SOLVABLE GROUPS, FREE DIVISORS AND NONISOLATED MATRIX SINGULARITIES I: TOWERS OF FREE DIVISORS

JAMES DAMON[1] AND BRIAN PIKE[2]

Abstract. This paper is the first part of a two part paper which introduces a method for determining the vanishing topology of nonisolated matrix singularities. A foundation for this is the introduction in this first part of a method for obtaining new classes of free divisors from representations V of connected solvable linear algebraic groups G. For equidimensional representations where $\dim G = \dim V$, with V having an open orbit, we give sufficient conditions that the complement $\mathcal{E}$ of this open orbit, the "exceptional orbit variety", is a free divisor (or a slightly weaker free* divisor).

We do so by introducing the notion of a "block representation"which is especially suited for both solvable groups and extensions of reductive groups by them. This is a representation for which the matrix representing a basis of associated vector fields on V defined by the representation can be expressed using a basis for V as a block triangular matrix, with the blocks satisfying certain nonsingularity conditions. We use the Lie algebra structure of G to identify the blocks, the singular structure, and a defining equation for $\mathcal{E}$.

This construction naturally fits within the framework of towers of Lie groups and representations yielding a tower of free divisors which allows us to inductively represent the variety of singular matrices as fitting between two free divisors. We specifically apply this to spaces of matrices including $m \times m$ symmetric, skew-symmetric or general matrices, where we prove that both the classical Cholesky factorization of matrices and a further "modified Cholesky factorization"which we introduce are given by block representations of solvable group actions. For skew-symmetric matrices, we further introduce an extension of the method valid for a representation of a nonlinear infinite dimensional solvable Lie algebras.

In part II, we shall use these geometric decompositions and results for computing the vanishing topology for nonlinear sections of free divisors to compute the vanishing topology for matrix singularities for all of the classes.

Introduction

In this paper and part II [DP], we introduce a method for computing the "vanishing topology"of nonisolated matrix singularities. A matrix singularity arises from a holomorphic germ $f_0 : \mathbb{C}^n, 0 \to M, 0$, where M denotes a space of matrices. If $\mathcal{V} \subset M$ denotes the variety of singular matrices, then we require that f_0 be transverse to $\mathcal{V}$ off 0 in $\mathbb{C}^n$. Then, $V_0 = f_0^{-1}(\mathcal{V})$ is the corresponding matrix singularity. Matrix singularities have appeared prominently in the Hilbert–Burch theorem [Hi], [Bh] for the representation of Cohen–Macaulay singularities of codimension 2 and for their deformations by Schaps [Sh], by Buchsbaum-Eisenbud [BE] for Gorenstein singularities of codimension 3, and in the defining support for Cohen-Macaulay

(1) Partially supported by the National Science Foundation grants DMS-0706941 and DMS-1105470 and (2) This paper contains work from this author's Ph. D. dissertation at Univ. of North Carolina.

modules, see e.g. Macaulay[Mc] and Eagon-Northcott [EN]. Considerable recent work has concerned the classification of various types of matrix singularities, including Bruce [Br], Haslinger [Ha], Bruce–Tari [BrT], and Goryunov–Zakalyukin.[GZ] and for Cohen–Macaulay singularities by Frühbis–Krüger–Neumer [FN].

The goal of this first part of the paper is to use representation theory for connected solvable linear algebraic groups to place the variety of singular matrices in a geometric configuration of divisors whose union is a free divisor. In part two, we then show how to use the resulting geometric configuration and an extension of the method of Lê-Greuel [LGr] to inductively compute the "singular Milnor number" of the matrix singularities in terms of a sum of lengths of determinantal modules associated to certain free divisors (see [DM] and [D1]). This will lead, for example, in part II to new formulas for the Milnor numbers of Cohen-Macaulay surface singularities. Furthermore, the free divisors we construct in this way are distinguished topologically by both their complements and Milnor fibers being $K(\pi, 1)$'s [DP2].

In this first part of the paper, we identify a special class of representations of linear algebraic groups (especially solvable groups) which yield free divisors. Free divisors arising from representations are termed "linear free divisors" by Mond, who with Buchweitz first considered those that arise from representations of reductive groups using quivers of finite type [BM]. While reductive groups and their representations (which are completely reducible) are classified, this is not the case for either solvable linear algebraic groups nor their representations (which are not completely reducible). We shall see that this apparent weakness is, in fact, an advantage.

We consider an equidimensional (complex) representation of a connected linear algebraic group $\rho : G \to \mathrm{GL}(V)$, so that $\dim G = \dim V$, and for which the representation has an open orbit $\mathcal{U}$. Then, the complement $\mathcal{E} = V \backslash \mathcal{U}$, the "exceptional orbit variety", is a hypersurface formed from the positive codimension orbits. We introduce the condition that the representation is a "block representation", which is a refinement of the decomposition arising from the Lie-Kolchin theorem for solvable linear algebraic groups. This is a representation for which the matrix representing a basis of associated vector fields on V defined by the representation, using a basis for V, can be expressed as a block triangular matrix, with the blocks satisfying certain nonsingularity conditions. We use the Lie algebra structure of G to identify the blocks and obtain a defining equation for $\mathcal{E}$.

In Theorem 2.7 we give a criterion that such a block representation yields a linear free divisor and for a slightly weaker version, we still obtain a free* divisor structure (where the exceptional orbit variety is defined with nonreduced structure). We shall see more generally that the result naturally extends to "towers of groups acting on a tower of representations" to yield a tower of free divisors in Theorem 4.3. This allows us to inductively place determinantal varieties of singular matrices within a free divisor by adjoining a free divisor arising from a lower dimensional representation.

We apply these results to representations of solvable linear algebraic groups associated to Cholesky-type factorizations for the different types of complex matrices. We show in Theorem 6.2 that the conditions for the existence of Cholesky–type factorizations for the different types of complex matrices define the exceptional orbit varieties which are either free divisors or free* divisors. For those cases with only free* divisors, we next introduce a modified form of Cholesky factorization which modifies the solvable groups to obtain free divisors still containing the varieties of singular matrices. This method extends to factorizations for $(n-1) \times n$ matrices (Theorem 7.1).

A new phenomena arises in §8 for skew-symmetric matrices. We introduce a modification of a block representation which applies to infinite dimensional nonlinear solvable Lie algebras. Such algebras are examples of "holomorphic solvable Lie algebras" not generated by finite dimensional solvable Lie algebras. We again prove in Theorem 8.1 that the exceptional orbit varieties for these block representations are free divisors.

Moreover, in §3 we give three operations on block representations which again yield block representations: quotient, restriction, and extension. In §9 the restriction and extension operations are applied to block representations obtained from (modified) Cholesky–type factorizations to obtain auxiliary block representations which will play an essential role in part II in computing the vanishing topology of the matrix singularities.

The representations we have considered so far for matrix singularities are induced from the simplest representations of $\mathrm{GL}_m(\mathbb{C})$. These results will as well apply to representations of solvable linear algebraic groups obtained by restrictions of representations of reductive groups to solvable subgroups and extensions by solvable groups. These results are presently under investigation.

1. Preliminaries on Free Divisors Arising from Representations of Lie Groups

The basic objects of investigation will be free divisors arising from representations of linear algebraic groups, especially solvable ones. Quite generally for a hypersurface germ $\mathcal{V}, 0 \subset \mathbb{C}^p, 0$ with defining ideal $I(\mathcal{V})$, we let

$$\mathrm{Derlog}(\mathcal{V}) \;=\; \{\zeta \in \theta_p : \text{ such that } \zeta(I(\mathcal{V})) \subseteq I(\mathcal{V})\}$$

where θ_p denotes the module of germs of holomorphic vector fields on $\mathbb{C}^p, 0$. Saito [Sa] defines $\mathcal{V}$ to be a *free divisor* if $\mathrm{Derlog}(\mathcal{V})$ is a free $\mathcal{O}_{\mathbb{C}^p,0}$-module (necessarily of rank p).

Saito also gave two fundamental criteria for establishing that a hypersurface germ $\mathcal{V}, 0 \subset \mathbb{C}^p, 0$ is a free divisor. Suppose $\zeta_i \in \theta_p$ for $i = 1, \dots, p$. Then, for coordinates $(y_1, \dots, y_p)$ for $\mathbb{C}^p, 0$, we may write a basis

$$\zeta_i \;=\; \sum_{j=1}^{p} a_{j,i} \frac{\partial}{\partial y_j} \qquad i = 1, \dots, p \tag{1.1}$$

with $a_{j,i} \in \mathcal{O}_{\mathbb{C}^p,0}$. We refer to the $p \times p$ matrix $A = (a_{j,i})$ as a *coefficient matrix* for the p vector fields $\{\zeta_i\}$ and the determinant $\det(A)$ as the *coefficient determinant.*

Saito's Criterion.

A sufficient condition that $\mathcal{V}, 0$ is a free divisor is given by Saito's criterion [Sa] which has two forms.

Theorem 1.1 (Saito's criterion)**.**

(1) *The hypersurface germ $\mathcal{V}, 0 \subset \mathbb{C}^p, 0$ is a free divisor if there are p elements $\zeta_1, \dots, \zeta_p \in \mathrm{Derlog}(\mathcal{V})$ and a basis $\{w_j\}$ for $\mathbb{C}^p$ so that the coefficient matrix $A = (a_{i\,j})$ has determinant which is a reduced defining equation for $\mathcal{V}, 0$. Then, $\zeta_1, \dots, \zeta_p$ is a free module basis for* $\mathrm{Derlog}(\mathcal{V})$.

Alternatively,

(2) *Suppose the set of vector fields $\zeta_1, \dots, \zeta_p$ is closed under Lie bracket, so that for all i and j*

$$[\zeta_i, \zeta_j] \;=\; \sum_{k=1}^{p} h_k^{(i,j)} \zeta_k$$

for $h_k^{(i,j)} \in \mathcal{O}_{\mathbb{C}^p,0}$. If the coefficient determinant is a reduced defining equation for a hypersurface germ $\mathcal{V}, 0$, then $\mathcal{V}, 0$ is a free divisor and $\zeta_1, \dots, \zeta_p$ form a free module basis of $\mathrm{Derlog}(\mathcal{V})$.

We make several remarks regarding the definition and criteria. First, given $\mathcal{V}, 0$ there are two choices of bases involved in the definition, the basis $\dfrac{\partial}{\partial y_i}$ and $\zeta_1, \dots, \zeta_p$. Hence the coefficient matrix is highly nonunique. However, the coefficient determinant is well-defined up to multiplication by a unit.

Second, $\mathrm{Derlog}(\mathcal{V})$ is more than a just finitely generated module over $\mathcal{O}_{\mathbb{C}^p,0}$; it is also a Lie algebra. However, with the exception of the $\{\zeta_i\}$ being required to be closed under Lie bracket in the second criteria, the Lie algebra structure of $\mathrm{Derlog}(\mathcal{V})$ does not enter into consideration.

In Saito's second criterion, if we let $\mathcal{L}$ denote the $\mathcal{O}_{\mathbb{C}^p,0}$–module generate by $\{\zeta_i, i = 1, \dots, p\}$, then $\mathcal{L}$ is also a Lie algebra. More generally we shall refer to any finitely generated $\mathcal{O}_{\mathbb{C}^p,0}$–module $\mathcal{L}$ which is also a Lie algebra as a *(local) holomorphic Lie algebra*. We will consider holomorphic Lie algebras defined for certain distinguished classes of representations of linear algebraic groups and use the Lie algebra structure to show that the coefficient matrix has an especially simple form.

Prehomogeneous Spaces and Linear Free Divisors.

Suppose that $\rho : G \to GL(V)$ is a rational representation of a connected complex linear algebraic group. If there is an open orbit $\mathcal{U}$ then such a space with group action is called a prehomogeneous space and has been studied by Sato and Kimura [So] [SK] but from the point of view of harmonic analysis. They have effectively determined the possible prehomogeneous spaces arising from irreducible representations of reductive groups.

If $\mathfrak{g}$ denotes the Lie algebra of G, then for each $v \in \mathfrak{g}$, there is a vector field on V defined by

$$\xi_v(x) \;=\; \frac{\partial}{\partial t}(\exp(t \cdot v) \cdot x)_{|t=0} \qquad \text{for } x \in V\,. \tag{1.2}$$

In the case $\dim G = \dim V = n$, Mond observed that if $\{v_i\}_{i=1}^n$ is a basis of the Lie algebra $\mathfrak{g}$ and the coefficient matrix of these vector fields with respect to coordinates for V has reduced determinant, then Saito's criterion can be applied to conclude $\mathcal{E} = V \backslash \mathcal{U}$ is a free divisor with $\mathrm{Derlog}(\mathcal{E})$ generated by the $\{\xi_{v_i}, i = 1, \dots, n\}$. This idea was applied by Buchweitz–Mond to reductive groups arising from quiver representations of finite type [BM] and more general quiver representations in [GMNS]. In the case that $\mathcal{E}$ is a free divisor, we follow Mond and call it a *linear free divisor*.

We shall call a representation with $\dim G = \dim V$ an *equidimensional representation*. Also, the variety $\mathcal{E} = V \backslash \mathcal{U}$ has been called the singular set or discriminant. We shall be considering in part II mappings into V, which also have singular sets and discriminants. To avoid confusion, we shall refer to $\mathcal{E}$, which is the union of the orbits of positive codimension, as the *exceptional orbit variety*.

Remark 1.2. In the case of an equidimensional representation with open orbit, if there is a basis $\{v_i\}$ for $\mathfrak{g}$ such that the determinant of the coefficent matrix defines $\mathcal{E}$ but with nonreduced structure, then we refer to $\mathcal{E}$ as being a *linear free* divisor*. A free* divisor structure can still be used for determining the topology of nonlinear sections as is done in [DM], except correction terms occur due to the presence of "virtual singularites"(see [D3]). However, by [DP2], the free* divisors that occur in this paper will have complements and Milnor fibers with the same topological properties as free divisors.

In contrast with the preceding results, we shall be concerned with nonreductive groups, and especially connected solvable linear algebraic groups. The representations of such groups G cannot be classified as in the reductive case. Instead, we will make explicit use of the Lie algebra structure of the Lie algebra $\mathfrak{g}$ and special properties of its representation on V. We do so by identifying it with its image in $\theta(V)$, which denotes the $\mathcal{O}_{V,0}$-module of germs of holomorphic vector fields on $V, 0$, which is also a Lie algebra. We will view it as the Lie algebra of the group $\mathrm{Diff}(V, 0)$ of germs of diffeomorphisms of $V, 0$, even though it is not an infinite dimensional Lie group in the usual sense.

However, there is an exponential map in terms of one–parameter subgroups. Let $\xi \in m \cdot \theta(V)$ (with m denoting the maximal ideal of $\mathcal{O}_{V,0}$). Integrating ξ gives a local one-parameter group of diffeomorphism germs $\varphi_t : V, 0 \to V, 0$ defined for $|t| < \varepsilon$ which satisfy $\dfrac{\partial \varphi_t}{\partial t} = \xi \circ \varphi_t$ and $\varphi_0 = id$. We define

$$\exp : m \cdot \theta(V) \to \mathrm{Diff}(V, 0) \qquad \text{where} \qquad \exp(s\,\xi) = \varphi_{st}\,.$$

Second, we have the natural inclusion $i : \mathrm{GL}(V) \hookrightarrow \mathrm{Diff}(V, 0)$ (where a linear transformation φ is viewed as a germ of a diffeomorphism of $V, 0$). There is a corresponding map

$$\tilde{i} : \mathfrak{gl}(V) \longrightarrow m \cdot \theta(V) \qquad A \mapsto \xi_A \tag{1.3}$$

where the $\xi_A(x) = A(x)$ are "linear vector fields", whose coefficients are linear functions. Then, $\tilde{i}$ is a bijection between $\mathfrak{gl}(V)$ and the subspace of linear vector fields. A straightforward calculation shows that $\tilde{i}$ is a Lie algebra homomorphism provided we use the negative of the usual Lie bracket for $m \cdot \theta(V)$.

