and if

$$dJ(u)(v-u) \geq 0 \quad \text{for all } v \in U,$$

then

$$J(v) - J(u) \geq 0 \quad \text{for all } v \in U,$$

as claimed.

(4) If U is open, then for every $u \in U$ we can find an open ball B centered at u of radius ϵ small enough so that $B \subseteq U$. Then for any $w \neq 0$ such that $\|w\| < \epsilon$, we have both $v = u + w \in B$ and $v' = u - w \in B$, so Condition (3) implies that

$$dJ(u)(w) \geq 0 \quad \text{and} \quad dJ(u)(-w) \geq 0,$$

which yields

$$dJ(u)(w) = 0.$$

Since the above holds for all $w \neq 0$ such such that $\|w\| < \epsilon$ and since $dJ(u)$ is linear, we leave it to the reader to fill in the details of the proof that $dJ(u) = 0$. $\square$

Example 40.7. Theorem 40.13 can be used to rederive the fact that the least squares solutions of a linear system $Ax = b$ (where A is an $m \times n$ matrix) are given by the normal equation

$$A^\top A x = A^\top b.$$

For this, we consider the quadratic function

$$J(v) = \frac{1}{2}\left\|Av - b\right\|_2^2 - \frac{1}{2}\left\|b\right\|_2^2,$$

and our least squares problem is equivalent to finding the minima of J on $\mathbb{R}^n$. A computation reveals that

$$\begin{aligned}
J(v) &= \frac{1}{2}\left\|Av - b\right\|_2^2 - \frac{1}{2}\left\|b\right\|_2^2 \\
&= \frac{1}{2}(Av - b)^\top (Av - b) - \frac{1}{2} b^\top b \\
&= \frac{1}{2}(v^\top A^\top - b^\top)(Av - b) - \frac{1}{2} b^\top b \\
&= \frac{1}{2} v^\top A^\top A v - v^\top A^\top b,
\end{aligned}$$

and so

$$dJ(u) = A^\top A u - A^\top b.$$

Since $A^\top A$ is positive semidefinite, the function J is convex, and Theorem 40.13(4) implies that the minima of J are the solutions of the equation

$$A^\top A u - A^\top b = 0.$$