Raphaël Bajet

Math 104 HW 1

1. induction → base case: $1^3=1$, and $1^2=1$; indeed, $1=1$

inductive step: assume $1^3+2^3+\ldots+n^3=(1+2+\ldots+n)^2$.

ACTUALLY, Lemma 1: $1+2+\ldots+n=\frac{n(n+1)}{2}$

proof: base case $n=1$: $\frac{1(2)}{2}=1=1$ ✓

inductive step: assume $1+2+\ldots+n=\frac{n(n+1)}{2}$. adding $n+1$ gives $1+2+\ldots+n+(n+1)=\frac{n(n+1)}{2}+(n+1)=\frac{n(n+1)+2(n+1)}{2}=\frac{(n+1)(n+2)}{2}$, and we've shown $P(n)\to P(n+1)$. by induction, $1+2+\ldots+n=\frac{n(n+1)}{2}$ ■

using lemma 1, we can restate the problem to be proving that $1^3+2^3+\ldots+n^3=\left(\frac{n(n+1)}{2}\right)^2=\frac{n^2(n+1)^2}{4}=\frac{n^2(n^2+2n+1)}{4}$
$=\frac{n^4+2n^3+n^2}{4}$. we've already shown the base case, now for the inductive step: if $1^3+2^3+\ldots+n^3=\frac{n^4+2n^3+n^2}{4}$ then
$1^3+2^3+\ldots+n^3+(n+1)^3=\frac{n^4+2n^3+n^2}{4}+(n+1)^3=\frac{n^4+2n^3+n^2+4n^3+12n^2+12n+4}{4}$
$=\frac{n^4+6n^3+13n^2+12n+4}{4}$. so we must show $n^4+6n^3+13n^2+12n+4$
$=(n+1)^2(n+2)^2$.

$(n+1)^2(n+2)^2=(n^2+2n+1)(n^2+4n+4)=n^4+4n^3+4n^2+2n^3+8n^2+8n+n^2+4n+4$
$=n^4+6n^3+13n^2+12n+4$, as desired. ■

2a. base case: $n=2$. $4>3$, as desired.

inductive step: if $n^2>n+1$, then $n^2+2n+1>n+1+2n+1$
$\to(n+1)^2>(n+1)+1+2n>(n+1)+1$, since $2n>0$ $(n\geq 2)$ ■

b) base case: $n=4$. $4!=120>4^2=16$, as desired.

inductive step: if $n!>n^2$, $n!(n+1)>n^2(n+1)$
$\to(n+1)!>n^3+n^2$. we ~~want~~ want $(n+1)!>(n+1)^2=n^2+2n+1$, so we must show that $n^3+\cancel{n^2}>\cancel{n^2}+2n+1$ for $n\geq 4$:

base case: $n=4$, $64>9$, as desired. inductive step: if $n^3>2n+1$, $n^3+3n^2+3n+1>3n^2+3n+1+2n+1$, and $(n+1)^3>3n^2+5n+2$
$>5n+2=4n+2+n>4n+2+1$ (since $n\geq 4$) $=2(2n+1)+1>2(n+1)+1$. ■

