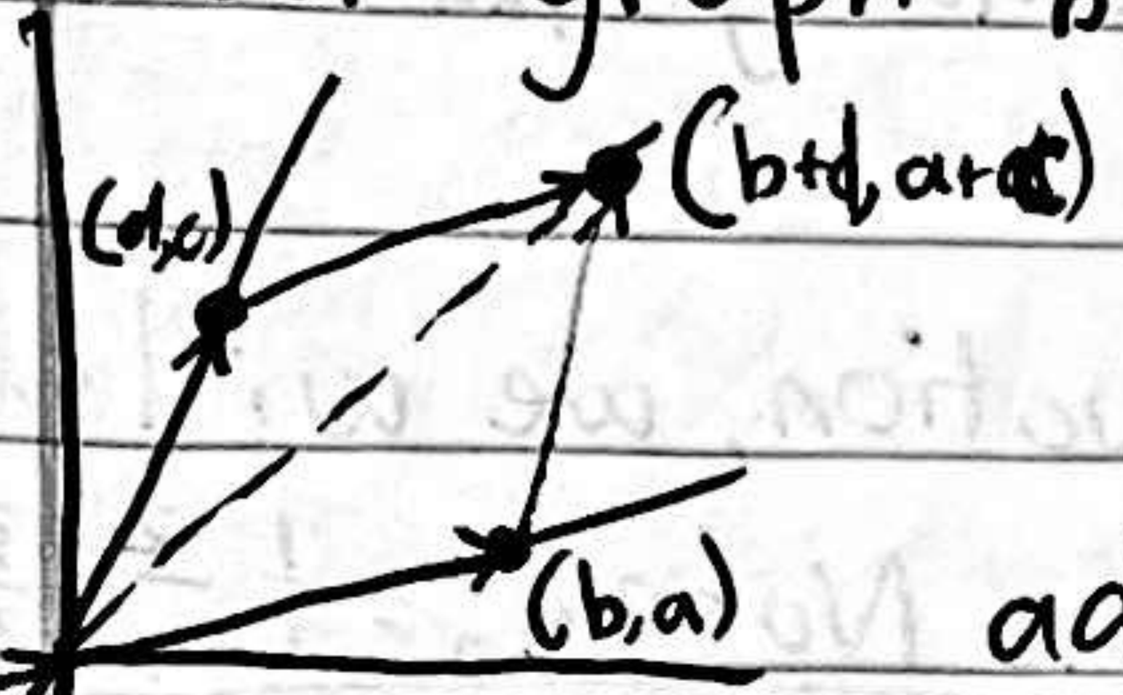
1. Lemma 1: $\forall a,b,c,d \in \mathbb{N}$, $\frac{a}{b} < \frac{c}{d} \longrightarrow \frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$ "drunk fractions"

Proof: graph $\frac{a}{b}$ and $\frac{c}{d}$ as coordinates (b,a) and (d,c):



since slope is defined $\frac{a-0}{b-0}=0$, "being smaller" here is represented by having a smaller slope.

adding the vectors corresponds to "drunk addition," which clearly results in a slope between the two originals.

so $\frac{a}{b} < \frac{c}{d} \longrightarrow \frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$, as desired ∎

Now, consider $q_n = \frac{F_n}{F_{n+1}}$. by fibonacci, this $= \frac{F_{n-1}+F_{n-2}}{F_{n-2}+F_{n-3}}$, which would be the result of "drunk adding" q_{n-1} and q_{n-2}.

so: $q_n < q_{n-1} \longrightarrow q_n < q_{n+1} < q_{n-1}$, and

$q_{n-1} < q_n \longrightarrow q_{n-1} < q_{n+1} < q_n$. by induction, we will show $q_n < q_{n-1}$ $\forall n$ odd, and $q_{n-1} < q_n$ $\forall n$ even. Note $q_n = \frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \dots$

