Finally, we show that τ is unique. Assume two decompositions (g_1, τ_1) and (g_2, τ_2). Since $\overrightarrow{f} = \overrightarrow{g_1}$, we have $\mathrm{Ker}\,(\overrightarrow{g_1} - \mathrm{id}) = \mathrm{Ker}\,\big(\overrightarrow{f} - \mathrm{id}\big)$. Since g_1 has some fixed point b, we get

$$f(b) = g_1(b) + \tau_1 = b + \tau_1,$$

that is, $\overrightarrow{bf(b)} = \tau_1$, and $\overrightarrow{bf(b)} \in \mathrm{Ker}\,\big(\overrightarrow{f} - \mathrm{id}\big)$, since $\tau_1 \in \mathrm{Ker}\,\big(\overrightarrow{f} - \mathrm{id}\big)$. Similarly, for some fixed point c of g_2, we get $\overrightarrow{cf(c)} = \tau_2$ and $\overrightarrow{cf(c)} \in \mathrm{Ker}\,\big(\overrightarrow{f} - \mathrm{id}\big)$. Then we have

$$\tau_2 - \tau_1 = \overrightarrow{cf(c)} - \overrightarrow{bf(b)} = \overrightarrow{cb} - \overrightarrow{f(c)f(b)} = \overrightarrow{cb} - \overrightarrow{f}\,(\overrightarrow{cb}),$$

which shows that

$$\tau_2 - \tau_1 \in \mathrm{Ker}\,\big(\overrightarrow{f} - \mathrm{id}\big) \cap \mathrm{Im}\,\big(\overrightarrow{f} - \mathrm{id}\big),$$

and thus that $\tau_2 = \tau_1$, since we have shown that

$$\overrightarrow{E} = \mathrm{Ker}\,\big(\overrightarrow{f} - \mathrm{id}\big) \oplus \mathrm{Im}\,\big(\overrightarrow{f} - \mathrm{id}\big).$$

The fact that (a) holds is a consequence of the uniqueness of g and τ, since f and 0 clearly satisfy the required conditions. That (b) holds follows from Lemma 27.8 (2), since the affine map f has a unique fixed point iff $E\big(1, \overrightarrow{f}\big) = \mathrm{Ker}\,\big(\overrightarrow{f} - \mathrm{id}\big) = \{0\}$. $\square$

The determination of x is illustrated in Figure 27.8.

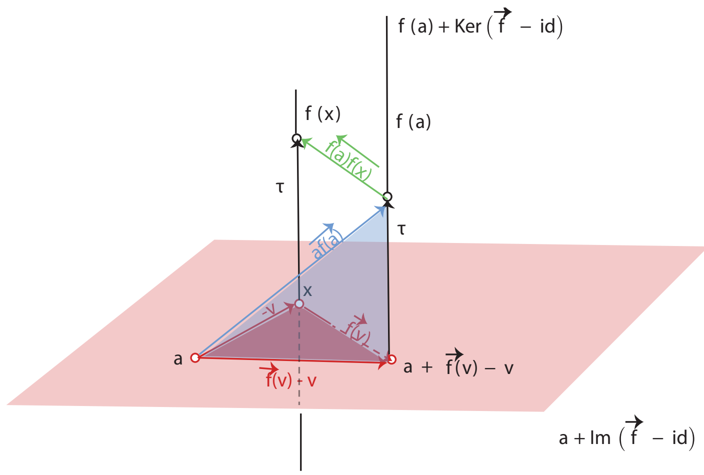

Figure 27.8: Affine rigid motion as $f = t \circ g$, where g has some fixed point x.

Remarks: