A new formula for the nth prime

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A new formula for the n-th prime:

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Using the below expression for the Characteristic Function of Prime Numbers:

$$\left\lfloor \frac{lcm(1,2,\ldots,j)}{j \cdot lcm(1,2,\ldots,j-1)} \right\rfloor = \begin{cases} 1 & \text{if } j \text{ is prime} \\ 0 & \text{if } j \text{ is composite} \end{cases}$$

(This function is complementary to the Smarandache Prime Function **[1],** defined as: P(n) = 0, if n is prime, and P(n) = 1 otherwise.
It is easy to prove this expression studying it in detail.
I have obtained this expression this last month (March 2004) but I do not know if already it is known.)

we obtain the following formula for the n-th prime **[2],[3],[4],[5],[6]:**

$$p_n = 1 + \sum_{k=1}^{\lfloor 2n\log n + 2 \rfloor} \left(1 - \left\lfloor \frac{1}{n} \sum_{j=2}^{k} \left\lfloor \frac{lcm(1,2,\ldots,j)}{j \cdot lcm(1,2,\ldots,j-1)} \right\rfloor \right\rfloor \right)$$

The Proof is the same of the previous articles.
It is necessary to see the references for a complete comprehension of the formulas.

We can see that this formula is faster than the previous:

Comparative table of times:

Prime	Prob 38	Lcm	Prob 38 mod	Lcm mod
P10=29	0,2 sec	0,06 sec	0,04 sec	0,02 sec
P20=71	2,5 sec	0,8 sec	0,4 sec	0,1 sec
P30=113	12,2 sec	3,7 sec	1,3 sec	0,4 sec
P40=173	36,5 sec	10,5 sec	2,9 sec	0,7 sec
P50=229	84 sec	24 sec	5,5 sec	1,3 sec
P100=541			41 sec	6,9 sec
P200=1223			299 sec	39 sec

*Prob 38: **[6]** It is the original formula without modifying.
The time complexity of this algorithm is O(nlog n)^3
*Lcm: Is the new formula.
*Prob 38 Mod: **[6]** It is the original formula with the modifications.
The time complexity of this algorithm is O(n log n)^(3/2).
*Lcm mod: It is the new formula calculating lcm (1,2, …, j) of way recurrent.

Which is the time complexity of this algorithm?

The code in Mathematica:

```
L[1]=1;
L[n_]:=L[n]=LCM[L[n-1],n]
LG[n_]:=L[n]/L[n-1]
FL[n_]:=Quotient[LG[n],n]
Pii[n_ ]:=Sum[FL[i],{i,2,n}]
PrimeLCM[n_]:=1+Sum[1-Quotient[Pii[k],n],{k,1,Floor[2*n*Log[n]+2]}]
Do[Print[n," ", Timing[PrimeLCM[n]]," ",Prime[n]],{n,200,200}]
200   1223     1223
{39.438 Second}
```

We can accelerate it more enough of the following form:

Using the bound of Rosser and Schoenfeld for p_n **[7]**:

$c_n = n\log n + n(\log(\log n) - 1/2)$

and modifying the formula considering that $p_n > \lfloor n\log n \rfloor$ **[7]** we obtain for n> 1:

$$p_n = \lfloor n\log n \rfloor + \sum_{k=\lfloor n\log n \rfloor}^{\lfloor C_n+3 \rfloor} \left(1 - \left\lfloor \frac{1}{n} \sum_{j=2}^{k} \left\lfloor \frac{lcm(1,2,\ldots,j)}{j \cdot lcm(1,2,\ldots,j-1)} \right\rfloor \right\rfloor \right)$$

This is a nice expression that relates the n-th prime number with the approximation obtained with the prime number theorem $n\log n$ adding a term of error.

The new times are:

Prime	Prob 38 mod	Lcm mod	RS acceleration
P10=29	0,04 sec	0,02 sec	0 sec
P20=71	0,4 sec	0,1 sec	0,02 sec
P30=113	1,3 sec	0,4 sec	0,05 sec
P40=173	2,9 sec	0,7 sec	0,09 sec
P50=229	5,5 sec	1,3 sec	0,15 sec
P100=541	41 sec	6,9 sec	0,86 sec
P200=1223	299 sec	39 sec	4,59 sec

Which is the time complexity of this algorithm?

References:

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