Investments Spring 2020 - Assignment 6

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1 Exercise 1

n° 6

$$R_i = a_i + \sum_{k=1}^{K} \beta_{ik} F_k + \varepsilon_i \Rightarrow R_i = a_i + \beta_i^{T} F + \varepsilon_i \quad (1)$$

a)

No arbitrage theory implies if $E[\varepsilon_i] = 0$

$$R_i^e = \beta_i^{T} R_F^e$$

R_F = vector of dim K made of the returns of the factors

$$\text{so } E[R_i] - R_0 = \beta_i^T \left[E[R_F - 1_k R_0] \right]$$

1_k = K-vector of ones

Taking expectation of the model (1)

$$E[R_i] = a_i + \beta_i^T E[R_F] + 0$$

So in order to be consistent with APT

$$a_i = R_0 - \beta_i^T \cdot (1_k R_0)$$

b) CAPM: $E[R_i] = R_0 + \beta_{im} E[R_m - R_0]$ where $\beta_{im} = \frac{Cov(R_i, R_m)}{\sigma^2_m}$

If $R_m = \sum_{k=1}^{K} w_k F_k = w_m^T F_k \Rightarrow E[R_m] = w_m^T E[F]$

$$E[R_i] = R_0 + \beta_{im} E[w_m^T R_F - R_0]$$

But we also know

$E[R_i] = a_i + \beta_i^T E[R_F]$ If APT hold $a_i = R_0 - \beta_i^T (1_k \cdot R_0)$

$$E[R_i] = a_i + \beta_i^T E(R_F) \Rightarrow E[R_i] = R_0 + \beta_i^T E[R_F - 1_k \cdot R_0]$$

