we obtain the equations

$$(\dagger_2) \quad \begin{cases} \nabla J_{u^k} + C^\top \lambda^k = 0 \\ \lambda^{k+1} = p_+(\lambda^k + \rho\varphi(u^k)). \end{cases}$$

Step 3. By subtracting the first of the two equations of $(\dagger_1)$ and $(\dagger_2)$ we obtain

$$\nabla J_{u^k} - \nabla J_u + C^\top(\lambda^k - \lambda) = 0,$$

and by subtracting the second of the two equations of $(\dagger_1)$ and $(\dagger_2)$ and using Proposition 48.6, we obtain

$$\left\|\lambda^{k+1} - \lambda\right\| \leq \left\|\lambda^k - \lambda + \rho C(u^k - u)\right\|.$$

In summary, we proved

$$(\dagger) \quad \begin{cases} \nabla J_{u^k} - \nabla J_u + C^\top(\lambda^k - \lambda) = 0 \\ \left\|\lambda^{k+1} - \lambda\right\| \leq \left\|\lambda^k - \lambda + \rho C(u^k - u)\right\|. \end{cases}$$

Step 4. Convergence of the sequence $(u^k)_{k\geq 0}$ to u.

Squaring both sides of the inequality in $(\dagger)$ we obtain

$$\left\|\lambda^{k+1} - \lambda\right\|^2 \leq \left\|\lambda^k - \lambda\right\|^2 + 2\rho\langle C^\top(\lambda^k - \lambda), u_k - u\rangle + \rho^2 \left\|C(u^k - u)\right\|^2.$$

Using the equation in $(\dagger)$ and the inequality

$$\langle \nabla J_{u^k} - \nabla J_u, u^k - u\rangle \geq \alpha \left\|u^k - u\right\|^2,$$

we get

$$\begin{aligned} \left\|\lambda^{k+1} - \lambda\right\|^2 &\leq \left\|\lambda^k - \lambda\right\|^2 - 2\rho\langle \nabla J_{u^k} - \nabla J_u, u^k - u\rangle + \rho^2 \left\|C(u^k - u)\right\|^2 \\ &\leq \left\|\lambda^k - \lambda\right\|^2 - \rho(2\alpha - \rho \left\|C\right\|_2^2) \left\|u^k - u\right\|^2. \end{aligned}$$

Consequently, if

$$0 \leq \rho \leq \frac{2\alpha}{\left\|C\right\|_2^2},$$

we have

$$\left\|\lambda^{k+1} - \lambda\right\| \leq \left\|\lambda^k - \lambda\right\|, \quad \text{for all } k \geq 0. \tag{$*_5$}$$

By $(*_5)$, the sequence $(\left\|\lambda^k - \lambda\right\|)_{k\geq 0}$ is nonincreasing and bounded below by 0, so it converges, which implies that

$$\lim_{k\mapsto\infty} \left(\left\|\lambda^{k+1} - \lambda\right\| - \left\|\lambda^k - \lambda\right\|\right) = 0,$$

and since

$$\left\|\lambda^{k+1} - \lambda\right\|^2 \leq \left\|\lambda^k - \lambda\right\|^2 - \rho(2\alpha - \rho \left\|C\right\|_2^2) \left\|u^k - u\right\|^2,$$