FIN 405: Investments (Due: 05/05/20)

Assignment #9

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Problem 1: Mean-var investing with fixed, lin-prop, and quadratic transaction costs

Initial dollar position X_0 and

$$\max_{X_1} \left[X_1 \mu - \frac{\gamma}{2} X_1^2 \sigma^2 - \left\{ \mathbb{1}_{X_1 \neq 0} b_0 + |X_1 - X_0| b_1 + \frac{1}{2}\lambda (X_1 - X_0)^2 \right\} \right] \quad (\cdot)$$

X_1 : terminal position

Total transaction cost paid for trading $X_1 - X_0$ are

$$TC = \underbrace{\mathbb{1}_{X_1 \neq X_0} b_0}_{\text{fixed cost}} + \underbrace{|X_1 - X_0| b}_{\text{linear proportional bid-ask spread component}} + \underbrace{\frac{1}{2}\lambda (X_1 - X_0)^2}_{\text{quadratic price impact component}}$$

a) To find the optimal strategy we have to maximize $(\cdot)$. In order to do that we find the FOC depending on X_1:

$$\frac{d}{dX_1}\left[X_1 \mu - \frac{\gamma}{2} X_1^2 \sigma^2 - \mathbb{1}_{X_1 \neq 0} b_0 - |X_1 - X_0| b_1 - \frac{1}{2}\lambda (X_1 - X_0)^2 \right] = 0$$

$$\mu - \gamma\sigma^2 X_1 - \underbrace{\frac{d}{dX_1}|X_1 - X_0|}\, b_1 - \lambda (X_1 - X_0) = 0$$

$$\frac{d|x|}{dx} = \frac{|x|}{x} \Rightarrow \frac{d}{dx_1}|X_1 - X_0| = \frac{d}{d(X_1 - X_0)}|X_1 - X_0| = \frac{|X_1 - X_0|}{(X_1 - X_0)}$$

$$\mu - \gamma\sigma^2 X_1 - b_1 \frac{|X_1 - X_0|}{X_1 - X_0} - \lambda (X_1 - X_0) = 0$$

$$X_1 (\gamma\sigma^2 - \lambda) = \mu + \lambda X_0 - b_1 \frac{|X_1 - X_0|}{X_1 - X_0} \Rightarrow X_1 = \frac{\mu + \lambda X_0 - b_1 \frac{|X_1 - X_0|}{X_1 - X_0}}{\gamma\sigma^2 + \lambda}$$

$$X_1 = \begin{cases} \dfrac{\mu + \lambda X_0 - b_1}{\gamma\sigma^2 + \lambda} & X_1 \geq X_0 \\[2ex] \dfrac{\mu + \lambda X_0 + b_1}{\gamma\sigma^2 + \lambda} & X_1 < X_0 \end{cases} \quad (*)$$

From the lecture notes we know that, for a one period linear proportional model

$$\max_x \{ x^T \mu - h\, |\Delta x|^T b - \frac{\gamma}{2} \underbrace{x^T \Sigma x}_{\text{one asset } \sigma^2 x^2} \}$$

the optimal position is given by

$$x_{t+1}^* = \begin{cases} (\gamma\Sigma)^{-1}(\mu - hb) & \text{if sign } |\Delta x| > 0 \\ (\gamma\Sigma)^{-1}(\mu + hb) & \text{" " } < 0 \\ x_t & \text{else} \end{cases}$$

and that the non trade region is when $(\gamma\Sigma)^{-1}(\mu - hb) < x_t < (\gamma\Sigma)^{-1}(\mu + hb)$

In our case $X_t = X_0$, $X_{t+1} = X_1$ and we are not in the case where the cost are only linear proportional

$\Rightarrow$ The trading strategy can be optimal only if $X_0 \notin \underbrace{[\underline{X}', \overline{X}']}$

for a certain interval defined by $\underline{X}', \overline{X}'$

In our model we also have to consider a fixed cost b_0.
The goal of our strategy is to obtain a gain which is sufficently large to compensate the costs.
Also for this fixed costs we can identify a non trading interval (slides):

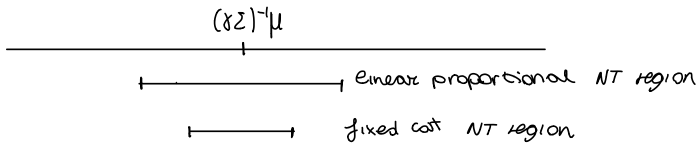
$$\max_x \left\{ x^T\mu - \lambda\, \mathbb{1}^T_{\Delta x \neq 0}\, b - \frac{\gamma}{2} x^T \Sigma x \right\} \qquad x^*_{t+1} = \begin{cases} (\gamma\Sigma)^{-1}\mu & \text{if } \frac{1}{2}\mu^T(\gamma\Sigma)^{-1}\mu - \lambda b > x^T\mu - \frac{\gamma}{2} x_t^T \Sigma x_t \\ x_t & \text{else} \end{cases}$$

$$\text{NT if} \quad (\gamma\Sigma)^{-1}\mu - \sqrt{\frac{2\lambda b}{\gamma\Sigma}} \leq x_t \leq (\gamma\Sigma)^{-1}\mu + \sqrt{\frac{2\lambda b}{\gamma\Sigma}}$$

In our model trading is optimal only if $x_0 \notin \underbrace{[\underline{x}'', \overline{x}'']}$
for a certain interval defined by $\underline{x}'', \overline{x}''$

$\Rightarrow$ We have identified 2 NT regions $[\underline{x}', \overline{x}']$, $[\underline{x}'', \overline{x}'']$.

From what we have shown from lecture slides, we have that both non trading regions contain $(\gamma\Sigma)^{-1}\mu$



The important thing is that the intersection of the two non-trading regions is still an interval, because they contain $(\gamma\Sigma)^{-1}\mu$, which in our case is $(\gamma\sigma^2)^{-1}\mu$.

$\Rightarrow$ The NT region is defined by the interval $[\underline{x}', \overline{x}'] \cap [\underline{x}'', \overline{x}'']$.
The minimum between $\underline{x}'$ and $\underline{x}''$ is $\underline{x}$, the same for the upper bound

$$\Rightarrow \text{NT} = [\underline{x}, \overline{x}] \quad \text{where} \quad \begin{aligned} \underline{x} &= \min(\underline{x}', \underline{x}'') \\ \overline{x} &= \max(\overline{x}', \overline{x}'') \end{aligned}$$

Moreover, we have that if $x \in NT \Rightarrow$ the optimal is given by $x_1 = x_0$.
This can be easily see in the lecture notes, where both for fixed costs and linear proportional costs, we have that $x^*_{t+1} = x_t$ iff we are in the NT region (see •). In our model $x^*_{t+1} = x_1$ and $x_t = x_0$ $\Rightarrow$ $x_1 = x_0$ if $x_0 \in NT$

If we are in the trading region there is optimal to trade toward a particular aim portfolio at a specific trading speed.

Considering the slides' theoretical model the speed is defined as:
$$\tau = \frac{1}{1 + \lambda\lambda/\gamma}$$

Then, in our model $$\tau = \frac{1}{1 + \frac{1 \cdot \lambda}{\gamma\sigma^2}}$$

In fact : $x_1\mu - \frac{\gamma}{2}\sigma^2 x_1^2 - \left\{ 1 \cdot \mathbb{1}_{x_1 \neq 0}\, b_0 + 1 \cdot |x_1 - x_0| b_1 + \frac{1}{2}\lambda (x_1 - x_0)^2 \right\}$

$$\Rightarrow \quad \Upsilon = \frac{\gamma\sigma^2}{\gamma\sigma^2 + \lambda}$$

Show that Υ is such that $X_1 = \Upsilon\, aim + (1-\Upsilon) X_0$ (for a certain aim)

From (*) $\quad X_1 = \frac{1}{\gamma\sigma^2+\lambda}\left[\mu - \mathrm{sgn}\,|X_1 - X_0|\, b_1 + \lambda X_0\right]$

$$= \underbrace{\frac{\mu - \mathrm{sgn}\,|X_1-X_0|\, b_1}{\gamma\sigma^2+\lambda}}_{\Upsilon\, aim} + \underbrace{\frac{\lambda}{\gamma\sigma^2+\lambda}}_{(1-\Upsilon)} X_0$$

$$\Rightarrow \Upsilon\, aim = \frac{\mu - \mathrm{sgn}\,|X_1-X_0|\, b_1}{\gamma\sigma^2+\lambda}$$

$$aim = \frac{\mu - \mathrm{sgn}\,|X_1-X_0|\, b_1}{\cancel{\gamma\sigma^2+\lambda}} \cdot \frac{\cancel{\gamma\sigma^2+\lambda}}{\gamma\sigma^2} \quad \Rightarrow \quad aim = \frac{\mu - \mathrm{sgn}\,|X_1-X_0|\, b_1}{\gamma\sigma^2}$$

b) Explain how the strategy changes

- When you turn off fixed costs ($b_0 = 0$): if trading is optimal, nothing changes in the strategy (*).
  However, the NT region may change. In particular, if the investor has no fixed costs, we can intuitively conclude that his/her trading region is larger (NT region smaller).

- When you turn off linear proportional costs ($b_1 = 0$): In this case, not only we influence the NT region, but also the trading strategy, when trading is optimal.
  If the investor is not subject to lin. prop. costs, it is likely that she/he will trade more.

- When you turn off linear proportional costs and fixed costs ($b_0 = b_1 = 0$): also in this case, both the NT region and the trading strategy (when trading is optimal) may change. The NT region shrink and also for this case, the investor is trading more aggressively.

- When you turn-off quadratic costs ($\lambda = 0$): when you cut off quadratic costs, as for the other costs, the NT region may change and the investor is trading more aggressively.
  The investor has bigger opportunities to better off trading.

Problem 2

Assignment 9 PS 2

Initial dollar position X_0 and seeks the vector of terminal position X_1 so as to maximize the mean-variance objective function

$$\max_{X_1} R_f + X_1^T(\mu - R_f) - \frac{\gamma}{2} X_1^T \Sigma X_1 - |X_1 - X_0|^T b$$

$R_0 = 2\%$

	μ_i	σ_i	ρ_{12}	b_i
Asset 1	5%	15%	30%	3%
	15%	25%		3%

a) Solve for the optimal portfolio when there is only a risky asset and R_0

Since this is a model with linear proportional costs we have two scenarios:

1) $X_1 - X_0 > 0$ so optimal to increase position in risky asset

In this case

$$\max_{x_1} R_f + x_1(\mu - R_f) - \frac{\gamma}{2} \Sigma x_1^2 - (x_1 - x_0) b$$

FOCs

$$0 = \mu - R_f - b - \gamma \Sigma X_1$$

$$X_{1B}^* = (\gamma \Sigma)^{-1}(\mu - R_f - b)$$

So we increase our position if X_1^* respect the initial condition $(X_1 - X_0 > 0)$ under a long position in the risky asset.

So we will increase our position to X_1^* as long as $X_0 < X_{1B}^*$ so

$$X_0 < X_{1B}^* \rightarrow X_0 < (\gamma \Sigma)^{-1}(\mu - R_f - b)$$

If $X_0 > X_{1B}^*$ no increase in the position of the risky asset

2) $X_1 - X_0 < 0$ so it's optimal to decrease position in the risky asset.

So

$$\max_{x_1} R_f + x_1(\mu - R_f) - \frac{\gamma}{2} \Sigma x_1^2 - (x_0 - x_1) b$$

focs

$$0 = \mu - R_f + b - \gamma \Sigma x_1$$

$$x_{1S}^* = (\gamma\Sigma)^{-1} (\mu - R_f + b)$$

optimal to reduce position if $x_0 > x_{1S}^*$

so

$$x_0 > (\gamma\Sigma)^{-1} (\mu - R_f + b)$$

so optimal to ~~#~~ do not reduce position if $x_0 < x_{1S}^*$

so ~~Be~~ if $x_{1B}^* < x_0 < x_{1S}^*$ optimal to not change our initial position

$$x_1^* = \begin{cases} (\gamma\Sigma)^{-1}(\mu - R_f - b) & \text{if } x_0 < x_{1B}^* \\ (\gamma\Sigma)^{-1}(\mu - R_f + b) & \text{if } x_0 > x_{1S}^* \\ x_0 & \text{else} \end{cases}$$

so no trade interval:

$$(\gamma\Sigma)^{-1}(\mu - R_f - b) \leq x_0 < (\gamma\Sigma)^{-1}(\mu - R_f + b)$$

b) optimal portfolio where there are two MSRY assets

3 possible actions buy sell nothing for each MSRY assets.

1° region investor trade both assets.

~~so~~ if define s as the strategy vector

$$s = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\max_{x_1} \; R_f + x_1^T(\mu - R_f) - \frac{\gamma}{2} x_1^T \Sigma x_1 - (x_1 - x_0)^T (b \circ s)$$

$b \circ s$ is the element-wise product

focs

$$0 = \mu - R_f - \gamma \Sigma x_1 - b \circ s$$

$$x_1^*(s) = (\gamma\Sigma)^{-1} (\mu - R_f - b \circ s)$$

So if it is optimal to buy both

$$X_{LL} = X_1^*\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{pmatrix}X_{1LL}\\X_{2LL}\end{pmatrix}$$

is optimal to increase the position to X_{LL}
if $X_0 \leq X_{LL}$

so we will have the first region as

$$LL = \{X_0 \in \mathbb{R}^2 \mid X_{10} \leq X_{1LL} \text{ and } X_{20} \leq X_{2LL}\}$$

2° Region
Buy asset 1 sell asset 2

$$X_{LS} = X_1^*\left(\begin{bmatrix}1\\-1\end{bmatrix}\right)$$

This strategy will be optimal only if $X_{10} \leq X_{1LS}$ and $X_{20} \geq X_{2LS}$

$$LS = \{X_0 \in \mathbb{R}^2 \mid X_{10} \leq X_{1BS} \text{ and } X_{20} \geq X_{2BS}\}$$

3° Region

$$X_{SL} = X_1^*\left(\begin{bmatrix}-1\\1\end{bmatrix}\right)$$

$$SL = \{X_0 \in \mathbb{R}^2 \mid X_{10} \geq X_{1SL} \text{ and } X_{20} \leq X_{2SL}\}$$

4° Region

$$X_{SS} = X_1^*\left(\begin{bmatrix}-1\\-1\end{bmatrix}\right)$$

$$SS = \{X_0 \in \mathbb{R}^2 \mid X_{10} \geq X_{1SS} \text{ and } X_{20} \geq X_{2SS}\}$$

