The reason for the terminology *coordinate form* is as follows: If E has finite dimension and if $(u_1, \ldots, u_n)$ is a basis of E, for any vector

$$v = \lambda_1 u_1 + \cdots + \lambda_n u_n,$$

we have

$$\begin{aligned} u_i^*(v) &= u_i^*(\lambda_1 u_1 + \cdots + \lambda_n u_n) \\ &= \lambda_1 u_i^*(u_1) + \cdots + \lambda_i u_i^*(u_i) + \cdots + \lambda_n u_i^*(u_n) \\ &= \lambda_i, \end{aligned}$$

since $u_i^*(u_j) = \delta_{i\,j}$. Therefore, u_i^* is the linear function that returns the ith coordinate of a vector expressed over the basis $(u_1, \ldots, u_n)$.

The following theorem shows that in finite-dimension, every basis $(u_1, \ldots, u_n)$ of a vector space E yields a basis $(u_1^*, \ldots, u_n^*)$ of the dual space E^*, called a *dual basis*.

Theorem 3.23. *(Existence of dual bases) Let E be a vector space of dimension n. The following properties hold: For every basis $(u_1, \ldots, u_n)$ of E, the family of coordinate forms $(u_1^*, \ldots, u_n^*)$ is a basis of E^* (called the dual basis of $(u_1, \ldots, u_n)$).*

Proof. (a) If $v^* \in E^*$ is any linear form, consider the linear form

$$f^* = v^*(u_1)u_1^* + \cdots + v^*(u_n)u_n^*.$$

Observe that because $u_i^*(u_j) = \delta_{i\,j}$,

$$\begin{aligned} f^*(u_i) &= (v^*(u_1)u_1^* + \cdots + v^*(u_n)u_n^*)(u_i) \\ &= v^*(u_1)u_1^*(u_i) + \cdots + v^*(u_i)u_i^*(u_i) + \cdots + v^*(u_n)u_n^*(u_i) \\ &= v^*(u_i), \end{aligned}$$

and so f^* and v^* agree on the basis $(u_1, \ldots, u_n)$, which implies that

$$v^* = f^* = v^*(u_1)u_1^* + \cdots + v^*(u_n)u_n^*.$$

Therefore, $(u_1^*, \ldots, u_n^*)$ spans E^*. We claim that the covectors $u_1^*, \ldots, u_n^*$ are linearly independent. If not, we have a nontrivial linear dependence

$$\lambda_1 u_1^* + \cdots + \lambda_n u_n^* = 0,$$

and if we apply the above linear form to each u_i, using a familar computation, we get

$$0 = \lambda_i u_i^*(u_i) = \lambda_i,$$

proving that $u_1^*, \ldots, u_n^*$ are indeed linearly independent. Therefore, $(u_1^*, \ldots, u_n^*)$ is a basis of E^*. $\square$