# ADDITIVE RELATIVE INVARIANTS AND THE COMPONENTS OF A LINEAR FREE DIVISOR

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ABSTRACT. A prehomogeneous vector space is a rational representation  $\rho:G \to \operatorname{GL}(V)$  of a connected complex linear algebraic group G that has a Zariski open orbit  $\Omega \subset V$ . M. Sato showed that the hypersurface components of  $D:=V \setminus \Omega$  are related to the characters  $H \to \operatorname{GL}(\mathbb C)$  of H, an algebraic abelian quotient of G. Mimicking this work, we investigate the additive functions of H, homomorphisms  $\Phi: H \to (\mathbb C, +)$ . Each such  $\Phi$  is related to an additive relative invariant, a rational function h on V such that  $h \circ \rho(g) - h = \Phi(g)$  on  $\Omega$  for all  $g \in G$ . Such an h is homogeneous of degree 0, and describes the behavior of certain subsets of D under the G-action.

For those prehomogeneous vector spaces with D a type of hypersurface called a linear free divisor, we prove there are no nontrivial additive functions of H, and hence H is an algebraic torus. From this we gain insight into the structure of such representations and prove that the number of irreducible components of D equals the dimension of the abelianization of G. For some special cases (G abelian, reductive, or solvable, or D irreducible) we simplify proofs of existing results. We also examine the homotopy groups of  $V \setminus D$ .

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## Introduction

A prehomogeneous vector space is a rational representation  $\rho: G \to \operatorname{GL}(V)$  of a connected complex linear algebraic group G that has a (unique) Zariski open orbit  $\Omega$  in V. These representations have been much studied from the viewpoint of number theory and harmonic analysis (e.g., [Kim03]). A particularly useful tool is to let  $v_0 \in \Omega$  and then consider the rational characters  $\chi: H \to \mathbb{G}_{\mathrm{m}} := \operatorname{GL}(\mathbb{C})$  of the abelian quotient  $H = G/[G,G] \cdot G_{v_0}$ , where  $G_{v_0}$  is the isotropy subgroup at  $v_0$ . There is a correspondence between these characters and relative invariants, certain

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rational functions on V, with zeros and poles on  $V \setminus \Omega$ , that behave in a specified manner under the G-action. In turn, the relative invariants are closely related to the hypersurface components of the algebraic set  $V \setminus \Omega$ . Thus, for example, the number of hypersurface components of  $V \setminus \Omega$  equals the rank of the free abelian group of characters of H.

In this paper we study the additive functions of H, the homomorphisms  $\Phi: H \to \mathbb{G}_{\mathbf{a}} := (\mathbb{C}, +)$ . This complements the classical study of the characters, as up to isomorphism  $\mathbb{G}_{\mathbf{m}}$  and  $\mathbb{G}_{\mathbf{a}}$  are the only 1-dimensional connected complex linear algebraic groups. Also, since H is connected and abelian the standard structure theory implies that  $H \cong \mathbb{G}_{\mathbf{m}}^{\ \ k} \times \mathbb{G}_{\mathbf{a}}^{\ \ell}$ , with k the rank of the character group of H and  $\ell$  the dimension of the vector space of additive functions of H; the characters and additive functions thus describe H completely. Of particular interest, it follows that the number of irreducible hypersurface components of  $V \setminus \Omega$  equals

$$(0.1) k = \dim(H) - \ell$$

(see Corollary 2.3), where  $\dim(H)$  is easily computable.

The content of the paper is the following. After reviewing the basic properties of prehomogeneous vector spaces in §1 and abelian linear algebraic groups in §2, we study the additive functions of H in §3. Just as the characters of H are related to relative invariants, we show that the additive functions of H are related to additive relative invariants, rational functions on V, with poles on  $V \setminus \Omega$ , that behave in another specified manner under the G-action (see Definition 3.1). More precisely, by Proposition 3.4 a rational function k on V is an additive relative invariant if and only if  $k(\rho(g)(v)) - k(v)$  is independent of  $v \in \Omega$  for all  $g \in G$ ; then this difference gives the additive function of G associated to k. These additive relative invariants are homogeneous of degree 0 and of the form  $\frac{h}{f}$  for a polynomial relative invariant f (Proposition 3.5). Geometrically, an additive relative invariant  $\frac{h}{f}$  describes a G-invariant subset f = h = 0 of f = 0, with G permuting the sets

$$V_{\epsilon} = \{x : f(x) = h(x) - \epsilon = 0\}, \quad \epsilon \in \mathbb{C},$$

by  $\rho(g)(V_{\epsilon}) = V_{\chi(g)\cdot\epsilon}$  for  $\chi$  the character corresponding to the relative invariant f (see Corollary 3.8); the converse also holds if  $\deg(\frac{h}{f}) = 0$  and f is reduced (Proposition 3.9). In §3.6 we investigate which characters and additive functions vanish on which isotropy subgroups, a key ingredient for §5. Finally, in §3.7 we conjecture that every additive relative invariant may be written as a sum of additive relative invariants of the form  $\frac{h_i}{f_i}$ , where  $f_i$  is a power of an irreducible polynomial relative invariant; this holds in all examples we have studied. We prove this splitting behavior under some strong additional hypotheses (Proposition 3.16).

In  $\S 4$  we describe some examples of prehomogeneous vector spaces and their additive relative invariants.

In §5, we study prehomogeneous vector spaces with the property that  $D := V \setminus \Omega$  is a type of hypersurface called a *linear free divisor*. Such objects have been of much interest recently (e.g., [BM06, GMNRS09, GMS11, DP12a, DP12b]), and were our original motivation. Using a criterion due to Brion and the results of §3.6, we show in Theorem 5.6 that such prehomogeneous vector spaces have no nontrivial additive relative invariants or nontrivial additive functions. Then by (0.1) the number of irreducible components of D equals  $\dim(H)$ , and this may be easily computed using only the Lie algebra  $\mathfrak{g}$  of G (Corollary 5.8; see also Remark 5.11). We also

gain insight into the structure of G and the behavior of the G-action (§5.3, 5.4, 5.5). In §5.6, we study the special cases where G is abelian, reductive, or solvable, and where D is irreducible, simplifying proofs of several known results. In §5.7 we observe that, although the number of components of a linear free divisor is always computable from the abstract Lie algebra structure  $\mathfrak g$  of G, the degrees of the polynomials defining these components are not; to find the degrees requires the representation  $\mathfrak g \to \mathfrak g\mathfrak l(V)$ . Lemma 5.27 describes several methods of computing the degrees, but these all require data that are themselves difficult to compute. Finally, in §5.8 we use our earlier results to study the homotopy groups of the complement of a linear free divisor.

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#### 1. Prehomogeneous vector spaces

We begin by briefly reviewing prehomogeneous vector spaces. Our reference is [Kim03], although the results we describe were developed by M. Sato [Sat90] in the 1960's.

In the whole article, we shall study only complex linear algebraic groups. For such a K, let  $K^0$  denote the connected component of K containing the identity, L(K) the Lie algebra of K, and  $K_v$  the isotropy subgroup at v of a particular K-action.

Let V be a finite-dimensional complex vector space, and G a connected complex linear algebraic group. Let  $\rho: G \to \operatorname{GL}(V)$  be a rational representation of G, i.e., a morphism of linear algebraic groups. When G has an open orbit  $\Omega$  in V, then we call  $(G, \rho, V)$  a prehomogeneous vector space. Then  $\Omega$  is unique and Zariski open, so that the complement  $\Omega^c = V \setminus \Omega$  is an algebraic set in V. We call  $\Omega^c$  the exceptional orbit variety as it is the union of the non-open orbits of G; others use discriminant or singular set.

One of the basic theorems of prehomogeneous vector spaces is that the hypersurface components of  $\Omega^c$  may be detected from certain multiplicative characters of G. More precisely, for any complex linear algebraic group K let X(K) denote the set of rational (multiplicative) characters, that is, the homomorphisms  $K \to \mathbb{G}_m := \operatorname{GL}(\mathbb{C})$  of linear algebraic groups. Then a rational function f on Vthat is not identically 0 is a relative invariant if there exists a  $\chi \in X(G)$  such that

$$(1.1) f(\rho(g)(v)) = \chi(g) \cdot f(v)$$

for all  $v \in \Omega$  and  $g \in G$ , in which case we<sup>1</sup> write  $f \stackrel{\text{m}}{\longleftrightarrow} \chi$ . By (1.1), the zeros and poles of f may occur only on  $\Omega^c$ .

Actually, f and  $\chi$  provide almost the same information when  $f \stackrel{\text{m}}{\longleftrightarrow} \chi$ . Using f, we may choose any  $v_0 \in \Omega$  and then recover  $\chi$  by defining

$$\chi(g) = \frac{f(\rho(g)(v_0))}{f(v_0)} \in (\mathbb{C}^*, \cdot) \cong \mathbb{G}_{\mathrm{m}}, \qquad g \in G.$$

Conversely, using  $\chi$  we may recover a nonzero constant multiple h of f by choosing any  $v_0 \in \Omega$ , and any nonzero value  $h(v_0) \in \mathbb{C}$ , and then defining h on  $\Omega$  by

<sup>&</sup>lt;sup>1</sup>Usually this is written  $f \longleftrightarrow \chi$ , but we shall use similar notation for the relationship between additive functions and additive relative invariants.

 $h(\rho(g)(v_0)) = \chi(g)h(v_0)$ . Since  $\Omega$  is dense in V, there is a unique way to extend h to a rational function on V.

Let  $X_1(G)$  be the set of  $\chi \in X(G)$  for which there exists an f with  $f \stackrel{\text{m}}{\longleftrightarrow} \chi$ . Then  $X_1(G)$  is an abelian group: if  $f_i \stackrel{\text{m}}{\longleftrightarrow} \chi_i$  and  $n_i \in \mathbb{Z}$  for i = 1, 2, then  $(f_1)^{n_1}(f_2)^{n_2} \stackrel{\text{m}}{\longleftrightarrow} (\chi_1)^{n_1}(\chi_2)^{n_2} \in X_1(G)$ . The group  $X_1(G)$  contains information about the hypersurface components of  $\Omega^c$ .