5° Region

Buy Asset 1 more

and asset 2 unchanged $X_{21} = X_{20}$

problem : using $\Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$

$$\max_{X_{11}} R_f + X_{11}(\mu_1 - R_f - b) + X_{20}(\mu_1 - R_f) - \frac{\gamma}{2}\left(\sigma_1^2 X_{11}^2 + \sigma_2^2 X_{20}^2 + 2\rho\sigma_1\sigma_2 X_{11} X_{20}\right)$$

FOC

$$0 = \mu_1 - R_f - b - \gamma\sigma_1^2 X_{11} - \rho\sigma_1\sigma_2 X_{20}$$

$$X_{1LN} = \frac{\mu_1 - R_f - b}{\gamma\sigma_1^2} - \frac{\rho\sigma_2}{\gamma\sigma_1} X_{20}$$

It is optimal to keep X_{20} constant and purchase more X_1 if

$$X_{2LL} < X_{20} \leq X_{2LS}$$

so

$$LN = \{X \in \mathbb{R}^2 \mid X_{2LL} \leq X_{20} \leq X_{2LS} \text{ and } X_{10} \leq X_{1LN}\}$$

6°

Sell asset 1

and asset 2 unchanged $\Rightarrow$ $X_{2SL} \leq X_{20} \leq X_{2SS}$

using the above reasoning

$$X_{1SN} = \frac{\mu_1 - R_f + b}{\gamma\sigma_1^2} - \frac{\rho\sigma_2}{\gamma\sigma_1} X_{20}$$

$$SN = \{X \in \mathbb{R}^2 \mid X_{2SL} \leq X_{20} \leq X_{2SS} \text{ and } X_{10} \geq X_{1SN}\}$$

7°

keep unchanged asset 1 $X_{1NL} = X_{10}$

Buy more asset 2 $X_{1LL} \leq X_{10} \leq X_{1SL}$

$$X_{2NL} = \frac{\mu_2 - R_f - b}{\gamma\sigma_2^2} - \frac{\rho\sigma_1}{\gamma\sigma_2} X_{10}$$

$$NL = \{X \in \mathbb{R}^2 \mid X_{1LL} \leq X_{10} \leq X_{1SL} \text{ and } X_{20} \leq X_{2NL}\}$$

8°)

keep unchanged asset 1 $x_{1N} = x_{10}$

$x_{1LS} \le x_{10} \le x_{1SS}$

sell asset 2

$$x_{2NS} = \frac{\mu_2 - R_f + b}{\gamma \sigma_2^2} - \frac{\rho \sigma_1}{\gamma \sigma_2} x_{10}$$

$$NS = \{ X_0 \in \mathbb{R}^2 \mid X_{1LS} \le X_{10} < X_{1SS} \text{ AND } X_{20} \ge X_{2NS} \}$$

9°)

No trade region
is the remaining part of the plan

$$NN = \{ X_0 \in \mathbb{R}^2 \setminus (LL \cup LS \cup SL \cup SS \cup LN \cup SN \cup NL \cup NS$$

c) How does the shape of the no-trade region change as you increase correlation?

If $\rho = \emptyset$ x_{1LN} and x_{1SN} are constant function since the do not depend on x_{20}
same for x_{2NL} and x_{2NS} that does not depend on x_{10}.
In particular the optimal strategy in the 2 assets model is linked strongly to the strategy in 1 asset model.
So we have

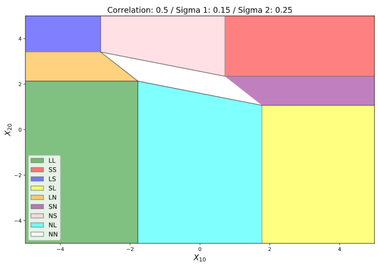
$$\left.\begin{aligned} x_{1LL} &= x_{1LS} = x_{1B}^* \\ x_{2LL} &= x_{2SL} = x_{2B}^* \\ x_{1SL} &= x_{1SS} = x_{1S}^* \\ x_{2LS} &= x_{2SS} = x_{2S}^* \end{aligned}\right\| \Rightarrow$$

so the no trading area is a rectangle

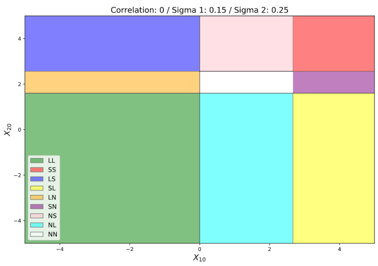
Moreover, when $\rho = \emptyset$, the optimal strategy of an asset does not depend on the current position in the other asset

(c: graphs)

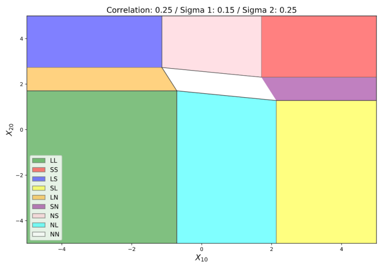
From Figure 1 is clear that the shape of the not trade zone is a perfect rectangle when $\rho = 0$ and hence when the the optimal strategy of an asset does not depend on the other. While This is no longer true when $\rho \neq 0$. The reason is that the increase in portfolio variance due to an increase in the position in asset 1 depends on the current position on asset 2.
Moreover, the increase in portfolio variance, due the increase in the position of one asset is higher, the higher the current position on asset 2, and the higher the correlation between both assets. This result that when $\rho \neq 0$ the shape of the not trade zone will change depending on ρ. as a matter of fact, we do not have anymore a rectangle but a parallelogram.
the same conclusion is valid if we consider an increase in volatility of asset how we can see from 2



