

Dirac, Majorana and Weyl fermions

Palash B. Pal

Saha Institute of Nuclear Physics

1/AF Bidhan-Nagar, Calcutta 700064, India

Abstract

This is a pedagogical article which discusses various kinds of fermion fields: Dirac, Majorana and Weyl. The definitions and motivations for introducing each kind of fields is discussed, along with the connections between them. It is pointed out that these definitions have to do with the proper Lorentz group, and not with respect to any discrete symmetry. The action of discrete symmetries like charge conjugation and CP on various types of fermion fields, particularly important for Majorana fermions, has also been clarified.

1 Introduction

When Dirac first wrote down his relativistic equation for a fermion field, he had primarily electrons in mind. It doesn't require much mind-reading to deduce this conclusion, because his first article [1] on this issue was entitled "The Quantum Theory of the Electron". Electrons have mass and charge. In his solutions, Dirac found the antiparticle, which has the same mass as the electron but is different from the electron because it has opposite charge.

Dirac's paper was published in 1928. The very next year, Weyl [2] showed that for massless fermions, a simpler equation would suffice, involving two-component fields as opposed to the four-component field that Dirac had obtained.

And then, in 1930, Pauli [3] proposed the neutrinos to explain the continuous energy spectrum of electrons coming out in beta decay. The neutrinos had to be uncharged because of conservation of electric charge, and they seemed to have vanishing mass from the analysis of beta decay data. It was therefore conjectured that the neutrinos are massless. Naturally, it was assumed that the neutrinos are therefore Weyl fermions, i.e., their properties are described by Weyl's theory.

There was also the possibility that neutrinos are the antiparticles of themselves, since they are uncharged. Description of such fermion fields was pioneered by Majorana [4] in 1937. The question was not taken seriously because, at that time, everybody was convinced that neutrinos are Weyl fermions.

The question became important much later, beginning in the 1960s, when people started examining the consequences of small but non-zero neutrino masses, and possibilities of detecting them. If neutrinos have mass, they cannot be Weyl fermions.

This opened the discussion of whether the neutrinos are Dirac fermions or Majorana fermions.

Majorana fermions became important in Particle Physics for other reasons as well. Supersymmetric theories require Majorana fermions as partners of spin-0 or spin-1 bosonic fields. One might also add that supersymmetric theories are best described in superspace, obtained by augmenting the usual spacetime variables with some fermionic parameters which transform as Majorana spinors.

With all the experience of working with Dirac fermions, working with Majorana fermions produced some **hiccups**. Even now, it is not uncommon to see fantastic claims about Majorana particles or fields in the literature that come out of strange **jugglery** or gymnastics with these objects: complicated operations that often have no rational or analytical basis [5].

The uneasiness can be compared to a feeling that Alice had experienced during her travels in Wonderland. At one point, she drank something and became very small. Then she saw a small cake with the words ‘EAT ME’ marked on it. She wondered whether she would shrink further, or grow back to her original size if she ate it. Finally, she

ate a little bit, and said anxiously to herself, ‘Which way? Which way?’,
holding her hand on the top of her head to feel which way it was growing,
and she was quite surprised to find that she remained the same size...

After this, Lewis Carroll comments that in fact, there was nothing to be surprised about. “This generally happens when one eats cake”. But Alice, by that time, got so much accustomed to seeing the extraordinary that she was getting surprised by seeing an ordinary thing happening to her.

Majorana fermions are quite simple objects, simpler than Dirac fermions. But we are so much accustomed to Dirac particles that we try to understand Majorana particles through Dirac particles. This is a **roundabout way**, and creates problems. In this article, we will introduce Majorana fermions through an imaginary journey in which we will pretend that we do not know about Dirac fermions. Dirac fermions will be also be mentioned, for the sake of completeness. And, although it is now known that the neutrinos are indeed massive, **Weyl fermions will appear in our journey as well. Although no known (or even conjectured) particle can possibly be a Weyl fermion, we will see that the concept is very useful, because Weyl fermions can be seen as building blocks of any fermion field.**

Our journey will be anachronistic. In the title of the paper, we have listed the three kinds in alphabetical order. That is also not the order we will follow in the article.

2 The Klein-Gordon equation and its solutions

To begin this journey, let us not even worry about fermions. In relativistic physics, the Hamiltonian of a free particle of mass m must satisfy the equation

$$H^2 = \mathbf{p}^2 + m^2, \quad (2.1)$$

written in the natural unit in which we have chosen $c = 1$. When we try to build a quantum theory, we can let both sides act on the wavefunction. Using the standard co-ordinate space operator for the momentum and setting $\hbar = 1$ by the choice of units, we obtain the equation for the wavefunction ϕ to be

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2 \right) \phi = 0. \quad (2.2)$$

This is the Klein-Gordon equation.

The differential operator acting on ϕ is real. So, if we choose an initial condition in which ϕ is real everywhere, the evolution through the equation will keep it real. **This will give us a real solution of the Klein-Gordon equation.**

Plane waves of the form $e^{-ip^\mu x_\mu}$ are solutions to the Klein-Gordon equation provided

$$p^\mu p_\mu = m^2. \quad (2.3)$$

We can use them to expand any other solution. For real solutions, such a Fourier expansion will be:

$$\phi(x) = \int_p \left(a(p) e^{-ip \cdot x} + a^*(p) e^{+ip \cdot x} \right), \quad (2.4)$$

We have divided the Fourier terms into two parts by imposing the condition

$$p^0 > 0, \quad (2.5)$$

so that the reality condition is transparent. The measure of the integral over p has been left undefined, and will be kept so, because it is not important for our discussion. For quantum fields, the Fourier co-efficients $a(p)$ become operators, and $a^*(p)$ should be understood to be the hermitian conjugate of $a(p)$.

3 The Dirac equation and its solutions

3.1 The equation

After this preamble, let us discuss the Dirac equation, which is

$$\left(i\gamma^\mu \partial_\mu - m \right) \Psi = 0. \quad (3.1)$$

The equation can be seen as the Schrödinger equation,

$$i \frac{\partial \Psi}{\partial t} = H \Psi, \quad (3.2)$$

arising from the Hamiltonian

$$H = \gamma^0 (\gamma^i p^i + m). \quad (3.3)$$

Alternatively, it can be seen as the Euler-Lagrange equation coming from the Lagrangian

$$\mathcal{L} = \bar{\Psi} (i\gamma^\mu \partial_\mu - m) \Psi, \quad (3.4)$$

where $\bar{\Psi} \equiv \Psi^\dagger \gamma^0$.

In these equations, γ^μ denotes a collection of four matrices, each 4×4 , which satisfy the conditions

$$[\gamma^\mu, \gamma^\nu]_+ = 2g^{\mu\nu}, \quad (3.5)$$

$$\gamma_0 \gamma_\mu \gamma_0 = \gamma_\mu^\dagger, \quad (3.6)$$

where $[A, B]_+ = AB + BA$ denotes the anticommutator. The first one, which has an implied unit matrix on the right hand side, is necessary so that the Dirac equation complies with the energy-momentum relation of Eq. (2.1). The second equation, which is necessary so that the Hamiltonian implied by the Dirac equation is hermitian, can more explicitly be written as

$$\gamma_0^\dagger = \gamma_0, \quad \gamma_i^\dagger = -\gamma_i. \quad (3.7)$$

Let us set up some notational rules that will be helpful for avoiding confusion.

Any solution of Eq. (3.1) will be called a *fermion field* and will be denoted by $\Psi(x)$. For specific solutions, we will use different notations. For example, for Majorana fields, we will use the notation $\psi(x)$, whereas for Weyl fields, $\chi(x)$. The word *spinor* will be used to denote any column-like function of energy and momentum which, when multiplied by a factor $\exp(ip \cdot x)$ or $\exp(-ip \cdot x)$, becomes a solution of the Dirac equation.

3.2 Real solutions

Is the Dirac equation a real equation like the Klein-Gordon equation? The answer depends on what the γ^μ 's are. If all non-zero elements of all four γ^μ 's are purely imaginary, then Eq. (3.1) is real. So the question is: can we define the γ^μ 's, subject to their basic properties encrypted in Eqs. (3.5) and (3.6), so that they are purely imaginary?

Indeed, we can. This was first found by Majorana. The representation, denoted by a tilde on the matrices,¹ is this:

$$\begin{aligned} \tilde{\gamma}^0 &= \begin{bmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{bmatrix}, & \tilde{\gamma}^1 &= \begin{bmatrix} i\sigma^1 & 0 \\ 0 & i\sigma^1 \end{bmatrix}, \\ \tilde{\gamma}^2 &= \begin{bmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{bmatrix}, & \tilde{\gamma}^3 &= \begin{bmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{bmatrix}, \end{aligned} \quad (3.8)$$

¹Throughout, we use the notation that whenever an array will be enclosed in square brackets, each entry should be thought of as a block of length 2, i.e., a 2×2 matrix for square arrays, and a 2×1 column for a column array.

where the σ^i 's are the usual Pauli matrices, written such that σ^2 is imaginary while the other two are real. Clearly,

$$\tilde{\gamma}_\mu^* = -\tilde{\gamma}_\mu \quad (3.9)$$

as proposed. These matrices constitute the Majorana representation of the γ -matrices.

So now suppose that we have written down the Dirac equation in detail, using the matrices from Eq. (3.8). That will be a real equation, just like the Klein-Gordon equation. Therefore, one should be able to find real solutions to this equation. In other words, we will find solutions which satisfy

$$\tilde{\psi} = \tilde{\psi}^*. \quad (3.10)$$

Such solutions will represent Majorana fermions. We emphasize that Eq. (3.10) is valid in the Majorana representation, a fact that is remembered by the presence of the tilde on top.

And now, this is the problem: Majorana representation is not unique in any sense. There are infinitely many choices of the Dirac matrices which satisfy Eqs. (3.5) and (3.6).² An important theorem says that if there are two choices of Dirac matrices, both satisfying Eqs. (3.5) and (3.6), they will be related by a similarity transformation involving a unitary matrix. In other words, the general solution of Eqs. (3.5) and (3.6) can be obtained from the Majorana representation as

$$\gamma^\mu = U \tilde{\gamma}^\mu U^\dagger \quad (3.11)$$

where U is a unitary matrix. If $\tilde{\Psi}$ is a solution of the Dirac equation in the Majorana representation of the Dirac matrices, a solution in this general representation will be given by

$$\Psi = U \tilde{\Psi}, \quad (3.12)$$

as can be checked easily from Eq. (3.1).

