4.4 The Effect of a Change of Bases on Matrices

The effect of a change of bases on the representation of a linear map is described in the following proposition.

Proposition 4.5. *Let E and F be vector spaces, let $\mathcal{U} = (u_1, \ldots, u_n)$ and $\mathcal{U}' = (u'_1, \ldots, u'_n)$ be two bases of E, and let $\mathcal{V} = (v_1, \ldots, v_m)$ and $\mathcal{V}' = (v'_1, \ldots, v'_m)$ be two bases of F. Let $P = P_{\mathcal{U}',\mathcal{U}}$ be the change of basis matrix from $\mathcal{U}$ to $\mathcal{U}'$, and let $Q = P_{\mathcal{V}',\mathcal{V}}$ be the change of basis matrix from $\mathcal{V}$ to $\mathcal{V}'$. For any linear map $f \colon E \to F$, let $M(f) = M_{\mathcal{U},\mathcal{V}}(f)$ be the matrix associated to f w.r.t. the bases $\mathcal{U}$ and $\mathcal{V}$, and let $M'(f) = M_{\mathcal{U}',\mathcal{V}'}(f)$ be the matrix associated to f w.r.t. the bases $\mathcal{U}'$ and $\mathcal{V}'$. We have*

$$M'(f) = Q^{-1}M(f)P,$$

or more explicitly

$$M_{\mathcal{U}',\mathcal{V}'}(f) = P^{-1}_{\mathcal{V}',\mathcal{V}} M_{\mathcal{U},\mathcal{V}}(f) P_{\mathcal{U}',\mathcal{U}} = P_{\mathcal{V},\mathcal{V}'} M_{\mathcal{U},\mathcal{V}}(f) P_{\mathcal{U}',\mathcal{U}}.$$

Proof. Since $f \colon E \to F$ can be written as $f = \mathrm{id}_F \circ f \circ \mathrm{id}_E$, since $P = P_{\mathcal{U}',\mathcal{U}}$ is the matrix of id_E w.r.t. the bases $(u'_1, \ldots, u'_n)$ and $(u_1, \ldots, u_n)$, and $Q^{-1} = P^{-1}_{\mathcal{V}',\mathcal{V}} = P_{\mathcal{V},\mathcal{V}'}$ is the matrix of id_F w.r.t. the bases $(v_1, \ldots, v_m)$ and $(v'_1, \ldots, v'_m)$ as illustrated by the following diagram

$$\begin{tikzcd}
\mathcal{U}, E \arrow[r, "f", "M_{\mathcal{U},\mathcal{V}}(f)"'] & \mathcal{V}, F \arrow[d, "P^{-1}_{\mathcal{V}',\mathcal{V}}"', "\mathrm{id}_F"] \\
\mathcal{U}', E \arrow[u, "P_{\mathcal{U}',\mathcal{U}}", "\mathrm{id}_E"'] \arrow[r, "M_{\mathcal{U}',\mathcal{V}'}(f)", "f"'] & \mathcal{V}', F,
\end{tikzcd}$$

by Proposition 4.2, we have $M'(f) = Q^{-1}M(f)P$. ☐

As a corollary, we get the following result.

Corollary 4.6. *Let E be a vector space, and let $\mathcal{U} = (u_1, \ldots, u_n)$ and $\mathcal{U}' = (u'_1, \ldots, u'_n)$ be two bases of E. Let $P = P_{\mathcal{U}',\mathcal{U}}$ be the change of basis matrix from $\mathcal{U}$ to $\mathcal{U}'$. For any linear map $f \colon E \to E$, let $M(f) = M_{\mathcal{U}}(f)$ be the matrix associated to f w.r.t. the basis $\mathcal{U}$, and let $M'(f) = M_{\mathcal{U}'}(f)$ be the matrix associated to f w.r.t. the basis $\mathcal{U}'$. We have*

$$M'(f) = P^{-1}M(f)P,$$

or more explicitly,

$$M_{\mathcal{U}'}(f) = P^{-1}_{\mathcal{U}',\mathcal{U}} M_{\mathcal{U}}(f) P_{\mathcal{U}',\mathcal{U}} = P_{\mathcal{U},\mathcal{U}'} M_{\mathcal{U}}(f) P_{\mathcal{U}',\mathcal{U}},$$