we have

$$g(x) = f(x) - \tau = x,$$

since $\overrightarrow{xf(x)} = \tau$ is equivalent to $x = f(x) - \tau$. As a composition of affine isometries, g is an affine isometry, x is a fixed point of g, and since $\tau \in \mathrm{Ker}\left(\overrightarrow{f} - \mathrm{id}\right)$, we have

$$\overrightarrow{f}(\tau) = \tau,$$

and since

$$g(b) = f(b) - \tau$$

for all $b \in E$, we have $\overrightarrow{g} = \overrightarrow{f}$. Since g has some fixed point x, by Lemma 27.8, $\mathrm{Fix}(g)$ is an affine subspace of E with direction $\mathrm{Ker}\left(\overrightarrow{g} - \mathrm{id}\right) = \mathrm{Ker}\left(\overrightarrow{f} - \mathrm{id}\right)$. We also have $f(b) = g(b) + \tau$ for all $b \in E$, and thus

$$(g \circ t_\tau)(b) = g(b + \tau) = g(b) + \overrightarrow{g}(\tau) = g(b) + \overrightarrow{f}(\tau) = g(b) + \tau = f(b),$$

and

$$(t_\tau \circ g)(b) = g(b) + \tau = f(b),$$

which proves that $t \circ g = g \circ t$.

To prove the existence of x as above, pick any arbitrary point $a \in E$. Since

$$\overrightarrow{E} = \mathrm{Ker}\left(\overrightarrow{f} - \mathrm{id}\right) \oplus \mathrm{Im}\left(\overrightarrow{f} - \mathrm{id}\right),$$

there is a unique vector $\tau \in \mathrm{Ker}\left(\overrightarrow{f} - \mathrm{id}\right)$ and some $v \in \overrightarrow{E}$ such that

$$\overrightarrow{af(a)} = \tau + \overrightarrow{f}(v) - v.$$

For any $x \in E$, since we also have

$$\overrightarrow{xf(x)} = \overrightarrow{xa} + \overrightarrow{af(a)} + \overrightarrow{f(a)f(x)} = \overrightarrow{xa} + \overrightarrow{af(a)} + \overrightarrow{f}(\overrightarrow{ax}),$$

we get

$$\overrightarrow{xf(x)} = \overrightarrow{xa} + \tau + \overrightarrow{f}(v) - v + \overrightarrow{f}(\overrightarrow{ax}),$$

which can be rewritten as

$$\overrightarrow{xf(x)} = \tau + \left(\overrightarrow{f} - \mathrm{id}\right)(v + \overrightarrow{ax}).$$

If we let $\overrightarrow{ax} = -v$, that is, $x = a - v$, we get

$$\overrightarrow{xf(x)} = \tau,$$

with $\tau \in \mathrm{Ker}\left(\overrightarrow{f} - \mathrm{id}\right)$.