If APT holds

$$\mu_j = R_0 + \sum_{k=1}^{K} \beta_{jk} (\mu_{F_k} - R_0) \quad (2)$$

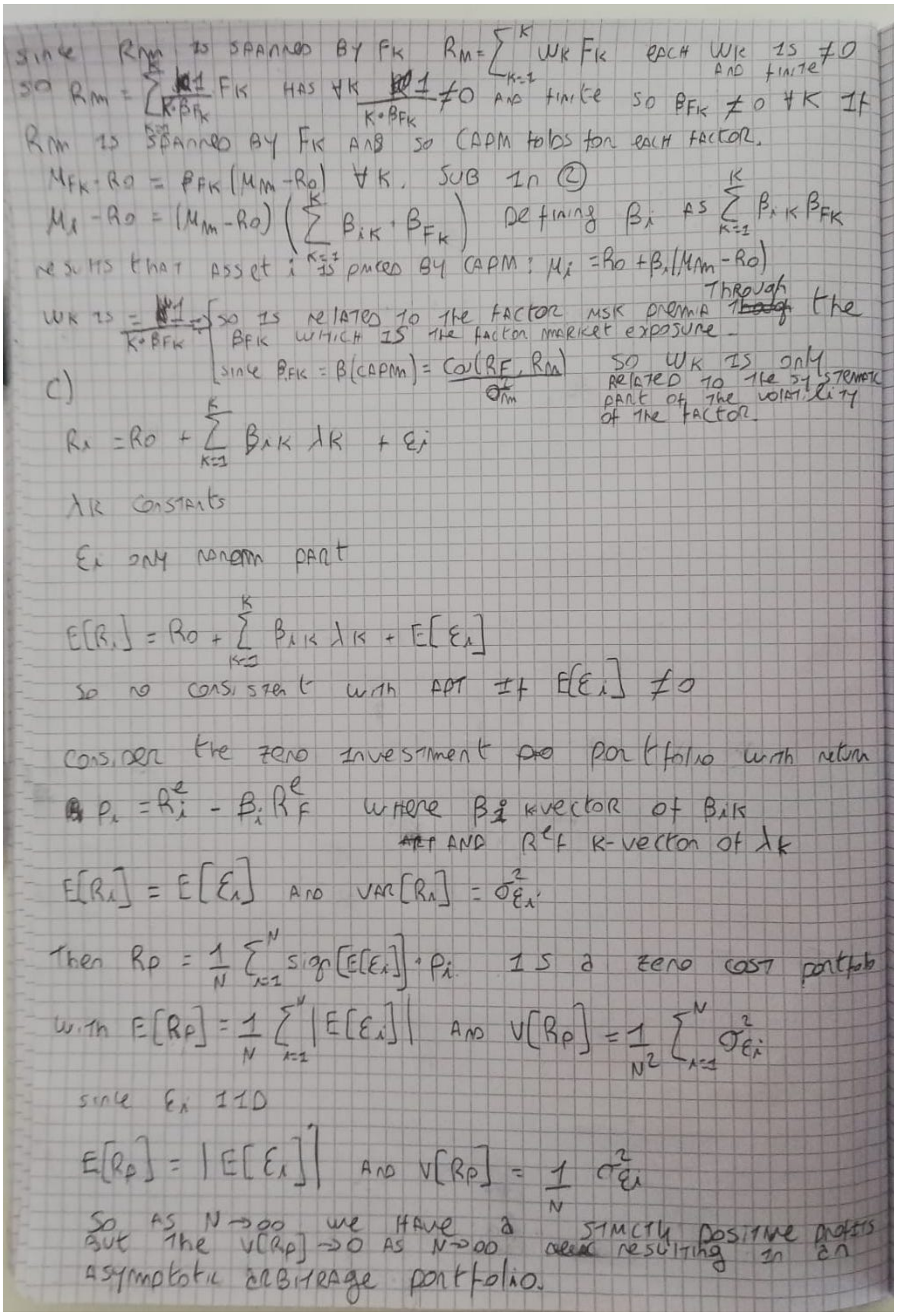
If CAPM hold

$$\mu_{F_k} = \cancel{R_0 (\mu_m - R_0} \; R_0 + \beta_{F_k} (\mu_m - R_0) \qquad \mu_m = E[R_m]$$

$$\text{So } \frac{\mu_{F_k} - R_0}{\beta_{F_k}} = (\mu_m - R_0) \Rightarrow \sum_{k=1}^{K} \frac{1}{\beta_{F_k}} (\mu_{F_k} - R_0) = \sum_{k=1}^{K} (\mu_m - R_0)$$

$$\sum_{k=1}^{K} \frac{1}{\beta_{F_k}} \cancel{\mu} F_k = \cancel{w} K \cdot R_m \to R_m = \sum_{k=1}^{K} \frac{\cancel{w} \cdot 1}{K \cdot \beta_{F_k}} F_k$$

Figure 1: Exercise 1a



since R_m is spanned by F_K $R_M = \sum_{K=1}^{K} W_K F_K$ each W_K is $\neq 0$ and finite

so $R_m = \sum_{K=1}^{K} \frac{1}{K \cdot \beta_{F_K}} F_K$ has $\forall K$ $\frac{1}{K \cdot \beta_{F_K}} \neq 0$ and finite so $\beta_{F_K} \neq 0$ $\forall K$ if R_m is spanned by F_K and so CAPM holds for each factor.

$\mu_{F_K} - R_0 = \beta_{F_K}(\mu_m - R_0)$ $\forall K$. sub in ②

$\mu_i - R_0 = (\mu_m - R_0)\left(\sum_{K=1}^{K} \beta_{iK} \cdot \beta_{F_K}\right)$ defining β_i as $\sum_{K=1}^{K} \beta_{iK}\beta_{FK}$

results that asset i is priced by CAPM: $\mu_i = R_0 + \beta_i(\mu_m - R_0)$

W_K is $= \frac{1}{K \cdot \beta_{F_K}} \Rightarrow$ so is related to the factor risk premia through the β_{F_K} which is the factor market exposure.

since $\beta_{F_K} = \beta(CAPM) = \frac{Cov(R_F, R_m)}{\sigma^2_m}$ so W_K is only related to the systematic part of the volatility of the factor.

c)

$$R_i = R_0 + \sum_{K=1}^{K} \beta_{iK} \lambda_K + \varepsilon_i$$

λ_K constants

ε_i only random part

$$E[R_i] = R_0 + \sum_{K=1}^{K} \beta_{iK}\lambda_K + E[\varepsilon_i]$$

so no consistent with APT if $E[\varepsilon_i] \neq 0$

Consider the zero investment ~~po~~ portfolio with return

$P_i = R^e_i - \beta_i R^e_F$ where β_i K-vector of β_{iK} and R^e_F K-vector of λ_K

$E[R_i] = E[\varepsilon_i]$ and $VAR[R_i] = \sigma^2_{\varepsilon_i}$

Then $R_P = \frac{1}{N}\sum_{i=1}^{N} \text{sign}\left[E[\varepsilon_i]\right] \cdot P_i$ is a zero cost portfolio

with $E[R_P] = \frac{1}{N}\sum_{i=1}^{N} \left|E[\varepsilon_i]\right|$ and $V[R_P] = \frac{1}{N^2}\sum_{i=1}^{N} \sigma^2_{\varepsilon_i}$

since ε_i IID

$E[R_P] = \left|E[\varepsilon_i]\right|$ and $V[R_P] = \frac{1}{N}\sigma^2_{\varepsilon_i}$

So as $N \to \infty$ we have a strictly positive profits but the $V[R_P] \to 0$ as $N \to \infty$ resulting in an asymptotic arbitrage portfolio.

