2. The canonical isomorphism of Proposition 33.16 holds under more general conditions. Namely, that K is a commutative ring with identity and that the E_i are finitely-generated projective K-modules (see Definition 35.7). See Bourbaki, [25] (Chapter III, §11, Section 5, Proposition 7).

We prove another useful canonical isomorphism that allows us to treat linear maps as tensors.

Let E and F be two vector spaces and let $\alpha\colon E^* \times F \to \mathrm{Hom}(E, F)$ be the map defined such that

$$\alpha(u^*, f)(x) = u^*(x)f,$$

for all $u^* \in E^*$, $f \in F$, and $x \in E$. This map is clearly bilinear, and thus it induces a linear map $\alpha_\otimes\colon E^* \otimes F \to \mathrm{Hom}(E, F)$ making the following diagram commute

$$\begin{tikzcd} E^* \times F \arrow[r, "\iota_\otimes"] \arrow[dr, "\alpha"'] & E^* \otimes F \arrow[d, "\alpha_\otimes"] \\ & \mathrm{Hom}(E, F), \end{tikzcd}$$

such that

$$\alpha_\otimes(u^* \otimes f)(x) = u^*(x)f.$$

Proposition 33.17. *If E and F are vector spaces (not necessarily finite dimensional), then the following properties hold:*

(1) The linear map $\alpha_\otimes\colon E^* \otimes F \to \mathrm{Hom}(E, F)$ is injective.

(2) If E is finite-dimensional, then $\alpha_\otimes\colon E^* \otimes F \to \mathrm{Hom}(E, F)$ is a canonical isomorphism.

(3) If F is finite-dimensional, then $\alpha_\otimes\colon E^* \otimes F \to \mathrm{Hom}(E, F)$ is a canonical isomorphism.

Proof. (1) Let $(e_i^*)_{i\in I}$ be a basis of E^* and let $(f_j)_{j\in J}$ be a basis of F. Then we know that $(e_i^* \otimes f_j)_{i\in I, j\in J}$ is a basis of $E^* \otimes F$. To prove that $\alpha_\otimes$ is injective, let us show that its kernel is reduced to (0). For any vector

$$\omega = \sum_{i\in I', j\in J'} \lambda_{ij}\, e_i^* \otimes f_j$$

in $E^* \otimes F$, with I' and J' some finite sets, assume that $\alpha_\otimes(\omega) = 0$. This means that for every $x \in E$, we have $\alpha_\otimes(\omega)(x) = 0$; that is,

$$\sum_{i\in I', j\in J'} \alpha_\otimes(\lambda_{ij}\, e_i^* \otimes f_j)(x) = \sum_{j\in J'} \left(\sum_{i\in I'} \lambda_{ij} e_i^*(x) \right) f_j = 0.$$