importantly $(x_1, x_2, x_3, x_4) = (1/3, 0, 1/3, 2)$ is an optimal solution for our original linear program and provides an optimal value of $-10/3$.

The primal-dual algorithm for linear programming doesn't seem to be the favorite method to solve linear programs nowadays. But it is important because its basic principle, to use a restricted (simpler) primal problem involving an objective function with fixed weights, namely 1, and the dual problem to provide feedback to the primal by improving the objective function of the dual, has led to a whole class of combinatorial algorithms (often approximation algorithms) based on the primal-dual paradigm. The reader will get a taste of this kind of algorithm by consulting Papadimitriou and Steiglitz [134], where it is explained how classical algorithms such as Dijkstra's algorithm for the shortest path problem, and Ford and Fulkerson's algorithm for max flow can be derived from the primal-dual paradigm.

47.7 Summary

The main concepts and results of this chapter are listed below:

- Strictly separating hyperplane.
- Farkas–Minkowski proposition.
- Farkas lemma, version I, Farkas lemma, version II.
- Distance of a point to a subset.
- Dual linear program, primal linear program.
- Dual variables, primal variables.
- Complementary slackness conditions.
- Dual simplex algorithm.
- Primal-dual algorithm.
- Restricted primal linear program.

47.8 Problems

Problem 47.1. Let $(v_1, \ldots, v_n)$ be a sequence of n vectors in $\mathbb{R}^d$ and let V be the $d \times n$ matrix whose j-th column is v_j. Prove the equivalence of the following two statements:

(a) There is no nontrivial positive linear dependence among the v_j, which means that there is no nonzero vector, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, with $y_j \geq 0$ for $j = 1, \ldots, n$, so that

$$y_1 v_1 + \cdots + y_n v_n = 0$$

or equivalently, $Vy = 0$.