THE NUMBER OF IRREDUCIBLE COMPONENTS OF A LINEAR FREE DIVISOR

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Abstract. A prehomogeneous vector space (ρ, G, V) is a rational representation $\rho : G \to \mathrm{GL}(V)$ of a connected complex linear algebraic group G having a Zariski open orbit $\Omega \subset V$. The hypersurface components of $D = V \setminus \Omega$ are related to the characters $H \to \mathrm{GL}(\mathbb{C})$ of H, an algebraic abelian quotient of G. Mimicking this classical work, we investigate the additive functions of H, homomorphisms $\Phi : H \to (\mathbb{C}, +)$. We prove that such Φ are related to certain rational functions on V which behave simply with respect to the action of G. These rational functions are homogeneous of degree 0 and describe certain G–invariant subsets of the hypersurface components of D.

For those prehomogeneous vector spaces for which D is a type of hypersurface called a linear free divisor, we prove there are no such additive functions. This gives a formula for the number of components of D in terms of the Lie algebra of G, and gives insight into the structure of G. For some special cases (G abelian, reductive, or solvable, or D irreducible) we give simpler proofs of existing results or additional insight.

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TODO: consistent use of subscripts of $\mathrm{Der}(-\log blah)$?
TODO: Ω^c?
TODO: can we improve the normal form of the lie alg. of a gp corr. LFD?

Introduction

A prehomogeneous vector space is a rational representation $\rho : G \to \mathrm{GL}(V)$ of a connected complex linear algebraic group G having a (unique) Zariski open orbit Ω. These objects have been much studied from the viewpoint of harmonic analysis (e.g.,[Kim03]). A basic fact is that the hypersurface components of the algebraic set $V\setminus\Omega$ may be understood from the (multiplicative) characters $\chi : H \to \mathrm{GL}(\mathbb{C}) = \mathbb{G}_{\mathrm{m}}$ of an abelian algebraic quotient H of G. In particular, the number of hypersurface components of $V \setminus \Omega$ is the rank of the free abelian group of characters of H.

In this work, we study the additive functions of H, homomorphisms $\Phi : H \to (\mathbb{C}, +) = \mathbb{G}_{\mathrm{a}}$. (As $\mathbb{G}_{\mathrm{m}}$ and $\mathbb{G}_{\mathrm{a}}$ are not isomorphic as linear algebraic groups, χ, Φ are members of distinct classes of homomorphisms.) Since over $\mathbb{C}$, connected abelian algebraic groups are of the form $(\mathbb{G}_{\mathrm{m}})^k \times (\mathbb{G}_{\mathrm{a}})^\ell$, these two types of homomorphisms completely determine H. In fact, an elementary relationship between $\dim(H)$, the characters of H, and the additive functions of H help describe the number of hypersurface components of $V\setminus\Omega$ (see Corollary 2.4); this is one of our motivations.

After reviewing the basic properties of prehomogeneous vector spaces and abelian linear algebraic groups in §1 and §2, we study the additive functions of H in §3. The (multiplicative) characters of H are related to *relative invariants*, certain rational functions on V, having zeros and poles on $V \setminus \Omega$, and which behave simply under the G–action. Similarly, the additive functions of H are related to *additive relative invariants* on V, rational functions on V, having zeros and poles on $V \setminus \Omega$, which behave in another simple manner under the G–action. We prove that additive relative invariants are homogeneous of degree 0 and of the form $\frac{h}{f}$ for a relative invariant f (Propositions 3.5, 3.5). Geometrically, an additive relative invariant $\frac{h}{f}$ describes a G–invariant codimension 1 subset of $f = 0$, with G permuting the sets $f = h - \epsilon = 0$, $\epsilon \in \mathbb{C}$, in a certain manner (Corollary 3.8). The generic isotropy subgroups on the hypersurface components of $V \setminus \Omega$ are in the kernel of certain characters and additive functions of H (Lemma 3.9). Nevertheless, some elementary questions remain (see §3.6).

In §4, we describe some examples of prehomogeneous vector spaces and their additive relative invariants.

In §5, we study prehomogeneous vector spaces for which $D = V \setminus \Omega$ is a type of hypersurface called a *linear free divisor*. Such objects have been of much study recently (TODO cites), and were our original motivation. Using a criterion due to Brion and our lemma on the generic isotropy subgroups on the hypersurface components of D, we show that such prehomogeneous vector spaces have no additive relative invariants (Theorem 5.5). Consequently, we may compute the number of irreducible components of D using only the Lie algebra of G, and we gain insight into the structure of G (Corollaries 5.7, 5.10). In §5.5, we study the special cases where G is abelian, reductive, or solvable, and where D is irreducible; this gives simpler proofs of several previous results. Intriguingly, we observe in §5.6 that the degrees of the components of a linear free divisor are not always detectable from the abstract Lie algebra of G. Many questions remain.

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1. Prehomogeneous vector spaces

We first review the basic facts about prehomogeneous vector spaces. Our reference for this is [Kim03], although the work we describe is due to Sato [Sat90].

In the whole article, we shall study only complex linear algebraic groups. For such a K, K^0 will denote the connected component of K containing the identity, $L(K)$ will be the Lie algebra of K, and K_v will be the isotropy subgroup at v of some K–action.

Let V be a finite-dimensional complex vector space, and G a connected complex linear algebraic group. Let $\rho : G \to \mathrm{GL}(V)$ be a rational representation of G. When G has an open orbit Ω in V, then we call (G, ρ, V) a *prehomogeneous vector space*. In this situation, Ω is unique and Zariski open, so that $\Omega^c = V \setminus \Omega$ is an algebraic set in V. We call Ω^c the *exceptional orbit variety* as it is the union of the orbits of G of dimension $< \dim(V)$ (although others use *discriminant* or *singular set*).

One of the basic theorems of prehomogeneous vector spaces is that the hypersurface components of Ω^c can be detected from certain multiplicative characters of G. More precisely, for any complex linear algebraic group K let $X(K)$ denote the set of *rational (multiplicative) characters*, that is, the homomorphisms $K \to \mathrm{GL}(\mathbb{C}) = \mathbb{G}_\mathrm{m}$ of linear algebraic groups. Then a rational function f on V which is not identically 0 is a *relative invariant* if there exists $\chi \in X(G)$ such that

$$f(\rho(g)(v)) = \chi(g) \cdot f(v) \tag{1.1}$$

for all $v \in \Omega$ and $g \in G$, in which case we[1] write $f \xleftrightarrow{\mathrm{m}} \chi$. Immediately from (1.1) we see that the zeros or poles of f can only occur on Ω^c.

Moreover, f and χ provide almost the same information. Knowing f, we may choose $v_0 \in \Omega$ and recover χ by defining

$$\chi(g) = \frac{f(\rho(g)(v_0))}{f(v_0)}, \qquad g \in G.$$

Conversely, knowing χ, we may recover a nonzero constant multiple h of f by choosing $v_0 \in \Omega$, and a nonzero value $h(v_0) \in \mathbb{C}$, and defining h on Ω by $h(\rho(g)(v_0)) = \chi(g)h(v_0)$. Since Ω is dense in V, there is a unique way to extend h to a rational function on V.

Let $X_1(G)$ be the set of $\chi \in X(G)$ for which there exists an f with $f \xleftrightarrow{\mathrm{m}} \chi$. ($X_1(G)$ depends on ρ, although our notation does not communicate this.) Notice that $X_1(G)$ is an abelian group, since if $f_i \xleftrightarrow{\mathrm{m}} \chi_i$ and $n_i \in \mathbb{Z}$ for $i = 1, 2$, then $(f_1)^{n_1}(f_2)^{n_2} \xleftrightarrow{\mathrm{m}} (\chi_1)^{n_1}(\chi_2)^{n_2} \in X_1(G)$. $X_1(G)$ contains information about the hypersurface components of Ω^c.

Theorem 1.1 (e.g., [Kim03, Theorem 2.9]). *Let (G, ρ, V) be a prehomogeneous vector space with exceptional orbit variety Ω^c, and $S_i = \{x \in V : f_i(x) = 0\}$, $i =$*

[1] Usually this is written $f \longleftrightarrow \chi$, but we shall use similar notation for the relationship between additive functions and additive relative invariants.

$1, \dots, r$ the distinct irreducible hypersurface components of Ω^c. Then the irreducible polynomials $f_1, \dots, f_r$ are relative invariants which are algebraically independent, and any relative invariant is of the form $c \cdot (f_1)^{m_1} \cdots (f_r)^{m_r}$ for nonzero $c \in \mathbb{C}$, and $m_i \in \mathbb{Z}$. Moreover, if each $f_i \overset{\mathrm{m}}{\longleftrightarrow} \chi_i$, then $X_1(G)$ is a free abelian group of rank r generated by $\chi_1, \dots, \chi_r$.

The (homogeneous) polynomials $f_1, \dots, f_r$ of the Theorem are called the *basic relative invariants* of (G, ρ, V).

Remark 1.2. Ω^c may be defined (as a set) by an ideal generated by homogeneous polynomials of degree $\dim(V)$. See §5.1 for more information.

