# PROBLEM SET 3

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## 1. Efficient Portfolios

$$R_p = R_f + w^T (R - R_f \mathbb{1}) \qquad U(w) = \mu_p - \frac{\gamma}{2}\sigma_p^2$$

a) Show that $\mu_i - R_f = \gamma \operatorname{cov}[R_i, R_p]$ $\forall i = 1, \dots N$

Proof

$$U(w) = \mathbb{E}(R_p) - \frac{\gamma}{2} \operatorname{var}(R_p) =$$

$$= \mathbb{E}(R_f + w^T(R - R_f\mathbb{1})) - \frac{\gamma}{2}\operatorname{var}(R_f + w^T(R - R_f \mathbb{1})) \underset{\operatorname{var}(R_f)=0}{=}$$

$$= R_f + \mathbb{E}\left(\sum_{i=1}^N w_i (R_i - R_f)\right) - \frac{\gamma}{2}\underbrace{\operatorname{var}\left(\sum_{i=1}^N w_i R_i\right)}_{\sum_{i=1}^N w_i^2 \sigma_i^2 + 2\sum_{i=1}^N \sum_{\substack{j=1 \\ j\neq i}}^N \operatorname{Cov}(R_i, R_j) = \sum_{i=1}^N \sum_{j=1}^N 2 w_i w_j \operatorname{Cov}(R_i, R_j)} =$$

$\operatorname{Cov}(R_i, R_i) = \sigma_i^2$

$$= R_f + \sum_{i=1}^N w_i(\mu_i - R_f) - \frac{\gamma}{\cancel{2}} \sum_{i=1}^N \sum_{j=1}^N \cancel{2} w_i w_j \operatorname{Cov}(R_i, R_j)$$

First order condition:

$$(\mu_i - R_f) - \frac{\gamma}{\cancel{2}} \sum_{j=1}^N \cancel{2} w_j \operatorname{Cov}(R_i, R_j) = 0$$

$$\mu_i - R_f - \gamma \operatorname{Cov}\Big(R_i, \underbrace{\sum_{j=1}^N w_j R_j}_{R_p}\Big) = 0$$

$$\mu_i - R_f = \gamma \operatorname{Cov}(R_i, R_p) \qquad \square$$

b) Show that $\mu_i - R_f = \beta_{i,p}(\mu_p - R_f)$ $\qquad \beta_{i,p} = \dfrac{\operatorname{Cov}(R_i, R_p)}{\sigma_p^2}$

Proof

$\mu_i - R_f \underset{a)}{=} \gamma \operatorname{Cov}(R_i, R_p)$ and we want that $\mu_i - R_f = \dfrac{\operatorname{Cov}(R_i, R_p)}{\sigma_p^2}(\mu_p - R_f)$

So we want to show that $\gamma \operatorname{cov}(R_i, R_p) = \dfrac{\operatorname{Cov}(R_i, R_p)}{\sigma_p^2}(\mu_p - \mu_f)$

$\Rightarrow$ we have to prove $\left[\gamma = \dfrac{\mu_p - R_f}{\sigma_p^2}\right]$ (*)

From $\mu_p := R_f + w^T(\mu - R_f \mathbb{1}) \rightarrow u(w) = \mu_p - \frac{\gamma}{2}\sigma_p^2 =$
$= R_f + w^T(\mu - R_f\mathbb{1}) - \frac{\gamma}{2} w' \Sigma w$

FOC: $\mu - R_f \mathbb{1} - \gamma \Sigma w = 0 \rightarrow \gamma = \dfrac{\mu - R_f \mathbb{1}}{\Sigma w} = \dfrac{w'\mu - w' R_f \mathbb{1}}{w' \Sigma w} = \dfrac{\mu_p - R_f}{\sigma^2}$

efficient portfolio (for $\Sigma w = 0$); multiply for $w$ (for the second equality)

$\Rightarrow$ (*) is true $\qquad \square$

It would have been possible to proceed differently.

