(1) If we denote the columns of B by $b_1, \ldots, b_n$, prove that

$$\begin{aligned} (n-3)b_1 - (b_2 + \cdots + b_n) &= 2(n-2)e_1 \\ b_1 - b_2 &= 2(e_1 + e_2) \\ b_1 - b_3 &= 2(e_1 + e_3) \\ \vdots \quad & \quad \vdots \\ b_1 - b_n &= 2(e_1 + e_n), \end{aligned}$$

where $e_1, \ldots, e_n$ are the canonical basis vectors of $\mathbb{R}^n$.

(2) Prove that B is invertible and that its inverse $A = (a_{ij})$ is given by

$$a_{11} = \frac{(n-3)}{2(n-2)}, \quad a_{i1} = -\frac{1}{2(n-2)} \quad 2 \leq i \leq n$$

and

$$\begin{aligned} a_{ii} &= -\frac{(n-3)}{2(n-2)}, \quad 2 \leq i \leq n \\ a_{ji} &= \frac{1}{2(n-2)}, \quad 2 \leq i \leq n, j \neq i. \end{aligned}$$

(3) Show that the n diagonal $n \times n$ matrices D_i defined such that the diagonal entries of D_i are equal the entries (from top down) of the ith column of B form a basis of the space of $n \times n$ diagonal matrices (matrices with zeros everywhere except possibly on the diagonal). For example, when $n = 4$, we have

$$D_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad D_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$D_3 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad D_4 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Problem 3.16. Given any $m \times n$ matrix A and any $n \times p$ matrix B, if we denote the columns of A by $A^1, \ldots, A^n$ and the rows of B by $B_1, \ldots, B_n$, prove that

$$AB = A^1 B_1 + \cdots + A^n B_n.$$

Problem 3.17. Let $f \colon E \to F$ be a linear map which is also a bijection (it is injective and surjective). Prove that the inverse function $f^{-1} \colon F \to E$ is linear.