So, how will the Majorana condition, Eq. (3.10), appear if we choose to work with some other representation of the Dirac matrices except the Majorana representation? From Eqs. (3.10) and (3.12), we can easily find that the condition would be

$$U^\dagger \psi = \left(U^\dagger \psi \right)^* \quad (3.13)$$

or

$$\psi = U U^\top \psi^*. \quad (3.14)$$

²In fact, there are other representations in which all four Dirac matrices are purely imaginary. These can be obtained by any interchange of the matrices for γ^1 , γ^2 and γ^3 that are given in Eq. (3.8), with the option of changing the overall sign of any number of them.

Note that since U is unitary, the combination UU^\top is also unitary. Instead of using U directly, it is customary to use another unitary matrix C which is defined by

$$UU^\top = \gamma_0 C, \quad (3.15)$$

and create a compact notation for denoting the kind of combination that the right hand side of Eq. (3.14):

$$\widehat{\Psi} \equiv \gamma_0 C \Psi^*. \quad (3.16)$$

While this notation can be used for any fermion field, a Majorana fermion field is defined through the condition

$$\psi = \widehat{\psi}. \quad (3.17)$$

3.3 Fourier expansion

In the Majorana representation, the solution $\widetilde{\psi}$ is real. In analogy with Eq. (2.4), we can write down its Fourier expansion:

$$\widetilde{\psi}(x) = \sum_s \int_p \left(a_s(p) \widetilde{u}_s(p) e^{-ip \cdot x} + a_s^\dagger(p) \widetilde{u}_s^*(p) e^{+ip \cdot x} \right). \quad (3.18)$$

Notice that the solution involves some basis objects $\widetilde{u}_s(p)$ and $\widetilde{u}_s^*(p)$, which are spinors. Since the Dirac matrices are 4×4 , we will need four basis spinors. Two basis spinors $\widetilde{u}_s(p)$ and their complex conjugates should be able to do the job. This is why there is a sum appearing in Eq. (3.18): the index s takes two values, corresponding to the two independent basis spinors. It has to be said that the independent variables in the spinors are the components of the spatial vector \mathbf{p} , since the energy is related to the 3-momentum. However, we will write $u(p)$ and so on when no confusion arises.

Notice that the two terms of Eq. (3.18) are obviously conjugates of each other, which is how the expression should give a real $\widetilde{\psi}$. The question is, how will this Fourier expansion look in an arbitrary representation for Dirac matrices? Using Eq. (3.12), we obtain

$$\psi(x) = \sum_s \int_p \left(a_s(p) U \widetilde{u}_s(p) e^{-ip \cdot x} + a_s^\dagger(p) U \widetilde{u}_s^*(p) e^{+ip \cdot x} \right). \quad (3.19)$$

Let us now define the basis spinors for the arbitrary representation through the relation

$$u_s(p) = U \widetilde{u}_s(p), \quad (3.20)$$

which mimics Eq. (3.12) for the field operator. Obviously then,

$$U \widetilde{u}_s^*(p) = U \left(U^\dagger u_s(p) \right)^* = U U^\top u_s^*(p). \quad (3.21)$$

We can therefore write Eq. (3.19) as

$$\psi(x) = \sum_s \int_p \left(a_s(p) u_s(p) e^{-ip \cdot x} + a_s^\dagger(p) v_s(p) e^{+ip \cdot x} \right), \quad (3.22)$$

introducing the notation

$$v_s(p) = \gamma_0 C u_s^*(p), \quad (3.23)$$

where the matrix C was defined in Eq. (3.15). Taking the complex conjugate of both sides and multiplying by UU^\top , it is easy to see that this definition also implies

$$u_s(p) = \gamma_0 C v_s^*(p). \quad (3.24)$$

In the Majorana representation, Eq. (3.23) and Eq. (3.24) means the same thing, viz., the u and the v spinors are complex conjugates of each other.

3.4 Some properties of the matrix C

The matrix C has some interesting properties which we want to derive now. Using the definition of Eq. (3.15), we obtain

$$\begin{aligned} C^{-1} \gamma_\mu C &= U^* U^\dagger \gamma_0 \gamma_\mu \gamma_0 U U^\top \\ &= U^* U^\dagger \gamma_\mu^\dagger U U^\top \\ &= U^* \left(U^\dagger \gamma_\mu U \right)^\dagger U^\top \\ &= U^* \tilde{\gamma}_\mu^\dagger U^\top \\ &= \left(U \tilde{\gamma}_\mu^* U^\dagger \right)^\top, \end{aligned} \quad (3.25)$$

where we have used Eqs. (3.6) and (3.11). Finally now, using Eq. (3.9), we obtain

$$C^{-1} \gamma_\mu C = - \left(U \tilde{\gamma}_\mu U^\dagger \right)^\top = -\gamma_\mu^\top, \quad (3.26)$$

which can also be taken as a definition for the matrix C . In this form, the definition does not refer to the Majorana representation at all. **Combining this with Eq. (3.16), we now obtain a definition of Majorana fermion that is independent of any representation. No matter which representation of Dirac matrices you are working with, you can find the matrix C in that representation through Eq. (3.26) and use it to define $\hat{\psi}$ through Eq. (3.16). A Majorana fermion satisfies Eq. (3.17), which is a generalized form of the straight-forward reality condition of Eq. (3.10).**

The second interesting property of the matrix C can be derived by noting that, since U is unitary,

$$U U^\top U^* U^\dagger = 1. \quad (3.27)$$

From Eq. (3.15), it can be rewritten as

$$\gamma_0 C (\gamma_0 C)^* = 1. \quad (3.28)$$

Using Eqs. (3.7) and (3.26), this can be written as

$$-CC^* = 1, \quad (3.29)$$

or equivalently as

$$C^* = -C^{-1}. \quad (3.30)$$

Using the unitarity of the matrix C , this relation can be cast into the form

$$C^\top = -C, \quad (3.31)$$

i.e., C must be an antisymmetric matrix in any representation of the Dirac matrices.

3.5 Lorentz invariance of the reality condition

Let us now go back to the reality condition of Eq. (3.17). The condition would be physically meaningful only if it holds irrespective of any reference frame, i.e., is Lorentz invariant. We now show that this is indeed the case.

Under infinitesimal Lorentz transformations which take the co-ordinate of a spacetime point from x^μ to $x'^\mu = x^\mu + \omega^{\mu\nu}x_\nu$, a fermion field transforms as follows:

$$\Psi'(x') = \exp\left(-\frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu}\right)\Psi(x), \quad (3.32)$$

where

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]. \quad (3.33)$$

Taking the complex conjugate of Eq. (3.32) and multiplying from the left by $\gamma_0 C$, we obtain

$$\begin{aligned} \widehat{\Psi}'(x') &= \gamma_0 C \exp\left(+\frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu}^*\right)\Psi^*(x) \\ &= \gamma_0 C \exp\left(+\frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu}^*\right)(\gamma_0 C)^{-1}\widehat{\Psi}(x). \end{aligned} \quad (3.34)$$

This contains the complex conjugate of the sigma-matrices. In order to tackle them, let us note that Eqs. (3.6) and (3.26) tell us that

$$\gamma_\mu^* \equiv \left(\gamma_\mu^\dagger\right)^\top = \gamma_0^\top \gamma_\mu^\top \gamma_0^\top = -(\gamma_0 C)^{-1} \gamma_\mu (\gamma_0 C) \quad (3.35)$$

This gives

$$\gamma_0 C \sigma_{\mu\nu}^* (\gamma_0 C)^{-1} = -\sigma_{\mu\nu}. \quad (3.36)$$

Using this, we can simplify Eq. (3.34) and write

$$\widehat{\Psi}'(x') = \exp\left(-\frac{i}{4}\omega^{\mu\nu}\sigma_{\mu\nu}\right)\widehat{\Psi}(x). \quad (3.37)$$

Apart from the hats on the fermion field, this equation is exactly the same as Eq. (3.32). In other words, this equation tells us that $\widehat{\Psi}$, defined in Eq. (3.16), transforms exactly the same way that the fermion field does under proper Lorentz transformations. The combination $\widehat{\Psi}$ can therefore be called the Lorentz-covariant conjugate, or LCC, of Ψ .

It is now obvious why a reality condition like Eq. (3.17) is Lorentz invariant. Both sides of this equation transform the same way under Lorentz transformations. So, if the condition is true in any one Lorentz frame, it would be true in all frames.

3.6 Generalization of the reality condition

One can also make the following observation on Eq. (3.17). It is Lorentz covariant of course, but if we put an extra numerical factor on one side of the equation, it will still be Lorentz covariant. Constants with modulus not equal to unity can be disallowed from normalization arguments, but we can still have a condition of the form

$$\psi = e^{i\alpha} \widehat{\psi}. \quad (3.38)$$

No doubt this will also define a Majorana field. The plane wave expansion of this field will contain the phase α . Instead of Eq. (3.22), we should now write

$$\psi(x) = \sum_s \int_p \left(a_s(p) u_s(p) e^{-ip \cdot x} + e^{i\alpha} a_s^\dagger(p) v_s(p) e^{+ip \cdot x} \right). \quad (3.39)$$

However, it is easy to see that the phase α cannot be physically relevant. Rather than working with the field ψ satisfying Eq. (3.38), we can $e^{-i\alpha/2}\psi$ as our field, and then this field will satisfy Eq. (3.17). Nevertheless, the freedom is sometimes useful in some manipulations.

4 Left or right?

This is one of the frequently asked questions (or FAQ's, an acronym made popular by internet sites), or maybe a frequently answered question (i.e., FAQ in a different sense) even when no one asks it. The literature seems to be replete with statements where a Majorana neutrino is called either a left-handed fermion or a right-handed one, thus volunteering an answer for its handedness, without even anyone asking for it.

There is, of course, nothing wrong in answering a question before it is asked, if someone feels that it is anticipated, and that the answer would be helpful for understanding the topic under discussion. The problem here is that the answer makes no sense, because the implicit question makes no sense. To explain this statement, we need to get into the definition of "handedness". For particles obeying the Dirac equation, there are two possible definitions, and we discuss both in turn.

4.1 Helicity

A definition of “handedness” that can be applied to any particle has to do with the relative orientation of its momentum and angular momentum. The definition hinges on a property called “helicity”. For a particle with 3-momentum \mathbf{p} , helicity is defined as

$$h_{\mathbf{p}} \equiv \frac{2\mathbf{J} \cdot \mathbf{p}}{p}, \quad (4.1)$$

where \mathbf{J} denotes the angular momentum of the particle, and $p = |\mathbf{p}|$. The orbital part of the angular momentum is perpendicular to the direction of momentum, and therefore does not contribute to helicity. Helicity can therefore be described as twice the value of the spin component of a particle along the direction of its momentum. The factor of 2 is inserted in the definition so that the eigenvalues of this operator come out to be integral for any particle.

