respect to the basis $\mathcal{P} = (p_1, p_2, p_3)$ and the coordinates of (q_1, q_2, q_3, q_4) are also expressed with respect to the basis $\mathcal{P} = (p_1, p_2, p_3)$. In practical situations, for example in computer vision, it is important to find necessary and sufficient conditions for the unique projective transformation mapping (p_1, p_2, p_3, p_4) to (q_1, q_2, q_3, q_4) to be defined on the convex hull of the points p_1, p_2, p_3, p_4.

Proposition 26.14. *The unique projective transformation mapping (p_1, p_2, p_3, p_4) to (q_1, q_2, q_3, q_4) (all points in the affine plane H of equation $z = 1$) is defined on the convex hull of the points p_1, p_2, p_3, p_4 iff the scalars in each of the pairs (α_1, λ_1), (α_2, λ_2) and (α_3, λ_3), have the same sign.*

Proof. With respect to the basis $\mathcal{P}$, the equation of the plane H is

$$x + y + z = 1,$$

so the image of $p = (x, y, 1 - x - y)$ under our linear map is

$$\begin{pmatrix} \frac{\lambda_1}{\alpha_1} x_1 & \frac{\lambda_2}{\alpha_2} x_2 & \frac{\lambda_3}{\alpha_3} x_3 \\ \frac{\lambda_1}{\alpha_1} y_1 & \frac{\lambda_2}{\alpha_2} y_2 & \frac{\lambda_3}{\alpha_3} y_3 \\ \frac{\lambda_1}{\alpha_1} & \frac{\lambda_2}{\alpha_2} & \frac{\lambda_3}{\alpha_3} \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 - x - y \end{pmatrix}.$$

The above point is a point at infinity iff

$$\left(\frac{\lambda_1}{\alpha_1} - \frac{\lambda_3}{\alpha_3}\right) x + \left(\frac{\lambda_2}{\alpha_2} - \frac{\lambda_3}{\alpha_3}\right) y + \frac{\lambda_3}{\alpha_3} = 0. \tag{$*$}$$

The unique projective transformation mapping (p_1, p_2, p_3, p_4) to (q_1, q_2, q_3, q_4) is defined on the convex hull of the points p_1, p_2, p_3, p_4 iff all four points p_1, p_2, p_3, p_4 are strictly contained in one of the two open half spaces determined by the line of equation $(*)$, which means that the affine form in $(*)$ must have the same sign on these four points.

When we evaluate the affine form in $(*)$ on the four points p_1, p_2, p_3, p_4 using coordinates $(x, y, 1 - x - y)$, w.r.t. the basis $\mathcal{P} = (p_1, p_2, p_3)$,

1. for $p_1 = (1, 0, 0)$ we get λ_1/α_1,
2. for $p_2 = (0, 1, 0)$ we get λ_2/α_2,
3. for $p_3 = (0, 0, 1)$ we get λ_3/α_3,
4. and for $p_4 = (\alpha_1, \alpha_2, \alpha_3)$ we get

$$\begin{aligned} \left(\frac{\lambda_1}{\alpha_1} - \frac{\lambda_3}{\alpha_3}\right) \alpha_1 + \left(\frac{\lambda_2}{\alpha_2} - \frac{\lambda_3}{\alpha_3}\right) \alpha_2 + \frac{\lambda_3}{\alpha_3} &= \lambda_1 + \lambda_2 + \frac{\lambda_3}{\alpha_3}(1 - \alpha_1 - \alpha_2) \\ &= \lambda_1 + \lambda_2 + \lambda_3 = 1. \end{aligned}$$