Assignment 2: ARMA Processes and Seasonal Processes

Instructions: The assignment is to be handed in via DTU Learn "FeedbackFruits" latest at March 24th at 23:59. You are allowed to hand in in groups of 1 to 4 persons. You must hand in a single pdf file presenting the results using text, math, tables and plots, do not include code in the report!. Arrange the report in sections and subsections according to the questions in this document. Please indicate your student numbers on the report.

Be aware that the model parametrizations in the assignment and in your favorite software package may not be identical. You will study how the choice of coefficients (of the operator polynomials in ARMA processes) affects the structure of the process, through simulated data and empirical autocorrelation functions (ACFs). The assignment is to some extent specified in terms of R commands. Users of other software packages should find similar functions to answer the questions.

This document includes a solution guide. In the peer-review process you must choose, for each section, one of the following four possibilities:

0: The group did not answer the questions or the answer was extremely flawed.

1: The group provided a partial answer to the questions, but some parts are uncorrect or missing.

2: The group provided a satisfactory answer to the questions (only few parts are missing and minor details may be missing or uncorrect).

3: The group provided an excellent answer to the questions.

However, the most important part of the review is that you give contructive feedback in comments, you must at least give the required number of comments set in the peer-review platformm.

1 Stability

Let the process $\{X_t\}$ be an AR(2) given by

$$X_t + \phi_1 X_{t-1} + \phi_2 X_{t-2} = \epsilon_t$$

where $\{\epsilon_t\}$ is a white noise process with $\sigma_\varepsilon = 1$.

Answer the following:

1.1. Determine if the process is stationary for $\phi_1 = -0.7$ and $\phi_2 = -0.2$ by analysing the roots of the characteristic equation.

> According to Theorem 5.9 ii) the process is stationary if the roots of $\phi(z^{-1})$ is within the unit circle. The roots are
>
> $$\phi(z^{-1}) = 0 \Leftrightarrow 1 + \phi_1 z^{-1} + \phi_2 z^{-2} = 0 \Leftrightarrow z^2 + \phi_1 z + \phi_2 = 0$$
>
> hence
>
> $$\frac{-\phi_1 \pm \sqrt{\phi_1^2 - 4\phi_2}}{2} = \frac{0.7 \pm \sqrt{0.49 + 0.8}}{2} \approx \{0.918, -0.218\}$$
>
> and thus they are within the unit cicle and the process is therefore stationary.
>
> Check with R
>
> ```
> abs(polyroot(c(-0.2,-0.7,1)))
>
> [1] 0.2179 0.9179
> ```

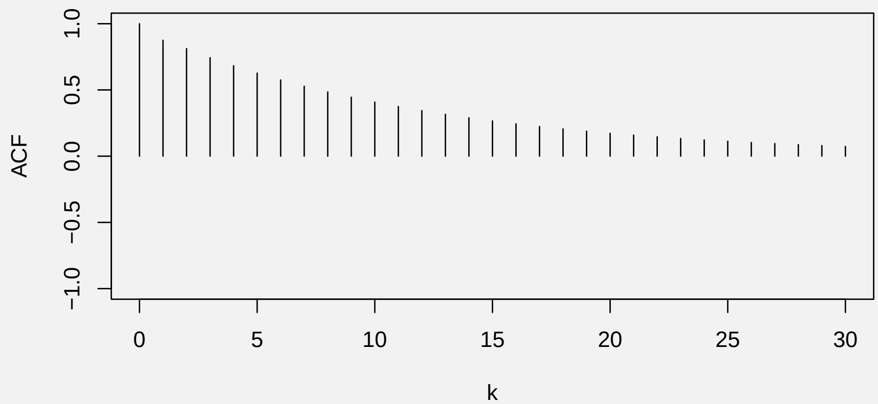
1.2. Is the process invertible?

> According to Theorem 5.9 i) an AR process is always invertible. Could also be shown using Definition 5.11.

