Linear Algebra Solution

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Prerequisite Lemmas (prove as needed)

Lemma 1 (Steinitz Exchange): If A is a linearly independent and B is a spaning set in a finite-dimensional space, then $|A| \leq |B|$. Moreover, one can exchange elements of A with those of B to get a basis.

PROOF. Let $A = (v_1, \ldots, v_m)$ be a linearly independent list, and let $B = (w_1, \ldots, w_n)$ be a spanning list of V. We aim to prove that $m \leq n$.

Since B spans V, each v_i can be written as a linear combination of vectors from B. In particular, v_1 is a linear combination of $w_1, \ldots, w_n$. Therefore, the list $(v_1, w_1, \ldots, w_n)$ is linearly dependent.

Because A is linearly independent, $v_1 \neq 0$. Hence, there must exist some $k > 0$ such that w_k is a linear combination of $v_1, w_1, \ldots, w_{k-1}$. Removing w_k from B, we still have a spanning list.

We continue this process: successively add each v_i into the list and eliminate one redundant vector in B each time to maintain a spanning list. Eventually, we obtain a list containing all of $v_1, \ldots, v_m$ and still spanning V. Since we remove one vector from B for each v_i added, we cannot perform this more than n times. Therefore, $m \leq n$. $\square$

Lemma 2 (Rank–Nullity): For a linear map $T : V \to W$ between finite-dimensional spaces. Then

$$\dim V = \operatorname{rank}(T) + \operatorname{nullity}(T)$$

PROOF. Let $(v_1, \ldots, v_n)$ be a basis of V and $(v_1, \ldots, v_k)$ is the basis of the kernel T ($k \leq n$). Let $v \in V$ be arbitrary vector that can be expressed in combination:

$$v = a_1 v_1 + \cdots + a_n v_n$$

Then

$$T(v) = T(a_1 v_1 + \cdots + a_n v_n) = a_{k+1} w_{k+1} + \cdots + a_n w_n$$

So the image of f is

$$\Im(f) = \text{span}(w_{k+1}, \ldots, w_n)$$

Moreover, the set $(w_{k+1}, \ldots, w_n)$ is linearly independent since from $T(v) = a_{k+1}w_{k+1} + \cdots + a_n w_n = 0$ then $v \in \text{kernel}(f)$. If there is some $a_i \neq 0$ for $i = k+1, \ldots, n$, then we can write

$$a_1 w_1 + \cdots + a_k w_k - (\text{sum of } a_i w_i \text{ for } a_i \neq 0) = w_{k+1} + \cdots + a_n w_n - (\text{sum of } a_i w_i \text{ for } a_i \neq 0) = 0$$

Hence, all the v_i whom $a_i \neq 0$ for $i > k$ is in kernel(f), which contradicts the fact that basis of the kernel spans itself already, moreover this set is linear independent. Hence $a_{k+1} = \cdots = a_n = 0$ then $(w_{k+1}, \ldots, w_n)$ is linear independent. Therefore $|\Im(f)| = \text{rank}(T) = n - k = \dim(v) - \text{nullity}(T)$. □

Lemma 0.1 (Cayley–Hamilton): Every square matrix A satisfies its characteristic polynomial $\chi_A(A) = 0$.

Lemma 0.2 (Schur Triangularization over $\mathbb{C}$): For $A \in M_n(\mathbb{C})$ there exists unitary U with U^*AU upper triangular.

Lemma 0.3 (Gram–Schmidt): Any linearly independent set in an inner-product space can be orthonormalized.

I. Vector Spaces and Linear Maps

Problem 1: Let V be a vector space with $\dim V = n$. Prove that any two bases of V have the same number of elements.

Solution. Let X, Y be basis of V. Since X is linearly independent, by the lemma 1, S there is a spanning list Z that $|X| \leq |Z|$. Moreover, Y is spanning in V then $|Y| \geq |Z|$, therefore $|Y| \geq |X|$. Conversely, by exchanging the role that $|X|$ is spanning list and $|Y|$ is linearly independent, it would follow that $|Y| \leq |X|$. Hence $|X| = |Y|$. □

Problem 2: Let $B = \{v_1, \ldots, v_n\}$ be a basis of V. Show that

$$\varphi : \mathbb{F}^n \to V, \quad \varphi(a_1, \ldots, a_n) = \sum_{i=1}^{n} a_i v_i$$

is a linear isomorphism.

Solution. Since B is a basis and $(a_1, \ldots, a_n) = \mathbb{F}^n$ then the linear combination

$$\varphi(a_1, \ldots, v_n) = a_1 v_1 + \cdots + a_n = v$$

for all $v \in V$, thus φ is surjective. Now suppose there is $(b_1, \ldots, b_n)$ in $\mathbb{F}$ such that

$$\varphi(a_1, \ldots, a_n) = \varphi(b_1, \ldots, b_n)$$

or equavilently

$$\sum_{i=1}^{n} a_i v_i = \sum_{i=1}^{n} b_i v_i$$

or

$$\sum_{i=1}^{n} (a_i - b_i) v_i = 0$$

Since B is linearly independent, it follows that

$$a_i - b_i = 0$$

for all i. Hence $(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$, consequently, φ is injective. Therefore, φ is a linear isomorphism. □

Problem 3: If $U, W \leq V$ are subspaces of a finite-dimensional V, prove

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W).$$

Solution. Let two bases $U = (v_1, \ldots, v_n)$ and $W = (w_1, \ldots, w_m)$. We move the basis vectors from W to U as considering the following process $P(U, W)$:

Step 1: If w_1 can be written as the combination of $v_1, \ldots, v_n$, then we let $w_1 \in T$. If not, we let $w_1 \in U$.

Step k: We put $w_k \in T$ if w_k is combination of $v_1, \ldots, v_n$ and in U if not.

The process stops after m steps. Since W contains every vector that is element of T, it follows that $T \subset W$. Also, every element of T is combination of basis of U, then $T \subset U$. Indeed T's element is linearly independent and moreover $U \cap (W \backslash T) = \emptyset$, therefore $T = U \cap W$. Suppose that $v \in U$ and $W \in W$ and $|T| = r$, we have

$$U + W = \sum_{i=1}^{n} a_i v_i + \sum_{i=1}^{m} b_i w_i$$

as w_i which is in T can be written in combination of the basis of U, equavilently

$$U + W = \sum_{i=1}^{n} c_i v_i + \sum_{i=1}^{m-r} d_i w_i$$

whom v_i and w_i is linearly independent. Hence $\dim(U+W) = m+n-r = \dim(U) + \dim(W) - \dim(u \cap W)$. $\square$

Problem 4: Define $T : \mathbb{R}^4 \to \mathbb{R}^3$ by $T(x_1, x_2, x_3, x_4) = (x_1 + x_2,\ x_2 + x_3,\ x_3 + x_4)$. Find bases of $\ker T$ and $\operatorname{Im} T$ and verify rank–nullity.