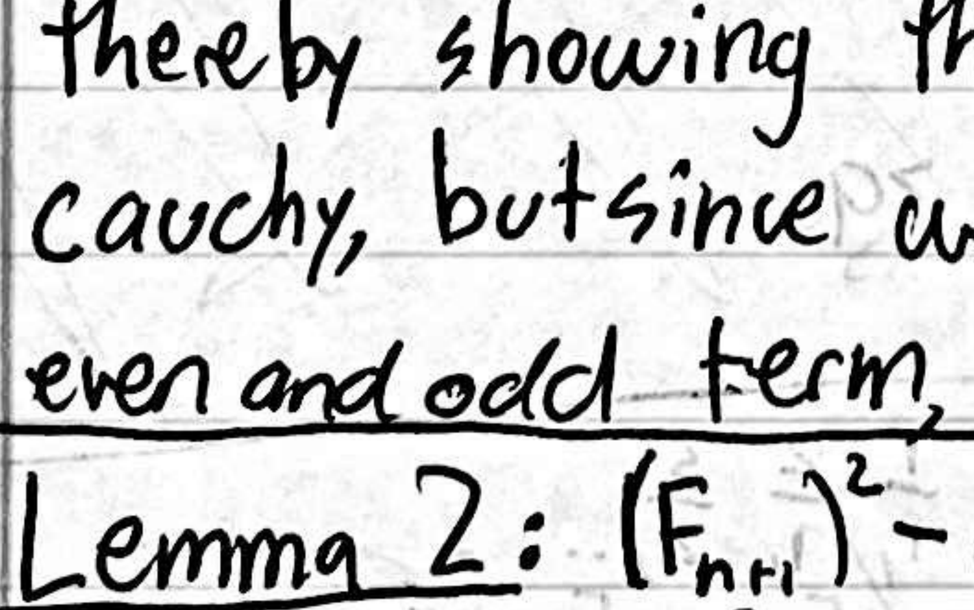
→ odd base case: $q_3 < q_2$, since $\frac{3}{2} < \frac{2}{1}$.

inductive step: assume $q_n < q_{n-1}$. then $q_n < q_{n+1}$, and $q_n < q_{n+2} < q_{n+1}$. so $q_{n+2} < q_{n+1}$ — since n odd $\rightarrow n+2$ odd, this holds ■

→ even base case: $q_1 < q_2$, since $1 < 2$.

inductive step: assume $q_{n-1} < q_n$. then $q_{n+1} < q_n$, and so $q_{n+1} < q_{n+2}$. since n even $\rightarrow n+2$ even, holds ■

These together mean q_n odd is monotonically ~~decreasing~~ increasing, and q_n even is monotonically ~~increasing~~ decreasing, like:



So clearly, $\sup\{s_n : n > N\}$ is the next even term, and Inf. the next odd. To prove the limit exists, I will show $\lim(\sup\{s_n : n>N\} - \inf\{s_n : n>N\}) = 0$, thereby showing that $\overline{\lim}(s_n) = \underline{\lim}(s_n)$. This is basically cauchy, but since we know sup + inf $\{s_n : n>N\}$ are the next even and odd term, it will suffice to show $|q_{N+1} - q_{N+2}| < \varepsilon$.

Lemma 2: $(F_{n+1})^2 - (F_n)(F_{n+2}) = 1$ for all even n.

base case: $(F_1)^2 - (F_0)(F_2) = 1^2 - 0 \cdot 1 = 1$ ✓

inductive step: suppose $(F_{n+1})^2 - (F_n)(F_{n+2}) = 1$.

Then $(F_{n+1}^2) + F_{n+1}F_{n+2} - F_nF_{n+2} - F_{n+1}F_{n+2} = 1$, so

$F_{n+1}F_{n+3} - F_{n+2}^2 = 1$, so $F_{n+1}F_{n+3} + F_{n+2}F_{n+3} - F_{n+2}^2 + F_{n+2}F_{n+3} = 1$

so $F_{n+3}(F_{n+1} + F_{n+2}) - F_{n+2}(F_{n+2} + F_{n+3}) = 1$, and

$F_{n+3}^2 - F_{n+2}F_{n+4} = 1$, as desired ■

remember, we wish to show $\forall \varepsilon > 0\ \exists N$ s.t. $|q_{N+1} - q_{N+2}| < \varepsilon$.
to simplify, consider only even N, so we wish to show:
$\left|\frac{F_{N+1}}{F_N} - \frac{F_{N+2}}{F_{N+1}}\right| < \varepsilon$, or $\left|\frac{F_{N+1}^2 - F_N F_{N+2}}{F_N F_{N+1}}\right| < \varepsilon$. That numerator look familiar? since N is even, we can apply Lemma 2 to get $\left|\frac{1}{F_N F_{N+1}}\right| < \varepsilon$. $\forall N$, $\left|\frac{1}{F_N F_{N+1}}\right| \in a_n$, where $a_n = \frac{1}{n}$. We've shown in class this converges to 0, and since $\frac{1}{F_N F_{N+1}}$ is a subsequence of a_n, it also converges.
Do the steps backward to get $\overline{\lim}(q_n) = \underline{\lim}(q_n)$, so q_n converges.
(and bounded by $[1, 2]$. i have to say this, otherwise $\underline{\lim}(q_n) = \overline{\lim}(q_n) = \lim(q_n)$ could be $\pm\infty$).