Given a representation $\rho : G \to GL(V)$ of a (complex) connected linear algebraic group G with associated Lie algebra homomorphism $\tilde{\rho}$, there is the following commutative exponential diagram.

Exponential Diagram for a Representation

$$\begin{array}{ccccc} \mathfrak{g} & \xrightarrow{\tilde{\rho}} & \mathfrak{gl}(V) & \xrightarrow{\tilde{i}} & m \cdot \theta(V) \\ {\scriptstyle \exp}\downarrow & & {\scriptstyle \exp}\downarrow & & {\scriptstyle \exp}\downarrow \\ G & \xrightarrow{\rho} & \mathrm{GL}(V) & \xrightarrow{i} & \mathrm{Diff}(V, 0) \end{array} \tag{1.4}$$

If ρ has finite kernel, then $\tilde{\rho}$ is injective. Even though it is not standard, we shall refer to such a representation as a *faithful representation*, as we could always divide by the finite group and obtain an induced representation which is faithful and does not alter the corresponding Lie algebra homomorphisms. Hence, $\tilde{i} \circ \tilde{\rho}$ is an isomorphism from $\mathfrak{g}$ onto its image, which we shall denote by $\mathfrak{g}_V$.

Hence, $\mathfrak{g}_V \subset m \cdot \theta(V)$ has exactly the same Lie algebra theoretic properties as $\mathfrak{g}$. For $v \in \mathfrak{g}$, we slightly abuse notation by more simply denoting $\xi_{\tilde{\rho}(v)}$ by $\xi_v \in \mathfrak{g}_V$, which we refer to as the associated *representation vector fields.* The $\mathcal{O}_{V,0}$ –module generated by $\mathfrak{g}_V$ is a holomorphic Lie algebra which has as a set of generators $\{\xi_{v_i}\}$, as v_i varies over a basis of $\mathfrak{g}$. Saito's criterion applies to the $\{\xi_{v_i}\}$; however, we shall use the correspondence with the Lie algebra properties of $\mathfrak{g}$ to deduce the properties of the coefficient matrix.

Naturality of the Representation Vector Fields.

The naturality of the exponential diagram leads immediately to the naturality of the constuction of representation vector fields. Let $\rho : G \to GL(V)$ and $\rho' : H \to GL(W)$ be representations of linear algebraic groups. Suppose there is a Lie group homomorphism $\varphi : G \to H$ and a linear transformation $\varphi' : V \to W$ such that when we view W as a G representation via φ, then φ' is a homomorphism of G-representations. We denote this by saying that $\Phi = (\varphi, \varphi') : (G, V) \to (H, W)$ is *homomorphism of groups and representations.*

Proposition 1.3. *The construction of representation vector fields is natural in the sense that if $\Phi = (\varphi, \varphi') : (G, V) \to (H, W)$ is a homomorphism of groups and representations, then for any $v \in \mathfrak{g}$, the representation vector fields ξ_v for G on V and $\xi_{\tilde{\varphi}(v)}$ for H on W are φ'–related.*

Proof. By (1.2), for $x \in V$

$$\begin{aligned} d\varphi'_x(\xi_v(x)) &= \frac{\partial}{\partial t}(\varphi'(\exp(t \cdot v) \cdot x))_{|t=0} = \frac{\partial}{\partial t}(\varphi(\exp(t \cdot v)) \cdot \varphi'(x))_{|t=0} \\ &= \frac{\partial}{\partial t}(\exp(t \cdot \tilde{\varphi}(v) \cdot \varphi'(x))_{|t=0} = \xi_{\tilde{\varphi}(v)}(\varphi'(x)). \end{aligned} \tag{1.5}$$

Hence, ξ_v and $\xi_{\tilde{\varphi}(v)}$ are φ'–related as asserted. □

2. Block Representations of Linear Algebraic Groups

We consider representations V of connected linear algebraic groups G which need not be reductive. These may not be completely reducible; hence, there may be invariant subspaces $W \subset V$ without invariant complements. It then follows that we may represent the elements of G by block upper triangular matrices; however, importantly, it does not follow that the corresponding coefficient matrix for a basis of representation vector fields need be block triangular. There is condition which we identify, which will lead to this stronger property and be the basis for much that follows. To explain it, we first examine the form of the representation vector fields for G. We choose a basis $\{\xi_{v_i}\}$ for $\mathfrak{g}_V$, and a basis for V formed from a basis $\{w_i\}$ for W and a complementary basis $\{u_j\}$ to W.

Lemma 2.1. *In the preceding situation, any representation vector field ξ_v has the form*

$$\xi_v = \sum_{\ell} b_\ell u_\ell + \sum_j a_j w_j \tag{2.1}$$

where $a_j \in \mathcal{O}_{V,0}$ and $b_\ell \in \pi^*\mathcal{O}_{V/W,0}$ for $\pi : V \to V/W$ the natural projection.

Proof. First, we know $(id, \pi) : (G, V) \to (G, V/W)$ is a homomorphism of groups and representations. By Proposition 1.3, the representation vector fields ξ_v on V and ξ'_v on V/W for $v \in \mathfrak{g}$ are π–related. Hence, the representation vector field ξ'_v on V/W has the form the first sum on the RHS of (2.1). The coefficients for the w_j will be function germs in $\mathcal{O}_{V,0}$. □

Next we introduce a definition.

Definition 2.2. Let G be a connected linear algebraic group which acts on V and which has a G–invariant subspace $W \subset V$ with $\dim W = \dim G$ such that G acts trivially on V/W. We say that G has a *relatively open orbit in* W if there is an orbit of G whose generic projection onto W is Zariski open.

This condition can be characterized in terms of the representation vector fields of G. We choose a basis $\{\xi_{v_i} : i = 1, \dots, k\}$ for $\mathfrak{g}_V$. Then, as G acts trivially on V/W, by Lemma 2.1 it follows that we can represent

$$\xi_{v_i} \;=\; \sum_j a_{ji} w_j \tag{2.2}$$

where $a_{ji} \in \mathcal{O}_{V,0}$. We refer to the matrix (a_{ji}) as a *relative coefficient matrix* for G and W. We also refer to $\det(a_{ji})$ as the *relative coefficient determinant for G and W*.

Lemma 2.3. *The action of G on V has a relatively open orbit if and only if the relative coefficient determinant is not zero.*

Proof. At any point $x \in V$, the orbit through x has tangent space spanned by a basis $\{\xi_{v_i} : i = 1, \dots, k\}$ for $\mathfrak{g}_V$. The projection onto W is a local diffeomorphism at x if and only if the projection of the subspace spanned by $\{\xi_{v_i}(x) : i = 1, \dots, k\}$ onto W is an isomorphism. As $\dim W = k$ and the projection sends the vectors to the vectors on the RHS of (2.2), we obtain a local diffeomorphism if and only if the matrix $(a_{ji}(x))$ is nonsingular.

Now, the composition of mappings $G \to V \to W$, where the first map is the orbit mapping $g \mapsto g \cdot x$ and the second denotes projection, is a rational map, so the image is constructible. If the mapping is a local diffeomorphism at a point then the image contains a metric neighborhood so the Zariski closure is W, and the image is Zariski open, and conversely. Hence, the result follows. □

Now we are in a position to introduce a basic notion for us, that of a block representation.

Definition 2.4. A equidimensional representation V of a connected linear algebraic group G will be called a *block representation* if:

i) there exists a sequence of G-invariant subspaces
$$V = W_k \supset W_{k-1} \supset \cdots \supset W_1 \supset W_0 = (0).$$
ii) for the induced representation $\rho_j : G \to GL(V/W_j)$, we let $K_j = \ker(\rho_j)$, then $\dim K_j = \dim W_j$ for all j and the equidimensional action of K_j/K_{j-1} on W_j/W_{j-1} has a relatively open orbit (in V/W_{j-1}) for each j.
iii) the relative coefficient determinants p_j for the representations of K_j/K_{j-1} on W_j/W_{j-1} are all reduced and relatively prime in pairs in $\mathcal{O}_{V,0}$ (by Lemma 2.1, $p_j \in \mathcal{O}_{V/W_{j-1},0}$ and we obtain $p_j \in \mathcal{O}_{V,0}$ via pull-back by the projection map from V).

We also refer to the decomposition of V using the $\{W_j\}$ and G by the $\{K_j\}$ with the above properties as the *decomposition for the block representation*.

Furthermore, if each p_j is irreducible, then we will refer to it as a *maximal block representation*.

If in the preceding both i) and ii) hold, and the relative coefficient determinants are nonzero but may be nonreduced or not relatively prime in pairs, then we say that it is a *nonreduced block representation*.

Block Triangular Form: We deduce for a block representation $\rho : G \to GL(V)$ (with representation as in Definition 2.4) a special block triangular form for its coefficient matrix with repect to bases respecting the W_j and the K_j. Specifically, we first choose a basis $\{w_i^{(j)}\}$ for W such that $\{w_1^{(j)}, \dots, w_{m_j}^{(j)}\}$ is a complementary basis to W_{j-1} in W_j, for each j. Second, letting $\mathbf{k}_j$ denote the Lie algebra for K_j, we choose a basis $\{v_i^{(j)}\}$ for $\mathfrak{g}$ such that $\{v_1^{(j)}, \dots, v_{m_j}^{(j)}\}$ is a complementary basis to $\mathbf{k}_{j-1}$ in $\mathbf{k}_j$.

Proposition 2.5. *Let $\rho : G \to GL(V)$ be a block representation with bases for $\mathfrak{g}$ and V as just described, with ordering of the bases first for each j and then all of the i for that j. Then, the coefficient matrix has a block triangular form. For example, if the vector fields form the columns with rows given by the basis for V and we use descending ordering on j, then the matrix is lower block triangular as in (2.3), where each D_j is a $m_j \times m_j$ matrix.*

Then, $p_j = \det(D_j)$ are the relative coefficient determinants.

$$\begin{pmatrix} D_k & 0 & 0 & 0 & 0 \\ * & D_{k-1} & 0 & 0 & 0 \\ * & * & \ddots & 0 & 0 \\ * & * & * & \ddots & 0 \\ * & * & * & * & D_1 \end{pmatrix} \tag{2.3}$$

In (2.3) if $p_1 = \det(D_1)$ is irreducible, then we will refer to the variety $\mathcal{D}$ defined by p_1 as the *generalized determinant variety* for the decomposition.

As an immediate corollary we have

Corollary 2.6. *For a block representation, the number of irreducible components in the exceptional orbit variety is at least the number of diagonal blocks in the corresponding block triangular form, with equality for a maximal block representation.*

Proof of Proposition 2.5. Since K_ℓ acts trivially on V/W_ℓ, by Lemma 2.1, for $v \in \mathbf{k}_\ell$ the associated representation vector field may be written as

$$\xi_v = \sum_{j=1}^{\ell} \sum_{i=1}^{m_j} a_{ij} w_i^{(j)} \tag{2.4}$$

where the basis for W_ℓ is given by $\{w_i^{(j)} : 1 \le j \le \ell, 1 \le i \le m_j\}$. Thus, for $\{v_i^{(\ell)} : i = 1, \dots, m_j\}$ a complementary basis to $\mathbf{k}_{\ell-1}$ in $\mathbf{k}_\ell$, the columns corresponding to $\xi_{v_i^{(\ell)}}$ will be zero above the block D_ℓ as indicated.

Furthermore, the quotient maps $(\varphi, \varphi') : (K_\ell, V) \to (K_\ell/K_{\ell-1}, V/W_{\ell-1})$ define a homomorphism of groups and representations. Thus, again by Lemma 2.1, the coefficients $a_{j\,i}$ of $w_j^{(\ell)}$, $j = 1, \dots, m_\ell$, for the $\xi_{v_i^{(\ell)}}$ are the same as those for the representation of $K_\ell/K_{\ell-1}$ on $V/W_{\ell-1}$. Thus, we obtain D_ℓ as the relative coefficient matrix for $K_\ell/K_{\ell-1}$ and $V/W_{\ell-1}$. Thus $p_\ell = \det(D_\ell)$. □

Exceptional Orbit Varieties as Free and Free* Divisors.

We can now easily deduce from Proposition 2.5 the basic result for obtaining linear free divisors from representations of linear algebraic groups.

Theorem 2.7. *Let $\rho : G \to \mathrm{GL}(V)$ be a block representation of a connected linear algebraic group G, with relative coefficient determinants $p_j, j = 1, \dots, k$. Then, the exceptional orbit variety $\mathcal{E}, 0 \subset V, 0$ is a linear free divisor with reduced defining equation $\prod_{j=1}^{k} p_j = 0$*

If instead $\rho : G \to \mathrm{GL}(V)$ is a nonreduced block representation, then $\mathcal{E}, 0 \subset V, 0$ is a linear free divisor and $\prod_{j=1}^{k} p_j = 0$ is a nonreduced defining equation for $\mathcal{E}, 0$.*

Proof. By Proposition 2.5, we may choose bases for $\mathfrak{g}$ and V so that the coefficient matrix has the form (2.3). Then, by the block triangular form, the determinant equals $\prod_{j=1}^{k} p_j$, which by condition iii) for block representations is reduced. Then, at any point where the determinant does not vanish, the orbit contains an open neighborhood. Since it is the image of G under a rational map it is Zariski open. Thus, all points where the determinant does not vanish belong to this single open orbit $\mathcal{U}$, and those points in the complement have positive codimension orbits defined by the vanishing of the determinant. This is the exceptional orbit variety $\mathcal{E}$. Hence, since the representation vector fields belong to $\mathrm{Derlog}(\mathcal{E})$, the first form of Saito's Criterion (Theorem 1.1) implies that $\mathcal{E}$ is a free divisor.

In the second case, if either the determinants of the relative coefficient matrices p_j are either nonreduced or not relatively prime in pairs then, although $\prod_{j=1}^{k} p_j = 0$ still defines $\mathcal{E}$, it is nonreduced. Hence, $\mathcal{E}$ is then only a linear free* divisor. □

The usefulness of this result comes from several features: its general applicability to nonreductive linear algebraic groups, especially solvable groups; the behavior of block representations under basic operations considered in §3; the simultaneous and inductive applicability to a tower of groups and corresponding representations in §4; and most importantly for applications the abundance of such representations especially those appearing in complex versions of classical Cholesky–type factorization theorems §6, their modifications §7, §8, and restrictions §9.

Remark 2.8.

For quiver representations of finite type studied by Buchweitz-Mond [BM] the block structure consists of a single block. $G = \prod_{i=1}^{m} G_i$ acting on $V = \prod_{i=1}^{m} V_i$ by the product representation defines a block representation with $W_j = \prod_{i=1}^{j} V_i$ and $K_j = \prod_{i=1}^{j} G_i$. In this case the coefficient matrix is just block diagonal. If each action of G_i on V_i defines a linear free divisor $\mathcal{E}_i$, then G acting on V defines a linear free divisor which is a product union of the $\mathcal{E}_i$ in the sense of [D2].

Representations of Linear Solvable Algebraic Groups. The most important special case for us will concern representations of connected solvable linear algebraic groups. Recall that a linear algebraic group G is *solvable* if there is a series of algebraic subgroups $G = G_0 \supset G_1 \supset G_2 \supset \cdots G_{k-1} \supset G_k = \{e\}$ with G_{j+1} normal in G_j such that G_j/G_{j+1} is abelian for all j. Equivalently, if $G^{(1)} = [G, G]$ is the (closed) commutator subgroup of G, and $G^{(j+1)} = [G^{(j)}, G^{(j)}]$, then for some j, $G^{(j)} = \{1\}$.

Unlike reductive algebraic groups, representations of solvable linear algebraic groups need not be completely reducible. Moreover, neither the representations nor the groups themselves can be classified. Instead, the important property of solvable groups for us is given by the Lie-Kolchin Theorem, which asserts that a finite dimensional representation V of a connected solvable linear algebraic group G has a flag of G–invariant subspaces

$$V = V_N \supset V_{N-1} \supset \cdots \supset V_1 \supset V_0 = \{0\}\,,$$

where $\dim V_j = j$ for all j. We shall be concerned with nontrivial block representations for the actions of connected solvable linear algebraic groups where the W_j form a special subset of the flag of G–invariant subspaces. Then, not only will we give the block representation, but we shall see that the diagonal blocks D_j will be given very naturally in terms of certain submatrices. These will be examined in §§ 6, 7, 8, and 9.

