Proposition 34.6. *Given any two linear maps $f\colon E \to E'$ and $f'\colon E' \to E''$, we have*

$$(f' \circ f) \wedge (f' \circ f) = (f' \wedge f') \circ (f \wedge f).$$

The generalization to the alternating product $f \wedge \cdots \wedge f$ of $n \geq 3$ copies of the linear map $f\colon E \to E'$ is immediate, and left to the reader.

34.2 Bases of Exterior Powers

Definition 34.4. Let E be any vector space. For any basis $(u_i)_{i \in \Sigma}$ for E, we assume that some total ordering $\leq$ on the index set Σ has been chosen. Call the pair $((u_i)_{i \in \Sigma}, \leq)$ an *ordered basis*. Then for any nonempty finite subset $I \subseteq \Sigma$, let

$$u_I = u_{i_1} \wedge \cdots \wedge u_{i_m},$$

where $I = \{i_1, \ldots, i_m\}$, with $i_1 < \cdots < i_m$.

Since $\bigwedge^n(E)$ is generated by the tensors of the form $v_1 \wedge \cdots \wedge v_n$, with $v_i \in E$, in view of skew-symmetry, it is clear that the tensors u_I with $|I| = n$ generate $\bigwedge^n(E)$ (where $((u_i)_{i \in \Sigma}, \leq)$ is an ordered basis). Actually they form a basis. To gain an intuitive understanding of this statement, let $m = 2$ and E be a 3-dimensional vector space lexicographically ordered basis $\{e_1, e_2, e_3\}$. We claim that

$$e_1 \wedge e_2, \qquad e_1 \wedge e_3, \qquad e_2 \wedge e_3$$

form a basis for $\bigwedge^2(E)$ since they not only generate $\bigwedge^2(E)$ but are linearly independent. The linear independence is argued as follows: given any vector space F, if w_{12}, w_{13}, w_{23} are any vectors in F, there is an alternating bilinear map $h\colon E^2 \to F$ such that

$$h(e_1, e_2) = w_{12}, \qquad h(e_1, e_3) = w_{13}, \qquad h(e_2, e_3) = w_{23}.$$

Because h yields a unique linear map $h_\wedge\colon \bigwedge^2 E \to F$ such that

$$h_\wedge(e_i \wedge e_j) = w_{ij}, \quad 1 \leq i < j \leq 3,$$

by Proposition 33.4, the vectors

$$e_1 \wedge e_2, \qquad e_1 \wedge e_3, \qquad e_2 \wedge e_3$$

are linearly independent. This suggests understanding how an alternating bilinear function $f\colon E^2 \to F$ is expressed in terms of its values $f(e_i, e_j)$ on the basis vectors (e_1, e_2, e_3). Using bilinearity and alternation, we obtain

$$\begin{aligned} f(u_1e_1 + u_2e_2 + u_3e_3, v_1e_1 + v_2e_2 + v_3e_3) &= (u_1v_2 - u_2v_1)f(e_1, e_2) + (u_1v_3 - u_3v_1)f(e_1, e_3) \\ &\quad + (u_2v_3 - u_3v_2)f(e_2, e_3). \end{aligned}$$