$XI - M$ is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & X(X-2) & 0 \\ 0 & 0 & 0 & X(X-2) \end{pmatrix}.$$

Thus the invariant factors of f are $q_1 = 1 = q_2$, $q_3 = X(X-2) = q_4$, and Theorem 36.5 implies that

$$\mathbb{R}^4 = E_1 \oplus E_2,$$

where $E_1 = Z(u_1, f) \cong \mathbb{R}[X]/(X(X-2))$ and $E_2 = Z(u_2, f) \cong \mathbb{R}[X]/(X(X-2))$. The subspace E_1 has basis (u_1, Mu_1) where $u_1 = (1,0,1,0)$ and $Mu_1 = (1,1,1,1)$, while the subspace E_2 has basis (u_2, Mu_2) where $u_2 = (0,0,1,0)$ and $Mu_2 = (0,1,1,0)$. Theorem 36.6 implies that rational canonical form of $M(f)$ is

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

By Proposition 36.2, two linear maps f and f' are similar iff there is an isomorphism between E_f and $E'_{f'}$, and thus by the uniqueness part of Theorem 35.31, iff they have the same similarity invariants $q_1, \ldots, q_n$.

Proposition 36.7. *If E and E' are two finite-dimensional vector spaces and if $f\colon E \to E$ and $f'\colon E' \to E'$ are two linear maps, then f and f' are similar iff they have the same similarity invariants.*

The effect of extending the field K to a field L is the object of the next proposition.

Proposition 36.8. *Let $f\colon E \to E$ be a linear map on a K-vector space E, and let $(q_1, \ldots, q_n)$ be the similarity invariants of f. If L is a field extension of K (which means that $K \subseteq L$), and if $E_{(L)} = L \otimes_K E$ is the vector space obtained by extending the scalars, and $f_{(L)} = 1_L \otimes f$ the linear map of $E_{(L)}$ induced by f, then the similarity invariants of $f_{(L)}$ are $(q_1, \ldots, q_n)$ viewed as polynomials in $L[X]$.*

Proof. We know that E_f is isomorphic to the direct sum

$$E_f \approx K[X]/(q_1K[X]) \oplus \cdots \oplus K[X]/(q_nK[X]),$$

so by tensoring with $L[X]$ and using Propositions 35.12 and 33.13, we get

$$\begin{aligned} L[X] \otimes_{K[X]} E_f &\approx L[X] \otimes_{K[X]} (K[X]/(q_1K[X]) \oplus \cdots \oplus K[X]/(q_nK[X])) \\ &\approx L[X] \otimes_{K[X]} (K[X]/(q_1K[X])) \oplus \cdots \oplus L[X] \otimes_{K[X]} (K[X]/(q_nK[X])) \\ &\approx (K[X]/(q_1K[X])) \otimes_{K[X]} L[X] \oplus \cdots \oplus (K[X]/(q_nK[X])) \otimes_{K[X]} L[X]. \end{aligned}$$