3. Operations on Block Representations

We next give several propositions which describe how block representations behave under basic operations on representations. These will concern taking quotient representations, restrictions to subrepresentations and subgroups, and extensions of representations. We will give an immediate application of the extension property Proposition 3.3 in the next section. We will also apply the restriction and extension properties in §9 to obtain auxilliary block representations which will be needed to carry out calculations in Part II.

Let $\rho : G \to GL(V)$ be a block representation with decomposition

$$V = W_k \supset W_{k-1} \supset \cdots \supset W_1 \supset W_0 = (0)$$

and normal algebraic subgroups

$$G = K_k \supset K_{k-1} \supset \cdots \supset K_1 \supset K_0\,,$$

with $K_j = \ker(\rho_j : G \to GL(V/W_j))$ and $\dim K_j = \dim W_j$ (so K_0 is a finite group). We also let p_j be the relative coefficient determinant for the action of K_j/K_{j-1} on W_j/W_{j-1} in V/W_{j-1}.

We first consider the induced quotient representation of G/K_ℓ on V/W_ℓ.

Proposition 3.1 (Quotient Property). *For the block representation $\rho : G \to GL(V)$ with its decomposition as above, the induced quotient representation $G/K_\ell \to GL(V/W_\ell)$ is a block representation with decomposition*

$$V/W_\ell \;=\; \bar{W}_\ell \supset \bar{W}_{\ell-1} \supset \cdots \supset \bar{W}_1 \supset \bar{W}_0 = (0) \qquad \textit{and}$$
$$\bar{G} \;=\; G/K_\ell \;=\; \bar{K}_\ell \supset \bar{K}_{\ell-1} \supset \cdots \supset \bar{K}_1 \supset \bar{K}_0$$

where $\bar{W}_j = W_{j+\ell}/W_\ell$ and $\bar{K}_j = K_{j+\ell}/K_\ell$. Then, the coefficient determinant is given by $\prod_{i=\ell+1}^{k} p_i$.

If ρ is only a nonreduced block representation then the quotient representation is a (possibly) nonreduced block representation.

Proof. By the basic isomorphism theorems $K_{j+\ell}/K_\ell = \ker(G/K_\ell \to GL(W_{j+\ell}/W_\ell))$, $\dim K_{j+\ell}/K_\ell = \dim W_{j+\ell}/W_\ell$, and the representations of $\bar{K}_j/\bar{K}_{j-1} \simeq K_{j+\ell}/K_{j+\ell-1}$ on $\bar{W}_j/\bar{W}_{j-1} \simeq W_{j+\ell}/W_{j+\ell-1}$ have the same relative coefficient determinants p_j

(these are polynomials defined on $W_{j+\ell}/W_{j+\ell-1}$). Thus, the relative coefficient determinants for the blocks in the quotient representations are reduced and relatively prime. Hence, the quotient representation is a block representation.

If the relative coefficient determinants for ρ are not necessarily reduced or relatively prime, then neither need be those for the quotient representation. $\square$

The second operation is that of restricting to an invariant subspace and subgroup.

Proposition 3.2 (Restriction Property). *Let $\rho : G \to GL(V)$ be a block representation with its decomposition as above. Also, let W be a G–invariant subspace with $W_\ell \supset W \supset W_{\ell-1}$ and K a connected linear algebraic subgroup with*

$K_\ell \supset K \supset K_{\ell-1}$ and $\dim K = \dim W$. Suppose that the relative coefficient determinant p of $K/K_{\ell-1}$ on $W/W_{\ell-1}$ together with the restrictions of the relative coefficient determinants $p_{j\,|W}$ for the actions of K_j/K_{j-1} on W_j/W_{j-1} in W/W_{j-1} for $j = 1, \dots, \ell-1$ are reduced and relatively prime. Then, the restricted representation $\bar{\rho} : K \to GL(W)$ is a block representation with decomposition

$$
\begin{aligned}
W \;&=\; \bar{W}_\ell \supset \bar{W}_{\ell-1} \supset \cdots \supset \bar{W}_1 \supset \bar{W}_0 = (0) \qquad \textit{and} \\
K \;&=\; \bar{K}_\ell \supset \bar{K}_{\ell-1} \supset \cdots \supset \bar{K}_1 \supset \bar{K}_0
\end{aligned}
$$

where for $0 \leq j < \ell$, $\bar{W}_j = W_j$ and $\bar{K}_j$ contains K_j as an open subgroup.

Proof. The decomposition is given in the statement of the proposition where

$$\bar{K}_j \;=\; \ker(K \to GL(W/\bar{W}_j))\,(=\, \ker(K \to GL(W/W_j))).$$

To see it has the desired properties, we first claim that K_j is an open subgroup of $\bar{K}_j$. It is a subgroup by the properties of ρ. If $\dim \bar{K}_j > \dim K_j$, then there would exist $v \in \bar{\mathfrak{k}}_j \backslash \mathfrak{k}_j$ (the Lie algebras of $\bar{K}_j$ and K_j), so that $\exp(t\,v) \in \ker(K \to GL(W/W_j))$. Then, by Lemma 2.1, the corresponding representation vector field $\xi_v(w) \in W_j$ for all $w \in W$. If we compute the relative coefficient matrix for the action of K/K_j on W/W_j, by including a $v \in \bar{\mathfrak{k}}_j$ in a complementary basis to $\mathfrak{k}_j$ in $\mathfrak{k}$ (the Lie algebras of K_j and K), then the relative coefficient matrix would have a column identically zero, and so the relative coefficient determinant would be 0. This contradicts it being equal to the product of nonzero relative coefficient determinants appearing in the statement. Thus, $\dim \bar{K}_j = \dim K_j$ so they have the same Lie algebra; and for all j, $\dim \bar{K}_j = \dim \bar{W}_j$ by either the assumption on ρ or $\dim K = \dim W$.

Then, it also follows that for $0 \leq j < \ell$, the relative coefficient determinant for $\bar{K}_j/\bar{K}_{j-1}$ on $\bar{W}_j/\bar{W}_{j-1}$ in $W/\bar{W}_{j-1}$ is the restriction to W of that the relative coefficient determinant for K_j/K_{j-1} on W_j/W_{j-1} in W/W_{j-1}. By assumption, these together with the relative coefficient determinant for $\bar{K}_\ell \supset \bar{K}_{\ell-1}$ on $\bar{W}_\ell \supset \bar{W}_{\ell-1}$ are reduced and relatively prime. Thus the coefficient determinant for the representation of $\bar{\rho} : K \to GL(W)$ is the product of the relative coefficient determinants, and hence is nonzero on a Zariski open subset. Hence, the kernel of the representation is finite, and it is a block representation. $\square$

Third, we have the following proposition which allows for the extension of a block representation yielding another block representation, providing a partial converse to Proposition 3.1.

Proposition 3.3 (Extension Property). *Let $\rho : G \to GL(V)$ be a representation of a connected linear algebraic group, so that $W \subset V$ is a G–invariant subspace and*

$K = \ker(G \to GL(V/W))$ with $\dim(K) = \dim(W)$. Suppose that the quotient representation $\bar{\rho} : G/K \to GL(V/W)$ is a block representation with decomposition

$$\begin{aligned} V/W \;&=\; \bar{W}_\ell \supset \bar{W}_{\ell-1} \supset \cdots \supset \bar{W}_1 \supset \bar{W}_0 = (0) \qquad \textit{and} \\ \bar{G} \;=\; G/K \;&=\; \bar{K}_\ell \supset \bar{K}_{\ell-1} \supset \cdots \supset \bar{K}_1 \supset \bar{K}_0\,, \end{aligned}$$

for which the relative coefficient determinant for the action of K on W is reduced and relatively prime to the coefficient determinant for $\bar{\rho}$. Then, ρ is a block representation with decomposition

$$\begin{aligned} V \;&=\; W_{\ell+1} \supset W_\ell \supset \cdots \supset W_1 \supset W_0 = (0) \qquad \textit{and} \\ G \;&=\; K_{\ell+1} \supset K_\ell \supset \cdots \supset K_1 \supset K_0 = \{Id\}\,. \end{aligned}$$

Here $W_1 = W$, $K_1 = K$, and for $j = 1, \ldots, \ell$, $W_{j+1} = \pi^{-1}(\bar{W}_j)$ and $K_{j+1} = \pi'^{-1}(\bar{K}_j)$ for $\pi : V \to V/W$ and $\pi' : G \to G/K$ the projections.

If instead $\bar{\rho}$ has a nonreduced block structure or the relative coefficient determinant for the action of K on W is nonreduced or not relatively prime to the coefficient determinant for $\bar{\rho}$, then, ρ is a nonreduced block representation.

Proof. Again the proposition gives the form of the decomposition, provided we verify the properties. By our assumptions, $\dim K_j = \dim W_j$ for all j. For $1 \leq j \leq \ell$, with $\pi' : G \to G/K$ as above,

$$\ker(G \to GL(V/W_j)) = \pi'^{-1}(\ker(G/K \to GL(V/\bar{W}_{j-1}))) = \pi'^{-1}(\bar{K}_{j-1}) = K_j\,.$$

Finally, using the stated decomposition, the coefficient matrix has a lower triangular block form. Then, the coefficient determinant for the representation of $\rho : G \to GL(V)$ is the product of the relative coefficient determinants, which equals the product of the relative coefficient determinant of K acting on W and the coefficient determinant of G/K acting on V/W (pulled back to V). Hence it is nonzero on a Zariski open subset. Thus, the kernel of the representation is finite, and it is a block representation. ☐

4. Towers of Linear Algebraic Groups and Representations

The two key questions concerning block representations are:

i) How do we find the G-invariant subspaces W_j?

ii) Given the $\{W_j\}$, what specifically are the diagonal blocks D_j?

The first question becomes more approachable when we have a series of groups with a corresponding series of representations.

Towers of Linear Algebraic Groups and Representations.

Rather than consider individual block representations, we consider simultaneously towers of linear algebraic groups and representations.

Definition 4.1. A *tower of linear algebraic groups* $\mathbf{G}$ is a sequence of such groups

$$\{e\} = G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_k \subset \cdots\,.$$

Such a tower has a *tower of representations* $\mathbf{V} = \{V_j\}$ if

$$(0) = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_k \subset \cdots$$

where each V_j is a representation of G_j, and for the inclusion maps $i_j : G_j \hookrightarrow G_{j+1}$, and $j_j : V_j \hookrightarrow V_{j+1}$, the mapping $(i_j, j_j) : (G_j, V_j) \to (G_{j+1}, V_{j+1})$ is a homomorphism of groups and representations.

Then, we identify within towers when the block representation structures are related.

Definition 4.2. A tower of connected linear algebraic groups and representations $(\mathbf{G}, \mathbf{V})$ has a *block structure* if: for all $\ell \geq 0$ the following hold:

i) Each V_ℓ is a block representation of G_ℓ via the decompositions
$$G_\ell = K_k^\ell \supset K_{k-1}^\ell \supset \cdots \supset K_1^\ell \supset K_0^\ell$$
where K_0^ℓ is a finite group, and
$$V_\ell = W_k^\ell \supset W_{k-1}^\ell \supset \cdots \supset W_1^\ell \supset W_0^\ell = (0).$$

ii) For each $\ell > 0$ the composition of the natural homomorphisms of representations
$$(G_{\ell-1}, V_{\ell-1}) \to (G_\ell, V_\ell) \to (G_\ell/K_1^\ell, V_\ell/W_1^\ell)$$
is an isomorphism of representations.

If instead in i) we only have nonreduced block representations, then we say that the tower has a *nonreduced block structure*.

We will use the properties of §3 to show that it sufficient to analyze each stage of the tower to deduce that it has a block structure. Before doing so we first deduce an important consequence for the collection of exceptional orbit varieties.

Then, for such a tower of representations with a block structure (or nonreduced block structure) we have the following basic theorem which will yield the results for many spaces of matrices.

Theorem 4.3. *Suppose $(\mathbf{G}, \mathbf{V})$ is a tower of connected linear algebraic groups and representations which has a block structure. Let $\mathcal{E}_\ell$ be the exceptional orbit variety for the action of G_ℓ on V_ℓ. Then,*

i) *For each ℓ, $\mathcal{E}_\ell$ is a linear free divisor.*

ii) *The quotient space V_ℓ/W_1^ℓ can be naturally identified with $V_{\ell-1}$ as $G_{\ell-1}$–representations.*

iii) *The generalized determinant variety $\mathcal{D}_\ell$ for the action of G_ℓ on V_ℓ satisfies $\mathcal{E}_\ell = \mathcal{D}_\ell \cup \pi_\ell^* \mathcal{E}_{\ell-1}$, where π_ℓ denotes a projection $V_\ell \to V_{\ell-1}$.*

If instead $(\mathbf{G}, \mathbf{V})$ has a nonreduced block structure, then each $\mathcal{E}_j$ is a linear free divisor.*

Remark 4.4. Once we have established that the actions of solvable groups on spaces of matrices form a tower of such representations, then this theorem allows us to place the determinant varieties, which will be the varieties of singular matrices, within a configuration of free divisors.

Proof. First, it is immediate from Theorem 2.7 that each $\mathcal{E}_\ell$ is a linear free divisor. Furthermore, by property ii) the composition $V_{\ell-1} \to V_\ell \to V_\ell/W_1^\ell$ is an isomorphism. This defines for each ℓ a projection $\pi_\ell : V_\ell \to V_{\ell-1}$with kernel W_1^ℓ which is equivariant for the isomorphism given by the composition of inclusion with the projection $G_{\ell-1} \to G_\ell \to G_\ell/K_1^\ell$. This establishes ii).

To show iii), we first specifically choose a basis $\{w_i^{(j)}\}$ for V_ℓ such that $\{w_1^{(\ell-1)}, \dots, w_{m_{\ell-1}}^{(\ell-1)}\}$ is a complementary basis to W_1^ℓ in V_ℓ (and projects to a basis for $V_{\ell-1}$), and $\{w_1^{(\ell)}, \dots, w_{m_\ell}^{(\ell)}\}$ is a basis for W_1^ℓ. Second, we let $\mathbf{k}_1$ denote the Lie algebra for K_1^ℓ, we choose a basis $\{v_i^{(j)}\}$ for $\mathfrak{g}_\ell$ such that $\{v_1^{(\ell-1)}, \dots, v_{m_{\ell-1}}^{(\ell-1)}\}$ is a complementary basis to $\mathbf{k}_1$ in $\mathfrak{g}_\ell$, and $\{v_1^{(\ell)}, \dots, v_{m_\ell}^{(\ell)}\}$ is a basis for $\mathbf{k}_1$.

Then, the coefficient matrix with respect to these bases will have the form

$$\begin{pmatrix} A & 0 \\ * & B \end{pmatrix} \tag{4.1}$$

where A denotes the coefficient matrix for the representation of $G_{\ell-1}$ on $V_{\ell-1}$ and B is the relative block for K_1^ℓ acting on W_1^ℓ. Thus, $\det(A)$ is the reduced defining equation for $\mathcal{E}_{\ell-1}$, except that we are viewing it as defined on V_ℓ via composition with projection π_ℓ. Hence on V_ℓ it defines $\pi_\ell^*(\mathcal{E}_{\ell-1})$. Also, $\det(B)$ defines the generalized determinantal variety $\mathcal{D}_\ell$. Thus, the determinant of the matrix (4.1), which defines $\mathcal{E}_\ell$, has defining variety $\pi_\ell^*(\mathcal{E}_{\ell-1}) \cup \mathcal{D}_\ell$ as claimed. □

We can also give a levelwise criterion in the tower that it have a block representation.

