so the spectral radius of $J = B$ is

$$\rho(J) = \cos\left(\frac{\pi}{5}\right) = 0.8090 < 1.$$

By Theorem 10.3, Jacobi's method converges for the matrix of this example.

Observe that we can try to "speed up" the method by using the new value u_1^{k+1} instead of u_1^k in solving for u_2^{k+2} using the second equations, and more generally, use $u_1^{k+1}, \ldots, u_{i-1}^{k+1}$ instead of $u_1^k, \ldots, u_{i-1}^k$ in solving for u_i^{k+1} in the ith equation. This observation leads to the system

$$\begin{array}{rcccccccc}
a_{11}u_1^{k+1} & = & & -a_{12}u_2^k & -a_{13}u_3^k & \cdots & & -a_{1n}u_n^k & +b_1 \\
a_{22}u_2^{k+1} & = & -a_{21}u_1^{k+1} & & -a_{23}u_3^k & \cdots & & -a_{2n}u_n^k & +b_2 \\
\vdots & \vdots & \vdots & & & & & & \\
a_{n-1\,n-1}u_{n-1}^{k+1} & = & -a_{n-1\,1}u_1^{k+1} & \cdots & -a_{n-1\,n-2}u_{n-2}^{k+1} & & & -a_{n-1\,n}u_n^k & +b_{n-1} \\
a_{n\,n}u_n^{k+1} & = & -a_{n\,1}u_1^{k+1} & -a_{n\,2}u_2^{k+1} & \cdots & & -a_{n\,n-1}u_{n-1}^{k+1} & & +b_n,
\end{array}$$

which, in matrix form, is written

$$Du_{k+1} = Eu_{k+1} + Fu_k + b.$$

Because D is invertible and E is lower triangular, the matrix $D - E$ is invertible, so the above equation is equivalent to

$$u_{k+1} = (D - E)^{-1}Fu_k + (D - E)^{-1}b, \quad k \geq 0.$$

The above corresponds to choosing M and N to be

$$\begin{aligned} M &= D - E \\ N &= F, \end{aligned}$$

and the matrix B is given by

$$B = M^{-1}N = (D - E)^{-1}F.$$

Since $M = D - E$ is invertible, we know that $I - B = M^{-1}A$ is also invertible.

The method that we just described is the *iterative method of Gauss–Seidel*, and the matrix B is called the *matrix of Gauss–Seidel* and denoted by $\mathcal{L}_1$, with

$$\mathcal{L}_1 = (D - E)^{-1}F.$$

One of the advantages of the method of Gauss–Seidel is that is requires only half of the memory used by Jacobi's method, since we only need

$$u_1^{k+1}, \ldots, u_{i-1}^{k+1}, u_{i+1}^k, \ldots, u_n^k$$

to compute u_i^{k+1}. We also show that in certain important cases (for example, if A is a tridiagonal matrix), the method of Gauss–Seidel converges faster than Jacobi's method (in this case, they both converge or diverge simultaneously).

In `Matlab` one step of Gauss–Seidel iteration is achieved by the following function: