Linear Algebra

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1. Vector Spaces

Definition 1.1: Vector Space

A triple $(V, +, \cdot)$ consisting of a set V, an addition map

$$+ : V \times V \to V, \quad (x, y) \mapsto x + y$$

and a scalar multiplication map

$$\cdot : \mathbb{F} \times V \to V, \quad (\lambda, x) \mapsto \lambda x$$

is called a real vector space if for all $x, y, z \in V$ and $\lambda, \mu \in \mathbb{R}$, the following axioms hold:

1. $(x + y) + z = x + (y + z)$
2. $x + y = y + x$
3. $x + 0 = x$
4. $x + (-x) = 0$
5. $\lambda(\mu x) = (\lambda\mu)x$
6. $1x = x$
7. $\lambda(x + y) = \lambda x + \lambda y$
8. $(\lambda + \mu)x = \lambda x + \mu x$

Note: 0 and 1 here denote the additive and multiplicative identities in $\mathbb{R}$.

Definition 1.2: Subspace

A subset U of V is a subspace if:

- $0 \in U$
- $u, w \in U \Rightarrow u + w \in U$
- $a \in \mathbb{F},\ u \in U \Rightarrow au \in U$

Definition 1.3: Sum of Subspaces

If $V_1, V_2, \ldots, V_m$ are subspaces of V, their sum is

$$V_1 + V_2 + \cdots + V_m = \{v_1 + v_2 + \cdots + v_m : v_i \in V_i\}$$

Theorem 1.1

The sum $V_1 + \cdots + V_m$ is the smallest subspace of V containing all V_i.

Proof. Because 0 is in $V_1, \ldots, V_m$, and subspaces are closed under addition, we get $0 + 0 + \cdots + 0 = 0 \in V_1 + V_2 + \cdots + V_m$. Then if x is in any V_i we can always put $0 + \cdots + x + 0 + \cdots = x \in V_1 + V_2 + \cdots + V_m$ Therefore, $V_1, V_2 + \cdots + V_m \in V_1 + V_2 + \cdots + V_m$. Now we want to prove that it's the smallest subspace. Suppost that there is a subspace of V, namely u which containing $V_1, \ldots, V_m$ and $U \subset V_1 + V_2 + \cdots + V_m$. Let x be a vector such that $x \in V_1 + V_2 + \cdots + V_m$ but not in u. For all i, there exists an $v_i \in V_i$ such that

$$u = v_1 + \cdots + v_m$$

And since U contains $V_1, V_2, \ldots, V_m$ then

$$\begin{aligned} v_1 + v_2 &\in U \\ (v_1 + v_2) + v_3 &\in U \\ &\cdots \\ (v_1 + v_2 + \cdots + v_{n-1}) + v_n &\in U \end{aligned}$$

Hence, $u \in U$, which contradicts our assumption that x is not in U. Therefore $V_1 + \cdots + V_m$ is the smallest subspace of V containing $V_1, \ldots, V_m$. □ □

Definition 1.4: Direct Sum

The sum $V_1 + \cdots + V_m$ is a *direct sum*, denoted $V_1 \oplus \cdots \oplus V_m$, if each element of the sum can be written uniquely as $v_1 + \cdots + v_m$ with $v_i \in V_i$.

Theorem 1.2

The sum $V_1 + \cdots + V_m$ is a direct sum if and only if the only way to write 0 as $v_1 + \cdots + v_m$ with $v_i \in V_i$ is when all $v_i = 0$.

Proof. It suffices to prove the converse, that is if

$$v_1 + v_2 + \cdots + v_m = 0$$

then $v_i = 0$, where each $v_i \in V_i$. Now we suppose that there is an element in the sum can be written in two different ways. Suppose that a vector v can be written as

$$v = x_1 + \cdots + x_m = y_m + \cdots + y_m$$

where each $x_i, y_i \in V_i$. By subtracting the two expressions we get

$$(x_1 - y_1) + \cdots + (y_1 - y_m) = 0$$

Let $v_i = (x_i - y_i)$ we get $v_1 + \cdots + v_m = 0$. By the hypothesis, this implies v_i must be equal to zero, which means $x_i = y_i$ for all i, This contradicts the assumption that the representations are distinct, that is $(x_1, \ldots, x_m)$ is different from $(y_1, \ldots, y_m)$