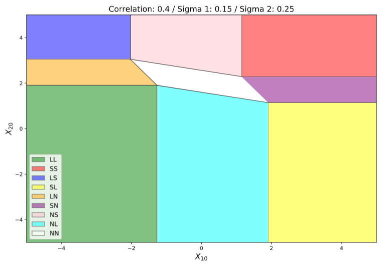
(a) Initial parameters



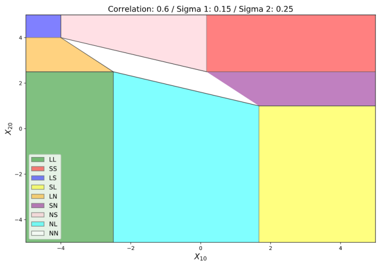
(b) correlation:0, sigma1:0.15, sigma2:0.25



(c) correlation:0.25, sigma1:0.15, sigma2:0.25

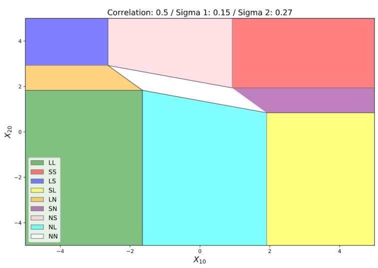


(d) correlation:0.4, sigma1:0.15, sigma2:0.25

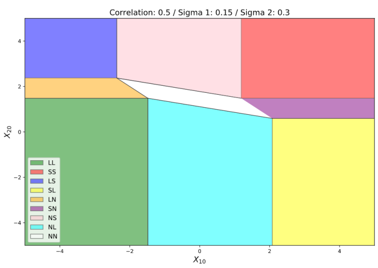


(e) correlation:0.6, sigma1:0.15, sigma2:0.25

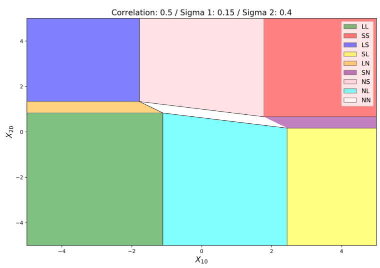
Figure 1: Impact of an increase in correlation, X_{10} on x axis and X_{20} on y axis



(a) correlation:0.5, sigma1:0.15, sigma2:0.27



(b) correlation:0.5, sigma1:0.15, sigma2:0.30



(c) correlation:0.5, sigma1:0.15, sigma2:0.40

Figure 2: Impact of volatility of asset 2, X_{10} on x axis and X_{20} on y axis

Problem 3: Quadratic transaction costs

$$V(k, n_{k-1}) = max_{n_k, t \leq k} E\left[\sum_{t=0}^{T} \rho^t \{n_t \mu - \frac{\lambda}{2}(n_t - n_{t-1})^2 - \frac{\gamma}{2} n_t^2 \sigma^2\}\right]$$

(a)

Starting at time T, we have that:

$$\begin{aligned}
V(T, n_{T-1}) &= max_{n_T} \left(n_T \mu - \frac{\lambda}{2}(n_T - n_{T-1})^2 - \frac{\gamma}{2} n_T^2 \sigma^2 \right) \\
&= max_{n_T} \left(n_T \mu - \frac{\lambda}{2} n_T^2 + \lambda n_T n_{T-1} - \frac{\lambda}{2} n_{T-1}^2 - \frac{\gamma}{2} n_T^2 \sigma^2 \right) \\
&= max_{n_T} \left(\frac{-1}{2} n_T^2 (\lambda + \gamma \sigma^2) + n_T(\mu + \lambda n_{T-1}) - \frac{1}{2} n_{T-1}^2 \right)
\end{aligned}$$

Solving the FOC w.r.t n_T, we get:

$$\begin{aligned}
\frac{\partial V}{\partial n_T} &= -n_T(\lambda + \gamma\sigma^2) + (\mu + \lambda n_{T-1}) = 0 \\
\Leftrightarrow n_T &= \frac{\mu + \lambda n_{T-1}}{\lambda + \gamma \sigma^2}
\end{aligned}$$

