

The Lagrangian is given by

$$\begin{aligned}
L(w, \epsilon, \xi, b, \eta, \lambda, \mu, \gamma) &= \frac{1}{2}w^\top w - \nu\eta + K_s(\epsilon^\top \epsilon + \xi^\top \xi) + w^\top X \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \\
&\quad - \epsilon^\top \lambda - \xi^\top \mu + b(\mathbf{1}_p^\top \lambda - \mathbf{1}_q^\top \mu) + \eta(\mathbf{1}_p^\top \lambda + \mathbf{1}_q^\top \mu) - \gamma\eta \\
&= \frac{1}{2}w^\top w + w^\top X \begin{pmatrix} \lambda \\ \mu \end{pmatrix} + \eta(\mathbf{1}_p^\top \lambda + \mathbf{1}_q^\top \mu - \nu - \gamma) \\
&\quad + K_s(\epsilon^\top \epsilon + \xi^\top \xi) - \epsilon^\top \lambda - \xi^\top \mu + b(\mathbf{1}_p^\top \lambda - \mathbf{1}_q^\top \mu).
\end{aligned}$$

To find the dual function  $G(\lambda, \mu, \gamma)$  we minimize  $L(w, \epsilon, \xi, b, \eta, \lambda, \mu, \gamma)$  with respect to  $w, \epsilon, \xi, b$ , and  $\eta$ . Since the Lagrangian is convex and  $(w, \epsilon, \xi, b, \eta) \in \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R} \times \mathbb{R}$ , a convex open set, by Theorem 40.13, the Lagrangian has a minimum in  $(w, \epsilon, \xi, b, \eta)$  iff  $\nabla L_{w, \epsilon, \xi, b, \eta} = 0$ , so we compute  $\nabla L_{w, \epsilon, \xi, b, \eta}$ . The gradient  $\nabla L_{w, \epsilon, \xi, b, \eta}$  is given by

$$\nabla L_{w, \epsilon, \xi, b, \eta} = \begin{pmatrix} w + X \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \\ 2K_s\epsilon - \lambda \\ 2K_s\xi - \mu \\ \mathbf{1}_p^\top \lambda - \mathbf{1}_q^\top \mu \\ \mathbf{1}_p^\top \lambda + \mathbf{1}_q^\top \mu - \nu - \gamma \end{pmatrix}.$$

By setting  $\nabla L_{w, \epsilon, \xi, b, \eta} = 0$  we get the equations

$$\begin{aligned}
w &= -X \begin{pmatrix} \lambda \\ \mu \end{pmatrix}, & (*_w) \\
2K_s\epsilon &= \lambda \\
2K_s\xi &= \mu \\
\mathbf{1}_p^\top \lambda &= \mathbf{1}_q^\top \mu \\
\mathbf{1}_p^\top \lambda + \mathbf{1}_q^\top \mu &= \nu + \gamma.
\end{aligned}$$

The last two equations are identical to the last two equations obtained in Problem (SVM<sub>s2'</sub>). We can use the other equations to obtain the following expression for the dual function  $G(\lambda, \mu, \gamma)$ ,

$$\begin{aligned}
G(\lambda, \mu, \gamma) &= -\frac{1}{4K_s}(\lambda^\top \lambda + \mu^\top \mu) - \frac{1}{2} \begin{pmatrix} \lambda^\top & \mu^\top \end{pmatrix} X^\top X \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \\
&= -\frac{1}{2} \begin{pmatrix} \lambda^\top & \mu^\top \end{pmatrix} \left( X^\top X + \frac{1}{2K_s} I_{p+q} \right) \begin{pmatrix} \lambda \\ \mu \end{pmatrix}.
\end{aligned}$$

Consequently the dual program is equivalent to the minimization program