Sort Race

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Abstract

Sorting is one of the oldest computing problems and is still very important in the age of big data. Numerous algorithms and implementation techniques have been proposed. In this study, we focus on comparison based, internal sorting algorithms. We created 12 types of data of various sizes for experiments and tested extensively various implementations in a single setting. Using some effective techniques, we found that *quicksort* is adaptive to nearly sorted inputs and is still the best overall sorting algorithm. We also identified techniques that are effective in *timsort*. It turns out to be one of the most popular and efficient sorting methods based on natural *mergesort*. In addition, we created a version of our own *mergesort*, which performs better than *timsort* on nearly sorted instances. Our implementations of *quicksort* and *mergesort* are different from other implementations reported in all textbooks or research articles, and are faster than any version of the C library *qsort* functions not only for randomly generated data, but also for various types of nearly sorted data. This work can aid the user to choose the best sorting algorithm for the hard sorting job at hand, and provides a platform for anyone to test their own sorting algorithm against the best in the field.

KEYWORDS:
Sorting algorithms; Merge sort; Quick Sort

1 | INTRODUCTION

What is the best sorting algorithm? The answer to this question is perhaps "it depends". If you are asked to recommend a sorting algorithm for an unknown problem, what is your recommendation? According to a typical textbook on algorithms, the answer would be *heapsort* or *mergesort*, the two asymptotically optimal algorithms, which take $\mathcal{O}(n \log(n))$ time in the worst case, where n is the number of elements to be sorted. In practice, neither *heapsort* nor *mergesort* is a good choice. The most popular choice is a variant of *mergesort* called natural *mergesort*. An equally good choice is *quicksort*. Of course, our claim is based on experiments, not on asymptotical analysis, as nobody can do better than an optimal method.

Sorting is arguably the most studied computing problem. Various algorithms and implementation techniques have been leveraged to demonstrate various algorithm design techniques. On the sorting algorithms animations website [1], eight different sorting methods, that is, insertion sort, selection sort, bubble sort, shellsort, mergesort, heapsort, and two versions of *quicksort* (both two-way and three-way splitting), are animated on four types of initial instances: Random, Nearly Sorted, Reversed, and Few Unique. Their conclusion is that "there is no best sorting algorithm", and "the initial condition (input order and key distribution) affects performance as much as the algorithm choice." Apparently, this conclusion does not provide any clue to the aforementioned recommendation. From now on, we will use Demo to refer to the sorting algorithms animations website [1].

TABLE 1 Summary of implementations of the C-library *qsort* function.

#	Year	OS	Author	Algorithm
1	1972	Unix	Lee Mcmahon	CAR Hoare's quick sort[2]
2	1983	BSD	Lee Mcmahon	CAR Hoare's quick sort[2]
3	1991	BSD	Douglas Schmidt	Bentley & McIlroy (B&M)'s quick sort[3]
4	1991	GNU	Douglas Schmidt[4]	B&M's quick sort[3]
5	1993	GNU	Mike Haertel[5]	John von Neumann's merge sort[6]

TABLE 2 The average running time of seven sorting methods on 4 types of inputs.

n=2000000	best time	insert	heap	shell	merge	quick	qsort	timsort
random	225876	–	3.47	1.73	1.16	**1.00**	1.18	1.15
reverse sorted	4728	–	113.78	26.34	**1.00**	1.78	19.74	1.07
nearly sorted	93185	1.62	5.73	1.92	**1.00**	1.54	1.50	1.11
few unique	128097	–	5.88	2.39	1.64	**1.00**	1.73	1.61
average time	126509	–	5.18	1.98	1.15	**1.00**	1.42	1.13

In a decathlon competition, the winner is not necessarily the fastest runner or the highest jumper, but is determined by the combined performance in all events. If we choose the best implementation for each sorting algorithm and put them in a rigorous sort race, the winner can be decided by the combined performance in all tests. As in any relay competition, the fairest performance metric would be the total time taken by each method, which will also be used in our sort race.

To have a simple sort race, only comparison-based internal sorting algorithms will be used. We will call an input array of elements for sorting as an instance and use instances of at least one million elements. The same instance and the same comparison function will be used by all participants, and all implementations will be compiled into one single program. Since selection sort and bubble sort perform poorly on any large instance, they are disqualified from the race. To be fair, the copy of *quicksort* with the best performance is used. Finally, five different sorting methods are chosen as participants in this race: insertion sort, shellsort[6], mergesort[7], heapsort[8], and quicksort[2].

Table 2 shows the experimental results of our implementations of the five sorting methods plus the GNU C library function *qsort* (Table 1, row 5) and *timsort*[9], on four types of integer lists as used in Demo, i.e., random, reverse sorted, nearly sorted, and few unique. All the methods are implemented in C and compiled as a single executable by gcc with optimization "-O3". The executable takes two parameters, the size and the type of the instance, and generates one instance and then run all the sorting methods on this instance. This way, the same set of instances is used for each method.

In Table 2, the column "best time" is the fastest average time (in microseconds) over 100 instances (each has 2,000,000 elements) for each of the seven methods. Under each method, the ratio of its running time to the best time is given. That is, the actual running time of that method is the multiplication of the ratio and the best time. If the ratio is 1.00, then the best time is produced by that method. The row "average time" represents the average time (in microseconds) of all 400 runs for each method, again in the format of ratio to the best time.

It is clear from the table that *quicksort* (nicknamed hyb2 in section 4) is the best overall method according to the best average time. The second best method is *mergesort* (nicknamed mer6 in section 3), which has the best performance for reverse sorted and nearly sorted instance. The weak performance of *qsort* on reverse sorted instances is typical for the standard recursive *mergesort* algorithm on such inputs, as the the default GNU C-library function *qsort* as of 2022 (Table 1, row 5) uses *mergesort* but does not use natural *mergesort*.

Why the conclusion from our experiment is different from the one in Demo? There are two reasons: (1) size matters, and (2) implementation matters. In Demo, up to 50 elements are used to test the idea of each sorting method, and this size is too small to demonstrate the strength of each method. For example, the overhead for choosing a good pivot in *quicksort* only pays off with a large size array. Moreover, it is very important to know that a sorting method works well or not for very large size problems. Every experienced programmer knows that different implementations of the same algorithm give different performances. For

Olympics, each country sends the best athletes to compete. For sort race, we should choose the best implementation possible for each method.

When talking about the best sorting implementation, we cannot ignore the existence of the *qsort* function in the C Standard Library. The *qsort* function has long been considered the "best" sorting function and being implemented in different libraries (see Table 1). Nowadays, the most popular *qsort* is the GNU project's libc implementation which powers most GNU/Linux distributions and several other operating systems. This *qsort* is interesting in that it shuns *quicksort* in favor of conventional topdown *mergesort*, due to the poor performance of *quicksort* on some types of instances [5]. Tim Peters created *timsort* based on Peter McIlroy's idea of natural mergesort [7], which consists of two phases: in phase 1, sorted subarrays (called runs) are identified from the input; in phase 2, two runs are merged into one run, until only one run remains. The two phases can be interleaved. There are several implementations based on natural *mergesort*, such as *timsort* [9], and *neat sort* [10]. Different techniques are used in these implementations. As the first goal of this article, we will try to identify which techniques are more effective and which are less effective through extensive experiments. From now on, by *mergesort*, we mean natural *mergesort* [7].

The BSD *qsort* (Table 1, row 3) is a version of *quicksort* based on B&M's *quicksort* algorithm. A significant deviation from B&M's *quicksort* is that the BSD *qsort* resorts to insertion sort, not only for small arrays, but also whenever a partitioning round is completed without having moved any elements. The latter is intended to capture nearly sorted inputs, since insertion sort will handle these cases efficiently. However, this technique may missfire. We found by accident that the BSD *qsort* performs poorly on a set of instances. By extensive experiments, we will try to identify which techniques used in the past [11] are effective and what new techniques can be used so that *quicksort* can handle better nearly sorted inputs and has better overall performances over all kinds of inputs. This is the second goal of this article.

The third goal of this article is to establish a software platform available in the public domain so that anyone can test their own sorting algorithm against other best implementations. Currently, the platform, called sortrace, is a C program which provides 12 types of inputs and over 20 implementations of sorting algorithms. We have tested over a dozen of different sequences in the shellsort algorithm to see which one works best. We hope that anyone who claims to have found a better sorting algorithm, can use this platform to sustain their claim by going through the sort race.

2 | TWELVE CLASSES OF INPUTS

Twelve classes of input instances are provided on the platform. For the first five classes of inputs, each array is randomly generated and each array element is one of the following five types: long integer, double, lists of integers of 16, 64, and 256 bytes, respectively. For the other seven classes, besides the array length n, it takes another integer k ($1 \leq k \leq n$) indicating the degree of presortness.

Many researchers prefer to use the number of inversions, $Inv(A)$, to measure how far (or close) the array is from being sorted. If the array is already sorted, then the inversion count is 0, but if the array is sorted in the reverse order, the inversion count is the maximum which is equal to $n(n-1)/2$. Formally speaking, two elements a_i and a_j form an inversion if $a_i > a_j$ and $i < j$. Let $Inv(A)$ denote the number of inversions in the array A, then

$$Inv(A) = |\{(i,j) : 1 \leq i < j \leq n, a_i > a_j\}|$$

If we break the array A at every position i, where $a_i > a_{i+1}$, into a collection of disjoint subarrays, then each subarray is a run, i.e., a sorted sequence of elements. Let $Run(A)$ be the number of such subarrays:

$$Run(A) = |\{i : 1 \leq i < n, a_i > a_{i+1}\}| + 1$$

When A is sorted, $Run(A) = 1$. When A is reverse sorted, the number of runs is maximal ($Run(A) = n$); in this case, reversing the array, which takes $\mathcal{O}(n)$ time, will make A sorted. In other words, neither $Inv(A)$ nor $Run(A)$ reflects the difficulty of making A sorted. Hence, we prefer another measure over $Inv(A)$ and $Run(A)$. For instance, we are interested in the minimal number k such that A can be broken into k subarrays, each of which is monotonic (i.e., either sorted or reverse sorted).

$$Mono(A) = min\{k : A = w_1 w_2 \ldots w_k, w_i \text{ is a monotonic subarray}, 1 \leq i \leq k\}$$

Intuitively, if $Mono(A) = k$, then A is the concatenation of k monotonic subarray (either sorted or reverse sorted). For instance, if A is sorted or reverse sorted, then $Mono(A) = 1$. If the first half of A is reverse sorted and the second half of A is sorted, A is called an "organ-pipe" array and $Mono(A) = 2$.

Here are the five classes of inputs with an additional parameter k:

- **k-limited**: A simple way to create "few unique" instances is to restrict elements in a certain range. An array of integers is said to be *k-limited* if each element x in the array is a randomly generated number and $0 \leq x \leq 2k$. When $k = 0$, all elements of the array are the same, i.e., 0. When $k = 1$, the array is a randomized binary list. The "few unique" instance used in Demo can be mapped to a 2-limited array of 50 elements. If we keep the same ratio of $2^2/50 = 2^k/2,000,000$, we get $k = 12$. For the "few unique" instance in Table 2, we used 12-limited arrays. When $k \geq 31$, it is the same as randomized integers. The best sorting method for the k-limited is the 3-way *quicksort* used in Demo and is related to Dijkstra's Dutch National Flag problem [12] (more details can be found in section 4).