Figure 2: exercise 1b

2 Exercise 2

a

We imported the data and we deleted stocks with less than 240 observations. In fact, we checked the number of stocks after the cancellation and we have that

```
We have deleted 3479 and we are left with 639 stocks.
```

In particular, for every stock we have 240 observations (20 years * 12 months for year). See Python code for details.

b

We estimated the market meta for each stocks; so we found 639 betas. The first 5 are:

```
              beta
10145.0   1.200190
10294.0   0.758752
10308.0   0.443625
```

```
10516.0  0.497710
10517.0  0.642811
```

After that, we added a new column for the betas in the DataFrame:

```
                      ret    shrout      prc     beta
permno  date
10145.0 2000-01 -0.167931  789233.0  48.0000  1.20019
        2000-02  0.006510  795134.0  48.1250  1.20019
        2000-03  0.094805  796591.0  52.6875  1.20019
        2000-04  0.062871  796591.0  56.0000  1.20019
        2000-05 -0.020089  798161.0  54.6875  1.20019
```

Then, we sorted stocks by beta into 10 decile portfolios:

```
                 decile
10145.0  (1.151, 1.296]
10294.0  (0.744, 0.882]
10308.0  (0.425, 0.607]
10516.0  (0.425, 0.607]
10517.0  (0.607, 0.744]
```

At this point we should have ten portfolios, so we checked it. We calculated them and print the result:

```
We have 10 portfolios.
```

For each of these portfolios, we had to calculate the equal weighted average return. For theory, we know that this means that every portfolio has the same weight. So, in order to do that, we used the mean.
We also computed the beta of the portfolio excess return. Our results, which are obviously ten, are:

	ew_ret	avg_beta
0	0.010578	0.285315
1	0.011674	0.515390
2	0.011796	0.685130
3	0.012387	0.818324
4	0.012547	0.944647
5	0.011389	1.072443
6	0.011606	1.221790
7	0.013521	1.361476
8	0.012934	1.539029
9	0.013058	1.975603

Next, we plotted our portfolio returns versus their beta (See Figure 3). In Figure 4 we also plotted the CAPM, to understand the relation between the two lines.

Both lines try to fit individual asset risk premiums as functions of asset risk (beta). From Figure 4 we can see that the slope of the EW line is flatter than the SML line. So, the slope coefficient of the EW line is smaller then the slope coefficient of the SML, which represents the mean excess return on the market portfolio.
The fact that the slope of our EW portfolios is smaller than the SML ones

does not indicate that they are not sufficiently profitable portfolios. In fact, the SML provides the required rate of return necessary to compensate investors for risk as well as the time value of money and, for each beta, our EW portfolios returns are above the SML.

In Figure 4 we can observe that the SML does not pass through the EW line, even if the SML slope is relatively larger, because the risk free ratio is low. A risk free ratio (i.e. the SML intercept) near to zero compels the SML line to start from a really low point. Contrariwise, the EW line intercept (i.e. when beta is zero) is sufficiently high to leave itself above the SML line even if the slope is smaller.