SOLUTION. The image vector can be rewritten as

$$\begin{bmatrix} x_1 + x_2 + 0x_3 + 0x_4 \\ 0x_1 + x_2 + x_3 + 0x_4 \\ 0x_1 + 0x_2 + x_3 + x_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = Ax$$

By using elementary transform we have

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Hence $\ker(T) = 1$, $\operatorname{rank}(T) = 3$. Solving the equation $Ax = 0$ we get

$$x_1 = -x_2 = x_3 = -x_4 = t$$

The solution can be written in matrix form as

$$x = \begin{bmatrix} x_1 \\ -x_1 \\ x_1 \\ -x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

Then $\ker(T) = \operatorname{span}\{[1, -1, 1, -1]^T\}$ and

$$\Im(T) = \operatorname{span}\left\{[1,0,0]^T, [0,1,0]^T, [0,0,1]^T\right\} = \mathbb{R}^3$$

$\square$

Problem 5: Let $V = P_2(\mathbb{F})$ with basis $B = \{1, x, x^2\}$. Define $D : V \to V$ by $D(p) = p'$. Compute $[D]_B^B$, then use it to compute $D^2(3x^2 - 5x + 1)$.

SOLUTION. Let

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

be the basis matrix of $\{1, x, x^2\}$. Then the matrix of D with respect to B is given by

$$D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$D^2 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

then

$$D^2(3x^2 - 5x + 1) = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$$

Hence $D^2(3x^2 - 5x + 1) = 6$. □

Problem 6: Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be $T(x, y, z) = (x - y,\ 2x + z,\ y - z)$. Compute $[T]$ in the standard basis, its rank and nullity, and decide if T is invertible.

SOLUTION. Let A be the matrix of T with respect to the standard basis, then A is given by

$$A = [T] = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

Using elementary row operations, we can transform A into the reduced row echelon form as follows:

$$A \to \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 0 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence $\operatorname{rank}(T) = 3$, $\ker(T) = 0$ and T is invertable. □

Problem 7: Let $V = \mathbb{R}^3$, $U = \operatorname{span}\{(1, 1, 1)\}$. Find a basis of V/U and describe the class of $(2, 0, 1)$.

SOLUTION. We have $\dim(V) = 3$ and $\dim(U) = 1$, then $\dim(V/U) = 2$. Choose additional vectors $\{[1,0,0]^T, [0,1,0]^T\}$ so the the coset

$$\{[1,0,0]^T + U, [0,1,0]^T + U\}$$

is a basis of V/U. Define the class $l = (2,0,1)$, we find the set of T such that for $v \in T$, we have

$$l - v \in U$$

which means if we let $v = (x,y,z)$ then

$$(2,0,1) - (x,y,z) = (\lambda,\lambda,\lambda) \text{ for } \lambda \in \mathbb{R}$$

which means $x = 2 - \lambda$, $y = -\lambda$ and $z = 1 - \lambda$. The class of $(2,0,1)$ is

$$T = \{(2-\lambda, -\lambda, 1-\lambda) \mid \lambda \in \mathbb{R}\}$$

□

Problem 8: Let $B = \{(1,0),(0,1)\}$ in $\mathbb{R}^2$, and $\{\varphi_1, \varphi_2\}$ its dual basis. For $f = 3\varphi_1 - \varphi_2$, compute $f(4,5)$ and write f in canonical coordinates.

SOLUTION. We have

$$[B] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The the matrix of dual basis vectors is

$$[v] = [B]^{-1}v = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

□

Hence $\varphi_1(x,y) = x$, $\varphi_2(x,y) = y$. Then

$$f = 3\varphi_1 - \varphi_2 = 3x - y$$

and $f(4,5) = 3.4 - 5$. The canonical coordinates of f are $[3,-1]$.

Problem 9: Let $T_1(x,y,z) = (x+y, y+z, z+x)$ and $T_2(x,y,z) = (x,y,0)$. Find $[T_1 \circ T_2]$ and $[(T_1 \circ T_2)^2]$ in the standard basis.

Solution. We have

$$T_1 \circ T_2 = (x+y, y+z, 0)$$

and

$$(T_1 \circ T_2) \circ (T_1 \circ T_2) = (x+y+y+z, y+z+0, 0) = (x+2y+z, y+z, 0)$$

Hence

$$[T_1 \circ T_2] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad [(T_1 \circ T_2)^2] = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

□

Problem 10: For linear maps $S : U \to V$ and $T : V \to W$, prove $\operatorname{rank}(T \circ S) \leq \min\{\operatorname{rank} S, \operatorname{rank} T\}$.

Solution. Since for any $u \in U$ we have

$$TS(u) = T(S(u)) \in \Im(T)$$

Hence

$$\operatorname{rank}(TS) = \dim[\Im(TS)] \leq \dim[\Im(T)] \leq \operatorname{rank}(T)$$

Let $L : \Im(S) \to V$ and $L = TS$, hence

$$\operatorname{rank}(TS) = \operatorname{rank}(L) \leq \dim[\Im(S)] = \operatorname{rank}(S)$$

Therefore $\operatorname{rank}(T \circ S) \leq \min\{\operatorname{rank} S, \operatorname{rank} T\}$ □

Problem 11: Let $P : \mathbb{R}^3 \to \mathbb{R}^3$ be the projection onto the plane $x+y+z=0$ along direction $(1,1,1)$. Find $[P]$ and check $P^2 = P$.