For a fermion obeying the Dirac equation, we can write the helicity as

$$h_{\mathbf{p}} = \frac{\boldsymbol{\Sigma} \cdot \mathbf{p}}{p}, \quad (4.2)$$

where $\frac{1}{2}\boldsymbol{\Sigma}$ denote the spin matrices, which satisfy the same commutation relations as the general angular momentum operators. These matrices are given by

$$\Sigma^i = \frac{1}{2}\varepsilon^{ijk}\sigma_{jk}, \quad (4.3)$$

where σ_{jk} are the space-space components of the set of matrices $\sigma_{\mu\nu}$ defined in Eq. (3.33).³ It can be easily seen that the eigenvalues of h are ± 1 . An eigenstate with eigenvalue -1 is usually called “left-handed”, whereas an eigenstate with eigenvalue $+1$ is called “right-handed”. In what follows, we will often use the terms “left-helical” and “right-helical” instead, in order to avoid confusion.

The interesting point is that h commutes with the Dirac Hamiltonian, a fact that can be checked with very little effort from Eq. (3.3), using the anticommutation relation of the Dirac matrices. For a free Dirac particle, helicity is therefore conserved: it does not change with time.

Helicity is also invariant under rotations, as the dot product in its definition clearly implies. In other words, if an observer works with a spatial co-ordinate system that is rotated with respect to that of another observer, both of them will infer the same value of helicity of a given particle.

However, helicity is not invariant under boosts. This can be easily seen by considering a simple example. Consider a fermion whose spin and momentum are both in the same direction, which we call the x -direction. Its helicity will be $+1$ in this case. Now consider the same particle from the point of view of a different observer who is moving also along the x -direction, faster than the particle with respect to the original frame. For this observer, the particle is moving in the opposite

³We take the convention $\varepsilon^{ijk} = +1$. In order to avoid any possible confusion, we use neither the antisymmetric tensor nor the components of $\boldsymbol{\Sigma}$ with lower indices.

direction, so the unit vector along the particle momentum is in the negative x -direction. The spin, however, does not change, since Eq. (4.3) tells us that the x -component of spin is really the yz -component of a rank-2 antisymmetric tensor, and components perpendicular to the frame velocity remain unaffected in a Lorentz transformation. The result is that, in the frame of this new observer, the helicity of the same fermion turns out to be -1 . And the lesson is this: a massive fermion cannot be exclusively left-helical or right-helical. Helicity depends on the observer who is looking at it.

We want to point out that in expounding this lesson, we have used the phrase “massive fermion”. One might wonder where the question of mass came into the argument. The answer is that, in our simple example, the different value of helicity is obtained from the point of view of an observer who moves faster than the particle in the original frame. For a massless particle, such a frame is impossible since the massless particle would always move at the speed of light. Hence, for a massless particle, the value of helicity should be Lorentz invariant. This is an issue that will be discussed later.

4.2 Chirality

The Greek word “chiros” means “hand”. From this word, the word “chirality” has been coined. Etymologically, “chirality” therefore means “handedness”. The meaning assigned to this technical word is associated with the matrix γ_5 which anticommutes with all Dirac matrices:

$$[\gamma_5, \gamma_\mu]_+ = 0 \quad \forall \mu. \quad (4.4)$$

From the anticommutation relation between the γ -matrices, it can be easily seen that the matrix

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \quad (4.5)$$

satisfies Eq. (4.4). An overall factor can be arbitrarily chosen in this definition, and we have chosen it in such a way that the matrix γ_5 has the properties

$$\gamma_5^\dagger = \gamma_5, \quad (\gamma_5)^2 = 1. \quad (4.6)$$

The last property guarantees that the matrices

$$L = \frac{1}{2}(1 - \gamma_5), \quad R = \frac{1}{2}(1 + \gamma_5), \quad (4.7)$$

can act as projection matrices on fermion fields and spinors. Such projections are also often called “left-handed” and “right-handed”, but we will use the terms “left-chiral” and “right-chiral” in order to avoid confusion. So, given any object $\Psi(x)$ that satisfies the Dirac equation, we can break it up into a left-chiral and a right-chiral part,

$$\Psi = \Psi_L + \Psi_R, \quad (4.8)$$

where

$$\Psi_L = L\Psi, \quad \Psi_R = R\Psi. \quad (4.9)$$

Alternatively, we can say that

$$L\Psi_L = \Psi_L, \quad R\Psi_L = 0, \quad (4.10)$$

and a similar set of equations for Ψ_R .

It is important to note that if we consider a left-chiral solution of the Dirac equation, it remains left-chiral under Lorentz transformations. This is guaranteed by the fact that

$$[\gamma_5, \sigma_{\mu\nu}] = 0 \quad \forall \mu, \nu, \quad (4.11)$$

which follows easily from Eq. (4.4), in conjunction with Eq. (3.33). Thus, chiral projections can be made in a Lorentz covariant way. However, chirality is not conserved even for a free particle, because γ_5 does not commute with the mass term in the Dirac Hamiltonian. This can be seen from the fact that the mass term in the Dirac Hamiltonian contains only one Dirac matrix, and therefore anticommutes, rather than commutes, with γ_5 . There is no problem with the derivative term. It contains two Dirac matrices (one of them hidden in the definition of $\bar{\Psi}$) and therefore commutes with γ_5 .

In this way, chirality and helicity have somewhat opposite characteristics: helicity is conserved for a free particle but is not Lorentz invariant, whereas chirality is Lorentz invariant but not conserved. Therefore, none of these properties is appropriate for characterizing a fermion that has mass. If a particle is branded left-helical, i.e., left-handed in the helicity sense, it will not appear to be so to a suitably boosted observer. If, at one time, a particle is found left-chiral, i.e., left-handed in the chirality sense, it will not remain so at another time.

5 Weyl fermions

It has been noted that the problem with assigning a frame-independent helicity to a fermion disappears if the fermion is massless. The problem with a conserved value of γ_5 also disappears in this limit, since γ_5 does indeed commute with the mass-independent term in the Dirac Hamiltonian. This shows that, without any ambiguity, one can talk about a positive or negative helicity fermion or of a left or right chiral fermion when one talks about massless fermions.

5.1 Irreducible fermion fields

Indeed, it is very convenient to use such objects in any discussion regarding fermions. A general solution of the Dirac equation is not an irreducible representation of the Lorentz group. This is best seen by the existence of the matrix γ_5 that commutes

with all generators of the representation, a fact that was summarized in Eq. (4.11). By Schur's lemma, no matrix other than the unit matrix should have this property if the generators pertain to an irreducible representation. We have already seen that a left-chiral fermion field retains its chirality under Lorentz transformations, implying that such fields are irreducible.⁴ So are right-chiral fields, of course. It is known that the proper Lorentz algebra is isomorphic to $SU(2) \times SU(2)$, so that any representation of the Lorentz algebra can be identified by its transformation properties under each of the $SU(2)$ factors. In this language, a left-chiral fermion would be a doublet under one of the $SU(2)$'s and singlet under the other, a fact that is summarized by denoting the representation as $(\frac{1}{2}, 0)$. A right-chiral fermion is a $(0, \frac{1}{2})$ representation. Either of them is called a Weyl fermion. A general fermion field transforms like a reducible representation $(\frac{1}{2}, 0) + (0, \frac{1}{2})$. This tells us that a general field can be described by two Weyl fields: one left-chiral and one right-chiral. This is the advantage of talking in terms of Weyl fields: they can be seen as the building blocks for any fermion field.

We could have said the same things in terms of helicity instead of chirality, because there is a connection between the two in the massless limit. For massless particles, the Dirac equation for an eigenstate of 3-momentum is given by

$$\left(\gamma^0 \mathbf{p} - \boldsymbol{\gamma} \cdot \mathbf{p}\right) w_p = 0, \quad (5.1)$$

where w_p can stand for $u(p)$ or $v(p)$. This can be written as

$$\left(1 - \gamma^0 \boldsymbol{\gamma} \cdot \frac{\mathbf{p}}{p}\right) w_p = 0. \quad (5.2)$$

Further, it can be shown that [6]

$$\gamma^0 \gamma^i = \gamma_5 \Sigma^i, \quad (5.3)$$

so that we can write

$$\left(1 - \gamma_5 \frac{\boldsymbol{\Sigma} \cdot \mathbf{p}}{p}\right) w_p = 0. \quad (5.4)$$

Multiplying throughout by γ_5 and using Eq. (4.6), we obtain

$$\gamma_5 w_p = \frac{\boldsymbol{\Sigma} \cdot \mathbf{p}}{p} w_p, \quad (5.5)$$

showing that helicity and chirality coincide for massless spinors. Thus, we can talk about handedness of Weyl spinors without any hesitation about the meaning of the term.

⁴Strictly speaking, this implies that left-chiral and right-chiral fields fall into different irreducible representations. It does not preclude the possibility that either of these can be further reduced. We show in Sec. 6 that the chiral fields are indeed irreducible.

5.2 Fourier expansion

Let us therefore talk about a **left-handed Weyl fermion field**. In order to keep a clear distinction with other kinds of fermions talked about earlier, we will denote it by the symbol χ . It is left-handed if it satisfies the relations

$$L\chi = \chi, \quad R\chi = 0. \quad (5.6)$$

The plane-wave expansion can now contain only left-chiral spinors $u_L \equiv Lu$ and $v_L \equiv Lv$, and can be written as

$$\chi(x) = \int_p \left(a(p) u_L(p) e^{-ip \cdot x} + \hat{a}^\dagger(p) v_L(p) e^{+ip \cdot x} \right). \quad (5.7)$$

Note that there is no sum over different solutions for u -type and v -type spinors. Since we have chosen one particular chirality, there can be only one solution of each kind. Because of this reason, we have not put any index on the creation and annihilation operators denoting spin projection or chirality.

