so that

$$2f(\ldots, u_i, u_i, \ldots) = 0,$$

and in every characteristic except 2, we conclude that $f(\ldots, u_i, u_i, \ldots) = 0$, namely f is alternating. $\square$

Proposition 34.1 shows that in every characteristic except 2, alternating and skew-symmetric multilinear maps are identical. Using Proposition 34.1 we easily deduce the following crucial fact.

Proposition 34.2. *Let $f\colon E^n \to F$ be an alternating multilinear map. For any families of vectors, $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$, with $u_i, v_i \in E$, if*

$$v_j = \sum_{i=1}^{n} a_{ij} u_i, \qquad 1 \leq j \leq n,$$

then

$$f(v_1, \ldots, v_n) = \left(\sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma)\, a_{\sigma(1),1} \cdots a_{\sigma(n),n}\right) f(u_1, \ldots, u_n) = \det(A) f(u_1, \ldots, u_n),$$

where A is the $n \times n$ matrix, $A = (a_{ij})$.

Proof. Use Property (ii) of Proposition 34.1. $\square$

We are now ready to define and construct exterior tensor powers.

Definition 34.2. An *n-th exterior tensor power* of a vector space E, where $n \geq 1$, is a vector space A together with an alternating multilinear map $\varphi\colon E^n \to A$, such that for every vector space F and for every alternating multilinear map $f\colon E^n \to F$, there is a unique linear map $f_\wedge\colon A \to F$ with

$$f(u_1, \ldots, u_n) = f_\wedge(\varphi(u_1, \ldots, u_n)),$$

for all $u_1, \ldots, u_n \in E$, or for short

$$f = f_\wedge \circ \varphi.$$

Equivalently, there is a unique linear map $f_\wedge$ such that the following diagram commutes:

$$\begin{tikzcd} E^n \arrow[r, "\varphi"] \arrow[dr, "f"'] & A \arrow[d, "f_\wedge"] \\ & F. \end{tikzcd}$$

The above property is called the *universal mapping property* of the exterior tensor power (A, φ).