(1) For any $y \in \mathbb{R}^V$, consider the Rayleigh ratio

$$R = \frac{y^\top L_{\mathrm{sym}}\, y}{y^\top y}.$$

Prove that if $x = D^{-1/2}y$, then

$$R = \frac{x^\top L x}{(D^{1/2}x)^\top D^{1/2}x} = \frac{\displaystyle\sum_{u\sim v}(x(u) - x(v))^2}{\displaystyle\sum_v d_v x(v)^2}.$$

(2) Prove that the second eigenvalue ν_2 of L_{sym} is given by

$$\nu_2 = \min_{\mathbf{1}^\top Dx = 0, x\neq 0} \frac{\displaystyle\sum_{u\sim v}(x(u) - x(v))^2}{\displaystyle\sum_v d_v x(v)^2}.$$

(3) Prove that the largest eigenvalue ν_m of L_{sym} is given by

$$\nu_m = \max_{x\neq 0} \frac{\displaystyle\sum_{u\sim v}(x(u) - x(v))^2}{\displaystyle\sum_v d_v x(v)^2}.$$

Problem 20.5. Let G be a graph with a set of nodes V with $m \geq 2$ elements, without isolated nodes. If $0 = \nu_1 \leq \nu_1 \leq \ldots \leq \nu_m$ are the eigenvalues of L_{sym}, prove the following properties:

(1) We have $\nu_1 + \nu_2 + \cdots + \nu_m = m$.

(2) We have $\nu_2 \leq m/(m-1)$, with equality holding iff $G = K_m$, the complete graph on m nodes.

(3) We have $\nu_m \geq m/(m-1)$.

(4) If G is not a complete graph, then $\nu_2 \leq 1$

Hint. If a and b are nonadjacent nodes, consider the function x given by

$$x(v) = \begin{cases} d_b & \text{if } v = a \\ -d_a & \text{if } v = b \\ 0 & \text{if } v \neq a, b, \end{cases}$$

and use Problem 20.4(2).