There exists another description of $X_1(G)$. Fix an element $v_0 \in \Omega$ with isotropy subgroup G_{v_0}. (Note that the orbit of v_0 is open if and only if $\dim(G_{v_0}) = \dim(G) - \dim(V)$.) Define the algebraic groups

$$G_1 = [G, G] \cdot G_{v_0} \qquad \text{and} \qquad H = G/G_1.$$

The group G_1 does not depend on the choice of $v_0 \in \Omega$ as all such isotropy subgroups are conjugate, and $[G, G] \subseteq G_1$. An examination of (1.1) shows that every $\chi \in X_1(G)$ has $G_1 \subseteq \ker(\chi)$, and thus gives a corresponding $\tilde{\chi} \in X(H)$. In fact,

Proposition 1.3 (e.g., Proposition 2.12 of [Kim03])**.** *$X_1(G) \cong X(H)$.*

Consequently, the structure of the abelian group H contains information on the number of irreducible hypersurface components of $V \setminus \Omega$.

2. Abelian complex linear algebraic groups

Let (G, ρ, V) be a prehomogeneous vector space, and use the notation of the previous section. To understand the rank of $X(H)$ and thus the number of irreducible hypersurface components of the exceptional orbit variety Ω^c, we must understand H. Since H is abelian and connected, it has a very simple structure. Recall that there are two distinct 1-dimensional connected complex linear algebraic groups, the multiplicative group $\mathbb{G}_{\mathrm{m}} = \mathbb{C}^* = \mathrm{GL}(\mathbb{C}^1)$ and the additive group $\mathbb{G}_{\mathrm{a}} = (\mathbb{C}, +)$.

Proposition 2.1 (e.g., [OV90, §3.2.5])**.** *An abelian connected complex linear algebraic group K is isomorphic to $(\mathbb{G}_{\mathrm{m}})^k \times (\mathbb{G}_{\mathrm{a}})^\ell$ for nonnegative integers k and ℓ, where the exponent denotes a repeated direct product.*

Remark 2.2. This Proposition is not true for all fields!

For any linear algebraic group K, let $\mathscr{A}(K)$ be the set of rational homomorphisms $\Phi : K \to \mathbb{G}_{\mathrm{a}}$, which are often called *additive functions* ([Spr98, §3.3]). $\mathscr{A}(K)$ is a complex vector space. When K is connected, $\Phi \in \mathscr{A}(K)$ is determined by $d\Phi_{(e)}$; since K has finite dimension, it follows that $\mathscr{A}(K)$ is finite-dimensional. When K has the decomposition of Proposition 2.1, the rank of $X(K)$ and the dimension of $\mathscr{A}(K)$ are related to k and ℓ.

Lemma 2.3. *If $K = (\mathbb{G}_{\mathrm{m}})^k \times (\mathbb{G}_{\mathrm{a}})^\ell$, then $X(K)$ is free of rank k and $\dim_{\mathbb{C}}(\mathscr{A}(K)) = \ell$.*

Proof. By the Jordan decomposition, any $\chi \in X(K)$ will have $\{1\}^k \times (\mathbb{G}_{\mathrm{a}})^\ell \in \ker(\chi)$, so χ factors through to an element of $X((\mathbb{G}_{\mathrm{m}})^k)$. Thus $X(K) \cong X((\mathbb{G}_{\mathrm{m}})^k)$, and similarly $\mathscr{A}(K) \cong \mathscr{A}((\mathbb{G}_{\mathrm{a}})^\ell)$. Finally, an easy exercise shows that $\mathrm{rank}(X((\mathbb{G}_{\mathrm{m}})^k)) = k$, and $\dim(\mathscr{A}((\mathbb{G}_{\mathrm{a}})^\ell)) = \ell$. □

Thus, for a prehomogeneous vector space (G, ρ, V), the additive characters of $H = G/[G,G] \cdot G_{v_0}$ reduce the potential number of irreducible hypersurface components of Ω^c from $\dim(H)$.

Corollary 2.4. *For a prehomogeneous vector space as in §1, the number of irreducible hypersurface components of Ω^c equals $\dim(H) - \dim(\mathscr{A}(H))$, where $H = G/([G,G] \cdot G_{v_0})$, and $v_0 \in \Omega$.*

Proof. By Theorem 1.1, Proposition 1.3, Proposition 2.1, and Lemma 2.3, the number of irreducible components equals $\operatorname{rank}(X(H)) = \dim(H) - \dim(\mathscr{A}(H))$. $\square$

It is natural, then, to study the geometric meaning of additive characters.

3. Additive relative invariants

In this section, we adapt the multiplicative theory from §1 for additive characters. Let (G, ρ, V) be a prehomogeneous vector space, let $v_0 \in \Omega$, and let $H = G/[G,G] \cdot G_{v_0}$.

3.1. Definition.

We define additive relative invariants analogously to (multiplicative) relative invariants.

Definition 3.1. A rational function h on V is an *additive relative invariant* if there exists a $\Phi \in \mathscr{A}(G)$ so that

$$h(\rho(g)(v)) - h(v) = \Phi(g) \tag{3.1}$$

for all $v \in \Omega$ and $g \in G$. In this situation, write $h \overset{\mathrm{a}}{\longleftrightarrow} \Phi$.

By (3.1), the poles of such an h occur only on Ω^c.

3.2. Basic properties.

We now establish some basic facts about additive relative invariants. First we investigate the uniqueness of the relationship $h \overset{\mathrm{a}}{\longleftrightarrow} \Phi$.

Proposition 3.2. *Let (ρ, G, V) be a prehomogeneous vector space.*

1. *If $h_1 \overset{\mathrm{a}}{\longleftrightarrow} \Phi$ and $h_2 \overset{\mathrm{a}}{\longleftrightarrow} \Phi$, then there exists $\alpha \in \mathbb{C}$ with $h_1 = \alpha + h_2$.*
2. *If $h \overset{\mathrm{a}}{\longleftrightarrow} \Phi_1$ and $h \overset{\mathrm{a}}{\longleftrightarrow} \Phi_2$, then $\Phi_1 = \Phi_2$.*

Proof. For (1), fix $v_0 \in \Omega$ and let $\alpha = h_1(v_0) - h_2(v_0)$, so $\Phi(g) + h_1(v_0) = \alpha + \Phi(g) + h_2(v_0)$ for all $g \in G$. Applying (3.1) shows that $h_1 = \alpha + h_2$ on Ω, and thus on V.

(2) is immediate from (3.1). $\square$

Let $\mathscr{A}_1(G)$ be the set of $\Phi \in \mathscr{A}(G)$ for which there exists an h with $h \overset{\mathrm{a}}{\longleftrightarrow} \Phi$. We now identify the characters in $\mathscr{A}_1(G)$, analogous to Proposition 1.3.

Proposition 3.3. *As vector spaces, $\mathscr{A}_1(G) \cong \mathscr{A}(H)$, where $H = G/[G,G]G_{v_0}$ and $v_0 \in \Omega$.*

Proof. Let $\Phi \in \mathscr{A}_1(G)$ with $h \overset{\mathrm{a}}{\longleftrightarrow} \Phi$. Evaluating (3.1) at $v_0 \in \Omega$ and $g \in G_{v_0}$ shows that $G_{v_0} \subseteq \ker(\Phi)$. Since $\mathbb{G}_{\mathrm{a}}$ is abelian, $[G,G] \subseteq \ker(\Phi)$. Thus any $\Phi \in \mathscr{A}_1(G)$ factors through the quotient $\pi : G \to H$ to a unique $\overline{\Phi} \in \mathscr{A}(H)$, such that $\Phi = \overline{\Phi} \circ \pi$. The map $\rho : \mathscr{A}_1(G) \to \mathscr{A}(H)$ defined by $\rho(\Phi) = \overline{\Phi}$ is $\mathbb{C}$-linear.

Conversely, define the $\mathbb{C}$-linear map $\sigma : \mathscr{A}(H) \to \mathscr{A}(G)$ by $\sigma(\overline{\Phi}) = \overline{\Phi} \circ \pi$. For $\overline{\Phi} \in \mathscr{A}(H)$, by the argument[2] of [Kim03, Proposition 2.11] we may define a regular

[2] This uses the fact that $\mathbb{C}$ has characteristic 0.

function $\overline{h}$ on Ω by $\overline{h}(\rho(g)(v_0)) = (\overline{\Phi} \circ \pi)(g)$, and then extend $\overline{h}$ to a rational function h on V. By construction $h(\rho(g)(v)) - h(v) = (\overline{\Phi} \circ \pi)(g)$ for $v = v_0$, but in fact this equation holds for all $v \in \Omega$. Thus $h \overset{\mathrm{a}}{\longleftrightarrow} \overline{\Phi} \circ \pi$, and $\sigma : \mathscr{A}(H) \to \mathscr{A}_1(G)$.

Finally, check that ρ and σ are inverses using the uniqueness of the factorization through π. For $\overline{\Phi} \in \mathscr{A}(H)$, $\rho(\sigma(\overline{\Phi})) = \rho(\overline{\Phi} \circ \pi) = \overline{\Phi}$. For $\Phi \in \mathscr{A}_1(G)$, $\sigma(\rho(\Phi)) = \sigma(\overline{\Phi}) = \overline{\Phi} \circ \pi$, for a $\overline{\Phi}$ with $\overline{\Phi} \circ \pi = \Phi$. $\square$

Additive relative invariants are the rational functions on V having the following property.

