RESEARCH STATEMENT

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1. Introduction

My research concerns the singularities of complex analytic varieties, the singularity theory of holomorphic maps, and connections between these topics and representation theory. Of particular interest are modules of 'logarithmic vector fields' and a class of hypersurfaces called 'free divisors'.

Let X be a smooth complex manifold with $p \in X$, and let $\mathscr{O}_{X,p}$ denote the ring of germs at p of holomorphic functions on X. Let $(\mathscr{V}, p)$ be a reduced analytic germ in X, with $\mathscr{V}$ defined by the vanishing of the functions in an $\mathscr{O}_{X,p}$-ideal I. Let $\mathrm{Der}_{X,p}$ denote the germs at p of holomorphic vector fields on X, an $\mathscr{O}_{X,p}$-module. It is natural to consider those (germs of) vector fields on the ambient space X that are tangent to $\mathscr{V}$; these *logarithmic vector fields* are formally defined by the condition

$$\mathrm{Der}_{X,p}(-\log \mathscr{V}) := \{\eta \in \mathrm{Der}_{X,p} : \eta(I) \subseteq I\}.$$

This $\mathscr{O}_{X,p}$-module is also an infinite-dimensional Lie algebra, closed under the Lie bracket of vector fields, and may be thought of as the Lie algebra of the group of biholomorphic diffeomorphisms of (X, p) that leave the set $\mathscr{V}$ invariant. It is thus natural to expect the algebraic properties of the logarithmic vector fields to strongly reflect the algebraic and geometric properties of $(\mathscr{V}, p)$.

The module $\mathrm{Der}_{X,p}(-\log \mathscr{V})$ always requires $\geq \dim(X)$ generators. Following Kyoji Saito [Sai80], we call a nonempty germ $(\mathscr{V}, p) \neq (X, p)$ a *free divisor* if $\mathrm{Der}_{X,p}(-\log \mathscr{V})$ requires only $\dim(X)$ generators, or equivalently, if it is a free $\mathscr{O}_{X,p}$–module, necessarily of rank equal to $\dim(X)$. A free divisor $(\mathscr{V}, p)$ is a hypersurface germ that is either smooth, or maximally singular in the sense that the singular locus $\mathrm{Sing}(\mathscr{V})$ has codimension 1 in $\mathscr{V}$. Algebraically, Aleksandrov [Ale90] showed that if a reduced $f \in \mathscr{O}_{X,p}$ defines a hypersurface $\mathscr{V}$ and J_f is the ideal generated by the partial derivatives of f, then $(\mathscr{V}, p)$ is a free divisor if and only if the Tjurina algebra $\mathscr{O}_{X,p}/(J_f, f)$ is a Cohen–Macaulay $\mathscr{O}_{X,p}$–module of codimension 2. Examples of free divisors include the *free hyperplane arrangements*, where $\mathscr{V}$ is a union of hyperplanes; all reduced plane curve singularities; and all discriminants[1] of versal unfoldings of isolated hypersurface and isolated complete intersection singularities.

Despite being studied since 1980, free divisors themselves remain mysterious. For instance, it is not completely understood which hyperplane arrangements are free. More generally, free divisors may serve as algebraically nice test cases for highly singular hypersurface, discriminants, and logarithmic vector fields.

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[1]For instance, the classical discriminant of a monic degree n polynomial in 1 variable is also the discriminant of a versal unfolding of the isolated hypersurface singularity defined by $f(x) = x^n$.

2. Research Objectives

In my dissertation, I studied topological spaces arising from certain deformations of analytic singularities. My method involved finding a number of free divisors, and has led directly to my more recent work on logarithmic vector fields, free divisors, and the class of 'linear' free divisors.

2.1. Linear free divisors and prehomogeneous vector spaces.

Free divisors classically arose as various types of discriminants, but also have connections to representation theory and harmonic analysis through the study of 'prehomogeneous vector spaces' (see [Sat90]). A hypersurface V in a vector space W is called a *linear free divisor* if $\mathrm{Der}_W(-\log V)$ has a free basis consisting of *linear* vector fields such as $2x\partial_x - z\partial_y$ or $(x-y)\partial_y$. Linear free divisors are a topic of active research (e.g., [GS10, GMS11]).