We now find out the helicities of the states that are produced from the vacuum by a^\dagger and \hat{a}^\dagger . For this, we need to invert Eq. (5.7), and we need the explicit form of the momentum integration implied in that equation. Let us suppose that

$$\int_p \equiv \int d^3p I_p, \quad (5.8)$$

where the factor I_p depends only on the magnitude of the 3-momentum, or equivalently on the energy. Let us also suppose that the u -type and the v -type spinors have been normalized according to the conditions

$$\begin{aligned} u_s^\dagger(\mathbf{p}) u_{s'}(\mathbf{p}) &= N_p \delta_{ss'}, & v_s^\dagger(\mathbf{p}) v_{s'}(\mathbf{p}) &= N_p \delta_{ss'}, \\ u_s^\dagger(\mathbf{p}) v_{s'}(-\mathbf{p}) &= v_s^\dagger(\mathbf{p}) u_{s'}(-\mathbf{p}) = 0. \end{aligned} \quad (5.9)$$

It is then easy to see that

$$\begin{aligned} a(k) &= \frac{1}{(2\pi)^3 I_k N_k} \int d^3x e^{ik \cdot x} u_L^\dagger(k) \chi(x), \\ \hat{a}^\dagger(k) &= \frac{1}{(2\pi)^3 I_k N_k} \int d^3x e^{-ik \cdot x} v_L^\dagger(k) \chi(x). \end{aligned} \quad (5.10)$$

The first of these equations implies

$$a^\dagger(k) = \frac{1}{(2\pi)^3 I_k N_k} \int d^3x e^{-ik \cdot x} \chi^\dagger(x) u_L(k). \quad (5.11)$$

For any field $\Psi(x)$ satisfying the Dirac equation, the angular momentum operator should satisfy a relation of the form

$$\left[\Psi(x), J_{\mu\nu} \right] = \left(i(x_\mu \partial_\nu - x_\nu \partial_\mu) + \frac{1}{2} \sigma_{\mu\nu} \right) \Psi(x) \quad (5.12)$$

The purely spatial part involving derivatives on the right hand side is the orbital angular momentum, and the second part the spin. **From this, it follows that**

$$\left[\Psi(x), h_{\mathbf{p}} \right] = \frac{\boldsymbol{\Sigma} \cdot \mathbf{p}}{p} \Psi(x), \quad (5.13)$$

since the orbital part does not contribute, as remarked earlier. Taking the hermitian conjugate of this equation, we obtain

$$\left[\Psi^\dagger(x), h_{\mathbf{p}} \right] = -\Psi^\dagger(x) \frac{\boldsymbol{\Sigma} \cdot \mathbf{p}}{p}, \quad (5.14)$$

using the fact that $h_{\mathbf{p}}$ is hermitian because Eqs. (3.6) and (4.3) imply that the matrices $\boldsymbol{\Sigma}$ are hermitian. In particular, this equation is valid for the Weyl field $\chi(x)$. From Eq. (5.11), we then obtain

$$\begin{aligned} \left[a^\dagger(k), h_{\mathbf{k}} \right] &= \frac{1}{(2\pi)^3 I_k N_k} \int d^3x e^{-ik \cdot x} \left[\chi^\dagger(x), h_{\mathbf{k}} \right] u_L(k) \\ &= -\frac{1}{(2\pi)^3 I_k N_k} \int d^3x e^{-ik \cdot x} \chi^\dagger(x) \frac{\boldsymbol{\Sigma} \cdot \mathbf{k}}{k} u_L(k). \end{aligned} \quad (5.15)$$

Since we are dealing with a massless field here, we can use Eq. (5.5), and then the fact that $\gamma_5 L = -L$, obtaining

$$\left[a^\dagger(k), h_{\mathbf{k}} \right] = \frac{1}{(2\pi)^3 I_k N_k} \int d^3x e^{-ik \cdot x} \chi^\dagger(x) u_L(k) = a^\dagger(k). \quad (5.16)$$

Applying both sides of this equation on the vacuum and noticing that the vacuum state does not have any momentum, we obtain

$$h_{\mathbf{k}} a^\dagger(k) |0\rangle = -a^\dagger(k) |0\rangle, \quad (5.17)$$

which shows that the state $a^\dagger(k)|0\rangle$ has helicity -1 . Performing an exactly similar calculation starting with Eq. (5.13), we can show that the state $\hat{a}^\dagger(k)|0\rangle$ has helicity $+1$. In other words, the Weyl field operator annihilates a negative helicity particle and creates a positive helicity antiparticle.

This could have been guessed from the CPT theorem. Under very general conditions, any field theory is CPT invariant. CPT invariance means that if we consider a process in which an initial state A consisting of some particles goes to some final state B , and another process involving the CP-conjugate of the final state particles going into the CP-conjugate of the initial state particles, the amplitude of the two processes should be equal. Therefore, a necessary condition for CPT invariance is that if a particle exists in a theory, its CP conjugate has to exist in the theory as well. The CP-conjugate of a left-chiral particle is a right-chiral antiparticle. Therefore, if the annihilation operator in a field operator annihilates a left-handed particle, the creation operator must create a right-handed antiparticle. A left-chiral Weyl fermion field has only these two states. If we consider a right-chiral Weyl fermion field, that will have a right-chiral particle and its CP-conjugate, a left-chiral antiparticle.

5.3 Majorana fermions from Weyl fermions

We have said earlier that Weyl fermions, being irreducible representations of the proper Lorentz group, can be used as building blocks of any kind of fermion field. We can now ask a specific question: how can a Majorana fermion field be built out of Weyl fields?

A Majorana fermion has mass. Therefore, it must have both left- and right-chiral components. It is therefore clear that we will need a left-chiral Weyl fermion field as well as a right-chiral one in order to obtain a Majorana field. However, the arrangement with the two chiralities must be such that the Majorana condition, Eq. (3.17), is satisfied. So the question boils down to this: how can one arrange to have two Weyl fields of two chiralities such that they satisfy the Majorana condition?

Before we can answer this question, we have some groundwork to do. A left-chiral Weyl field satisfies the equation

$$(1 + \gamma_5)\chi = 0, \quad (5.18)$$

as seen in Eq. (5.6). We take the complex conjugate of this equation and multiply to the left by $\gamma_0 C$, which gives

$$\gamma_0 C(1 + \gamma_5^*)\chi^* = 0. \quad (5.19)$$

Since γ_5 is hermitian, $\gamma_5^* = \gamma_5^\top$. Using Eqs. (3.26) and (4.5), we can show that

$$C^{-1}\gamma_5 C = \gamma_5^\top, \quad (5.20)$$

or

$$C\gamma_5^\top = \gamma_5 C. \quad (5.21)$$

Thus, Eq. (5.19) can be written as

$$\gamma_0(1 + \gamma_5)C\chi^* = 0. \quad (5.22)$$

Since γ_0 anticommutes with γ_5 , this is equivalent to

$$(1 - \gamma_5)\widehat{\chi} = 0, \quad (5.23)$$

using the definition of the Lorentz-covariant complex conjugation given in Eq. (3.16). This shows that if $\chi(x)$ is a left-chiral Weyl field, $\widehat{\chi}(x)$ is a right-chiral Weyl field.

The rest is obvious. The complex conjugation operation is a toggle operation, i.e., applying it twice is the same as not applying it ever. In other words,

$$\widehat{(\widehat{\Psi})} = \Psi \quad (5.24)$$

for any kind of fermion field. Thus, if we define a field by

$$\psi(x) = \chi(x) + \widehat{\chi}(x), \quad (5.25)$$

it will obviously satisfy the reality condition of Eq. (3.17) and will constitute a Majorana field.

The construction raises an interesting question. A Weyl fermion is massless whereas a Majorana fermion has mass. How, by adding two Weyl fermions with two opposite chiralities, we have also generated a mass?

The point is that we have not really ‘generated’ a mass: we have only created an arrangement where mass can be allowed. The mass term in the Dirac Lagrangian is of the form $\bar{\Psi}\Psi$. Using chiral projections, it can be written as $\bar{\Psi}_L\Psi_R + \bar{\Psi}_R\Psi_L$. There is no term like $\bar{\Psi}_L\Psi_L$ or $\bar{\Psi}_R\Psi_R$, because these combinations vanish. For a Weyl fermion field which has a specific chirality, the mass term must therefore vanish. In other words, the mass term must contain two different chiralities: a Weyl fermion is unable to meet this demand.

One might be tempted to argue that a term of the form $\chi^\top C^{-1}\chi$ is Lorentz invariant, is quadratic in the fields, and does not contain any derivatives. Moreover, it can be constructed with only one single chirality, and therefore should be considered as a possible mass term for chiral fermion fields. But this argument does not work because this term is not hermitian. One must add its hermitian conjugate to the Lagrangian as well, and this can be written as $\bar{\chi}\hat{\chi}$, or equivalently as $\hat{\chi}^\top C^{-1}\hat{\chi}$. Either way, it shows that we need the field $\hat{\chi}$ in order to write down a mass term, and this $\hat{\chi}$ is a right-chiral field, as shown before. In summary, a massive fermion must have a left-chiral as well as a right-chiral component. A left-handed Weyl fermion does not have a right-handed component, and hence cannot be massive. By adding a right-handed Weyl fermion $\hat{\chi}$ to the left-handed χ , we have fixed this shortcoming, and that is why a Majorana fermion, given in Eq. (5.25), can have a mass.

5.4 Dirac fermions from Weyl fermions

Finally, we come to Dirac fermions. These are also massive fermions, and therefore require Weyl fermions of both helicities. Also, these are in general complex fermions, i.e., they do not satisfy any reality condition like Majorana fermions do. When we wrote Eq. (5.25), we took the right-chiral Weyl field to be the LCC of the left-chiral field, which is why the resulting field satisfied the reality condition of Eq. (3.17). Instead, if we take two independent left-chiral Weyl fields $\chi_1(x)$ and $\chi_2(x)$, and make the combination

$$\Psi(x) = \chi_1(x) + \hat{\chi}_2(x), \quad (5.26)$$

this defines a Dirac field.

To summarize, a Dirac field is a completely unconstrained solution of the Dirac equation. Both Weyl and Majorana fields are simpler solutions, with some kind of constraints imposed on the solution. We have seen that there are two types of conditions that can be imposed in a Lorentz covariant manner on a solution of the Dirac equation. One is a reality condition, imposition of which produces a Majorana field. The other is a chirality condition, imposition of which produces a Weyl field.

We can ask whether we can impose both kinds of constraints at the same time. In other words, whether we can have a fermion field which is both Majorana and Weyl. It can be easily seen that it is not possible. The best way to see it is to use the Majorana representation of the Dirac matrices. In this representation, a Majorana field is real. On the other hand, a Weyl field satisfies either Eq. (5.6) or a similar

equation obtained by interchanging R and L . These two types of conditions can be written in an alternative form,

$$\gamma_5 \chi = \pm \chi. \quad (5.27)$$

In Majorana representation of the Dirac matrices, γ_5 is purely imaginary. Therefore, Eq. (5.27) cannot be satisfied by a real field χ , which shows that a Weyl field cannot be a Majorana field at the same time.

6 Two-component notation

In the Introduction, we mentioned that Weyl fermions can be represented in a more compact notation, viz. as two-component objects. In this section, we discuss how this notation works, and how much can be expressed with it.

