For example, if $A_1 = \{1, 2, 3\}$, we obtain the vector

$$\varphi(\{1, 2, 3\}) = (1, 1, 1, 1, 0, 1, 1, 0, 1, 0, 0, 1, 0, 0, 0, 0),$$

and if $A_2 = \{2, 3, 4\}$, we obtain the vector

$$\varphi(\{2, 3, 4\}) = (1, 0, 1, 1, 1, 0, 0, 0, 1, 1, 1, 0, 0, 0, 1, 0).$$

For any two subsets A_1 and A_2 of D, it is easy to check that

$$\langle \varphi(A_1), \varphi(A_2) \rangle = 2^{|A_1 \cap A_2|},$$

the number of common subsets of A_1 and A_2. For example, $A_1 \cap A_2 = \{2, 3\}$, and

$$\langle \varphi(A_1), \varphi(A_2) \rangle = 4.$$

Therefore, the function $\kappa\colon X \times X \to \mathbb{R}$ given by

$$\kappa(A_1, A_2) = 2^{|A_1 \cap A_2|}, \qquad A_1, A_2 \subseteq D$$

is a kernel function.

Kernels on collections of sets can be defined in terms of measures.

Example 53.7. Let $(D, \mathcal{A})$ be a measurable space, where D is a nonempty set and $\mathcal{A}$ is a σ-algebra on D (the measurable sets). Let X be a subset of $\mathcal{A}$. If μ is a positive measure on $(D, \mathcal{A})$ and if μ is finite, which means that $\mu(D)$ is finite, then we can define the map $\kappa_1\colon X \times X \to \mathbb{R}$ given by

$$\kappa_1(A_1, A_2) = \mu(A_1 \cap A_2), \qquad A_1, A_2 \in X.$$

We can show that κ is a kernel function as follows. Let $H = \mathrm{L}^2_\mu(D, \mathcal{A}, \mathbb{R})$ be the Hilbert space of μ-square-integrable functions with the inner product

$$\langle f, g \rangle = \int_D f(s)g(s)\, d\mu(s),$$

and let $\varphi\colon X \to H$ be the feature embedding given by

$$\varphi(A) = \chi_A, \qquad A \in X,$$

the characteristic function of A. Then we have

$$\begin{aligned}
\kappa_1(A_1, A_2) = \mu(A_1 \cap A_2) &= \int_D \chi_{A_1 \cap A_2}(s)\, d\mu(s) \\
&= \int_D \chi_{A_1}(s)\chi_{A_2}(s)\, d\mu(s) = \langle \chi_{A_1}, \chi_{A_2} \rangle \\
&= \langle \varphi(A_1), \varphi(A_2) \rangle.
\end{aligned}$$