Step 2: compute the limit

i claim $\lim q_n = \varphi$; that is, $\forall \varepsilon > 0, \exists N$ s.t. $n > N \rightarrow |q_n - \varphi| < \varepsilon$.
Consider $N = \log_3\left(\frac{1}{\varepsilon}\right) + 1$. then $n > N$ means $n - 1 > \log_3\left(\frac{1}{\varepsilon}\right)$, or $3^{n-1} > \frac{1}{\varepsilon}$, or $\frac{1}{3^{n-1}} < \varepsilon$. $\frac{1}{3^{n-1}} > \frac{1}{3^{n-1}(1-\frac{1}{3})} > \frac{1}{3^{n-1} - 3^{n-2}} > \frac{1}{3^{n-1} - 1} > \left|\frac{(\sqrt{5}-1)^{n-1}}{(\sqrt{5}+1)^{n-1} - (\sqrt{5}-1)^{n-1}}\right| > \left|\frac{\alpha^{n-1}}{\varphi^{n-1} - \alpha^{n-1}}\right|$
$> \left|\frac{-\alpha^{n-1}(\alpha - \varphi)}{\varphi^{n-1} - \alpha^{n-1}}\right| = \left|\frac{\varphi^n - \alpha^n}{\varphi^{n-1} - \alpha^{n-1}} - \frac{\varphi(\varphi^{n-1} - \alpha^{n-1})}{\varphi^{n-1} - \alpha^{n-1}}\right| = |q_n - \varphi|$, which thus $< \varepsilon$. (where $\alpha = 1 - \varphi$)
So $\lim(q_n) = \varphi$, as desired ∎

btw... this question took me ~5 hours, even with going to Vinh Kha's OH

Raphaël Bajet 9/21/24

Math 104 HW4

2.1: Since $(s_n) \to +\infty$, $\forall M \exists N$ s.t. $\forall n > N$, $s_n > M$. consider $M = \frac{M_1}{k}$ $\forall M_1$. then $s_n > \frac{M_1}{k}$, and since $k > 0$, $\forall M_1 \exists N$ s.t. $\forall n > N$, $ks_n > M_1$, and $(ks_n) \to +\infty$ ■

2: as above, but $M = \frac{M_2}{k}$ $\forall M_2$. Then $s_n > \frac{M_2}{k}$, and since $k < 0$, $ks_n < M_2$. as this defines divergence to $-\infty$, $(ks_n) \to -\infty$ ■

3: since t_n is bounded below, $\exists m$ s.t. $\forall n$, $m < t_n$. using the notation above, $s_n > M$. so, for the same N, $\forall M \exists N$ s.t. $\forall n > N$, $s_n + t_n > M + m = M_1$. ~~therefore $s_n + t_n$ diverges to $+\infty$, since $M > 0$ and m~~ if $m > 0$, then $M > 0 \wedge m > 0 \to M + m = M_1 > 0$, and $s_n + t_n$ diverges to $+\infty$. if $m < 0$, we still need to construct an $M_1 > 0$. we must simply choose an $M > -m$, which we can because "$\forall M > 0$" and $m < 0$. so if $M = -m + 1$, $M + m = (-m+1) + m = 1 > 0$, and thus $s_n + t_n$ diverges to $+\infty$ ■

3a) consider a such that $L<a<1$. We know that:
$\forall \varepsilon>0, \exists N$ s.t. $\forall n>N$, $\left|\left|\frac{s_{n+1}}{s_n}\right|-L\right|<\varepsilon$, so with $\varepsilon=a$, we have $\left|\frac{s_{n+1}}{s_n}\right|<\varepsilon+a$, and for this N, $|s_{n+1}|<a|s_n|$, for $n\geq N$. by induction, i will show that $|s_{n+1}|<a|s_n|$ implies $|s_n|<a^{n-N}|s_N|$:

base case: $n=N+1$; $|s_{N+1}|<a|s_N|,=a^{N+1-N}|s_N|$

inductivestep: if $|s_n|<a^{n-N}|s_N|$, $|s_{n+1}|<a|s_n|<a^{n-N+1}|s_N|$, so $|s_{n+1}|<a^{(n+1)-N}|s_N|$.

thus $n>N$ implies $|s_n|<a^{n-N}|s_N|$. Since $s_n\neq 0$, consider the inequality $0<|s_n|<a^{n-N}|s_N|$; squeezable! clearly $\lim 0=0$, so we simply have to show that $\lim(a^{n-N}|s_N|)=0$. since $|s_N|$ is a constant, by limit theorem this $=|s_N|\cdot\lim(a^{n-N})$ ~~$= \lim a^n \cdot a^N$~~

