Proposition 2.2. *In a monoid M with identity element e, if some element $a \in M$ has some left inverse $a' \in M$ and some right inverse $a'' \in M$, which means that*

$$a' \cdot a = e \tag{G3l}$$

and

$$a \cdot a'' = e, \tag{G3r}$$

then $a' = a''$.

Proof. Using (G3l) and the fact that e is an identity element, we have

$$(a' \cdot a) \cdot a'' = e \cdot a'' = a''.$$

Similarly, Using (G3r) and the fact that e is an identity element, we have

$$a' \cdot (a \cdot a'') = a' \cdot e = a'.$$

However, since M is monoid, the operation $\cdot$ is associative, so

$$a' = a' \cdot (a \cdot a'') = (a' \cdot a) \cdot a'' = a'',$$

as claimed. $\square$

Remark: Axioms (G2) and (G3) can be weakened a bit by requiring only (G2r) (the existence of a right identity) and (G3r) (the existence of a right inverse for every element) (or (G2l) and (G3l)). It is a good exercise to prove that the group axioms (G2) and (G3) follow from (G2r) and (G3r).

Another important property about inverse elements in monoids is stated below.

Proposition 2.3. *In a monoid M with identity element e, if a and b are invertible elements of M, where a^{-1} is the inverse of a and b^{-1} is the inverse of b, then ab is invertible and its inverse is given by $(ab)^{-1} = b^{-1}a^{-1}$.*

Proof. Using associativity and the fact that e is the identity element we have

$$\begin{aligned}
(ab)(b^{-1}a^{-1}) &= a(b(b^{-1}a^{-1})) && \text{associativity}\\
&= a((bb^{-1})a^{-1}) && \text{associativity}\\
&= a(ea^{-1}) && b^{-1} \text{ is the inverse of } b\\
&= aa^{-1} && e \text{ is the identity element}\\
&= e. && a^{-1} \text{ is the inverse of } a.
\end{aligned}$$