6.1 Weyl fermions

To see why a 2-component notation would work for a Weyl fermion, let us look at the Hamiltonian of Eq. (3.3). For massless particles, γ^0 and γ^i lose their separate identities: only the combinations $\gamma^0 \gamma^i$ appear in the Hamiltonian and therefore can have physical consequences. Let us use a shorthand to denote these three matrices:

$$\alpha^i = \gamma^0 \gamma^i. \quad (6.1)$$

Using Eq. (3.5), it is easy to deduce the anticommutation relations among the α^i 's:

$$[\alpha^i, \alpha^j]_+ = 2\delta^{ij}. \quad (6.2)$$

This set of relations can be satisfied by taking the α 's to be equal to the Pauli matrices. In this representation, the solution of the Dirac equation will be a 2-component object.

How are we going to define chirality of these objects, now that we don't have the services of the matrix γ_5 ? The answer is that we take the help of the fact that chirality coincides with helicity for massless spinors. Helicity eigenstate spinors can be defined in the 2-component notation by the equation

$$\frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{p} \xi_{\pm}(p) = \pm \xi_{\pm}(p), \quad (6.3)$$

The solution ξ_+ has helicity eigenvalue $+1$, and is a right-handed spinor. The solution ξ_- is left-handed.

The whole thing can also be viewed in terms of the 4×4 representation of the Dirac matrices. Only, for this purpose, the argument comes out clearly if we use a different representation of the Dirac matrices, called the *chiral representation* or the *Weyl representation*. In this representation, the Dirac matrices are given by

$$\tilde{\gamma}^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \tilde{\gamma}^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix}, \quad (6.4)$$

where we have put a crescent sign to indicate this representation. From Eq. (4.5), it follows that in this representation,

$$\check{\gamma}_5 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (6.5)$$

i.e., is block diagonal. This means

$$\check{L} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \check{R} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \quad (6.6)$$

Consider now the defining equation of a left-chiral spinor, Eq. (5.18). Clearly, in this representation, the lower two components of such a spinor must vanish. For a right-chiral spinor, the upper two components should vanish. Thus, for a spinor with a specific chirality, two of the four components are superfluous. The two-component notation described above essentially does away with the vanishing components explicitly and deals only with the non-trivial ones.

Surely, the Lagrangian as well as the equation of motion can be written in the 2-component notation. For this, we start from the 4-component notation, and note that in the chiral representation, the α -matrices are given by

$$\alpha^i = \begin{bmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{bmatrix}, \quad (6.7)$$

using Eqs. (6.1) and (6.4). Consider now a massless field Ψ . Eq. (3.4) tells us that its Lagrangian should be

$$\mathcal{L} = i\bar{\Psi}\gamma^\mu\partial_\mu\Psi = i\Psi^\dagger\left(\partial_0 + \alpha^k\partial_k\right)\Psi \quad (6.8)$$

in any representation of the Dirac matrices. We now introduce a notation that we will use for arbitrary fermion fields:

$$\check{\Psi} = \begin{bmatrix} \xi_t \\ \xi_b \end{bmatrix}, \quad (6.9)$$

where ξ_t and ξ_b are 2-component fields. They represent the top two and the bottom two components of the fermion field in the chiral representation. Using Eq. (6.7), we see that the Lagrangian can be written down as

$$\mathcal{L} = i\xi_t^\dagger\left(\partial_0 - \sigma^k\partial_k\right)\xi_t + i\xi_b^\dagger\left(\partial_0 + \sigma^k\partial_k\right)\xi_b. \quad (6.10)$$

If we are talking of a right-handed Weyl field, $\xi_t = 0$. The Lagrangian for ξ_b is often written in the more compact form

$$\mathcal{L} = i\xi_b^\dagger\sigma^\mu\partial_\mu\xi_b, \quad (6.11)$$

where one defines the set of four 2×2 matrices

$$\sigma^\mu \equiv (1, \boldsymbol{\sigma}). \quad (6.12)$$

For left-handed Weyl fields, the corresponding Lagrangian is

$$\mathcal{L} = i\xi_t^\dagger \bar{\sigma}^\mu \partial_\mu \xi_t, \quad (6.13)$$

where

$$\bar{\sigma}^\mu \equiv (1, -\boldsymbol{\sigma}). \quad (6.14)$$

Note that this use of the bar has nothing to do with the definition used for the fields, first introduced in Eq. (3.4).

The Dirac equation for Weyl fields can be written either from the Lagrangians in Eqs. (6.11) and (6.13) or starting from the massless Dirac equation in the 4-component formalism. Either way, one gets the equations for the 2-component fields to be

$$\sigma^\mu \partial_\mu \xi_t = 0, \quad \bar{\sigma}^\mu \partial_\mu \xi_b = 0. \quad (6.15)$$

or more explicitly

$$\left(\partial_0 - \sigma^i \partial_i\right) \xi_t = 0, \quad \left(\partial_0 + \sigma^i \partial_i\right) \xi_b = 0. \quad (6.16)$$

Lastly, let us write down the matrices $\sigma^{\mu\nu}$ in the chiral representation. It is straight forward to deduce, using Eq. (3.33), the results

$$\check{\sigma}^{0k} = \begin{bmatrix} i\sigma^k & 0 \\ 0 & -i\sigma^k \end{bmatrix}, \quad \check{\sigma}^{ij} = \varepsilon^{ijk} \begin{bmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{bmatrix}. \quad (6.17)$$

This shows that the generators are block diagonal, in two 2×2 blocks. This is an explicit demonstration of the fact that the 4×4 representation is reducible, a fact that we mentioned earlier. The 2×2 blocks are irreducible, since they contain the Pauli matrices, which cannot be diagonalized simultaneously.

The expressions of $\check{\sigma}^{ij}$ in Eq. (6.17) explains something that we have used but did not explain. Note that in Eq. (4.2) we used the matrices $\boldsymbol{\Sigma}$ to define helicity, but then in Eq. (6.3) we used the matrices $\boldsymbol{\sigma}$ in their place. This was done in anticipation of Eq. (6.17), which implies that $\check{\boldsymbol{\Sigma}}$ reduces to $\boldsymbol{\sigma}$ while operating either on left-chiral fields which have only the two upper components or on right-chiral fields which have only the two lower components.

6.2 Majorana fermions

A Majorana fermion field, in the Majorana representation, **has four real components.** Can it also be represented in terms of a 2-component field if we allow for complex components?

To answer this question, let us look at the matrix \check{U} that connects the chiral representation of the Dirac matrices to the Majorana representation in the sense of Eq. (3.11). From the explicit forms of the Dirac matrices in the two representations, it is straight forward to show that

$$\check{U} = \frac{1}{2} \begin{bmatrix} 1 + \sigma^2 & -i(1 - \sigma^2) \\ i(1 - \sigma^2) & 1 + \sigma^2 \end{bmatrix}. \quad (6.18)$$

If a Majorana field is represented by the components $\tilde{\psi}_1, \dots, \tilde{\psi}_4$ in the Majorana representation, in the chiral representation the field will be obtained by using Eq. (3.12), and the explicit form of \check{U} from Eq. (6.18). The result is

$$\check{\psi} = \frac{1}{2} \begin{pmatrix} (\tilde{\psi}_1 + \tilde{\psi}_4) - i(\tilde{\psi}_2 + \tilde{\psi}_3) \\ (\tilde{\psi}_2 - \tilde{\psi}_3) + i(\tilde{\psi}_1 - \tilde{\psi}_4) \\ -(\tilde{\psi}_2 - \tilde{\psi}_3) + i(\tilde{\psi}_1 - \tilde{\psi}_4) \\ (\tilde{\psi}_1 + \tilde{\psi}_4) + i(\tilde{\psi}_2 + \tilde{\psi}_3) \end{pmatrix}. \quad (6.19)$$

The components in the Majorana representation are real. We notice that, by taking real and imaginary parts of only the two upper components of $\check{\psi}$, we can obtain all the information that is there in $\tilde{\psi}$. For example, $\tilde{\psi}_1 = \text{Re } \check{\psi}_1 + \text{Im } \check{\psi}_2$. In a similar manner, we can determine all components of $\tilde{\psi}$ explicitly in terms of $\check{\psi}_1$ and $\check{\psi}_2$ only. The two lower components are not independent because $\check{\psi}$ should satisfy the Majorana condition. In the chiral representation,

$$\check{\gamma}^0 \check{C} = \begin{bmatrix} 0 & i\sigma^2 \\ -i\sigma^2 & 0 \end{bmatrix}, \quad (6.20)$$

which can be easily checked through Eqs. (3.15) and (6.18). Thus, if we use the right hand side of Eq. (6.9) to represent a Majorana field, we obtain the relation between the top and bottom 2-components in the form

$$\xi_b = -i\sigma^2 \xi_t^*, \quad \xi_t = i\sigma^2 \xi_b^*, \quad (6.21)$$

from the Majorana condition, Eq. (3.17). These relations can be explicitly checked in the expression of $\check{\psi}$ given in Eq. (6.19). Thus we can write a Majorana field in the chiral representation in the form

$$\check{\psi}(x) = \begin{bmatrix} \omega(x) \\ -i\sigma^2 \omega^*(x) \end{bmatrix}. \quad (6.22)$$

Clearly, everything about the Majorana field can be written down by using the upper two components only, which we have denoted by $\omega(x)$. This is also a 2-component representation like that used for Weyl fields earlier, only we use a different symbol in order to avoid confusion.

The Lagrangian of a Majorana field, in the 4-component notation, is given by

$$\mathcal{L} = \frac{1}{2} \left(\bar{\psi} i\gamma^\mu \partial_\mu \psi - m \bar{\psi} \psi \right). \quad (6.23)$$

The overall factor of $\frac{1}{2}$ compared to the general Dirac Lagrangian is usual for self-conjugate fields, introduced to ensure a consistent normalization of the field operators in quantum field theory. Using the representations of the Dirac matrices given in Eq. (6.4), one obtains

$$\mathcal{L} = \frac{i}{2} \left(\omega^\dagger \bar{\sigma}^\mu \partial_\mu \omega - m \omega^\top \sigma^2 \omega \right) + \text{h.c.}, \quad (6.24)$$

where “h.c.” means hermitian conjugate. The equation of motion that follows from this Lagrangian is given by

$$\bar{\sigma}^\mu \partial_\mu \omega + m \sigma^2 \omega^* = 0. \quad (6.25)$$

Let us give some details of the derivation of Eq. (6.24) which might be illuminating. Consider the term containing the derivative of ψ in Eq. (6.23). We already know, from Eqs. (6.10), (6.13) and (6.11), how this term will look like if written in the 2-component notation. For a Majorana field since the top and bottom two components are related by Eq. (6.21), we obtain

$$\bar{\psi} i \gamma^\mu \partial_\mu \psi = i \omega^\dagger \sigma^\mu \partial_\mu \omega + i \omega^\top \sigma^2 \bar{\sigma}^\mu \sigma^2 \partial_\mu \omega^*. \quad (6.26)$$

The first term on the right-hand side appears, as it is, in Eq. (6.24). As for the second term, let us write it in the generic form $a^\top X b$, where a and b are column matrices and X is a square matrix. Now let us put down explicit subscripts for the matrix elements, i.e., write $a^\top X b = a_\alpha X_{\alpha\beta} b_\beta$. Suppose now we want to write it with the component of b in front. We have to interchange the places of a_α and b_β , but it should be remembered that these are components of fermion fields, and therefore they anticommute. Thus we can write

$$a^\top X b = a_\alpha X_{\alpha\beta} b_\beta = -b_\beta X_{\alpha\beta} a_\alpha = -b^\top X^\top a. \quad (6.27)$$

Using this, and the relation

$$\left(\sigma^2 \bar{\sigma}^\mu \sigma^2 \right)^\top = \sigma^\mu \quad (6.28)$$

which can be easily checked to be true, we find

$$i \omega^\top \sigma^2 \bar{\sigma}^\mu \sigma^2 \partial_\mu \omega^* = -i (\partial_\mu \omega^\dagger) \sigma^\mu \omega, \quad (6.29)$$

which is the hermitian conjugate of the first term of the right-hand side of Eq. (6.26). The derivation of the mass term of Eq. (6.24) is very similar, and we do not give the details.

