49.2 Existence of Solutions of an Optimization Problem

We begin with the case where U is a closed but possibly unbounded subset of $\mathbb{R}^n$. In this case the following type of functions arise.

Definition 49.1. A real-valued function $J\colon V \to \mathbb{R}$ defined on a normed vector space V is *coercive* iff for any sequence $(v_k)_{k\geq 1}$ of vectors $v_k \in V$, if $\lim_{k\mapsto\infty} \|v_k\| = \infty$, then

$$\lim_{k\mapsto\infty} J(v_k) = +\infty.$$

For example, the function $f(x) = x^2 + 2x$ is coercive, but an affine function $f(x) = ax + b$ is not.

Proposition 49.1. *Let U be a nonempty, closed subset of $\mathbb{R}^n$, and let $J\colon \mathbb{R}^n \to \mathbb{R}$ be a continuous function which is coercive if U is unbounded. Then there is a least one element $u \in \mathbb{R}^n$ such that*

$$u \in U \quad \textit{and} \quad J(u) = \inf_{v\in U} J(v).$$

Proof. Since $U \neq \emptyset$, pick any $u_0 \in U$. Since J is coercive, there is some $r > 0$ such that for all $v \in \mathbb{R}^n$, if $\|v\| > r$ then $J(u_0) < J(v)$. It follows that J is minimized over the set

$$U_0 = U \cap \{v \in \mathbb{R}^n \mid \|v\| \leq r\}.$$

Since U is closed and since the closed ball $\{v \in \mathbb{R}^n \mid \|v\| \leq r\}$ is compact, U_0 is compact, but we know that any continuous function on a compact set has a minimum which is achieved. $\square$

The key point in the above proof is the fact that U_0 is compact. In order to generalize Proposition 49.1 to the case of an infinite dimensional vector space, we need some additional assumptions, and it turns out that the convexity of U and of the function J is sufficient. The key is that convex, closed and bounded subsets of a Hilbert space are "weakly compact."

Definition 49.2. Let V be a Hilbert space. A sequence $(u_k)_{k\geq 1}$ of vectors $u_k \in V$ *converges weakly* if there is some $u \in V$ such that

$$\lim_{k\mapsto\infty} \langle v, u_k\rangle = \langle v, u\rangle \quad \text{for every } v \in V.$$

Recall that a Hibert space is separable if it has a countable Hilbert basis (see Definition A.4). Also, in a Euclidean space (of finite dimension) V, the inner product induces an isomorphism between V and its dual V^*. In our case, we need the isomorphism $\sharp$ from V^* to V defined such that for every linear form $\omega \in V^*$, the vector $\omega^\sharp \in V$ is uniquely defined by the equation

$$\omega(v) = \langle v, \omega^\sharp\rangle \quad \text{for all } v \in V.$$