- **k-equal teeth**: A *k-equal teeth* (or *k-equal saw teeth*) array can be constructed as follows: Divide an array of n elements into k disjoint, equal-length subarrays and each subarray is filled with the numbers from 1 to n/k. Obviously, if A is a *k-equal teeth*, *$Run(A) = Mono(A) = k$*. If $k = 1$, then A is sorted. Among all comparison-based sorting algorithms, *mergesort* is the best one for this class of instances.

- **k-even teeth**: A *k-even teeth* array can be constructed from *k-equal teeth* as follows: Reverse each odd numbered subarray of a *k-equal teeth*. When $k = 1$, the array is reverse sorted. For $k = 2$, it is an "organ-pipe" array. Obviously, for $k \leq n/3$, if A is a k-even teeth, *$Mono(A) = k$*. When $k = n/2$, the list is a 2-limited list starting with $(2, 1, 1, 2, 2, 1, 1, 2, ...)$, and *$Mono(A) = n/4 + 1$*, because we may group the elements as $(2, 1, 1), (2, 2, 1, 1), (2, 2, 1, 1), ...$ The best sorting method for this class of instances is *mergesort*.

- **k-sharp teeth**: Initially, the array contains a sorted list from 1 to n. We divide the array into k disjoint, equal-length subarrays and reverse the odd numbered subarrays. When $k = 1$, the array is reverse sorted. If A is a *k-sharp teeth*, *$Mono(A) = k$*. The best sorting method for this class of instances is a special version of *mergesort* . We would like to point out that the BSD *qsort* performs very poorly on this type of inputs. For instance, when $n = 2,000,000$ and $k = 8$, *mergesort* takes 0.01 sec., B&M's *quicksort* function takes 0.1 sec., and the BSD qsort takes 279.4 sec. This might explain why *quicksort* was shunned from the C library.

- **k-shuffled teeth**: The array is first initialized with a *k-sharp teeth*, and then all k subarrays are randomly shuffled by preserving the relative order of the elements in each subarray. The concept of *k-shuffled teeth* lists is related to the SMS measure in [13], the least value k such that the input list can be formed by interleaving k monotone (either descending or ascending) sequences. Like the class of *k-sharp teeth*, *mergesort* is the best for this class when $k < n/10$; when $k > n/10$, *quicksort* is the best.

- **k-distance** An array $A = (a_1, a_2, ..., a_n)$ is said to be *k-distance* if $|rank(a_i) - i| \leq k$ for all $1 \leq i \leq n$, where $rank(a_i)$ is the position of a_i in the array after A is sorted. That is, for each a_i in A, the distance from its initial position to its final position is bound by k. To create a *k-distance* list, we first construct a sorted list and then divide the list into $n/(k + 1)$ subarrays, and shuffle each subarray of $k + 1$ elements randomly. If $k = n - 1$, the result is a random list. The "nearly sorted" instance used in Demo appears to be a 1-distance array of 50 elements. If we keep the same ratio of 1/50, we would use a 40000-distance array of 2,000,000 elements. Instead, we used 256-distance lists in Table 2. Indeed, the best performer for this class on 2,000,000 elements is insert sort if $k \leq 32$; for $k > 32$, *mergesort* is the best.

- **k-exchange**: An array A is said to be a *k-exchange* array if it becomes sorted after no more than exchanges of k pairs elements in A. In this case, $|\{a_i : rank(a_i) \neq i\}| \leq 2k$. To create a *$k$-exchange* array, we first create a sorted list, and then repeat the following operation k times: randomly choose two positions to exchange them. The model of *k-exchange* arrays is related to the Rem measure (the minimal number of items that must be removed to leave a sorted sequence) in [13]. It can be shown that if A is a *k-exchange*, then *$Mono(A) \leq 2k + 1$*. Like the class of *k-shuffled teeth*, *mergesort* is the best for this class when $k \leq n$; when $k > n$, *quicksort* is the best.

The last six classes described above are presorted to certain degree and can be called nearly sorted with proper k. We will present experimental results of various sorting methods on these instances in later sections. Roughly speaking, *quicksort* is the best method for the first six classes of inputs: five random arrays plus k-limited arrays. In our experiment, we found mergesort to be the best method for the remaining six classes of inputs, with the exception of k-distance arrays when $k \leq 32 (n = 2,000,000)$; in this case insertion sort is slightly faster than *mergesort*. Both *k-distance* and *k-exchange* arrays model nearly sorted arrays by several researchers [14]. Our experimental results show clearly that *mergesort* is the best choice for nearly sorted arrays. The second best is *quicksort*, which performs much better than the splaysort method reported in [13]. Whenever we use the above 12

classes of instances, we will always include the best performers for each class so that the ratios of each method's run time to the best time for that class will be kept the same.

3 | SELECT BEST MERGESORT

The first choice for selecting the best implementation of natural *mergesort* is perhaps *timsort* [9]. Timsort is a highly optimized implementation of natural *mergesort* with at least the following techniques:

(a) When scanning the input, if it finds a reverse sorted subsequence, it will reverse the subsequence to create a longer run. If the stable property is required, either the reverse sorted subsequence cannot contain equal elements (as done in timsort) or sub-arrays of equal elements are reversed twice (as done in our implementation).

(b) If a run in the input is less than the minimal length for a run, insertion sort is called to create a longer run.

(c) The minimal length of runs can be preset, or adjusted so that if each run keeps the minimal length, then the number of runs is (or slightly less than) a power of two.

(d) A binary search is used in the insertion sort to locate the position of the element to be inserted in the sorted list.

(e) Runs are pushed in a stack. Let X, Y, Z be the sizes of the top three runs in the stack, then it maintains that $X > Y + Z$ and $Y > Z$. If $X < Y + Z$, then Y is merged with the smaller of X and Z until the two conditions are satisfied.

