RESEARCH STATEMENT

BRIAN PIKE

1. Introduction

My research concerns relations between the local topology of analytic sets, the singularity theory of differentiable maps, and representations of Lie groups.

Consider an analytic germ of the form $\mathscr{V}_0 = f_0^{-1}(V)$, where $V, 0 \subset \mathbb{C}^p, 0$ is an analytic germ and $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ is a holomorphic germ. Although $\mathscr{V}_0$ is locally contractible, deformations of $\mathscr{V}_0$ produce interesting topological spaces which can provide invariants of $\mathscr{V}_0$. Let f_0 deform to a family $\{f_t\}$ of holomorphic germs, thereby obtaining deformations

$$\mathscr{V}_t = f_t^{-1}(V) \cap B_\epsilon(0), \qquad 0 < |t| \ll \epsilon,$$

of $\mathscr{V}_0$.

In the special case where $f_t = f_0 - t \cdot c$ for a generic $c \in \mathbb{C}^p$ and $V = \{0\}$, $\mathscr{V}_t = f_0^{-1}(t \cdot c) \cap B_\epsilon(0)$ is called the *Milnor fiber* of $\mathscr{V}_0$ (or f_0). The topology of the Milnor fiber has been studied for decades. For example, when $\mathscr{V}_0$ is an isolated hypersurface singularity (respectively, isolated complete intersection singularity), Milnor [Mil68] (resp., Hamm [Ham71]) showed that $\mathscr{V}_t$ has the homotopy type of a bouquet of $(n-p)$-spheres. Moreover, Milnor (respectively, Lê-Greuel [Lê 74, BG75, Gre75]) gave a computable algebraic formula for the number of spheres, called the *Milnor number of* $\mathscr{V}_0$. The Milnor number has been an incredibly useful invariant in the study of isolated singularities, and the homological properties of Milnor fibers have been a central theme in singularity theory. For instance, the "Milnor-Tjurina" relation $\mu \geq \tau$ connects the topologically defined Milnor number μ of $\mathscr{V}_0$ to the number of parameters needed for a versal unfolding of f_0.

If instead $\mathscr{V}_0$ has nonisolated singularities, then the connectivity of the Milnor fiber drops by the dimension of the singular set of $\mathscr{V}_0$ and the Milnor fiber may have a much more complicated topology ([KM75]). Only for very low dimensional singular sets has the topology of the Milnor fiber been computed (e.g., [Sie83, Sie87, Pel90, MS92, dJ90, dJdJ90, Zah94, Ném99, FM10]).

Instead, I study the case where f_0 is transverse off $0 \in \mathbb{C}^n$ to V and f_t is required to be fully transverse to V for $t \neq 0$. Then $\mathscr{V}_t$ is called the *singular Milnor fiber* of $\mathscr{V}_0$. In a generalization of the classical isolated singularity situation, Damon and Mond [DM91, Dam96a, Dam96b] showed that when V is a complete intersection, then $\mathscr{V}_t$ again has the homotopy type of a bouquet of $(n - \operatorname{codim}(V))$-spheres. The number of spheres is denoted by $\mu_V(f_0)$ and is called the *singular Milnor number of* $\mathscr{V}_0$. This is an ambient diffeomorphism invariant of $\mathscr{V}_0$ which is defined even when $\mathscr{V}_0$ has highly nonisolated singularities. Unfortunately, formulas for $\mu_V(f_0)$ are known in only a few cases; the most important of these is where $(V, 0)$ is a *free divisor*, a hypersurface germ for which the module of germs of holomorphic vector fields tangent to V is a free module.

2. Research Objectives

My research originates with the idea that $\mu_V(f_0)$ should be as useful an invariant for nonisolated singularities as the classical Milnor number has been for isolated singularities.

For example, Houston [Hou08] has used the singular Milnor number to generalize Teissier's "μ^*-constant implies equisingularity" result beyond isolated hypersurface singularities. Accordingly, there are a number of natural problems which parallel the classical theory.

