of G, the column of B corresponding to the oriented edge $\sigma(\{v_i, v_j\})$ has zero entries except for a $+1$ and a -1 in position i and position j or vice-versa, so we have

$$z_i = z_j.$$

An easy induction on the length of the path shows that if there is a path from v_i to v_j in G (unoriented), then $z_i = z_j$. Therefore, z has a constant value on any connected component of G. It follows that every vector $z \in \mathrm{Ker}\,(B^\top)$ can be written uniquely as a linear combination

$$z = \lambda_1 z^1 + \cdots + \lambda_c z^c,$$

where the vector z^i corresponds to the ith connected component K_i of G and is defined such that

$$z^i_j = \begin{cases} 1 & \text{iff } v_j \in K_i \\ 0 & \text{otherwise.} \end{cases}$$

This shows that $\dim(\mathrm{Ker}\,(B^\top)) = c$, and that $\mathrm{Ker}\,(B^\top)$ has a basis consisting of indicator vectors.

Since $B^\top$ is a $n \times m$ matrix, we have

$$m = \dim(\mathrm{Ker}\,(B^\top)) + \mathrm{rank}(B^\top),$$

and since we just proved that $\dim(\mathrm{Ker}\,(B^\top)) = c$, we obtain $\mathrm{rank}(B^\top) = m - c$. Since B and $B^\top$ have the same rank, $\mathrm{rank}(B) = m - c$, as claimed. $\square$

Definition 20.12. Following common practice, we denote by $\mathbf{1}$ the (column) vector (of dimension m) whose components are all equal to 1.

Since every column of B contains a single $+1$ and a single -1, the rows of $B^\top$ sum to zero, which can be expressed as

$$B^\top \mathbf{1} = 0.$$

According to Proposition 20.1, the graph G is connected iff B has rank $m-1$ iff the nullspace of $B^\top$ is the one-dimensional space spanned by $\mathbf{1}$.

In many applications, the notion of graph needs to be generalized to capture the intuitive idea that two nodes u and v are linked with a degree of certainty (or strength). Thus, we assign a nonnegative weight $w_{i\,j}$ to an edge $\{v_i, v_j\}$; the smaller $w_{i\,j}$ is, the weaker is the link (or similarity) between v_i and v_j, and the greater $w_{i\,j}$ is, the stronger is the link (or similarity) between v_i and v_j.

Definition 20.13. A *weighted graph* is a pair $G = (V, W)$, where $V = \{v_1, \ldots, v_m\}$ is a set of *nodes* or *vertices*, and W is a symmetric matrix called the *weight matrix*, such that $w_{i\,j} \geq 0$ for all $i, j \in \{1, \ldots, m\}$, and $w_{i\,i} = 0$ for $i = 1, \ldots, m$. We say that a set $\{v_i, v_j\}$ is an edge iff $w_{i\,j} > 0$. The corresponding (undirected) graph (V, E) with $E = \{\{v_i, v_j\} \mid w_{i\,j} > 0\}$, is called the *underlying graph* of G.