Solution. Let (x,y,z) is arbitrary coordinates. We want to find the projection that make (x,y,z) into (p,q,r) suchthat $p+q+r$, and the vector

$$\begin{bmatrix} x-p \\ y-q \\ z-r \end{bmatrix}$$

is parallel to $[1,1,1]^T$. We rewrite the projection into

$$[P] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

Since the above vector is parallel to $[1,1,1]^T$ then

$$x - p = y - p = z - r = t$$

plus three terms together we get

$$x + y + z = 3t$$

Then we can rewrite

$$p = \frac{2x - y - z}{3}, q = \frac{-x + 2y - z}{3}, r = \frac{-x - y + 2z}{3}$$

Therefore $[P]$ is given by

$$[P] = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

And since the output of P lines on the plane $x + y + z = 0$ already, hence $P^n = P$ for all $n \geq 1$. $\square$

Problem 12: Let R_θ be rotation in $\mathbb{R}^2$ by angle θ. Show $I - R_\theta$ is invertible for $\theta \not\equiv 0 \pmod{2\pi}$ and find $(I - R_\theta)^{-1}$ explicitly.

Problem 13 (Dual Space and Double Dual): Let V be a finite-dimensional vector space over a field $\mathbb{F}$.

1. Show that $\dim V^* = \dim V$.
2. Construct a natural linear map $\Phi : V \to V^{**}$ and prove that it is an isomorphism.
3. Give an example showing that if V is infinite-dimensional, Φ need not be surjective.

Problem 14 (Bilinear Forms and Orthogonality): Let $B : V \times V \to \mathbb{F}$ be a bilinear form on an n-dimensional vector space V.

1. Define the radical of B and prove it is a subspace of V.
2. Prove that B is non-degenerate if and only if the induced map $V \to V^*$, $v \mapsto B(v, -)$, is an isomorphism.
3. Suppose B is symmetric and non-degenerate over $\mathbb{R}$. Prove that V has an orthogonal basis with respect to B.

Problem 15 (Spectral Theorem): Let $T : V \to V$ be a linear operator on a finite-dimensional real inner product space.

1. Prove that if T is self-adjoint (i.e. $\langle Tv, w \rangle = \langle v, Tw \rangle$), then T has an orthonormal basis of eigenvectors.
2. Deduce that T is diagonalizable by an orthogonal matrix in any orthonormal basis.

Problem 16 (Minimal Polynomial and Structure): Let $T : V \to V$ be a linear operator on a finite-dimensional vector space.

1. Prove that the minimal polynomial $m_T(x)$ divides any polynomial $p(x)$ such that $p(T) = 0$.
2. Show that T is diagonalizable if and only if $m_T(x)$ has no repeated factors over $\mathbb{F}$.

Problem 17 (Cyclic Subspaces and Rational Canonical Form): Let $T : V \to V$ be a linear operator over $\mathbb{F}$.

1. Given $v \in V$, define the cyclic subspace $Z(v) = \operatorname{span}\{v, Tv, T^2v, \dots\}$ and show that it is T-invariant.
2. Show that $Z(v)$ is isomorphic to $\mathbb{F}[x]/(m_v(x))$ where m_v is the minimal polynomial of v with respect to T.
3. Outline how to decompose V into a direct sum of cyclic subspaces to obtain the rational canonical form of T.

Problem 18 (Jordan Decomposition – Theory): Let T be a linear operator on a finite-dimensional complex vector space V.

1. Prove that V can be decomposed as a direct sum of generalized eigenspaces of T.
2. Show that on each generalized eigenspace corresponding to λ, the operator T can be written as $\lambda I + N$ where N is nilpotent.

Problem 19 (Norms on Finite-Dimensional Spaces): Let $\| \cdot \|_a$ and $\| \cdot \|_b$ be norms on a finite-dimensional vector space V over $\mathbb{R}$.

1. Prove that there exist positive constants m, M such that

$$m\|v\|_a \leq \|v\|_b \leq M\|v\|_a$$

for all $v \in V$.

2. Conclude that all norms on a finite-dimensional space induce the same topology.

Problem 20 (Invariant Subspaces and Triangularization): Let $T : V \to V$ be a linear operator over $\mathbb{C}$.

1. Prove that T has at least one nontrivial invariant subspace.
2. Deduce that T is triangularizable (i.e. represented by an upper triangular matrix) in some basis.

II. Linear Systems: Inverses, Elimination, Cramer

A. Inverses

Problem 21: Invert by row-reducing $[A \mid I]$ for $A = \begin{pmatrix} 4 & 7 \\ 2 & 6 \end{pmatrix}$.

Problem 22: Invert by $[A \mid I]$: $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{pmatrix}$.

Problem 23: If $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 5 \end{pmatrix}$, compute A^{-1} efficiently.

Problem 24: Let $A(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. For which t is $A(t)$ invertible? Compute $A(t)^{-1}$.

Problem 25: Prove the 2×2 inverse formula: if $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Problem 26 (Sherman–Morrison): Let $A = I_n - uv^T$ with $v^T u \neq 1$. Prove $A^{-1} = I_n + \dfrac{uv^T}{1 - v^T u}$.

B. Gaussian Elimination

Problem 27: Row-reduce $A = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 2 & 4 & -2 & 1 \\ 0 & 1 & 3 & -1 \end{pmatrix}$ to RREF; find rank, pivot columns, and a basis of $\ker A$.

Problem 28: Solve $Ax = b$ and classify the solution set for

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}.$$

Problem 29:

$$\begin{cases} x + y + z = 1, \\ x + 2y + 3z = 2, \\ x + 2y + tz = 3. \end{cases}$$

Solve in terms of t, determine when the solution is unique, and when the system is inconsistent.

Problem 30: Compute the RREF of $A = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 2 & 1 & 3 & 0 \\ -1 & 1 & 0 & 1 \end{pmatrix}$, then read off a basis for the column space and for the kernel.

Problem 31: Solve and verify:

$$\begin{cases} 2x - y + 3z = 5, \\ -x + 4y + z = 6, \\ 3x + 2y - 2z = 1. \end{cases}$$

Problem 32: Solve the system

$$\begin{cases} x + 2y - z = 4, \\ 2x + y + z = 2, \\ 3x + 3y = 6 \end{cases}$$

1. Write it in augmented matrix form and reduce to RREF.
2. State whether the solution is unique, infinite, or nonexistent.
3. Express the solution set in parametric vector form.

Problem 33: Without fully solving, compute the rank of

$$B = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 2 & 1 & 5 & 0 \\ -1 & 1 & -1 & 2 \end{pmatrix}$$

by row reducing to echelon form.