1.3. Write the autocorrelation $\rho(k)$ for the AR(2) process as function of ϕ_1 and ϕ_2.

> We have actually the derivation of $\rho(k)$ for an AR(3) in Example 5.8, so we remove parts with ϕ_3 and get
>
> $$\rho(1) = \frac{-\phi_1}{1 + \phi_2} \tag{1}$$
> $$\rho(2) = -\phi_2 - \rho(1)\phi_1 \tag{2}$$
> $$\vdots \tag{3}$$
> $$\rho(k) = -\phi_1 \rho(k-1) - \phi_2 \rho(k-2) \tag{4}$$

1.4. Plot the autocorrelation $\rho(k)$ up to $n_{\text{lag}} = 30$ for the coefficient values above.



The plot should have exponential decaying pattern.

2 Simulating seasonal processes

A process $\{Y_t\}$ is said to follow a multiplicative $(p, d, q) \times (P, D, Q)_s$ seasonal model if

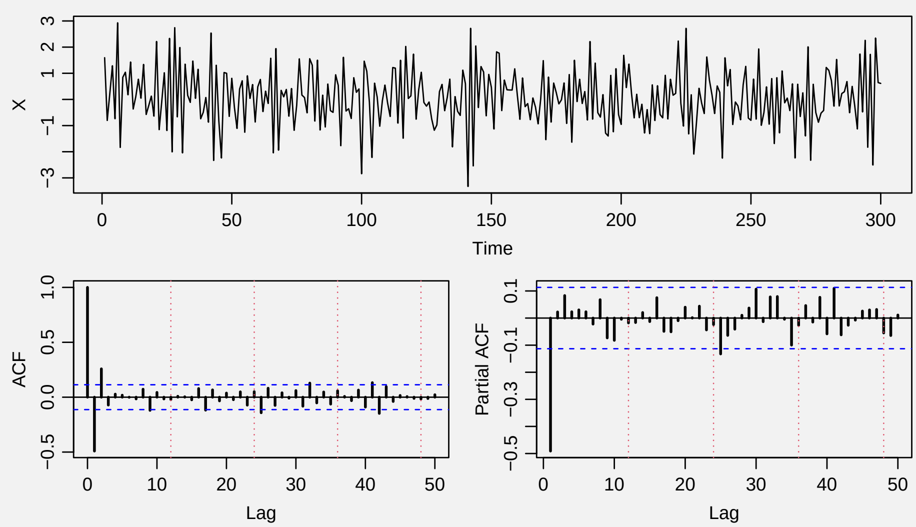
$$\phi(B)\Phi(B^s)\nabla^d\nabla_s^D Y_t = \theta(B)\Theta(B^s)\epsilon_t$$

where $\{\epsilon_t\}$ is a white noise process, and $\phi(B)$ and $\theta(B)$ are polynomials of order p and q, respectively. Furthermore, $\Phi(B^s)$ and $\Theta(B^s)$ are polynomials in B^s. All according to Definition 5.22 in the textbook.

Note: `arima.sim` does not have a seasonal module, so model formulations as standard ARIMA processes have to be made when using that function.

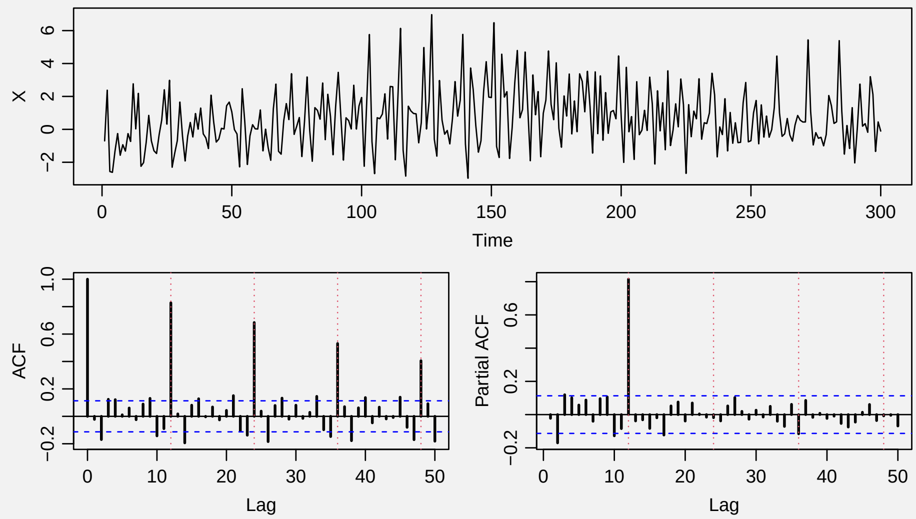
Simulate the following models. Plot the simulations and the associated autocorrelation functions (ACF and PACF). Comment on each result:

