Then

$$
\begin{aligned}
\mathrm{Ann}(\bigwedge^1 M) &= \mathrm{Ann}\, e_1 = (0) \\
\mathrm{Ann}(\bigwedge^2 M) &= \mathrm{Ann}\, e_1 \wedge e_2 = (0) \\
\mathrm{Ann}(\bigwedge^3 M) &= \mathrm{Ann}\, e_1 \wedge e_2 \wedge e_3 = (6) \\
\mathrm{Ann}(\bigwedge^4 M) &= \mathrm{Ann}\, e_1 \wedge e_2 \wedge e_3 \wedge e_4 = (2),
\end{aligned}
$$

and Proposition 35.29 provides another verification of

$$M \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/(6) \oplus \mathbb{Z}/(2).$$

Propostion 35.29 immediately implies the following crucial fact.

Proposition 35.30. *Let A be a commutative ring and let $\mathfrak{a}_1, \ldots, \mathfrak{a}_m$ be m ideals of A and $\mathfrak{a}'_1, \ldots, \mathfrak{a}'_n$ be n ideals of A such that $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots \subseteq \mathfrak{a}_m \neq A$ and $\mathfrak{a}'_1 \subseteq \mathfrak{a}'_2 \subseteq \cdots \subseteq \mathfrak{a}'_n \neq A$ If we have an isomorphism*

$$A/\mathfrak{a}_1 \oplus \cdots \oplus A/\mathfrak{a}_m \approx A/\mathfrak{a}'_1 \oplus \cdots \oplus A/\mathfrak{a}'_n,$$

then $m = n$ and $\mathfrak{a}_i = \mathfrak{a}'_i$ for $i = 1, \ldots, n$.

Proposition 35.30 yields the uniqueness of the decomposition in Theorem 35.25.

Theorem 35.31. *(Invariant Factors Decomposition) Let M be a finitely generated nontrivial A-module, where A a PID. Then, M is isomorphic to a direct sum of cyclic modules*

$$M \approx A/\mathfrak{a}_1 \oplus \cdots \oplus A/\mathfrak{a}_m,$$

where the $\mathfrak{a}_i$ are proper ideals of A (possibly zero) such that

$$\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots \subseteq \mathfrak{a}_m \neq A.$$

More precisely, if $\mathfrak{a}_1 = \cdots = \mathfrak{a}_r = (0)$ and $(0) \neq \mathfrak{a}_{r+1} \subseteq \cdots \subseteq \mathfrak{a}_m \neq A$, then

$$M \approx A^r \oplus (A/\mathfrak{a}_{r+1} \oplus \cdots \oplus A/\mathfrak{a}_m),$$

where $A/\mathfrak{a}_{r+1} \oplus \cdots \oplus A/\mathfrak{a}_m$ is the torsion submodule of M. The module M is free iff $r = m$, and a torsion-module iff $r = 0$. In the latter case, the annihilator of M is $\mathfrak{a}_1$. Furthermore, the integer r and ideals $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots \subseteq \mathfrak{a}_m \neq A$ are uniquely determined by M.

Proof. By Theorem 35.7, since $M_{\mathrm{tor}} = A/\mathfrak{a}_{r+1} \oplus \cdots \oplus A/\mathfrak{a}_m$, we know that the dimension r of the free summand only depends on M. The uniqueness of the sequence of ideals follows from Proposition 35.30. ∎