So we obtain:

$$\begin{aligned}
V(T, n_{T-1}) &= \frac{\mu + \lambda n_{T-1}}{\lambda + \gamma\sigma^2}\mu - \frac{\lambda + \gamma\sigma^2}{2}(\frac{\mu + \lambda n_{T-1}}{\lambda + \gamma\sigma^2})^2 + \lambda \frac{\mu + \lambda n_{T-1}}{\lambda + \gamma\sigma^2} n_{T-1} - \frac{\lambda}{2} n_{T-1}^2 \\
&= \frac{\mu^2}{\lambda + \gamma\sigma^2} + \frac{\lambda\mu}{\lambda + \gamma\sigma^2} n_{T-1} - \frac{1}{2}\frac{\mu^2}{\lambda + \gamma\sigma^2} - \frac{\lambda\mu}{\lambda + \gamma\sigma^2} n_{T-1} - \frac{1}{2}\frac{\lambda^2}{\lambda + \gamma\sigma^2} n_{T-1}^2 + \\
&\frac{\mu\lambda n_{T-1}}{\lambda + \gamma\sigma^2} + \frac{\lambda^2}{\lambda + \gamma\sigma^2} n_{T-1}^2 - \frac{\lambda}{2} n_{T-1}^2 \\
&= \frac{1}{2}\frac{\mu^2}{\lambda + \gamma\sigma^2} + \frac{\mu\lambda}{\lambda + \gamma\sigma^2} n_{T-1} + (\frac{1}{2}\frac{\lambda^2}{\lambda + \gamma\sigma^2} - \frac{\lambda}{2}) n_{T-1}^2 \\
&= -\frac{1}{2}(\lambda - \frac{\lambda^2}{\lambda + \gamma\sigma^2}) n_{T-1}^2 + \frac{\mu\lambda}{\lambda + \gamma\sigma^2} n_{T-1} + \frac{1}{2}\frac{\mu^2}{\lambda + \gamma\sigma^2}
\end{aligned}$$

Matching coefficients, we get the following:

$$\begin{aligned}
Q_T &= \lambda - \frac{\lambda^2}{\lambda + \gamma\sigma^2} \\
q_T &= \frac{\mu\lambda}{\lambda + \gamma\sigma^2} \\
c_T &= \frac{1}{2}\frac{\mu^2}{\lambda + \gamma\sigma^2}
\end{aligned}$$

(b)

Assume now that for $t < T$ we have $V(t+1,n) = \frac{-1}{2}n^2 Q_{t+1} + nq_{t+1} + c_{t+1}$, and that the Bellman equation is:

$$V(t, n_{t-1}) = max_{n_t}\{n_t\mu - \frac{\lambda}{2}(n_t - n_{t-1})^2 - \frac{\gamma}{2}n_t^2\sigma^2 + \rho E_t[V(t+1, n_t)]\}$$

Then the Bellman equation can be rewritten as:

$$\begin{aligned}
V(t, n_{t-1}) &= max_{n_t}\{n_t\mu - \frac{\lambda}{2}(n_t - n_{t-1})^2 - \frac{\gamma}{2}n_t^2\sigma^2 + \rho E_t[\frac{-1}{2}n_t^2 Q_{t+1} + n_t q_{t+1} + c_{t+1}]\} \\
&= max_{n_t}(n_t\mu - \frac{\lambda}{2}n_t^2 + \lambda n_{t-1}n_t - \frac{\lambda}{2}n_{t-1} - \frac{\gamma}{2}n_t^2\sigma^2 - \frac{1}{2}\rho n_t^2 E(Q_{t+1}) + \rho n_t E(q_{t+1}) + \rho E(c_{t+1})) \\
&= max_{n_t}(n_t(\mu + \lambda n_{t-1} + \rho E(q_{t+1})) - \frac{1}{2}n_t^2(\lambda + \gamma\sigma^2 + \rho E(Q_{t+1})) + \rho E(c_{t+1}) - \frac{\lambda}{2}n_{t-1}^2) \\
&= max_{n_t}(n_t(\mu + \lambda n_{t-1} + \rho q_{t+1}) - \frac{1}{2}n_t^2(\lambda + \gamma\sigma^2 + \rho Q_{t+1}) + \rho c_{t+1} - \frac{\lambda}{2}n_{t-1}^2)
\end{aligned}$$

Taking the FOC of the above w.r.t n_t, we get:

$$\begin{aligned}
&\mu + \lambda n_{t-1} + \rho q_{t+1} - n_t(\lambda + \gamma\sigma^2 + \rho Q_{t+1}) = 0 \\
&\Leftrightarrow n_t = \frac{\mu + \lambda n_{t-1} + \rho q_{t+1}}{\lambda + \gamma\sigma^2 + \rho Q_{t+1}} \\
&\Rightarrow V(t, n_{t-1}) = \frac{1}{2}\frac{(\mu + \lambda n_{t-1} + \rho q_{t+1})^2}{\lambda + \gamma\sigma^2 + \rho Q_{t+1}} + \rho c_{t+1} - \frac{\lambda}{2}n_{t-1}^2
\end{aligned}$$

Matching coefficients, we get the following recursive forms:

$$\begin{aligned}
Q_t &= \lambda - \frac{\lambda^2}{\lambda + \gamma\sigma^2 + \rho Q_{t+1}} \\
q_t &= \frac{\lambda(\mu + \rho q_{t+1})}{\mu + \gamma\sigma^2 + \rho Q_{t+1}} \\
c_t &= \frac{1}{2}\frac{(\mu + \rho q_{t+1})^2}{\lambda + \gamma\sigma^2 + \rho Q_{t+1}} + \rho c_{t+1}
\end{aligned}$$

