

the output is the matrix

$$\begin{pmatrix} 0.5774 & 0.4082 & 0.7071 \\ 0.5774 & -0.8165 & -0.0000 \\ 0.5774 & 0.4082 & -0.7071 \end{pmatrix},$$

which matches the result of Example 12.9.

Example 12.11. If we consider polynomials and the inner product

$$\langle f, g \rangle = \int_{-1}^1 f(t)g(t)dt,$$

applying the Gram–Schmidt orthonormalization procedure to the polynomials

$$1, x, x^2, \dots, x^n, \dots,$$

which form a basis of the polynomials in one variable with real coefficients, we get a family of orthonormal polynomials $Q_n(x)$ related to the *Legendre polynomials*.

The Legendre polynomials $P_n(x)$ have many nice properties. They are orthogonal, but their norm is not always 1. The Legendre polynomials $P_n(x)$ can be defined as follows. Letting f_n be the function

$$f_n(x) = (x^2 - 1)^n,$$

we define $P_n(x)$ as follows:

$$P_0(x) = 1, \quad \text{and} \quad P_n(x) = \frac{1}{2^n n!} f_n^{(n)}(x),$$

where $f_n^{(n)}$ is the n th derivative of f_n .

They can also be defined inductively as follows:

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= x, \\ P_{n+1}(x) &= \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n-1}(x). \end{aligned}$$

Here is an explicit summation for $P_n(x)$:

$$P_n(x) = \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{k} \binom{2n-2k}{n} x^{n-2k}.$$

The polynomials Q_n are related to the Legendre polynomials P_n as follows:

$$Q_n(x) = \sqrt{\frac{2n+1}{2}} P_n(x).$$