(f) Merge is always done among two consecutive sorted subarrays, say A and B. Any element in A no bigger than the first element of B is removed from the merge process. Similarly, any element of B bigger than the last element of A is also removed from the merge process. If A is shorter than B, then A is copied into a temporary memory; otherwise B is copied before the merge process

(g) Merge goes into the galloping mode if A (or B) wins for a consecutive number of times (a preset number). In the galloping mode, the binary search is used to locate the position in A where the current element in B should go.

Example: Suppose the input $A = (1, 15, 18, 19, 20, 16, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2)$, *Run(A)* $= 12$ and *Mono(A)* $= 2$, because A consists of one sorted subarray from 1 to 20 (a run) and one reverse sorted subarray from 16 to 2 (a reversed run). Timsort will reverse the second subarray and merge two runs into one. During the merge process, the first number in A, i.e., 1, will be skipped because it is smaller than 2, the first number of the second run. Then 15 through 20 in the first run will be copied into a temporary memory. Once 1, 2, 3, 4 are moved into the final position, since the second run won three times in a row, it goes into the galloping mode: using binary search, it finds that 15 falls in between 11 and 16 of the second run. So 5 through 11 will be moved to the final position without comparing them one by one against 15.

For neatsort [10], (a) and (f) are used; a heuristic different from (e) is used to select two runs to be merged. We have implemented our own *mergesort* and have tested (a) – (g) except (d). We found that (a), (f) and (g) have great impact to the performance of *mergesort*, so all of our implementations will use them. We also found that (b) provides a little performance improvement for most cases. However, we found that (c) has almost no impact. The motivation behind (c) is to avoid the merge of a very long run with a very short run. If the sizes of each run cannot be the same, the number of runs can hardly be an expected power of two. For (d), we found that using binary search in insertion sort, the impact to the running time is marginal, but it reduces the number of comparisons substantially. When expensive comparison functions are used, we still do not see the gain on running time. Instead of (e), we have tested two new strategies to select runs to merge:

(e_1) Merge of runs takes multiple rounds: in each round, all runs are paired and merged into one.

(e_2) Maintain a list of runs such that the length of each run is less than the half of the length of its predecessor. If the condition is not true, merge this run with its predecessor. Such a list is called "half-down list".

It turns out that (e), (e_1), (e_2) or *neatsort*'s heuristic have a small impact on the overall performance.

Table 3 provides the performance results of eight different implementations of *mergesort* for 12 classes of inputs. The value of k is 2^i, where $i = 0, 1, ..., 8$, for the last seven classes of inputs. The column "best" shows the best average running time of 100 executions, in microseconds. It may be produced by other implementations not listed in the table. The running times of each

TABLE 3 Performances of 8 different implementations of *mergesort* for 12 classes of inputs.

n=2000000	best	qsort	mer2	mer3	mer4	mer5	mer6	Tim	neat
random 16-list	764548	1.43	1.74	1.80	1.77	1.32	1.27	1.21	1.80
random 64-list	1279070	1.25	1.84	1.99	1.89	1.26	1.25	1.19	2.00
random 256-list	1530270	1.24	1.81	1.92	1.83	1.19	1.20	1.16	1.88
random double	249758	1.14	1.07	1.12	1.08	1.07	1.01	1.11	1.15
random integer	231084	1.17	1.11	1.16	1.12	1.12	1.06	1.13	1.17
k-limited	142186	1.53	1.30	1.39	1.30	1.37	1.26	1.35	1.51
k-equal teeth	25180	3.24	1.38	1.03	1.05	**1.00**	1.03	1.10	1.20
k-even teeth	24712	3.56	2.90	1.04	1.19	**1.00**	1.14	1.11	1.21
k-sharp teeth	13846	6.14	4.47	1.15	1.34	**1.00**	1.26	1.22	1.61
k-shuffled teeth	96920	1.54	1.04	1.14	**1.00**	1.16	1.01	1.07	1.49
k-distance	51051	2.15	**1.00**	1.43	1.05	1.49	1.10	1.27	1.38
k-exchange	10075	8.55	1.78	1.21	1.24	**1.00**	1.09	1.11	3.28
Average (12)	400872	1.24	1.56	1.64	1.57	1.11	1.09	1.08	1.65
Average (8)	82566	1.65	1.18	1.10	1.03	1.08	**1.00**	1.07	1.23

method are given as a ratio to the best average time. If it is 1.00, then this is the best performer among all sorting methods in our experiment. The column "qsort" gives the running time of the GNU qsort. The columns mer2–mer6 give the running times of our five versions of *mergesort* , which differ by the used techniques:

- mer2: uses (b), (f), (g), (e_1)

- mer3: uses (a), (f), (g), (e_1)

- mer4: uses (a), (b), (f), (g), (e_1)

- mer5: uses (a), (f), (g), (e_2)

- mer6: uses (a), (b), (f), (g), (e_2)

In fact, mer2 is not natural merge sort, as runs are not discovered from the input but created by insertion sort. The code of mer6 is given in the Appendix A for reference. The last row gives the average running times of the last 8 classes of inputs (not the average of ratios). The second last row gives the average time of all 12 classes. Mer6 is the best performer of the last 8 classes of inputs; mer4 is the second best (3% slower) and timsort is the third (7% slower). The GNU qsort is the worst performer in the table (65% slower).

When all the 12 classes are considered, *timsort* is the best among all *mergesort* implementations in our experiment, and is only 7% slower than the best *quicksort* implementation. Mer6 is the second best (1% slower than *timsort*). The advantage of *timsort* over mer6 is shown on the first three randomized classes where expensive comparison functions are used; for random integers or doubles, mer6 is still faster than *timsort* . The idea of (d) cannot reduce the running time of mer6 while it can reduce the number of comparisons. This may suggest that (e) is better than (e_1) and (e_2) for large size lists with high comparison costs.