Proposition 4.5. *Suppose that a tower of linear algebraic groups and representations $(\mathbf{G}, \mathbf{V})$ satisfies the following conditions: the representation of G_1 on V_1 is a block representation and for all $\ell \geq 1$ the following hold:*

i) *The representation V_ℓ of G_ℓ is an equidimensional representation and has an invariant subspace $W_\ell \subset V_\ell$ of the same dimension as K_ℓ, the connected component of the identity of $\ker(G_\ell \to GL(V_\ell/W_\ell))$.*
ii) *The action of K_ℓ on W_ℓ has a relatively open orbit in V_ℓ, and the relative coefficient determinant for K_ℓ on W_ℓ in V_ℓ is reduced and relatively prime to the coefficient determinant of $G_{\ell-1}$ acting on $V_{\ell-1}$ (pulled back to V_ℓ via projection along W_ℓ).*
iii) *The composition of the natural homomorphisms of representations*

$$(G_{\ell-1}, V_{\ell-1}) \to (G_\ell, V_\ell) \to (G_\ell/K_\ell, V_\ell/W_\ell)$$

is an isomorphism of representations.

Then, the tower $(\mathbf{G}, \mathbf{V})$ has the structure a of block representation. Furthermore, the decomposition for (G_ℓ, V_ℓ) is given by (4.3) and (4.4).

If the representation of G_1 on V_1 only has a nonreduced block representation or in condition ii) the relative coefficient determinants are not all reduced or not relatively prime then the tower has a nonreduced block structure.

Remark 4.6. If p_j denotes the relative coefficient determinant for the action of K_j on W_j in V_j, then it will follow by the proof of Theorem 4.3 that it is sufficent that p_ℓ is reduced and relatively prime to each of the p_j, $j = 1, \dots, \ell-1$ (pulled back to V_ℓ by the projection $\pi_i : V_\ell \to V_i$ consisting of the compositions of projections along the W_j).

Proof. We first show by induction on ℓ that each representation of G_ℓ on V_ℓ is a block representation. We begin by defining the decomposition for (G_ℓ, V_ℓ).

We let $\pi_j : G_j \to G_{j-1}$ denote the projection obtained from the composition of the projection $G_j \to G_j/K_j$ with the inverse of the isomorphism given by condition iii). We can analogously define $\pi_j' : V_j \to V_{j-1}$. Composing successively the π_j we obtain projections $\pi_\ell^j : G_\ell \to G_j$. Likewise we define $\pi_\ell^{j\,\prime} : V_\ell \to V_j$ by successive compositions of the π_j'.

Then, we define for $1 < j \leq \ell$,

$$W_j^\ell = \pi_\ell^{j\,\prime\,-1}(W_j) \quad \text{and} \quad K_j^\ell = \pi_\ell^{j\,-1}(K_j). \tag{4.2}$$

For $j = 1$, we let $W_1^\ell = W_\ell$ and $K_1^\ell = K_\ell$ (also $K_0^\ell = \ker(G_\ell \to GL(V_\ell))$).

Then, the decomposition is given by

$$V_\ell = W_\ell^\ell \supset W_{\ell-1}^\ell \supset \cdots \supset W_1^\ell \supset W_0^\ell = (0) \qquad \text{and} \tag{4.3}$$

$$G_\ell = K_\ell^\ell \supset K_{\ell-1}^\ell \supset \cdots \supset K_1^\ell \supset K_0^\ell\ . \tag{4.4}$$

Before continuing, we note that here we are using the trivial block decomposition for G_1 on V_1. However, given a full block representation for G_1 on V_1, we can refine the block representation given here by pulling it back via the π_ℓ^1 and $\pi_\ell^{1\,\prime}$.

Then, for $\ell = 1$, the decomposition given by (4.3) and (4.4) is that for G_1 on V_1. We assume it is true for all $j < \ell$, and consider the representation of G_ℓ on V_ℓ.

By assumption, $G_\ell/K_\ell \simeq G_{\ell-1}$ and $V_\ell/W_\ell \simeq V_{\ell-1}$ as $G_{\ell-1}$ representations. By the assumption, the relative coefficient determinant for the representation of K_ℓ on W_ℓ is reduced and relatively prime to the coefficient determinant of $G_{\ell-1}$ acting on $V_{\ell-1}$. Hence, we may apply Proposition 3.3 to conclude that the representation of G_ℓ on V_ℓ has a block representation obtained by pulling back that of $G_{\ell-1}$ on $V_{\ell-1}$ via the projections $\pi_\ell : G_\ell \to G_{\ell-1}$ and $\pi'_\ell : V_\ell \to V_{\ell-1}$. Specifically, for $j > 1$ we let

$$W_j^\ell = \pi_\ell'^{\,-1}(W_j^{\ell-1}) \quad \text{and} \quad K_j^\ell = \pi_\ell^{-1}(K_j^\ell)\ . \tag{4.5}$$

For $j = 1$, $W_1^\ell = W_\ell$ and $K_1^\ell = K_\ell$, while $K_0^\ell = \ker(G_\ell \to GL(V_\ell))$ is a finite group. However, by the inductive assumption, (4.5) gives exactly W_j^ℓ and K_j^ℓ defined for (4.2). This establishes the inductive step.

Then, assumption iii) establishes the second condition for the tower having a block structure.

If (G_1, V_1) only has a nonreduced block structure or the relative coefficient determinants are not reduced or not relatively prime, then the above proof only shows the (G_ℓ, V_ℓ) have nonreduced block structures. □

The use of this Proposition to establish that certain towers of representations have block structure will ultimately require that we establish that the relative coefficient determinants are irreducible and relatively prime. The following Lemma will be applied in later sections for each of the families that we consider.

Lemma 4.7. *Suppose $f \in \mathbb{C}[x_1, \dots, x_n, y]$, and $g = \frac{\partial f}{\partial y} \in \mathbb{C}[x_1, \dots, x_n]$.*

i) *If $\gcd(f, g) = 1$ then f is irreducible.*
ii) *If for each irreducible factor g_1 of g, there is a $(x_{10}, \dots, x_{n0}, y_0)$ so that $g_1(x_{10}, \dots, x_{n0}) = 0$ while $f(x_{10}, \dots, x_{n0}, y_0) \neq 0$, then f is irreducible.*

Proof of Lemma 4.7. For i), we let $R = \mathbb{C}[x_1, \dots, x_n]$ so $R[y] = \mathbb{C}[x_1, \dots, x_n, y]$. For $h \in R[y]$ we let $c(h)$ denote the content of h. By assumption we may write $f = g \cdot y + g_0$ with $g_0, g \in R$. Also, by assumption $\gcd(f, g) = 1$; hence $\gcd(g_0, g) = 1$.

Suppose $f = h_1 \cdot h_2$ in $R[y]$, then by the Gauss Lemma, $c(f) = c(h_1) \cdot c(h_2)$. Also, $\gcd(g_0, g) = 1$ implies $c(f) = 1$. In addition, $1 = \deg_y(f) = \deg_y(h_1) + \deg_y(h_2)$. Hence, one of the h_i, say $h_1 \in R$. Hence, $h_1 = c(h_1)$ divides 1 in R. This implies h_1 is a constant, and hence f is irreducible.

For ii), suppose $\gcd(f, g) \neq 1$. Let g_1 be a common irreducible factor. By assumption there is a $(x_{10}, \dots, x_{n0}, y_0)$ so that $g_1(x_{10}, \dots, x_{n0}) = 0$ while $f(x_{10}, \dots, x_{n0}, y_0) \neq 0$. This contradicts g_1 dividing f. Hence, $\gcd(f, g) = 1$, and by i), f is irreducible. $\square$

5. Basic Matrix Computations for Block Representations

To apply the results of the preceding sections, we must first perform several basic calculations for two basic families of representations. While the calculations themselves are straightforward, we collect them together in a form immediately applicable to the towers of representations we consider. We let $M_{m,p}$ denote the space of $m \times p$ complex matrices. We consider the following representations.

i) the *linear transformation representation* on $M_{m,p}$: defined by

$$\begin{aligned} \psi : GL_m(\mathbb{C}) \times GL_p(\mathbb{C}) &\to GL(M_{m,p}) \\ \psi(B, C)(A) &= B\,A\,C^{-1} \end{aligned} \tag{5.1}$$

ii) the *bilinear form representation* on $M_{m,m}$: defined by

$$\begin{aligned} \theta : GL_m(\mathbb{C}) &\to GL(M_{m,m}) \\ \theta(B)(A) &= B\,A\,B^T. \end{aligned} \tag{5.2}$$

We will then further apply these computations to the restrictions to families of solvable subgroups and subspaces which form towers $\rho_\ell : G_\ell \to GL(V_\ell)$ of representations. For these representations and their restrictions, we will carry out the following.

(1) identify a flag of invariant subspaces $\{V_j\}$;
(2) from among the invariant subspaces, identify distinguished subspaces W_j and the corresponding normal subgroups $K_j = \ker(G \to GL(V/W_j))$;
(3) compute the representation vector fields for a basis of the Lie algebra; and
(4) compute the relative coefficient matrix for the representation of K_j/K_{j-1} on W_j/W_{j-1} in V/W_{j-1} using special bases for the Lie algebra $\mathbf{k}_j/\mathbf{k}_{j-1}$ (the Lie algebra of K_j/K_{j-1}) and W_j/W_{j-1} to determine the diagonal blocks in the block representation.

Linear Transformation Representations :

Next, we let B_m denote the Borel subgroup of $GL(\mathbb{C}^m)$ consisting of invertible lower triangular matrices, and B_p^T denote the subgroup of $GL(\mathbb{C}^p)$ consisting of invertible upper triangular matrices (this is the transpose of B_p). We consider the representation ρ of $B_m \times B_p^T$ on $M_{m,p}$ obtained by restricting the linear tranformation representation ψ . Eventually we will be interested in the cases $p = m$ or $m + 1$.

Invariant Subspaces and Kernels of Quotient Representations :

To simplify notation, for fixed m and p we denote $M_{m,p}$ as M. We first define for given $0 \leq \ell \leq m$ and $0 \leq k \leq p$ the subspace $M^{(\ell,k)}$ of M which consists of matrices for which the upper left-hand $(m - \ell) \times (p - k)$ submatrix is 0. Thus, $\dim M^{(\ell,k)}$ decreases with decreasing ℓ and k. Given m and p we let $E_{i,j}$ denote the elementary $m \times p$ matrix with 1 in the i, j–th position, and 0 elsewhere.

We first observe

Lemma 5.1. *The subspaces $M^{(\ell,k)}$ are invariant subspaces for the representation of $B_m \times B_p^T$.*

Proof. We partition m into $m-\ell$ and ℓ and p into $p-k$ and k, and write our matrices in block forms with the rows and columns so partitioned. Then,

$$\begin{pmatrix} B' & 0 \\ * & * \end{pmatrix} \cdot \begin{pmatrix} A' & * \\ * & * \end{pmatrix} \cdot \begin{pmatrix} C'^{-1} & * \\ 0 & * \end{pmatrix} \;=\; \begin{pmatrix} B'\,A'\,C'^{-1} & * \\ * & * \end{pmatrix} . \tag{5.3}$$

Then, (5.3) shows that if $A' = 0$ then so is $B'\,A'\,C'^{-1}$. □

Then, we obtain an induced quotient representation

$$\rho_{\ell,k} : B_m \times B_p^T \to \mathrm{GL}(M/M^{(\ell,k)}) .$$

We consider the subgroup $K^{(\ell,k)}$ consisting of elements of $B_m \times B_p^T$ of the form

$$\left(\begin{pmatrix} \lambda \cdot I_{m-\ell} & 0 \\ * & * \end{pmatrix}, \begin{pmatrix} \lambda \cdot I_{p-k} & * \\ 0 & * \end{pmatrix} \right), \qquad \lambda \in \mathbb{C}^* . \tag{5.4}$$

This subgroup has the following role.

Lemma 5.2. *For the quotient representation $\rho_{\ell,k}$, $\ker(\rho_{\ell,k}) = K^{(\ell,k)}$.*

Proof of Lemma 5.2. We use the partition as in equation (5.3). The product is in $\ker(\rho_{\ell,k})$ if and only if

$$B'\,A'\,C'^{-1} \;=\; A' \tag{5.5}$$

for all $(m-\ell) \times (p-k)$ matrices A'. It follows that $K^{(\ell,k)} \subseteq \ker(\rho_{\ell,k})$.

For the reverse inclusion, we let $B' = (b_{i,j})$ and $C' = (c_{i,j})$ and examine (5.5) for $A' = E_{i,j}$, the $(m-\ell) \times (p-k)$–elementary matrices for $1 \leq i \leq m-\ell$, and $1 \leq j \leq p-k$. We see that $b_{i,j} = 0$ and $c_{i,j} = 0$ for $i \neq j$, and then $b_{i,i} = b_{j,j}$ and $c_{i,i} = c_{j,j}$ for all i and j. This implies $B' = \lambda I_{m-\ell}$, $C' = \kappa I_{p-k}$, and (5.3) implies $\lambda = \kappa$. □

We note that a consequence of Lemma 5.2, is that the representation ρ is not faithful, and hence cannot be an equidimensional representation. We shall see in the next section that by restricting to appropriate solvable subgroups we can overcome this in different ways. First, we determine the associated representation vector fields.

Representation Vector Fields :

The derivative of ρ at (I_m, I_p) is given by straightforward calculation to be

$$d\rho(B, C)(A) \;=\; B\,A - A\,C \tag{5.6}$$

for $(B, C) \in \mathfrak{gl}_m \oplus \mathfrak{gl}_p$ and $A \in M$. This computes $\dfrac{\partial}{\partial t}(\exp(tB)\,A\,\exp(tC)^{-1})_{|t=0}$, and hence is the representation vector field corresponding to (B, C) evaluated at A. We obtain two sets of vector fields

$$\xi_{i,j} = \xi_{(E_{i,j},0)} \quad \text{and} \quad \zeta_{i,j} = \xi_{(0,E_{i,j})} . \tag{5.7}$$

We calculate them using (5.6) to obtain for $A = (a_{i,j})$,

$$\xi_{k,\ell}(A) \;=\; E_{k,\ell}\, A \;=\; \sum_{s=1}^{p} a_{\ell,s} E_{k,s} \quad \text{and} \tag{5.8}$$
$$\zeta_{k,\ell}(A) \;=\; -A\, E_{k,\ell} \;=\; -\sum_{s=1}^{m} a_{s,k} E_{s,\ell}\ .$$

These can be described as follows: $\xi_{k,\ell}$ associates to the matrix A the matrix all of whose rows are zero except for the k–th which is the ℓ–row of A. Similarly $\zeta_{k,\ell}$ associates to the matrix A the matrix all of whose columns are zero except for the k–th column which is minus the ℓ–th column of A.

Bilinear Form Representations :

We next make analogous computations for the bilinear form representations.

Invariant Subspaces and Kernels of Quotient Representations : For the bilinear form representation θ on $M = M_{m,m}$, we observe that it is obtained by composition of ρ (for the case $p = m$) with the Lie group homomorphism $\sigma : B_m \to B_m \times B_m^T$ defined by $\sigma(B) = (B, (B^{-1})^T)$. Since $\theta = \rho \circ \sigma$, it is immediate that the invariant subspaces $M^{(\ell,k)}$ for $B_m \times B_m^T$ via ρ are also invariant for B_m via θ. Also, it immediately follows that for the quotient representation

$$\theta_{\ell,k} : B_m \to GL(M/M^{(\ell,k)})\,,$$

$\ker(\theta_{\ell,k}) = \sigma^{-1}(\ker(\rho_{\ell,k}))$. However, by Lemma 5.2, $\ker(\rho_{\ell,k}) = K^{(\ell,k)}$. Thus, an element $B \in \ker(\theta_{\ell,k}) = \sigma^{-1}(K^{(\ell,k)})$ has the form

$$B \;=\; \begin{pmatrix} \lambda \cdot I_\ell & 0 \\ * & * \end{pmatrix}, \qquad \lambda \in \mathbb{C}^*\ . \tag{5.9}$$

Also, by (5.4)

$$(B^{-1})^T \;=\; \begin{pmatrix} \lambda^{-1} \cdot I_\ell & * \\ 0 & * \end{pmatrix} \;=\; \begin{pmatrix} \lambda \cdot I_k & * \\ 0 & * \end{pmatrix}\ . \tag{5.10}$$

Hence, $\lambda = \pm 1$, and

$$B \;=\; \begin{pmatrix} \pm I_r & 0 \\ * & * \end{pmatrix}, \qquad \text{where } r = \max\{\ell, k\}\,. \tag{5.11}$$

We summarize this in the following Lemma.

