Investments Spring 2020 - Assignment 5

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Exercise 1

a)

- Before considering the tangency portfolio, let us introduce the variance minimization problem

$$\min_{w} \frac{1}{2} w'\Sigma w \quad / \quad \mu_p - R_0 = (\mu - R_0 \mathbb{1})'w$$

with a risk-free asset we do not have the contraint $\mathbb{1}'w = 1$

Lagrangian: $\mathcal{L} = \frac{1}{2} w'\Sigma w + \gamma \left(\mu_p - R_0 - (\mu - R_0\mathbb{1})'w\right)$

FOC: $\begin{cases} \frac{\partial \mathcal{L}}{\partial w} = \Sigma w - \gamma(\mu - R_0 \mathbb{1}) = 0 \\ \frac{\partial \mathcal{L}}{\partial \gamma} = \mu_p - R_0 - (\mu - R_0 \mathbb{1})'w = 0 \end{cases} \rightarrow \begin{cases} w = \gamma \Sigma^{-1}(\mu - R_0 \mathbb{1}) \\ \mu_p - R_0 = (\mu - R_0 \mathbb{1})'w \end{cases}$ (*)

Remember that: $A = \mathbb{1}'\Sigma^{-1}\mathbb{1} \qquad B = \mathbb{1}'\Sigma^{-1}\mu \qquad C = \mu'\Sigma^{-1}\mu$

we obtain $w = \gamma\Sigma^{-1}(\mu - R_0\mathbb{1})$

$$\mu_p - R_0 = (\mu - R_0\mathbb{1})' \cdot w = \gamma(\mu - R_0\mathbb{1})'\Sigma^{-1}(\mu - R_0\mathbb{1})$$

$$\gamma_0 = \frac{\mu_p - R_0}{\underbrace{(\mu - R_0\mathbb{1})'\Sigma^{-1}(\mu - R_0\mathbb{1})}}$$

$$\underbrace{\mu'\Sigma^{-1}\mu}_{C} \underbrace{-(R_0\mathbb{1})'\Sigma^{-1}\mu - \mu'\Sigma^{-1}R_0\mathbb{1}}_{-2R_0B} + \underbrace{(R_0\mathbb{1})'\Sigma^{-1}(R_0\mathbb{1})}_{R_0^2 A}$$

$$= \frac{\mu_p - R_0}{C - 2R_0B + R_0^2 A}$$

Now we can analize the specific case of the tangency portfolio → risky assets only $\Rightarrow \mathbb{1}'w_t = 1$, $w_0 = 0$

We have to impose the additional condition in (*) and we obtain

$$\begin{cases} \mu_p - \cancel{R_0} = \mu' w_t - \cancel{R_0} \qquad \mu_p = \mu' w \\ w_t = \gamma\Sigma^{-1}(\mu - R_0\mathbb{1}) \leftarrow \mathbb{1}'w_t = \gamma\mathbb{1}'\Sigma^{-1}(\mu - R_0\mathbb{1}) \end{cases}$$

$$1 = \gamma(B - AR_0)$$

$$\Rightarrow \gamma = \frac{1}{B - AR_0} \qquad \Rightarrow w_t = \frac{\Sigma^{-1}(\mu - R_0\mathbb{1})}{B - AR_0}$$

and $\mu_t = \mu' w_t = \dfrac{\mu'\Sigma^{-1}\mu - \mu'\Sigma^{-1}R_0\mathbb{1}}{B - AR_0}$

$$= \frac{C - BR_0}{B - AR_0}$$

$$\sigma_t^2 = W_t' \Sigma W_t = \frac{(\mu - R_0 \mathbb{1})' \Sigma^{-1}}{(B - AR_0)'} \Sigma \frac{\Sigma (\mu - R_0 \mathbb{1})}{B - AR_0} = \quad \Sigma^{-1}\Sigma = \mathbb{1}$$

$\Sigma' = \Sigma$

$$= \frac{\mu' \Sigma^{-1} \mu - 2R_0 (1' \Sigma^{-1} \mu)}{(B - AR_0)^2} = \frac{C - 2R_0 B + R_0^2 A}{(B - AR_0)^2}$$

$$SR_t = \frac{\mu_t - R_0}{\sigma_t} = \left[\frac{C - BR_0}{B - AR_0} - R_0 \right] \frac{B - AR_0}{(C - 2BR_0 + AR_0^2)^{1/2}} =$$

$$= \frac{C - BR_0 - BR_0 - AR_0^2)}{\cancel{B - AR_0}} \frac{\cancel{B - AR_0}}{(C - 2BR_0 + AR_0^2)^{1/2}} = ((C - 2BR_0 + AR_0^2)^{1/2}$$