For the three contrived classes, *k-equal teeth*, *k-even teeth*, and *k-sharp teeth*, mer5 is the best performer among all implementations, while the GNU *qsort* (Table 1, row 5) is at least three times slower than mer5. For k-exchange, the GNU *qsort* is almost eight times slower than mer5.

4 | SELECT BEST QUICKSORT

As the most popular sorting algorithm, *quicksort* has been extensively studied by many researchers and various techniques have been proposed. In this study, we consider B&M's *quicksort* implementation (Table 1, row 3). Let *n* be the input array size, the following techniques are in B&M *quicksort* function:

(a) If $n < \alpha$, call insertion sort. Bentley and McIlroy use $\alpha = 7$.

(b) If $n < \beta$, use the median of the first, middle and the last elements as pivot for splitting the array. Bentley and McIlory use $\beta = 40$.

(c) If $n \geq \beta$, use the pseudo-median of 9 elements (taken at every $(n/8)^{th}$) position plus the first element) as pivot.

(d) During the splitting process, all elements equal to the pivot are placed either at the beginning or end of the array. When the splitting is done, these elements equal to the pivot are moved in the middle of the array and are spared from the recursive calls.

(e) Do not identify the position of the pivot in the array. Bentley and McIlroy use this idea for all eight byte elements.

The use of (c) is to guarantee that a good pivot element is chosen, to avoid the quadratic worst-case time complexity. The use of (a) is to avoid the overhead of choosing a better pivot. The idea behind (d) is called 3-way splitting, and *quicksort* using (d) is called the 3-way *quicksort*. Evidently, when no repeated elements are present, 3-way splitting is slower than the classic 2-way splitting [15]. The idea of (e) is to avoid swapping the pivot with other elements so that the presortedness of the input is not disturbed.

For the 3-way splitting, at the end of splitting, the array is divided into three parts: the first part contains elements less than the pivot; the middle part contains elements equal to the pivot and the last part contains elements greater than the pivot. Recursive calls apply to the first and the last parts. The 2-way splitting in any standard textbook [15] also splits the array into three parts: the elements less than the pivot, the pivot, and the elements no less than the pivot. Using (e) without (d), the array is split into two parts: the elements no greater than the pivot and the elements no less than the pivot.

Besides the ideas used by Bentley and McIlroy, we also have tested the following two ideas:

(f) If $n \geq \alpha$, test if the array is already sorted; if yes, exit.

(g) At first, use the 2-way splitting. If the array contains more than $\gamma = 2$ elements equal to the pivot, use the 3-way splitting in the recursive calls.

The use of (f) is to avoid wasting time on already sorted subarrays, which occur often in nearly sorted inputs. The use of (g) is a typical example of hybrid algorithms. The program starts with the 2-way *quicksort*; when it identifies repeated elements, it turns to the 3-way *quicksort*. Combining (e) and (f), reverse sorted lists are no longer a problem for *quicksort* : Using the median of the input as pivot, the splitting will reverse the list and the two recursive calls will finish quickly with the use of (f). In other words, *quicksort* becomes adaptive as it takes O(n) time on sorted or reverse sorted lists. It also works well for some other troublesome instances including organ-pipe lists.

Table 4 shows the performance of seven implementations of *quicksort* on the same set of inputs used in Table 3. For reference, we also copied four versions of *mergesort* from Table 2: mer5, mer6, Tim and *qsort*. The seven quicksort implementations are: (1) B&M, Bentley and McIlroy's *quicksort* function. (2) 3-way: the 3-way *quicksort* uses the ideas of (a) to (f), and is almost identical to B&M except (f), i.e., test if the array is sorted as a preprocessing step. (3) 2-way: the 2-way *quicksort* using the ideas of (a), (b), (c), (e), and (f). (4)-(7): Four variants of hybrid *quicksort* using the ideas of (a)-(g), i.e., hyb1, hyb2, hyb3, and hyb4, with $(\alpha, \beta) = (16, 16), (32, 32), (32, 64)$, and $(16, 32)$ respectively. The BSD *qsort* is excluded here because it performs poorly on the k-sharp teeth instances when $k \geq 5$.

The performances of all methods on randomized inputs (the first five classes) are about the same, with the GNU *qsort* (Table 1, row 5) being the worst performer. The high cost on the class of random 256-lists is due to an expensive comparison function and high number of cache misses. Comparing to *mergesort* , *quicksort* as a family is slower on non-randomized inputs. In fact, the hybrid *quicksort* is the best performer over all the 12 classes of inputs. If we have to select one implementation of *quicksort*, then hyb2 ($\alpha = \beta = 32$) is the choice according to Table 4. The GNU *qsort* (Table 1, row 5) and B&M *quicksort* (Table 1, row 4) are about 23% slower than hyb2. The use of idea (f) is indispensible for this performance as it helps *quicksort* to save time on nearly sorted inputs.

Sarwar et al. [16] once conducted extensive experiments on *quicksort* and tried various methods for selecting a pivot (from 1, 3, 5, 9, and 17 elements). They concluded that *quicksort* performs best when the pivot is the median of three elements and there is no need to call insertion sort. Our experiment shows that their claims are true only for randomized inputs. For 2-equal teeth or 2-even teeth, which are called the organ-pipe shape inputs, if we let $\beta = \infty$, *quicksort* will always pick the median of three

TABLE 4 Performance of *quicksort* and *mergesort* implementations for 12 classes of inputs.

