Corollary 9.16. *The Frobenius norm of a matrix is given by the ℓ^2-norm of its vector of singular values;* $\|A\|_F = \|(\sigma_1, \ldots, \sigma_n)\|_2$.

In the case of a normal matrix if $\lambda_1, \ldots, \lambda_n$ are the (complex) eigenvalues of A, then

$$\sigma_i = |\lambda_i|, \quad 1 \leq i \leq n.$$

Proposition 9.17. *For every invertible matrix $A \in \mathrm{M}_n(\mathbb{C})$, the following properties hold:*

(1)

$$\begin{aligned} \mathrm{cond}(A) &\geq 1, \\ \mathrm{cond}(A) &= \mathrm{cond}(A^{-1}) \\ \mathrm{cond}(\alpha A) &= \mathrm{cond}(A) \quad \textit{for all } \alpha \in \mathbb{C} - \{0\}. \end{aligned}$$

(2) If $\mathrm{cond}_2(A)$ *denotes the condition number of A with respect to the spectral norm, then*

$$\mathrm{cond}_2(A) = \frac{\sigma_1}{\sigma_n},$$

where $\sigma_1 \geq \cdots \geq \sigma_n$ are the singular values of A.

(3) If the matrix A is normal, then

$$\mathrm{cond}_2(A) = \frac{|\lambda_1|}{|\lambda_n|},$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A sorted so that $|\lambda_1| \geq \cdots \geq |\lambda_n|$.

(4) If A is a unitary or an orthogonal matrix, then

$$\mathrm{cond}_2(A) = 1.$$

(5) The condition number $\mathrm{cond}_2(A)$ *is invariant under unitary transformations, which means that*

$$\mathrm{cond}_2(A) = \mathrm{cond}_2(UA) = \mathrm{cond}_2(AV),$$

for all unitary matrices U and V.

Proof. The properties in (1) are immediate consequences of the properties of subordinate matrix norms. In particular, $AA^{-1} = I$ implies

$$1 = \|I\| \leq \|A\| \left\|A^{-1}\right\| = \mathrm{cond}(A).$$

(2) We showed earlier that $\|A\|_2^2 = \rho(A^*A)$, which is the square of the modulus of the largest eigenvalue of A^*A. Since we just saw that the eigenvalues of A^*A are $\sigma_1^2 \geq \cdots \geq \sigma_n^2$, where $\sigma_1, \ldots, \sigma_n$ are the singular values of A, we have

$$\|A\|_2 = \sigma_1.$$