9/18/24

Math 104 HW3

1. Since S is not bounded above, $\nexists M$ s.t. $M > s$ $\forall s \in S$. now consider any product $s \cdot t \in S \cdot T$. since $s>0$ and $t>0$ $\forall s,t \in S,T$, $\frac{1}{s} > 0$, too. The archimedian property thus tells us that $\exists n$ s.t. $n(\frac{1}{s}) > t$, so $n > s \cdot t$.

1. i) $\underline{\sup(S \cdot T) = +\infty}$

Since S is not bounded above, $\nexists M$ s.t. $M > s$ $\forall s \in S$. Since $s>0$ and $t>0$ $\forall s,t \in S,T$, $t > 0$, and $\frac{1}{s} > 0$. Applying the archimedean property to $\frac{1}{s}$ and t, $\exists n$ s.t. $n \cdot \frac{1}{s} > t$, and so $n > t \cdot s$. i will claim $\forall t,s$, $\exists s_{big}$ s.t. $s_{big} > t \cdot s$. assume this is false – then $\exists t,s$ s.t. $\nexists s_{big}$ s.t. $s_{big} > t \cdot s$. then this $t \cdot s$ would be an upper bound of S — but $\nexists M$ s.t. $M > s$, so contradiction, and for any t,s, $\exists s_{big} \in S$ such that $s_{big} > t \cdot s$. since $s_{big} \in S$, and we know $\nexists M$ s.t. $M > s$ $\forall s \in S$, $\nexists M$ s.t. $M > t \cdot s$ $\forall t,s \in T,S$. therefore $T \cdot S$ is unbounded above, and so $\sup(T \cdot S) = +\infty$

ii) $\underline{\sup(S) \cdot \sup(T) = +\infty}$

since S is not bounded above, $\sup(S) = +\infty$. since $t > 0$ $\forall t \in T$, $\sup(T)$ is either a positive number or $+\infty$. either way, $\forall k > 0$ $k \cdot +\infty = +\infty$, and $+\infty \cdot +\infty = +\infty$. by cases, ii) holds.

so $\sup(T \cdot S) = +\infty = \sup(T) \cdot \sup(S)$ ∎

2. i) ~~lim~~ $\lim a_n = 2$

let $N = \log_2(\frac{1}{\varepsilon})$, and $\varepsilon > 0$. then $n > N$ implies $n > \log_2(\frac{1}{\varepsilon})$, so $2^n > 2^{\log_2(\frac{1}{\varepsilon})} = \frac{1}{\varepsilon}$, and so $\varepsilon > \frac{1}{2^n}$. since $\frac{1}{2^n} > 0\ \forall n$, $\frac{1}{2^n} = |-\frac{1}{2^n}| = |2 - \frac{1}{2^n} - 2| = |a_n - 2| > \varepsilon$. theefore, $\lim a_n = 2$, by definition of limit.

ii) let $\varepsilon > 0$. if ~~$N = \sqrt{\frac{1}{\varepsilon}}$~~, then $n > N$ implies $n > \sqrt{\frac{1}{\varepsilon}}$, $n^2 > \frac{1}{\varepsilon}$, $\varepsilon > \frac{1}{n^2}$. $N = \max(\sqrt{\frac{1}{\varepsilon}}, 3)$ since $n > 3$, $n^4 > n^2 + 3n$, and $-n^2 < -6n + 1$. continuing, we have $\varepsilon > \frac{1}{n^2} = |-\frac{1}{n^2}|$ (since $\frac{1}{n^2} > 0\ \forall n$) $= |\frac{-n^2}{n^4}| > |\frac{-6n+1}{n^2+3n}| = |\frac{(2n^2+1) - 2(n^2+3n)}{n^2+3n}| = |\frac{2n^2+1}{n^2+3n} - 2|$, which $< \varepsilon$. $\therefore \lim b_n = 2 = \lim a_n$

3. 1) if a_n converges to a, $\forall \varepsilon > 0\ \exists N$ s.t. $\forall n > N$, $|a_n - a| < \varepsilon$.

$+\infty$: ① if b_n diverges, ~~$\exists$~~ $\forall M > 0\ \exists N$ s.t. $\forall n > N$, $b_n > M$, or

$-\infty$: ② $\forall M < 0$, $\exists N$ s.t. $\forall n > N$, $b_n < M$. i) is for ①, ii) for ②

i) ~~I~~ will show $\lim(a_n + b_n) = +\infty$. by contradiction, assume $\exists M_b > 0$ s.t. $\forall N\ \exists n > N$ s.t. $a_n + b_n < M_b$. but in the definition ①, set $M = M_b - a_n$. we know $b_n > M_b - a_n$, so $a_n + b_n > M_b$, contra. therefore $\lim(a_n + b_n) = +\infty$; divergent.

ii) same as above for $\lim(a_n + b_n) = -\infty$. by contra assume $\exists M_c > 0$ s.t. $\forall N\ \exists n > N$ s.t. $a_n + b_n > M_c$. in definition ②, we ~~can~~ can set $M = M_c - a_n$, so $b_n < M = M_c - a_n$, and $a_n + b_n < M_c$, which contradics the claim. so $\lim(a_n + b_n) = -\infty$; divergent.

