(5) Assumption (2) does not imply that the system $Ax + Bz = c$ has any solution. For example, if

$$A = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

the system

$$\begin{aligned} x - z &= 1 \\ x - z &= 0 \end{aligned}$$

has no solution. However, since Assumption (3) implies that the program has an optimal solution, it implies that c belongs to the column space of the $p \times (n+m)$ matrix $\begin{pmatrix} A & B \end{pmatrix}$.

Here is an example where ADMM diverges for a problem whose optimum value is $-\infty$.

Example 52.6. Consider the problem given by

$$f(x) = x, \quad g(z) = 0, \quad x - z = 0.$$

Since $f(x) + g(z) = x$, and $x = z$, the variable x is unconstrained and the above function goes to $-\infty$ when x goes to $-\infty$. The augmented Lagrangian is

$$\begin{aligned} L_\rho(x, z, \lambda) &= x + \lambda(x - z) + \frac{\rho}{2}(x - z)^2 \\ &= \frac{\rho}{2}x^2 - \rho xz + \frac{\rho}{2}z^2 + x + \lambda x - \lambda z. \end{aligned}$$

The matrix

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

is singular and $L_\rho(x, z, \lambda)$ goes to $-\infty$ in when $(x, z) = t(1, 1)$ and t goes to $-\infty$. The ADMM steps are:

$$\begin{aligned} x^{k+1} &= z^k - \frac{1}{\rho}\lambda^k - \frac{1}{\rho} \\ z^{k+1} &= x^{k+1} + \frac{1}{\rho}\lambda^k \\ \lambda^{k+1} &= \lambda^k + \rho(x^{k+1} - z^{k+1}), \end{aligned}$$

and these equations hold for all $k \geq 0$. From the last two equations we deduce that

$$\lambda^{k+1} = \lambda^k + \rho(x^{k+1} - z^{k+1}) = \lambda^k + \rho(-\frac{1}{\rho}\lambda^k) = 0, \quad \text{for all } k \geq 0,$$

so

$$z^{k+2} = x^{k+2} + \frac{1}{\rho}\lambda^{k+1} = x^{k+2}, \quad \text{for all } k \geq 0.$$