's formula with integral remainder) Let E be a normed affine space, let A be an open subset of E, and let $f\colon A \to \mathbb{R}$ be a real-valued function on A. Given any $a \in A$ and any $h \neq 0$ in $\overrightarrow{E}$, if the closed segment $[a, a + h]$ is contained in A, and if f is a C^{m+1}-function on A, then we have*

$$\begin{aligned}f(a + h) = f(a) + \frac{1}{1!}\mathrm{D}^1 f(a)(h) + \cdots + \frac{1}{m!}\mathrm{D}^m f(a)(h^m)& \\ + \int_0^1 \frac{(1-t)^m}{m!}\Big[\mathrm{D}^{m+1} f(a + th)(h^{m+1})\Big]dt.&\end{aligned}$$