By block multiplication we get

$$\begin{aligned} TC + UW &= I_n \\ TV + \beta U &= 0_{n,1} \\ \alpha W &= 0_{1,n} \\ \alpha\beta &= 1. \end{aligned}$$

From the above equations we deduce that $\alpha, \beta \neq 0$ and $\beta = \alpha^{-1}$. Since $\alpha \neq 0$, the equation $\alpha W = 0_{1,n}$ yields $W = 0_{1,n}$, and so

$$TC = I_n, \quad TV + \beta U = 0_{n,1}.$$

It follows that T is invertible and C is its inverse, and since T is upper triangular, by the induction hypothesis, C is also upper triangular.

The above argument can be easily modified to prove that if A is invertible, then its diagonal entries are nonzero.

We are now ready to prove a very crucial result relating the rank and the dimension of the kernel of a linear map.

6.3 The Rank-Nullity Theorem; Grassmann's Relation

We begin with the following fundamental proposition.

Proposition 6.15. *Let E, F and G, be three vector spaces, $f\colon E \to F$ an injective linear map, $g\colon F \to G$ a surjective linear map, and assume that $\operatorname{Im} f = \operatorname{Ker} g$. Then, the following properties hold. (a) For any section $s\colon G \to F$ of g, we have $F = \operatorname{Ker} g \oplus \operatorname{Im} s$, and the linear map $f + s\colon E \oplus G \to F$ is an isomorphism.*[1]

(b) For any retraction $r\colon F \to E$ of f, we have $F = \operatorname{Im} f \oplus \operatorname{Ker} r$.[2]

$$\begin{tikzcd} E \arrow[r, shift left, "f"] & F \arrow[l, shift left, "r"] \arrow[r, shift left, "g"] & G \arrow[l, shift left, "s"] \end{tikzcd}$$

Proof. (a) Since $s\colon G \to F$ is a section of g, we have $g \circ s = \mathrm{id}_G$, and for every $u \in F$,

$$g(u - s(g(u))) = g(u) - g(s(g(u))) = g(u) - g(u) = 0.$$

Thus, $u - s(g(u)) \in \operatorname{Ker} g$, and we have $F = \operatorname{Ker} g + \operatorname{Im} s$. On the other hand, if $u \in \operatorname{Ker} g \cap \operatorname{Im} s$, then $u = s(v)$ for some $v \in G$ because $u \in \operatorname{Im} s$, $g(u) = 0$ because $u \in \operatorname{Ker} g$, and so,

$$g(u) = g(s(v)) = v = 0,$$

[1] The existence of a section $s\colon G \to F$ of g follows from Proposition 6.11.

[2] The existence of a retraction $r\colon F \to E$ of f follows from Proposition 6.11.