for all $p, q \in K[X]$ and all $u, v \in E$. Thus, with this new scalar multiplication, E is a $K[X]$-module denoted by E_f.

If $p = \lambda$ is just a scalar in K (a polynomial of degree 0), then

$$\lambda \cdot u = (\lambda \mathrm{id})(u) = \lambda u,$$

which means that K acts on E by scalar multiplication as before. If $p(X) = X$ (the monomial X), then

$$X \cdot u = f(u).$$

Since K is a field, the ring $K[X]$ is a PID.

If E is finite-dimensional, say of dimension n, since K is a subring of $K[X]$ and since E is finitely generated over K, the $K[X]$-module E_f is finitely generated over $K[X]$. Furthermore, E_f is a torsion module. This follows from the Cayley-Hamilton Theorem (Theorem 7.15), but this can also be shown in an elementary fashion as follows. The space $\mathrm{Hom}(E, E)$ of linear maps of E into itself is a vector space of dimension n^2, therefore the $n^2 + 1$ linear maps

$$\mathrm{id}, f, f^2, \ldots, f^{n^2}$$

are linearly dependent, which yields a nonzero polynomial q such that $q(f) = 0$.

We can now translate notions defined for modules into notions for endomorphisms of vector spaces.

1. To say that U is a submodule of E_f means that U is a subspace of E invariant under f; that is, $f(U) \subseteq U$.

2. To say that V is a cyclic submodule of E_f means that there is some vector $u \in V$, such that V is spanned by $(u, f(u), \ldots, f^k(u), \ldots)$. If E has finite dimension n, then V is spanned by $(u, f(u), \ldots, f^k(u))$ for some $k \leq n-1$. We say that V is a *cyclic subspace for f with generator u*. Sometimes, V is denoted by $Z(u; f)$.

3. To say that the ideal $\mathfrak{a} = (p(X))$ (with $p(X)$ a monic polynomial) is the annihilator of the submodule V means that $p(f)(u) = 0$ for all $u \in V$, and we call p the *minimal polynomial* of V.

4. Suppose E_f is cyclic and let $\mathfrak{a} = (q)$ be its annihilator, where

$$q(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0.$$

Then, there is some vector u such that $(u, f(u), \ldots, f^k(u))$ span E_f, and because q is the minimal polynomial of E_f, we must have $k = n-1$. The fact that $q(f) = 0$ implies that

$$f^n(u) = -a_0 u - a_1 f(u) - \cdots - a_{n-1} f^{n-1}(u),$$