2.1. A $(1, 0, 0) \times (0, 0, 0)_{12}$ model with the parameter $\phi_1 = 0.6$.



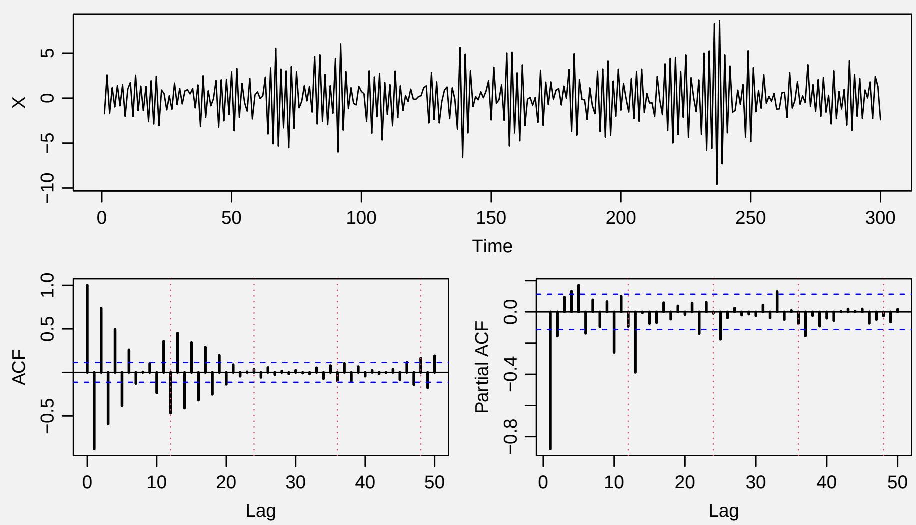
- Process (from which the time series is samples) is noisy and alternating around zero. The ACF has an alternating and exponential decay. PACF reveals the positive AR (ϕ) coefficient in lag 1 (often we would probably say "a positive AR coefficient", when put on other side, e.g. in `arima.sim`, but we should try keep the book definition...again, in real life things always get a bit mixed up...).

2.2. A $(0, 0, 0) \times (1, 0, 0)_{12}$ model with the parameter $\Phi_1 = -0.9$.



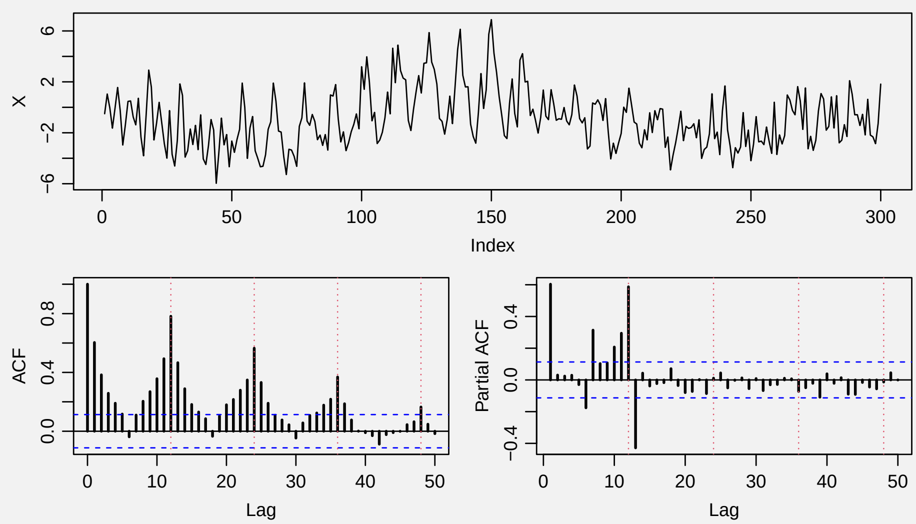
- Process with seasonal part. ACF is alternating and exponential decay in the seasonal lag. PACF only significant in the seasonal lag, hence the seasonal AR coefficient is reavealed.

