It is not hard to show that if the primal linear program with objective function $c^\top x$ and equational constraints $Ax = b$ and the dual program with objective function $b^\top y$ and inequality constraints $A^\top y \geq c$ have interior feasible points x and y, which means that $x > 0$ and $s > 0$ (where $s = A^\top y - c$), then the above system of equations has a unique solution such that x is the unique maximizer of f_μ on U; see Matousek and Gardner [123] (Section 7.2, Lemma 7.2.1).

A particularly important application of Proposition 50.9 is the situation where $\Omega = \mathbb{R}^n$.

50.4 Equality Constrained Minimization

In this section we consider the following Program (P):

$$\begin{aligned} &\text{minimize} && J(v) \\ &\text{subject to} && Av = b,\ v \in \mathbb{R}^n, \end{aligned}$$

where J is a convex differentiable function and A is an $m \times n$ matrix of rank $m < n$ (the number of equality constraints is less than the number of variables, and these constraints are independent), and $b \in \mathbb{R}^m$.

According to Proposition 50.9 (with $\Omega = \mathbb{R}^n$), Program (P) has a minimum at $x \in \mathbb{R}^n$ if and only if there exist some Lagrange multipliers $\lambda \in \mathbb{R}^m$ such that the following equations hold:

$$Ax = b \qquad \text{(pfeasibilty)}$$
$$\nabla J_x + A^\top \lambda = 0. \qquad \text{(dfeasibility)}$$

The set of linear equations $Ax = b$ is called the *primal feasibility equations* and the set of (generally nonlinear) equations $\nabla J_x + A^\top \lambda = 0$ is called the set of *dual feasibility equations*.

In general, it is impossible to solve these equations analytically, so we have to use numerical approximation procedures, most of which are variants of Newton's method. In special cases, for example if J is a quadratic functional, the dual feasibility equations are also linear, a case that we consider in more detail.

Suppose J is a convex quadratic functional of the form

$$J(x) = \frac{1}{2} x^\top P x + q^\top x + r,$$

where P is a $n \times n$ symmetric positive semidefinite matrix, $q \in \mathbb{R}^n$ and $r \in \mathbb{R}$. In this case

$$\nabla J_x = Px + q,$$