1. As a row operation, $P(i,k)$ permutes row i and row k.

2. As a column operation, $P(i,k)$ permutes column i and column k.

3. The inverse of $P(i,k)$ is $P(i,k)$ itself.

4. As a row operation, $E_{i,j;\beta}$ adds β times row j to row i.

5. As a column operation, $E_{i,j;\beta}$ adds β times column i to column j (note the switch in the indices).

6. The inverse of $E_{i,j;\beta}$ is $E_{i,j;-\beta}$.

7. As a row operation, $E_{i,\lambda}$ multiplies row i by λ.

8. As a column operation, $E_{i,\lambda}$ multiplies column i by λ.

9. The inverse of $E_{i,\lambda}$ is $E_{i,\lambda^{-1}}$.

We can define the notion of a reduced column echelon matrix and show that every matrix can be reduced to a unique reduced column echelon form. Now given any $m \times n$ matrix A, if we first convert A to its reduced row echelon form R, it is easy to see that we can apply elementary column operations that will reduce R to a matrix of the form

$$\begin{pmatrix} I_r & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{pmatrix},$$

where r is the number of pivots (obtained during the row reduction). Therefore, for every $m \times n$ matrix A, there exist two sequences of elementary matrices $E_1, \ldots, E_p$ and $F_1, \ldots, F_q$, such that

$$E_p \cdots E_1 A F_1 \cdots F_q = \begin{pmatrix} I_r & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{pmatrix}.$$

The matrix on the right-hand side is called the *rank normal form* of A. Clearly, r is the rank of A. As a corollary we obtain the following important result whose proof is immediate.

Proposition 8.21. *A matrix A and its transpose $A^\top$ have the same rank.*

8.15 Transvections and Dilatations ⊛

In this section we characterize the linear isomorphisms of a vector space E that leave every vector in some hyperplane fixed. These maps turn out to be the linear maps that are represented in some suitable basis by elementary matrices of the form $E_{i,j;\beta}$ (transvections) or $E_{i,\lambda}$ (dilatations). Furthermore, the transvections generate the group $\mathbf{SL}(E)$, and the dilatations generate the group $\mathbf{GL}(E)$.