2.1. **Making $\mu_V(f_0)$ computable.** In joint work with Damon [Pik10, DPc], we identify a general method to find, for a particular V, a computable algebraic formula for $\mu_V(f_0)$. In its most basic form, we find a suitable auxiliary germ $W \subset \mathbb{C}^p$ so that $\mu_{V\cup W}(f_0)$ and $\mu_W(f_0)$ are computable; then when these are all defined,

$$\mu_V(f) = (-1)^{d_1}\mu_{V\cup W}(f) + (-1)^{d_2}\mu_W(f) + (-1)^{d_3}\mu_{V\cap W}(f), \tag{1}$$

where each d_i is the dimension of a variety. Often $V \cap W$ is significantly simpler than V and $\mu_{V\cap W}(f_0)$ can be computed either directly, or inductively by this process.

2.1.1. *Matrix singularities.* Efforts thus far have focused on *matrix singularities*, $\mathscr{V}_0 = f_0^{-1}(V)$ where V lies in a vector space of matrices and consists of all those with less than maximal rank. Matrix singularities arise naturally in various contexts, for example the Hilbert-Burch Theorem, or the classification of codimension 3 Gorenstein singularities [BE77].

Although Goryunov-Mond [GM05] studied the Milnor numbers of isolated matrix singularities, matrix singularities almost always have nonisolated singularities, so that the Milnor number is not defined. Instead, we use the singular Milnor number. Since V is almost never a free divisor, $\mu_V(f_0)$ is not computable using previous methods. So far, we [DPc] have used the above method to find formulas for $\mu_V(f_0)$ where V is the hypersurface of singular matrices in the space of 2×2 and 3×3 symmetric matrices, 2×2 and 3×3 general matrices, and 4×4 skew-symmetric matrices.

In the future, I will find and simplify formulas for other matrix singularities. Although existing formulas compute $\mu_V(f_0)$ as a linear combination of the lengths of various modules, it is often possible to compose f_0 with a diffeomorphism to arrange for most terms to vanish. The remaining modules are always related, and preliminary calculations suggest that these terms may often be collapsed, so that $\mu_V(f_0)$ is given as the length of a single module. Such a formula may then hold beyond matrix singularities.

2.1.2. *Milnor-Tjurina relation for maps.* It has long been believed that an analogue of the Milnor-Tjurina relation $\mu \geq \tau$ for isolated singularities should hold for map germs, where μ is replaced by $\mu_{D(f_0)}(f_0)$ for the discriminant (or image) $D(f_0)$ of f_0, and τ remains the number of parameters for a versal unfolding of f_0. Indeed, Damon-Mond [DM91] originally defined the singular Milnor number to prove such a result for stable maps $f_0 : \mathbb{C}^n \to \mathbb{C}^p$ with $n \geq p$. A key step in the proof is that for such a map, $D(f_0)$ is a free divisor. When $n < p$, $D(f_0)$ is no longer free and $\mu_{D(f_0)}(f_0)$ is more difficult to compute; nevertheless, the relation has been shown for map germs $\mathbb{C}^n \to \mathbb{C}^{n+1}$ when $n = 1$ and $n = 2$. Since examples have suggested that $D(f_0)$ can often be rendered free by the addition of auxiliary hypersurfaces ([MS10]), I will apply the above method of computation to prove a Milnor-Tjurina relation for germs $\mathbb{C}^n \to \mathbb{C}^{n+1}$.

2.1.3. *Cohen-Macaulay codimension 2 singularities.* Let V be the variety of 2×3 matrices with rank < 2. Since V is not a complete intersection, $\mathscr{V}_t$ may not be a bouquet of spheres. Nevertheless, we have obtained a formula for the Euler characteristic of $\mathscr{V}_t$ and bounds on its Betti numbers (see [DPc]).

Of particular interest is when $n = 4$ and $\mathscr{V}_0$ is an isolated surface singularity in $\mathbb{C}^4$. Work by Wahl [Wah81] and Greuel-Steenbrink [GS83] leaves only the second Betti number β_2 of the Milnor fiber uncomputed, which they call the "Milnor number" of $\mathscr{V}_0$. Since here the singular Milnor fiber agrees with the Milnor fiber, our Euler characteristic formula reduces to a computable algebraic formula for β_2. Pereira [Per10] has used a classification of simple singularities of this type (due to [FKN10]) to prove a difficult "$\mu = \tau + 1$" result, which may follow easily from our formula for β_2. For isolated 3-folds in $\mathbb{C}^5$, we give bounds on the Betti numbers of the Milnor fiber. In future work, I will obtain similar results for other singularities described by the Hilbert-Burch theorem, and codimension 3 Gorenstein singularities.