**Theorem 1.1** (e.g., [Kim03, Theorem 2.9]). Let  $(G, \rho, V)$  be a prehomogeneous vector space with exceptional orbit variety  $\Omega^c$ , and  $S_i = \{x \in V : f_i(x) = 0\}$ ,  $i = 1, \ldots, r$  the distinct irreducible hypersurface components of  $\Omega^c$ . Then the irreducible polynomials  $f_1, \ldots, f_r$  are relative invariants which are algebraically independent, and any relative invariant is of the form  $c \cdot (f_1)^{m_1} \cdots (f_r)^{m_r}$  for nonzero  $c \in \mathbb{C}$ , and  $m_i \in \mathbb{Z}$ . Moreover, if each  $f_i \stackrel{\text{m}}{\longleftrightarrow} \chi_i$ , then  $X_1(G)$  is a free abelian group of rank r generated by  $\chi_1, \ldots, \chi_r$ .

The (homogeneous) irreducible polynomials  $f_1, \ldots, f_r$  of the Theorem are called the *basic relative invariants* of  $(G, \rho, V)$ .

**Remark 1.2.** Since  $v \in \Omega$  if and only if the corresponding orbit map  $g \mapsto \rho(g)(v)$  is a submersion at  $e \in G$ , and in coordinates this derivative consists of linear functions on V, the set  $\Omega^c$  may be defined by an ideal generated by homogeneous polynomials of degree  $\dim(V)$  (see §5.1).

There exists another description of  $X_1(G)$ . Fix an element  $v_0 \in \Omega$  with isotropy subgroup  $G_{v_0}$ . Since the orbit of  $v_0$  is open, we have  $\dim(G_{v_0}) = \dim(G) - \dim(V)$ . Define the algebraic groups

$$G_1 = [G, G] \cdot G_{v_0}$$
 and  $H = G/G_1$ .

The group  $G_1$  does not depend on the choice of  $v_0 \in \Omega$  as all such isotropy subgroups are conjugate, and  $[G, G] \subseteq G_1$ . An examination of (1.1) shows that every  $\chi \in X_1(G)$  has  $G_1 \subseteq \ker(\chi)$ , and thus factors through the quotient to give a corresponding  $\tilde{\chi} \in X(H)$ . In fact, the map  $\chi \mapsto \tilde{\chi}$  is an isomorphism:

**Proposition 1.3** (e.g., Proposition 2.12 of [Kim03]). Let  $(G, \rho, V)$  be a prehomogeneous vector space with open orbit  $\Omega$ . Then for any  $v_0 \in \Omega$ ,

$$X_1(G) \cong X(G/[G,G] \cdot G_{v_0}).$$

Consequently, the rank of the character group of this abelian quotient equals the number of irreducible hypersurface components of  $V \setminus \Omega$ .

### 2. Abelian complex linear algebraic groups

Let  $(G, \rho, V)$  be a prehomogeneous vector space, and use the notation of the previous section with  $H = G/[G, G] \cdot G_{v_0}$  for some  $v_0 \in \Omega$ . To understand the rank of X(H) and thus the number of irreducible hypersurface components of the exceptional orbit variety  $\Omega^c$ , we must understand H. Since H is abelian and connected, it has a very simple structure. Recall that over  $\mathbb C$  there are exactly two distinct 1-dimensional connected linear algebraic groups,  $\mathbb G_{\mathrm{m}} = \mathrm{GL}(\mathbb C^1) \cong (\mathbb C^*,\cdot)$  and  $\mathbb G_{\mathrm{a}} = (\mathbb C,+)$ .

**Proposition 2.1** (e.g., [OV90, §3.2.5]). An abelian connected complex linear algebraic group K is isomorphic to  $(\mathbb{G}_m)^k \times (\mathbb{G}_a)^\ell$  for nonnegative integers k and  $\ell$ , where the exponent denotes a repeated direct product.

Note that this Proposition can be false for other fields.

For any linear algebraic group K, let  $\mathscr{A}(K)$  be the set of rational homomorphisms  $\Phi: K \to \mathbb{G}_a$ , which are often called *additive functions* ([Spr98, §3.3]). Clearly,  $\mathscr{A}(K)$  is a complex vector space. When K is connected,  $\Phi \in \mathscr{A}(K)$  is determined by  $d\Phi_{(e)}$ , and hence  $\mathscr{A}(K)$  is finite-dimensional. When K has the decomposition of Proposition 2.1, the rank of X(K) and the dimension of  $\mathscr{A}(K)$  are related to k and  $\ell$ .

**Lemma 2.2.** If  $K = (\mathbb{G}_m)^k \times (\mathbb{G}_a)^\ell$ , then X(K) is free of rank k and  $\dim_{\mathbb{C}}(\mathscr{A}(K)) = \ell$ .

