(7) *Norm squared*: $f(x) = \frac{1}{2}\left\|x\right\|^2$ for any norm $\|\ \|$ on $\mathbb{R}^n$, with $\mathrm{dom}(f) = \mathbb{R}^n$. Since $|y^\top x| \leq \|x\|\,\|y\|^D$, we have

$$y^\top x - (1/2)\left\|x\right\|^2 \leq \|y\|^D \|x\| - (1/2)\left\|x\right\|^2.$$

The right-hand side is a quadratic function of $\|x\|$ which achieves its maximum at $\|x\| = \|y\|^D$, with maximum value $(1/2)(\|y\|^D)^2$. Therefore

$$y^\top x - (1/2)\left\|x\right\|^2 \leq (1/2)\Big(\|y\|^D\Big)^2$$

for all x, which shows that

$$f^*(y) \leq (1/2)\Big(\|y\|^D\Big)^2.$$

By definition of the dual norm and because the unit sphere is compact, for any $y \in \mathbb{R}^n$, there is some $x \in \mathbb{R}^n$ such that $\|x\| = 1$ and $y^\top x = \|y\|^D$, so multiplying both sides by $\|y\|^D$ we obtain

$$y^\top \|y\|^D x = \Big(\|y\|^D\Big)^2$$

and for $z = \|y\|^D x$, since $\|x\| = 1$ we have $\|z\| = \|y\|^D \|x\| = \|y\|^D$, so we get

$$y^\top z - (1/2)(\|z\|)^2 = \Big(\|y\|^D\Big)^2 - (1/2)\Big(\|y\|^D\Big)^2 = (1/2)\Big(\|y\|^D\Big)^2,$$

which shows that the upper bound $(1/2)\Big(\|y\|^D\Big)^2$ is achieved. Therefore,

$$f^*(y) = \frac{1}{2}\Big(\|y\|^D\Big)^2,$$

and $\mathrm{dom}(f^*) = \mathbb{R}^n$.

(8) *Log-sum-exp function*: $f(x) = \log\Big(\sum_{i=1}^n e^{x_i}\Big)$, where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. To determine the values of $y \in \mathbb{R}^n$ for which the maximum of $g(x) = y^\top x - f(x)$ over $x \in \mathbb{R}^n$ is attained, we compute its gradient and we find

$$\nabla f_x = \begin{pmatrix} y_1 - \dfrac{e^{x_1}}{\sum_{i=1}^n e^{x_i}} \\ \vdots \\ y_n - \dfrac{e^{x_n}}{\sum_{i=1}^n e^{x_i}} \end{pmatrix}.$$

Therefore, $(y_1, \ldots, y_n)$ must satisfy the system of equations

$$y_j = \frac{e^{x_j}}{\sum_{i=1}^n e^{x_i}}, \quad j = 1, \ldots, n. \tag{$*$}$$