2.2. **The ubiquity of free divisors.** To use (1) to compute $\mu_V(f_0)$, it is necessary to find a suitable auxiliary variety W; often, W and $V \cup W$ are chosen to be free divisors. Free divisors classically arise as discriminants of versal unfoldings for certain groups of equivalences. However, they also have connections to representation theory and harmonic analysis (see [Sat90]). If a representation $\rho : G \to \mathrm{GL}(\mathbb{C}^m)$ of a connected complex algebraic Lie group has a Zariski open orbit Ω and $\dim(G) = m$, then the complement Ω^c of the open orbit is a hypersurface. Buchweitz-Mond observed [BM06] that if Ω^c is reduced (in some sense), then a criterion of Saito [Sai80] concludes that Ω^c is a *linear* free divisor, a free divisor where the associated module is generated by "linear" vector fields on $\mathbb{C}^m$. Linear free divisors are now a topic of active research (e.g., [GS10, Sev09, dGMS09, GMS09]).

2.2.1. *Linear free divisors from representations of solvable groups.* Sato-Kimura [SK77] have classified irreducible reduced representations with open orbits, and Buchweitz-Mond [BM06] have studied indecomposable quiver representations. Instead, Damon and I [DPb] have shown that indecomposable representations of solvable groups give useful linear free divisors, including many of the auxiliary varieties necessary to compute the singular Milnor number of matrix singularities. We demonstrate infinite *towers* of linear free divisors coming from towers of representations of solvable groups. Since we have only used restrictions of two specific representations, in future work I will study solvable subgroups of other standard representations of reductive groups, and determine which of these yield linear free divisors.

We [DPb] have also shown that for these two representations, solvable group extensions often allow linear free divisors to be easily constructed from simpler linear free divisors. I expect similar results to hold for other representations. Such a construction operation may be the key to giving a structure theorem for linear free divisors.

2.2.2. *Deformations of linear free divisors.* I will study deformations of linear free divisors, perhaps using tools from Lie algebra cohomology. A family of subgroups G_t gives a family of representations and hypersurfaces Ω_t^c. By studying how Ω_t^c changes, we may determine when Ω_0^c may be perturbed to a linear free divisor. For example, I have conjectured that there is no linear free divisor which includes the singular 4×4 skew-symmetric matrices, even though this set is included in a non-reduced Ω^c. Work by Sato, Kimura, and others shows that this question is closely related to deformations of certain associated groups of rational characters.

2.2.3. *Free divisors whose complements are $K(\pi, 1)$'s.* Many questions have been asked about when the complement of a free divisor is a $K(\pi, 1)$. For instance, this includes a conjecture of Saito regarding free hyperplane arrangements (see [OT92]), as well as the classical "$K(\pi, 1)$ problem" for versal deformations of isolated hypersurface singularities. We have used solvability to show [DPa] that for a large class of examples of linear free divisors, that both the complements of the linear free divisors and the Milnor fiber of their defining equations are $K(\pi, 1)$'s. For both spaces, we identify π in terms of the rank of the Lie group and give explicit generators of the cohomology with complex coefficients. In contrast, very little is known about the topology of the Milnor fiber of nonisolated singularities in general (e.g., [NS09, Fer09, Fer06]). Our examples provide valuable data for this longstanding question, and future work may yield more information.

2.3. **Algorithms for singularity theory and algebraic geometry.** Computer algebra systems such as Macaulay 2 [GS] or Singular hold great promise to easily answer many of the routine questions in singularity theory and algebraic geometry. Extensions of these tools will make them more useful to others. I will develop and release extensions to compute singular Milnor numbers, to work with Lie algebras of vector fields, to compute the codimension of a germ under various groups of equivalences, and other tools.

I am particularly interested in a method to determine explicit Whitney stratifications. Certain *holonomic* algebraic sets have computable stratifications which agree with their Whitney stratification. By developing computable criteria for when an algebraic set is holonomic, I will be able to compute certain explicit Whitney stratifications. This method may be extended to stratify a non-holonomic algebraic set by viewing it as a transverse section of a holonomic algebraic set.

