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## COSMIC ABUNDANCES OF STABLE PARTICLES: IMPROVED ANALYSIS

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An exact relativistic single-integral formula for the thermal average of the annihilation cross section times velocity, the key quantity in the determination of the cosmic relic abundance of a species, is obtained. Since it does not require expansion of the cross section at low relative velocity, it can also be used when the cross section varies rapidly with energy, e.g. near the formation of a resonance or the opening of a new annihilation channel. We discuss approximate formulas in these cases, and we find that dips in the relic density near resonances are significantly broader and shallower than previously thought and that spurious reductions near thresholds disappear.

### 1. Introduction

We will examine relativistic effects on the thermal average of the annihilation cross section times the “relative velocity”  $\langle\sigma v\rangle$ , the key quantity in the determination of the cosmic relic abundance of a species.

For non-relativistic gases, the thermally averaged annihilation cross section  $\langle\sigma v\rangle$  has sometimes been approximated with the value of  $\sigma v$  at  $s = \langle s \rangle$ , the thermally averaged center-of-mass squared energy, and the expansion  $\langle s \rangle = 4m^2 + 6mT$  has been taken. Such an approximation is good only when  $\sigma v$  is almost linear in  $s$ , i.e. unfortunately almost never. A better approximation for non-relativistic gases is to expand  $\langle\sigma v\rangle$  in powers of  $x^{-1} \equiv T/m$ . A common way of doing this (cf. e.g. ref. [1]) is to write  $s = 4m^2 + m^2 v^2$ , expand  $\sigma v$  in powers of  $v^2$  and

take the thermal average to obtain

$$\begin{aligned}\langle\sigma v\rangle &= \langle a + bv^2 + cv^4 + \dots \rangle \\ &= a + \frac{3}{2}bx^{-1} + \frac{15}{8}cx^{-2} + \dots\end{aligned}\quad (1.1)$$

Most species are not completely non-relativistic at decoupling: when  $x$  is of order 20–25 (a typical value at freeze-out for weakly interacting particles) the mean rms velocity of the particles is of order  $c/4$ , and relativistic corrections of order 5–10% are expected. The relativistic expansion in powers of  $x^{-1}$  was first found in ref. [2].

All these formulations are based on the expansion of the cross section  $\sigma$  in powers of the “relative velocity”  $v$ . They become inappropriate either when the cross section is poorly approximated by its expansion, as near the formation of a resonance, or when its expansion diverges, as at the opening of a new annihilation channel.

We will propose a single-integral formula for  $\langle\sigma v\rangle$  (eq. (3.8) below) valid for all temperatures  $T \leq 3m$ , which does not require expansion of the cross section and which can be evaluated without the numerical difficulties associated with multiple integrals.

We will begin in sect. 2 by clarifying what the “relative velocity”  $v$  appearing in  $\langle\sigma v\rangle$  is in the relativistic context. For this purpose, we will briefly recall the derivation of the Boltzmann equation for the evolution of the spatial density of a species in the early universe. We will then derive the general single-integral formula for  $\langle\sigma v\rangle$  in sect. 3 and we will compare the expansions in powers of the temperature of the relativistic and non-relativistic thermal averages. The relativistic corrections are important only when a precision of a few per cent is required. If so, the Boltzmann equation should be solved at least with the same precision. This can be done either numerically or by an analytic approximation. In sects. 4 and 5 we will present a sufficiently precise relativistic generalization of the usual non-relativistic approximation to the solution of the Boltzmann equation. In particular, sect. 4 is devoted to the effective degrees of freedom in the early universe. We will then discuss the non-relativistic treatment of the thermal average in the presence of a resonance (sect. 6) and near the threshold of a new annihilation channel (sect. 7). Finally, we will present a specific example of resonances and thresholds: a light higgsino in the “minimally non-minimal” supersymmetric model, with mass in the region of the  $T$  resonances and the  $b\bar{b}$  threshold.

## 2. The Boltzmann equation

The evolution of the phase-space density  $f(\mathbf{p}, \mathbf{x}, t)$  of a particle species is described by the Boltzmann equation, which can be written as [1, 3–5]

$$L[f] = C[f], \quad (2.1)$$

where  $L$  is the Liouville operator giving the net rate of change in time of the particle phase-space density  $f$  and  $C$  is the collision operator representing the number of particles per phase-space volume that are lost or gained per unit time under collision with other particles.

In the Friedmann–Robertson–Walker cosmological model, the phase-space density is spatially homogeneous and isotropic so  $f$  depends only on the particle energy  $E$  and the time  $t$ ,  $f = f(E, t)$ . In this case the Liouville operator becomes

$$L[f] = \frac{\partial f}{\partial t} - H \frac{|p|^2}{E} \frac{\partial f}{\partial E}, \quad (2.2)$$

where  $H \equiv \dot{R}/R$  is the Hubble parameter,  $R$  the scale factor of the universe and a dot indicates a time derivative.

The particle number density  $n$  is the integral of the phase-space density over all momenta and the sum over all spins. With  $g$  spin degrees of freedom, each with the same distribution  $f(E, t)$ , we have

$$n = \int dn = \int f(E, t) \frac{g d^3p}{(2\pi)^3}. \quad (2.3)$$

In order to get an evolution equation for  $n$ , the Boltzmann equation (2.1) is integrated over the particle momenta and summed over the spin degrees of freedom.

For sake of clarity, we present the computations for the case of annihilations of two particles, 1 and 2, into two others, 3 and 4. At the end, it will be obvious how to sum over all the possible final channels. The Liouville term becomes

$$g_1 \int L[f_1] \frac{d^3p_1}{(2\pi)^3} = \frac{1}{R^3} \frac{d}{dt} (R^3 n_1) = \dot{n}_1 + 3Hn_1. \quad (2.4)$$

Integrating the collision term, only the inelastic contributions survive [1, 3–5]. So we have

$$\begin{aligned} g_1 \int C[f_1] \frac{d^3p_1}{(2\pi)^3} = & - \sum_{\text{spins}} \int [f_1 f_2 (1 \pm f_3)(1 \pm f_4) |\mathcal{M}_{12 \rightarrow 34}|^2 \\ & - f_3 f_4 (1 \pm f_1)(1 \pm f_2) |\mathcal{M}_{34 \rightarrow 12}|^2] \\ & \times (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \\ & \times \frac{d^3p_1}{(2\pi)^3 2E_1} \frac{d^3p_2}{(2\pi)^3 2E_2} \frac{d^3p_3}{(2\pi)^3 2E_3} \frac{d^3p_4}{(2\pi)^3 2E_4}, \quad (2.5) \end{aligned}$$

where the  $+$  ( $-$ ) sign in the statistical mechanical factors  $1 \pm f_i$  applies to bosons (fermions), the  $\mathcal{M}$  are the invariant polarized amplitudes obtained with the usual Feynman rules and the sum is over initial and final spins.

Eq. (2.5) is valid also when the particles 1 and 2 are identical (as is the case of Majorana fermions, for example). No additional factor of  $\frac{1}{2}$  should appear in eq. (2.5) in this case: there is a factor of  $\frac{1}{2}$  to avoid double counting of the particle states and a factor of 2 because two particles disappear in each annihilation. For massive particles which decouple in the early universe while they are a non-degenerate gas, we can neglect the statistical mechanical factors in eq. (2.5).

In order to proceed, it is necessary to assume that the annihilation products, 3 and 4, go quickly into equilibrium with the thermal background. This is certainly true if these particles are electrically charged, since they interact with the many thermal photons present. In most cases of interest it is also true for neutral particles as well. So we replace  $f_3$  and  $f_4$  with the (kinetic and chemical) equilibrium distributions  $f_3^{\text{eq}}$  and  $f_4^{\text{eq}}$  respectively.

Then, unitarity and the principle of detailed balance enable us to integrate over  $p_3$  and  $p_4$  independent of the distributions  $f_3$  and  $f_4$ . This is so because the detailed balance allows the replacement

$$f_3^{\text{eq}} f_4^{\text{eq}} = f_1^{\text{eq}} f_2^{\text{eq}}, \quad (2.6)$$

and unitarity yields the equation

$$\begin{aligned} & \sum_{\text{spins}} \int |\mathcal{M}_{34 \rightarrow 12}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \frac{d^3 p_3}{(2\pi)^3 2E_3} \frac{d^3 p_4}{(2\pi)^3 2E_4} \\ &= \sum_{\text{spins}} \int |\mathcal{M}_{12 \rightarrow 34}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \frac{d^3 p_3}{(2\pi)^3 2E_3} \frac{d^3 p_4}{(2\pi)^3 2E_4}, \quad (2.7) \end{aligned}$$

The definition of the unpolarized cross section  $\sigma_{12 \rightarrow 34}$  for the process  $12 \rightarrow 34$  can be written as

$$\sum_{\text{spins}} \int |\mathcal{M}_{12 \rightarrow 34}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \frac{d^3 p_3}{(2\pi)^3 2E_3} \frac{d^3 p_4}{(2\pi)^3 2E_4} = 4F g_1 g_2 \sigma_{12 \rightarrow 34}, \quad (2.8)$$

where  $F = [(p_1 \cdot p_2)^2 - m_1^2 m_2^2]^{1/2}$  and the spin factor  $g_1 g_2$  comes from the average over initial spins.

At this point, it is evident how to include all accessible final channels: one replaces  $\sigma_{12 \rightarrow 34}$  with the total annihilation cross section

$$\sigma = \sum_{\text{all } f} \sigma_{12 \rightarrow f}. \quad (2.9)$$

Using eqs. (2.5)–(2.8), the integrated collision term becomes

$$g_1 \int C[f_1] \frac{d^3 p_1}{(2\pi)^3} = - \int \sigma v_{\text{Mø}} (dn_1 dn_2 - dn_1^{\text{eq}} dn_2^{\text{eq}}), \quad (2.10)$$

where  $n_1, n_2$  are the particle number densities,  $n_1^{\text{eq}}$  and  $n_2^{\text{eq}}$  their equilibrium values, and the momentum-space differentials  $dn_i$  are defined as in eq. (2.3). We call  $v_{\text{Mø}} \equiv F/E_1 E_2$  the Møller velocity, after de Groot et al. [4]. It is defined in such a way that the product  $v_{\text{Mø}} n_1 n_2$  is invariant under Lorentz transformations and equals the product of the relative velocity  $v_{\text{lab}}$  and of the particle densities  $n_{1,\text{lab}} n_{2,\text{lab}}$  in the rest frame of one of the incoming particles (the so-called lab frame). This definition of  $v_{\text{Mø}}$  is such that the (invariant) interaction rate per unit volume per unit time can be written in any reference frame as [6]

$$\frac{dN}{dV dt} = \sigma v'_{\text{Mø}} n'_1 n'_2, \quad (2.11)$$

where  $\sigma$  is the invariant cross section and  $v'_{\text{Mø}}, n'_1, n'_2$  are given in the chosen reference frame\*. In our case, the densities and the Møller velocity in eq. (2.10) refer to the cosmic comoving frame. In terms of the particle velocities  $\boldsymbol{v}_1 = \boldsymbol{p}_1/E_1$  and  $\boldsymbol{v}_2 = \boldsymbol{p}_2/E_2$ , the Møller velocity  $v_{\text{Mø}}$  is given by

$$v_{\text{Mø}} = \left[ |\boldsymbol{v}_1 - \boldsymbol{v}_2|^2 - |\boldsymbol{v}_1 \times \boldsymbol{v}_2|^2 \right]^{1/2}. \quad (2.12)$$

The presence of the Møller velocity instead of the relative velocity  $|\boldsymbol{v}_1 - \boldsymbol{v}_2|$  in eq. (2.11) has been known for a long time [7], but has not been recognized in relation to computations of relic densities.

From symmetry considerations [5], the distributions in kinetic equilibrium are proportional to those in chemical equilibrium, with a proportionality factor independent of momentum. This is true if the species 1 and 2 are maintained in kinetic equilibrium through scattering with other particles in the thermal bath during all of their evolution, even after their decoupling when they are out of chemical

\* The cross section  $\sigma$  is actually the area of the target in its rest frame. If one would let the cross section transform under Lorentz transformations as an area,  $\sigma \rightarrow \sigma'$ , then one would have  $dN/dV dt = \sigma' v'_{\text{rel}} n'_1 n'_2$ , with  $v'_{\text{rel}} = |\boldsymbol{v}'_1 - \boldsymbol{v}'_2|$  the relative velocity and  $n'_1, n'_2$  the particle number densities in the new frame, without introduction of  $v'_{\text{Mø}}$ .

equilibrium. Thus eq. (2.10) can be written, both before and after decoupling, in the form

$$g_1 \int C[f_1] \frac{d^3 p_1}{(2\pi)^3} = -\langle \sigma v_{\text{Mø}} \rangle (n_1 n_2 - n_1^{\text{eq}} n_2^{\text{eq}}), \quad (2.13)$$

where the thermally averaged total annihilation cross section times Møller velocity is defined by

$$\langle \sigma v_{\text{Mø}} \rangle = \frac{\int \sigma v_{\text{Mø}} d n_1^{\text{eq}} d n_2^{\text{eq}}}{\int d n_1^{\text{eq}} d n_2^{\text{eq}}}. \quad (2.14)$$

Equating the integrated collision term, eq. (2.13), with the integrated Liouville term, eq. (2.4), we finally obtain

$$\dot{n}_1 + 3Hn_1 = -\langle \sigma v_{\text{Mø}} \rangle (n_1 n_2 - n_1^{\text{eq}} n_2^{\text{eq}}). \quad (2.15)$$

There is an analogous equation for  $n_2$  with the same right-hand side.

