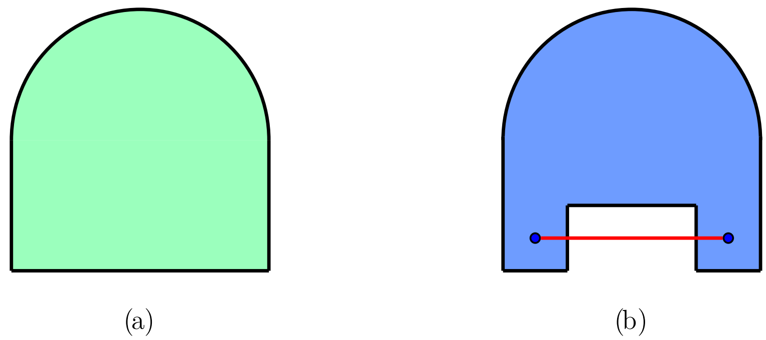

Figure 44.1: (a) A convex set; (b) A nonconvex set

Definition 44.2. An *affine subspace* A of $\mathbb{R}^n$ is any subset of $\mathbb{R}^n$ closed under affine combinations.

If A is a nonempty affine subspace of $\mathbb{R}^n$, then it can be shown that $V_A = \{a - b \mid a, b \in A\}$ is a linear subspace of $\mathbb{R}^n$ and that

$$A = a + V_A = \{a + v \mid v \in V_A\}$$

for *any* $a \in A$; see Gallier [72] (Section 2.5).

Definition 44.3. Given an affine subspace A, the linear space $V_A = \{a - b \mid a, b \in A\}$ is called the *direction of* A. The *dimension* of the nonempty affine subspace A is the dimension of its direction V_A.

Definition 44.4. *Convex combinations* are affine combinations $\lambda_1 x_1 + \cdots + \lambda_m x_m$ satisfying the extra condition that $\lambda_i \geq 0$ for $i = 1, \ldots, m$.

A convex set is defined as follows.

Definition 44.5. A subset V of $\mathbb{R}^n$ is *convex* if for any two points $a, b \in V$, we have $c \in V$ for every point $c = (1 - \lambda)a + \lambda b$, with $0 \leq \lambda \leq 1$ ($\lambda \in \mathbb{R}$). Given any two points a, b, the notation $[a, b]$ is often used to denote the line segment between a and b, that is,

$$[a, b] = \{c \in \mathbb{R}^n \mid c = (1 - \lambda)a + \lambda b,\ 0 \leq \lambda \leq 1\},$$

and thus a set V is convex if $[a, b] \subseteq V$ for any two points $a, b \in V$ ($a = b$ is allowed). The *dimension* of a convex set V is the dimension of its affine hull $\mathrm{aff}(A)$.

The empty set is trivially convex, every one-point set $\{a\}$ is convex, and the entire affine space $\mathbb{R}^n$ is convex.

It is obvious that the intersection of any family (finite or infinite) of convex sets is convex.