Therefore, every element in $V_1 + \cdots + V_m$ has a unique representation. Hence, $V_1 + \cdots + V_m$ is a direct sum. □

Theorem 1.3

For subspaces $U, W \subset V$,

$$U + W \text{ is a direct sum } \Leftrightarrow U \cap W = \{0\}$$

Proof. We begin by proving the forward direction, then proceed to the converse. Suppose first that $U + W$ is a direct sum and x is a non-zero element such that $x \in U \cap W$ and $x \in U \oplus W$. According to the properties of direct sum, there exists a unique pair (u, w) with$u \in U$ and $w \in W$ such that

$$x = u + w$$

Using the properties of vector subspaces, we have $x, u \in U$ implies $x - u = w \in U$. $x, w \in W$ implies $x - w = u \in W$. $x, w \in U$ implies $x + u = u + w + w \in U$ However, x can be rewritten as

$$x = (u + w + w) - u$$

for $u + w + w \in U$ and $-u \in W$. This contradicts the assumption that the representations are distinct. Hence, $x = 0$ and $U \cap W = \{0\}$ Secondly, suppose that $U \cap W = \{0\}$ and $x \in U + W$

such that x can be written as

$$x = x_1 + y_1 = x_2 + y_2$$

which $(x_1, y_1) \neq (x_2, y_2)$ and $x_1, x_2 \in U, y_1, y_2 \in W$. Notice that we have x_1, x_2 implies $x_1 - x_2 \in U$ y_1, y_2 implies $y_2 - y_1 \in W$ By subtracting the two above expression we get

$$0 = x - x = (x_1 - x_2) + (y_2 - y_1)$$

This implies $x_1 - x_2 = y_2 - y_1 = t$, which means $t \in U$ and $t \in W$, or $t \in U \cap W$. However, 0 is the only element that belongs to $U \cap W$, then $t = 0$ and in particular $x_1 = x_2$ and $y_1 = y_2$ implies $(x_1, y_1) = (x_2, y_2)$, this contradicts the hypothesis that $(x_1, y_1) \neq (x_2, y_2)$. Hence, every element in $U + W$ can only be writen as a unique representation, which means $U + W$ is a direct sum . □ □

2. Finite-Dimensional Vector Spaces

- A linear combination of a list $v_1, \ldots, v_m$ of vector in V is a vector of the form

$$a_1 v_1 + \cdots + a_m v_m$$

- Set of all linear combinations of a list of vectors is called

$$span(v_1 + \cdots + v_m) = \{a_1 v_1 + \cdots + a_m v_m : a_1, \ldots a_m \in \mathbb{F}\}$$

- The span of a list of vectors in V is the smallest subspace of V containing all vectors in the list - If $span(v_1, \ldots, v_m) = V$ we say that list spans V. - a list of vectors in V is called linearly independent if the only choice of $a_1, \ldots, a_m \in \mathbb{F}$ that makes

$$a_1 v_1 + \ldots a_m v_m = 0$$

is $a_1 = \cdots = a_m = 0$.

Lemma 2.1: Linear dependence removal

Suppose $v_1, \ldots, v_m$ is a linearly dependent list in V. Then there exists $k \in \{1, \ldots, m\}$ such that

$$v_k \in span(v_1, \ldots, v_k)$$

Furthermore, if k satisfies the condition above and the k^{th} term is removed from $v_1, \ldots, v_m$ then the span of remaning list equals $span(v_1, \ldots, v_m)$

Theorem 2.1

In a finite-dimensional vector space, every linearly independent list has length $\leq$ any spanning list.

Proof. Let $A = (v_1, \ldots, v_m)$ be a linearly independent list, and let $B = (w_1, \ldots, w_n)$ be a spanning list of V. We aim to prove that $m \leq n$.

Since B spans V, each v_i can be written as a linear combination of vectors from B. In particular, v_1 is a linear combination of $w_1, \ldots, w_n$. Therefore, the list $(v_1, w_1, \ldots, w_n)$ is linearly dependent.

Because A is linearly independent, $v_1 \neq 0$. Hence, there must exist some $k > 0$ such that w_k is a linear combination of $v_1, w_1, \ldots, w_{k-1}$. Removing w_k from B, we still have a spanning list.