References

[BE77] David A. Buchsbaum and David Eisenbud, *Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension* 3, Amer. J. Math. **99** (1977), no. 3, 447–485. MR 0453723 (56 #11983)

[BG75] E. Brieskorn and G.-M. Greuel, *Singularities of complete intersections*, Manifolds—Tokyo 1973 (Proc. Internat. Conf., Tokyo, 1973), Univ. Tokyo Press, Tokyo, 1975, pp. 123–129. MR 0367273 (51 #3515)

[BM06] Ragnar-Olaf Buchweitz and David Mond, *Linear free divisors and quiver representations*, Singularities and computer algebra, London Math. Soc. Lecture Note Ser., vol. 324, Cambridge Univ. Press, Cambridge, 2006, pp. 41–77. MR 2228227 (2007d:16028)

[Dam96a] James Damon, *Higher multiplicities and almost free divisors and complete intersections*, Mem. Amer. Math. Soc. **123** (1996), no. 589, x+113. MR 1346928 (97d:32050)

[Dam96b] ———, *Singular Milnor fibers and higher multiplicities*, J. Math. Sci. **82** (1996), no. 5, 3651–3656, Topology, 3. MR 1428721 (98f:32040)

[dGMS09] Ignacio de Gregorio, David Mond, and Christian Sevenheck, *Linear free divisors and Frobenius manifolds*, Compos. Math. **145** (2009), no. 5, 1305–1350. MR 2551998

[dJ90] Theo de Jong, *The virtual number of D_∞ points. I*, Topology **29** (1990), no. 2, 175–184. MR 1056268 (91f:32043)

[dJdJ90] Johan de Jong and Theo de Jong, *The virtual number of D_∞ points. II*, Topology **29** (1990), no. 2, 185–188. MR 1056269 (91f:32044)

[DM91] James Damon and David Mond, *$\mathcal{A}$-codimension and the vanishing topology of discriminants*, Invent. Math. **106** (1991), no. 2, 217–242. MR 1128213 (92m:58011)

[DPa] James Damon and Brian Pike, *Solvable group representations and free divisors whose complements are $K(\pi,1)$'s*, Topology Appl., To appear.

[DPb] ———, *Solvable groups, free divisors and nonisolated matrix singularities I: Towers of free divisors*, In preparation.

[DPc] ———, *Solvable groups, free divisors and nonisolated matrix singularities II: Vanishing topology*, In preparation.

[Fer06] J. Fernandez de Bobadilla, *Topological equisingularity of function germs with 1-dimensional critical set*, ArXiv Mathematics e-prints (2006).

[Fer09] ———, *On homotopy types of complements of analytic sets and Milnor fibres*, ArXiv e-prints (2009).

[FKN10] Anne Frühbis-Krüger and Alexander Neumer, *Simple Cohen-Macaulay codimension 2 singularities*, Comm. Algebra **38** (2010), no. 2, 454–495. MR 2598893

[FM10] J. Fernandez de Bobadilla and M. A. Marco-Buzunariz, *Topology of hypersurface singularities with 3-dimensional critical set*, ArXiv e-prints (2010).

[GM05] V. Goryunov and D. Mond, *Tjurina and Milnor numbers of matrix singularities*, J. London Math. Soc. (2) **72** (2005), no. 1, 205–224. MR 2145736 (2006e:32035)

[GMS09] M. Granger, D. Mond, and M. Schulze, *Free divisors in prehomogeneous vector spaces*, ArXiv e-prints (2009).

[Gre75] G.-M. Greuel, *Der Gauss-Manin-Zusammenhang isolierter Singularitäten von vollständigen Durchschnitten*, Math. Ann. **214** (1975), 235–266. MR 0396554 (53 #417)

[GS] Daniel R. Grayson and Michael E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, Available at http://www.math.uiuc.edu/Macaulay2/.

[GS83] Gert-Martin Greuel and Joseph Steenbrink, *On the topology of smoothable singularities*, Singularities, Part 1 (Arcata, Calif., 1981), Proc. Sympos. Pure Math., vol. 40, Amer. Math. Soc., Providence, R.I., 1983, pp. 535–545. MR 713090 (84m:14006)

[GS10] M. Granger and M. Schulze, *On the symmetry of b-functions of linear free divisors*, Publ. Res. Inst. Math. Sci. **46** (2010), no. 3, 479–506.

