The above discussion suggests that it might be useful to know when an alternating tensor is simple (decomposable). We will show in Section 34.7 that for tensors $\alpha \in \bigwedge^2(V)$, $\alpha\wedge\alpha = 0$ iff α is simple.

A general criterion for decomposability can be given in terms of some operations known as *left hook* and *right hook* (also called *interior products*); see Section 34.7.

It is easy to see that $\bigwedge(V)$ satisfies the following universal mapping property.

Proposition 34.13. *Given any K-algebra A, for any linear map $f\colon V \to A$, if $(f(v))^2 = 0$ for all $v \in V$, then there is a unique K-algebra homomorphism $\overline{f}\colon \bigwedge(V) \to A$ so that*

$$f = \overline{f} \circ i,$$

as in the diagram below.

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[dr, "f"'] & \bigwedge(V) \arrow[d, "\overline{f}"] \\ & A \end{tikzcd}$$

When E is finite-dimensional, recall the isomorphism $\mu\colon \bigwedge^n(E^*) \longrightarrow \mathrm{Alt}^n(E;K)$, defined as the linear extension of the map given by

$$\mu(v_1^* \wedge \cdots \wedge v_n^*)(u_1, \ldots, u_n) = \det(v_j^*(u_i)).$$

Now, we have also a multiplication operation $\bigwedge^m(E^*) \times \bigwedge^n(E^*) \longrightarrow \bigwedge^{m+n}(E^*)$. The following question then arises:

Can we define a multiplication $\mathrm{Alt}^m(E;K) \times \mathrm{Alt}^n(E;K) \longrightarrow \mathrm{Alt}^{m+n}(E;K)$ directly on alternating multilinear forms, so that the following diagram commutes?

$$\begin{tikzcd} \bigwedge^m(E^*) \times \bigwedge^n(E^*) \arrow[r, "\wedge"] \arrow[d, "\mu_m \times \mu_n"] & \bigwedge^{m+n}(E^*) \arrow[d, "\mu_{m+n}"] \\ \mathrm{Alt}^m(E;K) \times \mathrm{Alt}^n(E;K) \arrow[r, "\wedge"] & \mathrm{Alt}^{m+n}(E;K) \end{tikzcd}$$

As in the symmetric case, the answer is *yes*! The solution is to define this multiplication such that, for $f \in \mathrm{Alt}^m(E;K)$ and $g \in \mathrm{Alt}^n(E;K)$,

$$(f \wedge g)(u_1, \ldots, u_{m+n}) = \sum_{\sigma\in \mathrm{shuffle}(m,n)} \mathrm{sgn}(\sigma)\, f(u_{\sigma(1)}, \ldots, u_{\sigma(m)}) g(u_{\sigma(m+1)}, \ldots, u_{\sigma(m+n)}), \quad (**)$$

where $\mathrm{shuffle}(m,n)$ consists of all (m,n)-"shuffles;" that is, permutations σ of $\{1, \ldots m+n\}$ such that $\sigma(1) < \cdots < \sigma(m)$ and $\sigma(m+1) < \cdots < \sigma(m+n)$. For example, when $m = n = 1$, we have

$$(f \wedge g)(u, v) = f(u)g(v) - g(u)f(v).$$