and $\varphi(u_i) = 1 \otimes u_i$. Next, if ρ is injective, by definition of the scalar multiplication in the A-module $\rho_*(\rho^*(M))$, we have $\varphi(a_1u_1 + \cdots + a_nu_n) = 0$ iff

$$\rho(a_1)\varphi(u_1) + \cdots + \rho(a_n)\varphi(u_n) = 0,$$

and since $(\varphi(u_1), \ldots, \varphi(u_n))$ is a basis of $\rho^*(M)$, we must have $\rho(a_i) = 0$ for $i = 1, \ldots, n$, which (by injectivity of ρ) implies that $a_i = 0$ for $i = 1, \ldots, n$. Therefore, φ is injective. $\square$

In particular, if A is a subring of B, then ρ is the inclusion map and Proposition 35.41 shows that a basis of M becomes a basis of $M_{(B)}$ and that M is embedded into $M_{(B)}$. It is also easy to see that if M and N are two free A-modules and $f\colon M \to N$ is a linear map represented by the matrix X with respect to some bases $(u_1, \ldots, u_n)$ of M and $(v_1, \ldots, v_m)$ of N, then the B-linear map $\overline{f}$ is also represented by the matrix X over the bases $(\varphi(u_1), \ldots, \varphi(u_n))$ and $(\varphi(v_1), \ldots, \varphi(v_m))$.

Proposition 35.41 yields another proof of the fact that any two bases of a free A-module have the same cardinality. Indeed, if $\mathfrak{m}$ is a maximal ideal in the ring A, then we have the quotient ring homomorphism $\pi\colon A \to A/\mathfrak{m}$, and we get the $A/\mathfrak{m}$-module $\pi^*(M)$. If M is free, any basis $(u_1, \ldots, u_n)$ of M becomes the basis $(\varphi(u_1), \ldots, \varphi(u_n))$ of $\pi^*(M)$; but $A/\mathfrak{m}$ is a field, so the dimension n is uniquely determined. This argument also applies to an infinite basis $(u_i)_{i\in I}$. Observe that by Proposition 35.14, we have an isomorphism

$$\pi^*(M) = (A/\mathfrak{m}) \otimes_A M \approx M/\mathfrak{m}M,$$

so $M/\mathfrak{m}M$ is a vector space over the field $A/\mathfrak{m}$, which is the argument used in Theorem 35.1.

Proposition 35.42. *Given a ring homomomorphism $\rho\colon A \to B$, for any two A-modules M and N, there is a unique isomorphism*

$$\rho^*(M) \otimes_B \rho^*(N) \approx \rho^*(M \otimes_A N),$$

such that $(1 \otimes u) \otimes (1 \otimes v) \mapsto 1 \otimes (u \otimes v)$, for all $u \in M$ and all $v \in N$.

The proof uses identities from Proposition 33.13. It is not hard but it requires a little gymnastic; a good exercise for the reader.