Problem 34: Let

$$T : \mathbb{R}^3 \to \mathbb{R}^3, \quad T(x, y, z) = (x + y,\ y + z,\ x + z).$$

1. Find $\ker(T)$ and its dimension.
2. Find $\operatorname{im}(T)$ and its dimension.
3. Verify $\dim \ker(T) + \dim \operatorname{im}(T) = 3$.

Problem 35: Let

$$C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

1. Compute C^n for $n \geq 1$ by hand and guess the general formula.

2. Prove the formula by induction.

Problem 36: Given

$$D = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

1. Compute $\det(D)$.
2. Determine for which (a, b, c, d) the matrix D is invertible.
3. If invertible, write down D^{-1}.

Problem 37: Let $B = \{(1,0), (0,1)\}$ be the standard basis of $\mathbb{R}^2$ and let $B' = \{(2,1), (1,1)\}$.

1. Find the change-of-basis matrix P from B' to B.
2. Given $v' = (3,1)$ in B'-coordinates, find its coordinates in B.

C. Cramer's Rule

Problem 38: Solve the 2×2 system by Cramer's rule: $\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$ with $ad - bc \neq 0$.

Problem 39: Use Cramer's rule to solve $\begin{cases} x + 2y + 3z = 1 \\ 2x + y + z = 0 \\ 3x + 0 \cdot y + 2z = 4 \end{cases}$, and evaluate the three determinants explicitly.

Problem 40 (Parameter case): $\begin{cases} x + y + z = 1 \\ 2x + 3y + 4z = 2 \\ tx + 2y + 3z = 3 \end{cases}$. For which t does Cramer apply ($\det A \neq 0$)? Solve the exceptional cases by elimination.

III. Determinants, Eigenvalues, and Diagonalization

Problem 41: Compute $\det A$ using operations: $A = \begin{pmatrix} 2 & 1 & 0 & 3 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 3 & -1 \\ 1 & 0 & 0 & 2 \end{pmatrix}$.

Problem 42: If $A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$ with B, D square, show $\det A = \det B \cdot \det D$. Evaluate $\det \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 5 & -1 & 2 & 1 \\ 0 & 0 & 0 & 4 \end{pmatrix}$.

Problem 43 (Vandermonde): For distinct $x_1, \dots, x_n$,

$$\det \begin{pmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & x_n^{n-1} \end{pmatrix} = \prod_{1 \le i < j \le n} (x_j - x_i).$$

Give a proof (induction or polynomial argument).

Problem 44: Let $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$. Find χ_A, eigenvalues with algebraic and geometric multiplicities, decide diagonalizability, and compute A^k for $k \ge 1$.

Problem 45: For $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$, find eigenpairs, diagonalize $A = PDP^{-1}$, and compute A^{10}.

Problem 46: Let $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and $S = \operatorname{diag}(1, -1)$. Compute eigenvalues of R_θ and SR_θ. Over which fields ($\mathbb{R}/\mathbb{C}$) are they diagonalizable?

Problem 47: For $A(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, find eigenvalues, minimal polynomial, decide diagonalizability, and compute $A(t)^n$.

Problem 48 (Companion matrix): Let $C = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix}$. Show $\chi_C(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0$.

Problem 49 (Use Cayley–Hamilton): For $A = \begin{pmatrix} 3 & 1 \\ -4 & 0 \end{pmatrix}$, compute χ_A and use Cayley–Hamilton to express $A^{10} = \alpha A + \beta I$.

Problem 50: Diagonalize (if possible) and obtain a closed form for A^n: $A = \begin{pmatrix} 4 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 2 \end{pmatrix}$.

Problem 51: If A is diagonalizable over $\mathbb{C}$ with eigenvalues λ_i (with multiplicity), prove $\det A = \prod_i \lambda_i$ and $\operatorname{tr} A = \sum_i \lambda_i$. Verify for $A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

Problem 52: Are $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ similar over $\mathbb{R}$? If yes, find P with $P^{-1}AP = B$; if not, explain.

Problem 53: Compute the minimal polynomial of $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and the least-degree nonzero polynomial q with $q(A) = 0$.

Problem 54: For $A = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}$, find complex eigenvalues and a real invertible P such that $P^{-1}AP = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. Then compute e^{tA}.

Problem 55: Let A, B be invertible. Compute $\det\left((A^{-1}B^2A)^T\right)$ in terms of $\det A, \det B$.

Problem 56 (Matrix determinant lemma): Let $A = \lambda I$ and $u, v \in \mathbb{F}^n$. Show $\det(\mu I - (A + uv^T)) = (\mu - \lambda)^{n-1}\left((\mu - \lambda) - v^T u\right)$ and deduce the eigenvalues of $A + uv^T$.

IV. Markov Chains: Short Explainer + Tasks

What is a Markov chain? A Markov chain on a finite state space has a row-stochastic matrix P (rows sum to 1). If $x^{(0)}$ is a row vector of probabilities, then $x^{(n)} = x^{(0)}P^n$. A *stationary distribution* π satisfies $\pi = \pi P$ and $\sum_i \pi_i = 1$.

Two-state case. For $P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$ with $\alpha, \beta \in (0, 1)$,

$$\pi = \left(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}\right), \qquad P^n \to 1\pi \quad (n \to \infty),$$

provided P is irreducible and aperiodic (true here if $\alpha, \beta > 0$). The second eigenvalue is $1 - (\alpha + \beta)$, which controls the convergence rate.

Problem 57 (Compute and interpret): Let $P = \begin{pmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{pmatrix}$. Find π, diagonalize P, compute P^n, and $\lim_{n\to\infty} P^n$. Interpret π as the long-run fraction of time in each state.

Problem 58 (Hitting probability): For the same P, starting in state 1, compute the probability of visiting state 2 before returning to 1. (Solve a 2×2 linear system for the hitting probabilities.)

VII. Inner Products and Orthogonality

Problem 59: Use Gram–Schmidt to orthonormalize $\{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ in $\mathbb{R}^3$.

Problem 60: Show $T : \mathbb{R}^n \to \mathbb{R}^n$ preserves inner products iff its matrix is orthogonal.

Problem 61: Classify all 2×2 orthogonal matrices as rotations or reflections.

VIII. Canonical Forms and Jordan Theory

Problem 62: Find the Jordan form of $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$.

Problem 63: Let A be real 4×4 with minimal polynomial $(x-2)^2(x+1)^2$. List all possible Jordan canonical forms.