*Proof.* By the Jordan decomposition, any  $\chi \in X(K)$  will have  $\{1\}^k \times (\mathbb{G}_a)^\ell \subseteq \ker(\chi)$ , so  $\chi$  factors through to an element of  $X((\mathbb{G}_m)^k)$ . Thus  $X(K) \cong X((\mathbb{G}_m)^k)$ , and similarly  $\mathscr{A}(K) \cong \mathscr{A}((\mathbb{G}_a)^\ell)$ . Finally, an easy exercise shows that  $\operatorname{rank}(X((\mathbb{G}_m)^k)) = k$ , and  $\dim(\mathscr{A}((\mathbb{G}_a)^\ell)) = \ell$ .

Thus, for a prehomogeneous vector space we have:

Corollary 2.3. Let  $(G, \rho, V)$  be a prehomogeneous vector space with open orbit  $\Omega$ . Let  $v_0 \in \Omega$  and  $H = G/[G, G] \cdot G_{v_0}$ . Then the number of irreducible hypersurface components of  $\Omega^c$  equals  $\dim(H) - \dim(\mathscr{A}(H))$ .

*Proof.* By Theorem 1.1, Proposition 1.3, Proposition 2.1, and Lemma 2.2, the number of irreducible components equals  $\operatorname{rank}(X_1(G)) = \operatorname{rank}(X(H)) = \dim(H) - \dim(\mathcal{A}(H))$ .

It is natural, then, to study these additive functions and their geometric meaning.

# 3. Additive relative invariants

In this section, we develop for additive functions the analogue of the multiplicative theory from §1. Let  $(G, \rho, V)$  be a prehomogeneous vector space with open orbit  $\Omega$ . Let  $v_0 \in \Omega$  and  $H = G/[G, G] \cdot G_{v_0}$ .

3.1. **Definition.** We define additive relative invariants similarly to (multiplicative) relative invariants.

**Definition 3.1.** A rational function h on V is an additive relative invariant if there exists a  $\Phi \in \mathscr{A}(G)$  so that

(3.1) 
$$h(\rho(g)(v)) - h(v) = \Phi(g)$$

for all  $v \in \Omega$  and  $g \in G$ . In this situation, we write  $h \stackrel{\text{a}}{\longleftrightarrow} \Phi$ .

By (3.1), the poles of such an h occur only on  $\Omega^c$ , and h is constant on the orbits of  $\ker(\Phi)$ .

3.2. **Basic properties.** We now establish some basic facts about additive relative invariants. First we investigate the uniqueness of the relationship  $h \stackrel{a}{\longleftrightarrow} \Phi$ .

**Proposition 3.2.** For a prehomogeneous vector space  $(G, \rho, V)$ ,

- (1) If  $h_1 \stackrel{a}{\longleftrightarrow} \Phi$  and  $h_2 \stackrel{a}{\longleftrightarrow} \Phi$ , then there exists an  $\alpha \in \mathbb{C}$  with  $h_1 = \alpha + h_2$ .
- (2) If  $h \stackrel{a}{\longleftrightarrow} \Phi_1$  and  $h \stackrel{a}{\longleftrightarrow} \Phi_2$ , then  $\Phi_1 = \Phi_2$ .

*Proof.* For (1), fix  $v_0 \in \Omega$  and let  $\alpha = h_1(v_0) - h_2(v_0)$ , so  $\Phi(g) + h_1(v_0) = \alpha + \Phi(g) + h_2(v_0)$  for all  $g \in G$ . Applying (3.1) shows that  $h_1 = \alpha + h_2$  on  $\Omega$ , and thus on V.

(2) is immediate from 
$$(3.1)$$
.

Let  $\mathscr{A}_1(G)$  be the set of  $\Phi \in \mathscr{A}(G)$  for which there exists a rational function h on V with  $h \stackrel{\mathrm{a}}{\longleftrightarrow} \Phi$ . We now identify the additive functions in  $\mathscr{A}_1(G)$ , analogous to Proposition 1.3.

**Proposition 3.3.** As vector spaces,  $\mathscr{A}_1(G) \cong \mathscr{A}(H)$ , where  $H = G/[G,G] \cdot G_{v_0}$  and  $v_0 \in \Omega$ .

Proof. Let  $\Phi \in \mathscr{A}_1(G)$  with  $h \stackrel{\mathrm{a}}{\longleftrightarrow} \Phi$ . Evaluating (3.1) at  $v_0 \in \Omega$  and  $g \in G_{v_0}$  shows that  $G_{v_0} \subseteq \ker(\Phi)$ . Since  $\mathbb{G}_a$  is abelian,  $[G,G] \subseteq \ker(\Phi)$ . Thus any  $\Phi \in \mathscr{A}_1(G)$  factors through the quotient  $\pi : G \to H$  to a unique  $\overline{\Phi} \in \mathscr{A}(H)$ , with  $\Phi = \overline{\Phi} \circ \pi$ . The map  $\rho : \mathscr{A}_1(G) \to \mathscr{A}(H)$  defined by  $\rho(\Phi) = \overline{\Phi}$  is  $\mathbb{C}$ -linear.

