Proof. The trick is to construct projections π_i using the polynomials $p_j^{r_j}$ so that the range of π_i is equal to W_i. Let

$$g_i = m/p_i^{r_i} = \prod_{j \neq i} p_j^{r_j}.$$

Note that

$$p_i^{r_i} g_i = m.$$

Since $p_1, \ldots, p_k$ are irreducible and distinct, they are relatively prime. Then using Proposition 30.14, it is easy to show that $g_1, \ldots, g_k$ are relatively prime. Otherwise, some irreducible polynomial p would divide all of $g_1, \ldots, g_k$, so by Proposition 30.14 it would be equal to one of the irreducible factors p_i. But that p_i is missing from g_i, a contradiction. Therefore, by Proposition 30.15, there exist some polynomials $h_1, \ldots, h_k$ such that

$$g_1 h_1 + \cdots + g_k h_k = 1.$$

Let $q_i = g_i h_i$ and let $\pi_i = q_i(f) = g_i(f) h_i(f)$. We have

$$q_1 + \cdots + q_k = 1,$$

and since m divides $q_i q_j$ for $i \neq j$, we get

$$\begin{aligned} \pi_1 + \cdots + \pi_k &= \mathrm{id} \\ \pi_i \pi_j &= 0, \quad i \neq j. \end{aligned}$$

(We implicitly used the fact that if p, q are two polynomials, the linear maps $p(f) \circ q(f)$ and $q(f) \circ p(f)$ are the same since $p(f)$ and $q(f)$ are polynomials in the powers of f, which commute.) Composing the first equation with π_i and using the second equation, we get

$$\pi_i^2 = \pi_i.$$

Therefore, the π_i are projections, and E is the direct sum of the images of the π_i. Indeed, every $u \in E$ can be expressed as

$$u = \pi_1(u) + \cdots + \pi_k(u).$$

Also, if

$$\pi_1(u) + \cdots + \pi_k(u) = 0,$$

then by applying π_i we get

$$0 = \pi_i^2(u) = \pi_i(u), \quad i = 1, \ldots k.$$

To finish proving (a), we need to show that

$$W_i = \mathrm{Ker}\,(p_i^{r_i}(f)) = \pi_i(E).$$