2.3. A $(1, 0, 0) \times (0, 0, 1)_{12}$ model with the parameters $\phi_1 = 0.9$ and $\Theta_1 = -0.7$.



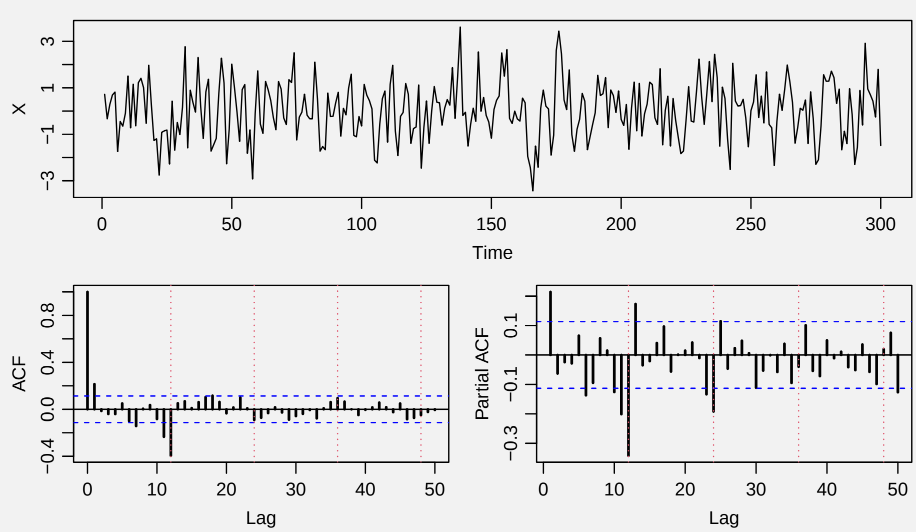
- "Strange process": Alternating becuase of both the AR coefficient and the "seasonal MA". It would be difficult to guess this process from the ACF and PACF!

2.4. A $(1, 0, 0) \times (1, 0, 0)_{12}$ model with the parameters $\phi_1 = -0.6$ and $\Phi_1 = -0.8$.



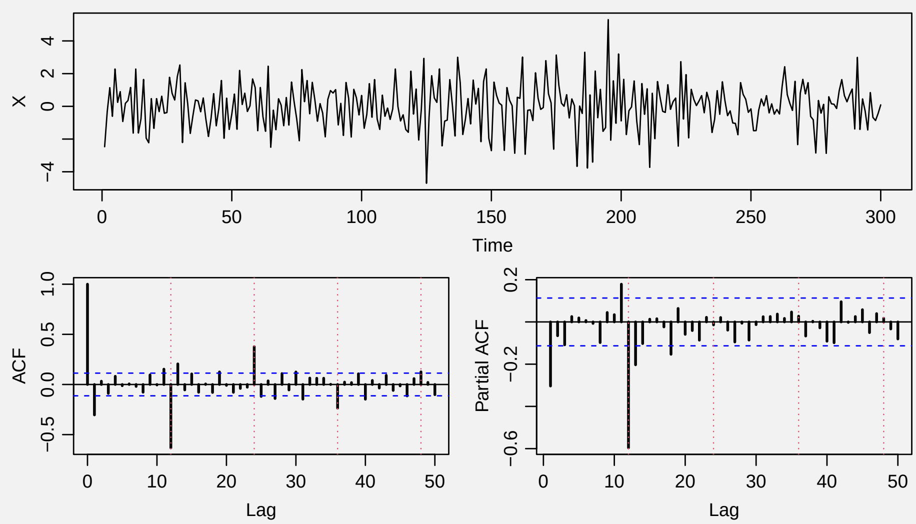
- The quite "high" AR and seasonal AR coefficients could lead to think, that this is a non-stationary process, however it turns out not to be.
- The process can be simulated with `arima.sim(list(ar=c(0.6,rep(0,10),0.8,-0.6*0.8)),n=300)`.
- Forgetting to include the 13 lag `-0.6*0.8` coefficient, will result in non-stationary process.
- We will give full points to everyone for this question, since we first gave a wrong solution (and also adviced about this solution on Ed)!: Non-stationary! `arima.sim()` won't simulate it! So either give up, or write your own function (or take the one from slides). Mad process: It simply explodes!