Proof 2: We start from a) $\mu_i - R_f = \gamma \, Cov(R_i, R_p) \quad \forall i$ (*)

$\Rightarrow$ with vectors:

$$\mu - R_f \mathbb{1} = \gamma \underset{\substack{\downarrow \\ \text{variance/covariance matrix}}}{\Sigma} w$$

Multiplying both sides for $w'$: $\underline{w'\mu} - w' R_f \mathbb{1} = \gamma \, \underline{\underline{w'\Sigma w}}$

$$\underline{\mu_p} - R_f = \gamma \, \underline{\underline{\sigma_p^2}} \quad (**)$$

Divide (*) by (**)

$$\frac{\mu_i - R_f}{\mu_p - R_f} = \frac{\cancel{\gamma} \, Cov(R_i, R_p)}{\cancel{\gamma} \, \sigma_p^2}$$

$$\mu_i - R_f = \underbrace{\frac{Cov(R_i, R_p)}{\sigma_p^2}}_{B_{i,p}} (\mu_p - R_f) \quad \emptyset$$

c) Show that $\forall i$: $R_i = R_f + \beta_i (R_p - R_f) + \varepsilon_t$ where $cov(R_p, \varepsilon_i) = 0 \quad \mathbb{E}(\varepsilon_i) = 0$

Proof From b) $\mu_i - R_f = B_{ip}(\mu_p - R_f) \quad \pm R_i$

$$R_i = R_f - \mu_i + B_{i,p}(\mu_p - R_f) + R_i \pm B_{i,p} R_p$$

$$= R_f + B_{ip}(R_p - R_f) \underbrace{- \mu_i + R_i - B_{i,p} R_p + B_{i,p}\mu_p}_{\varepsilon_i = B_{ip}(\mu_p - R_p) - \mu_i + R_i}$$

$$\mathbb{E}(\varepsilon_i) = \mathbb{E}\left[B_{ip}(\mu_p - R_p) - \mu_i + R_i\right]$$

$$= B_{i,p}\left[\mu_p - \underbrace{\mathbb{E}(R_p)}_{\mu_p}\right] - \mu_i + \underbrace{\mathbb{E}(R_i)}_{\mu_i} = 0$$

$$Cov(R_p, \varepsilon_i) = Cov(R_p, B_{ip}(\mu_p - R_p) - \mu_i + R_i) =$$

$$= \cancel{B_{ip}\, Cov(R_p, \mu_p)} - B_{i,p}\, Cov(R_p, R_p) - \cancel{Cov(R_p, \mu_i)} + Cov(R_p, R_i)$$

$$= -B_{ip}\,\sigma_p^2 + Cov(R_p, R_i) = -\frac{Cov(R_p, R_i)}{\cancel{\sigma_p^2}}\cancel{\sigma_p^2} + Cov(R_p, R_i) = 0 \quad \emptyset$$

Proof 2

Otherwise we can start from a general regression with $B_0$ and prove that $B_0 = R_f$, $B_1 = B_{i,p}$:

$$R_i = B_0 + B_1 (R_p - R_f) + \varepsilon_i$$

• About $B_1$, we do not have any doubt because we know that the $\overset{B_1}{\gamma}$ regression coefficient is given by the covariance divided by the variance of the variable $R_p - R_f$:

$$B_1 = \frac{Cov(R_i, R_p - R_f)}{Var(R_p - R_f)} = \frac{Cov(R_i, R_p) - 0}{\sigma_p^2 - 0} = B_{i,p}$$

• For $\beta_0$, we would like to use the previous points, which work on $\mu_i$.
So, we have to take the expectation :

$$\mu_i = \beta_0 + \beta_{i,p}(\mu_p - R_f) + 0$$

$E(\varepsilon_i) = 0$

From b)

$$\mu_i - R_f + R_f \overset{b)}{=} \beta_{i,p}(\mu_p - R_f) + R_f$$

$$\Rightarrow \quad R_f + \beta_{ip}(\mu_p - R_f) = \mu_i = \beta_0 + \beta_{ip}(\mu_p - R_f)$$

$$R_f = \beta_0$$

$\boxtimes$

d) Show that $SR_p = \frac{\mu_p - R_f}{\sigma_p}$ $\forall$ mean-variance efficient portfolios.

