If $n \geq m$, we proceed by induction on n. If $n = 0$, then $g = b_0$, $m = 0$, $f = a_0 \neq 0$, and we let $q = a_0^{-1} b_0$ and $r = 0$. Since $\deg(r) = \deg(0) = -\infty$ and $\deg(f) = \deg(a_0) = 0$ because $a_0 \neq 0$, we have $\deg(r) < \deg(f)$.

If $n \geq 1$, since $n \geq m$, note that

$$\begin{aligned} g_1(X) &= g(X) - b_n a_m^{-1} X^{n-m} f(X) \\ &= b_n X^n + b_{n-1} X^{n-1} + \cdots + b_0 - b_n a_m^{-1} X^{n-m} (a_m X^m + a_{m-1} X^{m-1} + \cdots + a_0) \end{aligned}$$

is a polynomial of degree $\deg(g_1) < n$, since the terms $b_n X^n$ and $b_n a_m^{-1} X^{n-m} a_m X^m$ of degree n cancel out. Now, since $\deg(g_1) < n$, by the induction hypothesis, we can find q_1 and r such that

$$g_1 = f q_1 + r \quad \text{and} \quad \deg(r) < \deg(f) = m,$$

and thus,

$$g_1(X) = g(X) - b_n a_m^{-1} X^{n-m} f(X) = f(X) q_1(X) + r(X),$$

from which, letting $q(X) = b_n a_m^{-1} X^{n-m} + q_1(X)$, we get

$$g = f q + r \quad \text{and} \quad \deg(r) < m = \deg(f).$$

We now prove uniqueness. If

$$g = f q_1 + r_1 = f q_2 + r_2,$$

with $\deg(r_1) < \deg(f)$ and $\deg(r_2) < \deg(f)$, we get

$$f(q_1 - q_2) = r_2 - r_1.$$

If $q_2 - q_1 \neq 0$, since the leading coefficient a_m of f is invertible, by Proposition 30.1, we have

$$\deg(r_2 - r_1) = \deg(f(q_1 - q_2)) = \deg(f) + \deg(q_2 - q_1),$$

and so, $\deg(r_2 - r_1) \geq \deg(f)$, which contradicts the fact that $\deg(r_1) < \deg(f)$ and $\deg(r_2) < \deg(f)$. Thus, $q_1 = q_2$, and then also $r_1 = r_2$. $\square$

It should be noted that the proof of Proposition 30.4 actually provides an algorithm for finding the *quotient* q and the *remainder* r of the division of g by f. This algorithm is called the *Euclidean algorithm, or division algorithm*. Note that the division of g by f is always possible when f is a monic polynomial, since 1 is invertible. Also, when A is a field, $a_m \neq 0$ is always invertible, and thus, the division can always be performed. We say that f *divides* g when $r = 0$ in the result of the division $g = fq + r$. We now draw some important consequences of the existence of the Euclidean algorithm.