We continue this process: successively add each v_i into the list and eliminate one redundant vector in B each time to maintain a spanning list. Eventually, we obtain a list containing all of $v_1, \ldots, v_m$ and still spanning V. Since we remove one vector from B for each v_i added, we cannot perform this more than n times. Therefore, $m \leq n$. $\square$

Theorem 2.2

Every subspace of a finite-dimensional vector space is finite-dimensional.

Proof. Let V be a finite-dimensional vector space, and let $U \subseteq V$ be a subspace. We aim to show that U is finite-dimensional.

If $U = \{0\}$, then U is clearly finite-dimensional. Otherwise, there exists a nonzero vector $u_1 \in U$. Let $A = \{u_1\}$.

Proceed inductively: suppose we have constructed a linearly independent list $A = \{u_1, u_2, \ldots, u_{k-1}\} \subseteq U$. If $U \subseteq \operatorname{span}(A)$, then U is spanned by finitely many vectors, and we are done.

Otherwise, there exists $u_k \in U$ such that

$$u_k \notin \operatorname{span}(u_1, u_2, \ldots, u_{k-1}).$$

Add u_k to A and repeat the process.

Since V is finite-dimensional, any linearly independent set in V contains at most $\dim V$ vectors. Therefore, this process must terminate after finitely many steps. At termination, we have a finite linearly independent set $A \subseteq U$ such that $U \subseteq \operatorname{span}(A)$, i.e., $U = \operatorname{span}(A)$.

Hence, U is finite-dimensional. $\square$

Definition 2.1: Basis

A basis of V is a linearly independent list that spans V.

Every $v \in V$ has a unique representation in a basis:

$$v = a_1 v_1 + \cdots + a_m v_m$$

Every spanning list can be reduced to a basis.

Every finite-dimensional space has a basis.

Every linearly independent list can be extended to a basis.

Theorem 2.3

Suppose V is finite-dimensional and U is a subspace of V. Then there is a subspace W of V such that $V = U \oplus W$.

Proof. Let $u_1, \ldots, u_n$ be a basis of U. This basis can be extended to $u_1, \ldots, u_n, w_1, \ldots, w_m$ which is a basis of V, and of course spans V. Now let $W = span(w_1, \ldots, w_n)$. We are processed to prove $V = U \oplus W$ and $U \cap W = \{0\}$. First of all let $v \in V$, by the above hypothesis

$$v = (a_1 w_1 + \cdots + a_n w_n) + (b_1 u_1 + \cdots + b_m u_m) = x + y$$

Indeed, $x \in U, y \in W$ and $u_1, \ldots, u_n, w_1, \ldots, w_m$ is linearly independent. Therefore, there is unique pair $(x, y) \in U \times W$ such that $v = x + y$. This means $v \in U \oplus W$, hence $V = U \oplus W$.

□

Definition 2.2: Dimension, $\dim V$

- The dimension of a finite-dimensional vector space is the length of any basis of the vector space.
- The dimension of a finite-dimensional vector space V is denoted by $\dim V$.

Theorem 2.4: Basis length does not depend on basis

Any two bases of a finite-dimensional vector space have the same length

Proof. Let X and Y be the basis set of V. Now we consider that X is linearly independent and Y is spanning list of V. By the lemma 2.1 $|X| \leq |Y|$. Conversely, X is spanning list of V and Y is linearly independent, then $|X| \geq |Y|$. Hence $|X| = |Y|$. □

Definition 2.3

We consider these properties

- If V is finite-dimensional and U is a subspace of V, then $\dim U \leq \dim V$.
- Every linearly independent list of vectors in V of length $\dim V$ is a basis of V.

- If U is subspace of V and $\dim U = \dim V$, then $U = V$.
- Every spanning list of vectors in V of length $\dim V$ is a basis of V.

Theorem 2.5: Dimension of a sum

f V_1 and V_2 are subspaces of a finite-dimensional vector space, then

$$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$$

Proof. Let $v_1, \ldots, v_m$ be a basis of $V_1 \cap V_2$. This basis can be extended to $v_1, \ldots, v_m, a_1, \ldots, a_k$ which is basis of V_1. We also have $v_1, \ldots, v_m, b_1, \ldots, a_l$ is a basis of V_2. $\square$