For the number solution look at the python code or at page 4 -

b)

- Find the zero Beta portfolio W_z (zero correl. with tang. portfolio), μ_z, σ_z^2, SR_z

Zero correlation $\Rightarrow$ $W_z' \Sigma W_t = 0$

$$W_z' \cancel{\Sigma} \frac{\cancel{\Sigma^{-1}} (\mu - R_0 \mathbb{1})}{B - AR_0} = \frac{W_z' (\mu - R_0 \mathbb{1})}{B - AR_0} = 0$$

Then, $W_z' (\mu - R_0 \mathbb{1}) = 0$

If W_z is an efficient portfolio (mean-variance) $\Rightarrow$ $\underset{\mu_z}{\mu_p} - R_0 = (\mu - R_0 \mathbb{1})' W_z = 0$

$$\Rightarrow \mu_z = R_0$$

This is consistent with the theory of a zero-beta port, which assumes that it has the same expected return as the risk-free rate.
- This has no market exposure $\Rightarrow$ underperfor a diversified market portfolio
- Zero systematic risk

W_z is a mean-variance efficient portfolio $\Rightarrow$ it is the composition of other 2 portfolios.

$$W_{min} = \frac{\Sigma^{-1} \mathbb{1}}{\mathbb{1}' \Sigma^{-1} \mathbb{1}} \qquad W_{slope} = \frac{\Sigma^{-1} \mu}{\mathbb{1}' \Sigma^{-1} \mu} \Rightarrow W_z = \frac{(A\mu_z - B) \cdot B}{\Delta} W_{slope} + \frac{(C - B\mu_z) A}{\Delta} W_{min}$$

where $\Delta = AC - B^2$

Finally $\sigma_z^2 = W_z' \Sigma W_z \qquad SR_z = \frac{\mu_z - R_0}{\sigma_z}$

For numerical results, see the python code or page 4.

c) d)

- Prove that it is better a combination of $\to$ risk free portfolio
  $\searrow$ risky asset only mean-variance efficient portfolio

The two mutual fund theorem states that any portfolio on the efficient frontier can be generated by holding a combination of any two given portfolios on the frontier. If the portfolio we want cannot be generated using only our wealth, one mutual fund can be sold short, while the other one can be hold in a quantity greater than the amount available for the investment (consider the limit m).

When we introduce the risk free portfolio, we add a possible component to improve the range of possible combinations.
=> In case we consider the one mutual fund theorem, which consider the tangency portfolio and the risk free portfolio.

Moreover, we have to notice that, due to the fact we have a zero-beta portfolio (made of risky assets), the agent can invest in all the combinations of the tangent portfolio and this zero beta portfolio. This zero-beta portfolio has the same expected return of the risk free asset. Therefore, instead of investing in the risk free asset which is subject to limitations, the agent can use the zero-beta portfolio and he/she still gets a mean-variance efficient portfolio.

Then our portfolio will be a combination of W_z and W_t : $W_p \overset{(*)}{=} x_t W_t + x_z W_z$

Optimal portfolio : $\max\limits_{w_p} \left(E(R_p) + \frac{a}{2} V(R_p)\right)$ s.t. $W_p' \mathbb{1} \le m$

$$W_p' \mathbb{1} \le m$$
$$(x_t W_t + x_z W_z)' \mathbb{1} \le m$$
$$x_t \underbrace{W_t' \mathbb{1}}_{1} + x_z \underbrace{W_z' \mathbb{1}}_{1} \le m$$
$$x_t + x_z \le m$$
↑ leverage constraint

where
$$\mu_p = x_t \mu_t + x_z \mu_z \qquad \sigma_p^2 = x_t^2 \sigma_t^2 + x_z^2 \sigma_z^2$$

We have to show that this implies that $R_p = R_0 + x_t(R_t - R_0) + x_z(R_z - R_0)$.

when we look at returns, from (*) we have $R_p = x_t R_t + x_z R_z$
so we can pass to excess returns: $R_p - R_0 = x_t(R_t - R_0) + x_z(R_z - R_0)$
$$R_p = R_0 + x_t(R_t - R_0) + x_z(R_z - R_0)$$

Our optimization problem becomes :
$$\begin{cases} \max\limits_{w_p} \left(E(R_p) - \frac{a}{2} V(R_p)\right) \\ \text{s.t.} \quad x_t + x_z \le m \\ R_p = R_0 + x_t(R_t - R_0) + x_z(R_z - R_0) \end{cases}$$

