Proposition 22.2 also holds for self-adjoint linear maps on a complex vector space with a Hermitian inner product. The proof is essentially the same and is left as an exercise to the reader.

The version of Proposition 22.2 for matrices follows immediately.

Proposition 22.3. *Let A be a real $n \times n$ symmetric matrix.*

(1) The eigenvalues of A are strictly positive iff

$$u^\top Au > 0 \quad \textit{for all } u \neq 0.$$

(2) The eigenvalues of A are nonnegative iff

$$u^\top Au \geq 0 \quad \textit{for all } u \neq 0.$$

It is important to note that Proposition 22.3 is false for nonsymmetric matrices.

Example 22.1. The matrix

$$A = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$$

has the positive eigenvalues $(1, 1)$, but

$$\begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} -3 \\ -1 \end{pmatrix} = -2.$$

Example 22.2. The matrix

$$A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$$

has the complex eigenvalues $1 + 2i, 1 - 2i$, and yet

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} x - 2y \\ 2x + y \end{pmatrix} = x^2 + y^2,$$

so $u^\top Au > 0$ for all $u \neq 0$.

Since $u^\top Au$ is a scalar, if A is a skew symmetric matrix ($A^\top = -A$), then we see that

$$u^\top Au = 0 \quad \text{for all } u \in \mathbb{R}.$$

Therefore, if A is a real $n \times n$ matrix then

$$u^\top Au = u^\top H(A)u \quad \text{for all } u \in \mathbb{R},$$

where $H(A) = (1/2)(A + A^\top)$ is the symmetric part of A. This explains why the notion of a positive definite matrix is only interesting for symmetric matrices. But but one should also be aware that even if a nonsymmetric matrix A has "well-behaved" eigenvalues, its symmetric part $H(A)$ may *not* be positive definite.