Conversely, define the  $\mathbb{C}$ -linear map  $\sigma: \mathscr{A}(H) \to \mathscr{A}(G)$  by  $\sigma(\overline{\Phi}) = \overline{\Phi} \circ \pi$ . For  $\overline{\Phi} \in \mathscr{A}(H)$ , we have  $G_{v_0} \subseteq \ker(\overline{\Phi} \circ \pi)$ , and hence we may define a function  $\overline{h}: \Omega \to \mathbb{C}$  by  $\overline{h}(\rho(g)(v_0)) = (\overline{\Phi} \circ \pi)(g)$ . By the argument of [Kim03, Proposition 2.11],  $\overline{h}$  is a regular function on  $\Omega$  that may be extended to a rational function h on V. By construction,  $h(\rho(g)(v)) - h(v) = (\overline{\Phi} \circ \pi)(g)$  for  $v = v_0$  and  $g \in G$ , but this implies that this equation holds for all  $v \in \Omega$  and  $g \in G$ . Thus  $h \overset{\text{a}}{\longleftrightarrow} \overline{\Phi} \circ \pi$ , and  $\sigma: \mathscr{A}(H) \to \mathscr{A}_1(G)$ .

Finally, check that  $\rho$  and  $\sigma$  are inverses using the uniqueness of the factorization through  $\pi$ .

Additive relative invariants are the rational functions on V having the following property.

**Proposition 3.4.** Let  $h: V \to \mathbb{C}$  be a rational function on V, holomorphic on  $\Omega$ . Then h is an additive relative invariant if, and only if, for all  $g \in G$ ,

$$h(\rho(q)(v)) - h(v)$$
 is independent of  $v \in \Omega$ .

*Proof.* By (3.1), additive relative invariants have this property.

Conversely, define  $\Phi: G \to \mathbb{C}$  by  $\Phi(g) = h(\rho(g)(v)) - h(v)$  for any  $v \in \Omega$ . As a composition of regular functions,  $\Phi$  is regular on G. Let  $g_1, g_2 \in G$ , and  $v \in \Omega$ . Then

$$\begin{split} \Phi(g_1g_2^{-1}) &= h\left(\rho(g_1g_2^{-1})(v)\right) - h(v) \\ &= h\left(\rho(g_1)(\rho(g_2^{-1})(v))\right) - h\left(\rho(g_2^{-1})(v)\right) + h\left(\rho(g_2^{-1})(v)\right) - h(v) \\ &= \left(h(\rho(g_1)(v_1)) - h(v_1)\right) + \left(h(v_1) - h(\rho(g_2)(v_1))\right), \end{split}$$

where  $v_1 = \rho(g_2^{-1})(v) \in \Omega$ . By our hypothesis,  $\Phi(g_1g_2^{-1}) = \Phi(g_1) - \Phi(g_2)$ . Thus  $\Phi: G \to \mathbb{G}_a$  is a homomorphism of algebraic groups, and since  $h \overset{a}{\longleftrightarrow} \Phi$ , we have  $\Phi \in \mathscr{A}_1(G)$ .

<sup>&</sup>lt;sup>2</sup>This uses the fact that  $\mathbb{C}$  has characteristic 0.

3.3. Homogeneity. Relative invariants are always homogeneous rational functions on V because  $\rho$  is a linear representation. Similarly, additive relative invariants are homogeneous of degree 0.

**Proposition 3.5.** If  $h \stackrel{\text{a}}{\leftarrow} \Phi$  and h is not the zero function, then h may be written as  $h = \frac{h_1}{f_1}$ , where  $h_1$  and  $f_1$  are homogeneous polynomials with  $\deg(h_1) = \deg(f_1)$ ,  $h_1$  and  $f_1$  have no common factors, and  $f_1$  is a relative invariant.

*Proof.* First note that by Remark 1.2, for any  $t \in \mathbb{C}^*$ , we have  $v \in \Omega$  if and only if  $t \cdot v \in \Omega$ ; we use this fact implicitly in the rest of the proof.

Let  $t \in \mathbb{C}^*$ , and define the rational function  $h_t$  on V by  $h_t(v) = h(t \cdot v)$ . Since  $\rho$  is a linear representation,

$$(3.2) h_t(\rho(g)(v)) = h(t \cdot \rho(g)(v)) = h(\rho(g)(t \cdot v)).$$

Applying (3.1) to (3.2) gives

(3.3) 
$$h_t(\rho(g)(v)) = \Phi(g) + h(t \cdot v) = \Phi(g) + h_t(v).$$

Since (3.3) holds for all  $g \in G$  and  $v \in \Omega$ , we have  $h_t \stackrel{\text{a}}{\longleftrightarrow} \Phi$ .