Lagrangean: $\mathcal{L} = E\left[R_0 + x_t(R_t - R_0) + x_z(R_z - R_0)\right] -$
$$- \frac{a}{2} V\left[R_0 - x_t(R_t - R_0) + x_z(R_z - R_0)\right] + \lambda \overbrace{(m - x_t - x_z)}^{m - x_t - x_z \ge 0}$$

$$\mathcal{L} = R_0 + x_t(\mu_t - R_0) + \cancel{x_z(\mu_z - R_0)}^{\mu_t = R_0} - \frac{a}{2}(x_t^2\sigma_t^2 + x_z^2\sigma_z^2) + \lambda(m - x_t - x_z)$$

FOC:

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial x_t} = (\mu_t - R_0) - a x_t \sigma_t^2 - \lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial x_z} = -a x_z \sigma_z^2 - \lambda = 0 \\ \lambda \geq 0 \quad m - x_t - x_z \geq 0 \\ \lambda(m - x_t - x_z) = 0 \end{cases}$$

KKT condition

Compute the optimal portfolio : x_t, x_z

Cases : 1* $\lambda = 0$ $m - x_t - x_z > 0$:

$$\begin{cases} \mu_t - R_0 - a x_t \sigma_t^2 = 0 \\ -a x_z \sigma_z^2 = 0 \end{cases}$$

$$\begin{cases} x_t = \frac{\mu_t - R_0}{a \sigma_t^2} \\ x_z = 0 \\ m - x_t - \cancel{x_z} > 0 \end{cases} \rightarrow m - \frac{\mu_t - R_0}{a\sigma_t^2} > 0$$

$$m > \frac{\mu_t - R_0}{a\sigma_t^2} \Rightarrow a > \underbrace{\frac{\mu_t - R_0}{m\sigma_t^2}}_{= a^*}$$

If $a > a^*$ $\Rightarrow$ unconstraint $\Rightarrow$ $x_z = 0$

in this unconstained case the investor does not invest in the zero-Beta portfolio (proof for point d) of the exercise)

$$x_{t_u} = \frac{\mu_t - R_0}{a\sigma_t^2} = \frac{SR_t}{a\sigma_t}$$

$$W_u = x_{t_u} W_t \qquad \mu_u = R_0 + x_t(\mu_t - R_0)$$

$$\sigma_u = x_t^2 \sigma_t^2$$

$$SR_u = \frac{\mu_p - R_0}{\sigma_p} = \frac{\cancel{x_t}(\mu_t - R_0)}{\cancel{x_t}\sigma_t} = SR_t$$

2* $\lambda > 0$ $m - x_t - x_z = 0$

Constraint

$$\begin{cases} x_t = m - x_z \\ \mu_t - R_0 - a(m - x_z)\sigma_t^2 - (\overbrace{-a x_z \sigma_z^2}^{\lambda}) \end{cases}$$

$$\begin{cases} x_t = m - x_z \\ \mu_t - R_0 - a m \sigma_t^2 + a x_z \sigma_t^2 + a x_z \sigma_z^2 = 0 \end{cases}$$

$$\begin{cases} x_t = m - x_z \\ \mu_t - R_0 - a m \sigma_t^2 + a x_z(\sigma_t^2 + \sigma_z^2) = 0 \end{cases}$$

$$\begin{cases} x_{t_c} = \frac{\mu_t - R_0 + a m \sigma_z^2}{a(\sigma_t^2 + \sigma_z^2)} \\ x_{z_c} = \frac{-\mu_t + R_0 + a m \sigma_t^2}{a(\sigma_t^2 + \sigma_z^2)} \end{cases} \Rightarrow W_c = x_{t_c} W_t + x_{z_c} W_z$$

$$\mu_c = R_0 + x_t(\mu_t - R_0) + \cancel{x_z(\mu_t - R_0)}$$

$$\sigma_c = x_t^2\sigma_t^2 + x_z^2\sigma_z^2 \qquad SR_c = \frac{\mu_c - R_0}{\sigma_c} = \frac{x_t(\mu_t - R_0)}{\sqrt{x_t^2\sigma_t^2 + x_t^2\sigma_z^2}}$$

What does the contraint in λ means in terms of a?

$$\lambda > 0 \iff -a x_z \sigma_z^2 > 0 \iff x_z < 0 \iff -\mu_t + R_0 + a\, m\, \sigma_t^2 < 0$$