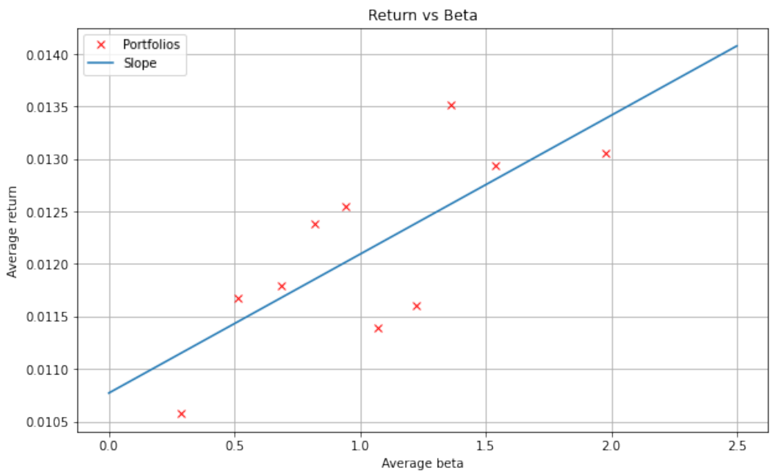


Figure 3: Plot of returns against betas

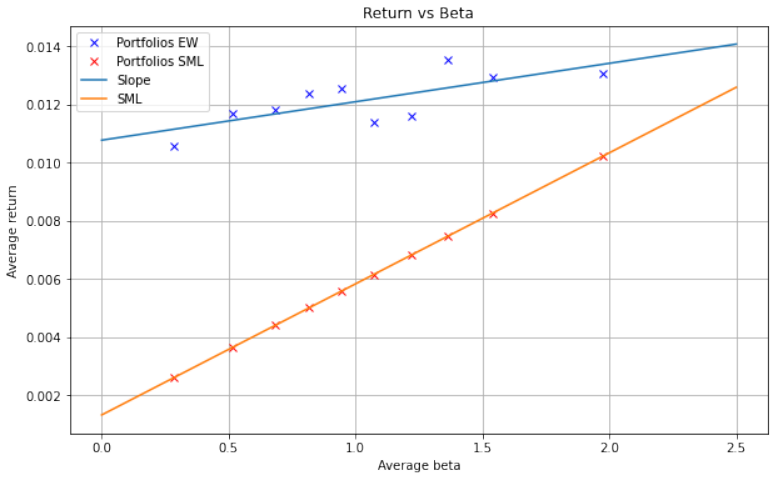


Figure 4: Plot of returns against betas with SML line

See Python code for details.

c

Note that in this exercise we interpret the "first sample" in the question as the the sample with period from 2000 to 2010 and take it as in-sample/training data.

Plot 5 shows that even though we can still draw a linear line through the points, we see a much weaker correlation between the average betas (taken from the first 10 years' sample i.e in-sample data) and the returns of the last 9 years than the ones observed in-sample. Furthermore, the downwards slope indicates a negative correlation between the betas obtained from training and the returns of the last-9-year's deciles.

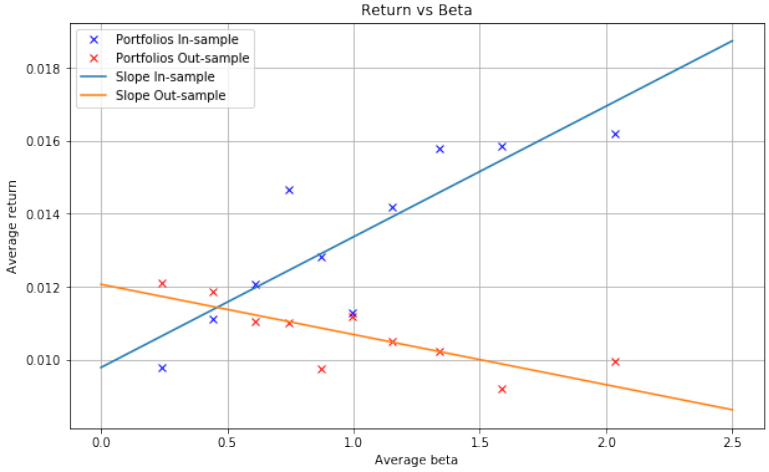


Figure 5: Plot of returns against betas for two samples

Plot 6 demonstrates the relationship between the average betas of the second sample (from 2011 to 2019) and the first one (from 2000 to 2010). We do see a correlation but it is not necessarily linear.

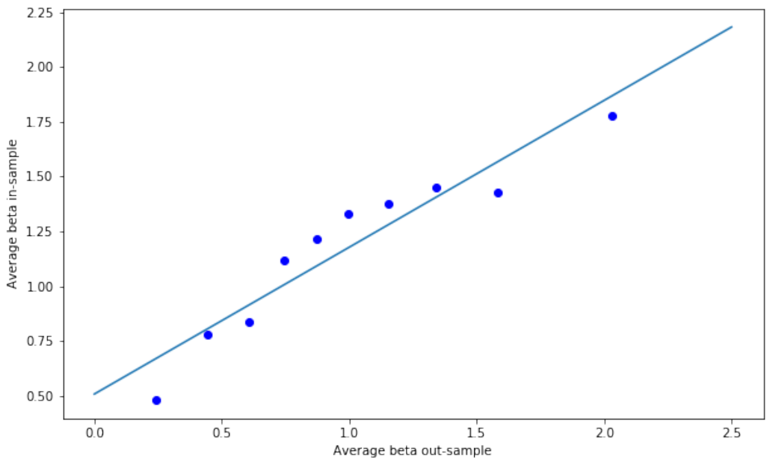


Figure 6: Plot of betas first sample against betas second sample