If particles 1 and 2 are identical, the density of the species  $n = n_1 = n_2$  satisfies

$$\dot{n} + 3Hn = -\langle \sigma v_{\text{Mø}} \rangle (n^2 - n_{\text{eq}}^2), \quad (2.16)$$

which is the familiar expression. The cross section that appears here is the usual one: summed over final and averaged over initial spins, with *no* factor of  $\frac{1}{2}$  for identical *initial* particles.

If particle 2 is the antiparticle of 1, the density of the species is  $n = n_1 + n_2$ . If the species have negligible chemical potential (the only case we consider in this paper), then  $n_1 = n_2$  and  $n = 2n_1$ . The equation for the particle (or antiparticle) density  $n_1$  is still eq. (2.16), but the equation for  $n$  contains a factor of  $\frac{1}{2}$  in front of the cross section. Thus there is a factor of  $\frac{1}{2}$  in front of  $\sigma$  for *non-identical* initial particles, and no extra factor for *identical* initial particles, contrary to a naive expectation. In the following, we will not write explicitly the factor of  $\frac{1}{2}$ , thus we write the formulas as for identical particles.

In eq. (2.16) it is useful to treat implicitly the decrease in density due to the expansion of the universe, considering the ratio of the number of particles to the entropy  $Y = n/s$ , with  $s$  the total entropy density of the universe. Dividing eq. (2.16) by  $S = R^3 s$ , the total entropy per comoving volume, which, in absence of entropy production, is constant, we obtain

$$\dot{Y} = -s \langle \sigma v_{\text{Mø}} \rangle (Y^2 - Y_{\text{eq}}^2). \quad (2.17)$$

Using the scale factor as a time variable, we obtain the following form for the

evolution equation:

$$\frac{dY}{dR} = - \frac{s \langle \sigma v_{\text{Mø}} \rangle}{RH} (Y^2 - Y_{\text{eq}}^2). \quad (2.18)$$

It is convenient to have  $Y$  as function of  $x = m/T$ , with  $T$  the photon temperature:

$$\frac{dY}{dx} = \frac{1}{3H} \frac{ds}{dx} \langle \sigma v_{\text{Mø}} \rangle (Y^2 - Y_{\text{eq}}^2). \quad (2.19)$$

The content of the universe enters eq. (2.19) through  $s$  contained in  $Y$  and through the factors written in front of  $\langle \sigma v_{\text{Mø}} \rangle$ . These can be rewritten in the following way. In the standard Friedmann–Robertson–Walker cosmology, the Hubble parameter is given by

$$H = \left( \frac{8}{3} \pi G \rho \right)^{1/2}, \quad (2.20)$$

where  $G$  is the gravitational constant and  $\rho$  is the total energy density of the universe. The effective degrees of freedom  $g_{\text{eff}}(T)$  and  $h_{\text{eff}}(T)$  for the energy and entropy densities are defined by

$$\rho = g_{\text{eff}}(T) \frac{\pi^2}{30} T^4, \quad s = h_{\text{eff}}(T) \frac{2\pi^2}{45} T^3, \quad (2.21), (2.22)$$

respectively, in such a way that  $g_{\text{eff}}(T) = h_{\text{eff}}(T) = 1$  for a relativistic species with one internal (or spin) degree of freedom. Substituting eqs. (2.20)–(2.22) into eq. (2.19), we arrive at the equation for the evolution of  $Y$ ,

$$\frac{dY}{dx} = - \left( \frac{45}{\pi} G \right)^{-1/2} \frac{g_*^{1/2} m}{x^2} \langle \sigma v_{\text{Mø}} \rangle (Y^2 - Y_{\text{eq}}^2), \quad (2.23)$$

where the content of the universe is given by the degrees of freedom parameter that we call  $g_*^{1/2}$ , defined as

$$g_*^{1/2} = \frac{h_{\text{eff}}}{g_{\text{eff}}^{1/2}} \left( 1 + \frac{1}{3} \frac{T}{h_{\text{eff}}} \frac{dh_{\text{eff}}}{dT} \right). \quad (2.24)$$

Let us point out again the assumptions under which the rate equation (2.23) is valid [5]: (i) Statistical mechanical factors are neglected in eq. (2.5), i.e. the Bose–Einstein or Fermi–Dirac distributions are approximated by the Maxwell–Boltzmann distribution, which is a very good approximation for temperatures  $T \lesssim 3m$ ; (ii) the annihilation products are in thermal equilibrium; (iii) the species



under consideration remain in kinetical equilibrium also after decoupling; (iv) the initial chemical potential of the species under consideration is negligible.

### 3. Thermal average

The thermal average of the annihilation cross section times velocity has usually been done by expanding the cross section at low relative velocity. As discussed in sect. 1, there are cases in which the cross section is poorly approximated by its expansion, or in which its expansion is even divergent. It is in these cases that an integral formula is particularly useful. We have seen in sect. 2 that the quantity that enters the Boltzmann equation for the evolution of a species is  $\langle \sigma v_{\text{Mø}} \rangle$ . Here we derive a general formula for the thermal average in the relativistic context, which involves a single integration and does not require expansion of the cross section.

As discussed before, we consider particle species whose equilibrium distribution function at temperature  $T$  is Maxwell-Boltzmann, i.e.  $f(E) \propto \exp(-E/T)$ , in the cosmic comoving frame, the frame where the gas is assumed to be at rest as a whole. In this case, the definition of  $\langle \sigma v_{\text{Mø}} \rangle$ , eq. (2.14), reads

$$\langle \sigma v_{\text{Mø}} \rangle = \frac{\int \sigma v_{\text{Mø}} e^{-E_1/T} e^{-E_2/T} d^3 p_1 d^3 p_2}{\int e^{-E_1/T} e^{-E_2/T} d^3 p_1 d^3 p_2}, \quad (3.1)$$

where  $p_1$  and  $p_2$  are the three-momenta and  $E_1$  and  $E_2$  the energies of the colliding particles in the cosmic comoving frame. The momentum-space volume element can be written as

$$d^3 p_1 d^3 p_2 = 4\pi p_1 dE_1 4\pi p_2 dE_2 \frac{1}{2} d\cos\theta, \quad (3.2)$$

where  $\theta$  is the angle between  $p_1$  and  $p_2$ ,  $p_1 = |p_1|$ ,  $p_2 = |p_2|$ , and we have used the relativistic relation between energy and momentum. Changing integration variables from  $E_1, E_2, \theta$  to  $E_+, E_-, s$ , given by

$$\begin{aligned} E_+ &= E_1 + E_2, & E_- &= E_1 - E_2, \\ s &= 2m^2 + 2E_1 E_2 - 2p_1 p_2 \cos\theta, \end{aligned} \quad (3.3)$$

the volume element becomes

$$d^3 p_1 d^3 p_2 = 2\pi^2 E_1 E_2 dE_+ dE_- ds, \quad (3.4)$$

and the integration region  $\{E_1 > m, E_2 > m, |\cos \theta| \leq 1\}$  transforms into

$$|E_-| \leq \sqrt{1 - \frac{4m^2}{s}} \sqrt{E_+^2 - s},$$

$$E_+ \geq \sqrt{s}, \quad s \geq 4m^2. \quad (3.5)$$

Thus, the numerator in eq. (3.1) can be computed as follows:

$$\begin{aligned} \int \sigma v_{\text{Mø}} e^{-E_1/T} e^{-E_2/T} d^3p_1 d^3p_2 &= 2\pi^2 \int dE_+ \int dE_- \int ds \sigma v_{\text{Mø}} E_1 E_2 e^{-E_+/T} \\ &= 4\pi^2 \int ds \sigma F \sqrt{1 - \frac{4m^2}{s}} \int dE_+ e^{-E_+/T} \sqrt{E_+^2 - s} \\ &= 2\pi^2 T \int ds \sigma (s - 4m^2) \sqrt{s} K_1(\sqrt{s}/T), \end{aligned} \quad (3.6)$$

where the integration over  $E_+$  can be performed because  $\sigma F = \sigma v_{\text{Mø}} E_1 E_2$  is a function of  $s$  only, and in the last step we have used  $F = \frac{1}{2} \sqrt{s(s - 4m^2)}$ . In a similar way, the denominator in eq. (3.1) becomes

$$\int e^{-E_1/T} e^{-E_2/T} d^3p_1 d^3p_2 = [4\pi m^2 T K_2(m/T)]^2. \quad (3.7)$$

The resulting special functions  $K_i$  are the modified Bessel functions of order  $i$  (see e.g. ref. [8]). Dividing eq. (3.6) by eq. (3.7), we get the thermal average in terms of a single integral

$$\langle \sigma v_{\text{Mø}} \rangle = \frac{1}{8m^4 T K_2^2(m/T)} \int_{4m^2}^{\infty} \sigma (s - 4m^2) \sqrt{s} K_1(\sqrt{s}/T) ds. \quad (3.8)$$

This is one of our primary results.

Eq. (3.8) has been obtained for particles with Maxwell–Boltzmann statistics. It is applicable to other statistics provided that  $T \lesssim 3m$ . When in the computation of relic abundances the relevant temperatures are less than the particle mass, we can safely use eq. (3.8) for all statistics. The insensitivity of the final abundance of heavy relics on the statistics of the particles was already noticed in ref. [9].

An expression similar to eq. (3.8) was obtained in ref. [10] for massless particles with Maxwell–Boltzmann statistics, but the formula was for  $\langle \sigma \rangle$  instead of  $\langle \sigma v_{\text{Mø}} \rangle = \langle \sigma(1 - \cos \theta) \rangle$ . In any case, the use of Maxwell–Boltzmann statistics

for massless particles is not correct: one must use the appropriate Fermi–Dirac or Bose–Einstein statistics.

Usually, either the relative velocity  $v_{\text{lab}} = |\mathbf{v}_{1,\text{lab}} - \mathbf{v}_{2,\text{lab}}|$  in the rest frame of one of the incoming particles or the relative velocity  $v_{\text{cm}} = |\mathbf{v}_{1,\text{cm}} - \mathbf{v}_{2,\text{cm}}|$  in the center-of-mass frame are used instead of  $v_{\text{Mø}}^*$ . It is therefore interesting to see what the relationship is between the thermal averages of  $\sigma v_{\text{Mø}}$ ,  $\sigma v_{\text{lab}}$  and  $\sigma v_{\text{cm}}$ .

We will show that the following property is fulfilled for a relativistic gas:

$$\langle \sigma v_{\text{Mø}} \rangle = \langle \sigma v_{\text{lab}} \rangle^{\text{lab}} \neq \langle \sigma v_{\text{cm}} \rangle^{\text{cm}}, \quad (3.9)$$

where the first average is taken in the cosmic comoving frame, and the meaning of the superscripts “lab” and “cm” on the second and third thermal averages will be explained below (eq. (3.14)).

To determine the relation between  $v_{\text{Mø}}$ ,  $v_{\text{lab}}$  and  $v_{\text{cm}}$ , we use the invariance of the product  $v_{\text{Mø}} dn_1 dn_2$ , where  $dn_i$  ( $i = 1, 2$ ) is the number density of particles with three-momentum in the element  $d^3p_i$  around  $\mathbf{p}_i$ . Under a Lorentz transformation, because of length contraction, the volume in coordinate-space is reduced by a factor  $\gamma$ . Thus, the particle number density  $dn_i$  changes in such a way that the ratio  $dn_i/E_i$  is invariant. Indicating quantities in a new reference frame with a prime, we have

$$\begin{aligned} v'_{\text{Mø}} dn'_1 dn'_2 &= v_{\text{Mø}} dn_1 dn_2 \\ &= v_{\text{Mø}} \frac{E_1}{E'_1} \frac{E_2}{E'_2} dn'_1 dn'_2, \end{aligned} \quad (3.10)$$

from which we read

$$\begin{aligned} v_{\text{Mø}} &= v'_{\text{Mø}} \frac{E'_1}{E_1} \frac{E'_2}{E_2} \\ &= v'_{\text{Mø}} \frac{1 - \mathbf{v}_1 \cdot \mathbf{v}_2}{1 - \mathbf{v}'_1 \cdot \mathbf{v}'_2}. \end{aligned} \quad (3.11)$$

In the last step we have multiplied and divided by the invariant  $\mathbf{p}_1 \cdot \mathbf{p}_2$ .

