23.2 Properties of the Pseudo-Inverse

We begin this section with a proposition which provides a way to calculate the pseudo-inverse of an $m \times n$ matrix A without first determining an SVD factorization.

Proposition 23.3. *When A has full rank, the pseudo-inverse A^+ can be expressed as $A^+ = (A^\top A)^{-1}A^\top$ when $m \geq n$, and as $A^+ = A^\top(AA^\top)^{-1}$ when $n \geq m$. In the first case ($m \geq n$), observe that $A^+A = I$, so A^+ is a left inverse of A; in the second case ($n \geq m$), we have $AA^+ = I$, so A^+ is a right inverse of A.*

Proof. If $m \geq n$ and A has full rank n, we have

$$A = V \begin{pmatrix} \Lambda \\ 0_{m-n,n} \end{pmatrix} U^\top$$

with Λ an $n \times n$ diagonal invertible matrix (with positive entries), so

$$A^+ = U \begin{pmatrix} \Lambda^{-1} & 0_{n,m-n} \end{pmatrix} V^\top.$$

We find that

$$A^\top A = U \begin{pmatrix} \Lambda & 0_{n,m-n} \end{pmatrix} V^\top V \begin{pmatrix} \Lambda \\ 0_{m-n,n} \end{pmatrix} U^\top = U\Lambda^2 U^\top,$$

which yields

$$(A^\top A)^{-1}A^\top = U\Lambda^{-2}U^\top U \begin{pmatrix} \Lambda & 0_{n,m-n} \end{pmatrix} V^\top = U \begin{pmatrix} \Lambda^{-1} & 0_{n,m-n} \end{pmatrix} V^\top = A^+.$$

Therefore, if $m \geq n$ and A has full rank n, then

$$A^+ = (A^\top A)^{-1}A^\top.$$

If $n \geq m$ and A has full rank m, then

$$A = V \begin{pmatrix} \Lambda & 0_{m,n-m} \end{pmatrix} U^\top$$

with Λ an $m \times m$ diagonal invertible matrix (with positive entries), so

$$A^+ = U \begin{pmatrix} \Lambda^{-1} \\ 0_{n-m,m} \end{pmatrix} V^\top.$$

We find that

$$AA^\top = V \begin{pmatrix} \Lambda & 0_{m,n-m} \end{pmatrix} U^\top U \begin{pmatrix} \Lambda \\ 0_{n-m,m} \end{pmatrix} V^\top = V\Lambda^2 V^\top,$$

which yields

$$A^\top(AA^\top)^{-1} = U \begin{pmatrix} \Lambda \\ 0_{n-m,m} \end{pmatrix} V^\top V\Lambda^{-2}V^\top = U \begin{pmatrix} \Lambda^{-1} \\ 0_{n-m,m} \end{pmatrix} V^\top = A^+.$$

Therefore, if $n \geq m$ and A has full rank m, then $A^+ = A^\top(AA^\top)^{-1}$. $\square$