map that is bijective. Thus, E_H is isomorphic to the affine hyperplane $w + H$. If we had chosen a different vector $w' \in E - H$ such that $E = Kw' \oplus H$, then E_H would be isomorphic to the affine hyperplane $w' + H$ parallel to $w + H$. But these two hyperplanes are clearly isomorphic by translation, and thus the affine structure on E_H depends only on H. $\square$

An affine space of the form E_H is called an *affine patch* on $\mathbf{P}(E)$. Proposition 26.16 allows us to view a projective space $\mathbf{P}(E)$ as the result of gluing some affine spaces together, at least when E is of finite dimension. For example, when E is of dimension 2, a hyperplane in E is just a line, and the complement of a point in the projective line $\mathbf{P}(E)$ can be viewed as an affine line. Thus, we can view $\mathbf{P}(E)$ as being covered by two affine lines glued together as illustrated by When $K = \mathbb{R}$, this shows that topologically, the projective line $\mathbb{RP}^1$ is equivalent to a circle. See Figure 26.18. When E is of dimension 3, a hyperplane in E is

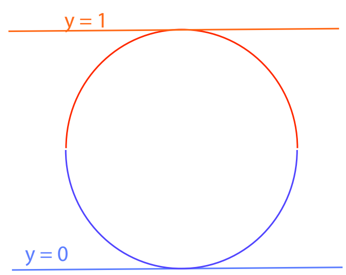

Figure 26.18: The covering of $\mathbb{RP}^1$ by the affine lines $y = 0$ and $y = 1$.

just a plane, and the complement of a projective line in the projective plane $\mathbf{P}(E)$ can be viewed as an affine plane. Thus, we can view $\mathbf{P}(E)$ as being covered by three affine planes glued together as illustrated by Figure 26.19.

However, even when $K = \mathbb{R}$, it is much more difficult to come up with a geometric embedding of the projective plane $\mathbb{RP}^2$ in $\mathbb{A}^3$, and in fact, this is impossible! Nevertheless, there are some fascinating immersions of the projective space $\mathbb{RP}^2$ as $3D$ surfaces with self-intersection, one of which is known as the Boy surface. We urge our readers to consult the remarkable book by Hilbert and Cohn-Vossen [92] for drawings of the Boy surface, and more. One should also consult Fischer's books [61, 60], where many beautiful models of surfaces are displayed, and the commentaries in Chapter 6 of [60] regarding models of $\mathbb{RP}^2$. More generally, when E is of dimension $n+1$, the projective space $\mathbf{P}(E)$ is covered by $n+1$ affine patches (hyperplanes) glued together. This idea is very fruitful, since it allows the treatment of projective spaces as manifolds, and it is essential in algebraic geometry.

We can now go back to the projective completion $\widetilde{E}$ of an affine space E.