Linear Algebra

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1. Vector Spaces

Definition 1.1: Vector Space

A triple $(V, +, \cdot)$ consisting of a set V, an addition map

$$+ : V \times V \to V, \quad (x, y) \mapsto x + y$$

and a scalar multiplication map

$$\cdot : \mathbb{F} \times V \to V, \quad (\lambda, x) \mapsto \lambda x$$

is called a real vector space if for all $x, y, z \in V$ and $\lambda, \mu \in \mathbb{R}$, the following axioms hold:

1. $(x + y) + z = x + (y + z)$
2. $x + y = y + x$
3. $x + 0 = x$
4. $x + (-x) = 0$
5. $\lambda(\mu x) = (\lambda\mu)x$
6. $1x = x$
7. $\lambda(x + y) = \lambda x + \lambda y$
8. $(\lambda + \mu)x = \lambda x + \mu x$

Note: 0 and 1 here denote the additive and multiplicative identities in $\mathbb{R}$.

Definition 1.2: Subspace

A subset U of V is a subspace if:

- $0 \in U$
- $u, w \in U \Rightarrow u + w \in U$
- $a \in \mathbb{F},\ u \in U \Rightarrow au \in U$

Definition 1.3: Sum of Subspaces

If $V_1, V_2, \ldots, V_m$ are subspaces of V, their sum is

$$V_1 + V_2 + \cdots + V_m = \{v_1 + v_2 + \cdots + v_m : v_i \in V_i\}$$

Theorem 1.1: T

e sum $V_1 + \cdots + V_m$ is the smallest subspace of V containing all V_i.

Proof. Because $0 \in V_i$, we have $0 + \cdots + 0 = 0 \in V_1 + \cdots + V_m$. For any $x \in V_i$, we can express it as $0 + \cdots + x + \cdots + 0$ so $x \in V_1 + \cdots + V_m$. Hence $V_i \subset V_1 + \cdots + V_m$.

Suppose U is a subspace containing all V_i. Let $x \in V_1 + \cdots + V_m$, then $x = v_1 + \cdots + v_m$ with $v_i \in V_i \subset U$. Since U is closed under addition, $x \in U$. So $V_1 + \cdots + V_m \subset U$. □

Definition 1.4: Direct Sum

The sum $V_1 + \cdots + V_m$ is a *direct sum*, denoted $V_1 \oplus \cdots \oplus V_m$, if each element of the sum can be written uniquely as $v_1 + \cdots + v_m$ with $v_i \in V_i$.

Theorem 1.2: T

e sum $V_1 + \cdots + V_m$ is a direct sum if and only if the only way to write 0 as $v_1 + \cdots + v_m$ with $v_i \in V_i$ is when all $v_i = 0$.

Proof. Suppose $v_1 + \cdots + v_m = 0$ implies $v_i = 0$. Assume $v = x_1 + \cdots + x_m = y_1 + \cdots + y_m$ with $x_i, y_i \in V_i$. Subtracting gives $(x_1 - y_1) + \cdots + (x_m - y_m) = 0$. By hypothesis, each $x_i = y_i$. Thus representation is unique. □

Theorem 1.3: F

r subspaces $U, W \subset V$,

$$U + W \text{ is a direct sum } \Leftrightarrow U \cap W = \{0\}$$

Proof. $(\Rightarrow)$ Suppose $x \in U \cap W$, then $x = u + w$ uniquely, with $u \in U$, $w \in W$. But also $x - u =$

$w \in U$, so $x \in U \Rightarrow w \in U$, hence $w \in U \cap W$. So $x = 0$.

($\Leftarrow$) Suppose $x = x_1 + y_1 = x_2 + y_2$ with $x_i \in U$, $y_i \in W$. Then $x_1 - x_2 = y_2 - y_1 \in U \cap W$. Since intersection is {0}, we get $x_1 = x_2$ and $y_1 = y_2$. □

2. Finite-Dimensional Vector Spaces

A linear combination of $v_1, \ldots, v_m \in V$ is any vector

$$a_1 v_1 + \cdots + a_m v_m$$

The span is

$$\text{span}(v_1, \ldots, v_m) = \{a_1 v_1 + \cdots + a_m v_m : a_i \in \mathbb{F}\}$$

Note: The span of a list of vectors is the smallest subspace containing them. If $\text{span}(v_1, \ldots, v_m) = V$, we say the list *spans* V.

A list $v_1, \ldots, v_m$ is linearly independent if the only solution to

$$a_1 v_1 + \cdots + a_m v_m = 0$$

is $a_1 = \cdots = a_m = 0$.

Lemma 2.1: I

$v_1, \ldots, v_m$ is a linearly dependent list, there exists k such that $v_k \in \text{span}(v_1, \ldots, v_{k-1})$. Removing v_k does not change the span.

Theorem 2.1: I

a finite-dimensional vector space, every linearly independent list has length $\leq$ any spanning list.

Proof. Let $A = (v_1, \ldots, v_m)$ be linearly independent, $B = (w_1, \ldots, w_n)$ spans V. Add v_1 to B: since B spans V, v_1 is a combination of w_i, making the new list dependent. Remove a dependent w_i, repeat for each v_i. Since B has n vectors, we cannot insert more than n independent v_i, so $m \leq n$. □

Theorem 2.2: E

ery subspace of a finite-dimensional vector space is finite-dimensional.

Proof. Let V finite-dimensional, $U \subseteq V$ a subspace. If $U = \{0\}$, done. Otherwise pick $u_1 \in U$ nonzero. Inductively build linearly independent list $\{u_1, \ldots, u_k\} \subset U$.

If $U \subset \operatorname{span}(u_1, \ldots, u_k)$, done. Otherwise pick $u_{k+1} \notin \operatorname{span}(\ldots)$. Since V is finite-dimensional, this process stops after $\dim V$ steps. Hence U is spanned by finitely many vectors. $\square$

Definition (Basis): A basis of V is a linearly independent list that spans V.

Every $v \in V$ has a unique representation in a basis:

$$v = a_1 v_1 + \cdots + a_m v_m$$

Every spanning list can be reduced to a basis. Every finite-dimensional space has a basis. Every linearly independent list can be extended to a basis.