and then λ will be a solution of the problem

$$\begin{aligned} & \text{find } \lambda \in \mathbb{R}^m_+ \text{ such that} \\ & G(\lambda) = \sup_{\mu \in \mathbb{R}^m_+} G(\mu), \end{aligned}$$

which is equivalent to the *Maximization Problem (D):*

$$\begin{aligned} & \text{maximize} \quad G(\mu) \\ & \text{subject to} \quad \mu \in \mathbb{R}^m_+. \end{aligned}$$

Definition 50.9. Given the Minimization Problem (P)

$$\begin{aligned} & \text{minimize} \quad J(v) \\ & \text{subject to} \quad \varphi_i(v) \leq 0, \quad i = 1, \ldots, m, \end{aligned}$$

where $J\colon \Omega \to \mathbb{R}$ and the constraints $\varphi_i\colon \Omega \to \mathbb{R}$ are some functions defined on some open subset Ω of some finite-dimensional Euclidean vector space V (more generally, a real Hilbert space V), the function $G\colon \mathbb{R}^m_+ \to \mathbb{R}$ given by

$$G(\mu) = \inf_{v \in \Omega} L(v, \mu) \quad \mu \in \mathbb{R}^m_+,$$

is called the *Lagrange dual function* (or simply *dual function*). The *Problem (D)*

$$\begin{aligned} & \text{maximize} \quad G(\mu) \\ & \text{subject to} \quad \mu \in \mathbb{R}^m_+ \end{aligned}$$

is called the *Lagrange dual problem*. The Problem (P) is often called the *primal problem*, and (D) is the *dual problem*. The variable μ is called the *dual variable*. The variable $\mu \in \mathbb{R}^m_+$ is said to be *dual feasible* if $G(\mu)$ is defined (not $-\infty$). If $\lambda \in \mathbb{R}^m_+$ is a maximum of G, then we call it a *dual optimal* or an *optimal Lagrange multiplier*.

Since

$$L(v, \mu) = J(v) + \sum_{i=1}^m \mu_i \varphi_i(v),$$

the function $G(\mu) = \inf_{v \in \Omega} L(v, \mu)$ is the pointwise infimum of some affine functions of μ, so it is *concave*, even if the φ_i are not convex. One of the main advantages of the dual problem over the primal problem is that it is a *convex optimization problem*, since we wish to maximize a concave objective function G (thus minimize $-G$, a convex function), and the constraints $\mu \geq 0$ are convex. In a number of practical situations, the dual function G can indeed be computed.

To be perfectly rigorous, we should mention that the dual function G is actually a *partial function*, because it takes the value $-\infty$ when the map $v \mapsto L(v, \mu)$ is unbounded below.