Proof. Since V is a neighborhood of a, there is some open subset, O, of V containing a. Then the complement, $K = E - O$, of O is closed and since E is compact, by Proposition 37.28, K is compact. Now, if we consider the family of all closed sets of the form, $K \cap F$, where F is any closed neighborhood of a, since $a \notin K$, this family has an empty intersection and thus, there is a finite number of closed neighborhood, $F_1, \ldots, F_n$, of a, such that $K \cap F_1 \cap \cdots \cap F_n = \emptyset$. Then, $U = F_1 \cap \cdots \cap F_n$ is closed and hence by Proposition 37.28, a compact neigborhood of a contained in $O \subseteq V$. See Figure 37.34. $\square$

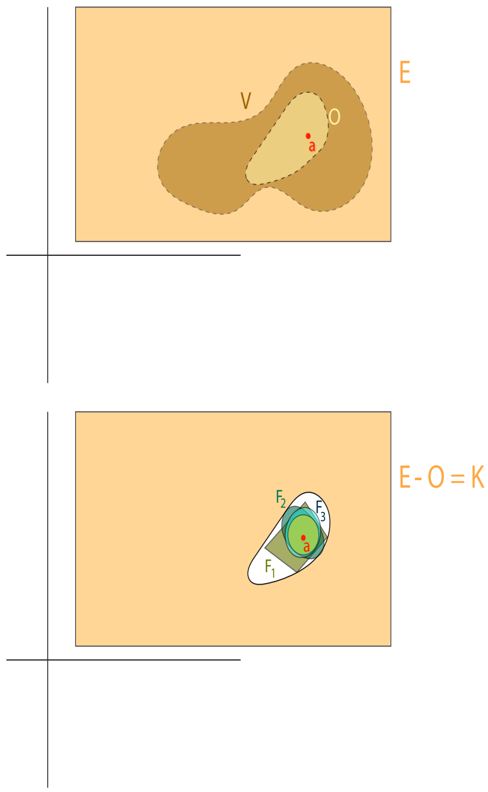

Figure 37.34: Let E be the peach square of $\mathbb{R}^2$. The compact neighborhood of a, U, is the intersection of the closed sets F_1, F_2, F_3, each of which are contained in the complement of K.

It can be shown that in a normed vector space of finite dimension, a subset is compact iff it is closed and bounded. For $\mathbb{R}^n$ the proof is simple.

In a normed vector space of infinite dimension, there are closed and bounded sets that are not compact!

More could be said about compactness in metric spaces but we will only need the notion of Lebesgue number, which will be discussed a little later. Another crucial property of compactness is that it is preserved under continuity.