However, since m_1 and m_2 are the slopes of the lines D_1 and D_2, it is well known that if θ is the (oriented) angle between D_1 and D_2, then

$$\tan\theta = \frac{m_2 - m_1}{m_1 m_2 + 1}.$$

Thus, we have

$$[D_1, D_2, D_I, D_J] = \frac{m_1 m_2 + 1 + i(m_2 - m_1)}{m_1 m_2 + 1 - i(m_2 - m_1)} = \frac{1 + i\tan\theta}{1 - i\tan\theta},$$

that is,

$$[D_1, D_2, D_I, D_J] = \cos 2\theta + i \sin 2\theta = e^{i2\theta}.$$

One can check that the formula still holds when $m_1 = \infty$ or $m_2 = \infty$, and also when $D_1 = D_2$. The formula

$$[D_1, D_2, D_I, D_J] = e^{i2\theta}$$

is known as *Laguerre's formula*.

If U denotes the group $\{e^{i\theta} \mid -\pi \leq \theta \leq \pi\}$ of complex numbers of modulus 1, recall that the map $\Lambda\colon \mathbb{R} \to U$ defined such that

$$\Lambda(t) = e^{it}$$

is a group homomorphism such that $\Lambda^{-1}(1) = 2k\pi$, where $k \in \mathbb{Z}$. The restriction

$$\Lambda\colon\ ]-\pi,\, \pi[\ \to (U - \{-1\})$$

of Λ to $\ ]-\pi,\, \pi[\ $ is a bijection, and its inverse will be denoted by

$$\log_U\colon (U - \{-1\}) \to\ ]-\pi,\, \pi[\ .$$

For stating Proposition 26.28 more conveniently, we extend $\log_U$ to U by letting $\log_U(-1) = \pi$, even though the resulting function is not continuous at -1!. Then we can write

$$\theta = \frac{1}{2}\log_U([D_1, D_2, D_I, D_J]).$$

If the orientation of the plane E is reversed, θ becomes $\pi - \theta$, and since

$$e^{i2(\pi-\theta)} = e^{2i\pi - i2\theta} = e^{-i2\theta},$$

$\log_U(e^{i2(\pi-\theta)}) = -\log_U(e^{i2\theta})$, and

$$\theta = -\frac{1}{2}\log_U([D_1, D_2, D_I, D_J]).$$

In all cases, we have

$$\theta = \frac{1}{2}|\log_U([D_1, D_2, D_I, D_J])|,$$

a formula due to Cayley. We summarize the above in the following proposition.