Proof. (1) Pick any $a \in M$ and any $b \in N$, which is possible, since M and N are nonempty. Since $\overrightarrow{M} = \{\overrightarrow{ax} \mid x \in M\}$ and $\overrightarrow{N} = \{\overrightarrow{by} \mid y \in N\}$, if $M \cap N \neq \emptyset$, for any $c \in M \cap N$ we have $\overrightarrow{ab} = \overrightarrow{ac} - \overrightarrow{bc}$, with $\overrightarrow{ac} \in \overrightarrow{M}$ and $\overrightarrow{bc} \in \overrightarrow{N}$, and thus, $\overrightarrow{ab} \in \overrightarrow{M} + \overrightarrow{N}$. Conversely, assume that $\overrightarrow{ab} \in \overrightarrow{M} + \overrightarrow{N}$ for some $a \in M$ and some $b \in N$. Then $\overrightarrow{ab} = \overrightarrow{ax} + \overrightarrow{by}$, for some $x \in M$ and some $y \in N$. But we also have

$$\overrightarrow{ab} = \overrightarrow{ax} + \overrightarrow{xy} + \overrightarrow{yb},$$

and thus we get $0 = \overrightarrow{xy} + \overrightarrow{yb} - \overrightarrow{by}$, that is, $\overrightarrow{xy} = 2\overrightarrow{by}$. Thus, b is the middle of the segment $[x, y]$, and since $\overrightarrow{yx} = 2\overrightarrow{yb}$, $x = 2b - y$ is the barycenter of the weighted points $(b, 2)$ and $(y, -1)$. Thus x also belongs to N, since N being an affine subspace, it is closed under barycenters. Thus, $x \in M \cap N$, and $M \cap N \neq \emptyset$.

(2) Note that in general, if $M \cap N \neq \emptyset$, then

$$\overrightarrow{M \cap N} = \overrightarrow{M} \cap \overrightarrow{N},$$

because

$$\overrightarrow{M \cap N} = \{\overrightarrow{ab} \mid a, b \in M \cap N\} = \{\overrightarrow{ab} \mid a, b \in M\} \cap \{\overrightarrow{ab} \mid a, b \in N\} = \overrightarrow{M} \cap \overrightarrow{N}.$$

Since $M \cap N = c + \overrightarrow{M \cap N}$ for any $c \in M \cap N$, we have

$$M \cap N = c + \overrightarrow{M} \cap \overrightarrow{N} \quad \text{for any } c \in M \cap N.$$

From this it follows that if $M \cap N \neq \emptyset$, then $M \cap N$ consists of a single point iff $\overrightarrow{M} \cap \overrightarrow{N} = \{0\}$. This fact together with what we proved in (1) proves (2).

(3) This is left as an easy exercise. $\square$

Remarks:

(1) The proof of Proposition 24.16 shows that if $M \cap N \neq \emptyset$, then $\overrightarrow{ab} \in \overrightarrow{M} + \overrightarrow{N}$ for all $a \in M$ and all $b \in N$.

(2) Proposition 24.16 implies that for any two nonempty affine subspaces M and N, if $\overrightarrow{E} = \overrightarrow{M} \oplus \overrightarrow{N}$, then $M \cap N$ consists of a single point. Indeed, if $\overrightarrow{E} = \overrightarrow{M} \oplus \overrightarrow{N}$, then $\overrightarrow{ab} \in \overrightarrow{E}$ for all $a \in M$ and all $b \in N$, and since $\overrightarrow{M} \cap \overrightarrow{N} = \{0\}$, the result follows from part (2) of the proposition.

We can now state the following proposition.

Proposition 24.17. *Given an affine space E and any two nonempty affine subspaces M and N, if S is the least affine subspace containing M and N, then the following properties hold:*