In the particular cases in which the primed system is the rest frame of one of the incoming particles (lab frame) or the center-of-mass frame (c.m. frame), the Møller velocity  $v'_{\text{Mø}}$  coincides with the relative velocity  $v_{\text{lab}}$  or  $v_{\text{cm}}$  respectively. In the lab frame, one of the  $\mathbf{v}_{i,\text{lab}}$  vanishes, and we get the relation

$$v_{\text{Mø}} = v_{\text{lab}} (1 - \mathbf{v}_1 \cdot \mathbf{v}_2), \quad (3.12)$$

\* Despite the possibly disturbing fact that  $v_{\text{cm}}$  can be larger than one, this definition of relative velocity is the natural one. Note that nothing travels at this velocity!

while in the center-of-mass frame,  $E_1^{\text{cm}} E_2^{\text{cm}} = s/4$ , and we obtain

$$v_{\text{M}\phi\text{I}} = v_{\text{cm}} \frac{1}{2} \left[ 1 - \boldsymbol{v}_1 \cdot \boldsymbol{v}_2 + \frac{m^2}{E_1 E_2} \right]. \quad (3.13)$$

We define the thermal average in the primed frame as

$$\langle \sigma v'_{\text{M}\phi\text{I}} \rangle' = \frac{\int \sigma v'_{\text{M}\phi\text{I}} dn'_1 dn'_2}{\int dn'_1 dn'_2}. \quad (3.14)$$

Its relation to  $\langle \sigma v_{\text{M}\phi\text{I}} \rangle$  in the cosmic comoving frame can be obtained by noticing that the numerator in eq. (3.14) is invariant under Lorentz transformations, while the denominator changes because of the contraction of volumes, as discussed above. Factoring out the change in normalization, we obtain

$$\langle \sigma v_{\text{M}\phi\text{I}} \rangle = \langle \sigma v'_{\text{M}\phi\text{I}} \rangle' \left\langle \frac{v_{\text{M}\phi\text{I}}}{v'_{\text{M}\phi\text{I}}} \right\rangle. \quad (3.15)$$

In particular, using eqs. (3.12) and (3.13) for the lab and c.m. frames respectively, we have

$$\left\langle \frac{v_{\text{M}\phi\text{I}}}{v_{\text{lab}}} \right\rangle = \langle 1 - \boldsymbol{v}_1 \cdot \boldsymbol{v}_2 \rangle = 1, \quad (3.16)$$

$$\left\langle \frac{v_{\text{M}\phi\text{I}}}{v_{\text{cm}}} \right\rangle = \frac{1}{2} \left[ 1 + \left\langle \frac{m^2}{E_1 E_2} \right\rangle \right] = \frac{1}{2} \left[ 1 + \frac{K_1^2(x)}{K_2^2(x)} \right]. \quad (3.17)$$

Thus we finally obtain the relation

$$\langle \sigma v_{\text{M}\phi\text{I}} \rangle = \langle \sigma v_{\text{lab}} \rangle^{\text{lab}} = \frac{1}{2} \left[ 1 + \frac{K_1^2(x)}{K_2^2(x)} \right] \langle \sigma v_{\text{cm}} \rangle^{\text{cm}}, \quad (3.18)$$

The factor  $\langle v_{\text{M}\phi\text{I}}/v_{\text{cm}} \rangle$  varies from  $\frac{1}{2}$  for a highly relativistic gas ( $x \rightarrow 0$ ) to one for a fully non-relativistic gas ( $x \rightarrow \infty$ ). Therefore, as indicated in eq. (3.9), the thermal averages taken in the cosmic comoving frame and in the center-of-mass frame are different in the relativistic regime. For a non-relativistic gas, the three thermal averages in eq. (3.18) coincide. For  $x \approx 20$ – $25$ , a typical value at freeze-out, the factor  $\langle v_{\text{M}\phi\text{I}}/v_{\text{cm}} \rangle$  amounts to 0.932–0.945, as can be seen from the asymptotic

expansion for low temperatures,  $x \gg 1$ ,

$$\left\langle \frac{v_{\text{MøI}}}{v_{\text{cm}}} \right\rangle = 1 - \frac{3}{2}x^{-1} + 3x^{-2} - \frac{75}{16}x^{-3} + \frac{45}{8}x^{-4} + O(x^{-5}). \quad (3.19)$$

Because  $\langle \sigma v_{\text{MøI}} \rangle = \langle \sigma v_{\text{lab}} \rangle^{\text{lab}}$ , it is convenient to perform the calculation of the thermal average in the lab frame. We therefore express our single-integral formula, eq. (3.8), in terms of the kinetic energy per unit mass in the lab frame,  $\epsilon$ , defined by

$$\epsilon = \frac{(E_{1,\text{lab}} - m) + (E_{2,\text{lab}} - m)}{2m} = \frac{s - 4m^2}{4m^2}. \quad (3.20)$$

Using  $v_{\text{lab}} = 2\epsilon^{1/2}(1 + \epsilon)^{1/2}/(1 + 2\epsilon)$ , we obtain

$$\langle \sigma v_{\text{MøI}} \rangle = \int_0^\infty d\epsilon \mathcal{K}(x, \epsilon) \sigma v_{\text{lab}}, \quad (3.21)$$

where the thermal kernel

$$\mathcal{K}(x, \epsilon) \equiv \frac{2x}{K_2^2(x)} \epsilon^{1/2} (1 + 2\epsilon) K_1(2x\sqrt{1 + \epsilon}) \quad (3.22)$$

represents the thermal distribution of  $\epsilon$ . Knowing  $\sigma v_{\text{lab}}$ , this integral can be done numerically as is. Since it is one dimensional, it does not present the difficulties connected with the numerical evaluation of multi-dimensional integrals. The result of a numerical integration of eq. (3.21) will be shown in sect. 8 for a specific example.

Another advantage of eq. (3.21) is that it does not require the expansion of  $\sigma v_{\text{lab}}$  in powers of  $\epsilon$ , as is usually done in the literature. Such an expansion can in fact fail if  $\sigma v_{\text{lab}}$  varies rapidly with  $\epsilon$ , as near resonances or thresholds. For example, for a two-particle final state, which is the case most commonly considered,  $\sigma v_{\text{lab}}$  is in general given by

$$\sigma v_{\text{lab}} = \frac{1}{64\pi^2(s - 2m^2)} \beta_{\text{f}} \int d\Omega |\overline{\mathcal{M}}|^2, \quad (3.23)$$

where  $\Omega$  is the center-of-mass scattering angle, and

$$\beta_{\text{f}} = \left[ 1 - \frac{(m_3 + m_4)^2}{s} \right]^{1/2} \left[ 1 - \frac{(m_3 - m_4)^2}{s} \right]^{1/2} \quad (3.24)$$

is an invariant that for  $m_3 = m_4$  coincides with the velocity of the final particles in the center-of-mass frame. The expansion of  $\sigma v_{\text{lab}}$  in powers of  $\epsilon$  diverges at threshold ( $m = m_3 + m_4$ ), because  $\beta_f$  has infinite derivatives at  $s = 4m^2$  (see fig. 3). The expansion of the cross section is inappropriate also near a resonance, where the propagator in  $\sigma$  leads to derivatives with alternating signs. Stopping the expansion at odd powers of  $\epsilon$  gives negative  $\sigma v_{\text{lab}}$  for energies slightly above a narrow resonance (as will be shown in fig. 2). Outside these problematic regions, the expansion of  $\langle \sigma v_{\text{Mø}} \rangle$  at large  $x$  is useful. We will next compare the two expansions that result from the relativistic and non-relativistic distribution functions.

Let us first expand the relativistic formula (3.21). The expansion of  $\sigma v_{\text{lab}}$  in powers of  $\epsilon$  is

$$\begin{aligned}\sigma v_{\text{lab}} &= \sum_{n=0}^{\infty} \frac{a^{(n)}}{n!} \epsilon^n \\ &= a^{(0)} + a^{(1)}\epsilon + \frac{1}{2}a^{(2)}\epsilon^2 + \dots,\end{aligned}\quad (3.25)$$

where  $a^{(n)}$  indicates the  $n$ th derivative of  $\sigma v_{\text{lab}}$  with respect to  $\epsilon$  evaluated at  $\epsilon = 0$ . The asymptotic expansion of the Bessel functions for large  $z$  is [8]

$$K_n(z) = \left( \frac{\pi}{2z} \right)^{1/2} e^{-z} P_n(z), \quad (3.26)$$

where  $P_n(z)$  is an asymptotic series given by

$$P_n(z) = 1 + \frac{(4n^2 - 1^2)}{1!(8z)} + \frac{(4n^2 - 1^2)(4n^2 - 3^2)}{2!(8z)^2} + \dots \quad (3.27)$$

Using eq. (3.26) in eq. (3.21), we get

$$\langle \sigma v_{\text{Mø}} \rangle \simeq \frac{2x^{3/2}}{\pi^{1/2}} \int_0^\infty \sigma v_{\text{lab}} \frac{1+2\epsilon}{(1+\epsilon)^{1/4}} \frac{P_1(2x\sqrt{1+\epsilon})}{P_2(x)} \epsilon^{1/2} \exp[-2x(\sqrt{1+\epsilon} - 1)] d\epsilon. \quad (3.28)$$

Expanding  $P_1(2x\sqrt{1+\epsilon})$  and  $\sigma v_{\text{lab}}$  in powers of  $\epsilon$ , and changing integration variable from  $\epsilon$  to  $y = 2x(\sqrt{1+\epsilon} - 1)$ , all the integrals can be performed analytically with the help of

$$\int_0^\infty y^{n+1/2} e^{-y} dy = \frac{(2n+1)!!}{2^n} \frac{\pi^{1/2}}{2}. \quad (3.29)$$

We finally obtain

$$\begin{aligned}
 \langle \sigma v_{\text{Mø}} \rangle &= a^{(0)} + \frac{3}{2} a^{(1)} x^{-1} + \left[ \frac{9}{2} a^{(1)} + \frac{15}{8} a^{(2)} \right] x^{-2} \\
 &+ \left[ \frac{15}{16} a^{(1)} + \frac{195}{16} a^{(2)} + \frac{35}{16} a^{(3)} \right] x^{-3} \\
 &+ \left[ \frac{675}{32} a^{(2)} + \frac{735}{32} a^{(3)} + \frac{315}{128} a^{(4)} \right] x^{-4} + O(x^{-5}). \quad (3.30)
 \end{aligned}$$

Up to order  $x^{-3}$ , this coincides with eq. (30) in ref. [2] (where the variable  $w(s)$  in ref. [2] is  $w(s) = m^2(1 + 2\epsilon)\sigma v_{\text{lab}}$ ).

It is convenient to define the non-relativistic thermal average as

$$\langle \sigma v_{\text{Mø}} \rangle_{\text{n.r.}} = \frac{\int \sigma v_{\text{lab}} e^{-p_1^2/2mT} e^{-p_2^2/2mT} d^3p_1 d^3p_2}{\int e^{-p_1^2/2mT} e^{-p_2^2/2mT} d^3p_1 d^3p_2}. \quad (3.31)$$

If we take as integration variables the total momentum  $\mathbf{p}_T = \mathbf{p}_1 + \mathbf{p}_2$  and the relative momentum  $\mathbf{p}_{\text{rel}} = \mathbf{p}_1 - \mathbf{p}_2$ , the integration over  $\mathbf{p}_T$  is trivial, since  $\sigma v_{\text{lab}}$  depends only on  $\mathbf{p}_{\text{rel}}$ . What remains is

$$\langle \sigma v_{\text{Mø}} \rangle_{\text{n.r.}} = \frac{2x^{3/2}}{\pi^{1/2}} \int_0^\infty \sigma v_{\text{lab}} \epsilon^{1/2} e^{-x\epsilon} d\epsilon, \quad (3.32)$$

where  $p_{\text{rel}}^2 = 4m^2\epsilon$ . Expanding  $\sigma v_{\text{lab}}$  in powers of  $\epsilon$  and using eq. (3.29), we obtain the expansion of the non-relativistic thermal average

$$\langle \sigma v_{\text{Mø}} \rangle_{\text{n.r.}} = a^{(0)} + \frac{3}{2} a^{(1)} x^{-1} + \frac{15}{8} a^{(2)} x^{-2} + \frac{35}{16} a^{(3)} x^{-3} + \frac{315}{128} a^{(4)} x^{-4} + O(x^{-5}). \quad (3.33)$$

For calculations of relic abundances, it is usually sufficient to keep only terms up to order  $x^{-1}$ . Comparing the expansions of the relativistic and non-relativistic averages, eqs. (3.33) and (3.30), we see that they coincide up to order  $x^{-1}$  in the regions where  $\sigma v_{\text{lab}}$  can be expanded. To obtain this property, it is crucial to use  $\sigma v_{\text{lab}}$  in the definition of the non-relativistic thermal average, eq. (3.31).