Lemma 5.3. *For the bilinear form representations,*

1. *The $M^{(\ell,k)}$ are invariant subspaces.*
2. *The kernel of the quotient representation $\theta_{\ell,k}$ consists of the elements of the form (5.11).*

Representation Vector Fields :

We can compute the representation vector fields either by using the naturality of the exponential diagram or by directly computing $d\theta$. In the first case, we see that corresponding to $E_{k,\ell}$ is the vector field $\xi_{E_{k,\ell}} = \xi_{k,\ell} - \zeta_{\ell,k}$ using the notation of (5.7).

Alternatively, the corresponding representation for Lie algebras $\mathbf{b}_m$ sends $B \in \mathbf{b}_m$ to the linear transformation sending $A \mapsto B\,A + A\,B^T$. This also defines the corresponding representation vector field ξ_B at A. Applied to $E_{k,\ell}$, we obtain

$$\xi_{E_{k,\ell}}(A) \; = \; E_{k,\ell}\,A \, + \, A\,E_{\ell,k} \; . \tag{5.12}$$

This action can be viewed as the action on bilinear forms defined by matrices A. We will eventually restrict this action to symmetric and skew–symmetric bilinear forms. We apply the above analysis to this representation.

To continue further, we next identify the solvable subgroups to which we will restrict the representations in order to obtain equidimensional representations.

6. Cholesky-Type Factorizations as Block Representations of Solvable Linear Algebraic Groups

In this section, we explain how the various forms of classical "Cholesky-type factorization" can be understood via representations of solvable groups on spaces of matrices leading to the construction of free (or free*) divisors containing the variety of singular matrices.

Traditionally, it is well–known that certain matrices can be put in normal forms after multiplication by appropriate matrices. The basic example is for symmetric matrices, where a symmetric matrix A can be diagonalized by composing it with an appropriate invertible matrix B to obtain $B \cdot A \cdot B^T$. The choice of B is highly nonunique. For real matrices, Cholesky factorization gives a unique choice for B provided A satisfies certain determinantal conditions. More generally, by "Cholesky-type factorization" we mean a general collection of results for factoring real matrices into products of upper and lower triangular matrices. These factorizations are traditionally used to simplify the solution of certain problems in applied linear algebra. For the cases of symmetric matrices and LU decomposition for general $m \times m$ matrices see [Dm] and for skew symmetric matrices see [BBW].

Here we state the versions of these theorems for complex matrices. The complex versions can be proven either by directly adapting the real proofs, as in [P], or they will also follow from Theorem 6.2.

Let $A = (a_{ij})$ denote an $m \times m$ complex matrix which may be symmetric, general, or skew-symmetric. We let $A^{(k)}$ denote the $k \times k$ upper left hand corner submatrix.

Theorem 6.1 (Complex Cholesky–Type Factorization)**.**

(1) Complex Cholesky factorization: *If A is a complex symmetric matrix with $\det(A^{(k)}) \neq 0$ for $k = 1, \dots, m$, then there exists a lower triangular matrix B, which is unique up to multiplication by a diagonal matrix with diagonal entries ± 1, so that $A = B \cdot B^T$.*

(2) Complex LU factorization: *If A is a general complex matrix with* $\det(A^{(k)}) \neq 0$ *for $k = 1, \dots, m$, then there exists a unique lower triangular matrix B and a unique upper triangular matrix C which has diagonal entries* $= 1$ *so that $A = B \cdot C$.*

(3) Complex Skew-symmetric Cholesky factorization : *If A is a skew-symmetric matrix for $m = 2\ell$ with* $\det(A^{(2k)}) \neq 0$ *for $k = 1, \dots, \ell$, then there exists a lower block triangular matrix B with 2×2–diagonal blocks of the form a) in (6.1) with complex entries r (i.e. $= r \cdot I$), so that $A = B \cdot J \cdot B^T$, for J the $2\ell \times 2\ell$ skew-symmetric matrix with 2×2–diagonal blocks of the form b) in (6.1). Then, B is unique up to multiplication by block diagonal matrices with a 2×2 diagonal blocks $= \pm I$. For $m = 2\ell + 1$, then there is again a unique factorization except now B has an additional entry of 1 in the last diagonal position, and J is replaced by J' which has J as the upper left corner $2\ell \times 2\ell$ submatrix, with remaining entries $= 0$.*

$$\text{(6.1)} \qquad a) \quad \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, \quad r > 0 \qquad and \quad b) \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Complex Cholesky Factorizations via Solvable Group Representations. We can view these results as really statements about representations of solvable groups on spaces of $m \times m$ complex matrices which will either be symmetric, general, or skew-symmetric (with m even). We consider for each of these cases the analogous representations of solvable linear algebraic groups which we shall show form towers of (possibly nonreduced) block representations for solvable groups.

General $m \times m$ Complex Matrices : As earlier $M_{m,m}$ denotes the space of $m \times m$ general complex matrices, with B_m the Borel subgroup of invertible lower triangular $m \times m$ matrices. We also let N_m be the nilpotent subgroup of B_m^T, consisting of the invertible upper triangular $m \times m$ matrices with 1's on the diagonal. The representation of $B_m \times N_m$ on $M_{m,m}$ is the restriction of the linear transformation representation (5.1). The inclusion homomorphisms $B_{m-1} \times N_{m-1} \hookrightarrow B_m \times N_m$ and inclusions $M_{m-1,m-1} \hookrightarrow M_{m,m}$ are defined as in (6.2).

$$\text{(6.2)} \qquad B \mapsto \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} \qquad C \mapsto \begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad A \mapsto \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$$

These then define a tower of representations of connected solvable algebraic groups.

Second we consider restrictions of the bilinear form representations. We may decompose $M_{m,m}$, viewed as a representation of the Borel subgroup B_m, as $M_{m,m} = Sym_m \oplus Sk_m$, where Sym_m denotes the space of $m \times m$ complex symmetric matrices and Sk_m the space of skew–symmetric matrices. Hence, we can restrict the respresentation to each summand.

Complex Symmetric Matrices : The representation of B_m on Sym_m is the restriction of the bilinear form representation. The inclusion homomorphisms $B_{m-1} \hookrightarrow B_m$ and inclusions $Sym_{m-1} \hookrightarrow Sym_m$ are defined as in (6.2) and define a tower of solvable group representations.

Complex Skew-Symmetric Matrices : If instead we consider the representation on the summand Sk_m, then we further restrict to a subgroup of B_m. For $m = 2\ell$ or $m = 2\ell + 1$, we let D_m denote the subgroup of B_m consisting of all lower triangle matrices of the type described in (3) of Theorem 6.1. The representation of D_m on Sk_m is the restricted representation. The inclusion homomorphism $D_{m-1} \hookrightarrow D_m$ and inclusion $Sk_{m-1} \hookrightarrow Sk_m$ are as in (6.2); and together these representations again form a tower of representations of connected solvable algebraic groups.

The representations in each of these cases are equidimensional representations. Simple counting arguments show the groups and vector spaces have the same dimension. Moreover, in each case the subgroups intersect the kernels of the representations ψ and θ in finite subgroups. Hence they are equidimensional.

The corresponding Cholesky–type factorization then asserts that the representation has an open orbit and that the exceptional orbit variety is defined by the vanishing of one of the conditions for the existence of the factorization. The open orbit is the orbit of one the basic matrices: the identity matrix in the first two cases, and J for the third.

We let $A = (a_{ij})$ denote an $m \times m$ complex matrix which may be symmetric, general, or skew-symmetric. As above, $A^{(k)}$ denotes the $k \times k$ upper left-hand corner submatrix. Then, these towers have the following properties.

Theorem 6.2.

i) *The tower of representations of $\{B_m\}$ on $\{Sym_m\}$ is a tower of block representations and the exceptional orbit varieties are free divisors defined by $\prod_{k=1}^{m} \det(A^{(k)}) = 0$.*
ii) *The tower of representations of $\{B_m \times N_m\}$ on $\{M_m\}$ is a tower of nonreduced block representations and the exceptional orbit varieties are free* divisors defined by $\prod_{k=1}^{m} \det(A^{(k)}) = 0$.*
iii) *The tower of representations of $\{D_m\}$ on $\{Sk_m\}$ is a tower of non-reduced block representations and the exceptional orbit varieties are free* divisors defined by $\prod_{k=1}^{\ell} \det(A^{(2k)}) = 0$, where $m = 2\ell$ or $2\ell + 1$.*

Remark 6.3. We make three remarks regarding this result.

1) Independently, Mond and coworkers [BM], [GMNS] in their work with reductive groups separately discovered the result for symmetric matrices by just directly applying the Saito criterion.

2) In the cases of general or skew-symmetric matrices, the exceptional orbit varieties are only free* divisors. We will see in Theorems 7.1 and 8.1 that we can modify the solvable groups so the resulting representation gives a modified Cholesky–type factorization with exceptional orbit variety which still contains the variety of singular matrices and which is a free divisor.

3) As a corollary of Theorem 6.2, we deduce Cholesky-type factorization in the complex cases as exactly characterizing the elements belonging to the open orbit in each case. The only point which has to be separately checked is the nonuniqueness, which is equivalent to determining the isotropy subgroup for the basic matrix in each case.

Proof of Theorem 6.2. The proof will be an application of Proposition 4.5 for each of the cases. We begin with the case for the linear transformation representation of $G_m = B_m \times N_m$ on $M_{m,m}$, the $m \times m$ matrices. We claim that the partial flag

$$M \;=\; M_{m,m} \supset M^{(m-1,m-1)} \supset \ldots M^{(1,1)} \supset 0 \tag{6.3}$$

(using the notation of §5) gives a nonreduced block representation. By Lemma 5.2, $K_\ell = K^{(\ell,\ell)}$ is the kernel of the quotient representation $\rho_{\ell,\ell} : G_m \to GL(M/M^{(\ell,\ell)})$. We claim that together these give a nonreduced block representation for $(G_m, M_{m,m})$.

To show this, it is sufficient to compute the relative coefficient matrix for the representation of $K_\ell/K_{\ell-1}$ on $M^{(\ell,\ell)}/M^{(\ell-1,\ell-1)}$. In fact, it is useful to introduce a refinement of the decomposition by introducing subrepresentations $M^{(\ell,\ell)} \supset$

$M^{(\ell,\ell-1)} \supset M^{(\ell-1,\ell-1)}$ in the sequence (6.3), and the corresponding kernels given by Lemma 5.2 $K_\ell \supset K^{(\ell,\ell-1)} \supset K_{\ell-1}$.

First, we consider the representation of $K_\ell/K^{(\ell,\ell-1)}$ on $M^{(\ell,\ell)}/M^{(\ell,\ell-1)}$. To simplify notation, we let $\ell' = m-\ell$. We use the complementary bases

$$\{E_{1\,\ell'+1}, E_{2\,\ell'+1}, \dots, E_{\ell'\,\ell'+1}\} \quad \text{to } M^{(\ell,\ell-1)} \text{ in } M^{(\ell,\ell)}, \text{ and}$$

$$\{(0,E_{1\,\ell'+1}), (0,E_{2\,\ell'+1}), \dots, (0,E_{\ell'\,\ell'+1})\} \quad \text{to } \mathbf{k}^{(\ell,\ell-1)} \text{ in } \mathbf{k}^{(\ell,\ell)}.$$

Here $\mathbf{k}^{(\ell,\ell)}/\mathbf{k}^{(\ell,\ell-1)}$ is the Lie algebra of the quotient group $N_m^{(\ell)}/N_m^{(\ell-1)}$, where $N_m^{(k)}$ denotes the subgroup of N_m consisting of matrices whose upper left $(m-k)\times(m-k)$ submatrix is the identity.

Using the notation of (5.7) and §6, the associated representation vector fields are $\zeta_{j,\ell'+1} = \xi_{(0,E_{j,\ell'+1})}$, $j = 1,\dots,\ell$. Then, by using (5.8), we compute the the relative coefficient matrix with respect to the given bases and $A = (a_{i\,j})$

$$\zeta_{j,\ell'+1}(A) \;=\; -\sum_{i=1}^{m} a_{i,j} E_{i\,\ell'+1}\,. \tag{6.4}$$

Using (6.4), we see that with respect to the relative basis for $M^{(\ell,\ell-1)}$ in $M^{(\ell,\ell)}$ we obtain the relative coefficient matrix $-(a_{i,j})$ for $i,j = 1,\dots \ell'$. For the $m\times m$ matrix $A = (a_{i,j})$, this is the matrix $-A^{(\ell')}$.

Second, we consider the representation of $K^{(\ell,\ell-1)}/K_{\ell-1}$ on $M^{(\ell,\ell-1)}/M^{(\ell-1,\ell-1)}$. Now we use the relative bases

$$\{E_{\ell'+1\,1}, E_{\ell'+1\,2}, \dots, E_{\ell'+1\,\ell'+1}\} \quad \text{to } M^{(\ell-1,\ell-1)} \text{ in } M^{(\ell,\ell-1)}, \text{ and}$$

$$\{(E_{\ell'+1\,1},0), (E_{\ell'+1\,2},0), \dots, (E_{\ell'+1\,\ell'+1},0)\} \quad \text{to } \mathbf{k}^{(\ell-1,\ell-1)} \text{ in } \mathbf{k}^{(\ell,\ell-1)}\,.$$

Now $\mathbf{k}^{(\ell-1,\ell-1)}/\mathbf{k}^{(\ell,\ell-1)}$ is the Lie algebra of the quotient group $B_m^{(\ell)}/B_m^{(\ell-1)}$, where $B_m^{(k)}$ denotes the subgroup of B_m consisting of matrices whose upper left $(m-k)\times(m-k)$ submatrix is the identity. By (5.7) the associated representation vector fields are $\xi_{\ell'+1,j} = \xi_{(E_{\ell'+1,j},0)}$, $j = 1,\dots,\ell'+1$. An argument analogous to the above using (5.8) gives the relative coefficient matrix to be the transpose of $A^{(\ell'+1)}$.

Hence, we see that there will be contributions to the coefficient determinant (up to a sign) of $\det A^{(\ell')}$ twice appearing for both $M^{(\ell,\ell-1)} \subset M^{(\ell,\ell)}$ and $M^{(\ell,\ell)}$ in $M^{(\ell+1,\ell)}$. Hence, the coefficient determinant is

$$\prod_{k=1}^{m-1} \det(A^{(k)})^2 \cdot \det(A)\,,$$

which is nonreduced.

Next, for (i), we let $Sym_m^{(j,j)} = Sym_m \cap M^{(j,j)}$. By Lemma 5.3, these are invariant subspaces. We claim that the partial flag

$$Sym_m \supset Sym_m^{(m-1,m-1)} \supset \cdots\cdots Sym_m^{(1,1)} \supset 0 \tag{6.5}$$

gives a block representation of B_m on Sym_m. By Lemma 5.3

$$K_\ell = \left\{ \begin{pmatrix} \pm I_{m-\ell} & 0 \\ * & * \end{pmatrix} \in B_m \right\}$$

is in the kernel of the quotient representation $\rho_{\ell,\ell} : L_m \to \mathrm{GL}(\mathrm{Sym}_m/\mathrm{Sym}_m^{(\ell,\ell)})$; and an argument similar to that in the proof of Lemma 5.2 shows it is the entire kernel.

$$\begin{array}{cc} & \begin{array}{ccc} \overbrace{\qquad\qquad\qquad}^{\ell'} & \overbrace{\qquad\quad}^{2} & \qquad \end{array} \\ \begin{array}{r} \ell' \text{ even}\left\{\vphantom{\begin{array}{c}a\\a\end{array}}\right. \\ 2\left\{\vphantom{\begin{array}{c}a\\a\end{array}}\right. \\ \\ \end{array} & \left(\begin{array}{ccc|cc|c} & & & & & \\ \hline * & \cdots & * & 1 & 0 & \\ * & \cdots & * & 0 & * & \\ \hline & & & & & \end{array}\right)\end{array}$$

FIGURE 1. The group G_m used in the proof of Theorem 6.2(iii).