Proposition 3.4. *Let $h : V \to \mathbb{C}$ be a rational function on V, holomorphic on Ω. If h has the property that, for all $g \in G$,*

$$h(\rho(g)(v)) - h(v) \text{ is independent of } v \in \Omega,$$

then there exists a $\Phi \in \mathscr{A}_1(G)$ with $h \overset{\mathrm{a}}{\longleftrightarrow} \Phi$.

Proof. Define $\Phi(g) = h(\rho(g)(v)) - h(v)$ for any $v \in \Omega$. As a composition of regular functions, $\Phi : G \to \mathbb{C}$ is regular. Let $g_1, g_2 \in G$, and $v \in \Omega$. Then

$$\begin{aligned} \Phi(g_1 g_2^{-1}) &= h\big(\rho(g_1 g_2^{-1})(v)\big) - h(v) \\ &= h\big(\rho(g_1)(\rho(g_2^{-1})(v))\big) - h\big(\rho(g_2^{-1})(v)\big) + h\big(\rho(g_2^{-1})(v)\big) - h(v) \\ &= (h(\rho(g_1)(v_1)) - h(v_1)) + (h(v_2) - h(\rho(g_2)(v_2))), \end{aligned}$$

where $v_1 = \rho(g_2^{-1})(v), v_2 = \rho(g_2^{-1})(v) \in \Omega$. By our hypothesis, $\Phi(g_1 g_2^{-1}) = \Phi(g_1) - \Phi(g_2)$. Thus $\Phi : G \to \mathbb{G}_{\mathrm{a}}$ is a homomorphism of algebraic groups, and since $h \overset{\mathrm{a}}{\longleftrightarrow} \Phi$, $\Phi \in \mathscr{A}_1(G)$. $\square$

3.3. **Homogeneity.** Relative invariants are always homogeneous rational functions on V. Similarly, additive relative invariants are homogeneous of degree 0.

Proposition 3.5. *If $h \overset{\mathrm{a}}{\longleftrightarrow} \Phi$, then h may be written as $h = \frac{h_1}{f_1}$, where h_1 and f_1 are homogeneous polynomials with $\deg(h_1) = \deg(f_1)$, h_1 and f_1 have no common factors, and f_1 is a relative invariant.*

Proof. First note that by Remark 1.2, for any $t \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $v \in \Omega$ if and only if $t \cdot v \in \Omega$; we will use this fact implicitly in the rest of the proof.

Let $t \in \mathbb{C}^*$, and define the rational function h_t on V by $h_t(v) = h(t \cdot v)$. Since ρ is a linear representation,

$$h_t(\rho(g)(v)) = h(t \cdot \rho(g)(v)) = h(\rho(g)(t \cdot v)). \tag{3.2}$$

Applying (3.1) to (3.2) gives

$$h_t(\rho(g)(v)) = \Phi(g) + h(t \cdot v) = \Phi(g) + h_t(v). \tag{3.3}$$

Since (3.3) is true for all $g \in G$ and $v \in \Omega$, we have $h_t \overset{\mathrm{a}}{\longleftrightarrow} \Phi$.

By Proposition 3.2(1), there exists an $\alpha : \mathbb{C}^* \to \mathbb{C}$ such that

$$h(t \cdot v) = h(v) + \alpha(t) \tag{3.4}$$

for all $t \in \mathbb{C}^*$ and $v \in \Omega$. If $s, t \in \mathbb{C}^*$ and $v \in \Omega$, using (3.4) repeatedly shows that

$$h(v) + \alpha(st) = h(s(t \cdot v)) = h(t \cdot v) + \alpha(s) = h(v) + \alpha(t) + \alpha(s),$$

or $\alpha(st) = \alpha(s) + \alpha(t)$. (3.4) also shows that $\alpha(1) = 0$, so that $\alpha : \mathbb{C}^* \to (\mathbb{C}, +)$ is a group homomorphism.

Fixing some $v \in \Omega$ and using (3.4) to define α shows that $\alpha : \mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{a}}$ is regular, hence a homomorphism of complex linear algebraic groups. By the Jordan decomposition (or Lemma 2.3), it follows that $\alpha = 0$. Then $h(t \cdot v) = h(v)$ for all nonzero $t \in \mathbb{C}$ and $v \in \Omega$; by density, h is homogeneous of degree 0.

Thus we may write $h = \frac{h_1}{f_1}$, with h_1 and f_1 homogeneous polynomials of equal degree, and without common factors. Since h may only have poles on Ω^c, f_1 defines a hypersurface in Ω^c. By Theorem 1.1, f_1 is a relative invariant. □

This gives an apparently nontrivial result about the structure of prehomogeneous vector spaces.

Corollary 3.6. *Let (G, ρ, V) be a prehomogeneous vector space, let $v_0 \in \Omega$, and let r be the number of irreducible hypersurface components of $V \setminus \Omega$. If $X_1(G) = \{1\}$ (equivalently, $r = 0$), then $\mathscr{A}_1(G) = \{0\}$ and $G = [G, G] \cdot G_{v_0}$. If $d = \dim(G/[G, G] \cdot G_{v_0}) \leq 1$, then $\mathscr{A}_1(G) = \{0\}$ and $r = d$.*

Proof. Suppose $X_1(G) = \{1\}$. Let $\Phi \in \mathscr{A}_1(G)$, and suppose $h \xleftrightarrow{\mathrm{a}} \Phi$. By Proposition 3.5, write $h = \frac{h_1}{f_1}$ for polynomials f_1 and h_1 with $\deg(h_1) = \deg(f_1)$, and f_1 a relative invariant. If $f_1 \xleftrightarrow{\mathrm{m}} \chi_1$, then by the hypothesis $\chi_1 = 1$. By (1.1), f_1 is a nonzero constant on Ω. Then $\deg(h_1) = 0$ too, so h itself is constant. By (3.1), $\Phi = 0$ and so $\mathscr{A}_1(G) = \{0\}$. If there are no nontrivial characters or additive functions, then by Proposition 2.1 and Lemma 2.3, $H = G/[G, G]G_{v_0}$ consists of a single point.

Now suppose $d \leq 1$. If $d = 0$, then $X_1(G) = \{1\}$. If $d = 1$, then we must have $\mathrm{rank}(X_1(G)) = 1$ by the first half of the statement. Apply Corollary 2.4. □

3.4. **Global equation.** Our original definition of an additive relative invariant only involves the behavior of the function on Ω. It is useful to have an equation which expresses the behavior on all of V.

Proposition 3.7. *Let $h \xleftrightarrow{\mathrm{a}} \Phi$. As in Proposition 3.5, write $h = \frac{h_1}{f_1}$ for polynomials h_1 and f_1, with $f_1 \xleftrightarrow{\mathrm{m}} \chi$. Then for all $g \in G$ and $v \in V$,*

$$h_1(\rho(g)(v)) = \chi(g) \cdot (f_1(v) \cdot \Phi(g) + h_1(v)) . \tag{3.5}$$

Proof. Define the regular functions $L, R : G \times V \to \mathbb{C}$ using the left, respectively, right sides of (3.5). By (3.1), these functions agree on $G \times \Omega$, and thus on $G \times V$. □

In particular, we may give a geometric interpretation of additive relative invariants.

Corollary 3.8. *In the situation of Proposition 3.7, let f be any irreducible factor of f_1. For $\epsilon \in \mathbb{C}$, let $V_\epsilon = \{x : f(x) = h_1(x) - \epsilon = 0\}$. Then V_0 is a codimension 2 G–invariant subset of V, and $\rho(g)(V_\epsilon) = V_{\chi(g) \cdot \epsilon}$.*

TODO: converse?

3.5. **Vanishing lemma.** We now prove an important and elementary lemma relating characters and additive functions to the generic isotropy subgroups on the hypersurface components of $V \setminus \Omega$.

Lemma 3.9. *Let $f_1, \ldots, f_r$ be the basic relative invariants of (G, ρ, V), with $f_i \xleftrightarrow{\mathrm{m}} \chi_i$. Let $v_i \in V(f_i)$, with $v_i \notin V(f_j)$ for $i \neq j$. Then*

(1) *For $i \neq j$, $G_{v_i} \subseteq \ker(\chi_j)$.*

(2) *Suppose* $h \xleftrightarrow{\mathrm{a}} \Phi$*, with* $h = \frac{h_1}{g_1}$ *as in Proposition 3.5, and* $g_1 = f_1^{k_1} \cdots f_r^{k_r}$ *the factorization of* g_1*. Then*
  (a) *If* $k_i > 0$ *and* $h_1(v_i) \neq 0$*, then* $G_{v_i}^0 \subseteq \ker(\chi_i)$*.*
  (b) *If* $k_i = 0$*, then* $G_{v_i} \subseteq \ker(\Phi)$*.*

Proof. If $g \in G_{v_i}$, then

$$f_j(v_i) = f_j(\rho(g)(v_i)) = \chi_j(g) f_j(v_i). \tag{3.6}$$

Since $v_i \notin V(f_j)$, (3.6) implies that $\chi_j(g) = 1$, proving (1).