2.5. A $(0,0,1) \times (0,0,1)_{12}$ model with the parameters $\theta_1 = 0.4$ and $\Theta_1 = -0.8$.



- Noisy process. Hard to see the pattern in the time series plot, but ACF and PACF do reveal the seasonality.

2.6. A $(0,0,1) \times (1,0,0)_{12}$ model with the parameters $\theta_1 = -0.4$ and $\Phi_1 = 0.7$.



- Again hard to see any pattern in the time series plot, but from the ACF and PACF both the short and seasonal parts can be seen, but hard to see if it's an MA or AR terms for either.

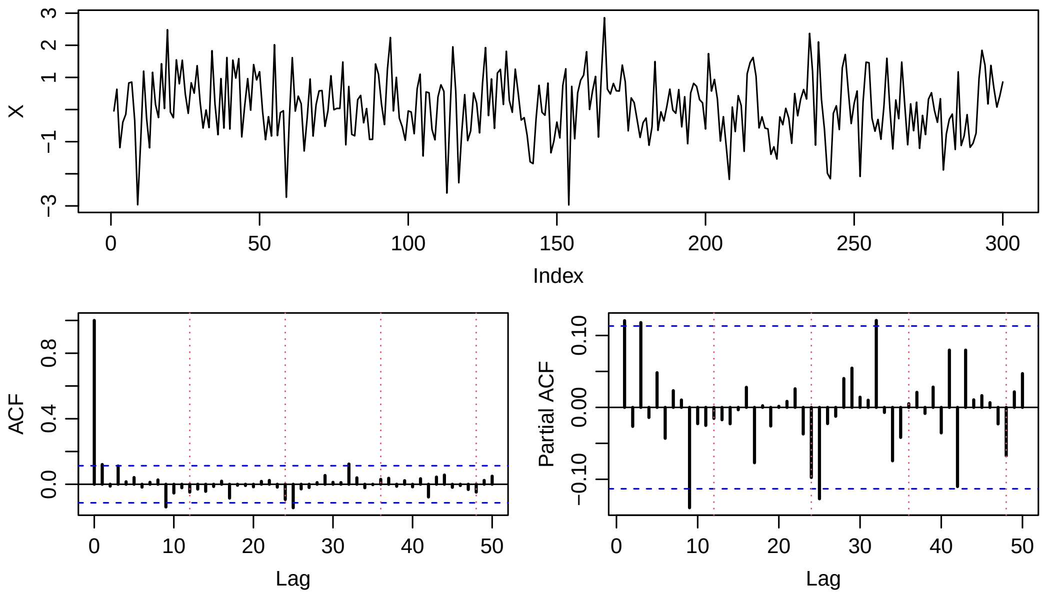
2.7. Summarize your observations on the processes and the autocorrelation functions. Which conclusions can you draw on the general behavior of the autocorrelation function for seasonal processes?

- When the process is only with either AR or MA lags, it's somewhat easier to recognize.
- Patterns cannot always be see in the time series plots, but ACF and PACF can often reveal them.
- When both AR and MA parts are included, the structure of the process becomes harder to see directly in the plots.

3 Identifying arma models

Below are plots of three simulated ARMA processes: time series plot, ACF and PACF. Guess the ARMA model structure for each of them, and give a short reasoning of your guess.

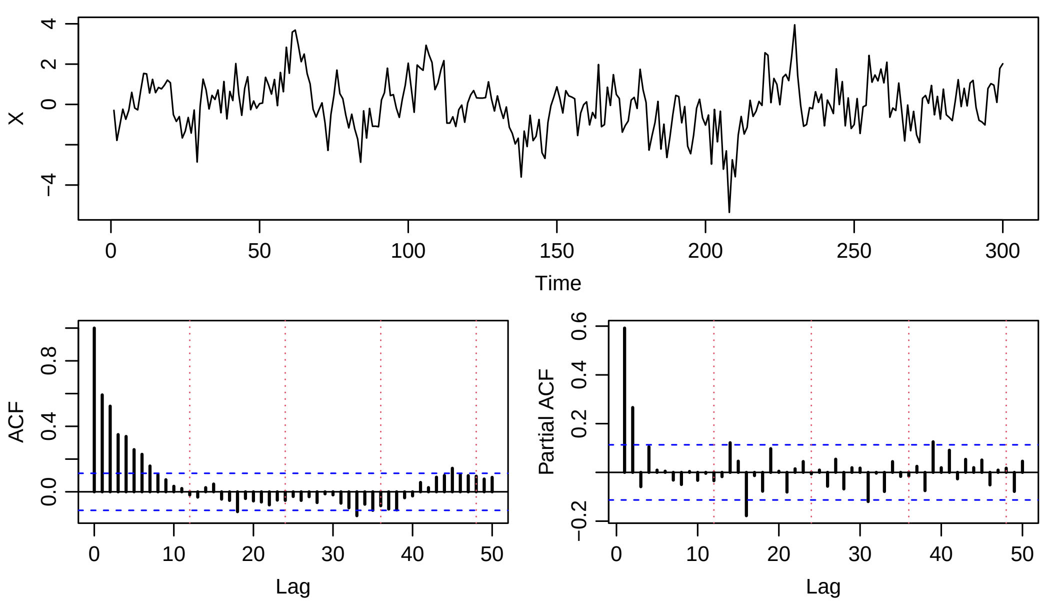
3.1. Process 1:



