where P is an orthogonal $n \times n$ matrix and R is an $m \times m$ invertible upper triangular matrix. If we write

$$x = P \begin{pmatrix} y \\ z \end{pmatrix},$$

where $y \in \mathbb{R}^m$ and $z \in \mathbb{R}^{n-m}$, the equation $C^\top x = t$ becomes

$$(R^\top \; 0) P^\top x = t,$$

that is,

$$(R^\top \; 0) \begin{pmatrix} y \\ z \end{pmatrix} = t,$$

which yields

$$R^\top y = t.$$

Since R is invertible, we get $y = (R^\top)^{-1} t$, and then it is easy to see that our original problem reduces to an unconstrained problem in terms of the matrix $P^\top A P$; the details are left as an exercise.

42.3 Maximizing a Quadratic Function on the Unit Sphere

In this section we discuss various quadratic optimization problems mostly arising from computer vision (image segmentation and contour grouping). These problems can be reduced to the following basic optimization problem: given an $n \times n$ real symmetric matrix A

$$\begin{aligned} &\text{maximize} && x^\top A x \\ &\text{subject to} && x^\top x = 1, \; x \in \mathbb{R}^n. \end{aligned}$$

In view of Proposition 23.10, the maximum value of $x^\top A x$ on the unit sphere is equal to the largest eigenvalue λ_1 of the matrix A, and it is achieved for any unit eigenvector u_1 associated with λ_1. Similarly, the minimum value of $x^\top A x$ on the unit sphere is equal to the smallest eigenvalue λ_n of the matrix A, and it is achieved for any unit eigenvector u_n associated with λ_n.

A variant of the above problem often encountered in computer vision consists in minimizing $x^\top A x$ on the ellipsoid given by an equation of the form

$$x^\top B x = 1,$$

where B is a symmetric positive definite matrix. Since B is positive definite, it can be diagonalized as

$$B = Q D Q^\top,$$