In the situation of (2), Proposition 3.7 shows that for all $g \in G$, $v \in V$,

$$h_1(\rho(g)(v)) = \chi_1^{k_1}(g) \cdots \chi_r^{k_r}(g) \left(f_1^{k_1}(v) \cdots f_r^{k_r}(v) \cdot \Phi(g) + h_1(v) \right). \tag{3.7}$$

If $k_i > 0$ and $g \in G_{v_i}$, then evaluating (3.7) at (g, v_i) and applying (1) gives

$$h_1(v_i) = \chi_i^{k_i}(g) \cdot h_1(v_i),$$

from which (2a) follows. If $k_i = 0$ and $g \in G_{v_i}$, then evaluating (3.7) at (g, v_i) and applying (1) gives

$$h_1(v_i) = f_1^{k_1}(v_i) \cdots f_r^{k_r}(v_i) \cdot \Phi(g) + h_1(v_i).$$

As $f_1^{k_1}(v_i) \cdots f_r^{k_r}(v_i) \neq 0$, we have $\Phi(g) = 0$. ☐

Remark 3.10. For (2a), since h is written in lowest terms, a generic $v_i \in V(f_i)$ has $h_1(v_i) \neq 0$. Hence, if f_i appears (nontrivially) in the denominator of an additive relative invariant, then by (1) and (2a), the identity component of the generic isotropy group on $V(f_i)$ lies in the kernel of every $\chi \in X_1(G)$.

Remark 3.11. In the situation of (2a), [Pik] implies that the ideal of Remark 1.2 is contained in $(f_i)^2$.

3.6. **Further questions.** A basic fact about prehomogeneous vector spaces is that any nontrivial factor of a relative invariant is itself a relative invariant; a crucial fact used in the proof is the unique factorization of a rational function on V.

The analogous statement for additive relative invariants would be

Conjecture 3.12. *There exists a basis for* $\mathscr{A}_1(G)$ *such that the corresponding additive relative invariants each have only one basic relative invariant in their denominator.*

It is natural to try to use a partial fraction expansion to prove this; unfortunately, for > 1 variable, such an expansion does not always exist. TODO

4. Examples

We now give some examples of prehomogeneous vector spaces and their additive relative invariants. Most of our examples have $\dim(G) = \dim(V)$; this is because for a fixed V, both $\dim(G_{v_0})$ and $\dim([G, G])$ generally increase with $\dim(G)$, and by Corollary 3.6 our examples should have $\dim(G/[G, G] \cdot G_{v_0}) \geq 2$.

Example 4.1. Let $G = \left\{ \left(\begin{smallmatrix} a & 0 & 0 \\ b & a & 0 \\ c & 0 & a \end{smallmatrix} \right) \in \mathrm{GL}_3(\mathbb{C}) \right\}$ act on $\mathbb{C}\{x, y, z\} = \mathbb{C}^3$ by multiplication. G has $D = V(x)$ as the exceptional orbit variety, with $f_1 = x \xleftrightarrow{\mathrm{m}} \chi_1 = a$. Since G is abelian and the generic isotropy subgroup at $(x, y, z) = (1, 0, 0)$ is trivial, G has 2 distinct additive functions. Namely, $h_1 = \frac{y}{x} \xleftrightarrow{\mathrm{a}} \Phi_1 = \frac{b}{a}$, and

$h_2 = \frac{z}{x} \xleftrightarrow{\text{a}} \Phi_2 = \frac{c}{a}$. On $V(x)$, radial subsets of the form $\alpha y + \beta z = 0$ are invariant, with the G–action multiplying both coordinates by $\chi_1 = a$.

This example may be generalized to have $n-1$ additive relative invariants in $\mathbb{C}^n$ by using $G \subseteq \mathrm{GL}(\mathbb{C}^n)$ having all diagonals equal, the first column unrestricted, and all other entries equal to zero.

Example 4.2. Let $G = \left\{ \left(\begin{smallmatrix} a & 0 & 0 \\ 0 & b & 0 \\ c & 0 & a \end{smallmatrix}\right) \in \mathrm{GL}_3(\mathbb{C}) \right\}$ act on $\mathbb{C}\{x,y,z\} = \mathbb{C}^3$ by multiplication. G has $f_1 = x \xleftrightarrow{\text{m}} \chi_1 = a$, $f_2 = y \xleftrightarrow{\text{m}} \chi_2 = b$. Since G is abelian and the generic isotropy subgroup at $(x,y,z) = (1,1,0)$ is trivial, G has 1 distinct additive function, namely, $h_1 = \frac{z}{x} \xleftrightarrow{\text{a}} \Phi_1 = \frac{c}{a}$. On $V(x)$, both $y=0$ and $z=0$ are G–invariant; however, $g \in G$ sends $y = \epsilon$ (respectively, $z = \epsilon$) to $y = \chi_2(g)\cdot\epsilon$ (resp., $z = \chi_1(g)\cdot\epsilon$), so only the latter has the behavior predicted by Corollary 3.8. Both $V(x)$ and $V(y)$ contain open orbits.

Example 4.3. Let $G = \left\{ \left(\begin{smallmatrix} a & 0 & 0 \\ 0 & b & 0 \\ c & 0 & \frac{a}{b} \end{smallmatrix}\right) \in \mathrm{GL}_3(\mathbb{C}) \right\}$ act on $\mathbb{C}\{x,y,z\} = \mathbb{C}^3$ by multiplication. G has $f_1 = x \xleftrightarrow{\text{m}} \chi_1 = a$, $f_2 = y \xleftrightarrow{\text{m}} \chi_2 = b$. G has trivial generic isotropy subgroup, but $\dim([G,G]) = 1$. Thus $G/[G,G]\cdot G_{v_0}$ has no nontrivial additive functions, and there are no additive relative invariants. The orbit structure of G agrees with that of the group in Example 4.2; however, neither G–invariant subset of $V(x)$ exhibits the behavior expected from Corollary 3.8.

Example 4.4. Fix a positive integer n and let $m \in \{n, n+1\}$. Let $L \subseteq \mathrm{GL}_n(\mathbb{C})$ (respectively, $U \subseteq \mathrm{GL}_m(\mathbb{C})$) consist of invertible lower triangular matrices (resp., upper triangular unipotent matrices). Let $G = L\times U$ act on the space $M(n,m,\mathbb{C})$ of $n \times m$ complex matrices by $(L,R)\cdot M = LMR^{-1}$. The classical "LU factorization" of a complex matrix asserts that this is a prehomogeneous vector space. For a matrix A, let $A^{(k)}$ denote the upper leftmost $k\times k$ submatrix of A. By [DP, §6], the basic relative invariants are of the form $f_i(M) = \det(M^{(i)})$, $i = 1,\dots,n$. We have $f_i \xleftrightarrow{\text{m}} \chi_i(L,R) = \det(L^{(i)})$. The identity matrix $v_0 = I$ (when $m = n$) or $v_0 = \begin{pmatrix} I & 0 \end{pmatrix}$ (when $m = n+1$) is in the open orbit and has a finite isotropy subgroup G_{v_0}.

Since $\dim(G/[G,G]\cdot G_{v_0}) = n+m-1$, we should expect $m-1$ distinct additive relative invariants. In fact, for $(L,R) \in G$ and $1 \le i \le m-1$, let $\Phi_i(L,R) = (R)_{i,i+1}$. A computation shows that Φ_i is an additive function of G with kernel containing $[G,G]\cdot G_{v_0}$. Less obvious is that

$$h_i(M) = \frac{\det((M \text{ with column } i \text{ deleted})^{(i)})}{\det(M^{(i)})} \xleftrightarrow{\text{a}} \Phi_i.$$

(TODO: write down a proof of this. Otherwise, only verified with Maple computations.)

5. Linear Free Divisors

We now specialize to prehomogeneous vector spaces (ρ, G, V) for which the complement of the open orbit Ω is a type of hypersurface called a linear free divisor. The main theorem is that these have no additive relative invariants, but this has significant consequences for their structure.

5.1. **Introduction.** Let $\mathscr{O}_{\mathbb{C}^n,p}$ denote the ring of germs of holomorphic functions at p, and $\mathrm{Der}_{\mathbb{C}^n,p}$ the $\mathscr{O}_{\mathbb{C}^n,p}$–module of germs of holomorphic vector fields on $\mathbb{C}^n$ at p. Associated to a germ (D,p) of an analytic set in $\mathbb{C}^n$ is the $\mathscr{O}_{\mathbb{C}^n,p}$–module of *logarithmic vector fields* defined by

$$\mathrm{Der}(-\log D) = \{\eta \in \mathrm{Der}_{\mathbb{C}^n,p} : \eta(I(D)) \subseteq I(D)\},$$

where $I(D)$ is the ideal of functions vanishing on D. These are the vector fields tangent to (D,p). $\mathrm{Der}(-\log D)$ is a Lie algebra using the Lie bracket of vector fields.

When $\mathrm{Der}(-\log D)$ is a free $\mathscr{O}_{\mathbb{C}^n,p}$–module, necessarily of rank n, then (D,p) is called a *free divisor*. Free divisors are always pure hypersurfaces.

Now let (ρ, G, V) be a prehomogeneous vector space. Differentiating the action of G gives a Lie algebra (anti-)homomorphism $\tau : \mathfrak{g} \to \mathrm{Der}_V$, where $\tau(X) = \xi_X$ is a vector field on V defined by

$$\tau(X)(v) = \xi_X(v) = \frac{d}{dt}\left(\rho(\exp(tX))(v)\right)|_{t=0}.$$

Each $\tau(X) = \xi_X$ is *linear*, that is, homogeneous of degree 0 (e.g., $x\frac{\partial}{\partial x} - 2z\frac{\partial}{\partial y}$), and tangent to the exceptional orbit variety $D = V \setminus \Omega$. Hence, $\tau(\mathfrak{g})$ is a finite-dimensional Lie subalgebra of $\mathrm{Der}(-\log D)$. (In fact, the ideal describing the $v \in V$ where $\dim(\tau(\mathfrak{g})(v)) < \dim(V)$ defines D as a set.)

