and by combining the above two equations, we get

$$\begin{aligned}2(\lambda-\overline{\lambda})\varphi(u,v) &= \lambda\varphi(u+v,u+v)-\lambda\varphi(u-v,u-v)\\ &\quad -\varphi(u+\lambda v,u+\lambda v)+\varphi(u-\lambda v,u-\lambda v).\end{aligned} \tag{$*$}$$

If the automorphism $\lambda\mapsto\overline{\lambda}$ is not the identity, then there is some $\lambda\in K$ such that $\lambda-\overline{\lambda}\neq 0$, and if K is not of characteristic 2, then we see that the sesquilinear form φ is completely determined by its restriction to the diagonal (that is, the set of values $\{\varphi(u,u)\mid u\in E\}$). In the special case where $K=\mathbb{C}$, we can pick $\lambda=i$, and we get

$$4\varphi(u,v)=\varphi(u+v,u+v)-\varphi(u-v,u-v)+i\varphi(u+\lambda v,u+\lambda v)-i\varphi(u-\lambda v,u-\lambda v).$$

Remark: If the automorphism $\lambda\mapsto\overline{\lambda}$ is the identity, then in general φ is not determined by its value on the diagonal, unless φ is symmetric.

In the sesquilinear setting, it turns out that the following two cases are of interest:

1. We have
$$\varphi(v,u)=\overline{\varphi(u,v)},\quad \text{for all } u,v\in E,$$
in which case we say that φ is *Hermitian*. In the special case where $K=\mathbb{C}$ and the involutive automorphism is conjugation, we see that $\varphi(u,u)\in\mathbb{R}$, for $u\in E$.

2. We have
$$\varphi(v,u)=-\overline{\varphi(u,v)},\quad \text{for all } u,v\in E,$$
in which case we say that φ is *skew-Hermitian*.

We observed that in characteristic different from 2, a sesquilinear form is determined by its restriction to the diagonal. For Hermitian and skew-Hermitian forms, we have the following kind of converse.

Proposition 29.8. *If φ is a nonzero Hermitian or skew-Hermitian form and if $\varphi(u,u)=0$ for all $u\in E$, then K is of characteristic 2 and the automorphism $\lambda\mapsto\overline{\lambda}$ is the identity.*

Proof. We give the proof in the Hermitian case, the skew-Hermitian case being left as an exercise. Assume that φ is alternating. From the identity

$$\varphi(u+v,u+v)=\varphi(u,u)+\varphi(u,v)+\overline{\varphi(u,v)}+\varphi(v,v),$$

we get
$$\varphi(u,v)=-\overline{\varphi(u,v)}\quad\text{for all } u,v\in E.$$

Since φ is not the zero form, there exist some nonzero vectors $u,v\in E$ such that $\varphi(u,v)=1$. For any $\lambda\in K$, we have

$$\lambda\varphi(u,v)=\varphi(\lambda u,v)=-\overline{\varphi(\lambda u,v)}=-\overline{\lambda}\,\overline{\varphi(u,v)},$$