implies that either there is some i_0 such that $\lambda_{i_0} > 0$ or there is some j_0 such that $\mu_{j_0} > 0$, so we have $\epsilon_{i_0} > 0$ or $\xi_{j_0} > 0$, which means that at least one point is misclassified. Thus Problem (SVM$_{s5}$) should only be used when the sets $\{u_i\}$ and $\{v_j\}$ are *not* linearly separable.

We can use the fact that the duality gap is 0 to find η. We have

$$\begin{aligned}\frac{1}{2}w^\top w + \frac{b^2}{2} - \nu\eta + K_s(\epsilon^\top\epsilon + \xi^\top\xi)&\\ &= -\frac{1}{2}\begin{pmatrix}\lambda^\top & \mu^\top\end{pmatrix}\left(X^\top X + \begin{pmatrix}\mathbf{1}_p\mathbf{1}_p^\top & -\mathbf{1}_p\mathbf{1}_q^\top \\ -\mathbf{1}_q\mathbf{1}_p^\top & \mathbf{1}_q\mathbf{1}_q^\top\end{pmatrix} + \frac{1}{2K_s}I_{p+q}\right)\begin{pmatrix}\lambda \\ \mu\end{pmatrix},\end{aligned}$$

so we get

$$\begin{aligned}\nu\eta &= K_s(\epsilon^\top\epsilon + \xi^\top\xi) + \begin{pmatrix}\lambda^\top & \mu^\top\end{pmatrix}\left(X^\top X + \begin{pmatrix}\mathbf{1}_p\mathbf{1}_p^\top & -\mathbf{1}_p\mathbf{1}_q^\top \\ -\mathbf{1}_q\mathbf{1}_p^\top & \mathbf{1}_q\mathbf{1}_q^\top\end{pmatrix} + \frac{1}{4K_s}I_{p+q}\right)\begin{pmatrix}\lambda \\ \mu\end{pmatrix}\\ &= \begin{pmatrix}\lambda^\top & \mu^\top\end{pmatrix}\left(X^\top X + \begin{pmatrix}\mathbf{1}_p\mathbf{1}_p^\top & -\mathbf{1}_p\mathbf{1}_q^\top \\ -\mathbf{1}_q\mathbf{1}_p^\top & \mathbf{1}_q\mathbf{1}_q^\top\end{pmatrix} + \frac{1}{2K_s}I_{p+q}\right)\begin{pmatrix}\lambda \\ \mu\end{pmatrix}.\end{aligned}$$

The above confirms that at optimality we have $\eta \geq 0$.

Remark: If we do not assume that $K_s = 1/(p+q)$, then the above formula must be replaced by

$$(p+q)K_s\nu\eta = \begin{pmatrix}\lambda^\top & \mu^\top\end{pmatrix}\left(X^\top X + \begin{pmatrix}\mathbf{1}_p\mathbf{1}_p^\top & -\mathbf{1}_p\mathbf{1}_q^\top \\ -\mathbf{1}_q\mathbf{1}_p^\top & \mathbf{1}_q\mathbf{1}_q^\top\end{pmatrix} + \frac{1}{2K_s}I_{p+q}\right)\begin{pmatrix}\lambda \\ \mu\end{pmatrix}.$$

There is a version of Theorem 54.8 stating that for a fixed K_s, the solution to Problem (SVM$_{s5}$) is unique and independent of the value of ν.

Theorem 54.9. *For K_s and ν fixed, if Problem* (SVM$_{s5}$) *succeeds then it has a unique solution. If Problem* (SVM$_{s5}$) *succeeds and returns $(\lambda, \mu, \eta, w, b)$ for the value ν and $(\lambda^\kappa, \mu^\kappa, \eta^\kappa, w^\kappa, b^\kappa)$ for the value $\kappa\nu$ with $\kappa > 0$, then*

$$\lambda^\kappa = \kappa\lambda,\;\; \mu^\kappa = \kappa\mu,\;\; \eta^\kappa = \kappa\eta,\;\; w^\kappa = \kappa w,\;\; b^\kappa = \kappa b.$$

As a consequence, $\delta = \eta/\left\|w\right\| = \eta^\kappa/\left\|w^\kappa\right\| = \delta^\kappa$, and the hyperplanes $H_{w,b}, H_{w,b+\eta}$ and $H_{w,b-\eta}$ are independent of ν.

Proof. The proof is an easy adaptation of the proof of Theorem 54.8 so we only give a sketch. The two crucial points are that the matrix

$$P = X^\top X + \begin{pmatrix}\mathbf{1}_p\mathbf{1}_p^\top & -\mathbf{1}_p\mathbf{1}_q^\top \\ -\mathbf{1}_q\mathbf{1}_p^\top & \mathbf{1}_q\mathbf{1}_q^\top\end{pmatrix} + \frac{1}{2K_s}I_{p+q}$$

is symmetric positive definite and that we have the single equational constraint

$$\mathbf{1}_p^\top\lambda + \mathbf{1}_q^\top\mu = (p+q)K_s\nu$$