For example, the graph Laplacian of graph G_1 is

$$L = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 3 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & -1 & 0 & -1 & 2 \end{pmatrix}.$$

Observe that each row of L sums to zero (because $(B^\sigma)^\top \mathbf{1} = 0$). Consequently, the vector $\mathbf{1}$ is in the nullspace of L.

Remarks:

1. With the unoriented version of the incidence matrix (see Definition 20.8), it can be shown that
$$BB^\top = D + A.$$

2. As pointed out by Evangelos Chatzipantazis, Proposition 20.2 in which the incidence matrix B^σ is replaced by the incidence matrix B of any *arbitrary* directed graph G does not hold. The problem is that such graphs may have both edges (v_i, v_j) and (v_j, v_i) between two distinct nodes v_i and v_j, and as a consequence, the inner product $b_i \cdot b_j = -2$ instead of -1. A simple counterexample is given by the directed graph with three vertices and four edges whose incidence matrix is given by
$$B = \begin{pmatrix} 1 & -1 & 0 & -1 \\ -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$
We have
$$BB^\top = \begin{pmatrix} 3 & -2 & -1 \\ -2 & 3 & -1 \\ -1 & -1 & 2 \end{pmatrix} \neq \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = D - A.$$

The natural generalization of the notion of graph Laplacian to weighted graphs is this:

Definition 20.16. Given any weighted graph $G = (V, W)$ with $V = \{v_1, \ldots, v_m\}$, the *(unnormalized) graph Laplacian $L(G)$ of G* is defined by

$$L(G) = D(G) - W,$$

where $D(G) = \mathrm{diag}(d_1, \ldots, d_m)$ is the degree matrix of G (a diagonal matrix), with

$$d_i = \sum_{j=1}^m w_{i\,j}.$$

As usual, unless confusion arises, we write D instead of $D(G)$ and L instead of $L(G)$.