for every finite subset I of K, we get

$$\begin{aligned}\|u-v\| &\leq \left\|u-\sum_{i\in I} c_i u_i\right\| + \left\|\sum_{i\in I} c_i u_i - v\right\| \\ &\leq \left\|u-\sum_{i\in I} c_i u_i\right\| + \left\|\sum_{i\in I} \lambda_i u_i - v\right\| \\ &\leq \left\|u-\sum_{i\in I} c_i u_i\right\| + \|v-w\| + \left\|w-\sum_{i\in I} \lambda_i u_i\right\|,\end{aligned}$$

and thus

$$\|u-v\| \leq \|v-w\| + 2\epsilon.$$

Since this holds for every $\epsilon > 0$, we have

$$\|u-v\| \leq \|v-w\|$$

for all $w \in V$, i.e. $\|v-u\| = d(v,V)$, with $u \in V$, which proves that $u = p_V(v)$. $\square$

A.2 The Hilbert Space $\ell^2(K)$ and the Riesz–Fischer Theorem

Proposition A.2 suggests looking at the space of sequences $(z_k)_{k\in K}$ (where $z_k \in \mathbb{C}$) such that $(|z_k|^2)_{k\in K}$ is summable. Indeed, such spaces are Hilbert spaces, and it turns out that every Hilbert space is isomorphic to one of those. Such spaces are the infinite-dimensional version of the spaces $\mathbb{C}^n$ under the usual Euclidean norm.

Definition A.3. Given any nonempty index set K, the space $\ell^2(K)$ is the set of all sequences $(z_k)_{k\in K}$, where $z_k \in \mathbb{C}$, such that $(|z_k|^2)_{k\in K}$ is summable, i.e., $\sum_{k\in K} |z_k|^2 < \infty$.

Remarks:

(1) When K is a finite set of cardinality n, $\ell^2(K)$ is isomorphic to $\mathbb{C}^n$.

(2) When $K = \mathbb{N}$, the space $\ell^2(\mathbb{N})$ corresponds to the space ℓ^2 of Example 2 in Section 14.1 . In that example, we claimed that ℓ^2 was a Hermitian space, and in fact, a Hilbert space. We now prove this fact for any index set K.

Proposition A.3. *Given any nonempty index set K, the space $\ell^2(K)$ is a Hilbert space under the Hermitian product*

$$\langle (x_k)_{k\in K}, (y_k)_{k\in K}\rangle = \sum_{k\in K} x_k \overline{y_k}.$$

The subspace consisting of sequences $(z_k)_{k\in K}$ such that $z_k = 0$, except perhaps for finitely many k, is a dense subspace of $\ell^2(K)$.