6.3 Fourier expansion

Since Weyl and Majorana fields can be written down in 2-component representation, it should also be possible to write their Fourier expansion in this representation. For a left-handed Weyl field, this expansion is

$$\chi(x) = \int_p \left(a(p) \xi_-(p) e^{-ip \cdot x} + \hat{a}^\dagger(p) \xi_-(p) e^{+ip \cdot x} \right). \quad (6.30)$$

For a right-handed Weyl field, basis spinors ξ_+ will appear in the expansion. The spinors ξ_\pm were defined in Eq. (6.3). The explicit components can be easily found out from the defining equation, viz.,

$$\xi_- = \begin{pmatrix} e^{-i\varphi} \sin \frac{\vartheta}{2} \\ -\cos \frac{\vartheta}{2} \end{pmatrix}, \quad \xi_+ = \begin{pmatrix} \cos \frac{\vartheta}{2} \\ e^{+i\varphi} \sin \frac{\vartheta}{2} \end{pmatrix}, \quad (6.31)$$

where the components of the 3-vector \mathbf{p} are denoted by

$$\begin{aligned} p_x &= p \sin \vartheta \cos \varphi, \\ p_y &= p \sin \vartheta \sin \varphi, \\ p_z &= p \cos \vartheta. \end{aligned} \quad (6.32)$$

The overall phases of ξ_{\pm} have been adjusted so that

$$\xi_+ = -i\sigma^2 \xi_-^*, \quad \xi_- = i\sigma^2 \xi_+^*. \quad (6.33)$$

For a Majorana field, both helicities should be present in the Fourier expansion, since a massive particle cannot have a Lorentz invariant value of helicity, as discussed earlier. Hence we start by writing the Fourier expansion in the form [7]

$$\omega(x) = \sum_{r=1,2} \int_p \left(a_r(p) \zeta_r(p) e^{-ip \cdot x} + a_r^\dagger(p) \eta_r(p) e^{+ip \cdot x} \right), \quad (6.34)$$

mimicking the 4-component expression of Eq. (3.18), using 2-component basis spinors $\zeta_r(p)$ and $\eta_r(p)$. Some conjugation relations exist between the 2-component ζ - and η spinors, similar to the relations between the 4-component u - and v -spinors, Eqs. (3.23) and (3.24). To find these relations, let us first note that

$$\bar{\sigma}^\mu \partial_\mu e^{\pm ip \cdot x} = \pm \left(E + \boldsymbol{\sigma} \cdot \mathbf{p} \right) e^{\pm ip \cdot x}, \quad (6.35)$$

so that, substituting Eq. (6.34) into Eq. (6.25) and equating the coefficients of a_1 and a_2 respectively, we obtain the relations [7]

$$\eta_r = \frac{E - \boldsymbol{\sigma} \cdot \mathbf{p}}{m} i\sigma^2 \zeta_r^*. \quad (6.36)$$

Equivalently, one can write

$$\zeta_r = -\frac{E - \boldsymbol{\sigma} \cdot \mathbf{p}}{m} i\sigma^2 \eta_r^*, \quad (6.37)$$

using the identities

$$\left(\frac{E - \boldsymbol{\sigma} \cdot \mathbf{p}}{m} \right)^{-1} = \frac{E + \boldsymbol{\sigma} \cdot \mathbf{p}}{m}, \quad (6.38)$$

and

$$\boldsymbol{\sigma}^* = -\sigma^2 \boldsymbol{\sigma} \sigma^2. \quad (6.39)$$

Any linearly independent choice of the ζ_r 's, along with the corresponding η_r 's defined from Eq. (6.36), will constitute the Fourier expansion of a Majorana field in the 2-component notation. It would be instructive to examine the nature of the Fourier modes by making specific choices for the basis spinors. For example, let us take

$$\zeta_1 = \xi_-, \quad \eta_2 = \xi_-. \quad (6.40)$$

Using the definition of ξ_- from Eq. (6.3) and the conjugation property from Eq. (6.33), we then obtain

$$\eta_1 = -\frac{E - \mathbf{p}}{m} \xi_+, \quad \zeta_2 = \frac{E - \mathbf{p}}{m} \xi_+. \quad (6.41)$$

Despite having a factor of m in the denominator, these expressions vanish rather than diverge in the limit of vanishing mass, because

$$\frac{E - \mathbf{p}}{m} = \frac{m}{E + \mathbf{p}}. \quad (6.42)$$

Thus, the Fourier expansion of Eq. (6.34) can be written explicitly as

$$\begin{aligned} \omega(x) = \int_p & \left(a_-(p) \xi_-(p) e^{-ip \cdot x} + \frac{m}{E + \mathbf{p}} a_+(p) \xi_+(p) e^{-ip \cdot x} \right. \\ & \left. + a_+^\dagger(p) \xi_-(p) e^{+ip \cdot x} + \frac{m}{E + \mathbf{p}} a_-^\dagger(p) \xi_+(p) e^{-ip \cdot x} \right). \end{aligned} \quad (6.43)$$

Note that the subscript on a^\dagger is the opposite of the subscript of ξ that multiplies it. This has to do with the fact that the helicity of the state produced by this part of the Fourier expansion is opposite to the helicity of the spinor that appears in the term, as proved in §5.2.

A look at the Fourier expansion of Eq. (6.43) shows something quite interesting. In the non-relativistic limit, $\mathbf{p} \approx 0$ and so $E \approx m$, so that $m/(E + \mathbf{p}) \approx 1$. In the static limit, this is an exact result. For such values of momenta, the two chiralities are produced and annihilated with the same amplitude. However, in the ultra-relativistic limit, $m \ll E$, so that the terms involving ξ_+ becoming vanishingly small and the Majorana field behaves very much the same as a left-chiral Weyl field. Had we made the opposite choices for ζ_1 and η_2 in Eq. (6.40), i.e., taken ξ_+ instead of ξ_- , the resulting Majorana field would have behaved like a right-chiral Weyl field in the ultra-relativistic limit.

We can try to construct the Fourier decomposition of the 4-component representation of the Majorana field from that of $\omega(x)$, using Eq. (6.22). The task is simple and straight forward. Take $\omega(x)$ as in Eq. (6.34), form $-i\sigma^2 \omega^*(x)$, and put one on top of the other. The resulting expression will have the form given in Eq. (3.22), with the 4-component spinors given by

$$\check{u}_r(p) = \begin{bmatrix} \zeta_r(p) \\ -i\sigma^2 \eta_r^*(p) \end{bmatrix}, \quad \check{v}_r(p) = \begin{bmatrix} \eta_r(p) \\ -i\sigma^2 \zeta_r^*(p) \end{bmatrix}. \quad (6.44)$$

Using the form of the matrix $\gamma^0 C$ in the chiral representation that was given in Eq. (6.20), it is straight forward to check that the u - and the v -spinors indeed satisfy the conjugation relations, Eqs. (3.23) and (3.24). Using the relations between the η and the ζ spinors, one can also write the u - and the v -spinors in the form

$$\check{u}_r(p) = \begin{bmatrix} \zeta_r(p) \\ \frac{E + \boldsymbol{\sigma} \cdot \mathbf{p}}{m} \zeta_r(p) \end{bmatrix}, \quad \check{v}_r(p) = \begin{bmatrix} \eta_r(p) \\ -\frac{E + \boldsymbol{\sigma} \cdot \mathbf{p}}{m} \eta_r(p) \end{bmatrix}. \quad (6.45)$$

With the choice of the basis spinors given in Eq. (6.40), this would read

$$\begin{aligned}\check{u}_-(p) &= \begin{bmatrix} \xi_- \\ \frac{m}{E+p} \xi_- \end{bmatrix}, & \check{u}_+(p) &= \begin{bmatrix} \frac{m}{E+p} \xi_+ \\ \xi_+ \end{bmatrix}, \\ \check{v}_+(p) &= \begin{bmatrix} -\frac{m}{E+p} \xi_+ \\ \xi_+ \end{bmatrix}, & \check{v}_-(p) &= \begin{bmatrix} \xi_- \\ -\frac{m}{E+p} \xi_- \end{bmatrix}.\end{aligned}\quad (6.46)$$

In the limit of vanishing mass, these spinors approach the eigenstates of the matrix γ_5 given in Eq. (6.5).

Finally, there should be a few words of caution about the use of the 2-component representation. First, a Dirac field does not have a similar 2-component representation. The point is that a Dirac field has in general four independent complex components. The number of independent parameters is half as much for a Weyl field because of the chirality condition, and also half as much for a Majorana field because of the reality condition, which can be accommodated in two complex components. But there is not enough room for the components of a Dirac field.

For Weyl and Majorana fields, even though the 2-component representation is more compact, for practical purposes of performing calculations, it is more convenient to use the 4-component representation given earlier. The reason is that there is hardly a physical process in which all particles are Weyl fermions or Majorana fermions. All charged fermions are Dirac fermions. If we have to use the 4-component representation of the Dirac matrices to deal with them, it is convenient to use the same for the other kinds of fermions as well [8].

7 Charge conjugation and CP

A field theory is called charge conjugation symmetric if its action remains invariant after substituting all fields by their complex conjugates (with a phase factor, if necessary). For a scalar field $\phi(x)$, the charge conjugation operation obviously implies replacement by a phase factor times $\phi^\dagger(x)$. For a fermion field, naturally, the Lorentz covariant conjugation has to be involved, because otherwise the resulting action will not even be Lorentz invariant. So, the operation of charge conjugation \mathcal{C} is given by

$$\mathcal{C}\Psi(x)\mathcal{C}^{-1} = \eta_C \widehat{\Psi}(x), \quad (7.1)$$

where η_C is a phase factor. This kind of symmetries can have particular importance for Majorana fields, for which ψ and $\widehat{\psi}$ are equal. We will address this point shortly.