#### 4. Degrees of freedom

In this section we show how to obtain the degrees of freedom parameter  $g_*^{1/2}$  defined in eq. (2.24). We follow the procedure discussed in ref. [2], but we introduce the compact expression, eq. (4.18), for the entropy degrees of freedom. We will consider separately two cases: the first when there is no entropy production and each species contributing to the total energy and entropy densities can be

considered as an ideal gas, and the second when these approximations break down, such as during the QCD quark-hadron phase transition, which occurs in the temperature range 150–400 MeV.

The computation of  $h_{\text{eff}}(T)$  and  $g_{\text{eff}}(T)$  in the second case requires a model for the phase transition. The study of such a model is out of the scope of this paper. We have taken the results of the model described in ref. [11] and have read the values of  $h_{\text{eff}}(T)$  and  $g_{\text{eff}}(T)$  during the QCD phase transition from the figures in ref. [2] for two extreme values of the transition temperature: 150 MeV and 400 MeV.

We now discuss the first case in more detail. We assume that each relevant species  $i$  is in kinetic equilibrium at temperature  $T_i$ . Thus its energy and entropy densities are given respectively by

$$\begin{aligned}\rho_i(T_i) &= \int E_i f_i(T_i, E_i) d^3 p_i, \\ s_i(T_i) &= \int \frac{3m_i^2 + 4p_i^2}{3E_i T_i} f_i(T_i, E_i) d^3 p_i,\end{aligned}\quad (4.1)$$

where  $\mathbf{p}_i$  is the three-momentum,  $E_i$  the energy,  $m_i$  the mass of the species in question, and the distribution function  $f_i(T_i, E_i)$  is

$$f_i(T_i, E_i) = \frac{g_i}{(2\pi)^3} \frac{1}{\exp(E_i/T_i) + \eta_i}, \quad (4.2)$$

with  $\eta_i = 1$  for Fermi-Dirac,  $\eta_i = -1$  for Bose-Einstein, and  $\eta_i = 0$  for Maxwell-Boltzmann statistics, and  $g_i$  is the number of internal (spin) degrees of freedom.

Analogous to eqs. (2.21) and (2.22), we define the effective degrees of freedom for energy and entropy for each species as

$$g_i(T) = \frac{30}{\pi^2 T^4} \rho_i(T) = \frac{15g_i}{\pi^4} x_i^4 \int_1^\infty \frac{y(y^2 - 1)^{1/2}}{\exp(x_i y) + \eta_i} y dy, \quad (4.3)$$

$$h_i(T) = \frac{45}{2\pi^2 T^3} s_i(T) = \frac{45g_i}{4\pi^4} x_i^4 \int_1^\infty \frac{y(y^2 - 1)^{1/2}}{\exp(x_i y) + \eta_i} \frac{4y^2 - 1}{3y} dy, \quad (4.4)$$

respectively, with  $x_i = m_i/T$ . They are functions of  $x_i$  only. The total energy and



entropy effective degrees of freedom are given by

$$g_{\text{eff}}(T) = \sum_i g^{(i)}(T), \quad h_{\text{eff}}(T) = \sum_i h^{(i)}(T), \quad (4.5), (4.6)$$

where the contributions of each species are

$$g^{(i)}(T) = g_i(T) \frac{T_i^4}{T^4}, \quad h^{(i)}(T) = h_i(T) \frac{T_i^3}{T^3}. \quad (4.7), (4.8)$$

We recall that the temperature  $T_i$  of a species decoupled from the photons is in general different from the temperature  $T$  of the photons.

Before the decoupling,  $h^{(i)}(T) = h_i(T)$ , where  $h_i(T)$  is defined in eq. (4.4). To compute  $h^{(i)}(T)$  after the decoupling, we use the conservation of the total entropy and of the entropy of each decoupled species separately. We make the assumption that there is no entropy production and that it is possible to define a sharp decoupling temperature  $T_{d_i}$  for each species, even if the decoupling process actually takes some time. In practice we choose  $T_{d_i} = m_i/20$  for the decoupling temperature. The above mentioned entropy conservations give

$$h^{(i)}(T) T^3 R(T)^3 = h_i(T_{d_i}) T_{d_i}^3 R(T_{d_i})^3, \quad (4.9)$$

and

$$h_{\text{eff}}(T) T^3 R(T)^3 = h_{\text{eff}}(T_{d_i}) T_{d_i}^3 R(T_{d_i})^3. \quad (4.10)$$

Dividing eq. (4.9) by eq. (4.10), we get the contribution after decoupling

$$h^{(i)}(T) = h_{\text{eff}}(T) \frac{h_i(T_{d_i})}{h_{\text{eff}}(T_{d_i})}. \quad (4.11)$$

We can separate in  $h(T)$  the contribution from all the species which are coupled at temperature  $T$ ,

$$h_c(T) = \sum_{T_{d_i} < T} h_i(T), \quad (4.12)$$

and the contribution from the species which are already decoupled at  $T$ , obtaining

$$\begin{aligned} h_{\text{eff}}(T) &= h_c(T) + \sum_{i \in \text{dec}} h^{(i)}(T) \\ &= h_c(T) + h_{\text{eff}}(T) \sum_{i \in \text{dec}} \frac{h_i(T_{d_i})}{h_{\text{eff}}(T_{d_i})}. \end{aligned} \quad (4.13)$$

Introducing  $r_i = h_i(T_{d_i})/h_c(T_{d_i})$  and  $r_c(T) = h_c(T)/h_{\text{eff}}(T)$ , we can rewrite eq. (4.13) as

$$r_c(T) = 1 - \sum_{i \in \text{dec}} r_i r_c(T_{d_i}). \quad (4.14)$$

Suppose now that when  $n$  species have decoupled the temperature is  $T_n$ . From eq. (4.14) we have

$$r_c(T_{n+1}) = r_c(T_n) - r_n r_c(T_{d_n}), \quad (4.15)$$

which evaluated at  $T = T_{d_n}$  gives

$$r_c(T_{d_n}) = r_c(T_n)(1 + r_n)^{-1}. \quad (4.16)$$

Substituting this back into eq. (4.15) we find the recurrence relation

$$r_c(T_{n+1}) = r_c(T_n)(1 + r_n)^{-1}, \quad (4.17)$$

which together with the initial condition that all particles were coupled at very high temperature, i.e.  $r_c(T) = 1$  for  $T > \max(T_{d_i})$ , permits us to obtain  $h_{\text{eff}}(T)$  in a very useful form

$$h_{\text{eff}}(T) = h_c(T) \prod_{i \in \text{dec}} \left( 1 + \frac{h_i(T_{d_i})}{h_c(T_{d_i})} \right). \quad (4.18)$$

Here the product extends to all the species decoupled and  $h_c(T)$  is the contribution from species coupled at temperature  $T$ , defined in eq. (4.12).

To compute  $g^{(i)}(T)$ , we compare eqs. (4.8) and (4.11) to obtain

$$\frac{T_i^3}{T^3} = \frac{h_{\text{eff}}(T)}{h_i(T)} \frac{h_i(T_{d_i})}{h_{\text{eff}}(T_{d_i})}, \quad (4.19)$$

for the temperature  $T_i$ . We substitute this into eqs. (4.7) and (2.21) and find

$$g_{\text{eff}}(T) = g_c(T) + \sum_{i \in \text{dec}} g_i(T) \frac{T_i^4}{T^4}, \quad (4.20)$$

with  $g_c(T)$  defined analogously to  $h_c(T)$ .

Finally, from eq. (2.24) we can compute

$$g_*^{1/2}(T) = \frac{h_{\text{eff}}(T)}{g_{\text{eff}}^{1/2}(T)} \left( 1 + \frac{1}{3} \frac{d \ln h_c(T)}{d \ln T} \right), \quad (4.21)$$

using eqs. (4.18)–(4.20) and inserting the results of ref. [2] for the QCD quark–hadron phase transition.

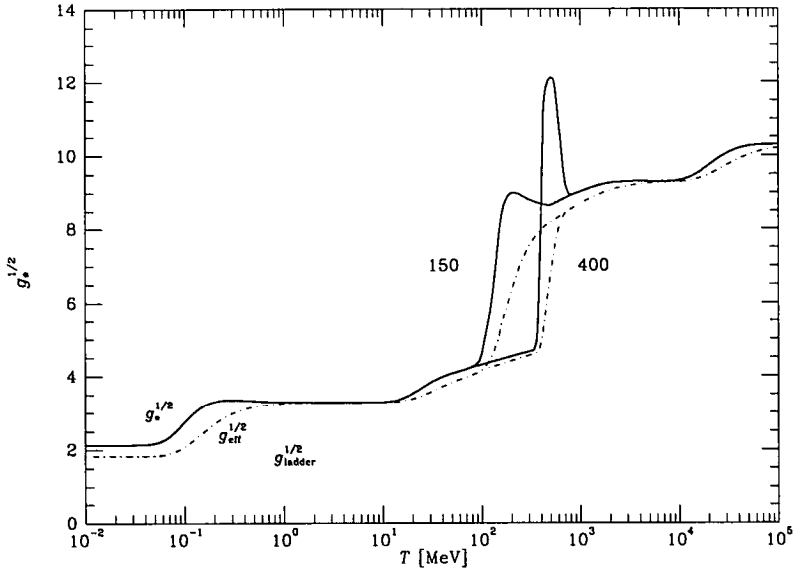


Fig. 1. The effective degrees of freedom in the early universe,  $g_*^{1/2}(T)$ . For comparison, we show the commonly used quantities  $g_{\text{eff}}^{1/2}(T)$  and  $g_{\text{ladder}}^{1/2}(T)$ .

In fig. 1 we plot  $g_*^{1/2}(T)$  so obtained for the two QCD phase transition temperatures 150 MeV and 400 MeV. For comparison, we also show  $g_{\text{eff}}^{1/2}(T)$ , which is the value many times assigned to  $g_*^{1/2}(T)$ , and the value  $g_{\text{ladder}}^{1/2}(T)$  obtained with the approximation of counting only the relativistic degrees of freedom and with a linear interpolation during the QCD phase transition.

### 5. Solution of the density equation

In general, the evolution equation (2.23) for the comoving abundance  $Y$  has to be solved numerically, because it is a form of the Riccati equation, of which no closed-form solutions are known in the general case.

We should integrate eq. (2.23) from  $x = -\infty$  to  $x = x_0 \equiv m/T_0$ , to obtain the value of the density  $Y_0 = Y(x_0)$  at the present photon temperature  $T_0$ . In practice it is sufficient to start at  $x = 1$ . In this regime, the particle statistics can be considered to be Maxwell-Boltzmann. The use of the correct statistics would amount to a correction of less than 1% on  $Y_0$  [9]. The equilibrium density  $Y_{\text{eq}}$  is given by

$$Y_{\text{eq}} = \frac{45g}{4\pi^4} \frac{x^2 K_2(x)}{h_{\text{eff}}(m/x)}, \quad (5.1)$$

where  $g$  is number of internal degrees of freedom of the species under consideration.

Here we present an accurate approximation to  $Y_0$  for heavy particles which does not require a numerical solution of the differential equation. This approximation is an extension of the usual one (see e.g. refs. [1, 9, 12]) to the case of the general dependence of  $g_*^{1/2}(T)$  and  $\langle \sigma v_{\text{Mø}} \rangle$  on the temperature  $T$  and the inclusion of relativistic effects.

The physical picture of the decoupling is well known [1, 9, 12]. The species in question remains in kinetic and chemical equilibrium with the background thermal bath until its annihilation rate becomes much smaller than the rate of expansion of the universe. After this point, it is decoupled from the other species and its density is affected only by the expansion of the universe.

