H. In particular, when $n = 3$, every improper orthogonal transformation is the product of a rotation with a reflection about a plane orthogonal to the axis of rotation.

Using Theorem 27.1, we can also give a rather simple proof of the classical fact that in a Euclidean space of odd dimension, every rotation leaves some nonnull vector invariant, and thus a line invariant.

If λ is an eigenvalue of f, then the following lemma shows that the orthogonal complement $E_\lambda(f)^\perp$ of the eigenspace associated with λ is closed under f.

Proposition 27.2. *Let E be a Euclidean space of finite dimension n, and let $f\colon E \to E$ be an isometry. For any subspace F of E, if $f(F) = F$, then $f(F^\perp) \subseteq F^\perp$ and $E = F \oplus F^\perp$.*

Proof. We just have to prove that if $w \in E$ is orthogonal to every $u \in F$, then $f(w)$ is also orthogonal to every $u \in F$. However, since $f(F) = F$, for every $v \in F$, there is some $u \in F$ such that $f(u) = v$, and we have

$$f(w) \cdot v = f(w) \cdot f(u) = w \cdot u,$$

since f is an isometry. Since we assumed that $w \in E$ is orthogonal to every $u \in F$, we have

$$w \cdot u = 0,$$

and thus

$$f(w) \cdot v = 0,$$

and this for every $v \in F$. Thus, $f(F^\perp) \subseteq F^\perp$. The fact that $E = F \oplus F^\perp$ follows from Lemma 12.11. $\square$

Lemma 27.2 is the starting point of the proof that every orthogonal matrix can be diagonalized over the field of complex numbers. Indeed, if λ is any eigenvalue of f, then $f(E_\lambda(f)) = E_\lambda(f)$, where $E_\lambda(f)$ is the eigenspace associated with λ, and thus the orthogonal $E_\lambda(f)^\perp$ is closed under f, and $E = E_\lambda(f) \oplus E_\lambda(f)^\perp$. The problem over $\mathbb{R}$ is that there may not be any real eigenvalues. However, when n is odd, the following lemma shows that every rotation admits 1 as an eigenvalue (and similarly, when n is even, every improper orthogonal transformation admits 1 as an eigenvalue).

Proposition 27.3. *Let E be a Euclidean space.*

(1) If E has odd dimension $n = 2m + 1$, then every rotation f admits 1 as an eigenvalue and the eigenspace F of all eigenvectors left invariant under f has an odd dimension $2p + 1$. Furthermore, there is an orthonormal basis of E, in which f is represented by a matrix of the form

$$\begin{pmatrix} R_{2(m-p)} & 0 \\ 0 & I_{2p+1} \end{pmatrix},$$

where $R_{2(m-p)}$ is a rotation matrix that does not have 1 as an eigenvalue.