By Proposition 3.2(1), there exists an  $\alpha: \mathbb{C}^* \to \mathbb{C}$  such that

$$(3.4) h(t \cdot v) = h(v) + \alpha(t)$$

for all  $t \in \mathbb{C}^*$  and  $v \in \Omega$ . If  $s, t \in \mathbb{C}^*$  and  $v \in \Omega$ , then using (3.4) repeatedly shows

$$h(v) + \alpha(st) = h(s(t \cdot v)) = h(t \cdot v) + \alpha(s) = h(v) + \alpha(t) + \alpha(s),$$

or  $\alpha(st) = \alpha(s) + \alpha(t)$ . By (3.4), we have  $\alpha(1) = 0$ , and hence  $\alpha : (\mathbb{C}^*, \cdot) \to (\mathbb{C}, +)$  is a group homomorphism.

Fixing some  $v \in \Omega$  and instead using (3.4) to define  $\alpha$  shows that  $\alpha : \mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{a}}$  is regular, hence a homomorphism of complex linear algebraic groups. By the Jordan decomposition,  $\alpha = 0$ . Then  $h(t \cdot v) = h(v)$  for all nonzero  $t \in \mathbb{C}$  and  $v \in \Omega$ ; by density, h is homogeneous of degree 0.

Thus we may write  $h = \frac{h_1}{f_1}$ , with  $h_1$  and  $f_1$  homogeneous polynomials of equal degree, and without common factors. Since h may only have poles on  $\Omega^c$ ,  $f_1$  defines a hypersurface in  $\Omega^c$ . By Theorem 1.1,  $f_1$  is a relative invariant.

This gives an apparently nontrivial result about the structure of prehomogeneous vector spaces for which the exceptional orbit variety has no hypersurface components.

Corollary 3.6. Let  $(G, \rho, V)$  be a prehomogeneous vector space, let  $v_0 \in \Omega$ , and let r be the number of irreducible hypersurface components of  $V \setminus \Omega$ . If  $X_1(G) = \{1\}$  (equivalently, r = 0), then  $\mathscr{A}_1(G) = \{0\}$  and  $G = [G, G] \cdot G_{v_0}$ . If  $d = \dim(G/[G, G] \cdot G_{v_0}) \leq 1$ , then  $\mathscr{A}_1(G) = \{0\}$  and r = d.

Proof. Suppose  $X_1(G)=\{1\}$ . Let  $\Phi\in\mathscr{A}_1(G)$ , and choose h with  $h\overset{\mathrm{a}}{\longleftrightarrow}\Phi$ . Assume that  $\Phi$  is nonzero and hence h is not constant. By Proposition 3.5, write  $h=\frac{h_1}{f_1}$  for polynomials  $f_1$  and  $h_1$  with  $\deg(h_1)=\deg(f_1)>0$ , and  $f_1$  a relative invariant. If  $f_1\overset{\mathrm{m}}{\longleftrightarrow}\chi_1$ , then by our hypothesis  $\chi_1=1$ . By (1.1),  $f_1$  is a nonzero constant on  $\Omega$ , contradicting the fact that  $\deg(f_1)>0$ . Thus  $\Phi=0$  and so  $\mathscr{A}_1(G)=\{0\}$ . Since there are no nontrivial characters or additive functions, by Proposition 2.1 and Lemma 2.2 we find that  $H=G/[G,G]\cdot G_{v_0}$  consists of a single point, and hence  $G=[G,G]\cdot G_{v_0}$ .

Now suppose  $d \leq 1$ . If d = 0, then by Corollary 2.3 and Proposition 3.3,  $r = 0 - \dim(\mathscr{A}_1(G)) \geq 0$ , implying the statement. If d = 1, then by the same reasoning either r = 0 or r = 1, but the former is impossible by the first half of the statement. Now apply Corollary 2.3 to show  $\mathscr{A}_1(G) = \{0\}$ .

3.4. Global equation. Our definition of an additive relative invariant only involves the behavior of the function on  $\Omega$ . We may describe the behavior on all of V.

**Proposition 3.7.** Let  $h \stackrel{a}{\longleftrightarrow} \Phi$  with h nonzero. As in Proposition 3.5, write  $h = \frac{h_1}{f_1}$  for polynomials  $h_1$  and  $f_1$ , with  $f_1 \stackrel{m}{\longleftrightarrow} \chi$ . Then for all  $g \in G$  and  $v \in V$ ,

(3.5) 
$$h_1(\rho(g)(v)) = \chi(g) \cdot (f_1(v) \cdot \Phi(g) + h_1(v)).$$

*Proof.* Define the regular functions  $L, R: G \times V \to \mathbb{C}$  using the left, respectively, right sides of (3.5). By (3.1) and (1.1), these functions agree on  $G \times \Omega$ , and thus on  $G \times V$ .

Of course, (3.5) also holds for  $h=0 \stackrel{\text{a}}{\longleftrightarrow} \Phi=0$  if  $h_1=0$  and  $f_1$  is a nonzero constant

3.5. **Geometric interpretation.** In particular, Proposition 3.7 provides the following geometric interpretation of additive relative invariants.

**Corollary 3.8.** In the situation of Proposition 3.7, let f be any factor of  $f_1$ . For  $\epsilon \in \mathbb{C}$ , let  $V_{\epsilon} = \{x : f(x) = h_1(x) - \epsilon = 0\}$ . Then  $\rho(g)(V_{\epsilon}) = V_{\chi(g) \cdot \epsilon}$  for all  $g \in G$  and  $\epsilon \in \mathbb{C}$ .