(denominator always positive)

$$\iff a < \frac{\mu_t - R_0}{m \sigma_t^2} = a^*$$

$$\Rightarrow \boxed{\lambda > 0 \iff a < a^*}$$

in this case $x_z \neq 0$ => the investor invests in the zero-beta portfolio (proof for point d) of the exercise)

Then $a < a^* \Rightarrow$ constrained case

$a \geq a^* \Rightarrow$ unconstrained case

Weight: $w_p = \begin{cases} w_c & \text{if } a \geq a^* \\ w_u & \text{if } a < a^* \end{cases}$

We have done all the calculations in python (see the code)

Python results for exercise 1:

a)

```
A is equal to 54.44444444444446
B is equal to 5.477778
C is equal to 0.583048

TANGENCY PORTFOLIO:
w_t is equal to
[0.4406682  0.28859447 0.27073733]

mu is equal to 0.112195
var_t is equal to 0.022571
sharp ratio_t is equal to 0.413982


ZERO BETA PORTFOLIO:
w_z is equal to
[ 1.90992134 -0.27480335 -0.63511799]

mu is equal to 0.050000
var_z is equal to 0.098628
sharp ratio_z is equal to 0.000000
```

Question e), Interpret the finding:

We see that, when we decrease the risk-aversion lever under a*, the agent becomes less risk averse and is willing to borrow to increase the returns. Under a*, he is unconstrained and he borrows at the risk free rate. In this case, the portfolio sharp ratio equals the tangency portfolio ones.

In general, when the risk-aversion level goes down, the investor increases the portion invested in the risky asset. When a reaches a certain cut off level, a*, the investor

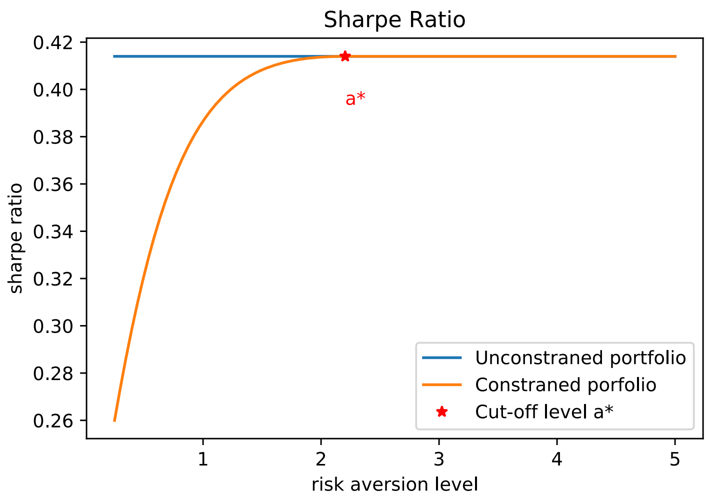

Figure 1: Sharpe ratio agaisnt risk aversion

starts to borrow to lever up his position in the risky assets. When he is under a*, he is unconstrained and can invest in the risky assets a portion bigger than m. While, if his risk-aversion exceeds a*, he is constrained and his leverage level is limited to m.
In this case, $a > a^*$, the agent decides to build a portfolio which is uncorrelated with the tangent portfolio and, thus, has no systematic risk. To increase his returns he uses the minimum mean-variance portfolio. We have to consider that the zero-beta portfolio is not a perfect substitute of the risk-free rate, we are only imposing that this constrained portfolio has no correlation with the tangency portfolio. It does not mean that it is not risky, it only does not have systematic risk.

Exercise 2: Arbitrage Pricing Theory

Given: R_M = common factor; $\beta_i = 1; \epsilon_i = 20\%; \alpha_i = \pm 1\%$. We invest in 10 stocks where we take short position (1M) in 5 stocks and long (1M) in 5 others.

This is thus a **zero-net investment** portfolio!

a. Find expectation and standard deviation of returns.

According to APT, we have that a security's excess return is:

$$R_i^e = \alpha_i + \beta_i R_M^e + \epsilon_i$$

Because $\beta_i = 1$ and risks later cancel out within the portfolio, and take i as index for a security that is shorted and j for one that is long-ed, we have:

$$R_i = \alpha_i + R_M = 1\% + R_M \Rightarrow R_{P_i} = 0.01 + R_M$$
$$R_j = \alpha_j + R_M = 1\% + R_M \Rightarrow R_{P_j} = -0.01 + R_M$$