First we determine the temperature of decoupling (freeze-out)  $T_f$  (an index  $f$  indicates the corresponding quantity at freeze-out). Before the decoupling, it is useful to work with the equation for  $\Delta = Y - Y_{\text{eq}}$ ,

$$\frac{d\Delta}{dx} = - \left( \frac{45}{\pi} G \right)^{-1/2} \frac{g_*^{1/2} m}{x^2} \langle \sigma v_{\text{Mø}} \rangle \Delta (\Delta + 2Y_{\text{eq}}) - \frac{dY_{\text{eq}}}{dx}. \quad (5.2)$$

Before the decoupling, we can neglect  $d\Delta/dx$ , because  $Y$  follows the equilibrium density  $Y_{\text{eq}}$ . We define  $T_f$  as the temperature at which  $\Delta = \delta Y_{\text{eq}}$ , with  $\delta$  a chosen number. Thus, the condition for freeze-out is

$$\left( \frac{45}{\pi} G \right)^{-1/2} \frac{g_*^{1/2} m}{x^2} \langle \sigma v_{\text{Mø}} \rangle Y_{\text{eq}} \delta (\delta + 2) = - \frac{d \ln Y_{\text{eq}}}{d \ln x}. \quad (5.3)$$

Inserting eq. (5.1) into eq. (5.3) results in the (transcendental) equation

$$\left( \frac{45}{\pi} G \right)^{-1/2} \frac{45g}{4\pi^4} \frac{K_2(x)}{h_{\text{eff}}(T)} g_*^{1/2} m \langle \sigma v_{\text{Mø}} \rangle \delta (\delta + 2) = \frac{K_1(x)}{K_2(x)} - \frac{1}{x} \frac{d \ln h_c(T)}{d \ln T}, \quad (5.4)$$

which must be solved numerically for  $x = x_f$ . The first term in the r.h.s. is relativistic in nature and goes to one in the usual non-relativistic version of this equation. The second term in the r.h.s. accounts for the variation in the comoving entropy density, and it is zero if one takes  $g_{\text{eff}}^{1/2}(T)$  instead of  $g_*^{1/2}(T)$ . The decoupling temperature  $T_f$ , and consequently  $Y_0$ , depend only logarithmically on  $\delta$ . Comparing the approximate results with those of the numerical integration of the differential equation for several cross sections, we found that  $\delta = 1.5$  is a good choice.

After the decoupling, we can neglect  $Y_{\text{eq}}$  in eq. (2.23), and integrate from  $T_f$  to  $T_0$ . We obtain

$$\frac{1}{Y_0} = \frac{1}{Y_f} + \left( \frac{45}{\pi} G \right)^{1/2} \int_{T_0}^{T_f} g_*^{1/2} \langle \sigma v_{\text{Mø}} \rangle dT. \quad (5.5)$$

This is the complete solution of the approximate density equation after freeze-out. In the literature, the term  $1/Y_f$  is usually omitted, but its presence is essential in achieving a very good approximation to  $Y_0$  (within a few per cent). Since  $T_0$  is very small compared to the mass of the particles usually under consideration, we will replace it by zero in eq. (5.5). Notice from eq. (5.5) that different annihilation channels contribute linearly to  $Y_0^{-1}$ , provided that  $x_f$  does not vary too much when an additional channel is added. This property will be useful when taking into account the formation of a resonance in one of the annihilation channels.

Knowing  $Y_0$ , we can compute  $\Omega = \rho_0/\rho_c = ms_0 Y_0/\rho_c$ , the present density of the species under consideration in terms of the critical density  $\rho_c = 3H^2/8\pi G$ . The result is

$$\Omega h^2 \theta^{-3} = 2.8282 \times 10^8 \frac{m}{\text{GeV}} Y_0, \quad (5.6)$$

where  $h$  is the Hubble constant in units of  $100 \text{ km sec}^{-1} \text{ Mpc}^{-1}$ ,  $\theta$  is the temperature of the cosmic microwave background in units of 2.75 K, and we have used  $h_{\text{eff}}(T_0) = 3.915$ .

To conclude this section, we present for completeness the usual approximation [1, 9, 12]. Neglecting variations in the comoving entropy density, the non-relativistic limit of the condition for freeze-out, eq. (5.4), is

$$\left( \frac{45}{\pi} G \right)^{-1/2} \frac{45g}{8\pi^4} \frac{\pi^{1/2} x^{-1/2} e^{-x}}{h_{\text{eff}}(T)} g_{\text{eff}}^{1/2} m \langle \sigma v_{\text{Mø}} \rangle \delta(\delta + 2) = 1. \quad (5.7)$$

Neglecting  $Y_f$  in eq. (5.5), we obtain

$$\Omega h^2 \theta^{-3} \approx 8.7661 \times 10^{-11} \text{ GeV}^{-2} \left[ \frac{1}{g_{\text{eff}}} \int_{T_0}^{T_f} \langle \sigma v_{\text{Mø}} \rangle \frac{dT}{m} \right]^{-1}, \quad (5.8)$$

where  $\overline{g_{\text{eff}}}^{1/2}$  is a typical value of  $g_{\text{eff}}^{1/2}(T)$  between  $T_0$  and  $T_f$ . In the case in which  $\langle \sigma v_{\text{Mø}} \rangle$  can be expanded in powers of  $x^{-1}$ ,

$$\langle \sigma v_{\text{Mø}} \rangle = \sum_{n=0}^{\infty} c_n x^{-n}, \quad (5.9)$$

as in eqs. (3.30) and (3.33), the integral in eq. (5.8) can be done term by term and

we get

$$\Omega h^2 \theta^{-3} \approx 8.7661 \times 10^{-11} \text{GeV}^{-2} \frac{x_f}{g_{\text{eff}}^{1/2}} \left[ \sum_{n=0}^{\infty} \frac{c_n}{n+1} x_f^{-n} \right]^{-1}, \quad (5.10)$$

which is the usual non-relativistic result.

## 6. Thermal average near a resonance

We now consider resonant cross sections of the Breit–Wigner form [13],

$$\sigma_{\text{res}} = \frac{4\pi\omega}{p^2} B_i B_f \frac{m_R^2 \Gamma_R^2}{(s - m_R^2)^2 + m_R^2 \Gamma_R^2}, \quad (6.1)$$

where the statistical factor  $\omega = (2J+1)/(2S+1)^2$ ,  $p = \frac{1}{2}(s - 4m^2)^{1/2}$  is the center-of-mass momentum,  $m$  and  $S$  are the mass and spin of the colliding particles,  $m_R$ ,  $\Gamma_R$  and  $J$  are the mass, total decay width and spin of the resonance, and  $B_i$  and  $B_f$  are the branching fractions of the resonance into the initial and final channel, respectively.

We can rewrite eq. (6.1) in terms of the kinetic energy per unit mass in the lab frame,  $\epsilon$ , defined in eq. (3.20). Introducing  $\gamma_R = m_R \Gamma_R / 4m^2$  and  $\epsilon_R = (m_R^2 - 4m^2)/4m^2$ , we have

$$\sigma_{\text{res}} = \frac{4\pi\omega}{m^2\epsilon} B_i B_f \frac{\gamma_R^2}{(\epsilon - \epsilon_R)^2 + \gamma_R^2}. \quad (6.2)$$

Using  $v_{\text{lab}} = 2\epsilon^{1/2}(1+\epsilon)^{1/2}/(1+2\epsilon)$ , and summing over final states  $f \neq i$ , we finally have

$$\sigma_{\text{res}} v_{\text{lab}} = \frac{8\pi\omega}{m^2} \frac{\gamma_R^2}{(\epsilon - \epsilon_R)^2 + \gamma_R^2} b_R(\epsilon). \quad (6.3)$$

The factor  $b_R(\epsilon)$ , given by

$$b_R(\epsilon) \equiv \frac{B_i(1-B_i)(1+\epsilon)^{1/2}}{\epsilon^{1/2}(1+2\epsilon)}, \quad (6.4)$$

is a slowly varying function of  $\epsilon$  near the resonance, and can be expanded in powers of  $\epsilon$  in the following way (cf. appendix A):

$$b_R(\epsilon) = \sum_{l=0}^{\infty} \frac{b_R^{(l)}}{l!} \epsilon^l. \quad (6.5)$$

We first examine the case of a very narrow resonance,  $\gamma_R \ll 1$ . Using the relation

$$\lim_{\gamma \rightarrow 0} \frac{\gamma}{x^2 + \gamma^2} = \pi \delta(x), \quad (6.6)$$

in eq. (6.3), we obtain

$$\sigma_{\text{res}} \nu_{\text{lab}} = \frac{8\pi\omega}{m^2} \gamma_R \pi \delta(\epsilon - \epsilon_R) b_R(\epsilon_R), \quad \gamma_R \ll 1. \quad (6.7)$$

The relativistic formula for the thermal average, eq. (3.21), can then be integrated, yielding the result

$$\begin{aligned} \langle \sigma_{\text{res}} \nu_{\text{Mø}} \rangle &= \frac{16\pi\omega}{m^2} \frac{x}{K_2^2(x)} \pi \gamma_R \epsilon_R^{1/2} (1 + 2\epsilon_R) \\ &\times K_1(2x\sqrt{1 + \epsilon_R}) b_R(\epsilon_R) \theta(\epsilon_R), \quad \gamma_R \ll 1, \end{aligned} \quad (6.8)$$

while integration of the non-relativistic formula, eq. (3.32), gives

$$\langle \sigma_{\text{res}} \nu_{\text{Mø}} \rangle_{\text{n.r.}} = \frac{16\pi\omega}{m^2} x^{3/2} \pi^{1/2} \gamma_R \epsilon_R^{1/2} e^{-x\epsilon_R} b_R(\epsilon_R) \theta(\epsilon_R), \quad \gamma_R \ll 1. \quad (6.9)$$

The profile of the  $l=0$  and  $l=1$  terms in the expansion of  $b_R(\epsilon)$  is shown at freeze-out ( $x_f = 25$ ) in fig. 2a by the heavy full (relativistic average) and dash-dot (non-relativistic average) lines. The non-relativistic average tends to overestimate the cross section near the peak of the resonance, and to underestimate it in the tail. We also see from the figure that a very narrow resonance contributes to the thermal averaged cross section only if  $\epsilon_R > 0$ , i.e. only if the particle mass  $m$  is less than half the resonance mass  $m_R$ . This is easily understood. When  $m > m_R/2$ , the center-of-mass rest energy (and consequently the total energy) exceeds the energy  $m_R/2$  necessary to produce the resonance. While if  $m < m_R/2$ , there are some collisions with enough center-of-mass kinetic energy to give a total energy  $m_R/2$ . The fraction of such collisions is of course governed by the thermal distribution, which eqs. (6.8) and (6.9) reproduce.

For a general value of  $\gamma_R$ , the relativistic average has to be computed numerically, while we have obtained a closed expression for the non-relativistic average.

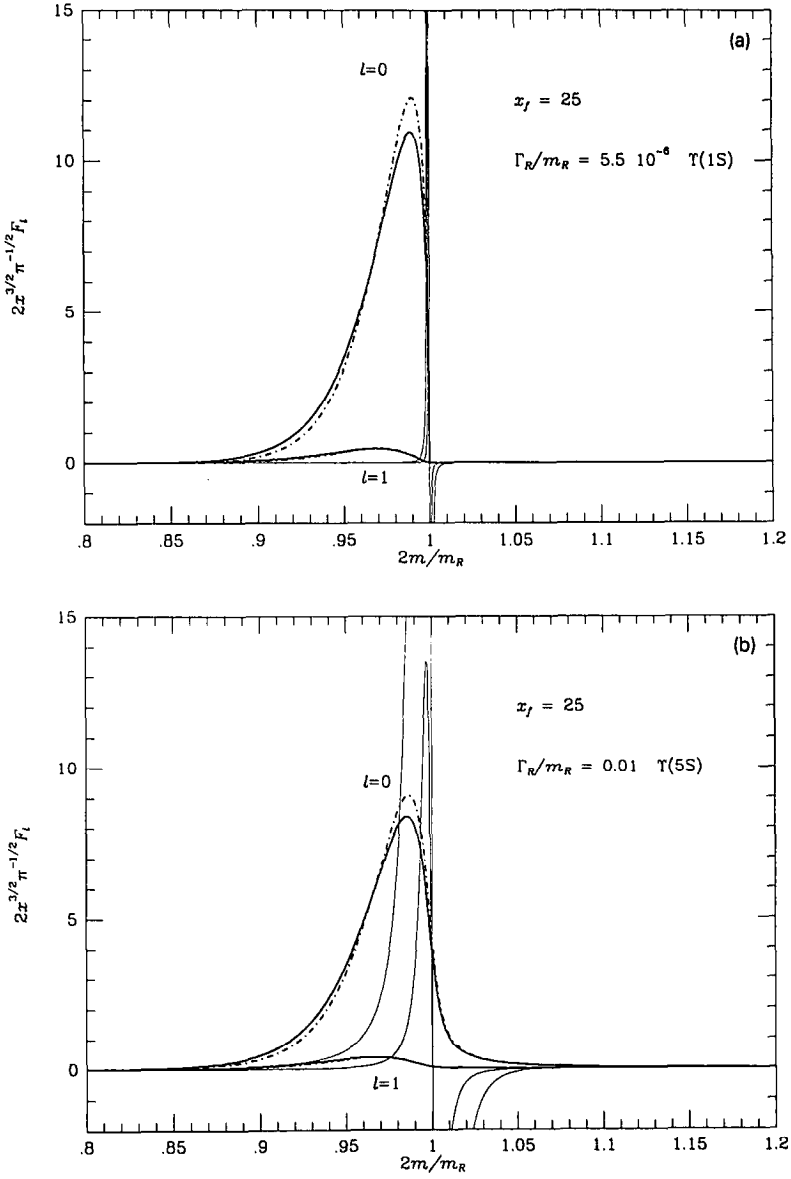


Fig. 2. The profile of the relativistic (full lines) and non-relativistic (dash-dot lines) thermal averages near a resonance, for  $l=0$  and  $l=1$ . For comparison, the light lines show the corresponding expansions in powers of  $x^{-l}$ .



