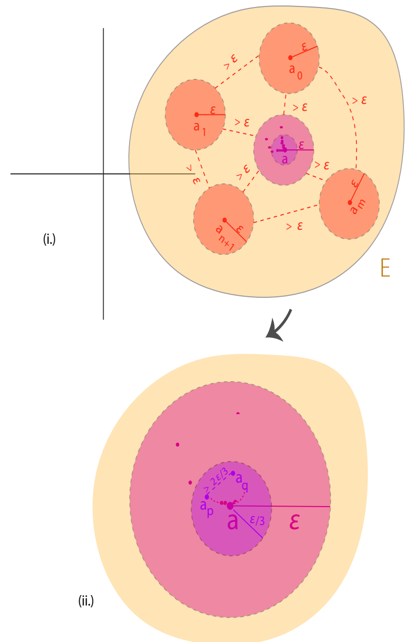

Figure 37.44: Let E be the peach region of $\mathbb{R}^2$. If E is not covered by a finite collection of orange balls with radius ϵ, the points of the sequence (a_n) are separated by a distance of at least ϵ. This contradicts the fact that a is the accumulation point of a, as evidenced by the enlargement of the plum disk in Figure (ii).

Theorem 37.47. *A metric space, E, is compact iff every sequence, (x_n), has an accumulation point.*

Proof. We already observed that the proof of Proposition 37.43 shows that for any compact space (not necessarily metric), every sequence, (x_n), has an accumulation point. Conversely, let E be a metric space, and assume that every sequence, (x_n), has an accumulation point. Given any open cover, $(U_i)_{i\in I}$ for E, we must find a finite open subcover of E. By Lemma 37.44, there is some $\delta > 0$ (a Lebesgue number for $(U_i)_{i\in I}$) such that, for every open ball, $B_0(a, \epsilon)$, of radius $\epsilon \leq \delta$, there is some open subset, U_j, such that $B_0(a, \epsilon) \subseteq U_j$. By Lemma 37.46, for every $\delta > 0$, there is a finite open cover, $B_0(a_0, \delta) \cup \cdots \cup B_0(a_n, \delta)$, of E by open balls of radius δ. But from the previous statement, every open ball, $B_0(a_i, \delta)$, is contained in some open set, U_{j_i}, and thus, $\{U_{j_1}, \ldots, U_{j_n}\}$ is an open cover of E. $\square$

37.8 Complete Metric Spaces and Compactness

Another very useful characterization of compact metric spaces is obtained in terms of Cauchy sequences. Such a characterization is quite useful in fractal geometry (and elsewhere). First