Therefore, $\tau_{v'-v,\lambda}(u) = u$, and $g = \tau_{v'-v,\lambda}$ does the job.

*Case 2*. $\varphi(v, v') = 0$.

First, check that $(u, u + v)$ is a also hyperbolic basis. Furthermore,

$$\varphi(v, u + v) = \varphi(v, u) + \varphi(v, v) = \varphi(v, u) = -1 \neq 0.$$

Thus, there is a symplectic transvection $\tau_{v,\lambda_1}$ such that $\tau_{u,\lambda_1}(v) = u + v$ and $\tau_{u,\lambda_1}(u) = u$. We also have

$$\varphi(u + v, v') = \varphi(u, v') + \varphi(v, v') = \varphi(u, v') = 1 \neq 0,$$

so there is a symplectic transvection $\tau_{v'-u-v,\lambda_2}$ such that $\tau_{v'-u-v,\lambda_2}(u + v) = v'$. Since

$$\varphi(u, v' - u - v) = \varphi(u, v') - \varphi(u, u) - \varphi(u, v) = 1 - 0 - 1 = 0,$$

we have $\tau_{v'-u-v,\lambda_2}(u) = u$. Then, the composition $g = \tau_{v'-u-v,\lambda_2} \circ \tau_{u,\lambda_1}$ is such that $g(u) = u$ and $g(v) = v'$. $\square$

We will use Proposition 29.37 in an inductive argument to prove that the symplectic transvections generate the symplectic group. First, make the following observation: If $U$ is a nondegenerate subspace of $E$, so that

$$E = U \overset{\perp}{\oplus} U^{\perp},$$

and if $\tau$ is a transvection of $H^{\perp}$, then we can form the linear map $\mathrm{id}_U \overset{\perp}{\oplus} \tau$ whose restriction to $U$ is the identity and whose restriction to $U^{\perp}$ is $\tau$, and $\mathrm{id}_U \overset{\perp}{\oplus} \tau$ is a transvection of $E$.

**Theorem 29.38.** *The symplectic group* $\mathbf{Sp}(2m, K)$ *is generated by the symplectic transvections. For every transvection* $f \in \mathbf{Sp}(2m, K)$*, we have* $\det(f) = 1$*.*

*Proof.* Let $G$ be the subgroup of $\mathbf{Sp}(2m, K)$ generated by the transvections. We need to prove that $G = \mathbf{Sp}(2m, K)$. Let $(u_1, v_1, \ldots, u_m, v_m)$ be a symplectic basis of $E$, and let $f \in \mathbf{Sp}(2m, K)$ be any symplectic map. Then, $f$ maps $(u_1, v_1, \ldots, u_m, v_m)$ to another symplectic basis $(u'_1, v'_1, \ldots, u'_m, v'_m)$. If we prove that there is some $g \in G$ such that $g(u_i) = u'_i$ and $g(v_i) = v'_i$ for $i = 1, \ldots, m$, then $f = g$ and $G = \mathbf{Sp}(2m, K)$.

We use induction on $i$ to prove that there is some $g_i \in G$ so that $g_i$ maps $(u_1, v_1, \ldots, u_i, v_i)$ to $(u'_1, v'_1, \ldots, u'_i, v'_i)$.

The base case $i = 1$ follows from Proposition 29.37.

For the induction step, assume that we have some $g_i \in G$ mapping $(u_1, v_1, \ldots, u_i, v_i)$ to $(u'_1, v'_1, \ldots, u'_i, v'_i)$, and let $(u''_{i+1}, v''_{i+1}, \ldots, u''_m, v''_m)$ be the image of $(u_{i+1}, v_{i+1}, \ldots, u_m, v_m)$ by $g_i$. If $U$ is the subspace spanned by $(u'_1, v'_1, \ldots, u'_m, v'_m)$, then each hyperbolic plane $W'_{i+k}$ given by $(u'_{i+k}, v'_{i+k})$ and each hyperbolic plane $W''_{i+k}$ given by $(u''_{i+k}, v''_{i+k})$ belongs to