Writing eq. (6.3) in the form

$$\sigma_{\text{res}} v_{\text{lab}} = \frac{8\pi\omega}{m^2} \gamma_R \operatorname{Re} \frac{i}{\epsilon_R + i\gamma_R - \epsilon} b_R(\epsilon), \quad (6.10)$$

and using the non-relativistic average, eq. (3.32), we obtain

$$\langle \sigma_{\text{res}} v_{\text{Mol}} \rangle_{\text{n.r.}} = \frac{16\pi\omega}{m^2} x^{3/2} \pi^{1/2} \gamma_R \sum_{l=0}^{\infty} \frac{b_R^{(l)}}{l!} F_l(z_R; x), \quad (6.11)$$

where  $z_R = \epsilon_R + i\gamma_R$  and we have defined the function

$$\begin{aligned} F_l(z_R; x) &= \operatorname{Re} \frac{i}{\pi} \int_0^{\infty} \frac{\epsilon'^{l+1/2} e^{-x\epsilon'}}{z_R - \epsilon'} d\epsilon' \\ &= (-1)^l \frac{\partial^l}{\partial x^l} F_0(z_R; x) \\ &= (-1)^l \frac{\partial^l}{\partial x^l} \operatorname{Re} \left[ z_R^{1/2} e^{-xz_R} \operatorname{erfc}(-ix^{1/2} z_R^{1/2}) \right]. \end{aligned} \quad (6.12)$$

For the first two terms,  $l=0$  and  $l=1$ , eqs. (6.11) and (6.12) give

$$\begin{aligned} \langle \sigma_{\text{res}} v_{\text{Mol}} \rangle_{\text{n.r.}} &= \frac{16\pi\omega}{m^2} x^{3/2} \pi^{1/2} \gamma_R \left\{ b_R^{(0)} \operatorname{Re} \left[ z_R^{1/2} e^{-xz_R} \operatorname{erfc}(-ix^{1/2} z_R^{1/2}) \right] \right. \\ &\quad \left. + b_R^{(1)} \left[ \gamma_R \pi^{-1/2} x^{-1/2} + \operatorname{Re} \left[ z_R^{3/2} e^{-xz_R} \operatorname{erfc}(-ix^{1/2} z_R^{1/2}) \right] \right] \right\}. \end{aligned} \quad (6.13)$$

In fig. 2 we plot the profile of the relativistic (full lines) and non-relativistic (dash-dot lines) thermal averages at freeze-out ( $x_f = 25$ ) for  $l=0$  and  $l=1$  near the  $\mathcal{T}(1S)$  [fig. 2a] and the  $\mathcal{T}(5S)$  [fig. 2b] resonances, with  $\Gamma_R/m_R = 5.5 \times 10^{-6}$  and  $\Gamma_R/m_R = 0.01$  respectively. For the  $\mathcal{T}(1S)$ , eqs. (6.8) and (6.9) for very narrow resonances can be used. For comparison, we have also shown (light lines) the results obtained using the expansion (3.30) with the cross section (6.3). The normalization of the plots is such that for  $\Gamma_R/m_R \ll 1$  the area under the curves with  $l=0$  is unity. The suppression of the  $l=1$  terms is evident. The non-relativistic average is larger than the relativistic one near the peak, and is smaller in the tail.

As expected, the peaks are broadened at finite temperature because of the thermal distributions. The position of the peak is shifted towards lower masses.

This is due to the fact that particles with mass  $m \geq (m_R + \Gamma_R)/2$  have always too much (rest) energy to produce a resonance, but those with  $m \leq (m_R - \Gamma_R)/2$  can have enough thermal energy to do so. For a very narrow resonance,  $\langle \sigma_{\text{res}} v_{M\emptyset l} \rangle$  just reproduces the thermal distribution of the particles “below resonance” ( $m < m_R/2$ ), while “above resonance”  $\langle \sigma_{\text{res}} v_{M\emptyset l} \rangle$  vanishes. For wider resonances, the low-energy wing permits to obtain non-zero values for  $\langle \sigma_{\text{res}} v_{M\emptyset l} \rangle$  also in the region  $m > m_R/2$ .

It is possible to obtain an approximate closed expression for the relic abundance in presence of a very narrow resonance in the non-relativistic case. In fact, substituting expression (6.9) into the approximation (5.5), the integration in temperature  $T = m/x$  can be performed analytically. We obtain, neglecting  $T_0$  with respect to  $m$ ,

$$\int_{T_0}^{T_f} \langle \sigma_{\text{res}} v_{M\emptyset l} \rangle_{\text{n.r.}} \frac{dT}{m} = \frac{16\pi\omega}{m^2} \pi \gamma_R \text{erfc}(x_f^{1/2} \epsilon_R^{1/2}) b_R(\epsilon_R) \theta(\epsilon_R), \quad \gamma_R \ll 1. \quad (6.14)$$

This expression allows us to compare the two different suppressions of the relic density expected from a proper account of thermal distributions and from a naive treatment. The naive resonant annihilation cross section at  $m = m_R/2$  is obtained by setting  $\epsilon = \epsilon_R = 0$  in eq. (6.3),

$$(\sigma_{\text{res}} v_{M\emptyset l})_{\text{naive}} = \frac{8\pi\omega}{m^2} b_R(0), \quad (6.15)$$

Integrating over  $T$ , we have naively

$$\int_{T_0}^{T_f} (\sigma_{\text{res}} v_{M\emptyset l})_{\text{naive}} \frac{dT}{m} = \frac{8\pi\omega}{m^2} \frac{b_R(0)}{x_f}. \quad (6.16)$$

The correct expression, eq. (6.14), evaluated at  $m = m_R/2$  is

$$\int_{T_0}^{T_f} \langle \sigma_{\text{res}} v_{M\emptyset l} \rangle_{\text{n.r.}} \Big|_{m=m_R} \frac{dT}{m} = \frac{8\pi\omega}{m_R^2} \pi^{3/2} \gamma_R b_R(0). \quad (6.17)$$

The ratio of the correct and the naively computed relic densities at the resonance is approximately given by the ratio of eq. (6.15) and eq. (6.17) as

$$\frac{\Omega}{\Omega_{\text{naive}}} \simeq \frac{1}{\pi^{3/2} \gamma_R x_f} = (0.72-0.90) \times 10^{-2} \frac{m_R}{\Gamma_R}, \quad (6.18)$$

where the numbers correspond to the range  $x_f = 20-25$ . The suppression in relic density due to a resonant annihilation could be overestimated by a large factor, even thousands, if the naive procedure is used for very narrow resonances, such as, for example, the  $T(1S)$  which has  $\Gamma_R/m_R = O(10^{-6})$ .

## 7. Thermal average near a threshold

Here we present an analytical approximation to the thermal average of an annihilation cross section near the opening of a new annihilation channel in the non-relativistic regime.

As we have already pointed out, the failure of the Taylor expansion of  $\sigma v_{\text{lab}}$ , eq. (3.23), is due to the presence of  $\beta_f$ . In order to avoid this difficulty, we factor out the term  $[s - (m_3 + m_4)^2]^{1/2}/2m$  that causes the trouble, and assume that the remaining part has a well-behaved Taylor expansion in the region of energies we are interested in. With the notation  $\xi \equiv m_{\text{th}}^2/m^2 \equiv (m_3 + m_4)^2/4m^2$ , we can write the contribution to  $\sigma v_{\text{lab}}$  of the channel with a threshold as

$$\sigma_{\text{th}} v_{\text{lab}} = (1 - \xi + \epsilon)^{1/2} \hat{a}(\epsilon), \quad (7.1)$$

where the function  $\hat{a}(\epsilon)$  can be safely expanded in powers of  $\epsilon$  as

$$\hat{a}(\epsilon) = \sum_{n=0}^{\infty} \frac{\hat{a}^{(n)}}{n!} \epsilon^{(n)}. \quad (7.2)$$

We could not find a closed form for the relativistic average, and therefore we integrated it numerically. However, a closed expression can be obtained for the non-relativistic average. From eq. (3.32) we have

$$\langle \sigma_{\text{th}} v_{\text{Mø}} \rangle_{\text{n.r.}} = \frac{2x^{3/2}}{\pi^{1/2}} \sum_{n=0}^{\infty} \frac{\hat{a}^{(n)}}{n!} I_n(x), \quad (7.3)$$

with

$$\begin{aligned} I_n(x) &\equiv \int_{\epsilon_{\text{th}}}^{\infty} (1 - \xi + \epsilon)^{1/2} \epsilon^{n+1/2} e^{-x\epsilon} d\epsilon \\ &= (-1)^n \frac{\partial^n}{\partial x^n} I_0(x), \end{aligned} \quad (7.4)$$

where the value of  $\epsilon_{\text{th}}$  depends on  $\xi$ :  $\epsilon_{\text{th}} = 0$  for  $\xi \leq 1$  ( $m \geq m_{\text{th}}$ , “above threshold”) and  $\epsilon_{\text{th}} = \xi - 1$  for  $\xi > 1$  ( $m < m_{\text{th}}$ , “below threshold”). The integrals  $I_n(x)$  can be

done analytically in terms of modified Bessel functions. Let us define  $\alpha = |1 - \xi|/2$ . "Above threshold" we get

$$\begin{aligned} I_0(x) &= \int_0^\infty (\epsilon + 2\alpha)^{1/2} \epsilon^{1/2} e^{-x\epsilon} d\epsilon \\ &= \frac{\alpha}{x} e^{+\alpha x} K_1(\alpha x), \quad \xi < 1, \end{aligned} \quad (7.5)$$

while "below threshold" we obtain

$$\begin{aligned} I_0(x) &= \int_{2\alpha}^\infty (\epsilon - 2\alpha)^{1/2} \epsilon^{1/2} e^{-x\epsilon} d\epsilon \\ &= \frac{\alpha}{x} e^{-\alpha x} K_1(\alpha x), \quad \xi > 1. \end{aligned} \quad (7.6)$$

The last equality follows from the substitution  $\epsilon \rightarrow \epsilon - 2\alpha$ . For  $\xi = 1$  ("at threshold"), we have  $\alpha = 0$  and eqs. (7.5) and (7.6) both approach the same limit

$$I_0(x) = 1/x^2, \quad \xi = 1. \quad (7.7)$$

In particular, for the first two terms, eq. (7.3) gives

$$\begin{aligned} \langle \sigma_{\text{th}} v_{\text{Mol}} \rangle_{\text{n.r.}} &= \frac{2\alpha x^{1/2}}{\pi^{1/2}} e^{\pm \alpha x} \{ \hat{a}^{(0)} K_1(\alpha x) \\ &\quad + \hat{a}^{(1)} \alpha [K_2(\alpha x) \mp K_1(\alpha x)] + \dots \}, \end{aligned} \quad (7.8)$$

where the upper signs correspond to  $\xi < 1$  ("above threshold"), and the lower signs to  $\xi > 1$  ("below threshold"). Using the expression  $K_n(z) \sim (n-1)! 2^{n-1} z^{-n}$  for small  $z$ , we obtain for  $\xi = 1$  ("at threshold")

$$\langle \sigma_{\text{th}} v_{\text{Mol}} \rangle_{\text{n.r.}} = \frac{2}{\pi^{1/2} x^{1/2}} [\hat{a}^{(0)} + 2\hat{a}^{(1)} x^{-1} + \dots], \quad \xi = 1. \quad (7.9)$$

For masses  $m$  "well above threshold",  $m \gg m_{\text{th}}$ , we have  $\xi \rightarrow 0$  and  $\alpha \rightarrow \frac{1}{2}$ . For  $x \gg 1$ , which is usually the case when computing relic abundances of heavy particles, we have  $\alpha x \gg \frac{1}{2}$ . Expanding in powers of  $x^{-1}$ , using the asymptotic

expansion of the Bessel functions for large arguments, eq. (3.26), and the relations

$$\begin{aligned}\hat{a}^{(0)} &= (2\alpha)^{-1/2} a^{(0)}, \\ \hat{a}^{(1)} &= (2\alpha)^{-1/2} \left( a^{(1)} - \frac{a^{(0)}}{4\alpha} \right),\end{aligned}\tag{7.10}$$

we recover eq. (3.33):

$$\langle \sigma_{\text{th}} \nu_{\text{M}\theta\text{l}} \rangle_{\text{n.r.}} = a^{(0)} + \frac{3}{2} a^{(1)} x^{-1} + \dots, \quad \xi \ll 1. \tag{7.11}$$

When the particle mass  $m$  becomes very small, “well below threshold”,  $\xi \rightarrow \infty$  and  $\alpha \rightarrow \infty$ , and using again the asymptotic expansion (3.26) for the Bessel functions, we obtain

$$\langle \sigma_{\text{th}} \nu_{\text{M}\theta\text{l}} \rangle_{\text{n.r.}} = e^{-(\xi-1)x} (\xi-1)^{1/2} \left[ \hat{a}^{(0)} + (\xi-1) \hat{a}^{(1)} + \frac{9}{4} \hat{a}^{(1)} x^{-1} + \dots \right], \quad \xi \gg 1, \tag{7.12}$$

which shows the exponential suppression expected from the fact that the channel is open only for the most energetic particles in the thermal distribution.