Hence, the expectation can be calculated as:

$$E[R_P] = 1M \times (0.01 + R_M) - 1M \times (-0.01 + R_M) = 0.02M = \$20,000$$

Now, as the investor invests in 10 stocks and with \$1M long/short position for each half, for each stock there is a position worth \$200,000 (\$0.2M), the variance of the portfolio is thus:

$$\begin{aligned}
\sigma^2[R_{P_i}] = \sigma^2[R_{P_j}] &= (w_i \times 0.2)^2 \\
&= (200,000 \times 0.2)^2 = \$1,600M \\
&\Rightarrow \sigma^2[R_P] = 10 \times 1,600 = \$1.6 \times 10^{10} \\
&\Rightarrow Std[R_P] = \$126,491.1064
\end{aligned}$$

b. If the analyst analyses 40 or 100 stocks instead of 10:

- The expected dollar return does not change because the portfolio is still equally weighted with zero-net investment.
- Variance/Std changes as follow:
  - For 40 stocks:

  $$\sigma^2[R_P] = 40 \times (50,000 \times 0.2)^2 = \$4B \Rightarrow Std[R_P] = \$63,245.5532$$

  - For 100 stocks:

  $$\sigma^2[R_P] = 100 \times (20,000 \times 0.2)^2 = \$1.6B \Rightarrow Std[R_P] = \$40,000$$

- Observing the results above, we can conclude that increasing stocks from 10 to 100 (a factor of 10) decreases the standard deviation by a factor of $\sqrt{10}$.

Exercise 3: Warren Buffet

a. See Python code.

b.

```
Berkshire Hathaway:
Berkshire Hathaway's annualized excess returns' mean: 0.182903
Berkshire Hathaway's execss annualized returns' standard deviation: 0.241099
Berkshire's annualized returns' Sharpe Ratio: 0.758621

Market:
The Market's annualized excess returns mean: 0.07785433526011559
The Market's annualized excess returns' standard deviation: 0.15140385938422957
The Market's annualized returns' Sharpe Ratio: 0.5142163190341038

SMB:
The SMB's annualized excess returns mean: 0.02288554913294799
The SMB's annualized excess returns' standard deviation: 0.10265327731674904
The SMB's annualized returns' Sharpe Ratio: 0.22294026777471382

HML:
The HML's annualized excess returns mean: 0.027479768786127196
The HML's annualized excess returns' standard deviation: 0.09918720408566224
The HML's annualized returns' Sharpe Ratio: 0.27704953516377484

MOM:
The MOM's annualized excess returns mean Mom: 0.074802
The MOM's annualized excess returns' standard deviation Mom: 0.15113
The MOM's annualized returns' Sharpe Ratio Mom: 0.494953
```

c. Run the following regressions:

i. regression of the excess return of Berkshire Hathaway (BRK) on the excess return of the market

```
                            OLS Regression Results
==============================================================================
Dep. Variable:                    BRK   R-squared:                       0.160
Model:                            OLS   Adj. R-squared:                  0.158
Method:                 Least Squares   F-statistic:                     98.19
Date:                Sun, 22 Mar 2020   Prob (F-statistic):           2.62e-21
Time:                        22:01:39   Log-Likelihood:                 673.82
No. Observations:                 519   AIC:                            -1344.
Df Residuals:                     517   BIC:                            -1335.
Df Model:                           1
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const          0.0124      0.003      4.234      0.000       0.007       0.018
Mkt-RF         0.6586      0.066      9.909      0.000       0.528       0.789
==============================================================================
Omnibus:                      376.984   Durbin-Watson:                   1.638
Prob(Omnibus):                  0.000   Jarque-Bera (JB):            11979.170
Skew:                           2.744   Prob(JB):                         0.00
Kurtosis:                      25.888   Cond. No.                         22.9
==============================================================================
The corresponding Information Ratio is 0.6508124085770653
```

ii. regression of the excess return of Berkshire Hathaway (BRK) on the excess return of the market, on SMB and HML.

```
                            OLS Regression Results
==============================================================================
Dep. Variable:                    BRK   R-squared:                       0.214
Model:                            OLS   Adj. R-squared:                  0.210
Method:                 Least Squares   F-statistic:                     46.79
Date:                Sun, 22 Mar 2020   Prob (F-statistic):           9.38e-27
Time:                        22:01:39   Log-Likelihood:                 691.24
No. Observations:                 519   AIC:                            -1374.
```

```
Df Residuals:                     515   BIC:                            -1357.
Df Model:                           3
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const          0.0110      0.003      3.834      0.000       0.005       0.017
Mkt-RF         0.7899      0.068     11.599      0.000       0.656       0.924
SMB           -0.2796      0.100     -2.804      0.005      -0.476      -0.084
HML            0.4839      0.104      4.655      0.000       0.280       0.688
==============================================================================
Omnibus:                      401.206   Durbin-Watson:                   1.641
Prob(Omnibus):                  0.000   Jarque-Bera (JB):            14131.260
Skew:                           2.983   Prob(JB):                         0.00
Kurtosis:                      27.857   Cond. No.                         39.6
==============================================================================
The corresponding Information Ratio is 0.5946632638274936
```

