What is important about Theorem 33.6 is not so much the construction itself but the fact that it produces a tensor product with the universal mapping property with respect to multilinear maps. Indeed, Theorem 33.6 yields a canonical isomorphism

$$\mathrm{L}(E_1 \otimes \cdots \otimes E_n, F) \cong \mathrm{L}(E_1, \ldots, E_n; F)$$

between the vector space of linear maps $\mathrm{L}(E_1 \otimes \cdots \otimes E_n, F)$, and the vector space of multilinear maps $\mathcal{L}(E_1, \ldots, E_n; F)$, *via* the linear map $- \circ \varphi$ defined by

$$h \mapsto h \circ \varphi,$$

where $h \in \mathrm{L}(E_1 \otimes \cdots \otimes E_n, F)$. Indeed, $h \circ \varphi$ is clearly multilinear, and since by Theorem 33.6, for every multilinear map $f \in \mathcal{L}(E_1, \ldots, E_n; F)$, there is a unique linear map $f_\otimes \in \mathrm{L}(E_1 \otimes \cdots \otimes E_n, F)$ such that $f = f_\otimes \circ \varphi$, the map $- \circ \varphi$ is bijective. As a matter of fact, its inverse is the map

$$f \mapsto f_\otimes.$$

We record this fact as the following proposition.

Proposition 33.7. *Given a tensor product $(E_1 \otimes \cdots \otimes E_n, \varphi)$, the linear map $h \mapsto h \circ \varphi$ is a canonical isomorphism*

$$\mathrm{L}(E_1 \otimes \cdots \otimes E_n, F) \cong \mathrm{L}(E_1, \ldots, E_n; F)$$

between the vector space of linear maps $\mathrm{L}(E_1 \otimes \cdots \otimes E_n, F)$, and the vector space of multilinear maps $\mathcal{L}(E_1, \ldots, E_n; F)$.

Using the "Hom" notation, the above canonical isomorphism is written

$$\mathrm{Hom}(E_1 \otimes \cdots \otimes E_n, F) \cong \mathrm{Hom}(E_1, \ldots, E_n; F).$$

Remarks:

(1) To be very precise, since the tensor product depends on the field K, we should subscript the symbol $\otimes$ with K and write

$$E_1 \otimes_K \cdots \otimes_K E_n.$$

However, we often omit the subscript K unless confusion may arise.

(2) For $F = K$, the base field, Proposition 33.7 yields a canonical isomorphism between the vector space $\mathrm{L}(E_1 \otimes \cdots \otimes E_n, K)$, and the vector space of multilinear forms $\mathcal{L}(E_1, \ldots, E_n; K)$. However, $\mathrm{L}(E_1 \otimes \cdots \otimes E_n, K)$ is the dual space $(E_1 \otimes \cdots \otimes E_n)^*$, and thus the vector space of multilinear forms $\mathcal{L}(E_1, \ldots, E_n; K)$ is canonically isomorphic to $(E_1 \otimes \cdots \otimes E_n)^*$.