Problem 64: Show that a nilpotent N on an n-dimensional space satisfies $N^n = 0$. (Use Jordan form or rank considerations.)

IX. Programming Problems

Problem 65 (Naïve matrix multiplication, $O(n^3)$): You are given $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{m\times p}$. Design an algorithm to compute $C = AB$.

Hints.

- Use the definition $C_{ij} = \sum_{k=1}^{m} A_{ik}B_{kj}$ with a *triple loop*.
- Loop order: prefer (i, k, j) so you reuse A_{ik} across the inner loop.
- Invariant: after processing k, you've accumulated $\sum_{t=1}^{k} A_{it}B_{tj}$ into C_{ij}.
- Complexity: count multiply+add operations, show $\Theta(nmp)$; for $n = m = p$ it's $\Theta(n^3)$.
- Numerical: prefer accumulating into a local register before writing to memory to reduce rounding and cache misses.

Problem 66 (Blocked (tiled) multiplication for cache locality): Modify your algorithm to multiply matrices by blocks of size $b \times b$.

Hints.

- Partition A, B, C into blocks so $C_{IJ} = \sum_K A_{IK}B_{KJ}$ where each block is $b \times b$.
- Choose b so that three $b \times b$ blocks fit in L1/L2 cache.

- Correctness follows from associativity and distributivity of matrix multiplication.

Problem 67 (Strassen's divide-and-conquer (the idea)): For $n = 2^k$, split A, B into 2×2 block matrices and compute $C = AB$ using 7 block multiplications instead of 8.

Hints.

- Remember the seven M_i combinations (you don't need to memorize them—derive by solving for $C_{11}, \dots, C_{22}$).
- Recursively apply on sub-blocks; stop at a threshold (hybrid with naïve).
- Show the recurrence $T(n) = 7T(n/2) + \Theta(n^2)$ and deduce $T(n) = \Theta(n^{\log_2 7})$.
- Note numerical stability trade-offs compared to naïve/blocked.

Problem 68 (Matrix exponentiation by squaring): Design an algorithm to compute A^k for $k \in \mathbb{N}$ efficiently.

Hints.

- Use binary expansion of k: if k is even, $A^k = (A^{k/2})^2$; if odd, $A^k = A \cdot A^{k-1}$.
- Maintain a running result R (start with I) and a power accumulator B (start with A).
- Complexity: $O(\log k)$ matrix multiplications; combine with blocked multiply for speed.

Problem 69 (Gaussian elimination with partial pivoting (algorithm sketch)): Given $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$, solve $Ax = b$.

Hints.

- For columns $k = 1$ to n: choose pivot row $p = \arg\max_{i \geq k} |A_{ik}|$, swap rows k and p, eliminate below.
- Keep an augmented matrix $[A \mid b]$; after forward elimination, back-substitute.
- Explain why partial pivoting helps numerical stability (growth factor).

Optional hints (peek only if stuck)

- Two-state Markov: eigenvalues are 1 and $1-(\alpha+\beta)$; use spectral decomposition to get P^n.
- A^k for upper-triangular Jordan blocks: powers produce binomial coefficients on superdiagonals.
- Normal equations: $X^\top X$ SPD $\Rightarrow$ Cholesky. QR avoids squaring condition number.
- Leontief: Neumann series $\sum_{k\geq 0} A^k$ converges iff $\rho(A) < 1$.
- Matrix power: binary exponentiation uses the identity $A^{2t} = (A^t)^2$ and $A^{2t+1} = A \cdot A^{2t}$.

X. Applied Linear Algebra Problems (Matrix-Free)

Problem 70 (Food Supply and Pricing): A bakery buys wheat and sugar from two suppliers. For each loaf of bread, the bakery uses 200 kg wheat and 50 kg sugar; for each cake, it uses 100 kg wheat and 120 kg sugar. The bakery produces B loaves of bread and C cakes per day.

It is known that:

$$\text{Total wheat used} = 2600 \text{ kg/day}, \quad \text{Total sugar used} = 1800 \text{ kg/day}.$$

Find B and C.

Hints: Form two linear equations from the wheat and sugar usage and solve for B and C.

Problem 71 (Food Market Customer Retention (Markov Model)): A customer shops at either Market A or Market B each week. If they go to A, there is a 70% chance they return to A next week; otherwise they switch to B. From B, there is a 60% chance they return to B; otherwise they switch to A.

Find:

1. The long-run fraction of customers at each market.
2. The expected return time to Market A.

Hints: Solve $\pi = \pi P$ with $\pi_1 + \pi_2 = 1$. Return time to A is $1/\pi_A$.

Problem 72 (Recipe Mixture Problem): You are making an energy drink from ingredients A, B, C:

A: 100 kcal protein, 50 kcal carbs
B: 60 kcal protein, 90 kcal carbs
C: 20 kcal protein, 150 kcal carbs

You want a 1000 kcal drink with 40% of calories from protein. Find the grams of each ingredient.

Hints: Convert the 40% condition to a linear equation in protein and carbs. The total calories condition gives a second equation.

Problem 73 (Graph Flow Conservation): Three cities X, Y, Z trade goods:

From X: 100 units to Y, 50 to Z
From Y: unknown to X, 60 to Z
From Z: 40 to X, unknown to Y

Every city's inflow equals its outflow. Find the unknown flows.

Hints: Flow conservation at each node gives a linear equation in the unknown flows.

Problem 74 (Seasonal Demand (Markov Model)): A restaurant's daily customer level is either Low (L), Medium (M), or High (H) with transition probabilities:

$$P = \begin{pmatrix} 0.6 & 0.3 & 0.1 \\ 0.2 & 0.5 & 0.3 \\ 0.1 & 0.3 & 0.6 \end{pmatrix}.$$

Find:

1. The stationary distribution (π_L, π_M, π_H).
2. The long-run average number of customers if L = 40, M = 70, H = 120.

Hints: Solve $\pi = \pi P$, $\pi_L + \pi_M + \pi_H = 1$. Average customers = $\pi_L \cdot 40 + \pi_M \cdot 70 + \pi_H \cdot 120$.

Problem 75 (Transportation Optimization): A farmer delivers apples from Farm F to Shops S1, S2, S3. Transport cost per crate is:

$$\text{F} \to S_1 : 2, \quad \text{F} \to S_2 : 3, \quad \text{F} \to S_3 : 4.$$

Demands: S_1: 50 crates, S_2: 40 crates, S_3: 60 crates. Farm capacity: 150 crates.

