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Math 104 Homework 2

3.6 a) since $(a+b) \in \mathbb{R}$, we know $|(a+b)+c| \le |a+b| + |c|$. (triangle inequality). applying it again to a and b, we have $|a+b| \le |a| + |b|$, and so replacing here gives $|a+b+c| \le |a| + |b| + |c|$, as desired. $\blacksquare$

b) base case: triangle inequality, $|a_1 + a_2| \le |a_1| + |a_2|$

inductive step: assume $|a_1 + a_2 + \ldots + a_n| \le |a_1| + |a_2| + \ldots + |a_n|$.

as with a), we apply the triangle inequality to $(a_1 + a_2 + \ldots + a_n)$ and $a_{n+1} \in \mathbb{R}$; $|a_1 + a_2 + \ldots + a_n + a_{n+1}| \le |a_1 + a_2 + \ldots + a_n| + |a_{n+1}|$. the inductive assumption + substitution shows $|a_1 + \ldots + a_{n+1}| \le |a_1| + \ldots + |a_{n+1}|$, as desired. $\blacksquare$

2. by cases: (1) $a<0, b<0$; (2) $a\geq 0, b<0$; (3) $a<0, b\geq 0$; (4) $a \geq 0, b \geq 0$.

(1) $||a|-|b|| = |-a+b|$. WLOG, set $a \leq b$. then $0 \leq -a+b$, and so $|-a+b| = -a+b$. now, let's manipulate $|a-b|$: since $-a+b \geq 0$, $a-b<0$, so $|a-b| = -a+b$. so $||a|-|b|| = |a-b|$ ✓

(2) secretly two cases: i) $a \geq -b$ and ii) $a < -b$:

first, $||a|-|b|| = |a+b|$.

i) since $a \geq -b$, $a+b \geq 0$ and $|a+b| = a+b$. also, $a-b \geq -2b \geq 0$, since $-2b>0$; so $|a-b|>0$ and $|a-b| = a-b$. so we want to show $a+b < a-b$. thus $b<-b$, and $2b<0$, which is true since $b<0$

ii) since $a<-b$, $a+b<0$ and $|a+b| = -(a+b)$; since $a \geq 0$, $a-b \geq -b > 0$, so $|a-b| = a-b$, and we want to compare $-(a+b)$ to $a-b$:

$-(a+b) = -a-b < -b < a-b$, and so $||a|-|b|| < |a-b|$ nee ✓

③ WLOG swap a + b — same as case ② ✓

④ $||a|-|b|| = |a-b|$, since $|a| = a$ and $|b| = b$, trivially ✓

therefore by cases, $||a|-|b|| \leq |a-b| \;\; \forall a,b \in \mathbb{R}$ ■

$(0,2)\cap(\frac{1}{2},\frac{3}{2})\cap(\frac{2}{3},\frac{4}{3})$

3. $\max A = 2$, no minimum, $\sup A = \max A = 2$, and $\inf A = 0$

no maximum, $\min B = \frac{-2+1}{2} = -\frac{1}{2}$. no supremum, $\inf B = \min B = -\frac{1}{2}$

$\max C = 4$, no minimum, $\sup C = 4$, $\inf C = 1$

no maximum, $\min D = 2$. no supremum, $\inf D = 2$

no maximum nor minimum, but $\sup E = \inf E = 1$

$\max F = \sup F = 1$, and $\min F = \inf F = -1$ $\blacksquare$

$\sup(A)+\sup(B)=\sup(A+B)$.