Each linear free divisor arises from a *prehomogeneous vector space*, a rational representation $\rho : G \to \mathrm{GL}(W)$ of a connected complex linear algebraic group G on a vector space W, such that ρ has an open orbit Ω in W. Then Ω is Zariski open, and its complement is an algebraic set. As all $\rho(g)$ leave invariant Ω^c, differentiating ρ gives a Lie algebra (anti-)homomorphism $d\rho_{(e)} : \mathfrak{g} \to \mathrm{Der}_W(-\log \Omega^c)$, where $\mathfrak{g}$ is the Lie algebra of G and the image is a finite-dimensional Lie algebra of linear logarithmic vector fields (e.g., [DP, §1]).

By Granger–Mond–Nieto-Reyes–Schulze [GMNRS09, §2], every linear free divisor $V \subset W$ is of the form Ω^c for a prehomogeneous vector space $\rho : G \to \mathrm{GL}(W)$ with $\dim(G) = \dim(W)$ and a "reduced" Ω^c, and conversely, and then $\mathrm{Der}_W(-\log \Omega^c)$ is generated by $d\rho_{(e)}(\mathfrak{g})$. Among all prehomogeneous vector spaces, those that give linear free divisors are the extremal class for which the group is of minimal dimension, and yet the group action generates all logarithmic vector fields.

2.1.1. *Finding linear free divisors.*

The initial examples of linear free divisors came primarily from quiver representations ([GMNRS09, BM06]). More recently, Granger–Mond–Schulze [GMS11] showed that up to 'castling transformation', the irreducible representations that give linear free divisors are known by a classification of certain prehomogeneous vector spaces by Sato–Kimura [SK77]. In these examples, the group is always reductive.

In contrast, Damon and I [DP] studied linear free divisors coming from representations of solvable linear algebraic groups. These are often arranged as infinite 'towers' of linear free divisors that come from 'towers' of representations of solvable groups. We also gave examples of linear free divisors for which the group is neither solvable nor reductive, and observed a pattern that solvable group extensions often allow new linear free divisors to be constructed from old, in some cases automatically producing an infinite tower from a single linear free divisor (e.g., [Pik10, §5.3]). It remains to thoroughly understand this phenomenon.

2.1.2. *The structure of linear free divisors.*

Granger–Mond–Schulze [GMS11] described the structure of linear free divisors with reductive groups; for instance, the number of irreducible components of the divisor equals the dimension of the center of $G \subset \mathrm{GL}(W)$, and the number of irreducible G–modules in W. More recently, I used a criterion of Brion [Bri06, GMS11] to show [Pika] that for a prehomogeneous vector space $\rho : G \to \mathrm{GL}(W)$ defining any linear free divisor V, there are no non-trivial rational representations $G \to (\mathbb{C}, +)$. Then by the theory of prehomogeneous vector spaces developed by Mikio Sato, and some structure theory for linear algebraic groups, the number of irreducible components of V is equal to $\dim(G/[G,G])$; this gives significant insight into the structure of the groups and representations that produce linear free divisors. For instance, the Lie algebra $\mathfrak{g}$ of G is the direct sum of $[\mathfrak{g},\mathfrak{g}]$ and an abelian subalgebra. Also, the isotropy subgroup at a generic point on an irreducible component $V(f_i)$ of a linear free divisor $V(f_1\cdots f_k)=\Omega^c$ permutes the level sets of f_i and leaves invariant the level sets of all f_j for $j\neq i$.

With further study, this may lead to a structure theorem for linear free divisors. All linear free divisors known to me take the form of a reductive linear free divisor that is then extended by a solvable group in a process described in [DP], to produce a 'mixed' linear free divisor on a larger space.

2.1.3. *Classifying linear free divisors.* Any conjecture on the structure of linear free divisors should be informed by a variety of examples. Although [GMNRS09] classified linear free divisors in $\mathbb{C}^k$ for $k\leq 4$, these examples are not sufficiently complicated; for example, they are all either solvable or reductive, never mixed. In 2011, I attempted to classify the linear free divisors in $\mathbb{C}^5$. Though this project has not yet been finished, it produced interesting new examples and led to the results in [Pika]. Brent Pym [Pym13] also used these examples to produce new examples of Poisson structures. My current project is to use the results of [Pika] to complete this classification in $\mathbb{C}^5$.