2) since $\lim a_n = 0$, $\forall \varepsilon > 0\ \exists N$ s.t. $\forall n > N$, $|a_n| < \varepsilon$. and since b_n is bounded, $\exists M > 0$ s.t. $\forall n$, $|b_n| < M$. the first guarantees $\exists N$ s.t. $\forall n > N$ $|a_n| < \frac{\varepsilon}{M}$. then $|a_n| M < \varepsilon$, and since $|b_n| < M$, $\varepsilon > |a_n| M > |a_n||b_n| > |a_n b_n - 0|$, and so by definition $\lim(a_n b_n) = 0$.

4a) $\forall \varepsilon > 0, \exists N_a$ s.t. $\forall n > N_a\ |a_n - s| < \varepsilon$

and $\forall \varepsilon > 0, \exists N_b$ s.t. $\forall n > N_b, |b_n - s| < \varepsilon$.

since $|a_n - s| < \varepsilon$, ① $-\varepsilon < a_n - s < \varepsilon$ and ② $-\varepsilon < b_n - s < \varepsilon$

since $a_n \leq s_n \leq b_n$, $a_n - s \leq s_n - s \leq b_n - s$. applying ① gives

$-\varepsilon < s_n - s \leq b_n - s$, and ② gives $-\varepsilon < s_n - s < \varepsilon$. so $|s_n - s| < \varepsilon$, QED ■

b) $|s_n| \leq t_n \rightarrow -t_n \leq s_n \leq t_n$. since $\lim t_n = \lim -t_n = 0$, $\lim s_n = 0$ ■ (from part a)

5a) if $\exists N_1$ s.t. $\forall n > N_1, s_n \geq a$, then $s_n \geq a$ for all but finitely many n. we wish to show that if $\forall \varepsilon > 0\ \exists N_2$ s.t. $\forall n > N_2 |s_n - s| < \varepsilon$, then $s \geq a$. consider $N = \max\{N_1, N_2\}$. then both properties hold for N.

since $|s_n - s| < \varepsilon$, $-\varepsilon < s_n - s < \varepsilon$. thus $-\varepsilon - a < s_n - a - s < \varepsilon - a$.

Since $s_n \geq a$, $s_n - a \geq 0$, so $(0) - s < \varepsilon - a$,

so $\varepsilon > a - s$. by contradiction, assume $\lim(s_n) = s < a$. then $0 < a - s$, so $\exists \varepsilon > 0$ s.t. $\varepsilon < a - s$ (since $(a-s)$ is fixed and $\neq 0$).

in particular, choose $\varepsilon = \frac{a-s}{2}$

contradiction! $\varepsilon > a - s\ \forall \varepsilon > 0 \not\leftrightarrow \exists \varepsilon > 0$ s.t. $\varepsilon < a - s$, so $s \not< a$ and $s \geq a$ ■

b) similar to a) — consider $N = \max\{N_1, N_2\}$, with N_2 defined as above and N_2 in $\exists N_2$ s.t. $\forall n > N_2, s_n \leq b$.

again, we know $-\varepsilon < s_n - s < \varepsilon$, so $-\varepsilon - b < s_n - b - s < \varepsilon - b$. since $s_n \leq b$, $s_n - b \leq 0$, and so $-\varepsilon - b < 0 - s = -s$, so $\varepsilon > s - b\ \forall \varepsilon > 0$. by contra, assume $\lim(s_n) = s > b$. then $s - b > 0$, and because $s - b$ is fixed, $\exists \varepsilon$ s.t. $\varepsilon < s - b$. but contra with $\forall \varepsilon > 0\ \varepsilon > s - b$, so assumption is false and $s \leq b$ ■

again, consider $\varepsilon = \frac{s-b}{2}$

c) $s_n \in [a,b] \leftrightarrow a \leq s_n \leq b$. by a), $\lim(s_n) \geq a$, and by b), $\lim(s_n) \leq b$. so $a \leq \lim(s_n) \leq b$, so $\lim(s_n) \in [a,b]$, QED ■

please leave an annotation, i'm actually curious

Note: since $\lim(a) = a$ and $\lim(s_n)$ is defined, can you just say $a \leq s_n \rightarrow \lim(a) \leq \lim(s_n) \rightarrow a \leq \lim(s_n)$??

