(1) The functions $f\colon \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ *and* $g\colon \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ *are proper and closed convex functions (see Section 51.1) such that* $\mathbf{relint}(\mathrm{dom}(f)) \cap \mathbf{relint}(\mathrm{dom}(g)) \neq \emptyset$.

(2) The $n \times n$ *matrix* $A^\top A$ *is invertible and the* $m \times m$ *matrix* $B^\top B$ *is invertible. Equivalently, the* $p \times n$ *matrix* A *has rank* n *and the* $p \times m$ *matrix has rank* m. *(This assumption is actually redundant, because it is implied by Assumption (3)).*

(3) The unaugmented Lagrangian $L_0(x, z, \lambda) = f(x) + g(z) + \lambda^\top (Ax + Bz - c)$ *has a saddle point, which means there exists* x^*, z^*, λ^* *(not necessarily unique) such that*

$$L_0(x^*, z^*, \lambda) \leq L_0(x^*, z^*, \lambda^*) \leq L_0(x, z, \lambda^*)$$

for all x, z, λ.

Then for any initial values (z^0, λ^0), *the following properties hold:*

(1) The sequence (r^k) *converges to* 0 *(residual convergence).*

(2) The sequence (p^k) *converge to* p^* *(objective convergence).*

(3) The sequences (x^k) *and* (z^k) *converge to an optimal solution* $(\widetilde{x}, \widetilde{z})$ *of Problem* (P_{admm}) *and the sequence* (λ^k) *converges an optimal solution* $\widetilde{\lambda}$ *of the dual problem (primal and dual variable convergence).*

Proof. The core of the proof is due to Boyd et al. [28], but there are new steps because we have the stronger hypothesis (2), which yield the stronger result (3).

The proof consists of several steps. It is not possible to prove directly that the sequences (x^k), (z^k), and (λ^k) converge, so first we prove that the sequence (r^{k+1}) converges to zero, and that the sequences (Ax^{k+1}) and (Bz^{k+1}) also converge.

Step 1. Prove the inequality (A1) below.

Consider the sequence of reals (V^k) given by

$$V^k = (1/\rho) \left\|\lambda^k - \lambda^*\right\|_2^2 + \rho \left\|B(z^k - z^*)\right\|_2^2.$$

It can be shown that the V^k satisfy the following inequality:

$$V^{k+1} \leq V^k - \rho \left\|r^{k+1}\right\|_2^2 - \rho \left\|B(z^{k+1} - z^k)\right\|_2^2. \tag{A1}$$

This is rather arduous. Since a complete proof is given in Boyd et al. [28], we will only provide some of the key steps later.

Inequality (A1) shows that the sequence (V^k) in nonincreasing. If we write these inequalities for $k, k-1, \ldots, 0$, we have

$$\begin{aligned}
V^{k+1} &\leq V^k - \rho \left\|r^{k+1}\right\|_2^2 - \rho \left\|B(z^{k+1} - z^k)\right\|_2^2 \\
V^k &\leq V^{k-1} - \rho \left\|r^k\right\|_2^2 - \rho \left\|B(z^k - z^{k-1})\right\|_2^2 \\
&\ \vdots \\
V^1 &\leq V^0 - \rho \left\|r^1\right\|_2^2 - \rho \left\|B(z^1 - z^0)\right\|_2^2,
\end{aligned}$$