We may provide a partial converse.

**Proposition 3.9.** Let f be a reduced polynomial relative invariant with  $f \stackrel{\text{m}}{\longleftrightarrow} \chi$ . Let  $h_1$  be a nonzero homogeneous polynomial on V, and for  $\epsilon \in \mathbb{C}$  let  $V_{\epsilon} = \{x : f(x) = h_1(x) - \epsilon = 0\}$ . If  $\deg(h_1) = \deg(f)$  and  $\rho(g)(V_{\epsilon}) = V_{\chi(g) \cdot \epsilon}$  for all  $g \in G$  and  $\epsilon \in \mathbb{C}$ , then  $h = \frac{h_1}{f}$  is an additive relative invariant.

*Proof.* As a function on  $G \times V$ ,

$$(g,v) \mapsto h_1(\rho(g)(v)) - \chi(g)h_1(v)$$

vanishes on  $G \times V(f)$ . Since f is reduced, by the Nullstellensatz we have

(3.6) 
$$h_1(\rho(g)(v)) - \chi(g)h_1(v) = f(v) \cdot \beta(g, v)$$

for some  $\beta \in \mathbb{C}[G \times V]$ . For each  $g \in G$ , both sides of (3.6) should be zero or homogeneous of degree  $\deg(h_1) = \deg(f)$  as functions on V; hence,  $\beta(g, -) \in \mathbb{C}[V]$  is zero or homogeneous of degree 0, and so  $\beta \in \mathbb{C}[G]$ . Finally, rearrange (3.6) and apply Proposition 3.4.

It is unclear whether the conclusion of Proposition 3.9 holds without the assumption that f is reduced and  $deg(h_1) = deg(f)$ .

3.6. Vanishing lemma. We now prove that certain characters and additive functions vanish on certain isotropy subgroups. This is a key observation used in §5.

Lemma 3.10. Let  $v \in V$ .

- (i) Let  $f_1$  be a relative invariant,  $f_1 \stackrel{\text{m}}{\longleftrightarrow} \chi$ . If  $f_1$  is defined at v and  $f_1(v) \neq 0$ , then  $G_v \subseteq \ker(\chi)$ .
- (ii) Let  $\frac{h_1}{f_1}$  be an additive relative invariant written as in Proposition 3.5, with  $h_1$  nonzero,  $\frac{h_1}{f_1} \stackrel{a}{\longleftrightarrow} \Phi$ , and  $f_1 \stackrel{m}{\longleftrightarrow} \chi$ . Then the following conditions are
  - $h_1(v) = 0$  or  $G_v \subseteq \ker(\chi)$ ,
  - $f_1(v) = 0$  or  $G_v \subseteq \ker(\Phi)$ .

*Proof.* Let  $g \in G_v$ . For (i), by (1.1) we have

$$f_1(v) = f_1(\rho(g)(v)) = \chi(g)f_1(v),$$

and (i) follows. For (ii), by Proposition 3.7 we have

$$h_1(v) = h_1(\rho(g)(v)) = \chi(g) (f_1(v)\Phi(g) + h_1(v)),$$

or

(3.7) 
$$h_1(v) \cdot (1 - \chi(g)) = \chi(g) \cdot f_1(v) \cdot \Phi(g).$$

Now consider the circumstances when one side of (3.7) vanishes for all  $g \in G_v$ ; the other side vanishes, too.

Lemma 3.10 is best understood when we consider generic points on the hypersurface components of  $\Omega^c$ .

**Lemma 3.11.** Let  $f_1, \ldots, f_r$  be the basic relative invariants of  $(G, \rho, V)$ , with  $f_i \stackrel{\mathrm{m}}{\longleftrightarrow} \chi_i$ . Let  $v_i \in V(f_i)$ , with  $v_i \notin V(f_j)$  for  $i \neq j$ . Then

- (1) If  $i \neq j$ , then  $G_{v_i} \subseteq \ker(\chi_j)$ .
- (2) Suppose  $h \stackrel{a}{\longleftrightarrow} \Phi$ , with h nonzero and  $h = \frac{h_1}{q_1}$  as in Proposition 3.5, and  $g_1 = f_1^{k_1} \cdots f_r^{k_r}$  the factorization of  $g_1$ . Then
  (a) If  $k_i > 0$  and  $h_1(v_i) \neq 0$ , then  $G_{v_i}^0 \subseteq \ker(\chi_i)$ .

  - (b) If  $k_i = 0$ , then  $G_{v_i} \subseteq \ker(\Phi)$ .

Proof. (1) follows from Lemma 3.10(i). For (2a), apply Lemma 3.10(ii) to show that  $G_{v_i} \subseteq \ker(\chi_1^{k_1} \cdots \chi_r^{k_r})$ . Then by (1),  $G_{v_i} \subseteq \ker(\chi_i^{k_i})$ , and this implies (2a). In the situation of (2b),  $g_1(v) \neq 0$  and by (1) we have  $G_{v_i} \subseteq \ker(\chi_1^{k_1} \cdots \chi_r^{k_r})$ . Applying Lemma 3.10(ii) then shows (2b).