When $\mathrm{Der}(-\log D)$ has a free basis of linear vector fields, $(D,0)$ (or D) is called a *linear free divisor*. Every linear free divisor comes from a prehomogeneous vector space (ρ, G, V) such that $\tau(\mathfrak{g})$ generates $\mathrm{Der}(-\log D)$, and $\dim(G) = \dim(V)$ ([GMNRS09, Lemma 2.3]). The prehomogeneous vector space may be constructed in the following way.

5.2. **Brion's criterion.** For a divisor $D \subset V$ in a finite-dimensional complex vector space, let $\mathrm{GL}(V)_D$ be the largest subgroup of $\mathrm{GL}(V)$ which preserves D. Then $\mathrm{GL}(V)_D$ is algebraic, and let $G = (\mathrm{GL}(V)_D)^0$. Then G is a connected complex linear algebraic group, and its Lie algebra $\mathfrak{g}$ is isomorphic to $\mathrm{Der}(-\log D)_0$, the subalgebra of linear vector fields in $\mathrm{Der}(-\log D)$.

Brion gave the following criterion for D to be a linear free divisor.

Theorem 5.1 ([Bri06], [GMS11, Theorem 2.1])**.** *The following are equivalent:*

(1) *D is a linear free divisor*
(2) *Both of these conditions hold:*
  (a) *$V \setminus D$ is a unique G-orbit, and the corresponding isotropy groups are finite.*
  (b) *Each irreducible component D_i of D contains an open G-orbit D_i^0, and the corresponding isotropy groups are extensions of finite groups by $\mathbb{G}_{\mathrm{m}}$.*

In this case, $\tau(\mathfrak{g})$ generates $\mathrm{Der}(-\log D)$.

Definition 5.2. When the conclusion of Theorem 5.1 holds, we shall say that G *defines the linear free divisor D*. In particular, G is connected, $\dim(G) = \dim(V)$, and with the inclusion $i : G \to \mathrm{GL}(V)$, (G, i, V) is a prehomogeneous vector space.

The proof of Theorem 5.1 shows the following.

Corollary 5.3. *Suppose that in the situation of Theorem 5.1, (2a) holds, D_i contains an open orbit D_i^0, and $v_i \in D_i^0$. Then G_{v_i} is an extension of a finite group by $\mathbb{G}_{\mathrm{m}}$ if and only if the induced representation of $G_{v_i}^0$ on the normal line to D_i at v_i is nontrivial.*

The representation on the normal line is actually a quite familiar object.

Lemma 5.4. *Let (ρ, G, V) be a prehomogeneous vector space with f as a basic relative invariant, and $v \in D = V(f)$. If H is a subgroup of $\rho(G)$ which fixes v and $v \in D$ is a smooth point, then the the representation $\rho_v : H \to \mathrm{GL}(L)$ of H on the "normal line" $L \cong T_vV/T_vD$ to D at v is*

$$\rho_v(h)(\ell) = \chi(h)\ell, \qquad \text{for all } \ell \in L,$$

where $\chi \overset{\mathrm{m}}{\longleftrightarrow} f$.

Geometrically, this makes sense. ρ_v acts on a normal slice to $f = 0$ at v, and these slices intersect all level sets of f. The action of G_v fixes v and translates between the level sets of f according to χ.

Proof. Since H fixes v and leaves invariant D, H leaves invariant T_vD. Then the normal line is $L = T_vV/T_vD$, and $\rho|_H$ induces the representation $\rho_v : H \to \mathrm{GL}(L)$, with $\rho_v(h)(w+T_vD) = \rho(h)(w)+T_vD$. Since D is reduced and v is a smooth point, $T_vD = \ker(df_{(v)} : T_vV \to T_0\mathbb{C} \cong \mathbb{C})$. In particular, $df_{(v)}$ induces a vector space isomorphism from L to $\mathbb{C}$. By definition, then, we have the following commutative diagram for all $h \in H$.

(5.1)
$$\begin{tikzcd} T_vV \arrow[r, "\rho(h)"] \arrow[d, "df_{(v)}"'] & T_vV \arrow[d, "df_{(v)}"] \\ L \arrow[r, "\rho_v(h)"] & L \end{tikzcd}$$

If $w \in T_vV$ and $\lambda \in \mathbb{C}$, then

$$f(v + \lambda \cdot \rho(h)(w)) = f(\rho(h)(v + \lambda v)) = \chi(h)f(v + \lambda w). \tag{5.2}$$

Differentiating (5.2) with respect to λ and evaluating at $\lambda = 0$, we have $df_{(v)}(\rho(h)(w)) = \chi(h) \cdot df_{(v)}(w)$, or by (5.1),

$$\rho_v(h)(df_{(v)}(w)) = \chi(h) \cdot df_{(v)}(w).$$

Thus $\rho_v(h)$ acts on L by multiplication by $\chi(h)$. □

5.3. **The main theorem.** Let (G, ρ, V) define the linear free divisor $D \subset V$ (see Definition 5.2). In particular, ρ is the inclusion of G into $\mathrm{GL}(V)$. Let $f_1, \dots, f_r$ be the basic relative invariants, so that $\cup_{i=1}^r V(f_i)$ is the irreducible decomposition of $D = V \setminus \Omega$. Let $f_i \overset{\mathrm{m}}{\longleftrightarrow} \chi_i$. Let v_i be a generic point on $V(f_i)$, and let $v_0 \in \Omega$.

Theorem 5.5. *Let G define the linear free divisor D, with the notation above. Then*

1. *The homomorphism $(\chi_1, \dots, \chi_r) : G \to (\mathbb{G}_{\mathrm{m}})^r$ is surjective and has kernel $[G,G] \cdot G_{v_0}$.*
2. *$G/[G,G] \cdot G_{v_0}$ has no additive characters or additive relative invariants.*
3. *For $i \neq j$, $G_{v_j}^0 \subseteq \ker(\chi_i)$.*
4. *$\chi_i|_{G_{v_i}^0} : G_{v_i}^0 \to \mathbb{G}_{\mathrm{m}}$ is surjective, with finite kernel.*

(5) $\ker(\chi_i|_{G^0_{v_i}}) = G^0_{v_i} \cap ([G,G] \cdot G_{v_0})$ *is finite.*
(6) *For* $i \neq j$, $G^0_{v_i} \cap G^0_{v_j}$ *is a finite subset of* $[G,G] \cdot G_{v_0}$.

Proof. Let $H = [G,G] \cdot G_{v_0}$. Since G/H is connected and abelian, by Proposition 2.1 $G/H \cong (\mathbb{G}_{\mathrm{m}})^k \times (\mathbb{G}_{\mathrm{a}})^\ell$. Choosing a free basis $\epsilon_1, \ldots, \epsilon_k$ of $X(G/H)$ and a basis $\Phi_1, \ldots, \Phi_\ell$ of $\mathscr{A}(G/H)$ gives an explicit isomorphism $\theta = (\epsilon_1, \ldots, \epsilon_k, \Phi_1, \ldots, \Phi_\ell) : G/H \to (\mathbb{G}_{\mathrm{m}})^k \times (\mathbb{G}_{\mathrm{a}})^\ell$. Theorem 1.1 and Proposition 1.3 show that we may let $\epsilon_i = \chi_i$. Then, (1) will follow from (2).

If $\Phi \in \mathscr{A}(G/H)$, then by Proposition 3.3 there exists a h with $h \xleftrightarrow{\mathrm{a}} \Phi$. By Proposition 3.5, write $h = \frac{h_1}{g_1}$, with g_1 a polynomial relative invariant, and $g_1 \xleftrightarrow{\mathrm{m}} \chi$. If f_i is a nontrivial factor of g_1, then by Lemma 3.9(2a) and the genericity of v_i, $G^0_{v_i} \subset \ker(\chi_i)$. (In fact, as this condition is invariant under conjugation and $\mathrm{smooth}(D) \cap V(f_i)$ is an orbit by Theorem 5.1 (see [GMS11]), v_i need only be a smooth point of D in $V(f_i)$.) Thus (4) will provide the contradiction to prove (2).

(3) is just Lemma 3.9(1).

By Theorem 5.1, Corollary 5.3, and Lemma 5.4, each $\chi_i|_{G^0_{v_i}}$ is nontrivial, thus surjective. As $\dim(G^0_{v_i}) = 1 = \dim(\mathrm{GL}(\mathbb{C}))$, the kernel is finite, giving (4), (2), and (1).

For (5), such elements are contained in the kernel by (1), and conversely by (3) and (1). By (4), the kernel is finite.

For (6), observe that by (3), such elements are in the kernel of the homomorphism of (1). For finiteness, use (5). $\square$

Remark 5.6. As 1–dimensional connected complex linear algebraic groups with a surjective homomorphism to $\mathbb{G}_{\mathrm{m}}$, for $i > 0$, $G^0_{v_i} \cong \mathbb{G}_{\mathrm{m}}$.