In practice, for  $\alpha x \geq 5$ , which corresponds to  $|\xi - 1| \geq 10/x$ , eq. (7.8) is already well approximated by its limiting form (7.11), and the limit below threshold, eq. (7.12), is already negligible. Since the important temperature is that of freeze-out, we can take  $x \approx x_f \approx 25$ , to obtain that a threshold is expected to affect at most the range  $0.6 \leq \xi \leq 1.4$ , corresponding to the mass range  $0.8 m_{\text{th}} \leq m \leq 1.2 m_{\text{th}}$ . For masses outside this range, we expect the usual treatment to be adequate.

In fig. 3 we show the shape of the relativistic (full lines) and non-relativistic (dash-dot lines) thermal averages for  $n = 0$  and  $n = 1$ . For comparison, the light lines show the corresponding expansions in powers of  $x^{-1}$ . The normalization is such that  $\langle \sigma \nu_{\text{M}\theta\text{l}} \rangle \rightarrow \hat{a}^{(0)}$  for  $m \rightarrow \infty$ . The spurious infinite peaks due to the infinite derivative of the factor  $\beta_f$  at  $m = m_{\text{th}}$  have disappeared. Notice that the non-relativistic average is always smaller than the relativistic one.

## 8. An example

We now want to apply the general formulas of the previous sections to a specific example. We have chosen a light dark matter candidate, recently proposed in ref. [14], in the minimal extension of the supersymmetric standard model (extended by the addition of a single chiral supermultiplet to induce Higgs mixing [15]). We have examined the values of masses where its relic density would be larger than the

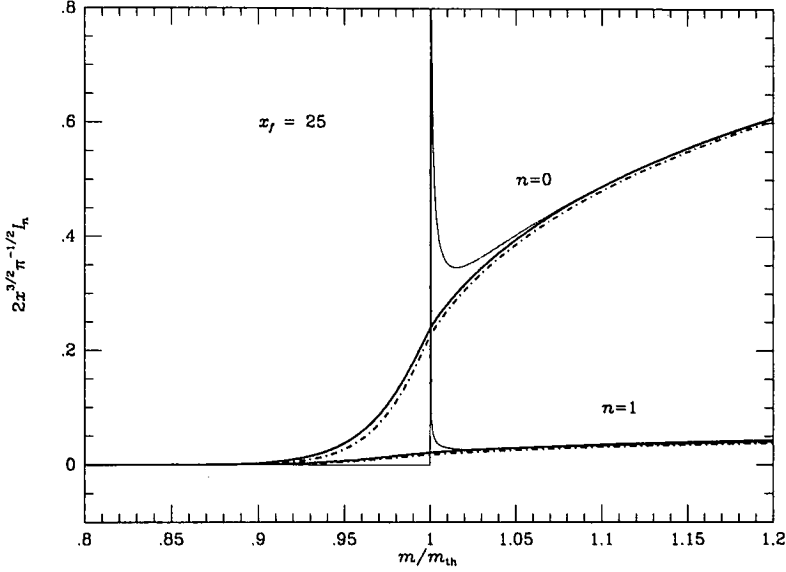


Fig. 3. The shape of the relativistic (full lines) and non-relativistic (dash-dot lines) thermal averages near a threshold, for  $n=0$  and  $n=1$ . For comparison, the light lines show the corresponding expansions in powers of  $x^{-1}$ .

critical density, and thus forbidden, except possibly at resonances of the annihilation cross section. We want to see if the reduction of the expected relic abundance due to resonances is enough to bring it below the critical density.

In the model of ref. [14], the lightest neutralino (assumed stable) is in general a linear combination of five current eigenstates: two gauginos,  $\tilde{W}^3$  and  $\tilde{B}$ , which are the spin- $\frac{1}{2}$  superpartners of the SU(2) and U(1) neutral gauge bosons respectively, and three higgsinos,  $\tilde{H}_1$ ,  $\tilde{H}_2$  and  $\tilde{N}$ , which are the spin- $\frac{1}{2}$  superpartners of the neutral components of the two Higgs doublets and of the Higgs singlet.

We consider here a particular higgsino combination, defined by

$$\tilde{D} = \frac{\sin \beta \tilde{H}_1 + \cos \beta \tilde{H}_2 - \text{sgn}(x) \sqrt{-\sin(2\beta) \lambda / 2k} \tilde{N}}{\sqrt{1 - \sin(2\beta) \lambda / 2k}}, \quad (8.1)$$

with mass

$$m_{\tilde{D}} = 2\lambda \sin 2\beta \frac{|x| - v \sqrt{-\sin(2\beta) \lambda / 2k}}{1 - \sin(2\beta) \lambda / 2k}. \quad (8.2)$$

Here,  $\tan \beta = v_2/v_1$  is the ratio of the two vacuum expectation values  $v_i = \langle H_i \rangle$  ( $i = 1, 2$ ) of the neutral components of the two Higgs doublets,  $x$  is the Higgs

singlet field vacuum expectation value  $x = \langle N \rangle$ , and  $\lambda$  and  $k$  are trilinear coupling constants in the superpotential  $W \ni \lambda H_1 H_2 N - \frac{1}{3} k N^3$ . As discussed in refs. [14, 16], the parameters  $v_1$ ,  $v_2$ ,  $x$ ,  $\lambda$  and  $k$  can be chosen real.

We see from eq. (8.2) that the  $\tilde{D}$  is a massless eigenstate for

$$|x| = x_0 \equiv v \sqrt{-\sin(2\beta)\lambda/2k} \quad (8.3)$$

if  $\lambda k < 0$ , and is a light mass eigenstate for values of  $|x|$  near  $x_0$ .

In ref. [14] it has been argued that it is possible to have a  $CP$ -conserving global minimum of the Higgs potential and positive squared masses for all the physical Higgs bosons with  $\lambda k < 0$  and  $x$  close to  $x_0$ , provided appropriate values for the soft supersymmetry-breaking parameters in the Higgs potential are chosen. They have analyzed the case  $|\lambda| = |2k| = g_2$ , the  $SU(2)$  gauge coupling constant, and  $\tan\beta = 2$ , and have found that the region where the  $\tilde{D}$  is light has not been excluded by present experimental data. This region lies at large  $SU(2)$  gaugino masses  $M_2 \geq 300$  GeV and at  $77 \text{ GeV} \leq |\lambda x| \leq 168 \text{ GeV}$ . (Note that with this choice of parameters,  $|\lambda x_0| = 103 \text{ GeV}$ .) The  $\tilde{D}$  mass in this region varies between zero and 65 GeV. They have also computed the  $\tilde{D}$  relic abundance, and concluded that the  $\tilde{D}$  could be a good dark matter candidate for  $m_{\tilde{D}} \geq 11 \text{ GeV}$ , while for smaller masses its density would exceed the critical value  $\Omega h^2 \theta^{-3} = \frac{1}{4}$ .

We will calculate the  $\tilde{D}$  relic abundance in the mass range  $m_{\tilde{D}} = 4\text{--}6 \text{ GeV}$ , where the resonances due to the  $\mathcal{T}$  particle and its excited states and the threshold for annihilation into bottom quark–antiquark pairs occur. In particular, we want to see if the enhancement in the  $\tilde{D}$  annihilation cross section due to the  $\mathcal{T}$  resonances is sufficient to reduce the relic density below its critical value. For simplicity in presenting this example, we have taken into account only  $Z$  boson exchange in the  $\tilde{D}\tilde{D} \rightarrow f\bar{f}$  annihilation cross section. For the masses of the scalar and pseudoscalar Higgs bosons obtained in ref. [14] in the  $\tilde{D}$  mass range we consider, we expect that the inclusion of Higgs exchange would not significantly modify our results. Thus, the annihilation cross section is

$$\sigma v_{\text{lab}} = \sigma_{\text{res}} v_{\text{lab}} + \sigma_{\text{non-res}} v_{\text{lab}}. \quad (8.4)$$

The non-resonant contribution is

$$\begin{aligned} \sigma_{\text{non-res}} v_{\text{lab}} = \sum_f \frac{(1 - \xi_f + \epsilon)^{1/2}}{(1 + \epsilon)^{1/2}} \frac{n_f}{32\pi} \frac{m_{\tilde{D}}^2}{m_W^4} \frac{g_2^4 \epsilon^2}{1 + 2\epsilon} \\ \times \left[ A_f^2 \xi_f + \frac{4}{3} A_f^2 \epsilon (1 - \xi_f + \epsilon) + \frac{4}{3} V_f^2 \epsilon (1 + \frac{1}{2} \xi_f + \epsilon) \right], \end{aligned} \quad (8.5)$$

where the sum extends over all fermions in the standard model,  $m_W$  is the  $W^\pm$  gauge boson mass,  $m_f$  the fermion mass,  $n_f$  the number of colors (1 for leptons,

3 for quarks),  $\xi_f \equiv m_f^2/m_D^2$ ,  $A_f = T_{3f}$  and  $V_f = T_{3f} - 2e_f \sin \theta_W$  are the standard axial and vector couplings of the fermion with the Z, and  $\zeta = \cos 2\beta / (1 - \sin(2\beta)\lambda/2k)$  is the coupling of the  $\tilde{D}$  with the Z.

For the resonant contribution,

$$\sigma_{\text{res}}^{U_{\text{lab}}} = \sum_R \frac{8\pi\omega}{m^2} \frac{\gamma_R^2}{(\epsilon - \epsilon_R)^2 + \gamma_R^2} b_R(\epsilon), \quad (8.6)$$

we consider the  $\mathcal{T}(nS)$  resonances ( $n = 1, \dots, 6$ ). We have taken their masses,  $m_T$ , total widths,  $\Gamma_T$ , and branching ratios into electrons  $B_{e^+e^-}^T$ , from ref. [17]. Their branching ratios into  $\tilde{D}\tilde{D}$  have been computed as explained in appendix A, with  $\omega = 3/4$  and  $e_b$  and  $b_R(\epsilon)$  corresponding to the bottom quark with constituent mass  $\bar{m}_b = m_T/2$ .

We will compare the  $\tilde{D}$  relic abundance obtained using three different methods to evaluate the thermal average  $\langle \sigma v_{\text{Mol}} \rangle$ : the relativistic single integral (3.21); the non-relativistic averages (6.9) and (6.13) for resonances and (7.8) for thresholds; and the relativistic expansion (3.30) up to order  $x^{-1}$ . We have performed the numerical integration of the density equation (2.23). We have computed the thermal average evaluating numerically the single-integral formula (3.21), but for the first three  $\mathcal{T}$  resonances, for which  $\Gamma_T/m_T \ll 1$  and it is faster but equally accurate to use the relativistic average for very narrow resonances, eq. (6.8). Our result is shown by the heavy full line in fig. 4 for QCD quark-hadron phase transition temperatures of (a) 150 MeV and (b) 400 MeV. The horizontal line corresponds to the critical value  $\Omega_{\tilde{D}} h^2 \theta^{-3} = 1/4$  for the relic density, above which the universe would be overclosed by the  $\tilde{D}$ -particles. The relic density we obtain is always higher than the critical density.