2.1.4. *Deforming linear free divisors.* Torielli [Tor12] studied deformations of linear free divisors, and proved that reductive linear free divisors are rigid and cannot be deformed to an inequivalent linear free divisor. Although he conjectured that this was true for all linear free divisors, my classification work produced a solvable counterexample in $\mathbb{C}^5$. In addition to furthering his work, it would be interesting to study the deformations of a prehomogeneous vector space that produce linear free divisors. For example, the variety V of singular 4×4 skew-symmetric matrices is a component of the complement of the open orbit of a particular prehomogeneous vector space ρ, but there is no known linear free divisor that contains V as a component; may ρ be deformed to produce such a linear free divisor?

2.1.5. *The topology of the complement of a linear free divisor.* One way to study a hypersurface is to study the topology of its complement. For free divisors specifically, there are at least two problems of interest. The first is to determine when the complement of a free divisor is an Eilenberg–MacLane space of type $K(\pi,1)$, that is, with trivial nth homotopy group for $n>1$. For instance, this includes a conjecture of Saito regarding free hyperplane arrangements (see [OT92]), as well as the classical "$K(\pi,1)$ problem" for versal deformations of isolated hypersurface singularities. The second is to determine when the *logarithmic comparison theorem* holds, that is, when the cohomology of the complement may be computed by the complex of *logarithmic differential forms* that have controlled poles along the divisor (e.g., [CJNMM96, GMNRS09]).

Since the complement of a linear free divisor is diffeomorphic to G/G_{v_0} for G_{v_0} a discrete isotropy subgroup, these problems are more accessible for linear free divisors. For instance, Damon and I [DP12] showed that for a large class of examples of solvable linear free divisors, both the complements of the linear free divisors and the Milnor fiber of their defining equations are $K(\pi,1)$'s. In [Pika], I showed that for $n>1$, the nth homotopy group of the complement of a linear free divisor is equal to the nth homotopy group of the semisimple part of a Levi subgroup of G. In future work, [Pika] should also provide insight into the logarithmic comparison theorem problem.

2.2. **The ubiquity of free divisors.** Linear free divisors are simplified test cases for many questions about arbitrary free divisors, and indeed that is the origin of many of the questions in §2.1. There are other questions specific to free divisors.

2.2.1. *Discriminants of deformations.* There are many known cases in which the discriminant of a suitable deformation of a certain type of singularity is a free divisor. For instance, free divisors classically arose as discriminants of versal unfoldings of isolated hypersurface and isolated complete intersection singularities. Damon [Dam98] gave sufficient conditions for a group of equivalences $\mathscr{G}$ to have the property that the discriminant of any $\mathscr{G}$-versal unfolding is a free divisor. A linear free divisor is in some sense the discriminant of a prehomogeneous vector space. Buchweitz has suggested that "every free divisor is the discriminant of something," and it would be very interesting to construct, from a free divisor $\mathscr{V}$, a deformation with discriminant equal to $\mathscr{V}$.

2.2.2. *Pulling back free divisors.* A natural question is to understand the behavior of free divisors under various operations. If $\varphi : X \to Y$ is a map between smooth spaces, and $\mathscr{V}$ is a free divisor in Y, when is $\varphi^{-1}(\mathscr{V})$ a free divisor? Buchweitz and I [BP] studied this very question, and showed that when all $\eta \in \mathrm{Der}_Y(-\log \mathscr{V})$ lift across φ, and the deformation module $T^1_{X/Y}$ of φ is a Cohen–Macaulay $\mathscr{O}_{X,p}$–module of codimension 2, then $\varphi^{-1}(\mathscr{V})$ is also a free divisor. (For instance, we found many instances from invariant theory where these hypotheses are satisfied for the algebraic quotient $\varphi : X \to X//G$, where G is a reductive group and X is a G-representation.) However, our hypotheses may be unnecessarily restrictive: Mond–Schulze [MS13] have identified cases in which pulling back a non-free divisor produces a free divisor. We should determine the exact conditions for $\varphi^{-1}(\mathscr{V})$ to be a free divisor.