n=2000000	best	B&M	3-way	2-way	hyb1	hyb2	hyb3	hyb4	mer5	mer6	Tim	qsort
random 16-list	764548	1.11	1.12	1.28	1.10	**1.00**	1.08	1.00	1.32	1.27	1.21	1.43
random 64-list	1279070	1.06	1.05	1.02	1.03	1.02	1.03	**1.00**	1.26	1.25	1.19	1.25
random 256-list	1530270	1.13	1.09	1.04	1.04	**1.00**	1.04	1.03	1.19	1.20	1.14	1.24
random double	249758	1.17	1.13	1.05	1.04	1.05	**1.00**	1.06	1.07	1.01	1.11	1.14
random integer	231084	1.18	1.11	1.03	1.03	1.05	**1.00**	1.05	1.12	1.06	1.13	1.17
k-limited	142186	1.13	1.07	1.10	1.03	1.04	**1.00**	1.04	1.37	1.26	1.35	1.53
k-equal teeth	25180	5.54	4.71	4.93	4.67	4.36	4.17	4.66	**1.00**	1.03	1.10	3.24
k-even teeth	24712	5.72	4.84	4.97	4.82	4.52	4.28	4.81	**1.00**	1.14	1.11	3.56
k-sharp teeth	13846	6.14	2.42	1.94	2.28	2.12	1.85	2.04	**1.00**	1.26	1.22	6.14
k-shuffled teeth	96920	2.37	2.09	1.95	1.96	1.97	1.88	1.98	1.16	1.01	1.07	1.54
k-distance	51051	2.53	2.41	2.04	2.09	2.17	1.85	2.21	1.49	1.10	1.27	2.15
k-exchange	10075	9.52	2.95	2.44	2.69	2.69	2.18	2.46	**1.00**	1.09	1.11	8.55
Average (12)	400872	1.23	1.06	1.04	1.02	**1.00**	1.01	1.01	1.11	1.09	1.08	1.24
Average (8)	82566	1.93	1.56	1.48	1.47	1.46	1.37	1.49	1.08	**1.00**	1.07	1.65

TABLE 5 Average number of comparisons per element (n = 2,000,000).

cmp per item	3-way	2-way	hyb2	hyb4	B&M	qsort	mer4	mer5	mer6	Tim
random 16-list	22.51	24.62	24.47	23.47	22.57	19.68	21.62	20.77	21.81	19.20
random 64-list	22.74	24.73	24.70	23.41	22.08	19.67	21.62	21.09	22.22	19.64
random 256-list	22.92	24.83	24.80	23.50	22.07	19.67	21.62	21.04	22.17	19.64
random double	22.79	24.83	24.79	23.51	22.08	19.67	21.62	21.09	22.16	19.64
random integer	22.95	24.80	24.77	23.48	22.13	19.67	21.62	21.03	22.15	19.64
k-limited	13.65	16.64	14.63	14.06	13.30	18.89	15.89	15.23	16.17	14.19
k-equal teeth	19.24	22.86	19.68	19.24	20.59	11.23	4.02	4.02	4.02	4.02
k-even teeth	18.78	22.51	19.36	18.91	20.44	11.31	4.67	4.05	4.49	4.16
k-sharp teeth	6.65	6.39	6.46	6.58	19.90	9.32	1.33	0.89	1.33	0.89
k-shuffled teeth	19.07	20.54	20.48	19.49	21.61	16.57	8.52	7.18	8.43	6.71
k-distance	20.26	20.62	20.53	20.85	20.63	11.72	5.40	5.31	5.35	4.99
k-exchange	6.07	6.07	6.07	6.07	19.69	12.36	1.00	1.00	1.00	1.00
Average cmp	17.64	19.70	18.56	17.96	20.09	15.81	12.41	11.89	12.61	11.14

elements as pivot and in this case, it exhibits the $\mathcal{O}(n^2)$ worst-case behavior. The idea of (c), i.e., use the pseudo-median of 9 elements as pivot, essentially prevents the worse-case from happening. In this case, the idea (a) helps to reduce the overhead of (c). When $\alpha = \beta$, (b) is excluded and (c) becomes the only strategy for selecting a pivot. Our experiment shows that as long as $16 \leq \alpha \leq \beta \leq 100$, the performance of *quicksort* has little difference on random inputs.

5 | COUNT NUMBERS OF COMPARISONS

In the previous sections, we used the running time as the metric to select the best performer. In the GNU C library *qsort* function, the comparison function must be passed as a parameter to *qsort* and we use the interface of *qsort* as a model for every sorting method. Since we have control over the comparison function, we can count how many comparisons a sorting algorithm performs, which is also a pretty good metric.

TABLE 6 Value of *s* in the number of comparisons function $f(n) = sn \log(n) + tn$.

value of *s*	3-way	2-way	hyb2	hyb4	B&M	qsort	mer4	mer5	mer6	Tim
random 16-list	1.03	1.06	1.01	1.04	1.10	1.00	1.13	1.02	1.01	1.02
random 64-list	1.14	1.11	1.11	1.11	1.13	1.01	1.13	1.02	1.01	1.02
random 256-list	1.09	1.07	1.07	1.07	1.07	1.01	1.13	1.03	1.05	1.02
random double	1.07	1.02	1.01	1.02	1.09	1.01	1.13	1.01	1.02	1.02
random integer	1.06	1.12	1.12	1.12	1.06	1.01	1.13	1.04	1.05	1.02
k-limited	0.46	0.50	0.46	0.48	0.48	0.96	0.54	0.49	0.50	0.51
k-equal teeth	1.24	1.27	1.23	1.23	1.31	0.41	0.01	0.01	0.01	0.00
k-even teeth	1.22	1.21	1.27	1.27	1.31	0.45	0.00	0.00	0.00	0.00
k-sharp teeth	0.04	0.03	0.03	0.02	0.98	0.45	0.00	0.00	0.00	0.00
k-shuffled teeth	0.79	0.77	0.78	0.79	1.07	0.84	0.00	0.00	0.00	0.00
k-distance	1.11	1.10	1.10	1.08	1.02	0.49	0.00	0.00	0.00	0.00
k-exchange	0.02	0.02	0.02	0.02	1.06	0.45	0.00	0.00	0.00	0.00

