Proof. Since N is free with basis $(v_1, \ldots, v_n)$, we have an isomorphism

$$N \approx Av_1 \oplus \cdots \oplus Av_n.$$

By Proposition 35.12, we obtain an isomorphism

$$M \otimes N \approx M \otimes (Av_1 \oplus \cdots \oplus Av_n) \approx (M \otimes Av_1) \oplus \cdots \oplus (M \otimes Av_n).$$

Because $(v_1, \ldots, v_n)$ is a basis of N, each v_j is torsion-free so the map $a \mapsto av_j$ is an isomorphism of A onto Av_j, and because $M \otimes A \approx M$, we have the isomorphism

$$M \otimes N \approx (M \otimes A) \oplus \cdots \oplus (M \otimes A) \approx M \oplus \cdots \oplus M = M^n,$$

as claimed. $\square$

Proposition 35.13 also holds for an infinite basis $(v_j)_{j\in J}$ of N. Obviously, a version of Proposition 35.13 also holds if M is free and N is arbitrary.

The next proposition will be also be needed.

Proposition 35.14. *Given any A-module M and any ideal $\mathfrak{a}$ in A, there is an isomorphism*

$$(A/\mathfrak{a}) \otimes_A M \approx M/\mathfrak{a}M$$

given by the map $(\overline{a} \otimes u) \mapsto au \pmod{\mathfrak{a}M}$, for all $\overline{a} \in A/\mathfrak{a}$ and all $u \in M$.

Sketch of proof. Consider the map $\varphi\colon (A/\mathfrak{a}) \times M \to M/\mathfrak{a}M$ given by

$$\varphi(\overline{a}, u) = au \pmod{\mathfrak{a}M}$$

for all $\overline{a} \in A/\mathfrak{a}$ and all $u \in M$. It is immediately checked that φ is well-defined because $au \pmod{\mathfrak{a}M}$ does not depend on the representative $a \in A$ chosen in the equivalence class $\overline{a}$, and φ is bilinear. Therefore, φ induces a linear map $\varphi\colon (A/\mathfrak{a}) \otimes M \to M/\mathfrak{a}M$, such that $\varphi(\overline{a} \otimes u) = au \pmod{\mathfrak{a}M}$. We also define the map $\psi\colon M \to (A/\mathfrak{a}) \otimes M$ by

$$\psi(u) = \overline{1} \otimes u.$$

Since $\mathfrak{a}M$ is generated by vectors of the form au with $a \in \mathfrak{a}$ and $u \in M$, and since

$$\psi(au) = \overline{1} \otimes au = \overline{a} \otimes u = 0 \otimes u = 0,$$

we see that $\mathfrak{a}M \subseteq \operatorname{Ker}(\psi)$, so ψ induces a linear map $\psi\colon M/\mathfrak{a}M \to (A/\mathfrak{a}) \otimes M$. We have

$$\begin{aligned}\psi(\varphi(\overline{a} \otimes u)) &= \psi(au) \\ &= \overline{1} \otimes au \\ &= \overline{a} \otimes u\end{aligned}$$