$x_n$ from the last equation, next plug this value of $x_n$ into the next to the last equation and compute $x_{n-1}$ from it, *etc.* This yields

$$\begin{aligned} x_n &= a_{n\,n}^{-1} b_n \\ x_{n-1} &= a_{n-1\,n-1}^{-1}(b_{n-1} - a_{n-1\,n} x_n) \\ &\vdots \\ x_1 &= a_{1\,1}^{-1}(b_1 - a_{1\,2} x_2 - \cdots - a_{1\,n} x_n). \end{aligned}$$

Note that the use of determinants can be avoided to prove that if $A$ is invertible then $a_{i\,i} \neq 0$ for $i = 1, \ldots, n$. Indeed, it can be shown directly (by induction) that an upper (or lower) triangular matrix is invertible iff all its diagonal entries are nonzero.

If $A$ is lower-triangular, we solve the system from top-down by *forward-substitution*.

Thus, what we need is a method for transforming a matrix to an equivalent one in upper-triangular form. This can be done by *elimination*. Let us illustrate this method on the following example:

$$\begin{array}{rcrcrcr} 2x & + & y & + & z & = & 5 \\ 4x & - & 6y & & & = & -2 \\ -2x & + & 7y & + & 2z & = & 9. \end{array}$$

We can eliminate the variable $x$ from the second and the third equation as follows: Subtract twice the first equation from the second and add the first equation to the third. We get the new system

$$\begin{array}{rcrcrcr} 2x & + & y & + & z & = & 5 \\ & - & 8y & - & 2z & = & -12 \\ & & 8y & + & 3z & = & 14. \end{array}$$

This time we can eliminate the variable $y$ from the third equation by adding the second equation to the third:

$$\begin{array}{rcrcrcr} 2x & + & y & + & z & = & 5 \\ & - & 8y & - & 2z & = & -12 \\ & & & & z & = & 2. \end{array}$$

This last system is upper-triangular. Using back-substitution, we find the solution: $z = 2$, $y = 1$, $x = 1$.

Observe that we have performed only *row operations*. The general method is to iteratively eliminate variables using simple row operations (namely, adding or subtracting a multiple of a row to another row of the matrix) while simultaneously applying these operations to the vector $b$, to obtain a system, $MAx = Mb$, where $MA$ is upper-triangular. Such a method is called *Gaussian elimination*. However, one extra twist is needed for the method to work in all cases: It may be necessary to permute rows, as illustrated by the following example:

$$\begin{array}{rcrcrl} x & + & y & + & z & = 1 \\ x & + & y & + & 3z & = 1 \\ 2x & + & 5y & + & 8z & = 1. \end{array}$$