Before that, it is useful to discuss what the charge conjugation operation means for chiral projections of fermion fields. We first note that the operation of charge conjugation is unitary, which implies that it will also have to be linear. Linearity of an operator \mathcal{O} implies that

$$\mathcal{O}(\alpha_1|a_1\rangle + \alpha_2|a_2\rangle) = \alpha_1\mathcal{O}|a_1\rangle + \alpha_2\mathcal{O}|a_2\rangle, \quad (7.2)$$

where $|a_1\rangle$ and $|a_2\rangle$ are arbitrary states, whereas α_1 and α_2 are arbitrary complex numbers. In other words, a linear operation does not affect the numerical co-efficients. The ‘states’, in this context, means anything that the operator \mathcal{O} acts on.

Take $\Psi_L \equiv L\Psi$ now. The chiral projection matrix L was defined in Eq. (4.7), and is a numerical matrix, i.e., a matrix whose elements are numbers, not fields. The operation \mathcal{C} acts on fields, and we should then write

$$\mathcal{C}\Psi_L\mathcal{C}^{-1} = \mathcal{C}L\Psi\mathcal{C}^{-1} = L\mathcal{C}\Psi\mathcal{C}^{-1} = \eta_C L\widehat{\Psi} \equiv \eta_C \widehat{\Psi}_L. \quad (7.3)$$

The co-ordinate x is unaffected by \mathcal{C} , so we have not written it in this equation.

To see an important feature of Eq. (7.3), let us also find out what is the LCC of Ψ_L . Using the definition of Eq. (3.16), we find,

$$\widehat{(\Psi_L)} \equiv \gamma_0 C(\Psi_L)^\star = \gamma_0 CL^\star \Psi^\star. \quad (7.4)$$

We now use the fact that γ_5 is hermitian, so that $L^\star = L^\top$, and then use Eq. (5.21) to write $CL^\top = LC$. Then, using the anticommutation of γ_0 and γ_5 , we obtain

$$\widehat{(\Psi_L)} = \gamma_0 LC\Psi^\star = R\gamma_0 C\Psi^\star = R\widehat{\Psi} \equiv \widehat{\Psi}_R. \quad (7.5)$$

In passing, we should also note that the same kind of relation holds between the spinors. For example, Eq. (3.24) implies that

$$u_L = L\gamma_0 C v^\star = \gamma_0 R C v^\star = \gamma_0 C R^\top v^\star = \gamma_0 C v_R^\star \quad (7.6)$$

and similar relations of this sort.

Let us review what we have obtained in Eqs. (7.1) and (7.5). On an unconstrained fermion field, we found that the charge conjugation operation and the LCC operation work the same way, apart from a possible phase factor. But these two operations are not at all the same thing on chiral projections of fermion fields. Nevertheless, because of the equivalence of these two operations on unconstrained fields, the literature is replete with instances [9] where the two have been confused for chiral fields as well. The confusion is enhanced by using the notation Ψ^c for the LCC and the charge conjugate of a field Ψ interchangeably, or sometimes without any clarification, even for a Weyl fermion. Elaborate statements are even made to the effect that charge conjugation changes chirality. This makes no sense whatsoever, and can be best seen with Weyl fields for which chirality is the same as helicity. Helicity involves spin and momentum, none of which changes under charge conjugation. Thus helicity is unaffected by charge conjugation, and so must be chirality.

To avoid such confusion, I think it is best not to use the notation Ψ^c at all, neither for charge conjugation nor for LCC. That is what I have been doing in this article: I have denoted LCC by a hat, and used the elaborate notation $\mathcal{C}\Psi(x)\mathcal{C}^{-1}$ when I wanted to denote the charge conjugate of a field $\Psi(x)$.

The situation is different if we consider not just charge conjugation but the combined operation CP . Since parity operation involves the matrix γ_0 , for unconstrained

fermion fields we can write

$$\mathcal{C}\mathcal{P}\Psi(t, \mathbf{x}) (\mathcal{C}\mathcal{P})^{-1} = \eta_{CP}\gamma_0\hat{\Psi}(t, -\mathbf{x}), \quad (7.7)$$

This is also a linear operation. So, on left-chiral fields, it implies

$$\begin{aligned} \mathcal{C}\mathcal{P}\Psi_L(t, \mathbf{x}) (\mathcal{C}\mathcal{P})^{-1} &= L\mathcal{C}\mathcal{P}\Psi(t, \mathbf{x}) (\mathcal{C}\mathcal{P})^{-1} = \eta_{CP}L\gamma_0\hat{\Psi}(t, -\mathbf{x}) \\ &= \eta_{CP}\gamma_0R\hat{\Psi}(t, -\mathbf{x}) = \eta_{CP}\gamma_0\hat{\Psi}_R(t, -\mathbf{x}), \end{aligned} \quad (7.8)$$

In this sense, $\hat{\Psi}_R$ is the CP-conjugate or the CP-transform of Ψ_L . This is what is naively expected: charge conjugation changes particles to antiparticles and vice-versa, whereas parity changes helicity, so that the CP-conjugate of a left-handed particle is a right-handed antiparticle, as is seen from Eq. (7.8).

Let us now discuss how C and CP symmetries can be employed in theories involving fermion fields. A free Majorana fermion is an eigenstate of charge conjugation symmetry, as is clear from Eqs. (3.17) and (7.1). If the interactions of a certain Majorana fermion are also invariant under this symmetry, then we can say that the physical Majorana particle is also an eigenstate of charge conjugation. This might work to a good approximation for supersymmetric partner of the photon. However, for neutrinos, the interactions violate charge conjugation symmetry badly, so it is useless to ask whether a Majorana neutrino can be an eigenstate of charge conjugation [10, 11].

The case of CP need not be the same. As far as we know, CP symmetry is respected to a good accuracy in nature. Thus, to the extent that we can ignore CP violation, we can think of a Majorana neutrino as a CP eigenstate, in the sense that

$$\mathcal{C}\mathcal{P}\psi(t, \mathbf{x}) (\mathcal{C}\mathcal{P})^{-1} = \eta_{CP}\gamma_0\psi(t, -\mathbf{x}). \quad (7.9)$$

This may not look like a typical eigenvalue equation because of the presence of the matrix γ_0 on the right hand side, but it can be shown [11, 12] that the particle states with vanishing 3-momentum are indeed eigenstates of CP in every sense of the term.

With a Weyl neutrino, it is not even possible to construct a theory that is invariant under charge conjugation. The reason should be obvious from our earlier discussion about the effect of this operation on Weyl fields. If we have a left-handed Weyl field, its charge conjugate should also be a left-handed field, with opposite internal quantum numbers. Existence of such an object is not implied or guaranteed by the existence of the left-handed field. If one wants charge conjugation symmetry, one will have to add this extra object in the theory. For CP symmetry, this problem does not exist. As we said, the CP conjugate of a left-handed Weyl particle is a right-handed Weyl particle, which is also the LCC. Whatever Lagrangian one writes with a field can also be written in terms of its LCC field, so the CP conjugate is in the theory anyway. This is guaranteed by CPT conservation, as has been argued earlier.

Once we take CP-violating effects into account, even Eq. (7.9) cannot be used to define a Majorana fermion field. In such cases, one can fall back on CPT [13]. However, we want to point out that even if CPT is not conserved, Majorana fermions can

still be defined. The definition does not depend on any of the discrete symmetries: it is a property of the proper Lorentz group that a covariant conjugation rule can be defined. Indeed, we have defined a Majorana field in Sec. 3, long before we have started the discussion on the discrete symmetries in this section. For scalar fields, it is obvious that one can impose the condition $\phi = \phi^\dagger$ without the assistance of any discrete symmetry. For a vector field like that of the photon or the Z -boson, we merely say that the field is real, without making any reference to any discrete symmetry. For fermion fields, imposing the condition $\psi = \hat{\psi}$ does not require anything more. The proper Lorentz group is respected by strong, weak and electromagnetic interactions, so if this condition is imposed on the free field, it will remain valid for the interactive field.

8 Feynman rules

Let us now discuss the Feynman rules for these different kinds of fermions. In view of the comments about the inconveniences encountered in using the 2-component formalism for Weyl and Majorana neutrinos, we will discuss the rules only in the 4-component formalism which are valid in arbitrary representations of the Dirac matrices.

8.1 Internal lines

For internal lines, the propagator has to be used. The propagator is the amplitude of propagating from one spacetime point x to another point y . In the language of quantum field theory, it is seen as the annihilation of a particle at x and its creation at y , and therefore depends on the field operators $\Psi(x)$ and $\bar{\Psi}(y)$. Of course the same operators can annihilate an antiparticle at y and produce it at x . Both possibilities are entailed, depending on whether x or y has a larger value of the time coordinate, in the propagator. The result depends on $x - y$, and we can take the Fourier transform of it and denote the Fourier component with momentum p by the notation $S_p(\Psi_a \bar{\Psi}_b)$, where a, b denote the component of the fermion field. For a Dirac field, the propagator is derived in every book of quantum field theory and is given by

$$S_p(\Psi_a \bar{\Psi}_b) = \frac{(\gamma^\mu p_\mu + m)_{ab}}{p^2 - m^2}, \quad (8.1)$$

where the mass multiplies an implied unit matrix, and the numerator is the a, b matrix element of the matrix sum. For Weyl fermions, the mass has to be set to zero: that's all.