iii. regression of the excess return of Berkshire Hathaway (BRK) on the excess return of the market, on SMB, HML and MOM.

```
                            OLS Regression Results
==============================================================================
Dep. Variable:                    BRK   R-squared:                       0.215
Model:                            OLS   Adj. R-squared:                  0.209
Method:                 Least Squares   F-statistic:                     35.25
Date:                Sun, 22 Mar 2020   Prob (F-statistic):           4.98e-26
Time:                        22:01:39   Log-Likelihood:                 691.60
No. Observations:                 519   AIC:                            -1373.
Df Residuals:                     514   BIC:                            -1352.
```

```
Df Model:                           4
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const          0.0105      0.003      3.602      0.000       0.005       0.016
Mkt-RF         0.8012      0.069     11.537      0.000       0.665       0.938
SMB           -0.2845      0.100     -2.848      0.005      -0.481      -0.088
HML            0.5068      0.108      4.712      0.000       0.295       0.718
Mom            0.0564      0.068      0.834      0.404      -0.076       0.189
==============================================================================
Omnibus:                      401.375   Durbin-Watson:                   1.641
Prob(Omnibus):                  0.000   Jarque-Bera (JB):            14109.195
Skew:                           2.986   Prob(JB):                         0.00
Kurtosis:                      27.835   Cond. No.                         40.7
==============================================================================
The corresponding Information Ratio is 0.5692419829409094
```

iv. regression of the excess return of Berkshire Hathaway (BRK) on the excess return of the market, on SMB, HML, MOM, RMW and CMA.

```
                            OLS Regression Results
==============================================================================
Dep. Variable:                    BRK   R-squared:                       0.222
Model:                            OLS   Adj. R-squared:                  0.213
Method:                 Least Squares   F-statistic:                     24.42
Date:                Sun, 22 Mar 2020   Prob (F-statistic):           1.78e-25
Time:                        22:01:39   Log-Likelihood:                 694.00
No. Observations:                 519   AIC:                            -1374.
Df Residuals:                     512   BIC:                            -1344.
Df Model:                           6
Covariance Type:            nonrobust
```

```
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const          0.0093      0.003      3.098      0.002       0.003       0.015
Mkt-RF         0.8210      0.073     11.178      0.000       0.677       0.965
SMB           -0.1594      0.109     -1.468      0.143      -0.373       0.054
HML            0.5351      0.145      3.679      0.000       0.249       0.821
RMW            0.3448      0.142      2.431      0.015       0.066       0.623
CMA           -0.0677      0.213     -0.317      0.751      -0.487       0.351
Mom            0.0328      0.069      0.476      0.634      -0.103       0.168
==============================================================================
Omnibus:                      411.161   Durbin-Watson:                   1.638
Prob(Omnibus):                  0.000   Jarque-Bera (JB):            15375.195
Skew:                           3.077   Prob(JB):                         0.00
Kurtosis:                      28.944   Cond. No.                         86.4
==============================================================================
The corresponding Information Ratio is 0.5026280066325507
```

d. How Robust-minus-Weak (RMW) and Conservative-minus-Aggressive (CMA) are constructed and how they capture sources of priced returns.

- RMW: the difference between the returns on diversified portfolios of stocks with robust and weak profitability. According to Fama and French (2015), profitability corresponds to the revenues minus cost of goods sold, other operation costs, interest expense, all divided by book equity.

$$RMW = 1/2(SmallRobust + BigRobust) - 1/2(SmallWeak + BigWeak)$$

- CMA: the difference between the returns on diversified portfolios of the stocks of low and high investment firms. Conservative firms are often those with low investment, while aggressive ones often correspond to high investments. Investment here is defined as the annual change in gross property, plant, and equipment (fixed assets) plus the annual change in inventories, all divided by the book value of total assets

(Fama and French, 2015).

$$CMA = 1/2(SmallConservative + BigConservative) - 1/2(SmallAggressive + BigAggressive)$$

e. What do the found β-estimates say about Buffet's investment strategy?