(c)

According to the finding in b for the optimal value of n_t:

$$n_t^* = \frac{\mu + \lambda n_{t-1} + \rho q_{t+1}}{\lambda + \gamma\sigma^2 + \rho Q_{t+1}} = \tau_t aim_t + (1 - \tau_t)n_{t-1}$$

Expand expression of the optimal value, we get:

$$\frac{\mu + \lambda n_{t-1} + \rho q_{t+1}}{\lambda + \gamma\sigma^2 + \rho Q_{t+1}} = \frac{\mu + \rho q_{t+1}}{\lambda + \gamma\sigma^2 + \rho Q_{t+1}} + \frac{\lambda n_{t-1}}{\lambda + \gamma\sigma^2 + \rho Q_{t+1}}$$

$$= \frac{\gamma\sigma^2 + \rho Q_{t+1}}{\gamma\sigma^2 + \lambda + \rho Q_{t+1}} aim_t + \frac{\lambda n_{t-1}}{\lambda + \gamma\sigma^2 + \rho Q_{t+1}}$$

Where $aim_t = \frac{\mu + \rho q_{t+1}}{\gamma\sigma^2 + \rho Q_{t+1}}$ and $\tau_t = \frac{\gamma\sigma^2 + \rho Q_{t+1}}{\gamma\sigma^2 + \lambda + \rho Q_{t+1}}$.

(d)

We plot the optimal trading for trading every 30min and every 10min and expected cumulative cost of trading for trading every 30min and 10min and obtain the following Figure 3, Figure 4:

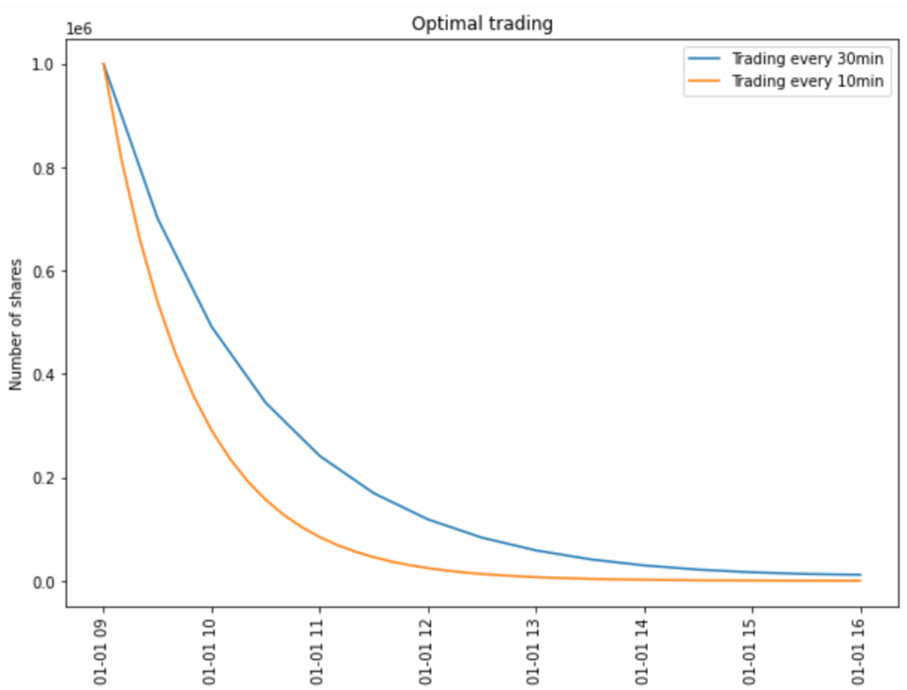


Figure 3: Optimal trading

(e)

We note that the model predicts that our optimal trading schedule does not depend on the realized price shocks, that is if the stock price goes up over the trading day or if it goes down does not affect our optimal trading rule. That does not seem very intuitive to me in reality, but in our model we can see that this is true, because the trading costs are the only one, which matters for our previous position n_{t-1} in function $V(t, n_{t-1})$ and they are the same no matter if the investor is buying or selling the stocks. Indeed, trading costs are $\frac{\lambda}{2}(n_t - n_{t-1})^2$, so it is the same if investor buys $n_t - n_{t-1}$ or if he sells $n_{t-1} - n_t$ stocks, because the sign does not matter. So trading schedule does not depend on the realized price shocks.

To change our optimal trading rule we could say that trading costs vary with different global economic conditions. For example if we have a lot of stocks with falling prices in the market, illiquidity problems could arise, so trading costs should be higher, when stock prices are falling.

