for all $v \in E$, and since by assumption,

$$f^*(u_1) \cdot v = u_1 \cdot f(v) \quad \text{and} \quad f^*(u_2) \cdot v = u_2 \cdot f(v),$$

for all $v \in E$. Thus we get

$$(f^*(u_1) + f^*(u_2)) \cdot v = (u_1 + u_2) \cdot f(v) = f^*(u_1 + u_2) \cdot v,$$

for all $v \in E$. Since our inner product is positive definite, this implies that

$$f^*(u_1 + u_2) = f^*(u_1) + f^*(u_2).$$

Similarly,

$$(\lambda u) \cdot f(v) = \lambda (u \cdot f(v)),$$

for all $v \in E$, and

$$(\lambda f^*(u)) \cdot v = \lambda (f^*(u) \cdot v),$$

for all $v \in E$, and since by assumption,

$$f^*(u) \cdot v = u \cdot f(v),$$

for all $v \in E$, we get

$$(\lambda f^*(u)) \cdot v = \lambda (u \cdot f(v)) = (\lambda u) \cdot f(v) = f^*(\lambda u) \cdot v$$

for all $v \in E$. Since $\flat$ is bijective, this implies that

$$f^*(\lambda u) = \lambda f^*(u).$$

Thus, f^* is indeed a linear map, and it is unique since $\flat$ is a bijection. $\square$

Definition 12.4. Given a Euclidean space E of finite dimension, for every linear map $f\colon E \to E$, the unique linear map $f^*\colon E \to E$ such that

$$f^*(u) \cdot v = u \cdot f(v), \quad \text{for all } u, v \in E$$

given by Proposition 12.8 is called the *adjoint of f (w.r.t. to the inner product)*. Linear maps $f\colon E \to E$ such that $f = f^*$ are called *self-adjoint* maps.

Self-adjoint linear maps play a very important role because they have real eigenvalues, and because orthonormal bases arise from their eigenvectors. Furthermore, many physical problems lead to self-adjoint linear maps (in the form of symmetric matrices).

Remark: Proposition 12.8 still holds if the inner product on E is replaced by a nondegenerate symmetric bilinear form φ.