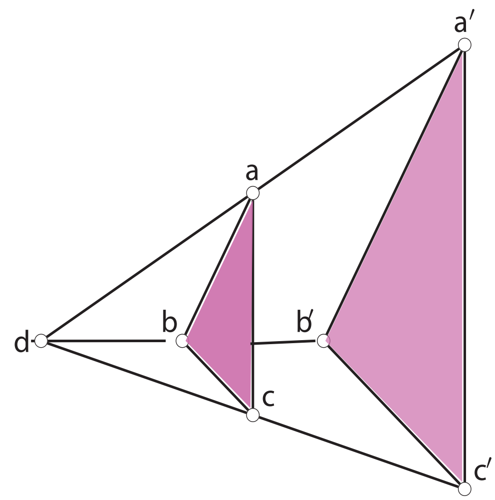

Figure 24.20: The effect of a central dilatation $H_{d,\lambda}(x)$.

Proposition 24.9. *Given any affine space E, for any affine bijection $f \in \mathbf{GA}(E)$, if $\overrightarrow{f} = \lambda\, \mathrm{id}_{\overrightarrow{E}}$, for some $\lambda \in \mathbb{R}^*$ with $\lambda \neq 1$, then there is a unique point $c \in E$ such that $f = H_{c,\lambda}$.*

Proof. The proof is straightforward, and is omitted. It is also given in Gallier [70]. □

Clearly, if $\overrightarrow{f} = \mathrm{id}_{\overrightarrow{E}}$, the affine map f is a translation. Thus, the group of affine dilatations $\mathbf{DIL}(E)$ is the disjoint union of the translations and of the dilatations of ratio $\lambda \neq 0, 1$. Affine dilatations can be given a purely geometric characterization.

Another point worth mentioning is that affine bijections preserve the ratio of volumes of parallelotopes. Indeed, given any basis $B = (u_1, \ldots, u_m)$ of the vector space $\overrightarrow{E}$ associated with the affine space E, given any $m+1$ affinely independent points $(a_0, \ldots, a_m)$, we can compute the determinant $\det_B(\overrightarrow{a_0a_1}, \ldots, \overrightarrow{a_0a_m})$ w.r.t. the basis B. For any bijective affine map $f\colon E \to E$, since

$$\det_B\Big(\overrightarrow{f}(\overrightarrow{a_0a_1}), \ldots, \overrightarrow{f}(\overrightarrow{a_0a_m})\Big) = \det\big(\overrightarrow{f}\big)\det_B(\overrightarrow{a_0a_1}, \ldots, \overrightarrow{a_0a_m})$$

and the determinant of a linear map is intrinsic (i.e., depends only on $\overrightarrow{f}$, and not on the particular basis B), we conclude that the ratio

$$\frac{\det_B\Big(\overrightarrow{f}(\overrightarrow{a_0a_1}), \ldots, \overrightarrow{f}(\overrightarrow{a_0a_m})\Big)}{\det_B(\overrightarrow{a_0a_1}, \ldots, \overrightarrow{a_0a_m})} = \det\big(\overrightarrow{f}\big)$$

is independent of the basis B. Since $\det_B(\overrightarrow{a_0a_1}, \ldots, \overrightarrow{a_0a_m})$ is the volume of the parallelotope spanned by $(a_0, \ldots, a_m)$, where the parallelotope spanned by any point a and the vectors