The first step is to embed a real vector space $E$ into a complex vector space $E_{\mathbb{C}}$. A quick but somewhat bewildering way to do so is to define the complexification of $E$ as the tensor product $\mathbb{C} \otimes E$. A more tangible way is to define the following structure.

**Definition 26.13.** Given a real vector space $E$, let $E_{\mathbb{C}}$ be the structure $E \times E$ under the addition operation

$$(u_1,\, u_2) + (v_1,\, v_2) = (u_1 + v_1,\, u_2 + v_2),$$

and let multiplication by a complex scalar $z = x + iy$ be defined such that

$$(x + iy) \cdot (u,\, v) = (xu - yv,\, yu + xv).$$

It is easily shown that the structure $E_{\mathbb{C}}$ is a complex vector space. It is also immediate that

$$(0,\, v) = i(v,\, 0),$$

and thus, identifying $E$ with the subspace of $E_{\mathbb{C}}$ consisting of all vectors of the form $(u,\, 0)$, we can write

$$(u,\, v) = u + iv.$$

Given a vector $w = u + iv$, its *conjugate* $\overline{w}$ is the vector $\overline{w} = u - iv$. Then conjugation is a map from $E_{\mathbb{C}}$ to itself that is an involution. If $(e_1, \ldots, e_n)$ is any basis of $E$, then $((e_1, 0), \ldots, (e_n, 0))$ is a basis of $E_{\mathbb{C}}$. We call such a basis a *real basis*.

Given a linear map $f\colon E \to E$, the map $f$ can be extended to a linear map $f_{\mathbb{C}}\colon E_{\mathbb{C}} \to E_{\mathbb{C}}$ defined such that

$$f_{\mathbb{C}}(u + iv) = f(u) + if(v).$$

We define the *complexification* of $\mathbf{P}(E)$ as $\mathbf{P}(E_{\mathbb{C}})$. If $\big(E, \overrightarrow{E}\big)$ is a real affine space, we define the *complexified projective completion of* $\big(E, \overrightarrow{E}\big)$ as $\mathbf{P}(\widehat{E}_{\mathbb{C}})$ and denote it by $\widetilde{E}_{\mathbb{C}}$. Then $\widetilde{E}$ is naturally embedded in $\widetilde{E}_{\mathbb{C}}$, and it is called the set of *real points* of $\widetilde{E}_{\mathbb{C}}$.

If $E$ has dimension $n+1$ and $(e_1, \ldots, e_{n+1})$ is a basis of $E$, given any homogeneous polynomial $P(x_1, \ldots, x_{n+1})$ over $\mathbb{C}$ of total degree $m$, because $P$ is homogeneous, it is immediately verified that

$$P(x_1, \ldots, x_{n+1}) = 0$$

iff

$$P(\lambda x_1, \ldots, \lambda x_{n+1}) = 0,$$

for any $\lambda \neq 0$. Thus, we can define the *hypersurface* $V(P)$ *of equation* $P(x_1, \ldots, x_{n+1}) = 0$ as the subset of $\widetilde{E}_{\mathbb{C}}$ consisting of all points of homogeneous coordinates $(x_1, \ldots, x_{n+1})$ such that $P(x_1, \ldots, x_{n+1}) = 0$. We say that the hypersurface $V(P)$ of equation $P(x_1, \ldots, x_{n+1}) = 0$ is *real* whenever $P(x_1, \ldots, x_{n+1}) = 0$ implies that $P(\overline{x}_1, \ldots, \overline{x}_{n+1}) = 0$.