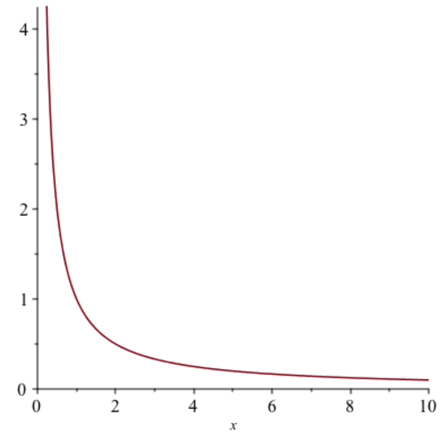

Figure 51.26: The graph of the partial function $f(x) = \frac{1}{x}$ for $x > 0$. The graph of this function decreases along the x-axis since 1 is a direction of recession.

Theorem 51.35. *Let f be a proper and closed convex function over $\mathbb{R}^n$. The following statements hold:*

(1) We have $\inf f = -f^*(0)$. Thus f is bounded below iff $0 \in \mathrm{dom}(f^*)$.

(2) The minimum set of f is equal to $\partial f^*(0)$. Thus the infimum of f is attained (which means that there is some $x \in \mathbb{R}^n$ such that $f(x) = \inf f$) iff f^* is subdifferentiable at 0. This condition holds in particular when $0 \in \mathbf{relint}(\mathrm{dom}(f^*))$. Moreover, $0 \in \mathbf{relint}(\mathrm{dom}(f^*))$ iff every direction of recession of f is a direction in which f is constant.

(3) For the infimum of f to be finite but unattained, it is necessary and sufficient that $f^*(0)$ be finite and $(f^*)'(0; y) = -\infty$ for some $y \in \mathbb{R}^n$.

(4) The minimum set of f is a nonempty bounded set iff $0 \in \mathrm{int}(\mathrm{dom}(f^*))$. This condition holds iff f has no directions of recession.

(5) The minimum set of f consists of a unique vector x iff f^* is differentiable at x and $x = \nabla f^*_0$.

(6) For each $\alpha \in \mathbb{R}$, the support function of $\mathrm{sublev}_\alpha(f)$ is the closure of the positively homogeneous convex function generated by $f^* + \alpha$. If f is bounded below, then the support function of the minimum set of f is the closure of the directional derivative map $y \mapsto (f^*)'(0; y)$.

In view of the importance of Theorem 51.35(4), we state this property as the following corollary.