4 a. first, i'll show $\sup(A)+\sup(B)$ is an upper bound of $A+B$. so we must show that $\forall s\in A+B$, $s\leq \sup(A)+\sup(B)$. since $s\in A+B$, $s=a+b$, where $a\in A$ and $b\in B$. then $\forall a,b\in A,B$, $a+b\leq \sup(A)+\sup(B)$. since $\forall a$, $a\leq \sup(A)$, and $\forall b$, $b\leq \sup(B)$, $a+b\leq \sup(A)+\sup(B)$, and $\sup(A)+\sup(B)$ is an upper bound of $A+B$. Now to use the secret lemma:

$\sup A+\sup B=\sup(A+B)\iff \forall\varepsilon>0\ \exists s\in A+B$ s.t. $s>\sup A+\sup B-\varepsilon$.

by contradiction, assume $\exists\varepsilon>0\ \forall s\in A+B\ \ s\leq \sup A+\sup B-\varepsilon$.

since $s\in A+B$, $\exists a,b\in A,B$ s.t. $s=a+b$, so $a+b\leq \sup A+\sup B-\varepsilon$. but then $(a-\sup A)+(b-\sup B)\leq -\varepsilon$, or $\varepsilon\leq (a-\sup A)\mp(b-\sup B)$ $=\cancel{\sup A+\sup B-a-b}\,(\sup A-a)+(\sup B-b)$. by ~~definition of sup A~~ the secret lemma, $\nexists\varepsilon>0$ such that $\forall a\in A\ \ a<\sup A-\varepsilon$, so $\nexists\varepsilon>0$ s.t. $\forall a\in A\ \ \varepsilon<\sup A-a$.

similarly, $\nexists\varepsilon>0$ s.t. $\forall b\in B\ \ \varepsilon<\sup B-b$. in better words:

$\forall\varepsilon>0\ \exists a\in A$ s.t. $\varepsilon\geq \sup A-a$, and

$\forall\varepsilon>0\ \exists b\in B$ s.t. $\varepsilon\geq \sup B-b$.

so, for any epsilon, $2\varepsilon\geq(\sup A-a)+(\sup B-b)$, and since 2ε is arbitrarily small, $\forall\varepsilon\ \ \varepsilon\geq(\sup A-a)+(\sup B-b)$. but we derived $\varepsilon<(\sup A-a)+(\sup B-b)$, so contradiction, and $\sup A+\sup B$ <u>is</u> a supremum of $A+B$ ∎.

4b. in 4a, we showed $\sup A + \sup B = \sup(A+B)$.
Consider $-A$, $-B$, and $-(A+B)$, whose elements are defined as the negatives of A, B, and $A+B$, respectively.

Lemma 1: for any bounded sets S, $\inf S = -\sup(-S)$

proof: $\sup S$ is a number such that
$\forall \varepsilon > 0, \exists s \in S$ s.t. $s > \sup S - \varepsilon$. so $\sup(-S)$ is:
$\forall \varepsilon > 0, \exists -s \in -S$ s.t. $-s > \sup(-S) - \varepsilon$. remember; any $-s \in -S$ has a corresponding unique $+s \in S$. so:
$\forall \varepsilon > 0, \exists +s \in S$ s.t. $-s > \sup(-S) - \varepsilon$. multiplying by -1:
$\forall \varepsilon > 0, \exists +s \in +S$ s.t. $+s < -\sup(-S) + \varepsilon$. this is the definition of $\inf(S)$, and so $\inf(S) = -\sup(-S)$ ■

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so (using Lemma 1), $\sup -A = -\inf(+A)$, $\sup -B = -\inf(+B)$, and $\sup(-(A+B)) = -\inf(+(A+B))$. then,
$-\inf(+A) - \inf(+B) = -\inf(+(A+B))$, and multiplying by -1 gives $\inf(A) + \inf(B) = \inf(A+B)$ ■

5. first, i'll show $(1) \to (2)$:

assume that $\forall a,b \in \mathbb{R}$, $a<b$, $\exists r \in \mathbb{Q}$ s.t. $a<r<b$. ~~then~~

