Proof. Since
$$f_{\mathbb{C}}(u+iv) = f(u) + if(v)$$
and
$$f_{\mathbb{C}}(u+iv) = (\lambda + i\mu)(u+iv) = \lambda u - \mu v + i(\mu u + \lambda v),$$
we have
$$f(u) = \lambda u - \mu v \quad \text{and} \quad f(v) = \mu u + \lambda v. \qquad \square$$

Using this fact, we can prove the following proposition.

Proposition 17.11. *Given a Euclidean space E, for any normal linear map $f\colon E \to E$, if $w = u + iv$ is an eigenvector of $f_{\mathbb{C}}$ associated with the eigenvalue $z = \lambda + i\mu$ (where $u, v \in E$ and $\lambda, \mu \in \mathbb{R}$), if $\mu \neq 0$ (i.e., z is not real) then $\langle u, v\rangle = 0$ and $\langle u, u\rangle = \langle v, v\rangle$, which implies that u and v are linearly independent, and if W is the subspace spanned by u and v, then $f(W) = W$ and $f^*(W) = W$. Furthermore, with respect to the (orthogonal) basis (u, v), the restriction of f to W has the matrix*
$$\begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix}.$$
If $\mu = 0$, then λ is a real eigenvalue of f, and either u or v is an eigenvector of f for λ. If W is the subspace spanned by u if $u \neq 0$, or spanned by $v \neq 0$ if $u = 0$, then $f(W) \subseteq W$ and $f^*(W) \subseteq W$.

Proof. Since $w = u + iv$ is an eigenvector of $f_{\mathbb{C}}$, by definition it is nonnull, and either $u \neq 0$ or $v \neq 0$. Proposition 17.10 implies that $u - iv$ is an eigenvector of $f_{\mathbb{C}}$ for $\lambda - i\mu$. It is easy to check that $f_{\mathbb{C}}$ is normal. However, if $\mu \neq 0$, then $\lambda + i\mu \neq \lambda - i\mu$, and from Proposition 17.4, the vectors $u + iv$ and $u - iv$ are orthogonal w.r.t. $\langle -, -\rangle_{\mathbb{C}}$, that is,
$$\langle u + iv, u - iv\rangle_{\mathbb{C}} = \langle u, u\rangle - \langle v, v\rangle + 2i\langle u, v\rangle = 0.$$
Thus we get $\langle u, v\rangle = 0$ and $\langle u, u\rangle = \langle v, v\rangle$, and since $u \neq 0$ or $v \neq 0$, u and v are linearly independent. Since
$$f(u) = \lambda u - \mu v \quad \text{and} \quad f(v) = \mu u + \lambda v$$
and since by Proposition 17.3 $u + iv$ is an eigenvector of $f^*_{\mathbb{C}}$ for $\lambda - i\mu$, we have
$$f^*(u) = \lambda u + \mu v \quad \text{and} \quad f^*(v) = -\mu u + \lambda v,$$
and thus $f(W) = W$ and $f^*(W) = W$, where W is the subspace spanned by u and v.

When $\mu = 0$, we have
$$f(u) = \lambda u \quad \text{and} \quad f(v) = \lambda v,$$
and since $u \neq 0$ or $v \neq 0$, either u or v is an eigenvector of f for λ. If W is the subspace spanned by u if $u \neq 0$, or spanned by v if $u = 0$, it is obvious that $f(W) \subseteq W$ and $f^*(W) \subseteq W$. Note that $\lambda = 0$ is possible, and this is why $\subseteq$ cannot be replaced by $=$. $\square$