When $F = K$, we call f an *n-linear form (or multilinear form)*. If $n \geq 2$ and $E_1 = E_2 = \ldots = E_n$, an n-linear map $f\colon E \times \ldots \times E \to F$ is called *symmetric*, if $f(x_1, \ldots, x_n) = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$ for every permutation π on $\{1, \ldots, n\}$. An n-linear map $f\colon E\times\ldots\times E \to F$ is called *alternating*, if $f(x_1, \ldots, x_n) = 0$ whenever $x_i = x_{i+1}$ for some i, $1 \leq i \leq n-1$ (in other words, when two adjacent arguments are equal). It does no harm to agree that when $n = 1$, a linear map is considered to be both symmetric and alternating, and we will do so.

When $n = 2$, a 2-linear map $f\colon E_1 \times E_2 \to F$ is called a *bilinear map*. We have already seen several examples of bilinear maps. Multiplication $\cdot\colon K \times K \to K$ is a bilinear map, treating K as a vector space over itself.

The operation $\langle -, - \rangle\colon E^* \times E \to K$ applying a linear form to a vector is a bilinear map.

Symmetric bilinear maps (and multilinear maps) play an important role in geometry (inner products, quadratic forms) and in differential calculus (partial derivatives).

A bilinear map is symmetric if $f(u, v) = f(v, u)$, for all $u, v \in E$.

Alternating multilinear maps satisfy the following simple but crucial properties.

Proposition 7.3. *Let $f\colon E \times \ldots \times E \to F$ be an n-linear alternating map, with $n \geq 2$. The following properties hold:*

(1)
$$f(\ldots, x_i, x_{i+1}, \ldots) = -f(\ldots, x_{i+1}, x_i, \ldots)$$

(2)
$$f(\ldots, x_i, \ldots, x_j, \ldots) = 0,$$
where $x_i = x_j$, and $1 \leq i < j \leq n$.

(3)
$$f(\ldots, x_i, \ldots, x_j, \ldots) = -f(\ldots, x_j, \ldots, x_i, \ldots),$$
where $1 \leq i < j \leq n$.

(4)
$$f(\ldots, x_i, \ldots) = f(\ldots, x_i + \lambda x_j, \ldots),$$
for any $\lambda \in K$, and where $i \neq j$.

Proof. (1) By multilinearity applied twice, we have

$$\begin{aligned} f(\ldots, x_i + x_{i+1}, x_i + x_{i+1}, \ldots) = {} & f(\ldots, x_i, x_i, \ldots) + f(\ldots, x_i, x_{i+1}, \ldots) \\ & + f(\ldots, x_{i+1}, x_i, \ldots) + f(\ldots, x_{i+1}, x_{i+1}, \ldots), \end{aligned}$$

and since f is alternating, this yields

$$0 = f(\ldots, x_i, x_{i+1}, \ldots) + f(\ldots, x_{i+1}, x_i, \ldots),$$