[Ham71] Helmut Hamm, *Lokale topologische Eigenschaften komplexer Räume*, Math. Ann. **191** (1971), 235–252. MR 0286143 (44 #3357)

[Hou08] K. Houston, *Equisingularity and The Euler Characteristic of a Milnor Fibre*, ArXiv e-prints (2008).

[KM75] Mitsuyoshi Kato and Yukio Matsumoto, *On the connectivity of the Milnor fiber of a holomorphic function at a critical point*, Manifolds—Tokyo 1973 (Proc. Internat. Conf., Tokyo, 1973), Univ. Tokyo Press, Tokyo, 1975, pp. 131–136. MR 0372880 (51 #9084)

[Lê 74] Lê Dũng Tráng, *Computation of the Milnor number of an isolated singularity of a complete intersection*, Funkcional. Anal. i Priložen. **8** (1974), no. 2, 45–49. MR 0350064 (50 #2557)

[Mil68] John Milnor, *Singular points of complex hypersurfaces*, Annals of Mathematics Studies, No. 61, Princeton University Press, Princeton, N.J., 1968. MR 0239612 (39 #969)

[MS92] D. B. Massey and D. Siersma, *Deformation of polar methods*, Ann. Inst. Fourier (Grenoble) **42** (1992), no. 4, 737–778. MR 1196092 (94h:32059)

[MS10] D. Mond and M. Schulze, *Adjoint divisors and free divisors*, ArXiv e-prints (2010).

[Ném99] András Némethi, *Hypersurface singularities with 2-dimensional critical locus*, J. London Math. Soc. (2) **59** (1999), no. 3, 922–938. MR 1709088 (2001a:32037)

[NS09] A. Nemethi and A. Szilard, *The boundary of the Milnor fibre of a non-isolated hypersurface surface singularity*, ArXiv e-prints (2009).

[OT92] Peter Orlik and Hiroaki Terao, *Arrangements of hyperplanes*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 300, Springer-Verlag, Berlin, 1992. MR 1217488 (94e:52014)

[Pel90] Ruud Pellikaan, *Deformations of hypersurfaces with a one-dimensional singular locus*, J. Pure Appl. Algebra **67** (1990), no. 1, 49–71. MR 1076748 (91m:32036)

[Per10] Miriam da Silva Pereira, *Determinantal varieties and singularities of matrices*, Ph.D. thesis, Universidade de São Paulo, 2010.

[Pik10] Brian Pike, *Singular Milnor numbers of non-isolated matrix singularities*, Ph.D. thesis, University of North Carolina at Chapel Hill, 2010.

[Sai80] Kyoji Saito, *Theory of logarithmic differential forms and logarithmic vector fields*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), no. 2, 265–291. MR 586450 (83h:32023)

[Sat90] Mikio Sato, *Theory of prehomogeneous vector spaces (algebraic part)—the English translation of Sato's lecture from Shintani's note*, Nagoya Math. J. **120** (1990), 1–34, Notes by Takuro Shintani, Translated from the Japanese by Masakazu Muro. MR 1086566 (92c:32039)

[Sev09] C. Sevenheck, *Bernstein polynomials and spectral numbers for linear free divisors*, ArXiv e-prints (2009).

[Sie83] Dirk Siersma, *Isolated line singularities*, Singularities, Part 2 (Arcata, Calif., 1981), Proc. Sympos. Pure Math., vol. 40, Amer. Math. Soc., Providence, RI, 1983, pp. 485–496. MR 713274 (85d:32017)

[Sie87] ———, *Singularities with critical locus a 1-dimensional complete intersection and transversal type* A_1, Topology Appl. **27** (1987), no. 1, 51–73. MR 910494 (88m:32015)

[SK77] M. Sato and T. Kimura, *A classification of irreducible prehomogeneous vector spaces and their relative invariants*, Nagoya Math. J. **65** (1977), 1–155. MR 0430336 (55 #3341)

[Wah81] Jonathan Wahl, *Smoothings of normal surface singularities*, Topology **20** (1981), no. 3, 219–246. MR 608599 (83h:14029)

[Zah94] Alexandru Zaharia, *Topological properties of certain singularities with critical locus a 2-dimensional complete intersection*, Topology Appl. **60** (1994), no. 2, 153–171. MR 1302470 (95k:32039)