Theorem 5.5 has a number of consequences, of which the most interesting is probably (1):

Corollary 5.7. *Let G define the linear free divisor D, with the notation above. Then*

(1) *D has*

$$r = \dim_{\mathbb{C}}(G/[G,G] \cdot G_{v_0}) = \dim_{\mathbb{C}}(G/[G,G]) = \dim_{\mathbb{C}}(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])$$

irreducible components, where $\mathfrak{g}$ is the Lie algebra of G.

(2) *Every element of G may be written in a finite number of ways as a product of elements from the subgroups $[G,G] \cdot G_{v_0}, G^0_{v_1}, \ldots, G^0_{v_r}$. The terms of such a product are unique modulo $[G,G] \cdot G_{v_0}$.*

(3) *Let $S \subseteq \{1, \ldots, r\}$. The subgroup of G which leaves invariant all level sets of f_i for $i \in S$ is the product of the subgroups $[G,G] \cdot G_{v_0}$, and $G^0_{v_j}$ for $j \notin S$.*

(4) *$[G,G] \cdot G_{v_0}$ contains all unipotent elements of G.*

Proof. For (1), combine Theorem 5.5(2) and Corollary 2.4. Note also that G_{v_0} is finite by Theorem 5.1.

Let $g \in G$. By Theorem 5.5(4), there exists $g_i \in G^0_{v_i}$ such that $\chi_i(g) = \chi_i(g_i)$. By Theorem 5.5(3) and (1), $g(g_1 \cdots g_r)^{-1}$ is in the kernel of each χ_i, and hence lies in $[G,G] \cdot G_{v_0}$. This proves existence.

To address uniqueness, let $g \in G$ and suppose $g = g^1_0 \cdot g^1_1 \cdots g^1_r = g^2_0 \cdot g^2_1 \cdots g^2_r$, with $g^j_0 \in [G,G] \cdot G_{v_0}$, and for $i > 0$, $g^j_i \in G^0_{v_i}$. By Theorem 5.5(1) and (3), for

$i > 0$, $\chi_i(g) = \chi_i(g_i^1) = \chi_i(g_i^2)$. Since each $\chi_i|_{G^0_{v_i}}$ has a finite kernel contained in $[G,G]\cdot G_{v_0}$ by Theorem 5.5(5), for $i > 0$, $g_i^1 = g_i^2$ modulo $\ker(\chi_i|_{G^0_{v_i}})$. Since g_0^j is uniquely determined by $g_1^j, \ldots, g_r^j$, there are precisely

$$\prod_{i=1}^{r} \#(\ker(\chi_i|_{G^0_{v_i}}))$$

ways to write g in this way. This proves (2).

By (1.1), $g \in G$ leaves invariant all level sets of f_i if and only if $g \in \ker(\chi_i)$. Then (2) and Theorem 5.5(3),(5) prove (3).

(4) follows from the Jordan decomposition of an algebraic group and Theorem 5.5(1). □

Example 5.8. On $\mathbb{C}^3$, use coordinates x, y, z. The linear free divisor $x(xz-y^2) = 0$ is defined by the solvable $G \subset \mathrm{GL}(\mathbb{C}^3)$ consisting of elements of the form

$$\begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ \frac{b^2}{a} & \frac{2bc}{a} & \frac{c^2}{a} \end{pmatrix}.$$

We have $f_1 = x \overset{\mathrm{m}}{\longleftrightarrow} a$ and $f_2 = xz-y^2 \overset{\mathrm{m}}{\longleftrightarrow} c^2$. For $v_0 = (1,0,1) \in \Omega$, $v_1 = (0,1,0)$, and $v_2 = (1,0,0)$, we have generic isotropy subgroups defined by, respectively, $(a,b,c) = (1,0,\pm 1)$, $(b,c) = (0,1)$, and $(a,b) = (1,0)$. As $[G,G]$ is defined by $a = c = 1$, $[G,G]\cdot G_{v_0}$ is defined by $a = 1$, $c = \pm 1$. Theorem 5.5 and Corollary 5.7 are easy to verify.

Let $L(K)$ denote the Lie algebra of an algebraic group K. On the level of Lie algebras, the analogous statement is the following.

Corollary 5.9. *As vector spaces,*

$$L(G) = L([G,G]) \oplus \bigoplus_{i=1}^{r} L(G_{v_i}).$$

We may choose a generator X_i for each $L(G_{v_i})$ such that $d(\chi_i)_{(e)}(X_j) = \delta_{ij}$, or equivalently, $\xi_{X_j}(f_i) = \delta_{ij} f_i$. For $X \in L([G,G])$, $\xi_X(f_j) = 0$.

Proof. Differentiating the homomorphism of Theorem 5.5(1) gives a homomorphism $\lambda : L(G) \to L((\mathbb{G}_{\mathrm{m}})^r) = \oplus_{i=1}^r L(\mathbb{G}_{\mathrm{m}})$ with kernel $L([G,G]G_{v_0}) = L([G,G])$. By Theorem 5.5(3) and (4), under λ each $L(G_{v_i})$ surjects onto the ith copy of $L(\mathbb{G}_{\mathrm{m}})$, and is zero on the rest. This gives the vector space decomposition, and proves that for $i \neq j$, $d(\chi_i)_{(e)}(L(G_{v_j})) = 0$. Now choose $X_i \in L(G_{v_i})$ such that $d(\chi_i)_{(e)}(X_i) = 1$.

The rest of the statement follows by differentiating $f_i(\rho(\exp(tX))(v)) = \chi_i(\exp(tX))f_i(v)$ to show $\xi_X(f_i) = d(\chi_i)_{(e)}(X)\cdot f_i$. □

5.4. **The structure of G.** We now use Theorem 5.5 and Corollary 5.7 to study the structure of algebraic groups defining linear free divisors.

Let G be a connected complex algebraic group. Let $\mathrm{Rad}(G)$ denote the radical of G, the maximal connected normal solvable subgroup. The *Levi decomposition* of G writes

$$G = \mathrm{Rad}_u(G) \rtimes L,$$

where the connected normal $\mathrm{Rad}_u(G)$ is the unipotent radical of G, the largest connected unipotent normal subgroup of G, consisting of all unipotent elements of $\mathrm{Rad}(G)$; and L is a Levi subgroup, a maximal connected reductive algebraic subgroup of G, unique up to conjugation [Bor91, 11.22]. Moreover, $L = \mathrm{Z}(L)^0 \cdot [L, L]$, where $\mathrm{Z}(\cdot)$ denotes the center, $\mathrm{Z}(L) \cap [L, L]$ is finite, $[L, L]$ is semisimple, and $(\mathrm{Z}(L))^0 = L \cap \mathrm{Rad}(G)$ is a maximal torus of $\mathrm{Rad}(G)$ [Bor91, 14.2,11.23].

Groups defining linear free divisors have the following structure.

Corollary 5.10. *Let G define the linear free divisor D and have the Levi decomposition above. Then*

1. *The number of irreducible components of D is $r = \dim(\mathrm{Z}(L))$.*
2. *$[G, G] = \mathrm{Rad}_u(G) \rtimes [L, L]$.*
3. *$G = [G, G] \cdot (\mathrm{Z}(L))^0$, with $[G, G] \cap (\mathrm{Z}(L))^0$ finite.*
4. *$(\chi_1, \ldots, \chi_r)|_{\mathrm{Z}(L)^0} : \mathrm{Z}(L)^0 \to (\mathbb{G}_\mathrm{m})^r$ is surjective, with a finite kernel.*

Proof. Let $R = \mathrm{Rad}_u(G)$. Since G is the semidirect product of R and L, a straightforward calculation shows that $[G, G] = [R, R] \cdot [R, L] \cdot [L, L]$. By Corollary 5.7(4) and connectedness, $R \subseteq [G, G]$. Since $[R, R], [R, L] \subseteq R$ as $R \trianglelefteq G$,

$$[G, G] = R \cdot [G, G] = R \cdot [R, R] \cdot [R, L] \cdot [L, L] = R \cdot [L, L].$$

As R consists of unipotent elements and $\mathrm{Rad}(G) \cap L$ consists of semisimple elements, $R \cap L = \{e\}$. Since $R \subseteq [G, G]$ are normal subgroups of G, $R \trianglelefteq [G, G]$. This proves the decomposition (2).

From this, we conclude

$$G = R \cdot [L, L] \cdot (Z(L))^0 = [G, G] \cdot (Z(L))^0; \tag{5.3}$$

moreover, (2) implies that $[G, G] \cap L = [L, L]$, and hence $[G, G] \cap (Z(L))^0 = [L, L] \cap (Z(L))^0$ is finite. This proves (3). Thus,

$$G/[G, G] \cong (Z(L))^0/([G, G] \cap Z(L)^0)$$

has dimension equal to $\dim(Z(L))$. Corollary 5.7(1) then proves (1).

Finally, (4) may be checked on the level of Lie algebras. The above implies that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{z}(\mathfrak{l})$, where $\mathfrak{g}$ and $\mathfrak{l}$ are the Lie algebras of G and L, and $\mathfrak{z}(\mathfrak{l})$ is the center of $\mathfrak{l}$. For $\chi = (\chi_1, \ldots, \chi_r)$, $\ker(d\chi_{(e)}) = [\mathfrak{g}, \mathfrak{g}]$ by Theorem 5.5(1), and hence $d\chi_{(e)}|_{\mathfrak{z}(\mathfrak{l})}$ is an isomorphism. In fact, by Theorem 5.5(1), the kernel in (4) is $(\mathrm{Z}(L))^0 \cap ([G, G] \cdot G_{v_0})$. $\square$

Consider the following examples of linear free divisors.