~~biggggg pencil~~

3. Let P_n be the proposition that $F_n = f(n)$ and $F_{n-1} = f(n-1)$.
base case P_1: $F_0 = 0$ and $F_1 = 1$ (given).
$f(0) = \frac{1}{\sqrt{5}}(1-1) = 0$, and $f(1) = \frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}+1}{2} - \left(1-\left(\frac{\sqrt{5}+1}{2}\right)\right)\right)$
$= \frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}+1-(2-\sqrt{5}-1)}{2}\right) = \frac{2\sqrt{5}+\cancel{2-2}}{2\sqrt{5}} = 1$. Thus P_1 is true.
inductive step: assuming P_n (for $n \geq 1$), show P_{n+1} is true.
if $F_n = \frac{1}{\sqrt{5}}(\varphi^n - (1-\varphi)^n)$ and $F_{n-1} = \frac{1}{\sqrt{5}}(\varphi^{n-1} - (1-\varphi)^{n-1})$, then
$F_n + F_{n-1} = F_{n+1}$ (def. of sequence) $= \frac{1}{\sqrt{5}}(\varphi^n + \varphi^{n-1} - (1-\varphi)^n - (1-\varphi)^{n-1})$
$= \frac{1}{\sqrt{5}}\left(\varphi^{n-1}(\varphi+1) - (\varphi-1)^{n-1}\left(\left(\frac{1-\varphi}{}\right)+1\right)\right)$.
For convenience, i will rename $1-\varphi$ to be α. We are given that φ solves $x^2 = x+1$. As it turns out, α does too:
$(1-\varphi)^2 = \varphi^2 - 2\varphi + 1 = (1-\varphi+1) \Rightarrow \varphi^2 - \varphi - 1 = 0$, and $\varphi^2 = \varphi+1$, so
$(\varphi+1) - \varphi - 1 = 0 = 0$, and $\alpha^2 = \alpha + 1$, too. Continuing above:
$F_{n+1} = \frac{1}{\sqrt{5}}(\varphi^{n-1}(\varphi^2) - \alpha^{n-1}(\alpha^2)) = \frac{1}{\sqrt{5}}(\varphi^{n+1} - (1-\varphi)^{n+1})$. by induction, the proof that $F_n = f(n)$ for $n \in \mathbb{N}$ (and 0) is complete ∎

4. Using the same P_n as problem 3, but with G_n instead of $f(n)$ base case:
if you can go up 1 or 2 stairs at a time, there ~~are no~~ is one ways of going up 0 stairs, and 1 way of going up 1. (trivial exhaustion). indeed, $F_1 = 0$ and $F_2 = 1$.
inductive step: assume $G_{n-1} = F_n$ and $G_n = F_{n+1}$. To get up $n+1$ stairs, you must either go up $n-1$ stairs and then go up 2, or up n steps then go up one. Since you can take only 1 or 2 steps, these are the only options, so $G_{n+1} = G_n + G_{n-1}$. Using the inductive assumption, $G_{n+1} = F_{n+1} + F_n = F_{n+2}$ by definition of Fibonacci sequence, and by induction $G_n = F_{n+1}$ for all $n \in \{0\} \cup \mathbb{N}$ ∎

$\sqrt{3} < 5$

5 let $x = \sqrt[3]{5-\sqrt{3}}$; $x^3 = 5-\sqrt{3}$, $x^3 - 5 = -\sqrt{3}$
and $(x^3-5)^2 = 3$, $x^6 - 10x^3 + 22 = 0$. by the
rational zeroes theorem, if x is rational it
can only be $= \pm 1, \pm 11, \pm 2, \pm 22$.
$5 > 5-\sqrt{3} > \sqrt[3]{5-\sqrt{3}} = x$, so 11 and 22 are eliminated.
Since $9 < 25$, $\sqrt{3} < 5$. and $0 < 5-\sqrt{3} < \sqrt[3]{5-\sqrt{3}} = x$,
leaving only $x=1$ and $x=2$. by exhaustion:
$(1) - 10(1) + 22 = 1 - 10 + 22 = 13 \neq 0$
$64 - 80 + 22 = 86 - 80 = 6 \neq 0$.
Since no rational possibilities hold, x is irrational. ∎

6 let $x = 30^{\frac{n}{m}}$. then $x^m = 30^n$.

Lemma ①: if $m \neq n$ and $m, n \in \mathbb{N}$, and $m > n$, $30^{\frac{n}{m}}$ is irrational.
proof: by contradiction, assume x is
rational, in which case it can be written as $\frac{a}{b}$ in
lowest terms. then $(\frac{a}{b})^m = 30^n$, and $a^m = b^m \cdot 30^n$.
therefore a^m is even $(30 = 2 \cdot 15)$, and so a is also
even. we'll write $a = 2c$, where $c \in \mathbb{N}^+$, and

$$b^m = \frac{2^m \cdot c^m}{30^n} = \frac{2^m}{2^n} \cdot \frac{c^m}{15^n} = 2^{m-n} \cdot \frac{c^m}{15^n}.$$

since $m > n$, $m - n > 0$, and
b^m is also even; b is even. but a and b cannot
both be even, since lowest terms, so here $30^{m/n} \notin \mathbb{Q}$ ∎
We've shown it's true when $m > n$. $m \neq n$, since $m \neq n$,
so we have to show for $m < n$:
Since $n > m$, write $n = xm + y$, where $x, y \in \mathbb{N}$ and $y < m$
then $30^{n/m} = 30^{\frac{xm+y}{m}} = 30^x \cdot 30^{y/m}$. 30^x is rational, and since
$y < m$, we know $30^{y/m}$ is irrational! (lemma ①)
$30^{n/m}$ is also irrational here. ∎

