Proof sketch. We only consider the second isomorphism. Since P is projective, we have some A-modules, P_1, F, with

$$P \oplus P_1 = F,$$

where F is some free module. Now, we know that for any A-modules, U, V, W, we have

$$\mathrm{Hom}_A(U \oplus V, W) \cong \mathrm{Hom}_A(U, W) \prod \mathrm{Hom}_A(V, W) \cong \mathrm{Hom}_A(U, W) \oplus \mathrm{Hom}_A(V, W),$$

so

$$P^* \oplus P_1^* \cong F^*, \qquad \mathrm{Hom}_A(P, Q) \oplus \mathrm{Hom}_A(P_1, Q) \cong \mathrm{Hom}_A(F, Q).$$

By tensoring with Q and using the fact that tensor distributes w.r.t. coproducts, we get

$$(P^* \otimes_A Q) \oplus (P_1^* \otimes_A Q) \cong (P^* \oplus P_1^*) \otimes_A Q \cong F^* \otimes_A Q.$$

Now, the proof of Proposition 33.17 goes through because F is free and finitely generated, so

$$\alpha_\otimes \colon (P^* \otimes_A Q) \oplus (P_1^* \otimes_A Q) \cong F^* \otimes_A Q \longrightarrow \mathrm{Hom}_A(F, Q) \cong \mathrm{Hom}_A(P, Q) \oplus \mathrm{Hom}_A(P_1, Q)$$

is an isomorphism and as $\alpha_\otimes$ maps $P^* \otimes_A Q$ to $\mathrm{Hom}_A(P, Q)$, it yields an isomorphism between these two spaces. $\square$

The isomorphism $\alpha_\otimes \colon P^* \otimes_A Q \cong \mathrm{Hom}_A(P, Q)$ of Proposition 35.11 is still given by

$$\alpha_\otimes(u^* \otimes f)(x) = u^*(x) f, \qquad u^* \in P^*,\ f \in Q,\ x \in P.$$

It is convenient to introduce the *evaluation map*, $\mathrm{Ev}_x \colon P^* \otimes_A Q \to Q$, defined for every $x \in P$ by

$$\mathrm{Ev}_x(u^* \otimes f) = u^*(x) f, \qquad u^* \in P^*,\ f \in Q.$$

We will need the following generalization of part (4) of Proposition 33.13.

Proposition 35.12. *Given any two families of A-modules $(M_i)_{i \in I}$ and $(N_j)_{j \in J}$ (where I and J are finite index sets), we have an isomorphism*

$$\Big(\bigoplus_{i \in I} M_i\Big) \otimes \Big(\bigoplus_{j \in I} M_j\Big) \approx \bigoplus_{(i,j) \in I \times J} (M_i \otimes N_j).$$

Proposition 35.12 also holds for infinite index sets.

Proposition 35.13. *Let M and N be two A-module with N a free module, and pick any basis $(v_1, \ldots, v_n)$ for N. Then, every element of $M \otimes N$ can expressed in a unique way as a sum of the form*

$$u_1 \otimes v_1 + \cdots + u_n \otimes v_n, \quad u_i \in M,$$

so that $M \otimes N$ is isomorphic to M^n (as an A-module).