Let $\ell' = m - \ell$. We let $e_{i\,j} = E_{i\,j} + E_{j\,i} \in Sym_m$ and use the complementary bases

$$\{e_{1\,\ell'+1}, \dots, e_{\ell'+1\,\ell'+1}\} \quad \text{to } \mathrm{Sym}_m(\mathbb{C})^{(\ell-1,\ell-1)} \text{ in } \mathrm{Sym}_m(\mathbb{C})^{(\ell,\ell)}, \text{ and}$$
$$\{E_{\ell'+1\,1}, \dots, E_{\ell'+1\,\ell'+1}\} \quad \text{to } \mathbf{k}_{\ell-1} \text{ in } \mathbf{k}_\ell\,.$$

By an analogue of (5.8), but applied to (5.12), the relative coefficient matrix with respect to these bases at $A \in \mathrm{Sym}_m(\mathbb{C})$ is $A^{(\ell')}$. Hence, the coefficient determinant is

$$\prod_{\ell=1}^{m} \det(A^{(\ell)})\,. \tag{6.6}$$

It only remains to show that (6.6) is reduced. We first show by induction on ℓ that each $p_\ell(A) = \det(A^{(\ell)})$ is irreducible. Since p_1 is homogeneous of degree 1, it is irreducible. Assume by the induction hypothesis that $p_{\ell-1}$ is irreducible. Expanding the determinant p_ℓ along the last column shows that its derivative in the $E_{\ell,\ell}$ direction is $p_{\ell-1}$. Since $p_{\ell-1}$ vanishes at (6.7) and p_ℓ does not, p_ℓ is irreducible by Lemma 4.7(ii).

$$\begin{pmatrix} I_{\ell-1} & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 0 \end{pmatrix} \in \mathrm{Sym}_m(\mathbb{C})\,. \tag{6.7}$$

As each p_ℓ is homogeneous of degree ℓ, the terms of (6.6) are relatively prime and (6.6) is reduced.

Lastly, consider (iii). Though D_m has a non-reduced block representation using invariant subspaces having even-sized zero blocks, it is easier to use a different group which has a finer non-reduced block representation and the same open orbit. Let G_m be defined in the same way as D_m but with 2×2 diagonal blocks of the form $\begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}$, $r \neq 0$. We claim that the partial flag

$$\mathrm{Sk}_m(\mathbb{C}) \supset \mathrm{Sk}_m(\mathbb{C})^{(m-1,m-1)} \supset \cdots \supset \mathrm{Sk}_m(\mathbb{C})^{(1,1)} \supset 0 \tag{6.8}$$

gives a non-reduced block representation of G_m. By Lemma 5.3, (6.8) are invariant subspaces and

$$K_\ell = \left\{ \begin{pmatrix} \pm I_{m-\ell} & 0 \\ * & * \end{pmatrix} \in G_m \right\} \tag{6.9}$$

is in the kernel of the quotient representation $\rho_{\ell,\ell} : G_m \to \mathrm{GL}(\mathrm{Sk}_m(\mathbb{C})/\mathrm{Sk}_m(\mathbb{C})^{(\ell,\ell)})$.

We let $\bar{e}_{i\,j} = E_{i\,j} - E_{j\,i} \in Sk_m(\mathbb{C})$ for $1 \leq i < j \leq m$ and let $\ell' = m - \ell$. We see in Figure 1 the form of G_m, and obtain the resulting complementary bases.

When ℓ' is even, we use the complementary bases

$$\{\bar{e}_{1\,\ell'+1}, \ldots, \bar{e}_{\ell'\,\ell'+1}\} \quad \text{to } \mathrm{Sk}_m(\mathbb{C})^{(\ell-1,\ell-1)} \text{ in } \mathrm{Sk}_m(\mathbb{C})^{(\ell,\ell)}, \text{ and}$$

$$\{E_{\ell'+1\,1}, E_{\ell'+1\,2}, \ldots, E_{\ell'+1\,\ell'}\} \quad \text{to } \mathbf{k}^{(\ell-1,\ell-1)} \text{ in } \mathbf{k}^{(\ell,\ell)}.$$

By an analogue of (5.8) for (5.12), we find that at $A = (a_{i\,j}) \in \mathrm{Sk}_m(\mathbb{C})$, the relative coefficient matrix for these bases is $A^{(\ell')}$. Its determinant is the square of the Pfaffian $\mathrm{Pf}(A^{(\ell')})$.

When ℓ' is odd, we use the complementary bases.

$$\{\bar{e}_{1\,\ell'+1}, \ldots, \bar{e}_{\ell'\,\ell'+1}\} \quad \text{to } \mathrm{Sk}_m(\mathbb{C})^{(\ell-1,\ell-1)} \text{ in } \mathrm{Sk}_m(\mathbb{C})^{(\ell,\ell)}, \text{ and}$$

$$\{E_{\ell'+1\,1}, \ldots, E_{\ell'+1\,\ell'-1}, E_{\ell'+1\,\ell'+1}\} \quad \text{to } \mathbf{k}^{(\ell-1,\ell-1)} \text{ in } \mathbf{k}^{(\ell,\ell)}.$$

We find that the resulting relative coefficient matrix for these bases is $A^{(\ell'+1)}$ with column ℓ' and row $\ell'+1$ deleted. Its determinant factors as the product of Pfaffians, $\mathrm{Pf}(A^{(\ell'+1)})\mathrm{Pf}(A^{(\ell'-1)})$ (see [MM], §406-415). Hence, the coefficient determinant is nonreduced with components as claimed.

We now show that G_m and D_m have the same open orbit. Let J be the matrix from Theorem 6.1 (3), an element of the open orbit of G_m. Let K be the group of invertible $m \times m$ diagonal matrices with 2×2 diagonal blocks in $\mathrm{SL}_2(\mathbb{C})$ (with a last entry of 1 if m is odd). Easy calculations show that K lies in the isotropy group at J, and that for all $A \in G_m$ (resp., all $B \in D_m$), there exists a $C \in K$ so that $AC \in D_m$ (resp., $BC \in G_m$); thus $AJA^T = ACJ(AC)^T$ (resp., $BJB^T = BCJ(BC)^T$), and G_m and D_m have the same open orbit. $\square$

7. Modified Cholesky–Type Factorizations as Block Representations

In the previous section we saw that for both general $m \times m$ matrices and skew–symmetric matrices, the corresponding exceptional orbit varieties are only free* divisors. In this section we address the first case by considering a modification of the Cholesky-type representation for general $m \times m$ matrices. This further extends to the space of $(m-1) \times m$ general matrices. In each case there will result a modified form of Cholesky–type factorization.

General $m \times m$ complex matrices :

For general $m \times m$ complex matrices we let C_m denote the subgroup of invertible upper triangular matrices with first diagonal entry $= 1$ and other entries in the first row 0. C_m is naturally isomorphic to B_{m-1}^T via

$$\begin{aligned} B_{m-1}^T &\longrightarrow C_m \\ B &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix} . \end{aligned} \tag{7.1}$$

We consider the action of $B_m \times C_m$ on $V = M_{m,m}$ by $(B, C) \cdot A = B\,A\,C^{-1}$. This is the restriction of the linear transformation representation. We again have the natural inclusions $M_{m,m} \hookrightarrow M_{m+1,m+1}$ and $B_m \times C_m \hookrightarrow B_{m+1} \times C_{m+1}$ where the inclusions (of each factor) are as in (6.2). These inclusions define a tower of representations of $\{B_m \times C_m\}$ on $\{M_{m,m}\}$.

General $(m-1) \times m$ complex matrices :

We modify the preceding action to obtain a representation of $B_{m-1} \times C_m$ on $V = M_{m-1,m}$ by $(B, C) \cdot A = B\,A\,C^{-1}$. We again have the natural inclusions $M_{m-1,m} \hookrightarrow M_{m,m+1}$ as in (6.2). Together with the natural inclusions $B_{m-1} \times C_m \hookrightarrow B_m \times C_{m+1}$, we again obtain a tower of representations of $\{B_{m-1} \times C_m\}$ on $\{M_{m-1,m}\}$.

To describe the exceptional orbit varieties, for an $m \times m$ matrix A, we let $\hat{A}$ denote the $m \times (m-1)$ matrix obtained by deleting the first column of A. If instead A is an $(m-1) \times m$ matrix, we let $\hat{A}$ denote the $(m-1) \times (m-1)$ matrix obtained by deleting the first column of A. In either case, we let $\hat{A}^{(k)}$ denote the $k \times k$ upper left submatrix of $\hat{A}$, for $1 \leq k \leq m-1$. Then, the towers of modified Cholesky–type representations given above have the following properties.

Theorem 7.1 (Modified Cholesky–Type Representation)**.**

(1) Modified LU decomposition: *The tower of representations* $\{B_m \times C_m\}$ *on* $\{M_{m,m}\}$ *has a block representation and the exceptional orbit varieties are free divisors defined by*

$$\prod_{k=1}^{m} \det(A^{(k)}) \cdot \prod_{k=1}^{m-1} \det(\hat{A}^{(k)}) \;=\; 0\,.$$

(2) Modified Cholesky representation for $(m-1) \times m$ matrices: *The tower of representations* $\{B_{m-1} \times C_m\}$ *on* $\{M_{m-1,m}\}$ *has a block representation and the exceptional orbit varieties are free divisors defined by*

$$\prod_{k=1}^{m-1} \det(A^{(k)}) \cdot \prod_{k=1}^{m-1} \det(\hat{A}^{(k)}) \;=\; 0\,.$$

Proof. First we let τ denote the restriction of ρ to $G_m = B_m \times C_m$. We will apply Proposition 4.5 using the same chain of invariant subspaces $\{W_j\}$ formed from $M^{(\ell,\ell)}$ and the refinements obtained by introducing the intermediate subspaces $M^{(\ell,\ell-1)}$ used in the proof of ii) in Theorem 6.2. Because the group B_m is unchanged the computation for the representation of $K^{(\ell,\ell-1)}/K_{\ell-1}$ on $M^{(\ell,\ell-1)}/M^{(\ell-1,\ell-1)}$ is the same as in ii) of Theorem 6.2.

We next have to replace the calculation for N_m by that for C_m for the representation of $K_\ell/K^{(\ell,\ell-1)}$ on $M^{(\ell,\ell)}/M^{(\ell,\ell-1)}$. We note that this changes exactly one vector in the basis, replacing $E_{\ell'+1\,1}$ by $E_{\ell'+1\,\ell'+1}$. When we compute the associated representation vector field, we obtain the column vector formed from the first ℓ' entries of the $\ell'+1$ column of A. Hence, we remove the first column and replace it by the $\ell'+1$-st column. This is exactly the matrix $-(\hat{A})^{(\ell')}$. Hence the coefficent determinant is (up to a sign)

$$\prod_{j=1}^{m} \det(A^{(j)}) \cdot \prod_{k=1}^{m-1} \det(\hat{A}^{(k)})\,. \tag{7.2}$$

We now show (7.2) is reduced. We proceed by induction on the size of the determinant. The functions $A \mapsto \det(A^{(1)})$ and $A \mapsto \det(\hat{A}^{(1)})$ are irreducible since they are homogeneous of degree 1. Suppose $A \mapsto \det(A^{(k)})$ (respectively, $A \mapsto \det(\hat{A}^{(k)})$) are irreducible for $k < j$. These determinants are related by differentiation:

$$\frac{\partial \det(A^{(j)})}{\partial a_{j\,j}} = \det(A^{(j-1)}) \qquad \text{and} \qquad \frac{\partial \det(\hat{A}^{(j)})}{\partial a_{j\,j+1}} = \det(\hat{A}^{(j-1)})\,.$$

Thus, we may apply Lemma 4.7(ii), using the induction hypothesis, to (7.3)a) (respectively, (7.3)b)) and deduce that the $j \times j$ determinants are irreducible.

$$(7.3) \qquad a) \begin{pmatrix} I_{j-2} & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 0 \end{pmatrix} \qquad b) \begin{pmatrix} 0_{(j-2)\times 1} & I_{j-2} & & & \\ & & 0 & 1 & \\ & & 1 & 0 & \\ & & & & 0 \end{pmatrix}$$

Thus, each factor of (7.2) is irreducible. Based on the (polynomial) degrees of $A \mapsto \det(A^{(j)})$ and $A \mapsto \det(\hat{A}^{(j)})$ and their values at (7.3)a), we conclude the factors are irreducible and distinct; hence, (7.2) is reduced.

Hence, the modifed Cholesky-type representation on $m \times m$ complex matrices is a block representation. Furthermore, the induced quotient representation of $G_m = B_m \times C_m$ on $M_{m,m}/M^{(1,1)}$ has kernel K_1 and it is easy to check that $G_m/K_1 \simeq G_{m-1}$. Hence, the $(G_m, M_{m,m})$ form a tower of block representations.

To obtain the second part of the theorem from the first, we observe that when using the intermediate subspaces $M^{(\ell,\ell-1)}$, the last nontrivial group and subspace in the block structure for $(G_m, M_{m,m})$ is $K^{(m-1,m)}$ and $M^{(1,0)}_{m,m}$, whose relative coefficient determinant is the determinant function. By Proposition 3.1, the representation of $G_m/K^{(1,0)}$ on the quotient $M_{m,m}/M^{(1,0)}_{m,m}$ gives a block representation isomorphic to the one described. In turn, the block representation for $M_{m-1,m}$ has $M^{(1,1)}_{m-1,m}$ as an invariant subspace with $K^{(1,1)}$ the kernel of the induced quotient representation. Forming the quotient $M_{m-1,m}/M^{(1,1)}_{m-1,m}$ gives a block representation of $G_m/K^{(1,1)}$ isomorphic to the one on $M_{m-2,m-1}$. Hence, we obtain a tower of block representations. $\square$

We have the following consequences for modified forms of Cholesky–type factorizations which follow from Theorem 7.1.

Theorem 7.2 (Modified Cholesky–Type Factorization)**.**

1. Modified LU decomposition: *If A is a general complex $m \times m$ matrix with* $\det(A^{(k)}) \neq 0$ *for* $k = 1, \dots, m$ *and* $\det(\hat{A}^{(k)}) \neq 0$ *for* $k = 1, \dots, m-1$, *then there exists a unique lower triangular matrix B and a unique upper triangular matrix C, which has first diagonal entry $= 1$, and remaining first row entries $= 0$ so that $A = B \cdot K \cdot C$, where K has the form of a) in (7.4).*
2. Modified Cholesky factorization for $(m-1) \times m$ matrices: *If A is an $(m-1) \times m$ complex matrix with* $\det(A^{(k)}) \neq 0$ *for* $k = 1, \dots, m-1$, $\det(\hat{A}^{(k)}) \neq 0$ *for* $k = 1, \dots, m-1$, *then there exists a unique $(m-1) \times (m-1)$ lower triangular matrix B and a unique $m \times m$ matrix C having the same form as in (1), so that $A = B \cdot K' \cdot C$, where K' has the form of b) in (7.4).*

Cholesky–type Factorization	Matrix Space	Solvable Group	Representation
symmetric matrices	Sym_m	B_m	Bil
general matrices	$M_{m,m}$	$B_m \times N_m$	LT
skew-symmetric	Sk_m	D_m	Bil
Modified Cholesky –type Factorization			
general $m \times m$	$M_{m,m}$	$B_m \times C_m$	LT
general $(m-1) \times m$	$M_{m-1,m}$	$B_{m-1} \times C_m$	LT

Table 1. Solvable groups and (nonreduced) Block representations for (modified) Cholesky–type Factorization arising from either the linear transformation representation (LT) or bilinear representation (Bil).

$$(7.4) \qquad a) \quad \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad and \quad b) \quad \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

The factorization theorem follows from Theorem 7.1 by directly checking that the matrices a), repectively b), in (7.4) are not in the exceptional orbit varieties.

