If $f\colon E \to E$ is a projection ($f^2 = f$), then

$$(\mathrm{id} - 2f)^2 = \mathrm{id} - 4f + 4f^2 = \mathrm{id} - 4f + 4f = \mathrm{id},$$

so $\mathrm{id} - 2f$ is an involution. As a consequence, we get the following result.

Proposition 14.25. *If $f\colon E \to E$ is a projection ($f^2 = f$), then* $\mathrm{Ker}\,(f)$ *and* $\mathrm{Im}(f)$ *are orthogonal iff $f^* = f$.*

Proof. Apply Proposition 14.24 to $g = \mathrm{id} - 2f$. Since $\mathrm{id} - g = 2f$ we have

$$U^+ = \mathrm{Ker}\left(\frac{1}{2}(\mathrm{id} - g)\right) = \mathrm{Ker}\,(f)$$

and

$$U^- = \mathrm{Im}\left(\frac{1}{2}(\mathrm{id} - g)\right) = \mathrm{Im}(f),$$

which proves the proposition. $\square$

A projection such that $f = f^*$ is called an *orthogonal projection*.

If $(a_1 \ldots, a_k)$ are k linearly independent vectors in $\mathbb{R}^n$, let us determine the matrix P of the orthogonal projection onto the subspace of $\mathbb{R}^n$ spanned by $(a_1, \ldots, a_k)$. Let A be the $n \times k$ matrix whose jth column consists of the coordinates of the vector a_j over the canonical basis $(e_1, \ldots, e_n)$.

Any vector in the subspace $(a_1, \ldots, a_k)$ is a linear combination of the form Ax, for some $x \in \mathbb{R}^k$. Given any $y \in \mathbb{R}^n$, the orthogonal projection $Py = Ax$ of y onto the subspace spanned by $(a_1, \ldots, a_k)$ is the vector Ax such that $y - Ax$ is orthogonal to the subspace spanned by $(a_1, \ldots, a_k)$ (prove it). This means that $y - Ax$ is orthogonal to every a_j, which is expressed by

$$A^\top (y - Ax) = 0;$$

that is,

$$A^\top A x = A^\top y.$$

The matrix $A^\top A$ is invertible because A has full rank k, thus we get

$$x = (A^\top A)^{-1} A^\top y,$$

and so

$$Py = Ax = A(A^\top A)^{-1} A^\top y.$$

Therefore, the matrix P of the projection onto the subspace spanned by $(a_1 \ldots, a_k)$ is given by

$$P = A(A^\top A)^{-1} A^\top.$$

The reader should check that $P^2 = P$ and $P^\top = P$.