Observe that for any nonempty finite subset $J \subseteq \Sigma$ with $|J| = n$, we have

$$L_I(u_J) = \begin{cases} 1 & \text{if } I = J \\ 0 & \text{if } I \neq J. \end{cases}$$

Note that when $\dim(E) = d$ and $n \leq d$, or when E is infinite-dimensional, the forms $u^*_{i_1}, \ldots, u^*_{i_n}$ are all distinct, so the above does hold. Since $L_I(u_I) = 1$, we conclude that $u_I \neq 0$. If we have a linear combination

$$\sum_I \lambda_I u_I = 0,$$

where the above sum is finite and involves nonempty finite subset $I \subseteq \Sigma$ with $|I| = n$, for every such I, when we apply L_I we get $\lambda_I = 0$, proving linear independence. ☐

As a corollary, if E is finite dimensional, say $\dim(E) = d$, and if $1 \leq n \leq d$, then we have

$$\dim(\bigwedge^n(E)) = \binom{n}{d},$$

and if $n > d$, then $\dim(\bigwedge^n(E)) = 0$.

Remark: When $n = 0$, if we set $u_\emptyset = 1$, then $(u_\emptyset) = (1)$ is a basis of $\bigwedge^0(V) = K$.

It follows from Proposition 34.7 that the family $(u_I)_I$ where $I \subseteq \Sigma$ ranges over finite subsets of Σ is a basis of $\bigwedge(V) = \bigoplus_{n \geq 0} \bigwedge^n(V)$.

As a corollary of Proposition 34.7 we obtain the following useful criterion for linear independence.

Proposition 34.8. *For any vector space E, the vectors $u_1, \ldots, u_n \in E$ are linearly independent iff $u_1 \wedge \cdots \wedge u_n \neq 0$.*

Proof. If $u_1 \wedge \cdots \wedge u_n \neq 0$, then $u_1, \ldots, u_n$ must be linearly independent. Otherwise, some u_i would be a linear combination of the other u_j's (with $j \neq i$), and then, as in the proof of Proposition 34.7, $u_1 \wedge \cdots \wedge u_n$ would be a linear combination of wedges in which two vectors are identical, and thus zero.

Conversely, assume that $u_1, \ldots, u_n$ are linearly independent. Then we have the linear forms $u^*_i \in E^*$ such that

$$u^*_i(u_j) = \delta_{i,j} \qquad 1 \leq i, j \leq n.$$

As in the proof of Proposition 34.7, we have a linear map $L_{u_1,\ldots,u_n} \colon \bigwedge^n(E) \to K$ given by

$$L_{u_1,\ldots,u_n}(v_1 \wedge \cdots \wedge v_n) = \det(u^*_j(v_i)) = \begin{vmatrix} u^*_1(v_1) & \cdots & u^*_1(v_n) \\ \vdots & \ddots & \vdots \\ u^*_n(v_1) & \cdots & u^*_n(v_n) \end{vmatrix},$$

for all $v_1 \wedge \cdots \wedge v_n \in \bigwedge^n(E)$. As $L_{u_1,\ldots,u_n}(u_1 \wedge \cdots \wedge u_n) = 1$, we conclude that $u_1 \wedge \cdots \wedge u_n \neq 0$. ☐