Chapter 13

QR-Decomposition for Arbitrary Matrices

13.1 Orthogonal Reflections

Hyperplane reflections are represented by matrices called Householder matrices. These matrices play an important role in numerical methods, for instance for solving systems of linear equations, solving least squares problems, for computing eigenvalues, and for transforming a symmetric matrix into a tridiagonal matrix. We prove a simple geometric lemma that immediately yields the QR-decomposition of arbitrary matrices in terms of Householder matrices.

Orthogonal symmetries are a very important example of isometries. First let us review the definition of projections, introduced in Section 6.1, just after Proposition 6.7. Given a vector space E, let F and G be subspaces of E that form a direct sum $E = F \oplus G$. Since every $u \in E$ can be written uniquely as $u = v + w$, where $v \in F$ and $w \in G$, we can define the two *projections* $p_F \colon E \to F$ and $p_G \colon E \to G$ such that $p_F(u) = v$ and $p_G(u) = w$. In Section 6.1 we used the notation π_1 and π_2, but in this section it is more convenient to use p_F and p_G.

It is immediately verified that p_G and p_F are linear maps, and that

$$p_F^2 = p_F,\ p_G^2 = p_G,\ p_F \circ p_G = p_G \circ p_F = 0, \quad \text{and} \quad p_F + p_G = \mathrm{id}.$$

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Definition 13.1. Given a vector space E, for any two subspaces F and G that form a direct sum $E = F \oplus G$, the *symmetry (or reflection) with respect to F and parallel to G* is the linear map $s \colon E \to E$ defined such that

$$s(u) = 2p_F(u) - u,$$

for every $u \in E$.