For comparison, the dash-dot line in fig. 4 represents the result of computing the non-relativistic thermal average taking into account resonances and thresholds with eqs. (6.9), (6.13) and (7.8), explicitly

$$\begin{aligned} \langle \sigma v_{\text{Mol}} \rangle_{\text{n.r.}} = & \frac{12\pi^{3/2}x^{3/2}}{m_D^2} \sum_{R=1,3} \theta(m_R - m_D) \frac{m_R \Gamma_R}{4m_D^2} \epsilon_R^{1/2} e^{-\epsilon_R x} [b_R^{(0)} + b_R^{(1)} \epsilon_R] \\ & + \frac{16\pi\omega}{m^2} x^{3/2} \pi^{1/2} \sum_{R=4,6} \gamma_R \{ b_R^{(0)} \text{Re}[z_R^{1/2} e^{-xz_R} \text{erfc}(-ix^{1/2} z_R^{1/2})] \\ & + b_R^{(1)} [\gamma_R \pi^{-1/2} x^{-1/2} + \text{Re}[z_R^{3/2} e^{-xz_R} \text{erfc}(-ix^{1/2} z_R^{1/2})]] \} \\ & + \frac{x^{1/2}}{\pi^{1/2}} \sum_f \alpha_f e^{\pm \alpha_f x} [\hat{a}_f^{(0)} K_1(\alpha_f x) + \hat{a}_f^{(1)} \alpha_f [K_2(\alpha_f x) \mp K_1(\alpha_f x)]] , \end{aligned} \quad (8.7)$$



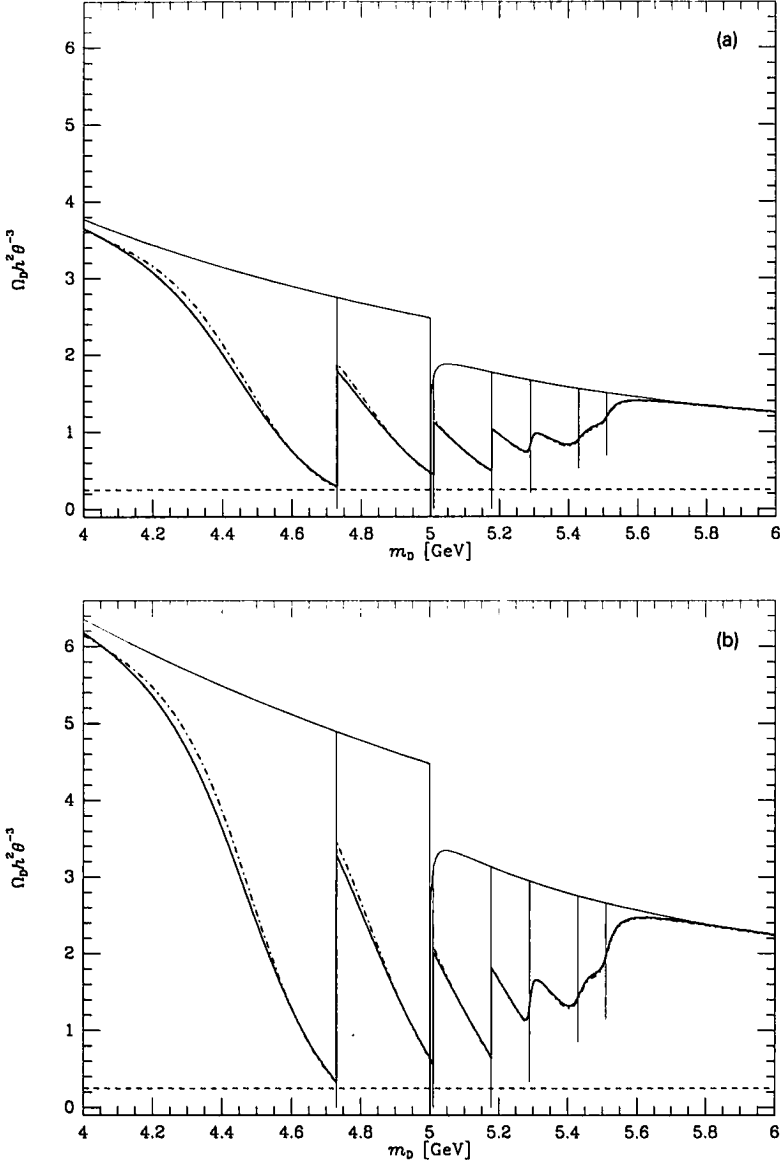


Fig. 4. The relic density  $\Omega_{\tilde{D}} h^2 \theta^{-3}$  of the light higgsino  $\tilde{D}$  as function of its mass  $m_{\tilde{D}}$ , for two values of the QCD quark-hadron phase transition temperature: (a) 150 MeV and (b) 400 MeV. The horizontal line indicates the critical value  $\Omega_{\tilde{D}} h^2 \theta^{-3} = 1/4$ , above which the universe would be overclosed by the  $\tilde{D}$ -particles. The heavy full line corresponds to the relativistic thermal average, and the dash-dot line to the non-relativistic one. For comparison, the light lines are obtained by the expansion in powers of temperature, and the vertical lines result from a naive treatment of resonances.

where the upper (lower) signs in the sum over fermions correspond to  $\xi_f < 1$  ( $\xi_f > 1$ , respectively) and

$$\alpha_f = \frac{|1 - \xi_f|}{2}, \quad \epsilon_R = \frac{m_R^2}{4m_D^2} - 1,$$

$$b_R^{(0)} = B_{e^+e^-}^R - \frac{4\pi m_R^2}{e^4 e_b^2} \hat{a}_b^{(0)} \quad b_R^{(1)} = B_{e^+e^-}^R - \frac{4\pi m_R^2}{e^4 e_b^2} \left[ \hat{a}_b^{(1)} - \frac{1}{2} \hat{a}_b^{(0)} \right].$$

The quantities  $\hat{a}_f^{(n)}$  can be extracted from eq. (8.4) using the definition in eqs. (7.1) and (7.2), and are given by

$$\begin{aligned} \hat{a}_f^{(0)} &= \frac{n_f}{32\pi} \frac{m_f^2}{m_W^4} g_{25}^4 \mathcal{A}_f^2, \\ \hat{a}_f^{(1)} &= \frac{n_f}{24\pi} \frac{m_D^2}{m_W^4} g_{25}^4 \mathcal{L}^2 \left[ \left(1 - \frac{1}{2}\xi_f\right) V_f^2 + (1 - \xi_f) \mathcal{A}_f^2 \right] - \frac{5}{2} \hat{a}_f^{(0)}. \end{aligned} \quad (8.8)$$

Also for comparison, the light line indicates the result obtained using the relativistic expansion (3.30) up to order  $x^{-1}$  for the non-resonant contributions to  $\sigma v_{\text{lab}}$ ,

$$\langle \sigma v_{\text{Mol}} \rangle = \sum_f \theta(m_{\tilde{D}} - m_f) \left[ a_f^{(0)} + \frac{3}{2} a_f^{(1)} x^{-1} \right] \quad (8.9)$$

(this is the prescription in ref. [2]), and the vertical lines show the naive expectation for a resonance, i.e. with no thermal distribution, eq. (6.15), as was done for example in ref. [18]. The coefficients  $a_f^{(n)}$  are obtained inverting eq. (7.10).

The effect of the proper treatment of the thermal distributions is evident.  $\tilde{D}$ -particles with mass just in correspondence with the first four  $T$ -resonances would have been claimed good dark matter candidates using the naive expression (6.15). On the contrary, resonant annihilation is thermally suppressed, and the relic abundance so obtained remains above the critical value. The correction factor is in agreement with eq. (6.18). Notice the asymmetry previously discussed in sect. 6.

The spurious infinite reduction in density at the  $b\bar{b}$  threshold ( $m_{\tilde{D}} = 5$  GeV) present in eq. (8.9) has been eliminated by the proper expansion of  $\sigma_{\text{th}} v_{\text{lab}}$  near a threshold, eq. (7.8). The densities at the two sides of the threshold smoothly join. The result is less prominent than in the case of resonances, and it could have consequences only when the opening of a new channel is the critical factor in reducing the relic density below the critical value. In such a case, the lower limit on the mass of a good dark matter candidate could change towards lower values. The exact amount of change depends of course on the strength of the new channel, but

we expect from the considerations at the end of sect. 7 that in general the lower limit would not decrease by more than 20%.

We would like to thank E.W. Kolb for discussions and for help in the numerical integration of the density evolution equation.

When this paper was almost completed, we received a preprint by G. Griest and S. Seckel [19], where the non-relativistic thermal average near a threshold and near the Z-pole are discussed, with results similar to ours.

### Appendix A

In the calculation of the annihilation cross section near a resonance, we need the partial decay rate of that resonance into two dark matter particles. We relate it to other measured partial decay rates into known channels, e.g. lepton pairs.

In the case of a (heavy) quarkonium we can use the naive quark model. The quarkonium, of mass  $m_R$ , total width  $\Gamma_R$  and spin  $J$ , is thought of as a quark-anti-quark non-relativistic bound state, described by the wave function  $\psi_{q\bar{q}}(r)$ . In this model, its branching fraction into a pair of particles,  $\chi^{(-)}\bar{\chi}^{(-)}$  say, can be obtained from the  $q\bar{q} \rightarrow \chi^{(-)}\bar{\chi}^{(-)}$  cross section  $\sigma(q\bar{q} \rightarrow \chi^{(-)}\bar{\chi}^{(-)})$  as

$$B_{\chi^{(-)}\bar{\chi}^{(-)}} = \frac{1}{\Gamma_R} \frac{4}{2J+1} |\psi_{q\bar{q}}(0)|^2 v_{q,\text{lab}} \sigma(q\bar{q} \rightarrow \chi^{(-)}\bar{\chi}^{(-)}), \quad (\text{A.1})$$

where the quarks have their constituent mass  $\bar{m}_q = m_R/2$ ,  $v_{q,\text{lab}}$  is the relative  $q\bar{q}$  velocity, and  $|\psi_{q\bar{q}}(0)|^2$  is the spatial probability density that  $q$  and  $\bar{q}$  meet at the same point.

From time-reversal invariance, we have the relation

$$\sigma(q\bar{q} \rightarrow \chi^{(-)}\bar{\chi}^{(-)}) = \frac{\beta_\chi^2}{\beta_q^2} \sigma(\chi^{(-)}\bar{\chi}^{(-)} \rightarrow q\bar{q}) \quad (\text{A.2})$$

where  $\beta_\chi$  and  $\beta_q$  are the center-of-mass velocities of  $\chi$  and  $q$  respectively. Substituting eq. (A.2) into eq. (A.1), and using the relation  $v_{q,\text{lab}} = 2\beta_q$  for non-relativistic quarks, we have

$$B_{\chi^{(-)}\bar{\chi}^{(-)}} = \frac{1}{\Gamma_R} \frac{4}{2J+1} |\psi_{q\bar{q}}(0)|^2 2\beta_\chi^2 \frac{\sigma(\chi^{(-)}\bar{\chi}^{(-)} \rightarrow q\bar{q})}{\beta_q}, \quad (\text{A.3})$$

Recalling now the definitions of  $\hat{a}(\epsilon)$ , eq. (7.1), and of  $b_R(\epsilon)$ , eq. (6.4), we obtain

$$b_R(\epsilon) = \frac{1}{\Gamma_R} \frac{4}{2J+1} |\psi_{q\bar{q}}(0)|^2 \frac{\hat{a}(\epsilon)}{(1+\epsilon)^{1/2}}. \quad (\text{A.4})$$

We have assumed here that  $B_{\chi^{(-)}\bar{\chi}^{(-)}} \ll 1$ .

We can estimate  $|\psi_{q\bar{q}}(0)|^2$  by means of the branching fraction into a known channel, e.g. into electron-positron pairs, which is given by

$$B_{e^+e^-} = \frac{1}{\Gamma_R} \frac{4}{2J+1} |\psi_{q\bar{q}}(0)|^2 \frac{e^4 e_q^2}{4\pi m_R^2}, \quad (\text{A.5})$$

with  $e_q$  the charge of the  $q$  quark in units of the positron charge  $e$ . We finally obtain

$$b_R(\epsilon) = B_{e^+e^-} \frac{4\pi m_R^2}{e^4 e_q^2} \frac{\hat{a}(\epsilon)}{(1+\epsilon)^{1/2}}, \quad (\text{A.6})$$

which can be expanded in  $\epsilon$  to obtain the coefficients  $b_R^{(l)}$  in terms of the  $\hat{a}^{(n)}$ .

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