Conversely, if $\varphi^{-1}(\mathscr{V})$ is a free divisor, what may be said about $\mathrm{Der}_Y(-\log \mathscr{V})$? One application for this would be in the study of castling transformations. If $M_{p,q}$ denotes the space of $p \times q$ complex matrices and $n > m$, then Granger–Mond–Schulze [GMS11] showed that under the castling operation of prehomogeneous vector spaces, a linear free divisor in $M_{n,n-m}$ transforms to a linear free divisor in $M_{n,m}$, and vice-versa; both linear free divisors are polynomials in the maximal minors of these spaces of matrices, and the castling transformation swaps corresponding maximal minors. To generalize this to arbitrary free divisors, as Buchweitz and I did [BP] in one direction of the $m = 1$ case, it would be very useful to describe $\mathrm{Der}_Y(-\log f)$ when $f \circ \varphi$ defines a free divisor, in particular for $\varphi : M_{n,m} \to \mathbb{C}^{\binom{n}{m}}$ that evaluates all maximal minors.

2.2.3. *Free completions.* Let $(\mathscr{V}_1, p)$ be a reduced hypersurface in (X, p). It is natural to ask whether every such hypersurface is part of some free divisor $(\mathscr{V}_1 \cup \mathscr{V}_2, p)$, that is, whether there exists a *free completion* of $\mathscr{V}_1$; sometimes, $\mathscr{V}_2$ is required to itself be a free divisor. Since $\mathrm{Der}_X(-\log(\mathscr{V}_1 \cup \mathscr{V}_2)) = \mathrm{Der}_X(-\log \mathscr{V}_1) \cap \mathrm{Der}_X(-\log \mathscr{V}_2)$, this is a question about how modules of logarithmic vector fields may intersect. Mond–Schulze [MS13] have found instances of free completions using data from the normalization of $\mathscr{V}_1$. We should describe those hypersurfaces that have a free completion, and do so constructively.

2.3. **The structure of the logarithmic vector fields.** For a particular $\mathscr{V} \subset X$ defined by an ideal I, $\mathrm{Der}_{X,p}(-\log \mathscr{V})$ may be readily computed by a computer as certain syzygies: for instance, a relation $\sum_i a_i \frac{\partial f}{\partial x_i} + bf = 0$ corresponds to $\sum_i a_i \frac{\partial}{\partial x_i} \in \mathrm{Der}_X(-\log V(f))$. However, many natural situations produce a submodule $L \subseteq \mathrm{Der}_{X,p}(-\log \mathscr{V})$, and an open problem is to find necessary and sufficient conditions for $L = \mathrm{Der}_{X,p}(-\log \mathscr{V})$. For free divisors, this is accomplished by a criterion of Saito [Sai80] where the condition is that the determinant of a presentation matrix M of L must be reduced in $\mathscr{O}_{X,p}$.

2.3.1. *Fitting ideals.* For an arbitrary $(\mathscr{V}, p)$, such a presentation matrix M is not square, and hence it is natural to instead consider the ideals generated by minors of M of a particular size, the *Fitting ideals* of $\mathrm{Der}_{X,p}/L$. In [Pikb] I found upper bounds for the Fitting ideals of logarithmic vector fields, and gave a geometric interpretation to these ideals. I also showed that for an arbitrary hypersurface $\mathscr{V}$, $L = \mathrm{Der}_{X,p}(-\log \mathscr{V})$ if and only if both the 0th Fitting ideal of $\mathrm{Der}_{X,p}/L$ is in some sense "reduced" with respect to all components of $\mathscr{V}$, and L is a reflexive module. (If L is free, then L is also reflexive and this simplifies to Saito's criterion.)

A simple example shows that for non-hypersurfaces, the Fitting ideals alone are insufficient to prove $L = \mathrm{Der}_{X,p}(-\log \mathscr{V})$. In future work, I will investigate whether some generalization of reflexivity holds for $\mathrm{Der}_{X,p}(-\log \mathscr{V})$ when $\mathrm{codim}(\mathscr{V}) > 1$; this may be the key to generalizing Saito's work further.

