Proof. Let $\varphi(0) = 0$, and for every nonnull polynomial $P(X) = a_0 + a_1X + \cdots + a_nX^n$, let

$$\varphi(P(X)) = h(a_0) + h(a_1)\beta + \cdots + h(a_n)\beta^n.$$

It is easily verified that φ is the unique homomorphism $\varphi\colon A[X] \to B$ extending h such that $\varphi(X) = \beta$. $\square$

Taking $A = B$ in Proposition 30.2 and $h\colon A \to A$ the identity, for every $\beta \in A$, there is a unique homomorphism $\varphi_\beta\colon A[X] \to A$ such that $\varphi_\beta(X) = \beta$, and for every polynomial $P(X)$, we write $\varphi_\beta(P(X))$ as $P(\beta)$ and we call $P(\beta)$ the *value of* $P(X)$ *at* $X = \beta$. Thus, we can define a function $P_A\colon A \to A$ such that $P_A(\beta) = P(\beta)$, for all $\beta \in A$. This function is called the *polynomial function induced by* P.

More generally, P_B can be defined for any (commutative) ring B such that $A \subseteq B$. In general, it is possible that $P_A = Q_A$ for distinct polynomials P, Q. We will see shortly conditions for which the map $P \mapsto P_A$ is injective. In particular, this is true for $A = \mathbb{R}$ (in general, any infinite integral domain). We now define polynomials in n variables.

Definition 30.3. Given $n \geq 1$ and a ring A, the set $\mathcal{P}_A(n)$ of *polynomials over* A *in* n *variables* is the set of functions $P\colon \mathbb{N}^{(n)} \to A$ such that $P(k_1, \ldots, k_n) \neq 0$ for finitely many $(k_1, \ldots, k_n) \in \mathbb{N}^{(n)}$. The polynomial such that $P(k_1, \ldots, k_n) = 0$ for all $(k_1, \ldots, k_n)$ is the *null (or zero) polynomial* and it is denoted by 0. We define addition of polynomials, multiplication by a scalar, and multiplication of polynomials, as follows: Given any three polynomials $P, Q, R \in \mathcal{P}_A(n)$, letting $a_{(k_1,\ldots,k_n)} = P(k_1, \ldots, k_n)$, $b_{(k_1,\ldots,k_n)} = Q(k_1, \ldots, k_n)$, $c_{(k_1,\ldots,k_n)} = R(k_1, \ldots, k_n)$, for every $(k_1, \ldots, k_n) \in \mathbb{N}^{(n)}$, we define $R = P + Q$ such that

$$c_{(k_1,\ldots,k_n)} = a_{(k_1,\ldots,k_n)} + b_{(k_1,\ldots,k_n)},$$

$R = \lambda P$, where $\lambda \in A$, such that

$$c_{(k_1,\ldots,k_n)} = \lambda a_{(k_1,\ldots,k_n)},$$

and $R = PQ$, such that

$$c_{(k_1,\ldots,k_n)} = \sum_{(i_1,\ldots,i_n)+(j_1,\ldots,j_n)=(k_1,\ldots,k_n)} a_{(i_1,\ldots,i_n)} b_{(j_1,\ldots,j_n)}.$$

For every $(k_1, \ldots, k_n) \in \mathbb{N}^{(n)}$, we let $e_{(k_1,\ldots,k_n)}$ be the polynomial such that

$$e_{(k_1,\ldots,k_n)}(k_1, \ldots, k_n) = 1 \quad \text{and} \quad e_{(k_1,\ldots,k_n)}(h_1, \ldots, h_n) = 0,$$

for $(h_1, \ldots, h_n) \neq (k_1, \ldots, k_n)$. We also denote $e_{(0,\ldots,0)}$ by 1. Given a polynomial P, the $a_{(k_1,\ldots,k_n)} = P(k_1, \ldots, k_n) \in A$, are called the *coefficients of* P. If P is not the null polynomial, there is a greatest $d \geq 0$ such that $a_{(k_1,\ldots,k_n)} \neq 0$ for some $(k_1, \ldots, k_n) \in \mathbb{N}^{(n)}$, with $d = k_1 + \cdots + k_n$, called the *total degree of* P and denoted by $\deg(P)$. Then, P is written uniquely as

$$P = \sum_{(k_1,\ldots,k_n)\in\mathbb{N}^{(n)}} a_{(k_1,\ldots,k_n)} e_{(k_1,\ldots,k_n)}.$$

When P is the null polynomial, we let $\deg(P) = -\infty$.