If we pick $\rho > 0$ such that $\rho < 2\alpha/C^2$, then

$$k^2 = 1 - 2\rho\alpha + \rho^2 C^2 < 1,$$

and then

$$\|F(v_1) - F(v_2)\| \leq k \|v_1 - v_2\|, \tag{$*_7$}$$

with $0 \leq k < 1$, which shows that F is a contraction. By Theorem 49.5, the map F has a unique fixed point $u \in U$, which concludes the proof of the first statement. If a is also symmetric, then the second statement is just the first part of Theorem 49.4. $\square$

Remark: Many physical problems can be expressed in terms of an unknown function u that satisfies some inequality

$$a(u, v - u) \geq h(v - u) \quad \text{for all } v \in U,$$

for some set U of "admissible" functions which is closed and convex. The bilinear form a and the linear form h are often given in terms of integrals. The above inequality is called a *variational inequality*.

In the special case where $U = V$ we obtain the Lax–Milgram theorem.

Theorem 49.7. *(Lax–Milgram's Theorem) Given a Hilbert space V, let $a\colon V \times V \to \mathbb{R}$ be a continuous bilinear form (not necessarily symmetric), let $h \in V'$ be a continuous linear form, and let J be given by*

$$J(v) = \frac{1}{2} a(v, v) - h(v), \quad v \in V.$$

If a is coercive, which means that there is some $\alpha > 0$ such that

$$a(v, v) \geq \alpha \|v\|^2 \quad \textit{for all } v \in V,$$

then there is a unique $u \in V$ such that

$$a(u, v) = h(v) \quad \textit{for all } v \in V.$$

If a is symmetric, then $u \in V$ is the unique element of V such that

$$J(u) = \inf_{v \in V} J(v).$$

The Lax–Milgram theorem plays an important role in solving linear elliptic partial differential equations; see Brezis [31].

We now consider various methods, known as gradient descents, to find minima of certain types of functionals.