we also have

$$\|y\| - \|x\| \leq \|x - y\|.$$

Therefore,

$$|\|x\| - \|y\|| \leq \|x - y\|, \quad \text{for all } x, y \in E. \tag{$*$}$$

Observe that setting $\lambda = 0$ in (N2), we deduce that $\|0\| = 0$ without assuming (N1). Then by setting $y = 0$ in $(*)$, we obtain

$$|\|x\|| \leq \|x\|, \quad \text{for all } x \in E.$$

Therefore, the condition $\|x\| \geq 0$ in (N1) follows from (N2) and (N3), and (N1) can be replaced by the weaker condition

(N1') For all $x \in E$, if $\|x\| = 0$, then $x = 0$,

A function $\| \, \| : E \to \mathbb{R}$ satisfying Axioms (N2) and (N3) is called a *seminorm*. From the above discussion, a seminorm also has the properties

$\|x\| \geq 0$ for all $x \in E$, and $\|0\| = 0$.

However, there may be nonzero vectors $x \in E$ such that $\|x\| = 0$.

Let us give some examples of normed vector spaces.

Example 9.1.

1. Let $E = \mathbb{R}$, and $\|x\| = |x|$, the absolute value of x.
2. Let $E = \mathbb{C}$, and $\|z\| = |z|$, the modulus of z.
3. Let $E = \mathbb{R}^n$ (or $E = \mathbb{C}^n$). There are three standard norms. For every $(x_1, \ldots, x_n) \in E$, we have the norm $\|x\|_1$, defined such that,

$$\|x\|_1 = |x_1| + \cdots + |x_n|,$$

we have the *Euclidean norm* $\|x\|_2$, defined such that,

$$\|x\|_2 = \left(|x_1|^2 + \cdots + |x_n|^2\right)^{\frac{1}{2}},$$

and the *sup*-norm $\|x\|_\infty$, defined such that,

$$\|x\|_\infty = \max\{|x_i| \mid 1 \leq i \leq n\}.$$

More generally, we define the ℓ^p*-norm* (for $p \geq 1$) by

$$\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p}.$$

See Figures 9.1 through 9.4.