Proposition 15.11 implies that for any diagonalizable matrix A, if we define $\Gamma(A)$ by

$$\Gamma(A) = \inf\{\mathrm{cond}(P) \mid P^{-1}AP = D\},$$

then for every eigenvalue λ of $A + \Delta A$, we have

$$\lambda \in \bigcup_{k=1}^{n} \{z \in \mathbb{C}^n \mid |z - \lambda_k| \leq \Gamma(A) \|\Delta A\|\}.$$

Definition 15.6. The number $\Gamma(A) = \inf\{\mathrm{cond}(P) \mid P^{-1}AP = D\}$ is called the *conditioning of A relative to the eigenvalue problem.*

If A is a normal matrix, since by Theorem 17.22, A can be diagonalized with respect to a unitary matrix U, and since for the spectral norm $\|U\|_2 = 1$, we see that $\Gamma(A) = 1$. Therefore, normal matrices are very well conditionned w.r.t. the eigenvalue problem. In fact, for every eigenvalue λ of $A + \Delta A$ (with A normal), we have

$$\lambda \in \bigcup_{k=1}^{n} \{z \in \mathbb{C}^n \mid |z - \lambda_k| \leq \|\Delta A\|_2\}.$$

If A and $A+\Delta A$ are both symmetric (or Hermitian), there are sharper results; see Proposition 17.28.

Note that the matrix $A(\epsilon)$ from the beginning of the section is not normal.

15.5 Eigenvalues of the Matrix Exponential

The Schur decomposition yields a characterization of the eigenvalues of the matrix exponential e^A in terms of the eigenvalues of the matrix A. First we have the following proposition.

Proposition 15.12. *Let A and U be (real or complex) matrices and assume that U is invertible. Then*

$$e^{UAU^{-1}} = Ue^AU^{-1}.$$

Proof. A trivial induction shows that

$$UA^pU^{-1} = (UAU^{-1})^p,$$

and thus

$$\begin{aligned} e^{UAU^{-1}} &= \sum_{p \geq 0} \frac{(UAU^{-1})^p}{p!} = \sum_{p \geq 0} \frac{UA^pU^{-1}}{p!} \\ &= U \left(\sum_{p \geq 0} \frac{A^p}{p!} \right) U^{-1} = Ue^AU^{-1}, \end{aligned}$$

as claimed. ∎