13.3 Summary

The main concepts and results of this chapter are listed below:

- *Symmetry (or reflection) with respect to F and parallel to G.*
- *Orthogonal symmetry (or reflection) with respect to F and parallel to G; reflections, flips.*
- Hyperplane reflections and *Householder matrices*.
- A key fact about reflections (Proposition 13.2).
- *QR-decomposition in terms of Householder transformations* (Theorem 13.4).

13.4 Problems

Problem 13.1. (1) Given a unit vector $(-\sin\theta, \cos\theta)$, prove that the Householder matrix determined by the vector $(-\sin\theta, \cos\theta)$ is

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}.$$

Give a geometric interpretation (i.e., why the choice $(-\sin\theta, \cos\theta)$?).

(2) Given any matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

Prove that there is a Householder matrix H such that AH is lower triangular, i.e.,

$$AH = \begin{pmatrix} a' & 0 \\ c' & d' \end{pmatrix}$$

for some $a', c', d' \in \mathbb{R}$.

Problem 13.2. Given a Euclidean space E of dimension n, if h is a reflection about some hyperplane orthogonal to a nonzero vector u and f is any isometry, prove that $f \circ h \circ f^{-1}$ is the reflection about the hyperplane orthogonal to $f(u)$.

Problem 13.3. (1) Given a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

prove that there are Householder matrices G, H such that

$$GAH = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix} = D,$$