and to the version of (D) given by

$$\begin{aligned} &\text{minimize} \quad y'b - y''b \\ &\text{subject to} \quad \begin{pmatrix} y' & y'' \end{pmatrix} \begin{pmatrix} A \\ -A \end{pmatrix} \geq c \text{ and } y', y'' \geq 0, \end{aligned}$$

where $y', y'' \in (\mathbb{R}^m)^*$, since the inequalities $Ax \leq b$ and $-Ax \leq -b$ together imply that $Ax = b$, we have equality for all these inequality constraints, and so the Conditions $(*_D)$ place no constraints at all on y' and y'', while the Conditions $(*_P)$ assert that

$$x_j = 0 \qquad \text{for all } j \text{ for which } \textstyle\sum_{i=1}^m (y'_i - y''_i) a_{ij} > c_j.$$

If we write $y = y' - y''$, the above conditions are equivalent to

$$x_j = 0 \qquad \text{for all } j \text{ for which } \textstyle\sum_{i=1}^m y_i a_{ij} > c_j.$$

Thus we have the following version of Theorem 47.11.

Theorem 47.13. *(Equilibrium Theorem, Version 2) For any Linear Program $(P2)$ in standard form (with $Ax = b$ where A is an $m \times n$ matrix, $x \geq 0$, and objective function $x \mapsto cx$) and its Dual Linear Program (D), for any feasible solution x of (P) and any feasible solution y of (D), x and y are optimal solutions iff*

$$x_j = 0 \qquad \text{for all } j \text{ for which } \textstyle\sum_{i=1}^m y_i a_{ij} > c_j. \tag{$*_P$}$$

Therefore, the slackness conditions applied to a Linear Program $(P2)$ in standard form and to its Dual (D) *only impose slackness conditions on the variables x_j of the primal problem.*

The above fact plays a crucial role in the primal-dual method.

47.5 The Dual Simplex Algorithm

Given a Linear Program $(P2)$ in standard form

$$\begin{aligned} &\text{maximize} \quad cx \\ &\text{subject to} \quad Ax = b \text{ and } x \geq 0, \end{aligned}$$

where A is an $m \times n$ matrix of rank m, if no obvious feasible solution is available *but if* $c \leq 0$, rather than using the method for finding a feasible solution described in Section 46.2 we may use a method known as the dual simplex algorithm. This method uses basic solutions (u, K) where $Au = b$ and $u_j = 0$ for all $u_j \notin K$, but does not require $u \geq 0$, so u may not be feasible. However, $y = c_K A_K^{-1}$ is required to be feasible for the dual program

$$\begin{aligned} &\text{minimize} \quad yb \\ &\text{subject to} \quad yA \geq c, \end{aligned}$$