(2) Prove that

$$\left\|A^{-1}\right\|_\infty \leq \delta^{-1}.$$

Hint. Prove that

$$\sup_{v\neq 0} \frac{\|A^{-1}v\|_\infty}{\|v\|_\infty} = \sup_{w\neq 0} \frac{\|w\|_\infty}{\|Aw\|_\infty}.$$

Problem 9.5. Let A be any invertible complex $n \times n$ matrix.

(1) For any vector norm $\| \;\|$ on $\mathbb{C}^n$, prove that the function $\| \;\|_A : \mathbb{C}^n \to \mathbb{R}$ given by

$$\|x\|_A = \|Ax\| \quad \text{for all} \quad x \in \mathbb{C}^n,$$

is a vector norm.

(2) Prove that the operator norm induced by $\| \;\|_A$, also denoted by $\| \;\|_A$, is given by

$$\|B\|_A = \left\|ABA^{-1}\right\| \quad \text{for every } n \times n \text{ matrix} \quad B,$$

where $\|ABA^{-1}\|$ uses the operator norm induced by $\| \;\|$.

Problem 9.6. Give an example of a norm on $\mathbb{C}^n$ and of a *real* matrix A such that

$$\|A\|_{\mathbb{R}} < \|A\|,$$

where $\|-\|_{\mathbb{R}}$ and $\|-\|$ are the operator norms associated with the vector norm $\|-\|$.

Hint. This can already be done for $n = 2$.

Problem 9.7. Let $\| \;\|$ be any operator norm. Given an invertible $n \times n$ matrix A, if $c = 1/(2\left\|A^{-1}\right\|)$, then for every $n \times n$ matrix H, if $\|H\| \leq c$, then $A + H$ is invertible. Furthermore, show that if $\|H\| \leq c$, then $\|(A + H)^{-1}\| \leq 1/c$.

Problem 9.8. Let A be any $m \times n$ matrix and let $\lambda \in \mathbb{R}$ be any positive real number $\lambda > 0$.

(1) Prove that $A^\top A + \lambda I_n$ and $AA^\top + \lambda I_m$ are invertible.

(2) Prove that

$$A^\top(AA^\top + \lambda I_m)^{-1} = (A^\top A + \lambda I_n)^{-1}A^\top.$$

Remark: The expressions above correspond to the matrix for which the function

$$\Phi(x) = (Ax - b)^\top(Ax - b) + \lambda x^\top x$$

achieves a minimum. It shows up in machine learning (kernel methods).

Problem 9.9. Let Z be a $q \times p$ real matrix. Prove that if $I_p - Z^\top Z$ is positive definite, then the $(p + q) \times (p + q)$ matrix

$$S = \begin{pmatrix} I_p & Z^\top \\ Z & I_q \end{pmatrix}$$

is symmetric positive definite.