Definition 3.2. Given a set A, an *I-indexed family* of elements of A, for short a *family*, is a function $a\colon I \to A$ where I is any set viewed as an index set. Since the function a is determined by its graph

$$\{(i, a(i)) \mid i \in I\},$$

the family a can be viewed as the set of pairs $a = \{(i, a(i)) \mid i \in I\}$. For notational simplicity, we write a_i instead of $a(i)$, and denote the family $a = \{(i, a(i)) \mid i \in I\}$ by $(a_i)_{i\in I}$.

For example, if $I = \{r, g, b, y\}$ and $A = \mathbb{N}$, the set of pairs

$$a = \{(r, 2), (g, 3), (b, 2), (y, 11)\}$$

is an indexed family. The element 2 appears twice in the family with the two distinct tags r and b.

When the indexed set I is totally ordered, a family $(a_i)_{i\in I}$ is often called an *I-sequence*. Interestingly, sets can be viewed as special cases of families. Indeed, a set A can be viewed as the A-indexed family $\{(a, a) \mid a \in I\}$ corresponding to the identity function.

Remark: An indexed family should not be confused with a multiset. Given any set A, a *multiset* is a similar to a set, except that elements of A may occur more than once. For example, if $A = \{a, b, c, d\}$, then $\{a, a, a, b, c, c, d, d\}$ is a multiset. Each element appears with a certain multiplicity, but the order of the elements does not matter. For example, a has multiplicity 3. Formally, a multiset is a function $s\colon A \to \mathbb{N}$, or equivalently a set of pairs $\{(a, i) \mid a \in A\}$. Thus, a multiset is an A-indexed family of elements from $\mathbb{N}$, but not a $\mathbb{N}$-indexed family, since distinct elements may have the same multiplicity (such as c an d in the example above). *An indexed family is a generalization of a sequence, but a multiset is a generalization of a set.*

We also need to take care of an annoying technicality, which is to define sums of the form $\sum_{i\in I} a_i$, where I is any *finite* index set and $(a_i)_{i\in I}$ is a family of elements in some set A equiped with a binary operation $+\colon A \times A \to A$ which is associative (Axiom (G1)) and commutative. This will come up when we define linear combinations.

The issue is that the binary operation $+$ only tells us how to compute $a_1 + a_2$ for two elements of A, but it does not tell us what is the sum of three of more elements. For example, how should $a_1 + a_2 + a_3$ be defined?

What we have to do is to define $a_1+a_2+a_3$ by using a sequence of steps each involving two elements, and there are two possible ways to do this: $a_1 + (a_2 + a_3)$ and $(a_1 + a_2) + a_3$. If our operation $+$ is not associative, these are different values. If it associative, then $a_1+(a_2+a_3) = (a_1 + a_2) + a_3$, but then there are still six possible permutations of the indices $1, 2, 3$, and if $+$ is not commutative, these values are generally different. If our operation is commutative, then all six permutations have the same value. *Thus, if $+$ is associative and commutative, it seems intuitively clear that a sum of the form $\sum_{i\in I} a_i$ does not depend on the order of the operations used to compute it.*