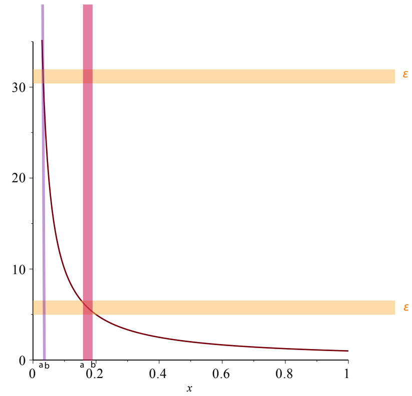

Figure 37.43: The real valued function $f(x) = 1/x$ is not uniformly continuous over $(0, \infty)$. Fix ϵ. In order for the y values to lie within the peach epsilon strip, the widths of the eta strips decrease as $x \to 0$.

Lemma 37.46. *Given a metric space, E, if every sequence, (x_n), has an accumulation point, then for every $\epsilon > 0$, there is a finite open cover, $B_0(a_0, \epsilon) \cup \cdots \cup B_0(a_n, \epsilon)$, of E by open balls of radius ϵ.*

Proof. Let a_0 be any point in E. If $B_0(a_0, \epsilon) = E$, then the lemma is proved. Otherwise, assume that a sequence, $(a_0, a_1, \ldots, a_n)$, has been defined, such that $B_0(a_0, \epsilon) \cup \cdots \cup B_0(a_n, \epsilon)$ does not cover E. Then, there is some a_{n+1} not in $B_0(a_0, \epsilon) \cup \cdots \cup B_0(a_n, \epsilon)$ and either

$$B_0(a_0, \epsilon) \cup \cdots \cup B_0(a_{n+1}, \epsilon) = E,$$

in which case the lemma is proved, or we obtain a sequence, $(a_0, a_1, \ldots, a_{n+1})$, such that $B_0(a_0, \epsilon) \cup \cdots \cup B_0(a_{n+1}, \epsilon)$ does not cover E. If this process goes on forever, we obtain an infinite sequence, (a_n), such that $d(a_m, a_n) > \epsilon$ for all $m \neq n$. Since every sequence in E has some accumulation point, the sequence, (a_n), has some accumulation point, a. Then, for infinitely many n, we must have $d(a_n, a) \leq \epsilon/3$ and thus, for at least two distinct natural numbers, p, q, we must have $d(a_p, a) \leq \epsilon/3$ and $d(a_q, a) \leq \epsilon/3$, which implies $d(a_p, a_q) \leq d(a_p, a) + d(a_q, a) \leq 2\epsilon/3$, contradicting the fact that $d(a_m, a_n) > \epsilon$ for all $m \neq n$. See Figure 37.44. Thus, there must be some n such that

$$B_0(a_0, \epsilon) \cup \cdots \cup B_0(a_n, \epsilon) = E. \qquad \square$$

Definition 37.37. A metric space E is said to be *precompact* (or *totally bounded*) if for every $\epsilon > 0$, there is a finite open cover, $B_0(a_0, \epsilon) \cup \cdots \cup B_0(a_n, \epsilon)$, of E by open balls of radius ϵ.

We now obtain the *Weierstrass–Bolzano* property.