Based on the results found in part c, we can see that across the regression controls, the factors that constantly hold the positive and high coefficients are market exposure, HML (value premium) and RMW, all of which are statistically significant at 5% level. Meanwhile, SMB's coefficient is negative and statistically significant for most of the time (apart from in the full model where the p-value is not that significant). These findings point to the fact that Buffet invested in stocks that have high market exposure, stocks with high value premium (high book-to-market ratios), stocks from robust firms (firms with higher profitability) (high RMW signals a large difference between returns of robust firms and those of weaker firms), and stocks from large firms (negative SMB shows that the smaller firms do not outperform the bigger ones). To elaborate on the high RMW, the article provided by the assignment (`https://www.forbes.com/sites/phildemuth/2013/06/27/the-mysterious-factor-p-charlie-munger-robert-novy-marx-and-the-profitability-factor/#567d0fba683f`) has also confirmed the success of the profitability factor through the research and findings of Munger and Novy-Marx.

On the other hand, momentum is a factor that demonstrates small coefficients across controls and are statistically insignificant. This means that Buffet does not have preferences towards recent trends of "winner" or "loser" stocks. Furthermore, the low and statistically insignificant CMA shows that Buffet is also not particularly interested in conservative (investment-wise) nor aggressive firms.

f. Whether exposure to common risk factors explains Warren Buffett's performance (what happens to the α and information ratio):

The α decreases from over 0.01 to over 0.009 from the first to the last model, whereas the information ratio is highest for the model that has only market exposure (first model) with ~ 0.65, then decreases from ~ 0.59 to ~ 0.50 from the second to the last model. The small fluctuation of α but significant drop in information shows that the portfolio's idiosyncratic risk (the denominator) increases after each control, implying that exposure to more risk factors does not help, but rather even explains the portions of low performance. Moreover, the R-squared is very low and does not significantly change in each model, again showing that the addition of new factors does not help explain better the returns. In fact, in the full model, the R squared implies that only 22% of the returns are explained by the independent variables.

g. optimal portfolio weight vector if you target a volatility of 20% for your portfolio: The optimal portfolio is the one that maximize the Sharpe ratio. In presence of an asset with a positive alpha we can follow the framework developed in class. Hence, invest in a combination of a zero-cost portfolio with zero factor exposure which return is given by alpha and the idiosyncratic risk, and the risks factors. The variance is given by the sum of the variance of the factors and the variance of the idiosyncratic risk. The variance of the idiosyncratic risk is estimated using the mean square error of a regression of the stock on the factors. It is an appropriate estimator since by definition the error term ols regression uncorrelated with the the factors and moreover since the regression does not show Heteroskedasticity (see ipython notebook for details on the test) the mse is an unbiased estimator. factor exposure.

	weight
Mkt-RF	1.23780360157084
SMB	1.02484529905805
HML	-0.278067668467825
RMW	2.39365325513903
CMA	2.96415348813385
Mom	0.865315587643230
rf	-0.339991760985985
BRK	0.570781741002945

h. Rerun all the regressions for data until 1995 and compare with the full-sample results.

```
Berkshire Hathaway  until 1995:
Berkshire Hathaway's annualized excess returns mean: 0.3233963700407888
Berkshire Hathaway's execss annualized returns' standard deviation:
0.30606055744907673
Berkshire's annualized returns' Sharpe Ratio: 1.0566417729099127
```

i. regression of the excess return of Berkshire Hathaway (BRK) on the excess return of the market.

```
                            OLS Regression Results
==============================================================================
Dep. Variable:                    BRK   R-squared:                       0.180
Model:                            OLS   Adj. R-squared:                  0.177
Method:                 Least Squares   F-statistic:                     50.38
Date:                Sun, 22 Mar 2020   Prob (F-statistic):           1.58e-11
Time:                        22:01:40   Log-Likelihood:                 255.70
No. Observations:                 231   AIC:                            -507.4
Df Residuals:                     229   BIC:                            -500.5
Df Model:                           1
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const          0.0215      0.005      4.020      0.000       0.011       0.032
Mkt-RF         0.8643      0.122      7.098      0.000       0.624       1.104
==============================================================================
Omnibus:                      189.915   Durbin-Watson:                   1.491
Prob(Omnibus):                  0.000   Jarque-Bera (JB):             4364.216
Skew:                           2.978   Prob(JB):                         0.00
Kurtosis:                      23.444   Cond. No.                         23.0
==============================================================================
The corresponding Information Ratio is 0.9258941346732417
```

