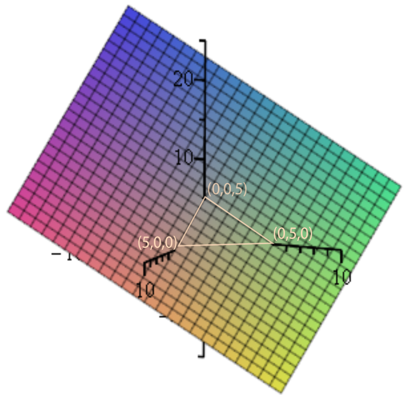

Figure 51.8: The affine hyperplane $H = \{x \in \mathbb{R}^3 \mid x + y + z - 2 = 0\}$. The half space H_+ faces the viewer and contains the point $(0, 0, 10)$, while the half space H_- is behind H and contains the point $(0, 0, 0)$.

Then if φ is the affine form given by $\varphi(x) = \langle x, u\rangle + c$, this affine form is nonconstant iff $u \neq 0$, and u is normal to the hyperplane H, in the sense that if $x_0 \in H$ is any fixed vector in H, and x is any vector in H, then $\langle x - x_0, u\rangle = 0$.

Indeed, $x_0 \in H$ means that $\langle x_0, u\rangle + c = 0$, and $x \in H$ means that $\langle x, u\rangle + c = 0$, so we get $\langle x_0, u\rangle = \langle x, u\rangle$, which implies $\langle x - x_0, u\rangle = 0$.

Here is an observation which plays a key role in defining the notion of subgradient. An illustration of the following proposition is provided by Figure 51.9.

Proposition 51.9. *Let $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ be a nonconstant affine form. Then the map $\omega\colon \mathbb{R}^{n+1} \to \mathbb{R}$ given by*

$$\omega(x, \alpha) = \varphi(x) - \alpha, \quad x \in \mathbb{R}^n,\ \alpha \in \mathbb{R},$$

is a nonconstant affine form defining a hyperplane $\mathcal{H} = \omega^{-1}(0)$ which is the graph of the affine form φ. Furthermore, this hyperplane is nonvertical in $\mathbb{R}^{n+1}$, in the sense that $\mathcal{H}$ cannot be defined by a nonconstant affine form $(x, \alpha) \mapsto \psi(x)$ which does not depend on α.

Proof. Indeed, φ is of the form $\varphi(x) = h(x) + c$ for some nonzero linear form h, so

$$\omega(x, \alpha) = h(x) - \alpha + c.$$

Since h is linear, the map $(x, \alpha) = h(x) - \alpha$ is obviously linear and nonzero, so ω is a nonconstant affine form defining a hyperplane $\mathcal{H}$ in $\mathbb{R}^{n+1}$. By definition,

$$\mathcal{H} = \{(x, \alpha) \in \mathbb{R}^{n+1} \mid \omega(x, \alpha) = 0\} = \{(x, \alpha) \in \mathbb{R}^{n+1} \mid \varphi(x) - \alpha = 0\},$$

which is the graph of φ. If $\mathcal{H}$ was a vertical hyperplane, then $\mathcal{H}$ would be defined by a nonconstant affine form ψ independent of α, but the affine form ω given by $\omega(x, \alpha) = \varphi(x) - \alpha$ and the affine form $\psi(x)$ can't be proportional, a contradiction. $\square$