which shows that the sequence $(\langle v, u_k\rangle)_{k\geq 0}$ is bounded. Since V is a separable Hilbert space, there is a countable family $(v_k)_{k\geq 0}$ of vectors $v_k \in V$ which is dense in V. Since the sequence $(\langle v_1, u_k\rangle)_{k\geq 0}$ is bounded (in $\mathbb{R}$), we can find a convergent subsequence $(\langle v_1, u_{i_1(j)}\rangle)_{j\geq 0}$. Similarly, since the sequence $(\langle v_2, u_{i_1(j)}\rangle)_{j\geq 0}$ is bounded, we can find a convergent subsequence $(\langle v_2, u_{i_2(j)}\rangle)_{j\geq 0}$, and in general, since the sequence $(\langle v_k, u_{i_{k-1}(j)}\rangle)_{j\geq 0}$ is bounded, we can find a convergent subsequence $(\langle v_k, u_{i_k(j)}\rangle)_{j\geq 0}$.

We obtain the following infinite array:

$$\begin{pmatrix} \langle v_1, u_{i_1(1)}\rangle & \langle v_2, u_{i_2(1)}\rangle & \cdots & \langle v_k, u_{i_k(1)}\rangle & \cdots \\ \langle v_1, u_{i_1(2)}\rangle & \langle v_2, u_{i_2(2)}\rangle & \cdots & \langle v_k, u_{i_k(2)}\rangle & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle v_1, u_{i_1(k)}\rangle & \langle v_2, u_{i_2(k)}\rangle & \cdots & \langle v_k, u_{i_k(k)}\rangle & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Consider the "diagonal" sequence $(w_\ell)_{\ell\geq 0}$ defined by

$$w_\ell = u_{i_\ell(\ell)}, \quad \ell \geq 0.$$

We are going to prove that for every $v \in V$, the sequence $(\langle v, w_\ell\rangle)_{\ell\geq 0}$ has a limit.

By construction, for every $k \geq 0$, the sequence $(\langle v_k, w_\ell\rangle)_{\ell\geq 0}$ has a limit, which is the limit of the sequence $(\langle v_k, u_{i_k(j)}\rangle)_{j\geq 0}$, since the sequence $(i_\ell(\ell))_{\ell\geq 0}$ is a subsequence of every sequence $(i_\ell(j))_{j\geq 0}$ for every $\ell \geq 0$.

Pick any $v \in V$ and any $\epsilon > 0$. Since $(v_k)_{k\geq 0}$ is dense in V, there is some v_k such that

$$\|v - v_k\| \leq \epsilon/(4C).$$

Then we have

$$\begin{aligned} |\langle v, w_\ell\rangle - \langle v, w_m\rangle| &= |\langle v, w_\ell - w_m\rangle| \\ &= |\langle v_k + v - v_k, w_\ell - w_m\rangle| \\ &= |\langle v_k, w_\ell - w_m\rangle + \langle v - v_k, w_\ell - w_m\rangle| \\ &\leq |\langle v_k, w_\ell\rangle - \langle v_k, w_m\rangle| + |\langle v - v_k, w_\ell - w_m\rangle|. \end{aligned}$$

By Cauchy–Schwarz and since $\|w_\ell - w_m\| \leq \|w_\ell\| + \|w_m\| \leq C + C = 2C$,

$$|\langle v - v_k, w_\ell - w_m\rangle| \leq \|v - v_k\| \, \|w_\ell - w_m\| \leq (\epsilon/(4C))2C = \epsilon/2,$$

so

$$|\langle v, w_\ell\rangle - \langle v, w_m\rangle| \leq |\langle v_k, w_\ell - w_m\rangle| + \epsilon/2.$$

With the element v_k held fixed, by a previous argument the sequence $(\langle v_k, w_\ell\rangle)_{\ell\geq 0}$ converges, so it is a Cauchy sequence. Consequently there is some ℓ_0 (depending on ϵ and v_k) such that

$$|\langle v_k, w_\ell\rangle - \langle v_k, w_m\rangle| \leq \epsilon/2 \quad \text{for all } \ell, m \geq \ell_0,$$