ii. regression of the excess return of Berkshire Hathaway (BRK) on the excess return of the market, on SMB and HML.

```
                            OLS Regression Results
==============================================================================
Dep. Variable:                    BRK   R-squared:                       0.200
Model:                            OLS   Adj. R-squared:                  0.189
Method:                 Least Squares   F-statistic:                     18.89
Date:                Sun, 22 Mar 2020   Prob (F-statistic):           5.69e-11
Time:                        22:01:40   Log-Likelihood:                 258.47
No. Observations:                 231   AIC:                            -508.9
Df Residuals:                     227   BIC:                            -495.2
Df Model:                           3
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const          0.0182      0.005      3.320      0.001       0.007       0.029
Mkt-RF         0.9822      0.140      7.009      0.000       0.706       1.258
SMB            0.1948      0.223      0.873      0.384      -0.245       0.634
HML            0.5369      0.239      2.242      0.026       0.065       1.009
==============================================================================
Omnibus:                      189.068   Durbin-Watson:                   1.523
Prob(Omnibus):                  0.000   Jarque-Bera (JB):             4180.392
Skew:                           2.975   Prob(JB):                         0.00
Kurtosis:                      22.973   Cond. No.                         47.8
==============================================================================
The corresponding Information Ratio is 0.7907517206515153
```

iii. regression of the excess return of Berkshire Hathaway (BRK) on the excess return of the market, on SMB, HML and MOM.

```
                            OLS Regression Results
==============================================================================
Dep. Variable:                    BRK   R-squared:                       0.200
Model:                            OLS   Adj. R-squared:                  0.186
Method:                 Least Squares   F-statistic:                     14.10
Date:                Sun, 22 Mar 2020   Prob (F-statistic):           2.74e-10
Time:                        22:01:40   Log-Likelihood:                 258.47
No. Observations:                 231   AIC:                            -506.9
Df Residuals:                     226   BIC:                            -489.7
Df Model:                           4
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const          0.0182      0.006      3.170      0.002       0.007       0.029
Mkt-RF         0.9817      0.142      6.927      0.000       0.702       1.261
SMB            0.1947      0.224      0.871      0.385      -0.246       0.635
HML            0.5381      0.243      2.214      0.028       0.059       1.017
Mom            0.0053      0.171      0.031      0.976      -0.332       0.342
==============================================================================
Omnibus:                      189.104   Durbin-Watson:                   1.523
Prob(Omnibus):                  0.000   Jarque-Bera (JB):             4181.790
Skew:                           2.976   Prob(JB):                         0.00
Kurtosis:                      22.976   Cond. No.                         48.5
==============================================================================
The corresponding Information Ratio is 0.7868623519832864
```

iv. regression of the excess return of Berkshire Hathaway (BRK) on the excess return of the market, on SMB, HML, MOM, RMW and CMA.

```
                            OLS Regression Results
==============================================================================
Dep. Variable:                    BRK   R-squared:                       0.201
Model:                            OLS   Adj. R-squared:                  0.179
Method:                 Least Squares   F-statistic:                     9.369
Date:                Sun, 22 Mar 2020   Prob (F-statistic):           3.57e-09
Time:                        22:01:40   Log-Likelihood:                 258.59
No. Observations:                 231   AIC:                            -503.2
Df Residuals:                     224   BIC:                            -479.1
Df Model:                           6
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const          0.0188      0.006      3.038      0.003       0.007       0.031
Mkt-RF         0.9682      0.146      6.646      0.000       0.681       1.255
SMB            0.1647      0.231      0.714      0.476      -0.290       0.620
HML            0.6926      0.365      1.896      0.059      -0.027       1.413
RMW           -0.0374      0.467     -0.080      0.936      -0.957       0.882
CMA           -0.3414      0.532     -0.641      0.522      -1.391       0.708
Mom            0.0252      0.177      0.142      0.887      -0.324       0.374
==============================================================================
Omnibus:                      185.920   Durbin-Watson:                   1.530
Prob(Omnibus):                  0.000   Jarque-Bera (JB):             3982.779
Skew:                           2.912   Prob(JB):                         0.00
Kurtosis:                      22.490   Cond. No.                         114.
==============================================================================
```

The corresponding Information Ratio is 0.8097058594889107

We can see that up to 1995 we have a stronger positive impact of the market excess returns the SMB (even if not significant now we have positive coefficient) and HML, while in this period RMW does not a significant impact. This can be due to the issue in the 90s of the first publications about the index model and the three factors model of Fame and French that have strongly influenced the portfolios strategy of many fund managers, resulting in fund returns that are positive correlated with these factors.