$=a^N|s_N|\lim a^n$, since a^N is another constant. consider $N=\log_a(\varepsilon)$;
$n>N$ implies $n>\log_a(\varepsilon)\rightarrow a^n<\varepsilon$ (b/c $0<a<1$), so $\lim a^n=0$ ∎

(and by blah blah blah, $\lim s_n=0$)

$L=$

b) Consider $t_n = \frac{1}{|s_n|}$: part a) tells us that $\lim\left|\frac{t_{n+1}}{t_n}\right| < 1 \to L = 0$.
$L = \lim\left|\frac{t_{n+1}}{t_n}\right| = \lim\left|\frac{\frac{1}{|s_{n+1}|}}{\frac{1}{|s_n|}}\right| = \lim\left|\frac{s_n}{s_{n+1}}\right|$. Note that:
$\lim\left|\frac{s_{n+1}}{s_n}\right| < 1 \iff \frac{1}{\lim\left|\frac{s_{n+1}}{s_n}\right|} > 1 \iff \lim\left|\frac{1}{\frac{s_{n+1}}{s_n}}\right| > 1 \iff \lim\left|\frac{s_n}{s_{n+1}}\right| > 1$, by ross limit theorem. Therefore, $\lim\left|\frac{t_{n+1}}{t_n}\right| > 1 \to \lim \frac{1}{\left|\frac{t_{n+1}}{t_n}\right|} < 1 \xrightarrow{(a)} \lim\left|\frac{1}{\frac{t_{n+1}}{t_n}}\right| = 0$.
Therefore, by ross theorem 9.10 (and that $|x| \geq 0 \ \forall x$), $L = +\infty$. ■

$(1, 2, \frac{1}{3}, 4, \frac{1}{5}, 6, \frac{1}{7}) + (1, \frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}, -1, -\frac{\sqrt{2}}{2}, \ldots)$

4.1: Consider (s_{n_k}) for $n_k = 3+4k$ for $k=1,2,\ldots$. the $n^{(-1)^n}$ term is $\frac{1}{3}, \frac{1}{7}, \ldots (\to 0)$ and the $\sin(\frac{(n+1)\pi}{4})$ term is always 0, so sum $\to 0$ by limit theorem.

2: consider $n_k = 1+8k$. The $n^{(-1)^n}$ term is $1, \frac{1}{9}, \frac{1}{17}, \ldots \to 0$, and $\sin(\frac{(n+1)\pi}{4})$ is always 1. By limit theorem, $0+1=1$.

3: consider $n_k = 5+8k$. The $n^{(-1)^n}$ term is $\frac{1}{5}, \frac{1}{13}, \ldots \to 0$, and the $\sin(\frac{(n+1)\pi}{4})$ term is always -1. By limit theorem, $0+(-1)=-1$.

4: consider $n_k = 2k$. Since $\sin x \in [-1,1]\ \forall x$, $-2 < \sin x$. so consider $n^{(-1)^n} - 2$ (clearly $< n^{(-1)^n} - \sin(\frac{(n+1)\pi}{4})$). Since $n^{(-1)^n}$ is $2, 4, 6, \ldots \to +\infty$, $n^{(-1)^n} - 2$ is $0, 2, 4, \ldots \to +\infty$ in the same way. since this sequence $\leq$ the one we care about, the real one is also $\to +\infty$

5. we wish to show that $\forall_\varepsilon > 0, \exists N$ s.t. $\forall n,m > N, |a_m - a_n| < \varepsilon$. WLOG, consider $m > n$; $|a_m - a_n| = |a_m - a_{m-1} + a_{m-1} + a_{m-2} - \dots - a_{n+1} + a_{n+1} - a_n|$, or $|a_m - a_n| = \left|\sum_{k=n+1}^{m} a_k - a_{k-1}\right| \le \sum_{k=n+1}^{m} |a_k - a_{k-1}|$, by Δ inequality. since $|x| \ge 0$,

m, N (these bounds stay)

this $\le \sum_{k=N}^{m} |a_k - a_{k-1}|$. Since $|a_{n+1} - a_n| \le C|a_n - a_{n-1}|$, and $0 < C < 1$, $|a_{n+1} - a_n| < |a_n - a_{n-1}|$, so the largest difference between terms is $|a_2 - a_1|$.

