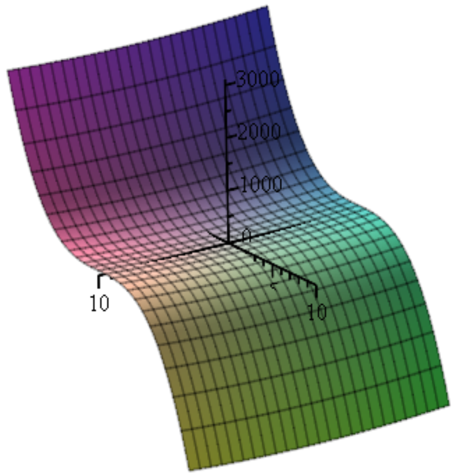

Figure 40.3: The graph of $f(x, y) = x^2 - 3y^3$. Note that $(0, 0)$ not a local extremum despite the fact that $df(0, 0) = 0$.

Theorem 40.6. *Let E be a normed vector space, let $J\colon \Omega \to \mathbb{R}$ be a function with Ω some open subset of E, and assume that J is differentiable in Ω and that $dJ(u) = 0$ at some point $u \in \Omega$. The following properties hold:*

(1) If $\mathrm{D}^2 J(u)$ exists and if there is some number $\alpha \in \mathbb{R}$ such that $\alpha > 0$ and

$$\mathrm{D}^2 J(u)(w, w) \geq \alpha \|w\|^2 \quad \textit{for all } w \in E,$$

then J has a strict local minimum at u.

(2) If $\mathrm{D}^2 J(v)$ exists for all $v \in \Omega$ and if there is a ball $B \subseteq \Omega$ centered at u such that

$$\mathrm{D}^2 J(v)(w, w) \geq 0 \quad \textit{for all } v \in B \textit{ and all } w \in E,$$

then J has a local minimum at u.

Proof. (1) Using the formula of Taylor–Young, for every vector w small enough, we can write

$$\begin{aligned} J(u + w) - J(u) &= \frac{1}{2}\mathrm{D}^2 J(u)(w, w) + \|w\|^2 \epsilon(w) \\ &\geq \left(\frac{1}{2}\alpha + \epsilon(w)\right) \|w\|^2 \end{aligned}$$

with $\lim_{w \mapsto 0} \epsilon(w) = 0$. Consequently if we pick $r > 0$ small enough that $|\epsilon(w)| < \alpha/2$ for all w with $\|w\| < r$, then $J(u + w) > J(u)$ for all $u + w \in B$, where B is the open ball of center u and radius r. This proves that J has a local strict minimum at u.