The proof of Lemma 27.4 yields two flips f_{2k-1} and f_{2k} such that

$$f_{2k}(e_{n-2}) = -e_{n-2} \quad \text{and} \quad s_{2k} \circ s_{2k-1} = f_{2k} \circ f_{2k-1},$$

since the $(n-2)$th diagonal entry in both matrices is -1, which means that $e_{n-2} \in F_{2k}^{\perp}$, where F_{2k} is the subspace of dimension $n-2$ determining f_{2k}. Since $u = \|u\| e_{n-2}$, we also have $u \in F_{2k}^{\perp}$. $\square$

Remarks:

(1) It is easy to prove that if f is a rotation in $\mathbf{SO}(3)$ and if D is its axis and θ is its angle of rotation, then f is the composition of two flips about lines D_1 and D_2 orthogonal to D and making an angle $\theta/2$.

(2) It is natural to ask what is the minimal number of flips needed to obtain a rotation f (when $n \geq 3$). As for arbitrary isometries, we will prove later that every rotation is the composition of k flips, where

$$k = n - \dim(\mathrm{Ker}\,(f - \mathrm{id})),$$

and that this number is minimal (where $n = \dim(E)$).

We now turn to affine isometries.

27.2 Affine Isometries (Rigid Motions)

In the remaining sections we study affine isometries. First, we characterize the set of fixed points of an affine map. Using this characterization, we prove that every affine isometry f can be written uniquely as

$$f = t \circ g, \quad \text{with} \quad t \circ g = g \circ t,$$

where g is an isometry having a fixed point, and t is a translation by a vector τ such that $\overrightarrow{f}(\tau) = \tau$, and with some additional nice properties (see Theorem 27.10). This is a generalization of a classical result of Chasles about (proper) rigid motions in $\mathbb{R}^3$ (screw motions). We prove a generalization of the Cartan–Dieudonné theorem for the affine isometries: Every isometry in $\mathbf{Is}(n)$ can be written as the composition of at most n affine reflections if it has a fixed point, or else as the composition of at most $n+2$ affine reflections. We also prove that every rigid motion in $\mathbf{SE}(n)$ is the composition of at most n affine flips (for $n \geq 3$). This is somewhat surprising, in view of the previous theorem.

Definition 27.1. Given any two nontrivial Euclidean affine spaces E and F of the same finite dimension n, a function $f\colon E \to F$ is *an affine isometry (or rigid map)* if it is an affine map and

$$\|\overrightarrow{f(a)f(b)}\| = \|\overrightarrow{ab}\|,$$

for all $a, b \in E$. When $E = F$, an affine isometry $f\colon E \to E$ is also called a *rigid motion*.