matrix to rref, we can apply elementary row operations to the matrix $[u_k\ \Gamma]$, which consists of rows $1, \ldots, m$ of the tableau.

Once the new matrix Γ^+ is obtained, the new reduced costs are given by the following proposition.

Proposition 46.2. *Given any Linear Program $(P2)$ in standard form*

$$\begin{aligned} &\text{maximize} && cx \\ &\text{subject to} && Ax = b \ \textit{and}\ x \geq 0, \end{aligned}$$

where A is an $m \times n$ matrix of rank m, if (u, K) is a basic (not necessarily feasible) solution of $(P2)$ and if $K^+ = (K - \{k^-\}) \cup \{j^+\}$, with $K = (k_1, \ldots, k_m)$ and $k^- = k_\ell$, then for $i = 1, \ldots, n$ we have

$$c_i - c_{K^+}\gamma^i_{K^+} = c_i - c_K\gamma^i_K - \frac{\gamma^i_{k^-}}{\gamma^{j^+}_{k^-}}(c_{j^+} - c_K\gamma^{j^+}_K).$$

Using the reduced cost notation, the above equation is

$$(\overline{c}_{K^+})_i = (\overline{c}_K)_i - \frac{\gamma^i_{k^-}}{\gamma^{j^+}_{k^-}}(\overline{c}_K)_{j^+}.$$

Proof. Without any loss of generality and to simplify notation assume that $K = (1, \ldots, m)$ and write j for j^+ and ℓ for k_m. Since $\gamma^i_K = A_K^{-1}A^i$, $\gamma^i_{K^+} = A_{K^+}^{-1}A^i$, and $A_{K^+} = A_K E(\gamma^j_K)$, we have

$$c_i - c_{K^+}\gamma^i_{K^+} = c_i - c_{K^+}A^{-1}_{K^+}A^i = c_i - c_{K^+}E(\gamma^j_K)^{-1}A^{-1}_K A^i = c_i - c_{K^+}E(\gamma^j_K)^{-1}\gamma^i_K,$$

where

$$E(\gamma^j_K)^{-1} = \begin{pmatrix} 1 & & & -(\gamma^j_\ell)^{-1}\gamma^j_1 & & & \\ & \ddots & & \vdots & & & \\ & & 1 & -(\gamma^j_\ell)^{-1}\gamma^j_{\ell-1} & & & \\ & & & (\gamma^j_\ell)^{-1} & & & \\ & & & -(\gamma^j_\ell)^{-1}\gamma^j_{\ell+1} & 1 & & \\ & & & \vdots & & \ddots & \\ & & & -(\gamma^j_\ell)^{-1}\gamma^j_m & & & 1 \end{pmatrix}$$

where the ℓth column contains the γs. Since $c_{K^+} = (c_1, \ldots, c_{\ell-1}, c_j, c_{\ell+1}, \ldots, c_m)$, we have

$$c_{K^+}E(\gamma^j_K)^{-1} = \left(c_1, \ldots, c_{\ell-1}, \frac{c_j}{\gamma^j_\ell} - \sum_{k=1, k\neq \ell}^{m} c_k\frac{\gamma^j_k}{\gamma^j_\ell}, c_{\ell+1}, \ldots, c_m\right),$$