and

$$(u^* \lrcorner z) \wedge z = \sum_{i=1}^{p} \alpha_i e_i \wedge e_1 \wedge \cdots \wedge e_i \wedge \cdots \wedge e_p = 0.$$

Now assume that $(u^* \lrcorner z) \wedge z = 0$ for all $u^* \in \bigwedge^{p-1} E^*$, and that $\dim(W) = m > p$, where W is the smallest subspace of E such that $z \in \bigwedge^p W$ If $e_1, \ldots, e_m$ is a basis of W, then we have $z = \sum_I \lambda_I e_I$, where $I \subseteq \{1, \ldots, m\}$ and $|I| = p$. Recall that $z \neq 0$, and so, some λ_I is nonzero. By Proposition 34.25, each e_i can be written as $u^* \lrcorner z$ for some $u^* \in \bigwedge^{p-1} E^*$, and since $(u^* \lrcorner z) \wedge z = 0$ for all $u^* \in \bigwedge^{p-1} E^*$, we get

$$e_j \wedge z = 0 \quad \text{for} \quad j = 1, \ldots, m.$$

By wedging $z = \sum_I \lambda_I e_I$ with each e_j, as $m > p$, we deduce $\lambda_I = 0$ for all I, so $z = 0$, a contradiction. Therefore, $m = p$ and Corollary 34.26 implies that z is decomposable. □

As a corollary of Proposition 34.27 we obtain the following fact that we stated earlier without proof.

Proposition 34.28. *Given any vector space E of dimension n, a vector $x \in \bigwedge^2 E$ is decomposable iff $x \wedge x = 0$.*

Proof. Recall that as an application of Proposition 34.19 we proved the formula (†), namely

$$u^* \lrcorner (x \wedge x) = 2((u^* \lrcorner x) \wedge x)$$

for all $x \in \bigwedge^2 E$ and all $u^* \in E^*$. As a consequence, $(u^* \lrcorner x) \wedge x = 0$ iff $u^* \lrcorner (x \wedge x) = 0$. By Proposition 34.27, the 2-vector x is decomposable iff $u^* \lrcorner (x \wedge x) = 0$ for all $u^* \in E^*$ iff $x \wedge x = 0$. Therefore, a 2-vector x is decomposable iff $x \wedge x = 0$. □

As an application of Proposition 34.28, assume that $\dim(E) = 3$ and that (e_1, e_2, e_3) is a basis of E. Then any 2-vector $x \in \bigwedge^2 E$ is of the form

$$x = \alpha e_1 \wedge e_2 + \beta e_1 \wedge e_3 + \gamma e_2 \wedge e_3.$$

We have

$$x \wedge x = (\alpha e_1 \wedge e_2 + \beta e_1 \wedge e_3 + \gamma e_2 \wedge e_3) \wedge (\alpha e_1 \wedge e_2 + \beta e_1 \wedge e_3 + \gamma e_2 \wedge e_3) = 0,$$

because all the terms involved are of the form $c\, e_{i_1} \wedge e_{i_2} \wedge e_{i_3} \wedge e_{i_4}$ with $i_1, i_2, i_3, i_4 \in \{1, 2, 3\}$, and so at least two of these indices are identical. Therefore, every 2-vector $x = \alpha e_1 \wedge e_2 + \beta e_1 \wedge e_3 + \gamma e_2 \wedge e_3$ is decomposable, although this not obvious at first glance. For example,

$$e_1 \wedge e_2 + e_1 \wedge e_3 + e_2 \wedge e_3 = (e_1 + e_2) \wedge (e_2 + e_3).$$

We now show that Proposition 34.27 yields an equational criterion for the decomposability of an alternating tensor $z \in \bigwedge^p E$.