2.3.2. *Lie algebra structure.* Another avenue for answering this question lies in the work of Hauser–Müller [HM93]. They investigate the Lie algebra structure of the logarithmic vector fields, and show that modules of the form $\cap_{i=1}^k \mathrm{Der}_{X,p}(-\log V_i)$ are distinguished by being the end of a chain of subalgebras having a "maximal balanced" property. It would be very interesting to find a practical means to check this property, and moreover to check if $k = 1$.

2.4. **Singular Milnor fibers.** Let $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ be a holomorphic germ, let $(V, 0)$ be an analytic germ in $\mathbb{C}^p$, and let $\mathscr{V}_0 = f_0^{-1}(V)$. Although $\mathscr{V}_0$ is locally contractible, deformations of $\mathscr{V}_0$ can produce nontrivial topological spaces that provide invariants of $\mathscr{V}_0$. Let f_0 deform to a family $\{f_t\}$ of holomorphic germs, thereby obtaining deformations

$$\mathscr{V}_t = f_t^{-1}(V) \cap B_\epsilon(0), \qquad 0 < |t| \ll \epsilon,$$

of $\mathscr{V}_0$ in some open ball near $0 \in \mathbb{C}^n$.

When $V = \{0\}$ and $f_t = f_0 - t \cdot c$ for a generic $c \in \mathbb{C}^p$, then $\mathscr{V}_t = f_0^{-1}(t \cdot c) \cap B_\epsilon(0)$ is the *Milnor fiber* of $\mathscr{V}_0$ (or f_0). The topology of the Milnor fiber has been studied for decades. For example, when f_0 defines an isolated hypersurface or isolated complete intersection singularity (ICIS), then $\mathscr{V}_t$ has the homotopy type of a bouquet of $(n-p)$-spheres by [Mil68, Ham71]. Moreover, Milnor and Lê–Greuel gave computable algebraic formulas for the number μ of spheres, called the *Milnor number of* $\mathscr{V}_0$. The Milnor number has been an incredibly useful invariant in the study of isolated singularities, and the homological properties of Milnor fibers have been a central theme in singularity theory. For instance, the "Milnor–Tjurina" relation $\mu \geq \tau$ connects the topologically defined Milnor number μ of $\mathscr{V}_0$ to the number of parameters τ needed for a versal unfolding of f_0.

If instead $\mathscr{V}_0$ has nonisolated singularities, then the connectivity of the Milnor fiber drops by the dimension of the singular set of $\mathscr{V}_0$ and the topology of the Milnor fiber may be much more complicated ([KM75]). Only for very low dimensional singular sets has the topology of the Milnor fiber been computed (e.g., [Sie87, Zah94, Ném99, FdBMB13]).

An alternate approach is to use a different generalization of the ICIS case. If f_0 is transverse off $0 \in \mathbb{C}^n$ to V and f_t is a deformation that is fully transverse to V for $t \neq 0$, then $\mathscr{V}_t$ is the *singular Milnor fiber* of $\mathscr{V}_0$. Damon and Mond [DM91, Dam96a, Dam96b] showed that when V is a complete intersection, then $\mathscr{V}_t$ again has the homotopy type of a bouquet of $(n - \mathrm{codim}(V))$-spheres; for $V = \{0\}$, this recovers the classical cases. The number $\mu_V(f_0)$ of spheres is called the *singular Milnor number of* $\mathscr{V}_0$. This is an ambient diffeomorphism invariant of $\mathscr{V}_0$ as a pullback of V that can be defined even when $\mathscr{V}_0$ has highly nonisolated singularities. Unfortunately, formulas for $\mu_V(f_0)$ are known in only a few cases; the most important of these is where $(V, 0)$ is a free divisor.

This $\mu_V(f_0)$ should be as useful an invariant for nonisolated singularities as the classical Milnor number has been for isolated singularities. Accordingly, a number of natural problems parallel the classical theory.