thus $\sum_{k=N}^{m} |a_k - a_{k-1}|$

thus $\sum_{k=N}^{m} |a_k - a_{k-1}| \le \sum_{k=N}^{m} |a_2-a_1|\cdot C^k$ (because $|a_3-a_2| \le C\cdot|a_2-a_1|$, and in this way $|a_4-a_3| \le C\cdot|a_3-a_2| \le C^2|a_2-a_1|$, and so on), whish is $\le \sum_{k=N}^{m} |a_2-a_1|\cdot C^k = |a_2-a_1|\sum_{k=N}^{m} C^k$. Remember, we wish to show this $<\varepsilon$; because $|a_2-a_1|$ is constant, it suffices to show $\exists N$ s.t. $\sum_{k=N}^{m} C^k < \varepsilon$. The hint tells us:

$$\underbrace{1+C+C^2+\dots+C^{N-2}+C^{N-1}}_{=\frac{1-C^{N-1}}{1-C}} + \underbrace{C^N+C^{N+1}+\dots+C^{m-1}+C^m}_{\sum_{k=N}^{m} C^k} = \frac{1-C^{m+1}}{1-C}$$

and so we wish to show $\forall \varepsilon>0, \exists N$ s.t. $\forall m>N, \left(\frac{1-C^{m+1}}{1-C} - \frac{1-C^{N-1}}{1-C}\right) < \varepsilon$.

Consider $N = 2+\log_c(\varepsilon(1-c)+C^{m+1})$. clearly $N > 1+\log_c(\varepsilon(1-C)+C^{m+1})$, so $N-1 > \log_c(\varepsilon(1-c)+C^{m+1})$, and since $0<c<1$, $C^{N-1} < C^{\log_c(\varepsilon(1-c)+C^{m+1})} = \varepsilon(1-C)+C^{m-1}$. so $C^{N-1}-C^{m-1} < \varepsilon(1-C)$, and $\frac{1-C^{m-1}-1+C^{N-1}}{1-C} < \varepsilon$, so $\frac{1-C^{m+1}}{1-C} - \frac{1-C^{N-1}}{1-C} < \varepsilon$, as we wanted. by overcomplification, we've show (a_n) is Cauchy, and so (a_n) converges

6.1: First, i'll show the ~~so~~ sequence converges. Since $0 \le \sin x \le x$ whenever $x \in [0,1]$, $\sin a_1 \le a_1$, and so $a_2 \le a_1$. Since $0 \le a_2$, the result will always stay in the range $[0,1]$, and we can apply this indefinitely to find $a_1 \ge a_2 \ge a_3 \dots$. So, a_n is monotonically decreasing. also, since $\sin x \in [-1,1]\ \forall x$, ~~the~~ a_n is bounded below by $\min(-2, a_1)$. Thus, a_n converges.

Now that we know a_n converges, let $L = \lim a_n$. Since $a_{n+1} = \sin(a_n)$, $\lim(a_{n+1}) = L = \lim(\sin(a_n)) = \sin(L)$, by the hint. Thus we want to solve $L = \sin L$ for $L \in [-1,1]$, and the only such candidate is 0. therefore if $x \in [0,1]$, $\lim a_n = 0$ ∎

2: First, i'll show s_n converges. Remember $\sin(x)$ is odd – this means $\sin(-a) = -\sin(a)\ \forall a$. also, if $x \in [-1,0]$, then $-x \in [0,1]$. as for a), assert that $0 \le \sin(-x) \le -x$. ~~then $\sin(-x) \le$~~

then $0 \le -\sin(x) \le -x$, so $0 \le -a_2 \le -x$, so $a_2 \ge x$ and $a_1 \in [-1,0]$. keep applying this to get $a_1 \le a_2 \le a_3 \le \dots$, so a_n is monotonically increasing. also, since $a_i \in [-1,0]\ \forall i \in \mathbb{N}$, a_n is bounded above by 1 – therefore it converges.

Since the limit exists, let $L = \lim a_n$. as for a), we can write $L = \sin(L)$, and again we're looking in the interval $[-1,1]$, for which $L = 0$ is the only answer ∎

3: <u>case 1:</u> $\sin(x) < 0$. then $a_2 \in [-1,0]$, and if we consider it the x of a), $\lim a_n = 0$.

<u>case 2:</u> $\sin(x) = 0$. $\overbrace{\sin(\sin(\sin\dots(\sin(0))\dots)}^{n} = 0\ \forall n \in \mathbb{N}$, so $\lim = 0$ (constant sequence)

<u>case 3:</u> ~~cos~~ $\sin(x) > 0$. case 1, but for b) instead of a).

~~Since~~ $\sin(x) \in [-1,1]\ \forall x \in \mathbb{R}$, these cases are sufficient ∎