We use the "Golden table" (Table 6.1).

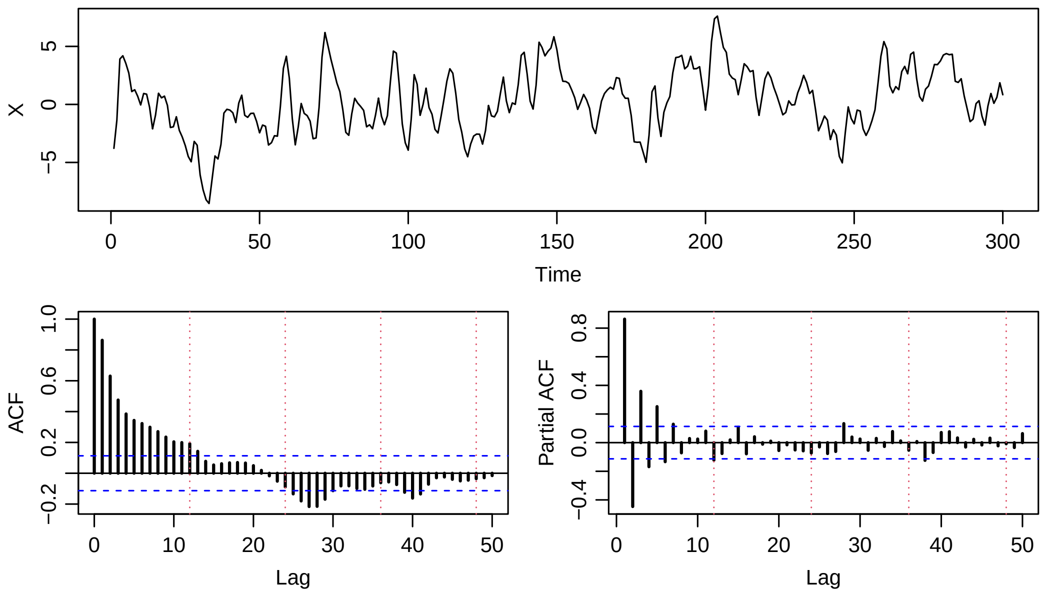
- No significant ACF and PACF, and no pattern seen in the time series plot, hence conclude it's white noise.
- The model is an ARMA(0,0), i.e. white noise.
- More than 5% of the lag correlations are just outside the CIs, but no clear pattern seen, however guess on some short AR part would be an option.

3.2. Process 2:



- Guess on ARMA(2,0).
- Seen from the exponential decay in ACF and only two significant lags in the PACF.
- Both ϕ coefficients seems to be negative.

3.3. Process 3:



- Guess on ARMA(1,1).
- Exponential decay in ACF indicates AR part with negative ϕ coefficient.
- Alternating exponential decay in PACF indicates MA part and negative θ coefficient
- However, it is actually the the other way around! i.e. $\phi_1 = 0.8$ and $\theta_1 = 0.9$. So AR part makes it alternating and the MA makes it decaying.

In conclusion: We can use ACF and PACF and the Golden table to get an idea of the ARMA structure, but we can't be certain, so with data "from real world processes" we use this in combination with model selection techniques to find a suitable structure.