2.4.1. *Computing* $\mu_V(f_0)$. In joint work with Damon [Pik10, DP14], we identified a general method to find, for a particular V, a computable algebraic formula for $\mu_V(f_0)$. In its most basic form, we find a suitable auxiliary germ $W \subset \mathbb{C}^p$ so that $\mu_{V\cup W}(f_0)$ and $\mu_W(f_0)$ are computable; often, W is a free completion of V. Then when all terms are defined, an Euler characteristic calculation yields

$$\mu_V(f_0) = (-1)^{d_1}\mu_{V\cup W}(f_0) + (-1)^{d_2}\mu_W(f_0) + (-1)^{d_3}\mu_{V\cap W}(f_0), \tag{1}$$

where each d_i is the dimension of a variety. Often $V \cap W$ is significantly simpler than V and $\mu_{V\cap W}(f_0)$ may be computed either directly, or inductively by this process.

2.4.2. *Matrix singularities.* Efforts thus far have focused on *matrix singularities*, $\mathscr{V}_0 = f_0^{-1}(V)$ where V lies in a vector space of matrices and consists of those with less than maximal rank. Matrix singularities arise naturally in various contexts, for example the Hilbert–Burch Theorem, or the classification of codimension 3 Gorenstein singularities.

Although Goryunov–Mond [GM05] studied the Milnor numbers of isolated matrix singularities, matrix singularities are usually nonisolated, with no classical Milnor number. Instead, we use the singular Milnor number. Since V is almost never a free divisor, $\mu_V(f_0)$ is not computable using previous methods. So far, we [DP14] have used the method of §2.4.1 to find formulas for $\mu_V(f_0)$ where V is the hypersurface of singular matrices in the space of 2×2 and 3×3 symmetric matrices, 2×2 and 3×3 general matrices, and 4×4 skew-symmetric matrices. Future work may simplify these formulas and generalize the results beyond low-dimensional cases.

2.4.3. *Milnor–Tjurina relation for maps.* It has long been believed that an analogue of the Milnor–Tjurina relation $\mu \geq \tau$ for isolated singularities should hold for map germs, where μ is replaced by $\mu_{D(f_0)}(f_0)$ for the discriminant (or image) $D(f_0)$ of f_0, and τ remains the number of parameters for a versal unfolding of f_0. Indeed, Damon–Mond [DM91] originally defined the singular Milnor number to prove such a result for stable maps $f_0 : \mathbb{C}^n \to \mathbb{C}^p$ with $n \geq p$. A key step in the proof is that for such a map, $D(f_0)$ is a free divisor. When $n < p$, then $D(f_0)$ is no longer free and $\mu_{D(f_0)}(f_0)$ is more difficult to compute; nevertheless, the relation has been shown for map germs $\mathbb{C}^n \to \mathbb{C}^{n+1}$ when $n = 1$ and $n = 2$. Since examples in [MS13] suggest that $D(f_0)$ can often be rendered free by the addition of auxiliary hypersurfaces, we should apply the above method of computation to try to prove a Milnor–Tjurina relation for germs $\mathbb{C}^n \to \mathbb{C}^{n+1}$.

2.4.4. *Cohen–Macaulay codimension 2 singularities.* Now let V be the variety of 2×3 matrices with rank < 2. Since V is not a complete intersection, $\mathscr{V}_t$ may not be a bouquet of spheres. Nevertheless, in [DP14] we have obtained a formula for the Euler characteristic of $\mathscr{V}_t$ and bounds on its Betti numbers.

Of particular interest is when $n = 4$ and $\mathscr{V}_0$ is an isolated surface singularity in $\mathbb{C}^4$. Work by Wahl [Wah81] and Greuel–Steenbrink [GS83] leaves only the second Betti number β_2 of the Milnor fiber uncomputed, which they call the "Milnor number" of $\mathscr{V}_0$. Since here the singular Milnor fiber agrees with the Milnor fiber, our Euler characteristic formula reduces to a computable algebraic formula for β_2. Pereira [Per10] has used a classification of simple singularities of this type (due to [FKN10]) to prove a difficult "$\mu = \tau + 1$" result that may follow easily from our formula for β_2.

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