then it is not hard to show that for any $a_1, \ldots, a_n \in \mathbb{Z}$,

$$x = a_1 t_1 m_1' + \cdots + a_n t_n m_n'$$

satisfies the congruences

$$x \equiv a_i \pmod{m_i}, \quad i = 1, \ldots, n.$$

Theorem 32.15 can be used to characterize rings isomorphic to finite products of quotient rings. Such rings play a role in the structure theorem for torsion modules over a PID.

Given n rings $A_1, \ldots, A_n$, recall that the product ring $A = A_1 \times \cdots \times A_n$ is the ring in which addition and multiplication are defined componenwise. That is,

$$\begin{aligned} (a_1, \ldots, a_n) + (b_1, \ldots, b_n) &= (a_1 + b_1, \ldots, a_n + b_n) \\ (a_1, \ldots, a_n) \cdot (b_1, \ldots, b_n) &= (a_1 b_1, \ldots, a_n b_n). \end{aligned}$$

The additive identity is $0_A = (0, \ldots, 0)$ and the multiplicative identity is $1_A = (1, \ldots, 1)$. Then, for $i = 1, \ldots, n$, we can define the element $e_i \in A$ as follows:

$$e_i = (0, \ldots, 0, 1, 0, \ldots, 0),$$

where the 1 occurs in position i. Observe that the following properties hold for all $i, j = 1, \ldots, n$:

$$\begin{aligned} e_i^2 &= e_i \\ e_i e_j &= 0, \quad i \neq j \\ e_1 + \cdots + e_n &= 1_A. \end{aligned}$$

Also, for any element $a = (a_1, \ldots, a_n) \in A$, we have

$$e_i a = (0, \ldots, 0, a_i, 0, \ldots, 0) = pr_i(a),$$

where pr_i is the projection of A onto A_i. As a consequence

$$\mathrm{Ker}\,(pr_i) = (1_A - e_i)A.$$

Definition 32.3. Given a commutative ring A, a *direct decomposition* of A is a sequence $(\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$ of ideals in A such that there is an isomorphism $A \approx A/\mathfrak{b}_1 \times \cdots \times A/\mathfrak{b}_n$.

The following theorem gives useful conditions characterizing direct decompositions of a ring.

Theorem 32.16. *Let A be a commutative ring and let $(\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$ be a sequence of ideals in A. The following conditions are equivalent:*

(a) *The sequence $(\mathfrak{b}_1, \ldots, \mathfrak{b}_n)$ is a direct decomposition of A.*