~~$a-r<0<b-r$, or just $a-r<0$. consider $\varepsilon>0$. clearly, $a-r>\varepsilon$ $\forall \varepsilon>0$ since $a-r<0$, $|a-r|=-a+r$~~

consider $b \in \mathbb{R} = a+\varepsilon$. by (1), $\exists r \in \mathbb{Q}$ s.t. $a<r<a+\varepsilon$, and so $a-r<0<(a-r)+\varepsilon$. by definition of absolute value (and since $a-r<0$), $|a-r| = -(a-r)$, so $-(a-r)>0>-(a-r)-\varepsilon$, so $|a-r|>\underbrace{0>|a-r|-\varepsilon}$ and implies that $|a-r|<\varepsilon$, as desired ■

now for $(2) \to (1)$:

assume $\forall a \in \mathbb{R}$, $\forall \varepsilon>0$, $\exists r \in \mathbb{Q}$ s.t. $|a-r|<\varepsilon$. ~~in fact,~~ WLOG limit $r \in \mathbb{Q}$ to be $\leq a$. if $r=a$, then $|a-r|=0$, which clearly $<\varepsilon$ $\forall \varepsilon>0$. and since $r<a$, $a-r>0$, so $|a-r| = a-r$. so $a-r<\varepsilon>0$. then $a<r+\varepsilon$ and $a-\varepsilon<r<b$. if $r<0$, let $b=-r$, and if $r \geq 0$, let $b=2r+1$. then clearly the inequality holds, and $\forall r \in \mathbb{Q}$ $-r$ and $2r+1 \in \mathbb{R}$. since $\forall a \in \mathbb{R}$ $\forall \varepsilon>0$, $a-\varepsilon \in \mathbb{R}$, we've shown any two reals have a rational between them; $(2) \to (1)$.

since $1 \to 2$ and $2 \to 1$, $1 \Leftrightarrow 2$ ~~and~~ ■

6.1: first, we must show $(\sup A)\cdot(\sup B)$ is an upper bound of $A\cdot B$ — that is, $\forall a\cdot b \in A\cdot B, (\sup A)\cdot(\sup B) \geq a\cdot b$, or $\left(\frac{\sup A}{a}\right)\cdot\left(\frac{\sup B}{b}\right) \geq 1$. since $\sup A \geq a$ and $\sup B \geq b$, we know $\left(\frac{\sup A}{a}\right) \geq 1$ and $\left(\frac{\sup B}{b}\right) \geq 1$. $\therefore \left(\frac{\sup A}{a}\right)\cdot\left(\frac{\sup B}{b}\right) \geq 1\cdot 1 = 1$, and $\sup A \cdot \sup B$ is an upper bound of $A\cdot B$. Secret Lemma time:

$\sup A \sup B = \sup(A\cdot B) \iff \forall \varepsilon > 0\ \exists a\cdot b \in A\cdot B$ s.t. $a\cdot b > \sup A \cdot \sup B - \varepsilon$.

dividing both sides by $\sup A \cdot \sup B$, we get:

$\left(\frac{a}{\sup A}\right)\cdot\left(\frac{b}{\sup B}\right) > \cancel{1} \frac{-\varepsilon}{\sup A \sup B}$. since $A + B$ are strictly positive, $\sup A > 0$ and $\sup B > 0$, so $\sup A \sup B > 0$ and we can rename $\frac{-\varepsilon}{\sup A \sup B}$ to another $-c$, with $c > 0$. since $\forall a, b \in A, B$, $a > 0$ and $b > 0$ (and $\sup A > 0$ and $\sup B > 0$), $\left(\frac{a}{\sup A}\right) > 0$ and $\left(\frac{b}{\sup B}\right) > 0$, so $\left(\frac{a}{\sup A}\right)\cdot\left(\frac{b}{\sup B}\right) > 0$, ~~sin~~ and since $c > 0$, $-c < 0$, and $\forall c > 0$ $\left(\frac{a}{\sup A}\right)\cdot\left(\frac{b}{\sup B}\right) > -c$ holds; we've shown $\sup A \cdot \sup B = \sup(A\cdot B)$ $\blacksquare$

2. No — consider $A = \{-1, -2\}$ and $B = \{1, 2\}$. then $A\cdot B = \{-4, -2, -1\}$ and $\sup(A\cdot B) = -1$. but $\sup A = -1$ and $\sup B = 2$, so $\sup A \cdot \sup B = -2 \neq \sup(A\cdot B)$ $\blacksquare$