**Definition 37.42.** Given normed vector spaces $E$, $F$, and $G$, for every continuous bilinear map $f\colon E \times F \to G$, we define the *norm* $\|f\|$ *of* $f$ as

$$\begin{aligned}\|f\| &= \inf\{k \geq 0 \mid \|f(x,y)\| \leq k\|x\|\|y\|, \text{ for all } x \in E, y \in F\} \\ &= \sup\{\|f(x,y)\| \mid \|x\|, \|y\| \leq 1\}.\end{aligned}$$

From Definition 37.41, for every continuous bilinear map $f \in \mathcal{L}_2(E, F; G)$, we have

$$\|f(x,y)\| \leq \|f\|\|x\|\|y\|,$$

for all $x, y \in E$. It is easy to verify that $\mathcal{L}_2(E, F; G)$ is a normed vector space under the norm of Definition 37.42.

Given a bilinear map $f\colon E \times F \to G$, for every $u \in E$, we obtain a linear map denoted $fu\colon F \to G$, defined such that, $fu(v) = f(u, v)$. Furthermore, since

$$\|f(x,y)\| \leq \|f\|\|x\|\|y\|,$$

it is clear that $fu$ is continuous. We can then consider the map $\varphi\colon E \to \mathcal{L}(F; G)$, defined such that, $\varphi(u) = fu$, for any $u \in E$, or equivalently, such that,

$$\varphi(u)(v) = f(u, v).$$

Actually, it is easy to show that $\varphi$ is linear and continuous, and that $\|\varphi\| = \|f\|$. Thus, $f \mapsto \varphi$ defines a map from $\mathcal{L}_2(E, F; G)$ to $\mathcal{L}(E; \mathcal{L}(F; G))$. We can also go back from $\mathcal{L}(E; \mathcal{L}(F; G))$ to $\mathcal{L}_2(E, F; G)$. We summarize all this in the following proposition.

**Proposition 37.60.** *Let $E, F, G$ be three normed vector spaces. The map $f \mapsto \varphi$, from $\mathcal{L}_2(E, F; G)$ to $\mathcal{L}(E; \mathcal{L}(F; G))$, defined such that, for every $f \in \mathcal{L}_2(E, F; G)$,*

$$\varphi(u)(v) = f(u, v),$$

*is an isomorphism of vector spaces, and furthermore, $\|\varphi\| = \|f\|$.*

As a corollary of Proposition 37.60, we get the following proposition which will be useful when we define second-order derivatives.

**Proposition 37.61.** *Let $E, F$ be normed vector spaces. The map app from $\mathcal{L}(E; F) \times E$ to $F$, defined such that, for every $f \in \mathcal{L}(E; F)$, for every $u \in E$,*

$$app(f, u) = f(u),$$

*is a continuous bilinear map.*