Formulate a linear optimization problem to minimize total transport cost and find the optimal distribution.

Hints: Constraints: total outflow = farm capacity, inflow per shop = demand. Minimize a linear cost function.

Review Questions (Multiple Choice)

1. (Finite dimension) For any finite-dimensional V, which is true?
   ☐ Any spanning set has size $\leq$ any LI set.
   ☐ Any LI set has size $\leq$ any spanning set.
   ☐ All subsets are bases.
   ☐ Dimension depends on the field but not on V.

2. (Rank–nullity) For $T : V \to W$ linear (finite-dim V), which identity holds?
   ☐ $\dim V = \operatorname{rank} T - \operatorname{nullity} T$.
   ☐ $\dim V = \operatorname{rank} T + \operatorname{nullity} T$.
   ☐ $\dim W = \operatorname{rank} T + \operatorname{nullity} T$.
   ☐ $\dim V = \dim W$.

3. (Determinant) For square A, B of same size:
   ☐ $\det(AB) = \det A + \det B$.
   ☐ $\det(AB) = \det A \cdot \det B$.
   ☐ $\det(AB) = \det A$.
   ☐ $\det(AB) = \det B$.

4. (Invertibility) A square matrix A is invertible iff:
   ☐ $\det A = 0$.
   ☐ $\det A \neq 0$.
   ☐ $\operatorname{rank}(A) < n$.
   ☐ A is upper triangular.

5. (Projection spectrum) If $P^2 = P$ (real), then the eigenvalues of P are:
   ☐ $\{0, 1\}$ only.
   ☐ $\{-1, 0, 1\}$.
   ☐ All reals in $[0, 1]$.
   ☐ Complex unit circle.

6. (Orthogonal) For real Q orthogonal:
   ☐ $Q^T Q = I$.

☐ $QQ^T = 2I$.
☐ Columns are arbitrary.
☐ $\det Q = 0$.

7. (Rotation eigenvalues) In $\mathbb{R}^2$, R_θ has complex eigenvalues:
☐ $1, 1$.
☐ $-1, -1$.
☐ $e^{\pm i\theta}$.
☐ $\cos\theta \pm i$.

8. (Invertibility of $I - R_\theta$) Over $\mathbb{R}$, $I - R_\theta$ is invertible iff:
☐ $\theta \equiv 0 \pmod{2\pi}$.
☐ $\theta \not\equiv 0 \pmod{2\pi}$.
☐ Always.
☐ Never.

9. (Poly space) $\{1, x, x^2\}$ in $P_2(\mathbb{F})$ is:
☐ Dependent.
☐ Spanning but not independent.
☐ Basis.
☐ None.

10. (Dual basis) For basis $\{v_i\}$ and dual $\{\varphi_i\}$:
☐ $\varphi_i(v_j) = \delta_{ij}$.
☐ $\varphi_i(v_j) = 1$.
☐ $\varphi_i(v_j) = 0$.
☐ Undefined.

11. (Derivative on P_2) $D : p \mapsto p'$ on P_2 has:
☐ Rank 3.
☐ Rank 2.
☐ Rank 1.
☐ Rank 0.

12. (Nullity of D above) The nullity of D is:
☐ 0.
☐ 1.
☐ 2.
☐ 3.

13. (Map $T(x, y, z) = (x + y, y + z, x + z)$) Then:
☐ rank $= 2$.
☐ rank $= 3$ (invertible).

☐ nullity $= 1$.
☐ Not linear.

14. (Kernel of derivative) For D on polynomials:
☐ $\ker D = \{0\}$.
☐ $\ker D$ are constants.
☐ $\ker D$ are linear polynomials.
☐ Empty.

15. (Triangular eigenvalues) For upper triangular A:
☐ Eigenvalues are diagonal entries.
☐ Eigenvalues are zeros of first row.
☐ Need not exist.
☐ Always all 1.

16. (Trace/eigenvalues) For any A over $\mathbb{C}$:
☐ $\mathrm{tr}(A)$ equals product of eigenvalues.
☐ $\det(A)$ equals sum of eigenvalues.
☐ $\mathrm{tr}(A)$ equals sum of eigenvalues.
☐ None.

17. (2×2 characteristic poly) For $A \in M_2$:
☐ $\chi_A(\lambda) = \lambda^2 - \det(A)\lambda + \mathrm{tr}(A)$.
☐ $\chi_A(\lambda) = \lambda^2 - \mathrm{tr}(A)\lambda + \det(A)$.
☐ $\chi_A(\lambda) = \lambda^2 + \mathrm{tr}(A)\lambda + \det(A)$.
☐ $\lambda^2 - \det(A)$.

18. (Cayley–Hamilton) A square A:
☐ Satisfies its characteristic polynomial.
☐ Satisfies its minimal polynomial only if diagonalizable.
☐ Never satisfies polynomials.
☐ Only for symmetric A.

19. (Nilpotent eigenvalues) If $N^k = 0$, then eigenvalues are:
☐ All 1.
☐ All 0.
☐ ± 1.
☐ Unit circle.

20. (Triangularizable) If A is similar to an upper triangular matrix, its diagonal entries are:
☐ Arbitrary.
☐ All zero.
☐ The eigenvalues (with multiplicity).
☐ The singular values.

21. (Gram–Schmidt) Applied to a LI set in an inner product space yields:
☐ Orthonormal basis of its span.
☐ Same set unchanged.
☐ Empty set.
☐ A basis of the orthogonal complement.

22. (Projection in $\mathbb{R}^2$) Nontrivial projection P onto a line has:
☐ $\det P = 1$.
☐ $\det P = 0$.
☐ $\det P = -1$.
☐ $\det P = 2$.

23. (Subspace dimension formula) For $U, W \leq V$:
☐ $\dim(U + W) = \dim U + \dim W - \dim(U \cap W)$.
☐ $\dim(U + W) = \dim U + \dim W$.
☐ $\dim(U \cap W) = 0$ always.
☐ None.

24. (Spanning $= n$ vectors in $\mathbb{R}^n$) Then the set is:
☐ Dependent.
☐ A basis.
☐ Not enough info.
☐ Orthogonal.

25. (Change of basis) The matrix from B' to B has columns:
☐ Coordinates of B in B'.
☐ Coordinates of B' in B.
☐ Eigenvectors.
☐ Orthonormal vectors.

