We now define the notion of affine tangent space in a very general way. Next, we will see what it means for manifolds $\Gamma(f)$, as in Definition 39.10.

Definition 39.11. Given a normed affine space E, given any nonempty subset M of E, given any point $a \in M$, we say that a vector $u \in \overrightarrow{E}$ is *tangent at a to M* if there exist a sequence $(a_n)_{n\in\mathbb{N}}$ of points in M converging to a, and a sequence $(\lambda_n)_{n\in\mathbb{N}}$, with $\lambda_i \in \mathbb{R}$ and $\lambda_n \geq 0$, such that the sequence $(\lambda_n(a_n - a))_{n\in\mathbb{N}}$ converges to u.

The set of all vectors tangent at a to M is called the *family of tangent vectors at a to M* and the set of all points of E of the form $a + u$ where u belongs to the family of tangent vectors at a to M is called the *affine tangent family at a to M*.

Clearly, 0 is always tangent, and if u is tangent, then so is every λu, for $\lambda \in \mathbb{R}$, $\lambda \geq 0$. If $u \neq 0$, then the sequence $(\lambda_n)_{n\in\mathbb{N}}$ must tend towards $+\infty$. We have the following proposition.

Proposition 39.18. *Let E and F be two normed affine spaces, let A be an open subset of E, let $a \in A$, and let $f\colon A \to F$ be a function. If $\mathrm{D}f(a)$ exists, then the family of tangent vectors at $(a, f(a))$ to Γ is a subspace $T_a(\Gamma)$ of $\overrightarrow{E} \times \overrightarrow{F}$, defined by the condition (equation)*

$$(u, v) \in T_a(\Gamma) \quad \textit{iff} \quad v = \mathrm{D}f(a)(u),$$

and the affine tangent family at $(a, f(a))$ to Γ is an affine variety $T_a(\Gamma)$ of $E \times F$, defined by the condition (equation)

$$(x, y) \in T_a(\Gamma) \quad \textit{iff} \quad y = f(a) + \mathrm{D}f(a)(x - a),$$

where Γ is the graph of f.

The proof is actually rather simple. We have $T_a(\Gamma) = a + T_a(\Gamma)$, and since $T_a(\Gamma)$ is a subspace of $\overrightarrow{E} \times \overrightarrow{F}$, the set $T_a(\Gamma)$ is an affine variety. Thus, the affine tangent space at a point $(a, f(a))$ is a familar object, a line, a plane, etc.

As an illustration, when $E = \mathbb{R}^2$ and $F = \mathbb{R}$, the affine tangent plane at the point (a, b, c) to the surface of equation $z = f(x, y)$, is defined by the equation

$$z = c + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b).$$

If $E = \mathbb{R}$ and $F = \mathbb{R}^2$, the tangent line at (a, b, c), to the curve of equations $y = g(x)$, $z = h(x)$, is defined by the equations

$$\begin{aligned} y &= b + \mathrm{D}g(a)(x - a), \\ z &= c + \mathrm{D}h(a)(x - a). \end{aligned}$$

Thus, derivatives and partial derivatives have the desired intended geometric interpretation as tangent spaces. Of course, in order to deal with this topic properly, we really would have to go deeper into the study of (differential) manifolds.

We now briefly consider second-order and higher-order derivatives.