We summarize in Table 1, each type of complex (modified) Cholesky–type representation, the space of complex matrices, the solvable group and the representation type.

8. Block Representations for Nonlinear Solvable Lie Algebras

In the preceding section we saw that the Cholesky-type representations for the spaces of general $m \times m$ and $m \times (m+1)$ matrices were nonreduced block representations, yielding free* divisors. However, by modifying the solvable groups and representations we obtained block representations, whose exceptional orbit varieties are free divisors and contain the determinantal varieties. In this section, we take a different approach to modifying the Cholesky representation on $\mathrm{Sk}_m(\mathbb{C})$ to obtain a representation whose exceptional orbit variety is a free divisor containing the Pfaffian variety. The underlying reason for this change is that factorization properties of determinants of submatrices of skew- symmetric matrices suggests that a reduced exceptional orbit variety may not be possible for a solvable linear algebraic group. However, the essential ideas of the block representation will continue to be valid if we replace the finite dimensional solvable Lie algebra by an infinite dimensional solvable holomorphic Lie algebra which has the analog of a block representation.

We will then obtain the exceptional orbit varieties which are "nonlinear"free divisors. The resulting sequence of free divisors on $\mathrm{Sk}_m(\mathbb{C})$ (for all m) have the tower-like property that they are formed by repeated additions of generalized determinantal and Pfaffian varieties (c.f., Theorem 4.3(iii)). We shall present the main ideas here, but we will refer to §5.2 of [P] for certain technical details of the computations.

We first consider the bilinear form representation on $\mathrm{Sk}_m(\mathbb{C})$ of the group

$$G_m = \left\{ \begin{pmatrix} T_2 & 0_{2,m-2} \\ 0_{m-2,2} & B_{m-2} \end{pmatrix} \right\}, \tag{8.1}$$

where T_2 is the group of 2×2 invertible diagonal matrices. Let $\mathfrak{g}_m$ be the Lie algebra of G_m. When $m = 3$, the exceptional orbit variety of this representation is the normal crossings linear free divisor on $\mathrm{Sk}_3(\mathbb{C})$. For $m > 3$, this representation cannot have an open orbit, as $\dim(\mathrm{Sk}_m(\mathbb{C})) - \dim(G_m) = m - 3$. Nonetheless, this is a representation of the finite dimensional solvable Lie algebra $\mathfrak{g}_m$ on $\mathrm{Sk}_m(\mathbb{C})$. The associated representation vector fields generate a solvable holomorphic Lie algebra $\mathcal{L}_m^{(0)}$. Our goal is to construct an extension of $\mathcal{L}_m^{(0)}$ by adjoining as generators $m-3$ nonlinear Pfaffian vector fields to obtain a solvable holomorphic Lie algebra $\mathcal{L}_m$ which is a free $\mathcal{O}_{s_m}$ module of rank $s_m = \dim_{\mathbb{C}} \mathrm{Sk}_m(\mathbb{C}) = \binom{m}{2}$, where we abbreviate $\mathcal{O}_{Sk_m(\mathbb{C}),0}$ as $\mathcal{O}_{s_m}$. Then we will apply Saito's criterion to deduce that the resulting "exceptional orbit variety" is a free divisor.

For $S \subseteq \{1, \dots, m\}$ and $A \in \mathrm{Sk}_m(\mathbb{C})$, we define $\mathrm{Pf}_S(A)$ to be the Pfaffian of the matrix obtained by deleting all rows and columns of A not indexed by S. For any $i \in \{2, \dots, m\}$, let $\epsilon(i)$ be either 1 or 2, so that $\epsilon(i)$ and i have opposite parity, and hence $\{\epsilon(i), \epsilon(i)+1, \dots, i\}$ has even cardinality. As in §6, we let $\bar{e}_{i,j} = E_{i,j} - E_{j,i} \in \mathrm{Sk}_m(\mathbb{C})$ for $1 \leq i < j \leq m$. Then for $2 \leq k \leq m-2$, define

$$\eta_k(A) = \sum_{k<p<q\leq m} \mathrm{Pf}_{\{\epsilon(k),\dots,k,p,q\}}(A) \cdot \bar{e}_{p,q}, \tag{8.2}$$

which is a (homogeneous) vector field on $\mathrm{Sk}_m(\mathbb{C})$ of degree $\lfloor \frac{k}{2} \rfloor$. Here $\bar{e}_{p,q}$, viewed as a constant vector field, denotes $\dfrac{\partial}{\partial a_{p,q}} - \dfrac{\partial}{\partial a_{q,p}}$ and hence has degree -1.

For example, if $m = 2\ell$, the degrees of the η_k form a sequence $1, 1, 2, 2, \dots,$ ending with a single top degree $\ell - 1$; while for $m = 2\ell + 1$, the sequence consists of successive pairs of integers. For m even, the top vector field is just $\mathrm{Pf}(A)\, \bar{e}_{m-1,m}$.

Then, $\mathcal{L}_m$ will be the $\mathcal{O}_{s_m}$-module generated by a basis $\{\xi_{E_{i,j}}\}$ of representation vector fields associated to G_m and $\{\eta_k,\ 2 \leq k \leq m-2\}$. Note this module has s_m generators so Saito's criterion may be applied. We let, as earlier, $\hat{A}$ denote the matrix A with the left column removed, and let $\hat{\hat{A}}$ be the matrix A with the two left columns deleted.

Then, the modification of the Cholesky-type representation for the $\mathrm{Sk}_m(\mathbb{C})$ is given by the following result.

Theorem 8.1. *The $\mathcal{O}_{s_m}$ module $\mathcal{L}_m$ is a solvable holomorphic Lie algebra for $m \geq 3$. In addition, it is a free $\mathcal{O}_{s_m}$ module of rank s_m, and it defines a free divisor on $\mathrm{Sk}_m(\mathbb{C})$ given by the equation*

$$\prod_{k=1}^{m-2} \det\left(\hat{\hat{A}}^{(k)}\right) \cdot \prod_{k=2}^{m} \mathrm{Pf}_{\{\epsilon(k),\dots,k\}}(A) \;=\; 0\ . \tag{8.3}$$

Remark 8.2. We note in (8.3), that when k is odd, $\epsilon(k) = 2$, so that $\mathrm{Pf}_{\{\epsilon(k),\dots,k\}}(A)$ is the Pfaffian of the $(k-1) \times (k-1)$ upper left-hand submatrix of the matrix obtained from A by first deleting the top row and first column.

Before proving this theorem, we illustrate it in the simplest nontrivial case of $\mathrm{Sk}_4(\mathbb{C})$.

Example 8.3. First, $\dim_{\mathbb{C}} \mathrm{Sk}_4(\mathbb{C}) = 6$; while $\dim G_4 = 5$, with Lie algebra $\mathbf{g}_4$ having basis $\{E_{1,1}, E_{2,2}, E_{3,3}, E_{4,3}, E_{4,4}\}$. For $\mathcal{L}_4$, we adjoin to the representation vector fields associated to the basis for $\mathbf{g}_4$ an additional generator $\eta_2 = \mathrm{Pf}(A) \cdot \bar{e}_{3,4}\, (= \mathrm{Pf}(A) \cdot (\frac{\partial}{\partial a_{3,4}} - \frac{\partial}{\partial a_{4,3}}))$. Then the coefficient matrix using the basis $\{\bar{e}_{1,2}, \bar{e}_{1,3}, \bar{e}_{2,3}, \bar{e}_{1,4}, \bar{e}_{2,4}, \bar{e}_{3,4}\}$ is

$$\begin{pmatrix} a_{12} & a_{12} & 0 & 0 & 0 & 0 \\ a_{13} & 0 & a_{13} & 0 & 0 & 0 \\ 0 & a_{23} & a_{23} & 0 & 0 & 0 \\ a_{14} & 0 & 0 & a_{13} & a_{14} & 0 \\ 0 & a_{24} & 0 & a_{23} & a_{24} & 0 \\ 0 & 0 & a_{34} & 0 & a_{34} & \mathrm{Pf}(A) \end{pmatrix}$$

which has block lower triangular form, with determinant

$$a_{1\,2}a_{1\,3}a_{2\,3}(a_{1\,3}a_{2\,4} - a_{1\,4}a_{2\,3}) \cdot \mathrm{Pf}(A)\,.$$

The term $a_{2\,3}$ is the Pfaffian $\mathrm{Pf}_{\{2,3\}}(A)$ as described in Remark 8.2. The determinant has degree 7 and, by the theorem, defines a free divisor, which is not a linear free divisor.

Proof of Theorem 8.1. To prove the theorem we will apply Saito's Criterion (Theorem 1.1(2)). For it, we first show that $\mathcal{L}_m$ is a holomorphic Lie algebra. Since $\mathfrak{g}_m$ is a Lie algebra, it is sufficient to show that both $[\xi, \eta_k]$ and $[\eta_k, \eta_l] \in \mathcal{L}_m$ for all $2 \leq \ell, k \leq m-2$ and any representation vector field ξ associated to G_m.

Proposition 8.4. *If $E_{p,q} \in \mathfrak{g}_m$, then*

$$[\xi_{E_{p,q}}, \eta_k] = \begin{cases} \eta_k & \text{if } p = q \text{ and } \epsilon(k) \leq p \leq k \\ 0 & \text{otherwise} \end{cases}.$$

If $k < l$, then

$$[\eta_k, \eta_l] = \frac{1}{2}(\delta_{\epsilon(k),\epsilon(l)} + l - k - 1)\mathrm{Pf}_{\{\epsilon(k),\ldots,k\}} \cdot \eta_l.$$

Proof. The full details are given in Appendix A of [P]. However, we remark that the computation of these Lie brackets is very lengthy, and makes repeated applications of the following Pfaffian identity of Dress-Wenzel. □

Theorem 8.5 (Dress-Wenzel [DW])**.** *Let $I_1, I_2 \subseteq \{1, \ldots, m\}$. Write the symmetric difference $I_1 \Delta I_2 = \{i_1, \ldots, i_\ell\}$ with $i_1 < \cdots < i_\ell$. Then*

$$\sum_{\tau=1}^{\ell} (-1)^\tau \mathrm{Pf}_{I_1 \Delta \{i_\tau\}} \mathrm{Pf}_{I_2 \Delta \{i_\tau\}} = 0.$$

We next show that $\mathcal{L}_m$ is free as an $\mathcal{O}_{s_m}$-module. To do this, we determine the coefficient matrix of the generators of $\mathcal{L}_m$.

By the discussion in §5 and §6.1, the bilinear form representation has the invariant subspaces $\mathrm{Sk}_m(\mathbb{C})^{(\ell,\ell)} = \mathrm{Sk}_m(\mathbb{C}) \cap M^{(\ell,\ell)}$, and the kernels of the induced quotient representations for $0 \leq \ell \leq m-3$ are

$$K_\ell = \left\{ \begin{pmatrix} \pm I_{m-\ell} & 0 \\ * & B_\ell \end{pmatrix} \in G_m \right\}. \tag{8.4}$$

(The kernels for $\ell = m-2, m-1$ do not take this form.) We denote the Lie algebras of K_ℓ by $\mathbf{k}_\ell$.

For the decomposition, we consider $\mathrm{Sk}_m(\mathbb{C})^{(\ell,\ell)}$ for $0 \leq \ell \leq m-3$ (together with $\mathrm{Sk}_m(\mathbb{C})$). First, the complementary basis for $\mathrm{Sk}_m(\mathbb{C})^{(m-3,m-3)}$ in $\mathrm{Sk}_m(\mathbb{C})$ is $\{\bar{e}_{1,2}, \bar{e}_{1,3}, \bar{e}_{2,3}\}$, and $\{E_{1,1}, E_{2,2}, E_{3,3}\}$ is a complementary basis for $\mathbf{k}_{m-3}$ in $\mathbf{g}_m$.

For $\ell \leq m-3$, as earlier we let $\ell' = m - \ell$, and use the complementary bases

$$\{\bar{e}_{1\,\ell'+1}, \dots, \bar{e}_{\ell'\,\ell'+1}\} \quad \text{to } \mathrm{Sk}_m(\mathbb{C})^{(\ell-1,\ell-1)} \text{ in } \mathrm{Sk}_m(\mathbb{C})^{(\ell,\ell)}.$$

For the subgroups K_ℓ, we use the corresponding complementary bases

$$\{E_{\ell'+1\,3}, \dots, E_{\ell'+1\,\ell'+1}\} \qquad \text{to } \mathbf{k}_{\ell-1} \text{ in } \mathbf{k}_\ell.$$

As

$$\dim\left(\mathrm{Sk}_m(\mathbb{C})^{(\ell,\ell)}/\mathrm{Sk}_m(\mathbb{C})^{(\ell-1,\ell-1)}\right) \;=\; \ell' \;=\; \dim \mathbf{k}_\ell/\mathbf{k}_{\ell-1} + 1\,,$$

we adjoin a single η_k with $k = m - \ell - 1 = \ell' - 1$. We note that just as for $\xi_{E_{\ell'+1\,j}}$, this $\eta_{\ell'-1}$ has 0 coefficients for the relative basis of $\mathrm{Sk}_m(\mathbb{C})/\mathrm{Sk}_m(\mathbb{C})^{(\ell,\ell)}$.

Proposition 8.6. *With the above relative bases (with the corresponding $\eta_{\ell'-1}$ adjoined to the appropriate relative bases as indicated) the coefficient matrix of $\mathcal{L}_m$ is block lower triangular with $m-2$ diagonal blocks $\{D_\ell\}$ (as in* (2.3)*), where at $A = (a_{ij}) \in \mathrm{Sk}_m(\mathbb{C})$,*

$$D_{m-2}(A) = \begin{pmatrix} a_{12} & a_{12} & 0 \\ a_{13} & 0 & a_{13} \\ 0 & a_{23} & a_{23} \end{pmatrix}$$

and for $1 \leq \ell \leq m-3$, with $\ell' = m - \ell$, there is the $\ell' \times \ell'$ diagonal block

$$D_\ell(A) = \begin{pmatrix} a_{1,3} & \cdots & a_{1,\ell'+1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ a_{\ell'-1,3} & \cdots & a_{\ell'-1,\ell'+1} & 0 \\ a_{\ell',3} & \cdots & a_{\ell',\ell'+1} & \mathrm{Pf}_{\{\epsilon(\ell'-1),\dots,\ell'-1\}}(A) \end{pmatrix}. \tag{8.5}$$

Hence, the coefficient determinant for this block is

$$\det(D_\ell(A)) \;=\; \det(\hat{\hat{A}}^{(\ell'-1)}) \cdot \mathrm{Pf}_{\{\epsilon(\ell'-1),\dots,\ell'-1\}}(A)\,. \tag{8.6}$$

Proof. We claim that the coefficient matrix with respect to the two sets of bases is block lower triangular with $m-2$ blocks. The first block corresponds to $\mathbf{g}_m/K_{m-3}$ and a direct calculation shows it is the 3×3 block D_{m-2} in the proposition. For the subsequent blocks, we note by Lemma 2.1 and the remark concerning η_k preceding the proposition, that the columns corresponding to $\{E_{\ell'+1\,3}, \dots, E_{\ell'+1\,\ell'+1}, \eta_{\ell'-1}\}$ will be 0 above the $\ell' \times \ell'$ diagonal block D_ℓ.

Moreover, for this block, by the calculations carried out in §6, the upper left $(\ell'-1) \times (\ell'-1)$ submatrix is $\hat{\hat{A}}^{(\ell'-1)}$ (because $E_{\ell'+1\,1}$ and $E_{\ell'+1\,2}$ are missing in the basis for $\mathbf{k}_\ell/\mathbf{k}_{\ell-1}$). Also, by the form of $\eta_{\ell'-1}$, the column for it will only have an entry $\mathrm{Pf}_{\{\epsilon(\ell'-1),\dots,\ell'-1\}}$ in the last row of the block. Thus, D_ℓ and $\det(D_\ell)$ have the forms as stated. $\square$

Then, applying Proposition 8.6 to each diagonal block yields as the coefficient determinant (up to sign) the left-hand side of (8.3). Lemma 4.7 can be used as in earlier cases to show that the determinant is reduced. Thus, by Saito's Criterion $\mathcal{L}_m$ is a free $\mathcal{O}_{s_m}$ module which defines a free divisor on $\mathrm{Sk}_m(\mathbb{C})$ with defining equation (8.3).

