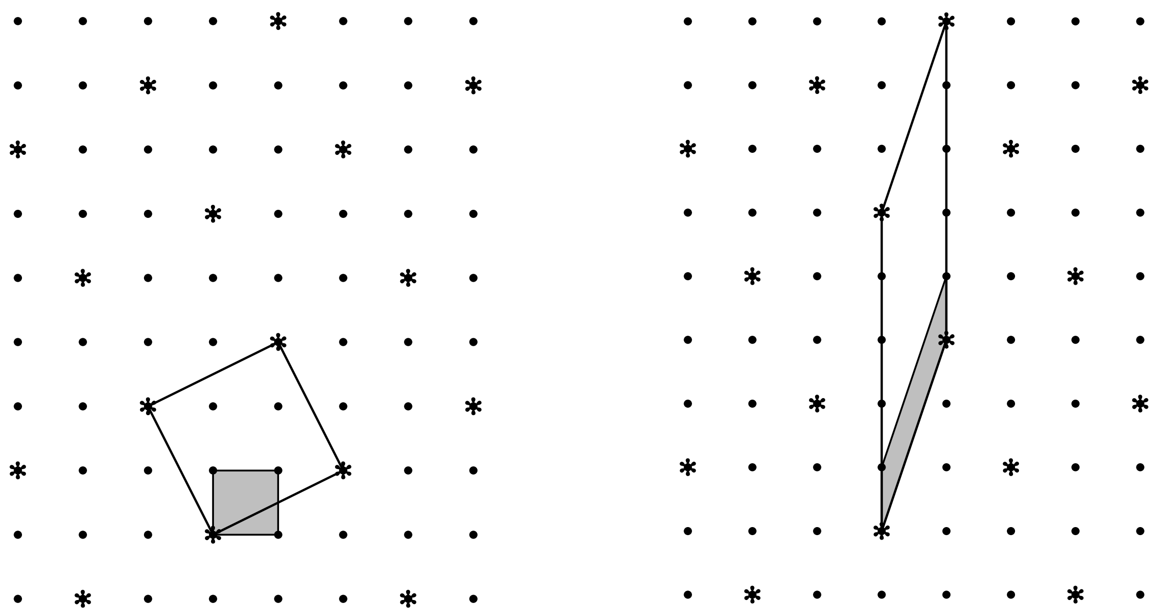

Figure 35.1: Diagonalization applied to a lattice

Theorem 35.38. *(Elementary Divisors Decomposition) Let M be a finitely generated nontrivial A-module, where A a PID. Then, M is isomorphic to the direct sum $A^r \oplus M_{\mathrm{tor}}$, where A^r is a free module and where the torsion module M_{tor} is a direct sum of cyclic modules of the form $A/p_i^{n_{i,j}}$, for some primes $p_1, \ldots, p_t \in A$ and some positive integers $n_{i,j}$, such that for each $i = 1, \ldots, t$, there is a sequence of integers*

$$1 \leq \underbrace{n_{i,1}, \ldots, n_{i,1}}_{m_{i,1}} < \underbrace{n_{i,2}, \ldots, n_{i,2}}_{m_{i,2}} < \cdots < \underbrace{n_{i,s_i}, \ldots, n_{i,s_i}}_{m_{i,s_i}},$$

with $s_i \geq 1$, and where $n_{i,j}$ occurs $m_{i,j} \geq 1$ times, for $j = 1, \ldots, s_i$. Furthermore, the irreducible elements p_i and the integers $r, t, n_{i,j}, s_i$, $m_{i,j}$ are uniquely determined.

Proof. By Theorem 35.31, we already know that $M \approx A^r \oplus M_{\mathrm{tor}}$, where r is uniquely determined, and where

$$M_{\mathrm{tor}} \approx A/\mathfrak{a}_{r+1} \oplus \cdots \oplus A/\mathfrak{a}_m,$$

a direct sum of cyclic modules, with $(0) \neq \mathfrak{a}_{r+1} \subseteq \cdots \subseteq \mathfrak{a}_m \neq A$. Then, each $\mathfrak{a}_i$ is a principal ideal of the form $\alpha_i A$, where $\alpha_i \neq 0$ and α_i is not a unit. Using the Chinese Remainder Theorem (Theorem 32.15), if we factor α_i into prime factors as

$$\alpha_i = u p_1^{k_1} \cdots p_h^{k_h},$$

with $k_j \geq 1$, we get an isomorphism

$$A/\alpha_i A \approx A/p_1^{k_1} A \oplus \cdots \oplus A/p_h^{k_h} A.$$