If π_H and π_D are the projections of E onto H and D, then we have

$$f(x) = \pi_H(x) + \alpha\pi_D(x).$$

The inverse of f is given by

$$f^{-1}(x) = \pi_H(x) + \alpha^{-1}\pi_D(x).$$

When $\alpha = -1$, we have $f^2 = \mathrm{id}$, and f is a symmetry about the hyperplane H in the direction D. This situation includes orthogonal reflections about H.

Case 2. $\alpha = 1$.

In this case,

$$f(x) - x = th,$$

that is, $f(x) - x \in Kh$ for all $x \in E$. Assume that the hyperplane H is given as the kernel of some linear form φ, and let $a = \varphi(v)$. We have $a \neq 0$, since $v \notin H$. For any $x \in E$, we have

$$\varphi(x - a^{-1}\varphi(x)v) = \varphi(x) - a^{-1}\varphi(x)\varphi(v) = \varphi(x) - \varphi(x) = 0,$$

which shows that $x - a^{-1}\varphi(x)v \in H$ for all $x \in E$. Since every vector in H is fixed by f, we get

$$\begin{aligned} x - a^{-1}\varphi(x)v &= f(x - a^{-1}\varphi(x)v) \\ &= f(x) - a^{-1}\varphi(x)f(v), \end{aligned}$$

so

$$f(x) = x + \varphi(x)(f(a^{-1}v) - a^{-1}v).$$

Since $f(z) - z \in Kh$ for all $z \in E$, we conclude that $u = f(a^{-1}v) - a^{-1}v = \beta h$ for some $\beta \in K$, so $\varphi(u) = 0$, and we have

$$f(x) = x + \varphi(x)u, \quad \varphi(u) = 0. \tag{$*$}$$

A linear map defined as above is denoted by $\tau_{\varphi,u}$.

Conversely for any linear map $f = \tau_{\varphi,u}$ given by Equation $(*)$, where φ is a nonzero linear form and u is some vector $u \in E$ such that $\varphi(u) = 0$, if $u = 0$, then f is the identity, so assume that $u \neq 0$. If so, we have $f(x) = x$ iff $\varphi(x) = 0$, that is, iff $x \in H$. We also claim that the inverse of f is obtained by changing u to $-u$. Actually, we check the slightly more general fact that

$$\tau_{\varphi,u} \circ \tau_{\varphi,w} = \tau_{\varphi,u+w}.$$

Indeed, using the fact that $\varphi(w) = 0$, we have

$$\begin{aligned} \tau_{\varphi,u}(\tau_{\varphi,w}(x)) &= \tau_{\varphi,w}(x) + \varphi(\tau_{\varphi,w}(x))u \\ &= \tau_{\varphi,w}(x) + (\varphi(x) + \varphi(x)\varphi(w))u \\ &= \tau_{\varphi,w}(x) + \varphi(x)u \\ &= x + \varphi(x)w + \varphi(x)u \\ &= x + \varphi(x)(u + w). \end{aligned}$$