<u>Proof</u> We want that $SR_p$ does not depend on $w$ (the composition of the portfolio)
$\Rightarrow$ we consider $w_a \neq w_b$ and whe show that $SR_a = SR_b$

$$SR_a = \frac{\mu_a - R_f}{\sigma_a}$$

$$SR_b = \frac{\mu_b - R_f}{\sigma_b} \quad \text{where} \quad \mu_b - R_f = \frac{\mu_a - R_f}{\sigma_a^2} Cov(R_a, R_b) = \frac{SR_a}{\sigma_a} Cov(R_a, R_b)$$

$$\Rightarrow SR_b = SR_a \cdot \frac{Cov(R_a, R_b)}{\sigma_a \sigma_b} = SR_a \cdot \rho_{a,b}$$

we have to show that $\rho_{ab} = 1$:

we know tha $\forall$ portfolio is a different combination of the same 2 portfolios (they are a kind of 'base')
$\Rightarrow$ they are all perfectly correlated $\Rightarrow$ $SR_a = SR_b$ $\forall$ $w_a \neq w_b$ $\boxtimes$

2) Show that only min var frontier portfolio can be rewritten as a convex combination of any two arbitrary min var frontier portfolios $w_a, w_b$ in the sense that $w = a w_a + (1-a) w_b$

Assume $w$ is not a min var frontier portfolio while $w_A$ and $w_B$ yes

$$w = a(w_a) + (1-a) w_b$$

$$w_a = C \cdot w_{TAN} + (1-C)\, w_{Rf}$$ from the two fund separation theorem

$w_{TAN}$ = tangency portfolio

$w_{Rf}$ = risk free asset

$C$ = weigh in each single asset

Same for $w_b$

$$w_b = D \cdot w_{TAN} + (1-D)\, w_{Rf}$$

so

$$w = a \cdot (C \cdot w_{TAN} + (1-C)\, w_{Rf}) + (1-a)\left[D \cdot w_{TAN} + (1-D)\, w_{Rf}\right]$$

$$= \left[a \cdot C + D(1-a)\right] w_{TAN} + \left[a(1-C) + (1-a)(1-D)\right] \cdot w_{Rf}$$

Since $a \cdot C + D(1-a) + a(1-C) + (1-a)(1-D) = 1$

$w$ is a convex combination of $w_{TAN}$ and $w_{RF}$ so for the two fund separation theorem is an portfolio on the minimum variance frontier but this goes in contraddiction to the initial. Hence only min var frontier portfolio can be written as the linear combination of two arbitrary minimum variance frontier portfolio.

$R_{min}$ global min-VAR portfolio

show

$$Cov(R, R_{min}) = VAR(R_{min})$$

Assume $R_p = wR + (1-w)R_{min}$

$$VAR(R_p) = w^2\sigma_R^2 + (1-w)^2\sigma^2_{min} + 2w(1-w)\,cov(R_{min}, R)$$

Minimize $VAR(R_p)$

foc wrt $w$: // relationship between $w$ that minimize $\sigma_p$ and parameters

$$2w\sigma_R^2 + (1-w)(-2)\sigma^2_{min} + 2cov(R_{min}, R) - 4w\,cov(R_{min}, R) = 0$$

However $VAR(R_p)$ is minimized when $= \sigma^2_{min}$ so when $w = 0$

Sub in the foc:

$$-2\sigma^2_{min} + 2cov(R_{min}, R) = 0$$

so $\sigma^2_{min} = cov(R_{min}, R)$