such that

$$\overrightarrow{xb} = \frac{\lambda_1}{\lambda_1 + \lambda_2}\overrightarrow{xa_1} + \frac{\lambda_2}{\lambda_1 + \lambda_2}\overrightarrow{xa_2}.$$

Since

$$\lambda_1\overrightarrow{xa_1} + \lambda_2\overrightarrow{xa_2} = (\lambda_1 + \lambda_2)\overrightarrow{xb},$$

we let

$$\langle a_1, \lambda_1\rangle \mathbin{\widehat{+}} \langle a_2, \lambda_2\rangle = \left\langle \frac{\lambda_1}{\lambda_1 + \lambda_2}a_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2}a_2, \lambda_1 + \lambda_2 \right\rangle,$$

the dilatation associated with the point b and the scalar $\lambda_1 + \lambda_2$.

Given a translation defined by u and a dilatation $\langle a, \lambda\rangle$, since $\lambda \neq 0$, we have

$$\lambda\overrightarrow{xa} + u = \lambda(\overrightarrow{xa} + \lambda^{-1}u),$$

and so, letting $b = a + \lambda^{-1}u$, since $\overrightarrow{ab} = \lambda^{-1}u$, we have

$$\lambda\overrightarrow{xa} + u = \lambda(\overrightarrow{xa} + \lambda^{-1}u) = \lambda(\overrightarrow{xa} + \overrightarrow{ab}) = \lambda\overrightarrow{xb},$$

and we let

$$\langle a, \lambda\rangle \mathbin{\widehat{+}} u = \langle a + \lambda^{-1}u, \lambda\rangle,$$

the dilatation of center $a + \lambda^{-1}u$ and ratio λ.

The sum of two translations u and v is of course defined as the translation $u + v$. It is also natural to define multiplication by a scalar as follows:

$$\mu \cdot \langle a, \lambda\rangle = \langle a, \lambda\mu\rangle,$$

and

$$\lambda \cdot u = \lambda u,$$

where λu is the product by a scalar in $\overrightarrow{E}$.

We can now use the definition of the above operations to state the following proposition, showing that the "hat construction" described above has allowed us to achieve our goal of embedding both E and $\overrightarrow{E}$ in the vector space $\widehat{E}$.

Proposition 25.1. *The set $\widehat{E}$ consisting of the disjoint union of the translations and the dilatations $H_{a,1-\lambda} = \langle a, \lambda\rangle$, $\lambda \in \mathbb{R}, \lambda \neq 0$, is a vector space under the following operations of addition and multiplication by a scalar: If $\lambda_1 + \lambda_2 = 0$, then*

$$\langle a_1, \lambda_1\rangle \mathbin{\widehat{+}} \langle a_2, \lambda_2\rangle = \lambda_1\overrightarrow{a_2a_1};$$

if $\lambda_1 + \lambda_2 \neq 0$, then

$$\begin{aligned}
\langle a_1, \lambda_1\rangle \mathbin{\widehat{+}} \langle a_2, \lambda_2\rangle &= \left\langle \frac{\lambda_1}{\lambda_1 + \lambda_2}a_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2}a_2, \lambda_1 + \lambda_2 \right\rangle, \\
\langle a, \lambda\rangle \mathbin{\widehat{+}} u &= u \mathbin{\widehat{+}} \langle a, \lambda\rangle = \langle a + \lambda^{-1}u, \lambda\rangle, \\
u \mathbin{\widehat{+}} v &= u + v;
\end{aligned}$$