26. (Composition matrices) If $S : U \to V$, $T : V \to W$ and bases B_U, B_V, B_W, then:
☐ $[T \circ S]_{B_U}^{B_W} = [S]_{B_U}^{B_V} [T]_{B_V}^{B_W}$.
☐ $[T \circ S]_{B_U}^{B_W} = [T]_{B_V}^{B_W} [S]_{B_U}^{B_V}$.
☐ Order doesn't matter.
☐ Product undefined.

27. (Quick rank) $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ has rank:
☐ 0.
☐ 1.
☐ 2.
☐ 3.

28. (2×2 det) $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ equals:
☐ $ab + cd$.
☐ $ad - bc$.
☐ $ac - bd$.
☐ $a + b + c + d$.

29. (2×2 inverse) If $ad - bc \neq 0$, then:
☐ $A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.
☐ $A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
☐ A^{-1} doesn't exist.
☐ $A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

30. (Reflection $S = \mathrm{diag}(1, -1)$) Eigenvalues are:
☐ $1, 1$.
☐ $-1, -1$.
☐ $1, -1$.
☐ $e^{\pm i\theta}$.

31. (Markov) For row-stochastic P:
☐ 1 is an eigenvalue.
☐ $\det P = 1$.
☐ All eigenvalues $= 1$.
☐ No eigenvalues.

32. (Stationary dist.) π satisfies:
☐ $P\pi = \pi$.
☐ $\pi = \pi P$ and $\sum_i \pi_i = 1$.
☐ $\pi P^2 = \pi$ only.
☐ π arbitrary.

33. (Convergence) Irreducible, aperiodic finite-state Markov P:
☐ $P^n \to 0$.
☐ $P^n \to I$.
☐ $P^n \to \mathbf{1}\pi$.
☐ Diverges.

34. (Companion matrix) Its characteristic polynomial equals:
☐ $x^n + a_{n-1}x^{n-1} + \cdots + a_0$.
☐ x^n.

☐ Minimal polynomial.
☐ Determinant.

35. (Vandermonde) For distinct x_i the determinant equals:
☐ $\prod_i x_i$.
☐ $\prod_{i<j}(x_j - x_i)$.
☐ 0.
☐ $\sum_i x_i$.

36. (Rank inequality) Always true:
☐ $\operatorname{rank}(TS) \geq \max\{\operatorname{rank} T, \operatorname{rank} S\}$.
☐ $\operatorname{rank}(TS) \leq \min\{\operatorname{rank} T, \operatorname{rank} S\}$.
☐ $\operatorname{rank}(TS) = \operatorname{rank} T + \operatorname{rank} S$.
☐ None.

37. (Kernels and composition) For $S : U \to V$, $T : V \to W$:
☐ $S^{-1}(\ker T) \subseteq \ker(T \circ S)$.
☐ $S^{-1}(\ker T) = \{0\}$ always.
☐ $\ker(T \circ S) = \{0\}$ always.
☐ No relation.

38. (Real symmetric) Then:
☐ Not diagonalizable.
☐ Diagonalizable by orthogonal matrix.
☐ Only upper-triangularizable.
☐ Needs complex field.

39. (Normal/complex) If A is normal ($AA^* = A^*A$):
☐ Unitary diagonalizable.
☐ Never diagonalizable.
☐ Only triangularizable.
☐ Needs real entries.

40. (Min vs char poly) On finite-dim spaces:
☐ m_A divides χ_A.
☐ χ_A divides m_A.
☐ Unrelated.
☐ Equal always.

41. (Diagonalizable criterion) A is diagonalizable over a splitting field iff:
☐ χ_A has distinct roots.
☐ m_A splits with distinct linear factors.
☐ $\det A \neq 0$.
☐ A is symmetric.

42. (Similarity invariants) If $A \sim B$:
    $\square$ $\mathrm{tr}A = \mathrm{tr}B$, $\det A = \det B$.
    $\square$ Determinant changes sign.
    $\square$ Trace doubles.
    $\square$ None.

43. (Planar orthogonal, $\det = 1$) Then:
    $\square$ Reflection.
    $\square$ Rotation.
    $\square$ Projection.
    $\square$ Shear.

44. (Projection trace) For projection P:
    $\square$ $\mathrm{tr}(P) = 0$.
    $\square$ $\mathrm{tr}(P) = \mathrm{rank}(P)$.
    $\square$ $\mathrm{tr}(P) = n$.
    $\square$ $\mathrm{tr}(P)$ arbitrary.

45. (Rank of $A^T A$) For any A:
    $\square$ $\mathrm{rank}(A^T A) = \mathrm{rank}(A)$.
    $\square$ $= \mathrm{rank}(A) + \mathrm{rank}(A^T)$.
    $\square$ $= n$ always.
    $\square$ $= 0$.

46. (Invertible A) Then:
    $\square$ $\ker A = \{0\}$.
    $\square$ $\ker A \neq \{0\}$.
    $\square$ $\mathrm{rank}A < n$.
    $\square$ $\det A = 0$.

47. (Orthonormal columns) For A with orthonormal columns:
    $\square$ $A^T A = I$.
    $\square$ $AA^T = I$ for any shape.
    $\square$ $\det A = 0$ always.
    $\square$ Not full rank.

48. (Triangular determinant) For upper triangular U:
    $\square$ $\det U = \prod$ diagonal.
    $\square$ $\det U = \sum$ diagonal.
    $\square$ $\det U = 0$.
    $\square$ Depends on off-diagonals.

49. (Block triangular determinant) For $A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$:

☐ $\det A = \det B + \det D$.
☐ $\det A = \det B \cdot \det D$.
☐ $\det A = \det C$.
☐ 0.

50. (Commuting with simple spectrum) If $AB = BA$ and A has n distinct eigenvalues:
☐ A, B are simultaneously diagonalizable.
☐ B is a scalar.
☐ B is nilpotent.
☐ No conclusion.

51. (Dimension of P_m) Over $\mathbb{F}$:
☐ m.
☐ $m + 1$.
☐ $2m$.
☐ Infinite.

52. (Interpolation map) $T : P_m \to \mathbb{R}^{m+1}$, $T(p) = (p(x_0), \ldots, p(x_m))$ with distinct x_i:
☐ Not linear.
☐ An isomorphism.
☐ Not injective.
☐ Not surjective.

