Proof. We have $r_q(x, y, z) = (x, y, z)$ iff

$$\varphi^{-1}(q(x\sigma_3 + y\sigma_2 + z\sigma_1)q^*) = (x, y, z)$$

iff

$$q(x\sigma_3 + y\sigma_2 + z\sigma_1)q^* = \varphi(x, y, z),$$

and since

$$\varphi(x, y, z) = x\sigma_3 + y\sigma_2 + z\sigma_1 = A$$

with

$$A = \begin{pmatrix} x & z - iy \\ z + iy & -x \end{pmatrix},$$

we see that $r_q(x, y, z) = (x, y, z)$ iff

$$qAq^* = A \quad \text{iff} \quad qA = Aq.$$

We have

$$qA = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \begin{pmatrix} x & z - iy \\ z + iy & -x \end{pmatrix} = \begin{pmatrix} \alpha x + \beta z + i\beta y & \alpha z - i\alpha y - \beta x \\ -\overline{\beta} x + \overline{\alpha} z + i\overline{\alpha} y & -\overline{\beta} z + i\overline{\beta} y - \overline{\alpha} x \end{pmatrix}$$

and

$$Aq = \begin{pmatrix} x & z - iy \\ z + iy & -x \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} = \begin{pmatrix} \alpha x - \overline{\beta} z + i\overline{\beta} y & \beta x + \overline{\alpha} z - i\overline{\alpha} y \\ \alpha z + i\alpha y + \overline{\beta} x & \beta z + i\beta y - \overline{\alpha} x \end{pmatrix}.$$

By equating qA and Aq, we get

$$\begin{aligned} i(\beta - \overline{\beta})y + (\beta + \overline{\beta})z &= 0 \\ 2\beta x + i(\alpha - \overline{\alpha})y + (\overline{\alpha} - \alpha)z &= 0 \\ 2\overline{\beta} x + i(\alpha - \overline{\alpha})y + (\alpha - \overline{\alpha})z &= 0 \\ i(\beta - \overline{\beta})y + (\beta + \overline{\beta})z &= 0. \end{aligned}$$

The first and the fourth equation are identical and the third equation is obtained by conjugating the second, so the above system reduces to

$$\begin{aligned} i(\beta - \overline{\beta})y + (\beta + \overline{\beta})z &= 0 \\ 2\beta x + i(\alpha - \overline{\alpha})y + (\overline{\alpha} - \alpha)z &= 0. \end{aligned}$$

Replacing α by $a + ib$ and β by $c + id$, we get

$$\begin{aligned} -dy + cz &= 0 \\ cx - by + i(dx - bz) &= 0, \end{aligned}$$