Proof of Claim. If Hypothesis (V) holds, then we have

$$\varphi(f(u), v) = \varphi(u, v) \quad \text{for all } u \in U \text{ and all } v \in V,$$

since $\varphi(f(u), v) - \varphi(u, v) = \varphi(f(u) - u, v) = 0$, with $f(u) - u \in D$ and $v \in V$ orthogonal to D. By Proposition 29.44 with $f_1 = f$ and f_2 the inclusion of V into E, we can extend f to an injective metric map on $U \oplus V$ leaving all vectors in V fixed. In this case, the set $\{f(w) - w \mid w \in U \oplus V\}$ is still the line D. ☐

We show below that the fact that f can be extended to $U \oplus V$ implies that f can be extended to the whole of E. There are two cases. In Case (a), $E = U \oplus V$ and we are done. In case (b), $D^\perp = U \oplus V$ where $D^\perp$ is a hyperplane in E and f is an isometry of $D^\perp$. By a subtle argument, we will show that f can be extended to an isometry of E.

We are reduced to proving that a subspace V as above exists. We distinguish between two cases.

Case (a). $U \not\subseteq D^\perp$.

Proof of Case (a). In this case, formula $(**)$ show that $f(U)$ is not contained in $D^\perp$ (check this!). Consequently,

$$U \cap D^\perp = f(U) \cap D^\perp = H.$$

We can pick V to be any supplement of H in $D^\perp$, and the above formula shows that $V \cap U = V \cap f(U) = (0)$. Since $U \oplus V$ contains the hyperplane $D^\perp$ (since $D^\perp = H \oplus V$ and $H \subseteq U$), and $U \oplus V \neq D^\perp$ (since U is not contained in $D^\perp$ and $V \subseteq D^\perp$), we must have $E = U \oplus V$, and as we showed as a consequence of hypothesis (V), f can be extended to an isometry of $U \oplus V = E$. ☐

Case (b). $U \subseteq D^\perp$.

Proof of Case (b). In this case, formula $(**)$ shows that $f(U) \subseteq D^\perp$ so $U + f(U) \subseteq D^\perp$, and since $D = \{f(u) - u \mid u \in U\}$, we have $D \subseteq D^\perp$; that is, the line D is isotropic.

We show that there exists a subspace V of $D^\perp$, such that

$$D^\perp = U \oplus V = f(U) \oplus V.$$

Thus, case (b) shows that we are reduced to the situation where $U = D^\perp$ and f is an isometry of U.

If $U = f(U)$ we pick V to be a supplement of U in $D^\perp$. Otherwise, let $x \in U$ with $x \notin H$, and let $y \in f(U)$ with $y \notin H$. Since $f(H) = H$ (pointwise), f is injective, and H is a hyperplane in U, we have

$$U = H \oplus Kx, \quad f(U) = H \oplus Ky.$$