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Lemma 2: rational · irrational = irrational

rational can be written in lowest form as $\frac{a}{b}$, w/ $a, b \in \mathbb{Z}$

Then rational · irrational $\lambda = \frac{a}{b} \cdot \lambda = \frac{a\lambda}{b}$. so we want to show multiplication by an integer = irrational.

by contradiction: if $c \cdot \lambda = \frac{x}{y}$ (with $c, x, y \in \mathbb{Z}$), then $\lambda = \frac{x}{cy}$, which is impossible since λ is irrational.

similarly for division: $\frac{\lambda}{c} = \frac{x}{y} \rightarrow \lambda = \frac{cx}{y}$, impossible.

so, back to $\frac{a\lambda}{b}$, $a\lambda$ is irrational α, and $\frac{\alpha}{b}$ is also irrational. therefore, rational · irrational = irrational ☒

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so we've shown if $n > m$, we can rewrite $30^{n/m}$ to $30^{x} \cdot 30^{y/m}$, where 30^{x} is rational and $30^{y/m}$ is not, by Lemma ①. By Lemma ②, this result is irrational.

Since we showed the result for $m > n$ and $m < n$ (and $m \neq n$ since $m \nmid n$) the proof is complete ☒

7(1). <u>First i'll show NS' → NS:</u>

Set $S \subseteq \mathbb{N}$; $1 \in S$; $n \in S \to n+1 \in S$. by contradiction, assume $S \neq \mathbb{N}$. then the complement of $\overline{S}$ w/respect to $\mathbb{N} \neq \emptyset$, and by NS' this complement $\overline{S}$ contains a least element. the contrapositive of $n \in S \to n+1 \in S$ is $n+1 \notin S \to n \notin S$, or $n+1 \in \overline{S} \to n \in \overline{S}$. therefore this smallest element of $\overline{S}$ cannot exist, since for any $n \in \overline{S}$, there is a smaller $n-1 \in \overline{S}$. so our assumption that $S \neq \mathbb{N}$ is false, and NS' → NS.

so $\overline{S} = \emptyset$, contradicting this

<u>Next, that NS → NS':</u>

if $(S \subseteq \mathbb{N} \wedge 1 \in S \wedge n \in S \to n+1 \in S) \to S = \mathbb{N}$, then we want to show that ① every nonempty subset has a least element and ② every $1 \neq n \in \mathbb{N}$ is the successor of another number in $\mathbb{N}$.

② we know that $n \in \mathbb{N} \to n+1 \in \mathbb{N}$. by contradiction, assume $\exists w \in \mathbb{N}$ s.t. w is not the succesor of another number in $\mathbb{N}$, and $w \neq 1$. That is, $w \in \mathbb{N}$ and $w-1 \notin \mathbb{N}$. i can construct a set S without w that is then $\mathbb{N}$. I start with $S = \{1\}$, and using $n \in S \to n+1 \in S$, build up the set as i go, adding each new successor: $S = \{1, 1+1, 1+1+1, \ldots\}$. since $w-1 \notin \mathbb{N}$, w will never show up, and thus i have constructed a set $S = \mathbb{N}$ ~~withou~~ where $w \notin S$. ↯. $w \notin \mathbb{N}$.

① by contradiction, assume i <u>CAN</u> make such a subset with no smallest element. The set S i made for ② contains <u>only</u> elements that are ordered — the bigger number is the one with more "+1"s, and no two different elements have a different # of "+1"s. Since $S = \mathbb{N}$, i know this property is thus also true of $\mathbb{N}$.

7(2). Consider the set $S = \{1, 1+1, 1+1+1, \ldots\} \cup \{\omega, \omega+1, \omega+1+1, \ldots\}$. clearly, $\{1, \omega\}$ is a subset of S, but since they are incomparable, there is no "smallest number" between 1 and ω. therefore, $\{1, \omega\} \subseteq S$ s.t. there is no smallest element $\boxtimes$