53. (Kernel of nonzero functional) For nonzero $f \in V^*$:
☐ $\ker f = V$.
☐ $\ker f$ is a hyperplane (codim 1).
☐ $\ker f = \{0\}$.
☐ Empty.

54. (Fundamental theorem of LA) For any real A:
☐ $\mathcal{N}(A^T) = \mathcal{C}(A)^\perp$.
☐ $\mathcal{N}(A) = \mathcal{C}(A)^\perp$.
☐ $\mathcal{C}(A) = \mathbb{R}^n$.
☐ None.

55. (Matrix exponential determinant) For any square A:
☐ $\det(e^A) = e^{\operatorname{tr} A}$.
☐ $\det(e^A) = \operatorname{tr}(e^A)$.
☐ $\det(e^A) = 1$ always.
☐ Undefined.

56. (Transpose/inverse) For invertible A:
☐ $(A^{-1})^T = (A^T)^{-1}$.

☐ $(A^{-1})^T = A$.
☐ $(A^T)^{-1} = A$.
☐ False.

57. (Rank-one) For $u, v \neq 0$:
☐ uv^T has rank 1.
☐ Rank 2.
☐ Rank 0.
☐ Not linear.

58. (Projection eigenspaces) For orthogonal projection onto U:
☐ Eigenvalues 1 on U, 0 on $U^\perp$.
☐ Eigenvalues $\pm i$.
☐ All 1.
☐ All 0.

59. ($I - R_\theta$ revisited) Over $\mathbb{R}$, $I - R_\theta$ is invertible iff:
☐ $\theta \not\equiv 0 \pmod{2\pi}$.
☐ $\theta \equiv 0 \pmod{2\pi}$.
☐ Always.
☐ Never.

60. (Two-state Markov stationary distribution) For $P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$ with $\alpha, \beta > 0$, the stationary distribution is:
☐ $\left(\frac{\alpha}{\alpha+\beta}, \frac{\beta}{\alpha+\beta}\right)$.
☐ $\left(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}\right)$.
☐ $(\frac{1}{2}, \frac{1}{2})$.
☐ $(1, 0)$.

XII. Additional Practice Problems

Problem 76: Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Compute $\det A$, $\operatorname{rank}(A)$, and check if A is invertible.

Problem 77: Find a basis for the kernel of the matrix $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{pmatrix}$.

Problem 78: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x, y) = (3x + y, x + 2y)$. Find $[T]$ in standard basis and decide if T is invertible.

Problem 79: Find all eigenvalues and eigenvectors of $A = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$.

Problem 80: Let V be the space of all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f'' = -f$. Show that V is a vector space. Find a basis.

Problem 81: Let $T : P_2 \to \mathbb{R}^3$ be defined by $T(p) = (p(0), p(1), p(2))$. Find the matrix of T in the basis $\{1, x, x^2\}$.

Problem 82: Determine if the vectors $v_1 = (1, 2, 3)$, $v_2 = (2, 4, 6)$, $v_3 = (3, 6, 9)$ form a basis.

Problem 83: Suppose A is a 3×3 matrix with eigenvalues $2, 2, -1$. Is A diagonalizable?

Problem 84: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ reflect points across the line $y = x$. Find $[T]$.

Problem 85: Find the matrix representing the projection onto the xy-plane in $\mathbb{R}^3$.

Problem 86: Let $f(x) = x^3 - 2x^2 + 3x - 1$. Use companion matrix to represent multiplication by x modulo f.

Problem 87: Given the food transition matrix $P = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}$ between rice and noodles, compute $\lim_{n\to\infty} P^n$.

Problem 88: Let G be a graph with adjacency matrix A. What does A^n_{ij} count?

Problem 89: Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by a matrix with rank 2. What is the nullity?

Problem 90: Let T be a linear transformation such that $T^2 = T$. Prove the image and kernel of T intersect trivially.

Problem 91: Write the algorithm for matrix multiplication $C = AB$, where A is $m \times n$, B is $n \times p$.

Problem 92: Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (x + z, y, z)$. Is T invertible? Compute its inverse if yes.

Problem 93: Suppose a vector v satisfies $A^3 v = 0$ but $A^2 v \neq 0$. What can be said about the minimal polynomial of A?

Problem 94: Let $f(x) = x^n$ be in P_n. Define $D(f) = f'$. Show D is a linear map and find its matrix in monomial basis.

Problem 95: Find the rank of matrix $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}$.

Problem 96: Let $v_1 = (1, 0, 0)$, $v_2 = (1, 1, 0)$, $v_3 = (1, 1, 1)$. Use Gram-Schmidt to orthonormalize.

Problem 97: Is the following system consistent? $\begin{cases} x + y + z = 1 \\ x + 2y + 3z = 2 \\ 2x + 3y + 4z = 3 \end{cases}$

Problem 98: Let V be vector space of 2×2 matrices. Find $\dim V$ and a basis.

Problem 99: Let T be defined on polynomials P_3 by $T(p) = p(1)$. What is the rank of T?

Problem 100: Let T be rotation by $\pi/4$ in $\mathbb{R}^2$. Is T diagonalizable over $\mathbb{C}$? Over $\mathbb{R}$?

Problem 101: Let $v_1 = (1, 2)$, $v_2 = (3, 6)$. Compute $\text{span}\{v_1, v_2\}$ and determine if it's 1D or 2D.

Problem 102: Let T be defined by $T(f) = f''$ on P_3. Find the matrix of T in monomial basis.

Problem 103: Let A be 2×2 with $\chi_A(x) = x^2 + 1$. Find A^4.

Answer Key (choice number 1–4)

1. 2	2. 2	3. 2	4. 2	5. 1	6. 1	7. 3	8. 2	9. 3	10. 1
11. 2	12. 2	13. 2	14. 2	15. 1	16. 3	17. 2	18. 1	19. 2	20. 3
21. 1	22. 2	23. 1	24. 2	25. 2	26. 2	27. 2	28. 2	29. 1	30. 3
31. 1	32. 2	33. 3	34. 1	35. 2	36. 2	37. 1	38. 2	39. 1	40. 1
41. 2	42. 1	43. 2	44. 2	45. 1	46. 1	47. 1	48. 1	49. 2	50. 1
51. 2	52. 2	53. 2	54. 1	55. 1	56. 1	57. 1	58. 1	59. 1	60. 2