TABLE 7 Value of *t* in the number of comparisons function $f(n) = sn \log(n) + tn$.

value of *t*	3-way	2-way	hyb2	hyb4	B&M	qsort	mer4	mer5	mer6	Tim
random 16-list	-1.03	1.36	1.31	0.56	-0.59	-1.34	-1.53	-0.43	0.16	-1.80
random 64-list	-1.09	1.47	1.44	0.13	-1.53	-1.40	-1.98	-0.38	1.08	-1.77
random 256-list	0.12	2.40	2.38	1.09	-0.34	-1.39	-1.97	-0.53	0.22	-1.77
random double	0.47	3.47	3.69	2.16	-0.76	-1.38	-1.94	-0.11	0.74	-1.81
random integer	0.83	1.33	1.40	0.01	-0.15	-1.39	-1.95	-0.80	0.13	-1.79
k-limited	3.93	6.27	4.93	4.00	3.29	-1.13	4.57	5.05	5.66	3.49
k-equal teeth	-6.66	-3.77	-6.03	-6.46	-6.93	2.72	3.80	3.80	3.80	4.02
k-even teeth	-6.85	-2.83	-7.33	-7.77	-7.06	1.86	4.67	4.05	4.49	4.16
k-sharp teeth	5.73	5.70	5.75	6.90	-0.55	-0.09	3.36	3.89	3.35	4.91
k-shuffled teeth	2.53	4.32	4.17	3.05	-0.71	-0.97	8.48	5.43	5.55	6.68
k-distance	-2.91	-2.46	-2.47	-1.67	-0.68	1.42	5.73	5.32	5.36	5.53
k-exchange	5.65	5.65	5.65	5.65	-2.41	2.92	1.05	1.05	1.05	1.05

Table 5 lists the average number of comparisons per item for the same inputs in Table 4 ($n = 2,000,000$) for the selected 10 sorting methods, the first five from *quicksort* , and the second five from *mergesort*. That is, each number is the average of 100 numbers of comparisons per item for randomized arrays. To compute the average number of comparisons for each *k*-class, we chose 9 values for *k*. The value of *k* is 2^i, where $i = 0, 1, \ldots, 8$. For each class, we took the average number of comparisons from 50 executions and then took the average for the whole class. The use of binary search in insertion sort can reduce the number of comparisons by about 10% (not shown here) for those methods using insertion sort.

For $n = 2,000,000$, hyb4 makes on average $19n$ comparisons for *k-equal teeth* and *k-even teeth* arrays, lower than $23n$ comparisons for randomized lists (and much lower than $38n$ of heapsort for *k-equal teeth* and *k-even teeth* arrays). The idea of testing whether an array is already sorted is used in the first four *quicksort* methods. However, Table 5 does not show any significant increases in the number of comparisons. In fact, the small values for the *k-sharp teeth* and *k-exchange* classes tell that this idea is very useful.

As an approximation, we may assume that the number of comparisons is defined by a function $f(n) = sn \log(n) + tn$ on an input array of *n* elements, where *s* and *t* are constants. The number of comparisons per item is $g(n) = f(n)/n = s \log(n) + t$, which happens to be a line on $\log(n)$. To determine the values of *s* and *t*, we run our implementations on the 12 classes of inputs of various sizes: there are five values for $n = j10^6$, where $j = 1, ..., 5$. We then used linear regression to compute the approximate values of *s* and *t* (each line is computed over 5 points).

Tables 6 and 7 give the values of *s* and *t* for the selected 10 sorting methods. For randomized inputs, the values of *s* for the 10 methods are close to 1, indicating these are very good sorting methods. For instance, for hyb4 over random doubles, the number of comparisons is $f(n) = 1.02n \log(n) + 2.16n$. These numbers also indicate that the relative strength of these method tend to be the same as comparison functions of different costs are used. For the classes of *k-equal teeth* and *k-even teeth* arrays, the *s* values of *quicksort* are slightly larger than 1, indicating that these are hard problems for *quicksort*. For the last six classes, with the exception of the GNU *qsort* (Table 1, row 5), the *s* values of the *mergesort* methods are (almost) zero. That is, these methods take linear time to sort these six classes. That explains why these methods are so fast on these inputs.

6 | CONCLUSIONS

Through extensive experiments, we have achieved our goals for this project.

- We have suggested 12 classes of inputs which exhibit different degree of presortedness and can be easily generated by a program. Using them, we can identify the best sorting methods and test other methods against them. For instance, we have tested a dozen of series for shellsort[6] and found that the best shellsort is still 180% slower than the best mergesort or quicksort over the 12 classes of inputs. The biggest disappointment is heapsort, which has the optimal complexity of $\mathcal{O}(n \log(n))$, but performs poorly. For the bottom-up heapsort[8], which is claimed to be much faster than *quicksort*[17], it is about three times slower over the 12 classes of inputs and 7 times slower over the six classes of nearly sorted inputs. Smoothsort[18], a variant of heapsort, runs about two times slower than *quicksort*. Splaysort[13] is also much slower than *mergesort* and *quicksort* on nearly sorted inputs. All the sorting methods mentioned above and in the previous sections are implemented in a C program called *sortrace*. The 12 inputs can also be generated from the same program. That is the software platform we plan to extend, as we will add more types of inputs into the platform. The C code of *sortrace* is publicly available[19].

- For sorting methods based on natural *mergesort*[7], we have identified three techniques of *timsort*[9] as important: (1) reversing revered runs; (2) the galloping merge, and (3) management of runs to decide which and when to merge two runs. The weakness of neatsort[10] is perhaps due to the missing of (2). We proposed a new way to manage runs (in mer5 and mer6 using the half-down list) which has better performance than *timsort* for nearly sorted inputs. The use of insertion sort is helpful as evident by the slight gain of mer6 over mer5. The use of binary search in insertion sort has high impact on the number of comparisons but little impact on running time. The adjustment of the minimal run length is easy to implement but does not have any significant impact on performance.

