However, if the basis functions are simple enough, this can be done "by hand." Otherwise, numerical integration methods must be used, but there are some good ones.

Let us also remark that the proof of Theorem 19.1 also shows that the unique solution of (DWF) is the unique minimizer of J over all functions in V_a. It is also possible to compare the approximate solution $u^{(a)} \in V_a$ with the exact solution $u \in V$.

Theorem 19.2. *Suppose $c(x) \geq 0$ for all $x \in [0,1]$. For every finite-dimensional subspace V_a ($\dim(V_a) = n$) of V, for every basis $(w_1, \ldots, w_n)$ of V_a, the following properties hold:*

(1) There is a unique function $u^{(a)} \in V_a$ such that

$$a(u^{(a)}, v) = \widetilde{f}(v), \quad \text{for all } v \in V_a, \tag{DWF}$$

and if $u^{(a)} = u_1 w_1 + \cdots + u_n w_n$, then $\mathbf{u} = (u_1, \ldots, u_n)$ is the solution of the linear system

$$A\mathbf{u} = b, \tag{$*$}$$

with $A = (a_{ij}) = (a(w_i, w_j))$ and $b_j = \widetilde{f}(w_j)$, $1 \leq i,j \leq n$. Furthermore, the matrix $A = (a_{ij})$ is symmetric positive definite.

(2) The unique solution $u^{(a)} \in V_a$ of (DWF) is the unique minimizer of J over V_a, that is,

$$J(u^{(a)}) = \inf_{v \in V_a} J(v),$$

(3) There is a constant C independent of V_a and of the unique solution $u \in V$ of (WF), such that

$$\left\| u - u^{(a)} \right\|_V \leq C \inf_{v \in V_a} \|u - v\|_V .$$

We proved (1) and (2), but we will omit the proof of (3) which can be found in Ciarlet [41].

Let us now give examples of the subspaces V_a used in practice. They usually consist of piecewise polynomial functions.

Pick an integer $N \geq 1$ and subdivide $[0,1]$ into $N+1$ intervals $[x_i, x_{i+1}]$, where

$$x_i = hi, \quad h = \frac{1}{N+1}, \quad i = 0, \ldots, N+1.$$

We will use the following fact: every polynomial $P(x)$ of degree $2m+1$ ($m \geq 0$) is completely determined by its values as well as the values of its first m derivatives at two distinct points $\alpha, \beta \in \mathbb{R}$.