Lastly, since the degree 0 subalgebra $\mathbf{g}_m$ of $\mathcal{L}_m$ is solvable, the solvability of $\mathcal{L}_m$ follows from the next lemma, completing the proof of the Theorem. □

Lemma 8.7. *A holomorphic Lie algebra $\mathcal{L}$ generated by homogeneous vector fields of degree ≥ 0 is solvable if and only if the degree 0 subalgebra is solvable.*

Proof. Let L_0 denote the Lie algebra of vector fields of degree zero (it is a linear Lie algebra). Also, let $\mathcal{L}^{(k)}$ denote the holomorphic sub-Lie algebra generated by the homogeneous vector fields of degree $\geq k$. Then, as $[\mathcal{L}^{(k)}, \mathcal{L}^{(j)}] \subset \mathcal{L}^{(k+j)}$, it follows that the Lie algebra $\mathcal{L}^{(k)}/\mathcal{L}^{(k+1)}$ is abelian for $k \geq 1$. Lastly, the projection induces an isomorphism $\mathcal{L}/\mathcal{L}^{(1)} = \mathcal{L}^{(0)}/\mathcal{L}^{(1)} \simeq L_0$. This is solvable by assumption. Hence, if we adjoin to $\{\mathcal{L}^{(k)}\}$ the pullback of the derived series of L_0 via the projection of $\mathcal{L}$ onto L_0, we obtain a filtration by subalgebras, each an ideal in the preceding, whose successive quotients are abelian. Hence, $\mathcal{L}$ is solvable.

For the reverse direction we just note that L_0, as a quotient of the solvable Lie algebra $\mathcal{L}$, is solvable. □

9. Block Representations by Restriction and Extension

In this section we apply the restriction and extension properties of block representations to obtain free divisors which will be used in part II.

Suppose $\rho : G \to GL(V)$ is a block representation with associated decomposition

$$V = W_k \supset W_{k-1} \supset \cdots \supset W_1 \supset W_0 = (0)$$

with $K_j = \ker(\rho_j)$ for the induced representation $\rho_j : G \to GL(V/W_j)$.

If we restrict to the representation of K_m on W_m, we will obtain a decomposition descending from W_m with corresponding normal subgroups K_j. We already know that the resulting coefficient matrix has the necessary block triangular form. There is a problem because the corresponding relative coefficient determinants are those for ρ restricted to the subspace W_m. Although the relative coefficient determinants were reduced and relatively prime as polynomials on V, this may not continue to hold on W_m.

A simple example illustrating this problem occurs for the bilinear form representation of B_2 on $Sym_2(\mathbb{C})$. Suppose we restrict to the subspace $W_1 \subset Sym_2(\mathbb{C})$ of symmetric matrices with upper left entry $= 0$. The corresponding normal subgroup of B_2 has upper left entry $= 1$. In terms of the basis used in §6, the coefficient matrix is $A = \begin{pmatrix} 0 & a_{1\,2} \\ a_{1\,2} & a_{2\,2} \end{pmatrix}$. Thus, the relative coefficient matrix is $a_{1\,2}^2$, so it is a nonreduced block representation.

Nonetheless, in many cases of interest we may restrict a tower of block representations by modifying the lowest degree one to obtain another tower of block representations.

Restricted Symmetric Representations

We consider several restrictions of the tower of representations $\{(B_m, Sym_m)\}$. First, for the subrepresentations $\{(G_m, W_{m-1})\}$ for $m \geq 3$. Here $G_m \subset B_m$ is the subgroup of matrices $B = (b_{i\,j}) \in B_m$ with entries $b_{2\,1} = 0$ so that the upper left

3×3-block has the form a) in (9.1).

$$(9.1) \qquad a) \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ * & * & * \end{pmatrix} \qquad b) \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & * & * & 0 \\ 0 & * & * & * \end{pmatrix} .$$

As in §6, we let $W_{m-1} = Sym_m^{(m-1,m-1)}(\mathbb{C}) \subset Sym_m(\mathbb{C})$, which is the subspace of symmetric matrices with the upper left entry equal to 0. With the same inclusions as for $\{(B_m, Sym_m(\mathbb{C}))\}$, $\{(G_m, W_{m-1})\}$ is again a tower of representations.

Second, we consider the restriction of the same tower $\{(B_m, Sym_m(\mathbb{C}))\}$ but to the subspace W_{m-2}, which consists of matrices with the upper left hand 2×2 block equal to 0. We only consider the tower beginning with $m \geq 4$. This time we choose G_m to be the subgroup of B_m consisting of matrices with upper left 4×4-block of the form b) in (9.1).

Proposition 9.1. *The two restrictions of the tower $\{(B_m, Sym_m(\mathbb{C}))\}$ define block representations of towers. Thus, the exceptional orbit varieties are free divisors and have defining equations given by: for the first case*

$$(9.2) \qquad - a_{1\,2}\, a_{2\,2} \cdot (a_{3\,3}a_{1\,2}^2 - 2a_{2\,3}a_{1\,2}a_{1\,3} + a_{2\,2}a_{1\,3}^2) \cdot \prod_{k=4}^{m} \det(A_1^{(k)}) \;=\; 0\,;$$

and for the second case

$$(9.3) \qquad -a_{1\,3}\, a_{2\,3} \cdot (a_{1\,3}a_{2\,4} - a_{1\,4}a_{2\,3}) \cdot (a_{3\,3}a_{2\,4}^2 - 2a_{3\,4}a_{2\,4}a_{2\,3} + a_{4\,4}a_{2\,3}^2) \cdot \prod_{k=5}^{m} \det(A_2^{(k)}) \;=\; 0\,.$$

where $A_r^{(k)}$ denotes the upper left $k \times k$ submatrix of A_r, which is obtained from A by setting $a_{i,j} = 0$ for $1 \leq i, j \leq r$.

Remark 9.2. The middle term in (9.2) is the determinant of the generic 3×3 symmetric matrix with $a_{1\,1} = 0$ and for (9.3) it is minus the determinant of the 3×3 lower-right submatrix of $A_1^{(4)}$ (so $a_{2\,2} = 0$), and it is reduced.

Proof. The proof of each statement is similar so we just consider the second case. It is the restriction of the tower $\{(B_m, Sym_m(\mathbb{C}))\}$ to the subspace W_{m-2}, which consists of matrices with the upper left hand 2×2 block equal to 0. Then, we will apply the Restriction Property, Proposition 3.2.

It is only necessary to consider the diagonal block corresponding to W_{m-2}/W_{m-4} and G_m/K_{m-4}. It is sufficient to consider the subrepresentation on $W_2 \subset Sym_4(\mathbb{C})$. We use the complementary bases

$$\{E_{1\,1}, E_{2\,2}, E_{3\,2}, E_{3\,3}, E_{4\,2}, E_{4\,3}, E_{4\,4}\} \quad \text{to } \mathbf{k}_{m-4} \text{ in } \mathbf{g}_m, \text{ and}$$

$$\{e_{1\,3}, e_{2\,3}, e_{3\,3}, e_{1\,4}, e_{2\,4}, e_{3\,4}, e_{4\,4}\} \quad \text{to } W_{m-4} \text{ in } W_{m-2}$$

(using the notation of §6).

The corresponding relative coefficient matrix has the form

$$\begin{pmatrix} a_{1\,3} & 0 & 0 & a_{1\,3} & 0 & 0 & 0 \\ 0 & a_{2\,3} & 0 & a_{2\,3} & 0 & 0 & 0 \\ 0 & 0 & a_{2\,3} & a_{3\,3} & 0 & 0 & 0 \\ a_{1\,4} & 0 & 0 & 0 & 0 & a_{1\,3} & a_{1\,4} \\ 0 & a_{2\,4} & 0 & 0 & 0 & a_{2\,3} & a_{2\,3} \\ 0 & 0 & a_{2\,4} & a_{3\,3} & a_{2\,3} & a_{3\,3} & a_{3\,4} \\ 0 & 0 & 0 & 0 & a_{2\,4} & 0 & a_{4\,4} \end{pmatrix}$$

This has for its determinant the reduced polynomial

$$-a_{1\,3}\, a_{2\,3} \cdot (a_{1\,3}a_{2\,4} - a_{1\,4}a_{2\,3}) \cdot (a_{3\,3}a_{2\,4}^2 - 2a_{3\,4}a_{2\,4}a_{2\,3} + a_{4\,4}a_{2\,3}^2)\,.$$

Then, the subsequent relative coefficient determinants are those for $(B_m, Sym_m(\mathbb{C}))$, but with $a_{1\,1} = a_{1\,2} = a_{2\,2} = 0$. Just as for the unrestricted case, we see using Lemma 4.7 that they are reduced and relatively prime. Hence, we obtain a tower of block representations. Thus, the exceptional orbit variety is free with defining equation the product of the relative coefficient determinants. □

Restricted General Representations

We second consider the restrictions of the tower of block representations formed from $(B_m \times C_m, M_{m,m})$ and $(B_{m-1} \times C_m, M_{m-1,m})$ as in §6. These together form a tower of block representations. We consider the restriction to the subspaces where $a_{1,1} = 0$ for $m \geq 3$. We replace B_m by the subgroup B'_m with upper left hand 2×2 matrix a diagonal matrix.

Proposition 9.3. *For restrictions of the tower formed from $(B_m \times C_m, M_{m,m})$ and $(B_{m-1} \times C_m, M_{m-1,m})$ define block representations of towers so the exceptional orbit varieties are free divisors and have defining equations given by: for $M_{m,m}$ with $m \geq 3$,*

$$a_{1\,2}\, a_{2\,1}\, a_{2\,2} \cdot (a_{1\,2}a_{2\,3} - a_{1\,3}a_{2\,2}) \cdot \prod_{k=3}^{m} \det(A_1^{(k)}) \cdot \prod_{k=3}^{m-1} \det(\hat{A}_1^{(k)}) \;=\; 0\,; \tag{9.4}$$

and for $M_{m-1,m}$, with $m \geq 3$,

$$a_{1\,2}\, a_{2\,1}\, a_{2\,2} \cdot (a_{1\,2}a_{2\,3} - a_{1\,3}a_{2\,2}) \cdot \prod_{k=3}^{m-1} \det(A_1^{(k)}) \cdot \prod_{k=3}^{m-1} \det(\hat{A}_1^{(k)}) \;=\; 0 \tag{9.5}$$

with $A_1^{(k)}$ as defined earlier.

Proof. The proof is similar to that for Proposition 9.1. It is sufficient to consider the (lowest degree) representation of $G_2 = B'_2 \times C_3$ on $M_{2,3}$, and then restrict the other relative coefficients determinants by evaluating those from Theorem 7.1 with $a_{1,1} = 0$ and use Lemma 4.7 to see that they are reduced and relatively prime.

We compute the coefficient matrix using the complementary bases

$$\{(E_{1\,1}, 0), (E_{2\,2}, 0), (0, E_{2\,2})\} \quad \text{to } \mathbf{k}_{m-2} \text{ in } \mathbf{g}_m, \text{ and}$$

$$\{E_{1\,2}, E_{2\,1}, E_{2\,2}\} \quad \text{to } W_{m-2} \text{ in } W_{m-1}\,.$$

The corresponding coefficient determinant will be, up to sign,

$$a_{1\,2}\, a_{2\,1}\, a_{2\,2} \cdot (a_{1\,2}a_{2\,3} - a_{1\,3}a_{2\,2})\,.$$

□

The preceding involve restrictions of block representations of solvable linear algebraic groups. We may also apply the Extension Property, Proposition 3.3, to extend block representations for a class of groups which extend both solvable and reductive groups.

Example 9.4 (Extension of a solvable group by a reductive group)**.** We consider the restriction of the bilinear form representation to the group

$$G_3 = \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \in \mathrm{GL}_3(\mathbb{C}) \right\}$$

and to the subspace $V_3 = \mathrm{Sym}_3^{(2,2)}(\mathbb{C}) \subset \mathrm{Sym}_3(\mathbb{C})$, consisting of matrices with upper left entry zero. We considered the restriction to this subspace in Proposition 9.1; however, now the group G_3 is reductive. This representation also will play a role in part II in the computations for 3×3 symmetric matrix singularities. A direct calculation shows that this equidimensional representation has coefficient determinant

$$-(a_{2\,2}a_{3\,3} - a_{2\,3}^2) \cdot (a_{3\,3}a_{1\,2}^2 - 2a_{2\,3}a_{1\,2}a_{1\,3} + a_{2\,2}a_{1\,3}^2)\,,$$

which defines the exceptional orbit variety as a linear free divisor on V_3. The second term in the product is the determinant of the 3×3 matrix with $a_{1\,1} = 0$.

The Extension Property, Proposition 3.3, now allows us to inductively extend the reductive group G_3 by a solvable group, and the representation to a representation of the extended group, obtaining a linear free divisor for the larger representation. We again use the notation of §6. For $m \geq 3$, we more generally let $V_m = \mathrm{Sym}_m^{(m-1,m-1)}(\mathbb{C}) \subset \mathrm{Sym}_m(\mathbb{C})$ (also the subspace considered in Proposition 9.1). However, the extended group

$$G_m = \left\{ \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \in \mathrm{GL}_m(\mathbb{C}) : A \in G_3, C \in B_{m-3}(\mathbb{C}) \right\} \tag{9.6}$$

is no longer reductive (nor solvable). We note that it is the extension of G_3 by the solvable subgroup K_{m-3} consisting of elements in G_m with $A = I$ in (9.6). These subgroups were used earlier in both §6 for the tower structure of $Sym_m(\mathbb{C})$ and also in Proposition 9.1. Then, (G_m, V_m) for the bilinear form representation restricts to G_m acting on V_m form a tower of representations using the same inclusions (6.2) as earlier.

Proposition 9.5. *The $\{(G_m, V_m)\}$ for $m \geq 3$ form a tower of block representations so the exceptional orbit varieties are linear free divisors and their defining equations are given by*

$$(a_{2\,2}a_{3\,3} - a_{2\,3}^2) \cdot \prod_{j=3}^{m} \det(A_1^{(j)}) \;=\; 0\,. \tag{9.7}$$

Proof. To verify this claim, we apply the extension property to the entire tower in the form of Proposition 4.5. The first group and representation are (G_3, V_3) which is a block representation with just one block.

Next, we let $W_1 = \mathrm{Sym}_m^{(1,1)}(\mathbb{C}) \subset V_m$. The kernel of the quotient representation $G_m \to \mathrm{GL}(V_m/W_1)$ is the product of a finite group with the subgroup $K_1 \subset G_m$. Then, G_m/K_1 is naturally identified with G_{m-1}, and V_m/W_1 with V_{m-1}. With these identifications, $G_m/K_m \to \mathrm{GL}(V_m/W_m)$ is isomorphic as a representation to $G_{m-1} \to \mathrm{GL}(V_{m-1})$. This establishes ii) of Proposition 4.5.

Lastly, the coefficient determinant for K_1 acting on V_m with $a_{1,1} = 0$ is $\det(A_1^{(m)})$. As this is not identically zero, K_1 has a relatively open orbit. Also, this polynomial is irreducible and relatively prime to the coefficient determinant for G_m/K_1. Thus, ii) of Proposition 4.5 follows and the claim for (G_m, V_m) follows. $\square$

It appears that linear free divisors can often be extended to larger linear free divisors using an extension of the original group by a solvable group. For more examples see §5.3 of [P].

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Department of Mathematics, University of North Carolina, Chapel Hill, NC 27599-3250, USA

Current address: B. Pike: Dept. of Computer and Mathematical Sciences, University of Toronto at Scarborough, 1265 Military Trail, Toronto, ON M1C 1A4,, Canada