Assignment 7

1. **Portfolio choice with liabilities (20 points)**

   There are N risky assets available for investment. R is the vector of asset returns, μ is the vector of asset expected returns, Σ is the variance-covariance matrix of returns, and w is the vector of portfolio weights (fractions of wealth invested in the N risky assets). There is a riskless asset with a rate of return of R_0 and no borrowing or short-sale constraints ($w_0 = 1 - w'\mathbf{1}$ is the fraction of wealth invested in the riskless asset).

   (a) Let R_p denote the return on the portfolio w and consider the following investment problem

   $$\max_w \left(E[R_p] - \frac{a}{2} Var(R_p) \right) \tag{1}$$

   Solve for the optimal portfolio w. Show that there is "two-fund separation" in the sense that all investors will choose to divide their wealth between the risk-free asset and the tangency portfolio, a portfolio with weights w_{tan} you should identify, that is fully invested in risky assets and independent of risk-aversion. What happens to the fraction invested in the risk-free asset when risk-aversion becomes infinitely large?

   Solution:

   The investors maximization problem is equivalent to

   $$\max_w \left(R_p + w'(\mu - R_0\mathbf{1}) - \frac{a}{2} w'\Sigma w \right) \tag{2}$$

   The first-order condition is

   $$\mu - R_0\mathbf{1} = a\Sigma w, \tag{3}$$

   implying

   $$w^* = \frac{1}{a}\Sigma^{-1}(\mu - R_0\mathbf{1}). \tag{4}$$

   Defining

   $$w_{tan} := \frac{\Sigma^{-1}(\mu - R_0\mathbf{1})}{\mathbf{1}'\Sigma^{-1}(\mu - R_0\mathbf{1})}, \tag{5}$$

one can rewrite the investors optimal portfolio as

$$w^* = \frac{\mathbf{1}'\Sigma^{-1}(\mu - R_0\mathbf{1})}{a} w_{tan}, \tag{6}$$

with

$$w_0 = 1 - \mathbf{1}'w^* = 1 - \frac{\mathbf{1}'\Sigma^{-1}(\mu - R_0\mathbf{1})}{a}. \tag{7}$$

As $a \to \infty$, one gets $w^* \to \mathbf{0}$ and $w_0 \to 1$.

(b) Suppose the investor also has a liability L with $E[L] = \mu_L$, $Var(L) = \sigma_L$, and $Cov(R, L) = \Sigma_L$ (a vector of covariances between the N returns and L). Consider the following investment problem in the presence of the liability:

$$\max_w \left(E[R_p - L] - \frac{a}{2} Var(R_p - L) \right) \tag{8}$$

Solve for the optimal portfolio w.

Solution:

Rewriting the investors maximization problem as

$$\max_w \left(R_p + w'(\mu - R_0\mathbf{1}) - \mu_L - \frac{a}{2}\sigma_L^2 - \frac{a}{2} w'(\Sigma w - \Sigma_L) \right), \tag{9}$$

one gets the first-order condition

$$\mu - R_0\mathbf{1} = a\Sigma w - \frac{a}{2}\Sigma_L, \tag{10}$$

implying

$$w^* = \frac{1}{a}\Sigma^{-1}(\mu - R_0\mathbf{1}) + \frac{1}{2}\Sigma^{-1}\Sigma_L. \tag{11}$$

(c) Show that there is now "three-fund separation" in the sense that all investors will choose to divide their wealth between the risk-free assets, the tangency portfolio w_{tan} and a portfolio w_L you should identify and that is identical for all investors and related to the liability risk. What happens to the optimal position in the risk-free asset when risk-aversion becomes infinitely large? Give some intuition.

Solution:
Defining w_{tan} as above and letting

$$w_L := \frac{1}{2}\Sigma^{-1}\Sigma_L, \tag{12}$$

one can rewrite the optimal portfolio as

$$w^* = \frac{\mathbf{1}'\Sigma^{-1}(\mu - R_0\mathbf{1})}{a} w_{tan} + w_L, \tag{13}$$

with

$$w_0 = 1 - \mathbf{1}'w^* = 1 - \frac{\mathbf{1}'\Sigma^{-1}(\mu - R_0\mathbf{1})}{a} - \frac{1}{2}\mathbf{1}'\Sigma^{-1}\Sigma_L. \tag{14}$$

As $a \to \infty$, one gets $w^* \to w_L$ and $w_0 \to 1 - \frac{1}{2}\mathbf{1}'\Sigma^{-1}\Sigma_L$. This means that a very risk-averse investor will try to hedge as much as possible against variation in wealth that comes from the liability L. Any remaining wealth will just be invested in the risk-free asset to avoid any unnecessary risk.

2. **Market capitalization and expected returns (40 points).** In this exercise we will test the CAPM using portfolio sorted based on beta.

   (a) Download data on the same stock you used in Problem Set 6, i.e. the 639 stocks that have been traded each day between 2000 and 2019. Also download the share price (prc) as well as the number of shares outstanding (shrout). Download the same market return and risk-free rate that you also used in the last assignment.

   (b) Calculate the market capitalization of each stock in each month (price times number of shares outstanding). Make sure to always use the absolute value of the share price (CRSP sometimes reports negative values). Form 10 groups of stocks based on the market capitalization on December 31, 2019, with first decile being in group 1 and so on... Then compute the following:

   - equally weighted returns for each month, the average excess return of each portfolio as well as the corresponding (equally weighted) alphas and betas, i.e. the regression coefficients of the portfolios' excess returns on the market risk premium.

- value-weighted returns for each month, the average excess return of each portfolio as well as the corresponding (value weighted) alphas and betas, i.e. the regression coefficients of the portfolios' excess returns on the market risk premium. **Note:** Value-weighting means computing a weighted average of the returns in each month, where the weights are proportional to the **lagged** market capitalizations.

(c) Based on the results in part b), how is market capitalization related to average excess returns? Can the capm explain the behavior of the stocks? Consider equally weighted and value-weighted returns separately.

(d) Instead of sorting stock based on market capitalization in the end of the sample, sort stocks in each month into 10 deciles based on their lagged market capitalization. As in part b), compute value-weighted and equally weighted returns for those 10 groups. Also compute the average excess return of each portfolio as well as the corresponding (value weighted) alphas and betas. How is market capitalization related to average excess returns? Can the capm explain the behavior of the stocks? Consider equally weighted and value-weighted returns separately.