with respect to which the matrix representing φ is a block diagonal matrix M of the form

$$M = \begin{pmatrix} J & & & & 0 \\ & J & & & \\ & & \ddots & & \\ & & & J & \\ 0 & & & & 0_{n-2r} \end{pmatrix},$$

with

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Proof. If $\varphi = 0$, then $E = E^{\perp}$ and we are done. Otherwise, there are two nonzero vectors $u, v \in E$ such that $\varphi(u, v) \neq 0$, so by Proposition 29.23, we obtain a hyperbolic plane W_2 spanned by two vectors u_1, v_1 such that $\varphi(u_1, v_1) = 1$. The subspace W_1 is nondegenerate (for example, $\det(J) = -1$), so by Proposition 29.21, we get a direct sum

$$E = W_1 \oplus W_1^{\perp}.$$

By Proposition 29.14, we also have

$$E^{\perp} = (W_1 \oplus W_1^{\perp}) = W_1^{\perp} \cap W_1^{\perp\perp} = \mathrm{rad}(W_1^{\perp}).$$

By the induction hypothesis applied to $W_1^{\perp}$, we obtain our theorem. $\square$

The following corollary follows immediately.

Proposition 29.25. *Let $\varphi\colon E \times E \to K$ be an alternating bilinear form on a space E of finite dimension n.*

(1) The rank of φ is even.

(2) If φ is nondegenerate, then $\dim(E) = n$ is even.

(3) Two alternating bilinear forms $\varphi_1\colon E_1 \times E_1 \to K$ and $\varphi_2\colon E_2 \times E_2 \to K$ are equivalent iff $\dim(E_1) = \dim(E_2)$ and φ_1 and φ_2 have the same rank.

The only part that requires a proof is part (3), which is left as an easy exercise.

If φ is nondegenerate, then $n = 2r$, and a basis of E as in Theorem 29.24 is called a *symplectic basis*. The space E is called a *hyperbolic space* (or *symplectic space*).

Observe that if we reorder the vectors in the basis

$$(u_1, v_1, \ldots, u_r, v_r, w_1, \ldots, w_{n-2r})$$

to obtain the basis

$$(u_1, \ldots, u_r, v_1, \ldots v_r, w_1, \ldots, w_{n-2r}),$$