If we write $h = g \circ f$, we need to find the maps $h_k \colon E_k \to G$, with

$$h_k(x_k) = h(x_k) = (g \circ f)(x_k),$$

and $h_{ik} \colon E_k \to G_i$ given by

$$h_{ik} = pr_i^G \circ h_k.$$

We have

$$\begin{aligned}
h_k(x_k) &= (g \circ f)(x_k) = g(f(x_k)) = g(f_k(x_k)) \\
&= g\bigg(\sum_{j=1}^n f_{jk}(x_k)\bigg) && \text{by } (*_1) \\
&= \sum_{j=1}^n g(f_{jk}(x_k)) = \sum_{j=1}^n g_j(f_{jk}(x_k)) && \text{since } g \text{ is linear} \\
&= \sum_{j=1}^n \sum_{i=1}^m g_{ij}(f_{jk}(x_k)) = \sum_{i=1}^m \sum_{j=1}^n g_{ij}(f_{jk}(x_k)), && \text{by } (*_2)
\end{aligned}$$

and since $\sum_{j=1}^n g_{ij}(f_{jk}(x_k)) \in G_i$, we conclude that

$$h_{ik}(x_k) = \sum_{j=1}^n g_{ij}(f_{jk}(x_k)) = \sum_{j=1}^n (g_{ij} \circ f_{jk})(x_k), \tag{$*_3$}$$

which can also be expressed as

$$h_{ik} = \sum_{j=1}^n g_{ij} \circ f_{jk}. \tag{$*_4$}$$

Equation $(*_4)$ is exactly the analog of the formula for the multiplication of matrices of scalars! We just have to replace multiplication by composition. The $m \times p$ matrix of linear maps (h_{ik}) is the "product" AB of the matrices of linear maps $A = (g_{ij})$ and $B = (f_{jk})$, except that multiplication is replaced by composition.

In summary we just proved the following result.

Proposition 6.12. *Let E, F, G be three vector spaces expressed as direct sums*

$$E = \bigoplus_{k=1}^p E_k, \quad F = \bigoplus_{j=1}^n F_j, \quad G = \bigoplus_{i=1}^m G_i.$$

For any two linear maps $f \colon E \to F$ and $g \colon F \to G$, let $B = (f_{jk})$ be the $n \times p$ matrix of linear maps associated with f (with respect to the decomposition of E and F as direct sums) and let $A = (g_{ij})$ be the $m \times n$ matrix of linear maps associated with g (with respect to the decomposition of F and G as direct sums). Then the $m \times p$ matrix $C = (h_{ik})$ of linear maps