Example 5.11. We continue Example 5.8. $\mathrm{Rad}_u(G)$ is defined by $a = c = 1$, and the maximal torus $L = \mathrm{Z}(L)$ is defined by $b = 0$. Corollary 5.10 is easy to check; in particular, the 2-element subgroup $L \cap G_{v_0}$ lies in the kernel of the homomorphism of (4).

The following example is neither reductive nor solvable.

Example 5.12 ([DP, Example 9.4])**.** Define the algebraic group

$$G = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & d & e & 0 \\ f & g & h & i \end{pmatrix} \in \mathrm{GL}(\mathbb{C}^4) \right\}.$$

Let S be the space of 4×4 symmetric matrices with the usual coordinates x_{ij}, $1 \le i \le j \le 4$. Let $V \subset S$ be the subspace where $x_{11} = 0$. Let $\rho : G \to \mathrm{GL}(V)$ be defined by $\rho(A)(M) = AMA^T$. Note that $\ker(\rho) = \{\pm I\}$. The Levi decomposition of G has L defined by $f = g = h = 0$, and $\mathrm{Rad}(G)$ defined by $c = b - e = 0$ (the group of lower-triangular $g \in G$ is not normal in G). Then $\mathrm{Rad}_u(G)$ is defined by $a = b = e = i = 1$ and $c = d = 0$, $[L, L]$ is defined by $f = g = h = 0$ and $be - cd = 1$, and $L \cap \mathrm{Rad}(G) = \mathrm{Z}(L)$ is defined by $b = e$ and $c = d = f = g = h = 0$. Corollary 5.10(2) is checked by verifying that $[G, G]$ is defined by $a = be - cd = i = 1$.

The exceptional orbit variety is the linear free divisor defined by $f_1 \cdot f_2 \cdot f_3$, where

$$f_1 = \begin{vmatrix} x_{22} & x_{23} \\ x_{23} & x_{33} \end{vmatrix}, \qquad f_2 = \begin{vmatrix} 0 & x_{12} & x_{13} \\ x_{12} & x_{22} & x_{23} \\ x_{13} & x_{23} & x_{33} \end{vmatrix}, \qquad f_3 = \begin{vmatrix} 0 & x_{12} & x_{13} & x_{14} \\ x_{12} & x_{22} & x_{23} & x_{24} \\ x_{13} & x_{23} & x_{33} & x_{34} \\ x_{14} & x_{24} & x_{34} & x_{44} \end{vmatrix},$$

corresponding to the characters $\chi_1 = (cd - be)^2$, $\chi_2 = a^2(cd - be)^2$, $\chi_3 = a^2(cd - be)^2 i^2$. Let

$$v_0 = \begin{pmatrix} 0&1&0&0 \\ 1&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}, \qquad v_1 = \begin{pmatrix} 0&1&0&0 \\ 1&0&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 0&0&1&1 \\ 0&0&1&0 \\ 1&1&0&0 \\ 1&0&0&0 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 0&1&0&0 \\ 1&1&0&0 \\ 0&0&1&0 \\ 0&0&0&0 \end{pmatrix}$$

be generic points in Ω and on each $V(f_i)$. Then G_{v_0} is 8 points, defined by $a = b = \pm 1, e = \pm 1, i = \pm 1, c = d = f = g = h = 0$. G_{v_1} is defined by $a = \frac{1}{b}, c = d = f = g = h = 0, e = \pm 1, i = \pm 1$; G_{v_2} by $a = b = \frac{1}{e}, c = d = f = g = h = 0, i = e$; and G_{v_3} by $a = b = \pm 1$, $e = \pm 1$, $c = d = f = g = h = 0$, $i \neq 0$. With these calculations, the conclusions of Theorem 5.5 and Corollaries 5.7 and 5.10 are straightforward to check. TODO: actually check this.

5.5. **Consequences in a few special cases.** We now apply Theorem 5.5 and its corollaries to several special types of linear free divisors.

The following observation will be useful.

Remark 5.13. Let $G \subseteq \mathrm{GL}(V)$ define the linear free divisor D. Since G generates all of the linear vector fields in $\mathrm{Der}(-\log D)$, including the Euler vector field, it must be the case that $(\mathbb{C} \setminus \{0\}) \cdot I = \mathbb{G}_{\mathrm{m}} \cdot I \subseteq G$.

5.5.1. *Abelian groups.* The *normal crossings divisor* is a divisor given by the union of the coordinate hyperplanes for some choice of coordinates. When we use linear forms as coordinates for a vector space, this is a linear free divisor, and it is the only linear free divisor defined by an abelian group.

Corollary 5.14 (Theorem 2.12 of [GMS11])**.** *Let V be a finite-dimensional complex vector space and suppose that a connected complex linear algebraic group $G \subseteq \mathrm{GL}(V)$ defines a linear free divisor D in V. Then G (or its Lie algebra $\mathfrak{g}$) is abelian if and only if D is equivalent (under a change of basis in V) to the normal crossings divisor.*

Proof. Let $n = \dim(G) = \dim(V)$.

If D is the normal crossings divisor, then after choosing a basis of V, G is the diagonal group in $\mathrm{GL}(V)$, isomorphic to $(\mathbb{G}_{\mathrm{m}})^n$. Thus G and $\mathfrak{g}$ are abelian.

If $\mathfrak{g}$ or G is abelian, then by Corollary 5.7(1), D has $\dim(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]) = \dim(\mathfrak{g}) = n$ irreducible components. (By Corollary 5.10(3), G is an algebraic torus.) Since the irreducible components $D_1, \ldots, D_n$ of D are defined by homogeneous polynomials of positive degree, each D_i is a hyperplane through 0 defined by a linear form. By the following Lemma, D must be equivalent to the normal crossings divisor. $\square$

Lemma 5.15. *If a central hyperplane arrangement $H = \cup_{i=1}^n H_i$ in V, $n = \dim(V)$, is a linear free divisor, then $\cap_{i=1}^n H_i = \{0\}$ and H is equivalent to the normal crossings divisor.*

Proof. Let $I = \cap_{i=1}^n H_i$, and let each α_i be a linear form defining H_i. Since I is the kernel of $\alpha = (\alpha_1, \ldots, \alpha_n) : V \to \mathbb{C}^n$, α is invertible if and only if $I = \{0\}$.

Suppose that $I \neq \{0\}$, so that I contains a line $L = \{\lambda v : \lambda \in \mathbb{C}\}$ for some nonzero $v \in V$. Define the vector field $\xi(x) = v$ (identifying $T_x V$ with V) on V. Since each hyperplane D_i contains the line L, ξ is tangent to each D_i (since, e.g., $\alpha_i(p + \lambda v) = \alpha_i(p) + \lambda \alpha_i(v) = \alpha_i(p) + \lambda \cdot 0$). Thus ξ is tangent to $D = \cup_{i=1}^n D_i$, so $\xi \in \mathrm{Der}(-\log D)$. Since ξ has degree -1, ξ cannot be expressed in terms of linear vector fields, contradicting the assumption that D is a linear free divisor.

Thus $I = \{0\}$, α is invertible, and in using α to identify V with $\mathbb{C}^n$ we see that D is the normal crossings divisor. □

5.5.2. *Irreducible linear free divisors.* We may be fairly explicit about which groups produce irreducible linear free divisors. Recall that an algebraic group H is called *perfect* if $[H, H] = H$. For instance, semisimple groups are perfect.

Corollary 5.16. *Let $D \subset V$ be a linear free divisor with group G. Let $H = G \cap \mathrm{SL}(V)$, and let $K = \mathbb{G}_{\mathrm{m}} \cdot I \subseteq \mathrm{GL}(V)$. The following are equivalent:*

1. *D has 1 irreducible component;*
2. *$H^0 = [G, G]$;*
3. *$G = K \cdot [G, G]$;*
4. *H^0 is perfect;*
5. *There exists a perfect connected codimension 1 algebraic subgroup J of G.*

Proof. By Remark 5.13, $K \subseteq G$. Also, $G = K \cdot H$, and there are a finite number of ways to write $g \in G$ as a product of elements of K and H. It follows that $\dim(H) = n - 1$. The multiplication morphism $K \times H^0 \to G$ has a connected image of dimension $\dim(G)$, which must be G; hence, $G = K \cdot H^0$.

If (1), then by Corollary 5.7(1), $\dim([G, G]) = n - 1$. Since $[G, G] \subseteq H$, $n - 1 = \dim([G, G]) \leq \dim(H) < n$, and $[G, G]$ is connected, we have $[G, G] = H^0$. This gives (2).

If (2), then by the above work, $G = K \cdot [G, G]$, giving (3).

Suppose (3). Then $\dim([G, G]) = n - 1$, as $[G, G] \subseteq H$ and necessarily $\dim(G) \leq \dim(K) + \dim([G, G])$. Since they are both connected, $[G, G] = H^0$. Since K is in the center of G, we have $[KN, KM] = [N, M]$ for any subgroups N and M of G. In particular,

$$H^0 = [G, G] = [K[G, G], K[G, G]] = [[G, G], [G, G]] = [H^0, H^0],$$

giving (4).

