Note that the symmetry of the bilinear form a played a crucial role. Also, the inequalities

$$a(u, v-u) \geq h(v-u) \quad \text{for all } v \in U$$

are sometimes called *variational inequalities*.

Definition 49.5. A bilinear form $a\colon V \times V \to \mathbb{R}$ such that there is some real $\alpha > 0$ such that

$$a(v, v) \geq \alpha \|v\|^2 \quad \text{for all } v \in V$$

is said to be *coercive*.

Theorem 49.4 is the special case of Stampacchia's theorem and the Lax–Milgram theorem when $U = V$, and where a is a symmetric bilinear form. To prove Stampacchia's theorem in general, we need to recall the *contraction mapping theorem*.

Definition 49.6. Let (E, d) be a metric space. A map $f\colon E \to E$ is a *contraction* (or a *contraction mapping*) if there is some real number k such that $0 \leq k < 1$ and

$$d(f(u), f(v)) \leq kd(u, v) \quad \text{for all } u, v \in E.$$

The number k is often called a *Lipschitz constant*.

The following theorem is proven in Section 37.10; see Theorem 37.54. A proof can be also found in Apostol [4], Dixmier [51], or Schwartz [150], among many sources. For the reader's convenience we restate this theorem.

Theorem 49.5. *(Contraction Mapping Theorem) Let (E, d) be a complete metric space. Every contraction $f\colon E \to E$ has a unique fixed point (that is, an element $u \in E$ such that $f(u) = u$).*

The contraction mapping theorem is also known as the *Banach fixed point theorem*.

Theorem 49.6. *(Lions–Stampacchia) Given a Hilbert space V, let $a\colon V \times V \to \mathbb{R}$ be a continuous bilinear form (not necessarily symmetric), let $h \in V'$ be a continuous linear form, and let J be given by*

$$J(v) = \frac{1}{2}\, a(v, v) - h(v), \quad v \in V.$$

If a is coercive, then for every nonempty, closed, convex subset U of V, there is a unique $u \in U$ such that

$$a(u, v-u) \geq h(v-u) \quad \textit{for all } v \in U. \tag{$*$}$$

If a is symmetric, then $u \in U$ is the unique element of U such that

$$J(u) = \inf_{v \in U} J(v).$$