which shows that z^{k+1} minimizes the function

$$z \mapsto g(z) + (\lambda^{k+1})^\top Bz.$$

Consequently, we have

$$g(z^{k+1}) + (\lambda^{k+1})^\top Bz^{k+1} \leq g(z^*) + (\lambda^{k+1})^\top Bz^*. \tag{B1}$$

Similarly, x^{k+1} minimizes $L_\rho(x, z^k, \lambda^k)$ iff

$$\begin{aligned} 0 &\in \partial f(x^{k+1}) + A^\top \lambda^k + \rho A^\top (Ax^{k+1} + Bz^k - c) \\ &= \partial f(x^{k+1}) + A^\top (\lambda^k + \rho r^{k+1} + \rho B(z^k - z^{k+1})) \\ &= \partial f(x^{k+1}) + A^\top \lambda^{k+1} + \rho A^\top B(z^k - z^{k+1}) \end{aligned}$$

since $r^{k+1} - Bz^{k+1} = Ax^{k+1} - c$ and $\lambda^{k+1} = \lambda^k + \rho(Ax^{k+1} + Bz^{k+1} - c) = \lambda^k + \rho r^{k+1}$.

Equivalently, the above derivation shows that

$$0 \in \partial f(x^{k+1}) + A^\top (\lambda^{k+1} - \rho B(z^{k+1} - z^k)), \tag{$\dagger_2$}$$

which shows that x^{k+1} minimizes the function

$$x \mapsto f(x) + (\lambda^{k+1} - \rho B(z^{k+1} - z^k))^\top Ax.$$

Consequently, we have

$$f(x^{k+1}) + (\lambda^{k+1} - \rho B(z^{k+1} - z^k))^\top Ax^{k+1} \leq f(x^*) + (\lambda^{k+1} - \rho B(z^{k+1} - z^k))^\top Ax^*. \tag{B2}$$

Adding up Inequalities (B1) and (B2), using the equation $Ax^* + Bz^* = c$, and rearranging, we obtain inequality (A2).

Step 8. Prove that $(x^k), (z^k)$, and (λ^k) converge to optimal solutions.

Recall that (r^k) converges to 0, and that (x^k), (z^k), and (λ^k) converge to limits $\widetilde{x}$, $\widetilde{z}$, and $\widetilde{\lambda}$. Since $r^k = Ax^k + Bz^k - c$, in the limit, we have

$$A\widetilde{x} + B\widetilde{z} - c = 0. \tag{$*_1$}$$

Using ($\dagger_1$), in the limit, we obtain

$$0 \in \partial g(\widetilde{z}) + B^\top \widetilde{\lambda}. \tag{$*_2$}$$

Since $(B(z^{k+1} - z^k))$ converges to 0, using ($\dagger_2$), in the limit, we obtain

$$0 \in \partial f(\widetilde{x}) + A^\top \widetilde{\lambda}. \tag{$*_3$}$$

From ($*_2$) and ($*_3$), we obtain

$$0 \in \partial f(\widetilde{x}) + \partial g(\widetilde{z}) + A^\top \widetilde{\lambda} + B^\top \widetilde{\lambda}. \tag{$*_4$}$$