Chapter 16

Unit Quaternions and Rotations in $\mathbf{SO}(3)$

This chapter is devoted to the representation of rotations in $\mathbf{SO}(3)$ in terms of unit quaternions. Since we already defined the unitary groups $\mathbf{SU}(n)$, the quickest way to introduce the *unit quaternions* is to define them as the elements of the group $\mathbf{SU}(2)$.

The skew field $\mathbb{H}$ of quaternions and the group $\mathbf{SU}(2)$ of unit quaternions are discussed in Section 16.1. In Section 16.2, we define a homomorphism $r\colon \mathbf{SU}(2) \to \mathbf{SO}(3)$ and prove that its kernel is $\{-I, I\}$. We compute the rotation matrix R_q associated with the rotation r_q induced by a unit quaternion q in Section 16.3. In Section 16.4, we prove that the homomorphism $r\colon \mathbf{SU}(2) \to \mathbf{SO}(3)$ is surjective by providing an algorithm to construct a quaternion from a rotation matrix. In Section 16.5 we define the exponential map $\exp\colon \mathfrak{su}(2) \to \mathbf{SU}(2)$ where $\mathfrak{su}(2)$ is the real vector space of skew-Hermitian 2×2 matrices with zero trace. We prove that exponential map $\exp\colon \mathfrak{su}(2) \to \mathbf{SU}(2)$ is surjective and give an algorithm for finding a logarithm. We discuss quaternion interpolation and prove the famous *slerp interpolation formula* due to Ken Shoemake in Section 16.6. This formula is used in robotics and computer graphics to deal with interpolation problems. In Section 16.7, we prove that there is no "nice" section $s\colon \mathbf{SO}(3) \to \mathbf{SU}(2)$ of the homomorphism $r\colon \mathbf{SU}(2) \to \mathbf{SO}(3)$, in the sense that any section of r is neither a homomorphism nor continuous.

16.1 The Group $\mathbf{SU}(2)$ of Unit Quaternions and the Skew Field $\mathbb{H}$ of Quaternions

Definition 16.1. The *unit quaternions* are the elements of the group $\mathbf{SU}(2)$, namely the group of 2×2 complex matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \quad \alpha, \beta \in \mathbb{C},\ \alpha\overline{\alpha} + \beta\overline{\beta} = 1.$$

The *quaternions* are the elements of the real vector space $\mathbb{H} = \mathbb{R}\,\mathbf{SU}(2)$.