Since the permutations of $\{a, b, c, d\}$ are generated by the above transpositions, the cross-ratio takes at most six values. Letting $\lambda = [a, b, c, d]$, if $\lambda \in \{\infty, 0, 1\}$, then any permutation of $\{a, b, c, d\}$ yields a cross-ratio in $\{\infty, 0, 1\}$, and if $\lambda \notin \{\infty, 0, 1\}$, then there are at most the six values

$$\lambda, \quad \frac{1}{\lambda}, \quad 1-\lambda, \quad 1-\frac{1}{\lambda}, \quad \frac{1}{1-\lambda}, \quad \frac{\lambda}{\lambda-1}.$$

It can be shown that the function

$$\lambda \mapsto 256\frac{(\lambda^2-\lambda+1)^3}{\lambda^2(1-\lambda)^2}$$

takes a constant value on the six values listed above.

We also define when four points form a harmonic division. For this, we need to assume that K is not of characteristic 2.

Definition 26.9. Given a projective line Δ, we say that a sequence of four collinear points (a, b, c, d) in Δ (where a, b, c are distinct) forms a *harmonic division* if $[a, b, c, d] = -1$. When $[a, b, c, d] = -1$, we also say that c and d are *harmonic conjugates* of a and b.

If a, b, c are distinct collinear points in some affine space, from

$$[a, b, c, \infty] = \frac{\overrightarrow{ca}}{\overrightarrow{cb}},$$

we note that c is the midpoint of (a, b) iff $[a, b, c, \infty] = -1$, that is, if (a, b, c, ∞) forms a harmonic division. Figure 26.22 shows a harmonic division (a, b, c, d) on the real line, where the coordinates of (a, b, c, d) are $(-2, 2, 1, 4)$.

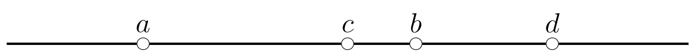

Figure 26.22: Four points forming a harmonic division.

If $\Delta = \mathbb{P}^1_K$ and a, b, c, d are all distinct from ∞, then we see immediately from the formula

$$[a, b, c, d] = \frac{c-a}{c-b} \Bigg/ \frac{d-a}{d-b}$$

that $[a, b, c, d] = -1$ iff

$$2(ab + cd) = (a + b)(c + d).$$

We also check immediately that $[a, b, c, \infty] = -1$ iff

$$a + b = 2c.$$