Otherwise, $a \neq \pm 1$ and $(b, c, d) \neq (0, 0, 0)$, and we are seeking some $A = \theta B \in \mathfrak{su}(2)$ with $\det(B) = 1$ and $0 < \theta < \pi$, such that, by Proposition 16.7,

$$q = e^{\theta B} = \cos\theta I + \sin\theta B.$$

Let

$$B = \begin{pmatrix} iu_1 & u_2 + iu_3 \\ -u_2 + iu_3 & -iu_1 \end{pmatrix},$$

with $u = (u_1, u_2, u_3)$ a unit vector. We must have

$$a = \cos\theta, \quad e^{\theta B} - (e^{\theta B})^* = q - q^*.$$

Since $0 < \theta < \pi$, we have $\sin\theta \neq 0$, and

$$2\sin\theta \begin{pmatrix} iu_1 & u_2 + iu_3 \\ -u_2 + iu_3 & -iu_1 \end{pmatrix} = \begin{pmatrix} \alpha - \overline{\alpha} & 2\beta \\ -2\overline{\beta} & \overline{\alpha} - \alpha \end{pmatrix}.$$

Thus, we get

$$u_1 = \frac{1}{\sin\theta} b, \quad u_2 + iu_3 = \frac{1}{\sin\theta}(c + id);$$

that is,

$$\begin{aligned} \cos\theta &= a \qquad (0 < \theta < \pi) \\ (u_1, u_2, u_3) &= \frac{1}{\sin\theta}(b, c, d). \end{aligned}$$

Since $a^2 + b^2 + c^2 + d^2 = 1$ and $a = \cos\theta$, the vector $(b, c, d)/\sin\theta$ is a unit vector. Furthermore if the quaternion q is of the form $q = [\cos\theta, \sin\theta u]$ where $u = (u_1, u_2, u_3)$ is a unit vector (with $0 < \theta < \pi$), then

$$A = \theta \begin{pmatrix} iu_1 & u_2 + iu_3 \\ -u_2 + iu_3 & -iu_1 \end{pmatrix} \tag{$*_{\log}$}$$

is a logarithm of q. $\square$

Observe that not only is the exponential map exp: $\mathfrak{su}(2) \to \mathbf{SU}(2)$ surjective, but the above proof shows that it is injective on the open ball

$$\{\theta B \in \mathfrak{su}(2) \mid \det(B) = 1, 0 \leq \theta < \pi\}.$$

Also, unlike the situation where in computing the logarithm of a rotation matrix $R \in \mathbf{SO}(3)$ we needed to treat the case where $\mathrm{tr}(R) = -1$ (the angle of the rotation is π) in a special way, computing the logarithm of a quaternion (other than $\pm I$) does not require any case analysis; no special case is needed when the angle of rotation is π.