If $b \neq 0$, then the inequality

$$\frac{\|\Delta x\|}{\|x\|} \leq \operatorname{cond}(A)\frac{\|\Delta b\|}{\|b\|}$$

holds and is the best possible. This means that for a given matrix A, there exist some vectors $b \neq 0$ and $\Delta b \neq 0$ for which equality holds.

Proof. We already proved the inequality. Now, because $\| \ \|$ is a subordinate matrix norm, there exist some vectors $x \neq 0$ and $\Delta b \neq 0$ for which

$$\left\|A^{-1}\Delta b\right\| = \left\|A^{-1}\right\| \|\Delta b\| \quad \text{and} \quad \|Ax\| = \|A\| \|x\|.$$

∎

Proposition 9.14. *Let A be an invertible matrix and let x and $x + \Delta x$ be the solutions of the two systems*

$$\begin{aligned} Ax &= b \\ (A + \Delta A)(x + \Delta x) &= b. \end{aligned}$$

If $b \neq 0$, then the inequality

$$\frac{\|\Delta x\|}{\|x + \Delta x\|} \leq \operatorname{cond}(A)\frac{\|\Delta A\|}{\|A\|}$$

holds and is the best possible. This means that given a matrix A, there exist a vector $b \neq 0$ and a matrix $\Delta A \neq 0$ for which equality holds. Furthermore, if $\|\Delta A\|$ is small enough (for instance, if $\|\Delta A\| < 1/\left\|A^{-1}\right\|$), we have

$$\frac{\|\Delta x\|}{\|x\|} \leq \operatorname{cond}(A)\frac{\|\Delta A\|}{\|A\|}(1 + O(\|\Delta A\|));$$

in fact, we have

$$\frac{\|\Delta x\|}{\|x\|} \leq \operatorname{cond}(A)\frac{\|\Delta A\|}{\|A\|}\left(\frac{1}{1 - \|A^{-1}\| \|\Delta A\|}\right).$$

Proof. The first inequality has already been proven. To show that equality can be achieved, let w be any vector such that $w \neq 0$ and

$$\left\|A^{-1}w\right\| = \left\|A^{-1}\right\| \|w\|,$$

and let $\beta \neq 0$ be any real number. Now the vectors

$$\begin{aligned} \Delta x &= -\beta A^{-1}w \\ x + \Delta x &= w \\ b &= (A + \beta I)w \end{aligned}$$