Corollary 47.10. *The Primal Program (P) has an optimal solution iff the following set of constraints is satisfiable:*

$$\begin{aligned} Ax &\leq b \\ yA &\geq c \\ cx &\geq yb \\ x &\geq 0,\ y \geq 0_m^\top. \end{aligned}$$

In fact, for any feasible solution (x^*, y^*) of the above system, x^* is an optimal solution of (P) and y^* is an optimal solution of (D)

47.3 Complementary Slackness Conditions

Another useful corollary of the strong duality theorem is the following result known as the *equilibrium theorem*.

Theorem 47.11. *(Equilibrium Theorem) For any Linear Program (P) and its Dual Linear Program (D) (with set of inequalities $Ax \leq b$ where A is an $m \times n$ matrix, and objective function $x \mapsto cx$), for any feasible solution x of (P) and any feasible solution y of (D), x and y are optimal solutions iff*

$$y_i = 0 \qquad \text{for all } i \text{ for which } \textstyle\sum_{j=1}^n a_{ij}x_j < b_i \tag{$*_D$}$$

and

$$x_j = 0 \qquad \text{for all } j \text{ for which } \textstyle\sum_{i=1}^m y_i a_{ij} > c_j. \tag{$*_P$}$$

Proof. First assume that $(*_D)$ and $(*_P)$ hold. The equations in $(*_D)$ say that $y_i = 0$ unless $\sum_{j=1}^n a_{ij}x_j = b_i$, hence

$$yb = \sum_{i=1}^m y_i b_i = \sum_{i=1}^m y_i \sum_{j=1}^n a_{ij}x_j = \sum_{i=1}^m \sum_{j=1}^n y_i a_{ij} x_j.$$

Similarly, the equations in $(*_P)$ say that $x_j = 0$ unless $\sum_{i=1}^m y_i a_{ij} = c_j$, hence

$$cx = \sum_{j=1}^n c_j x_j = \sum_{j=1}^n \sum_{i=1}^m y_i a_{ij} x_j.$$

Consequently, we obtain

$$cx = yb.$$

By weak duality (Proposition 47.6), we have

$$cx \leq yb = cx$$