6.1) if $\lim a_n = a$, by secret lemma $\forall \varepsilon > 0\ \exists N_1$ s.t. $\forall n > N_1\ |a_n - a| < \varepsilon$.

we wish to show $\forall \varepsilon > 0\ \exists N_2$ s.t. $\forall\ > N_2 \left|\frac{a_1 + a_2 + \dots + a}{\ } - a\right| < \varepsilon$.

first, some manipulation:

$\left|\frac{a_1 + \dots + a}{\ } - a\right| = \left|\frac{a_1 + \dots + a - a}{\ }\right| = \frac{|a_1 + \dots + a - a|}{\ } < \varepsilon \rightarrow |a_1 + \dots + a - a| < \varepsilon$.

since ε is arbitrarily small, it suffices to show $< \varepsilon$.

note that for $n > N$, $|a_n - a| < \varepsilon$ (from $\lim a_n = a$). but there are all the pesky a_i where $i \le N$. fret not:

Lemma ①: if $\forall \varepsilon > 0\ \exists N$ s.t. $\forall n > N\ |a_n - a| < \varepsilon$ ($\lim a_n = a$), $\lim b_n = a$, where $b_n = \frac{c + a_m + a_{m+1} + \dots + a_{m+n-1}}{n}$, where $c \in \mathbb{R}$ and $m > N$ (that is, a_m comes after the required place in the definition for $\lim a_n$). first, note $|a_n - a| < \varepsilon$ means $\exists x_i$ s.t.

→ there $|a_n - a| = \varepsilon - x_n$. that is, for $m > N$, $|a_m - a| = \varepsilon - x_m$.

We wish to show $\forall \varepsilon > 0\ \exists N$ s.t. $\forall n > N\ |b_n - a| < \varepsilon$.

that is, $\underbrace{|c + a_m + a_{m+1} + \dots + a_{m+n-1} - an|}$ $< \varepsilon n$ ~~we did here~~.

by triangle inequality, $\le |c| + |a_m - a| + |a_{m+1} - a| + \dots + |a_{m+n-1} - a| < \varepsilon n$.

plugging in from there, we get this

→ this $= |c| + (\varepsilon - x_m) + (\varepsilon - x_{m+1}) + \dots + (\varepsilon - x_{m+n-1}) = |c| + n \cdot \varepsilon - (x_m + x_{m+1} + \dots + x_{m+n-1})$

consider $x_{min} = \min\{x_m, x_{m+1}, \dots, x_{m+n-1}\}$. looking back at their definition there, we see all $x_i > 0$, so $x_{min} > 0$, too.

case 1: $|c| = 0$. then this $= n \cdot \varepsilon - (\text{positive } \#) < n \cdot \varepsilon$, as desired

case 2: $|c| > 0$. by the archimedean property, since $x_{min} > 0$, $\exists n \in \mathbb{N}$ such that $n \cdot x_{min} > |c|$. in fact, this n will be the one we use from the start! back to this,

this $\le |c| + n \cdot \varepsilon - n \cdot x_{min}$, since $x_{min} \le x_i\ \forall i$. so

$= n \cdot \varepsilon + (|c| - n \cdot x_{min}) < n \cdot \varepsilon$, as desired!

now, apply Lemma ① to all a_i where $i \le N_1$ (from def. of $\lim(a_n)$). for any ε, a finite a_m (where $m > N_1$) can be allocated to each a_i, since we didn't specify which a_m we were talking about in Lemma ①. Consider each a_i to be the c — we then have $|a_1 + a_2 + \dots + a_z - za|$, by Δ inequality,

$$\le |a_1 + a_m + a_{m+1} + \dots + a_{m+n-1} - \hat{n}a| + |a_2 + a_p + \dots + a_{p+n_2-1} - \hat{n}_2 a| + \dots + |a_{n-1} + a_y + \dots + a_{y+n_{n-1}} - \hat{n}_n a|$$
$$+ |a_{n+1} - a| + |a_{n+2} - a| + \dots + |a_z - a|.$$

by Lemma ①, the first line $< \frac{\varepsilon}{z} + \frac{\varepsilon}{z} + \dots + \frac{\varepsilon}{z} = \frac{n\varepsilon}{z}$ and by def. of $\lim(a_n)$ (or secret lemma) the bottom $< \frac{\varepsilon}{z} + \frac{\varepsilon}{z} + \dots + \frac{\varepsilon}{z} = \frac{(z-n)\varepsilon}{z}$

and their sum $= \frac{n\varepsilon}{z} + \frac{(z-n)\varepsilon}{z} = \frac{z\varepsilon - n\varepsilon + n\varepsilon}{z} = \varepsilon$, i've shown $\exists N$ (comically large, perhaps, but since N_1 is finite and each a_i where $i \in \{1, \dots, N_1\}$ needs a finite # of a_m's, $N > N_1 \cdot \max(m)$ is sufficient) such that $\forall n > N$ $|s_n - a| < \varepsilon$ ■

2) consider $a_n = (-1)^n$. Then $s_n = \frac{(-1)^1 + (-1)^2 + \dots + (-1)^n}{n} = \frac{-1+1-1+1-\dots+1}{n}$.

we can group by 2 to get $s_{even} = \frac{(1-1)+(1-1)+\dots+(1-1)}{n} = \frac{0}{n} = 0$ and $s_{odd} = \frac{(1-1)+(1-1)+\dots-1}{n} = -\frac{1}{n}$, which we know $\to 0$.

so $\lim s_n = 0$, but clearly $a_n = (-1)^n$ does not converge (this example is given in class). ∴ this is a counterexample of the converse ■

→ b/c monotonic increasing and lim sup = 0 = lim