If H^0 is perfect, then since $\dim(H^0) = n - 1$, we have (5).

If (5), then since $J = [J, J] \subseteq [G, G] \subseteq H$ and $\dim(H) = n - 1$, we have $\dim([G, G]) = n - 1$. By Corollary 5.7(1), D has 1 irreducible component. □

Remark 5.17. Let G satisfy Corollary 5.16 and let L be a Levi subgroup of G. Since K is not in $[G, G]$ and is invariant under conjugation, $K = \mathrm{Z}(L)^0$. Also, when $\dim(V) > 1$ L must have a nontrivial semisimple part: if $[L, L]$ is finite, then $\mathrm{Rad}_u(G) \subseteq [G, G]$ have the same dimension and are connected, so $\mathrm{Rad}_u(G) = [G, G] = H^0$ is solvable, not perfect.

Remark 5.18. The case when H is semisimple was thoroughly explored in [GMS11]; are there any other examples? Since an irreducible prehomogeneous vector space has H semisimple by [Kim03, Theorem 7.21], such an example must not be irreducible.

The Lie algebra version of this is that $\mathfrak{g}$ must be the direct sum of a 1–dimensional ideal and a perfect ideal $\mathfrak{h}$, or that $\mathfrak{g}$ contains a codimension 1 perfect subalgebra.

5.5.3. *Reductive linear free divisors.*

Corollary 5.19 ([GMS11, Lemma 2.6])**.** *For a linear free divisor D defined by a reductive group G, the number of irreducible components of D equals the dimension of the center of G. Let H be the subgroup of G leaving invariant the level sets of the product $f_1 \cdots f_r$. Then D is irreducible if and only if H^0 is semisimple.*

Proof. Let G have a Levi decomposition. Since $\mathrm{Rad}_u(G)$ is trivial and G is a Levi subgroup, apply Corollary 5.10(1) to get the first statement.

If D is irreducible, then $r = 1$ and so by Corollary 5.7(3), $H = [G,G] \cdot G_{v_0}$. In particular, $H^0 = [G,G]$, and this is semisimple by the structure theory.

Conversely, suppose H^0 is semisimple. Since $H = \ker(\chi_1 \cdots \chi_r)$, by Theorem 5.5(1) H^0 has codimension 1 in G. Since H^0 is perfect, by Corollary 5.16 D is irreducible. □

Granger–Mond–Schulze also show [GMS11, Theorem 2.7] that for a reductive group, the number of irreducible components is the same as the number of irreducible G-modules in V.

5.5.4. *Solvable linear free divisors.* By the Lie–Kolchin Theorem, for any solvable $G \subset \mathrm{GL}(V)$, there exists a basis of V making G lower triangular.

Corollary 5.20. *Let $D \subset V$ be a linear free divisor defined by a solvable group $G \subset \mathrm{GL}(V)$. Fix any basis which makes G lower triangular, and let $\phi : G \to (\mathbb{G}_\mathrm{m})^{\dim(V)}$ send g to the diagonal entries of g. Then*

1. *The number of components of D is the dimension of the maximal torus of G, or $\dim(\phi(G))$.*
2. *D has a hyperplane component.*
3. *$[G,G]$ equals the group G_u of unipotent elements of G, and consists of those $g \in G$ with ones on the diagonal.*
4. *Every $\chi \in X_1(G)$ factors through ϕ.*

Proof. The Lie–Kolchin Theorem guarantees an invariant complete flag in V, hence an invariant hyperplane H. H cannot intersect Ω and so H is a component of D, proving (2). In these coordinates, this H is defined by the "first coordinate".

Let G have a Levi decomposition. Since $\mathrm{Rad}(G) = G$, $\mathrm{Rad}_u(G) = G_u$ and L^0 is a maximal torus of G. By the structure theory for solvable groups, $[G,G] \subseteq G_u$. For G lower triangular, by definition G_u consists of the $g \in G$ with ones on the diagonal. By Corollary 5.7(4), $[G,G] = G_u^0 = G_u$, finishing (3).

By Corollary 5.10(1), the number of components of D is $\dim(Z(L)) = \dim(L)$. By (3), $\ker(\phi) = G_u$, and hence the number of components is also $\dim(\phi(G))$, proving (1).

Since both $\ker(\phi)$ and $\ker(\chi)$ contain G_u, we have induced maps $\overline{\phi} : G/G_u \to (\mathbb{G}_\mathrm{m})^{\dim(V)}$ and $\overline{\chi} : G/G_u \to \mathbb{G}_\mathrm{m}$. Since $\overline{\phi}$ is an isomorphism onto its image, there exists a homomorphism $\theta : \mathrm{Im}(\phi) \to \mathbb{G}_{\mathrm{m}}$ such that $\chi = \theta \circ \phi$. As θ is a character on a subtorus of $(\mathbb{G}_{\mathrm{m}})^{\dim(V)}$, θ extends to a character $\psi : (\mathbb{G}_{\mathrm{m}})^{\dim(V)} \to \mathbb{G}_{\mathrm{m}}$ by [Bor91, 8.2], with $\chi = \psi \circ \phi$. This proves (4). $\square$

Remark 5.21. By (4), the characters corresponding to the basic relative invariants are functions of the diagonal entries. (In fact, (4) holds for any connected lower-triangular G and any $\chi \in X(G)$.)

Remark 5.22. By Corollary 5.20(1) and [DP12, Theorems 3.1, 3.2], in this situation $V \setminus D$ is a $K(\pi, 1)$ space, with π isomorphic to an extension of $\mathbb{Z}^d$ by the generic isotropy subgroup, where d is the number of irreducible components of D.

The Lie algebra version of this is that for solvable G defining a linear free divisor, $\mathfrak{g}$ is a semidirect sum of $[\mathfrak{g}, \mathfrak{g}]$ and an abelian subalgebra.

5.6. **What else?** Let $G \subseteq \mathrm{GL}(V)$ define a linear free divisor D. What other properties of D may be determined from the Lie algebra of G?

5.6.1. *Degrees?* For instance, may we compute the degrees of the irreducible components of the divisor? The answer seems to be no.

Example 5.23. Consider the following two linear free divisors.

$$\begin{aligned} D_1 : &\quad (x_3x_5 - x_4^2) \begin{vmatrix} 0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{vmatrix} = 0 \\ D_2 : &\quad (x_2^2x_3^2 - 4x_1x_3^3 - 4x_2^3x_4 + 18x_1x_2x_3x_4 - 27x_4^2x_1^2)x_5 = 0 \end{aligned}$$

(D_1 is [DP, Example 9.4], while D_2 is the product-union of a hyperplane with [GMS11, Theorem 2.11(2)].) The degrees of the irreducible components of D_1 and D_2 are different, and will remain so under any linear change of coordinates.

Surprisingly, the groups which define D_1 and D_2 have the same Lie algebra structure: $\mathfrak{gl}_2(\mathbb{C}) \oplus \mathfrak{gl}_1(\mathbb{C})$. Thus, D_1 and D_2 are constructed from inequivalent representations of the same Lie algebra.

This example shows that it is not possible to use the abstract, finite-dimensional Lie algebra structure to determine the degrees of the defining equations of the irreducible components. However, since the representation of the Lie algebra determines the divisor, it contains the information necessary to compute these degrees; how may we do so easily?

5.6.2. *(Non-linear) free divisors?* Is it possible to find a formula for the number of components of a (non-linear) free divisor (D, p)? The natural guess, $\dim_{\mathbb{C}}(M/[M, M])$ for $M = \mathrm{Der}(-\log D)$, does not work[3]. For example, if D is a hyperplane in $\mathbb{C}^2$, then the number computed is 0; in fact, 0 is the answer whenever $\mathrm{Der}(-\log D) \not\subseteq \mathscr{M}_p \cdot \mathrm{Der}_{\mathbb{C}^n,p}$. Other examples give answers too large: for the plane curve $D = V((a^2 - b^3)(a^7 - b^{13}))$, the number computed is 45.

[3]Note that $[M, M]$ is the bracket of modules, not just the module generated by the brackets of the generators of M.

6. Homotopy groups of $V \setminus D$

TODO: write this.

Have an isomorphism $G/[G,G] \cdot G_{v_0} \to (\mathbb{G}_{\mathrm{m}})^r$, so $p : G/G_{v_0} \simeq V \setminus D \to (\mathbb{G}_{\mathrm{m}})^r$ has fiber $[G,G] \cdot G_{v_0}/G_{v_0}$? Then long exact sequence of fibration gives

Theorem 6.1. *For $n > 1$, $\pi_n(V \setminus D) \cong \pi_n([G,G] \cdot G_{v_0}/G_{v_0})$. Also, have*

$$0 \to \pi_1([G,G] \cdot G_{v_0}/G_{v_0}) \to \pi_1(V \setminus D) \to \pi_1(B) \cong \mathbb{Z}^r \to 0,$$

and $\pi_0(V \setminus D) = 0$

Question: Does there exist a formula for the number of components of a (not-necessarily-linear) free divisor?

Ex: where $M/[M,M]$ is not finite over $\mathbb{C}$ for $M = \mathrm{Der}(-\log X)$.

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