- For sorting methods based on *quicksort*, we confirmed that Bentley and McIlroy's method of choosing a pseudo-median over 9 elements as pivot is crucial to avoid the worst case complexity of $\mathcal{O}(n^2)$. We proposed two new ideas to improve *quicksort*: (1) testing sortedness as a preprocessing step in each recursive call; (2) a hybrid *quicksort* combing the 2-way *quicksort* with the 3-way *quicksort*. The idea of (1) is to make *quicksort* adaptive to nearly sorted inputs. The idea of (2) is to avoid the overhead of 3-way splitting when there are few identical elements. As a result, our *quicksort* is the overall champion for the 12 classes of inputs. For the 8 k-classes of inputs, our *quicksort* also beats the GNU *qsort*, even though the worst-case time complexity of *quicksort* is worse than that of *mergesort* (or heapsort). For the textbook version of the *quicksort* algorithm[15], the worst-case time complexity comes when the input list is already sorted or near-sorted. This property was regarded as a significant drawback of *quicksort*. Our experiment shows that using the proposed techniques in this article, we can not only avoid the worst-case complexity completely but also make *quicksort* adaptive to nearly sorted inputs. Apparently, the C library should not shun *quicksort*.

We have tried to avoid theoretic analysis and technical details in this article. As further research, we are interested in a formal analysis of the adaptiveness of *quicksort* when sortedness testing is used as a preprocessing step. Apparently such testing uses more comparisons. However, the experiment shows no significant increase in the number of comparisons. For instance, for hyb4 on *n* randomized doubles, the number of comparisons is $f(n) = 1.02n \log(n) + 2.16n$, while B&M's *quicksort* uses $f(n) = 1.09n \log(n) + 0.76n$. Moreover, we would like to see how the complexity of *mergesort* and *quicksort* is related to the metric *Mono(A)*.

One advantage of *mergesort* over *quicksort* is that *mergesort* is stable. In general, *mergesort* takes $\mathcal{O}(n)$ more working memory than *quicksort* as required by merging. In our experiments, we see that both methods slow down on long lists of heavy items.

Timsort works better than our *mergesort* on such case, but is still slower than *quicksort*. This is evident from Table 4 that hyb2 is shown to be the best for lists of 256 numbers. Our experiment has been run on a linux machine with an Intel(R) Xeon(R) CPU E5-2667 (20M Cache and 3.20GHz) and 64GB memory. Further experiments are needed to test the limits of both methods and search for better techniques to deal with memory shortage. We would also like to test our program on different machines and investigate the impact of cache memory.

Regarding the question raised in the beginning of this article, "if you are asked to recommend a sorting algorithm for an unknown problem, what is your recommendation?" What will be your answer after reading this article? Our answer is *quicksort*, especially our hybrid *quicksort*, if stable sorting is not required. When stable sorting is required, or the input contains some kind of presortedness and the memory is not a concern, then *mergesort*, either *timsort* or ours. When your task needs to sort short lists ($n \leq 2000$) many times, our recommendation is still *quicksort*, either 3-way or hybrid, as the top 3 performers all come from the *quicksort* family for small lists in our experiment.

7 | ACKNOWLEDGEMENTS

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APPENDIX

A OUR IMPLEMENTATION OF MERGE SORT MER6 IN C

```
1  // merge(arr, left, middle, right) merges arr[left, middle−1] and arr[middle, right−1] into arr[left, right−1]
2  // reverse(arr, left, right) reverse the subarray arr[left, right].
3
4  void mergesort6 (WORD *arr, size_t n, size_t es, int (*cmp)(const void *, const void *)) {
5      int *run, next, j, s, t, runnum, start;
6      run = (int *) malloc(64*sizeof(int));   // size of run is log(n) − 64 is more than enough.
7      runnum = run[0] = start = 0; next=1;
8      while (next < n) {
9          if (cmp(arr+next−1, arr+next)>0) {   // i.e., if (arr[next−1] > arr[next])
10
11             if (start +minrun > next) {      // make sure each run has at least minrun elements
12                 j = next+1; s = 0;   // look for reversed run and reverse equal elements
13                 while (j < n && ((t=cmp(arr+j−1, arr+j))>=0)) {
14                     if (t ==0) s++;
15                     else if (s > 0) { reverse(arr, j−1−s, j−1); s = 0; }   // i.e., reverse arr[j−1−s .. j−1].
16                     j++;
17                 }
18                 if (next > start +1) {
19                     reverse(arr, next, j−1);                 // reverse from next to j−1, then merge two parts.
20                     merge(arr, start, next, j);
21                 } else { reverse(arr, start, j−1); }   // reverse from start to j−1
22                 next = j;
23
24                 if (start +minrun > next) {      // still too short, use insertion sort to extend the run
25                     j = (start +minrun<n)? start+minrun : n;
26                     isort(arr, start, next, j);       // insertion sort arr[start .. j−1] where arr[next .. next−1] is sorted.
27                     next = j;
28             } else {  // of (start +minrun > next)
29                 // record a new run
30                 run[++runnum] = start = next++;
31
32                 // merge two runs if the current run > 1/2 of its predecessor
33                 while (runnum > 1 && (run[runnum]−run[runnum−1]) > (run[runnum−1]−run[runnum−2])>>1) {
34                     j = run[runnum−2];
35                     merge(arr, j, run[runnum−1], run[runnum]);
36                     run[runnum−1] = run[runnum];
37                     runnum−−;
38                 }
39             }
40         } else next++;
41     }
42
```

```
43        // merge all runs into one, from last to first .
44        while (runnum > 0) {
45            j = run[runnum−1];
46            merge(arr, j, run[runnum−−], n);
47        }
48        free (run);
49    }
```