There are other examples.

**Example 12.2.** For instance, let $E$ be a vector space of dimension 2, and let $(e_1, e_2)$ be a basis of $E$. If $a > 0$ and $b^2 - ac < 0$, the bilinear form defined such that

$$\varphi(x_1 e_1 + y_1 e_2,\, x_2 e_1 + y_2 e_2) = a x_1 x_2 + b(x_1 y_2 + x_2 y_1) + c y_1 y_2$$

yields a Euclidean structure on $E$. In this case,

$$\Phi(x e_1 + y e_2) = a x^2 + 2bxy + c y^2.$$

**Example 12.3.** Let $\mathcal{C}[a, b]$ denote the set of continuous functions $f\colon [a, b] \to \mathbb{R}$. It is easily checked that $\mathcal{C}[a, b]$ is a vector space of infinite dimension. Given any two functions $f, g \in \mathcal{C}[a, b]$, let

$$\langle f, g\rangle = \int_a^b f(t)g(t)dt.$$

We leave it as an easy exercise that $\langle -, -\rangle$ is indeed an inner product on $\mathcal{C}[a, b]$. In the case where $a = -\pi$ and $b = \pi$ (or $a = 0$ and $b = 2\pi$, this makes basically no difference), one should compute

$$\langle \sin px, \sin qx\rangle, \quad \langle \sin px, \cos qx\rangle, \quad \text{and} \quad \langle \cos px, \cos qx\rangle,$$

for all natural numbers $p, q \geq 1$. The outcome of these calculations is what makes Fourier analysis possible!

**Example 12.4.** Let $E = \mathrm{M}_n(\mathbb{R})$ be the vector space of real $n \times n$ matrices. If we view a matrix $A \in \mathrm{M}_n(\mathbb{R})$ as a "long" column vector obtained by concatenating together its columns, we can define the inner product of two matrices $A, B \in \mathrm{M}_n(\mathbb{R})$ as

$$\langle A, B\rangle = \sum_{i,j=1}^{n} a_{ij} b_{ij},$$

which can be conveniently written as

$$\langle A, B\rangle = \mathrm{tr}(A^\top B) = \mathrm{tr}(B^\top A).$$

Since this can be viewed as the Euclidean product on $\mathbb{R}^{n^2}$, it is an inner product on $\mathrm{M}_n(\mathbb{R})$. The corresponding norm

$$\|A\|_F = \sqrt{\mathrm{tr}(A^\top A)}$$

is the Frobenius norm (see Section 9.2).

Let us observe that $\varphi$ can be recovered from $\Phi$.