For Majorana fermions, the same expression is obtained as well. However, there are more combinations of operators that can create a particle at x and annihilate it at y . The reason is that a Majorana particle is the antiparticle of itself, so that the field operator $\psi(x)$ contains both the annihilation and the creation operator of

this particle. Thus $\psi(x)$ can create as well as annihilate, and so can $\bar{\psi}(x)$. So, a propagator can be formed even out of the field operators $\psi(x)\psi(y)$. To obtain the expression of such a propagator, note that Eq. (3.17) implies

$$\psi^\top = \bar{\psi}C, \quad (8.2)$$

or in terms of matrix elements,

$$\psi_b = \bar{\psi}_d C_{db}. \quad (8.3)$$

Therefore,

$$S_p(\psi_a \psi_b) = S_p(\psi_a \bar{\psi}_d) C_{db} = \frac{((\gamma^\mu p_\mu + m)C)_{ab}}{p^2 - m^2}, \quad (8.4)$$

using the expression that appears in Eq. (8.1). Similarly, there can also be the propagator with two $\bar{\psi}$ operators, and its expression can be similarly obtained:

$$S_p(\bar{\psi}_a \bar{\psi}_b) = S_p(\psi_d \bar{\psi}_b) (C^{-1})_{da} = \frac{(C^{-1}(\gamma^\mu p_\mu + m))_{ab}}{p^2 - m^2}. \quad (8.5)$$

There is an interesting property of the propagators given in Eqs. (8.4) and (8.5) that is worth noticing. As said earlier, the components of fermion field operators anticommute, i.e., $\psi_a(x)\psi_b(y) = -\psi_b(y)\psi_a(x)$. So, if we interchange x and y , and also the indices a and b , the propagator should change sign. The Fourier transform kernel is $\exp(-ip \cdot (x - y))$, so interchanging x and y implies changing the sign of p in the Fourier transform. Thus we should have

$$S_p(\psi_a \psi_b) = -S_{-p}(\psi_b \psi_a), \quad (8.6)$$

and a similar equation for the $\bar{\psi}\bar{\psi}$ propagator. In other words, the matrices appearing in the expression with even powers of p should be antisymmetric, and those with odd powers of p should be symmetric. The mass term is easily seen to satisfy this property since we have already proved that C is antisymmetric, and so C^{-1} must also be so. In addition, one can use Eqs. (3.26) and (3.31) to show that both $\gamma^\mu C$ and $C^{-1}\gamma^\mu$ are symmetric matrices.

8.2 External lines

As for the case of internal lines, we do not elaborate on the external line Feynman rules for Dirac fermions, since they are covered in any standard textbook. The rules for Weyl fermions resemble the rules for Dirac fermions, with the only difference that there is only one spinor for each case, and this spinor has a well-defined chirality.

For Majorana fermions, however, the rules are different. The reason for the difference has already been discussed while talking about the propagators: the field operator ψ can either create or annihilate a particle, and so can $\bar{\psi}$. The various possibilities that arise have been summarized, along with the rules for Dirac and Weyl particles, in Table 1.

Table 1: Feynman rules for external fermion lines. For Majorana fermions, the phase α has been defined in Eq. (3.38).

Type of fermion	Feynman rule for			
	incoming		outgoing	
	with ψ	with $\bar{\psi}$	with ψ	with $\bar{\psi}$
Dirac particle	$\sum_s u_s(\mathbf{p})$	0	0	$\sum_s \bar{u}_s(\mathbf{p})$
Dirac antiparticle	0	$\sum_s \bar{v}_s(\mathbf{p})$	$\sum_s v_s(\mathbf{p})$	0
Majorana	$\sum_s u_s(\mathbf{p})$	$e^{-i\alpha} \sum_s \bar{v}_s(\mathbf{p})$	$e^{i\alpha} \sum_s v_s(\mathbf{p})$	$\sum_s u_s(\mathbf{p})$
LH Weyl particle	$u_L(\mathbf{p})$	0	0	$\bar{v}_R(\mathbf{p})$
antiparticle of LH Weyl	0	$\bar{v}_R(\mathbf{p})$	$v_L(\mathbf{p})$	0

8.3 An example

The Feynman rules show why we need to be careful in dealing with Majorana fermions. The field operator can do multiple task which a Dirac field operator cannot. As an illustrative example, consider an interaction

$$\mathcal{L}_{\text{int}} = g\Phi\bar{\Psi}F\Psi, \quad (8.7)$$

where g is a coupling constant, Φ is a boson field, and F is some 4×4 numerical matrix sandwiched between the fermion field operators. For example, if Φ is a spinless particle, F can be the unit matrix or γ_5 , or a linear combination of the two.

If the boson is massive enough, it can decay into a final state containing two fermions. If the fermions are Dirac particles, the Feynman amplitude for the process will be

$$\mathcal{M} = g \sum_{s_1, s_2} \bar{u}_{s_1}(p_1) F v_{s_2}(p_2) \quad (8.8)$$

with an obvious notation about the spins and momenta of the final-state particles. Here, the operator Ψ creates the antiparticle in the final state, whereas the operator $\bar{\Psi}$ creates the particle.

If, on the other hand, a Majorana pair is produced in the final state, the amplitude will be different. The reason is that, now the operator ψ can create either of the two, and so can $\bar{\psi}$. So we should write

$$\mathcal{M} = g \sum_{s_1, s_2} \left(\bar{u}_{s_1}(p_1) F v_{s_2}(p_2) - \bar{u}_{s_2}(p_2) F v_{s_1}(p_1) \right), \quad (8.9)$$

omitting an overall factor of $e^{i\alpha}$ as dictated by Table 1, because it would disappear anyway when the absolute square of the amplitude will be taken to calculate any physical quantity. Notice also the relative minus sign between the two terms, which appears because of the anticommutation relation of the fermion fields.

This expression can also be written in an alternative form by using Eqs. (3.23) and (3.24). We note that, using shorthand notations like $u_1 \equiv u_{s_1}(p_1)$, we can write

$$\bar{u}_2 F v_1 = (\gamma_0 C v_2^*)^\dagger \gamma_0 F \gamma_0 C u_1^* = v_2^\top C^{-1} F \gamma_0 C u_1^*. \quad (8.10)$$

But the whole thing is a number, so we might as well write it as the transpose of the matrices involved. Thus,

$$\bar{u}_2 F v_1 = \left(v_2^\top C^{-1} F \gamma_0 C u_1^* \right)^\top = u_1^\dagger C \gamma_0^\top F^\top C^{-1} v_2 = -\bar{u}_1 C F^\top C^{-1} v_2, \quad (8.11)$$

using Eq. (3.26) on the way. The amplitude of Eq. (8.9) can now be written in the form

$$\mathcal{M} = g \sum_{s_1, s_2} \bar{u}_{s_1}(p_1) \left(F + C F^\top C^{-1} \right) v_{s_2}(p_2). \quad (8.12)$$

We started by saying that Majorana fermions are simpler objects compared to Dirac fermions. There cannot be any argument about this statement, at least between persons who would agree that real numbers are simpler than complex numbers, or a real scalar field is simpler than a complex scalar field. Yet, now we see that the amplitudes involving Majorana fermions can have more terms compared to a similar amplitude involving Dirac fermions, so there is a price to pay for the simplicity.

It should be noted that this price has nothing to do with the fermionic nature of the fields. This is true even for scalar fields, for example. Consider an interaction term $(\phi_1^\dagger \phi_2)(\phi_2^\dagger \phi_1)$ that drives a tree-level elastic scattering between two bosons ϕ_1 and ϕ_2 . If the fields are complex, there is only one way the creation and annihilation operators can work for this process, viz., ϕ_1 can annihilate the 1-particle in the initial state and ϕ_1^\dagger can create it in the final state, and similarly for ϕ_2 . However, if the fields are real, there are more cases to consider because then ϕ_1 is the same as ϕ_1^\dagger , and any of the two factors of ϕ_1 in the interaction term can annihilate the initial state particle as well as create the final state particle. For the scalar case, such possibilities would produce an overall factor, because everything else is the same. For fermion fields, because of the matrix structure, the different terms are not exactly the same, but they are related, as we can see in the example of Eq. (8.12).

Acknowledgements : Amol Dighe enthusiastically responded when I proposed to give a pedagogical talk in the NuGoa conference that he was organizing in Goa in April 2009. When I could not finish all I wanted to say in the designated one hour's time, he even arranged a special evening session where I could present the rest. Gustavo Branco, one of the people in that audience, asked me to give a similar set of lectures in Lisbon when I visited his group in November 2009. And Jorge Romão insisted that I write up the material. They all, along with the people in the audiences, made me feel that these talks can be useful to people. I thank them all. After the initial version of this write-up was submitted to the internet, I have enjoyed discussions with Kaushik Balasubramanian and Rainer Plaga.

References

- [1] P. A. M. Dirac, "The Quantum theory of electron," Proc. Roy. Soc. Lond. A **117** (1928) 610–624.

- [2] H. Weyl, “Electron and gravitation,” Z. Phys. **56** (1929) 330–352 [English translation: Surveys High Energ. Phys. **5** (1986) 261–267].
- [3] W. Pauli, in a letter dated 4 December 1930 to the participants of a conference in Tübingen. An English translation can be found, e.g., in “Dear radioactive ladies and gentlemen,” Phys. Today **31N9** (1978) 27.
- [4] E. Majorana, “Theory Of The Symmetry Of Electrons And Positrons,” Nuovo Cim. **14** (1937) 171–184.
- [5] As a recent example, see D. Singh, N. Mobed and G. Papini, “Can gravity distinguish between Dirac and Majorana neutrinos?,” Phys. Rev. Lett. **97** (2006) 041101[1–4] [arXiv:gr-qc/0605153]. The fallacy in the argument was pointed out in J. F. Nieves and P. B. Pal, “Comment on ‘Can gravity distinguish between Dirac and Majorana neutrinos?’,” Phys. Rev. Lett. **98** (2007) 069001 [arXiv:gr-qc/0610098].
- [6] Such identities can of course be easily verified by considering all possible cases. They can also be proved from the basic properties of Dirac matrices, Eqs. (3.5) and (3.6). See, e.g., P. B. Pal, “Representation-independent manipulations with Dirac matrices and spinors,” arXiv:physics/0703214 (<http://arxiv.org/abs/physics/0703214v2>).
- [7] K. M. Case, “Reformulation of the Majorana Theory of the Neutrino,” Phys. Rev. **107** (1957) 307–316.
- [8] We emphasize that the use of the 2-component notation in practical calculations is only *inconvenient*, not *impossible*. As an example of its use, see J. Schechter and J. W. F. Valle, “Majorana Neutrinos And Magnetic Fields,” Phys. Rev. D **24** (1981) 1883–1889 [Erratum-ibid. D **25** (1982) 283].
- [9] Such instances are so many that it is impossible and useless to have an exhaustive reference. Just to prove that the claim is not vacuous, see e.g., Eqs. (6.14) to (6.16) on page 274 of “Physics of neutrinos and application to astrophysics” by M. Fukugita and T. Yanagida (Springer, 2002).
- [10] B. Kayser and A. S. Goldhaber, “CPT and CP Properties Of Majorana Particles, And The Consequences,” Phys. Rev. D **28** (1983) 2341–2344.
- [11] B. Kayser, “CPT, CP, And C Phases And Their Effects In Majorana Particle Processes,” Phys. Rev. D **30** (1984) 1023–1033.
- [12] For a textbook exposition, see, e.g., R. N. Mohapatra and P. B. Pal, “Massive neutrinos in Physics and Astrophysics”, (World Scientific, 3rd edition, 2004).
- [13] J. F. Nieves, “Electromagnetic Properties Of Majorana Neutrinos,” Phys. Rev. D **26** (1982) 3152–3158.