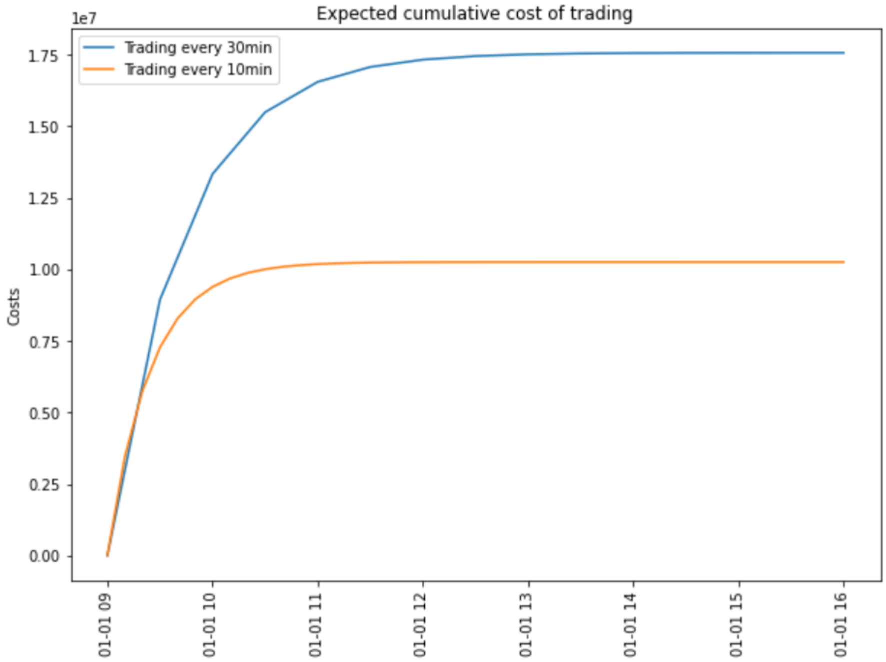
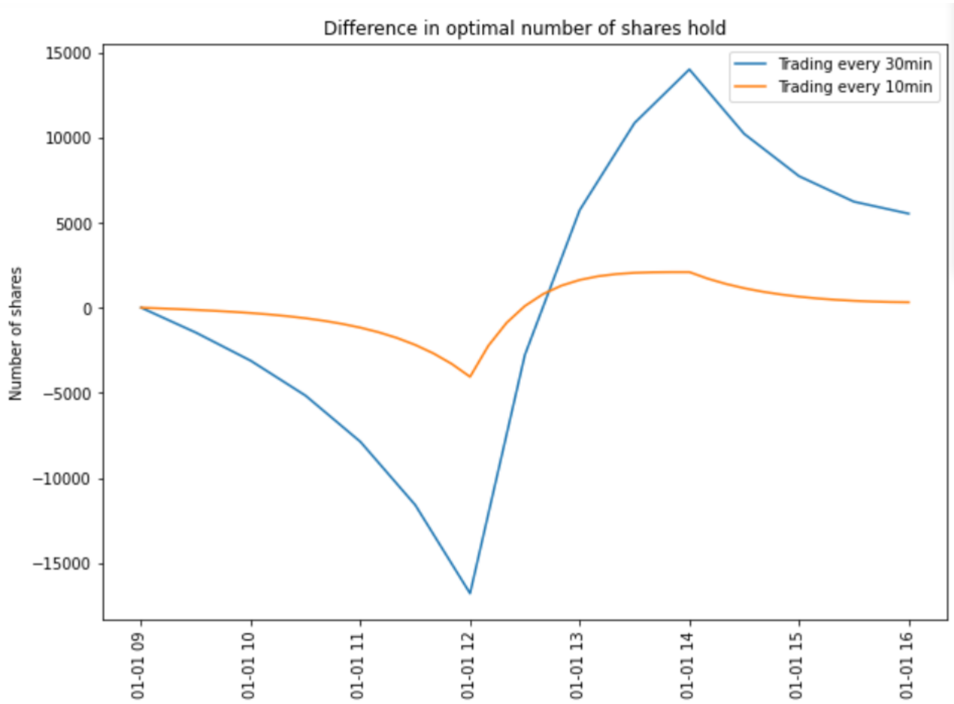


Figure 4: Expected cumulative cost of trading

Another example would be, if investors would have different risk aversion, when being rich or poor. So they wouldn't trade in the same way if they were poor or rich. Sometimes being less rich and holding more number of stock that the optimal target could mean bigger willingness to sell the stocks when their prices are falling.

(f)

If we assume that liquidity is lower during the lunch time ($12:00-14:00$), we could solve this problem by setting λ step constant, instead of constant as it was before. In this case we set λ to 2bps if the hour is between 12:00 and 14:00 and to 1bps otherwise. From the graph we can see that there is now more selling of stocks before the lunch time and more selling of stocks after the lunch time. It it intuitive that with this trading cost function, investors doesn't want to trade so much during the lunch break, because they are not willing to pay higher transaction costs. We obtain the following graph:


Difference in optimal number of shares hold
Trading every 30min
Trading every 10min
Number of shares
15000
10000
5000
0
−5000
−10000
−15000
01-01 09
01-01 10
01-01 11
01-01 12
01-01 13
01-01 14
01-01 15
01-01 16