Since A is symmetric positive definite, by Proposition 42.2, the function $v \mapsto L(v, \mu)$ has a unique minimum obtained for the solution u_μ of the linear system

$$Av = b - C^\top \mu;$$

that is,

$$u_\mu = A^{-1}(b - C^\top \mu).$$

This shows that the Problem (P_μ) has a unique solution which depends continuously on μ. Then for any solution λ of the dual problem, $u_\lambda = A^{-1}(b - C^\top \lambda)$ is an optimal solution of the primal problem.

We compute $G(\mu)$ as follows:

$$\begin{aligned} G(\mu) = L(u_\mu, \mu) &= \frac{1}{2} u_\mu^\top A u_\mu - u_\mu^\top (b - C^\top \mu) - \mu^\top d \\ &= \frac{1}{2} u_\mu^\top (b - C^\top \mu) - u_\mu^\top (b - C^\top \mu) - \mu^\top d \\ &= -\frac{1}{2} u_\mu^\top (b - C^\top \mu) - \mu^\top d \\ &= -\frac{1}{2} (b - C^\top \mu)^\top A^{-1} (b - C^\top \mu) - \mu^\top d \\ &= -\frac{1}{2} \mu^\top C A^{-1} C^\top \mu + \mu^\top (C A^{-1} b - d) - \frac{1}{2} b^\top A^{-1} b. \end{aligned}$$

Since A is symmetric positive definite, the matrix $CA^{-1}C^\top$ is symmetric positive semidefinite. Since A^{-1} is also symmetric positive definite, $\mu^\top C A^{-1} C^\top \mu = 0$ iff $(C^\top \mu)^\top A^{-1} (C^\top \mu) = 0$ iff $C^\top \mu = 0$ implies $\mu = 0$, that is, $\mathrm{Ker}\, C^\top = (0)$, which is equivalent to $\mathrm{Im}(C) = \mathbb{R}^m$, namely if C has rank m (in which case, $m \leq n$). Thus $CA^{-1}C^\top$ is symmetric positive definite iff C has rank m.

We showed just after Theorem 49.8 that the functional $v \mapsto (1/2) v^\top A v$ is elliptic iff A is symmetric positive definite, and Theorem 49.8 shows that an elliptic functional is coercive, which is the hypothesis used in Theorem 49.4. Therefore, by Theorem 49.4, if the inequalities $Cx \leq d$ have a solution, the primal problem has a unique solution. In this case, as a consequence, by Theorem 50.17(2), the function $-G(\mu)$ always has a minimum, which is unique if C has rank m. The fact that $-G(\mu)$ has a minimum is not obvious when C has rank $< m$, since in this case $CA^{-1}C^\top$ is not invertible.

We also verify easily that the gradient of G is given by

$$\nabla G_\mu = C u_\mu - d = -C A^{-1} C^\top \mu + C A^{-1} b - d.$$

Observe that since $CA^{-1}C^\top$ is symmetric positive semidefinite, $-G(\mu)$ is convex.

Therefore, if C has rank m, a solution of Problem (P) is obtained by finding the unique solution λ of the equation

$$-C A^{-1} C^\top \mu + C A^{-1} b - d = 0,$$