**Remark 3.12.** Since h is written in lowest terms in (2), nearly any choice of  $v_i$ in  $V(f_i)$  has  $h_1(v_i) \neq 0$ . Thus, if  $v_i$  is generic and  $f_i$  appears (nontrivially) in the denominator of an additive relative invariant, then by (1), (2a), and Theorem 1.1, the subgroup  $G_{v_i}^0$  lies in the kernel of every  $\chi \in X_1(G)$ .

Remark 3.13. In the situation of (2a), the results of [Pik, §5] imply that the ideal I of Remark 1.2 is contained in  $(f_i)^2$ . In particular, if  $I \nsubseteq (f_i)^2$  then there are no additive relative invariants with  $f_i$  appearing in the denominator.

Remark 3.14. Lemma 3.11 implies much about the orbits of certain algebraic subgroups H of G. If  $G_v^0 \subseteq H$  then  $H_v^0 = G_v^0 \cap H$  and hence  $\dim(H \cdot v) =$  $\dim(G \cdot v) - \operatorname{codim}(H)$ .

3.7. **Decomposition.** For prehomogeneous vector spaces, any nontrivial factor of a relative invariant is itself a relative invariant; a crucial fact used in the proof is the unique factorization of a rational function on V.

An analogous statement for additive relative invariants might be that any additive relative invariant may be expressed as a sum of additive relative invariants with 'simpler' denominators. It would be natural to use a partial fraction expansion to prove this, but such an expansion does not always exist for rational functions of several variables. Still, we conjecture:

**Conjecture 3.15.** There exists a basis  $\Phi_1, \ldots, \Phi_k$  for  $\mathscr{A}_1(G)$  such that if we choose reduced rational functions  $h_i$  with  $h_i \stackrel{a}{\longleftrightarrow} \Phi_i$ , then for each i the denominator of  $h_i$  is a positive power of some basic relative invariant.

The strongest result we have in this direction is the following.

**Proposition 3.16.** Let  $\frac{h_1}{f_1f_2}$  be an additive relative invariant written in lowest terms, with  $f_1$  and  $f_2$  polynomial relative invariants,  $f_1$  irreducible, and  $f_1f_2$  reduced. Let  $f_i \stackrel{\text{m}}{\longleftrightarrow} \chi_i$ . If  $I = (f_1, f_2)$  is a radical ideal, and every  $v^* \in V(f_1, f_2)$  is in the closure of an orbit of  $K = \ker(\chi_1)$  acting on  $V(f_1) \setminus V(f_2)$ , then there exist homogeneous polynomials  $\alpha, \beta \in \mathbb{C}[V]$  such that

$$\frac{h_1}{f_1 f_2} = \frac{\alpha}{f_1} + \frac{\beta}{f_2},$$

with  $\frac{\alpha}{f_1}$  and  $\frac{\beta}{f_2}$  reduced additive relative invariants.

*Proof.* The statement holds for  $h_1 = 0$ , so assume  $h_1$  is nonzero. By Proposition 3.7, for all (g, v) we have

(3.8) 
$$h_1(\rho(g)(v)) = \chi_1(g)\chi_2(g) \left( f_1(v)f_2(v)\Phi(g) + h_1(v) \right).$$

Let  $v^* \in V(f_1, f_2)$ . By hypothesis,  $v^* \in \overline{K \cdot v_1}$  for some  $v_1 \in V(f_1) \setminus V(f_2)$ , and hence there is a sequence  $(k_t)$  in K with  $\lim_{t\to\infty} \rho(k_t)(v_1) = v^*$ . Evaluating (3.8) at  $(k_t, v_1)$  gives

(3.9) 
$$h_1(\rho(k_t)(v_1)) = \chi_2(k_t)h_1(v_1),$$

as  $\chi_1(k_t) = 1$  and  $f_1(v_1) = 0$ . Since by (1.1)

$$\lim_{t \to \infty} \chi_2(k_t) = \lim_{t \to \infty} \frac{f_2(\rho(k_t)(v_1))}{f_2(v_1)} = \frac{f_2(v^*)}{f_2(v_1)} = 0$$

and  $h_1$  is continuous, (3.9) implies that  $h_1(v^*) = \lim_{t \to \infty} h_1(\rho(k_t)(v_1)) = 0$ .

Thus  $h_1$  vanishes on  $V(f_1, f_2)$ , hence  $h_1 \in \sqrt{I} = I$ , and so there exists  $\alpha, \beta \in \mathbb{C}[V]$  such that

$$(3.10) h_1 = \alpha \cdot f_2 + \beta \cdot f_1;$$

by taking homogeneous parts,  $\alpha$  and  $\beta$  may be chosen homogeneous with  $\deg(\alpha) = \deg(f_1)$  and  $\deg(\beta) = \deg(f_2)$ .

By (3.8) and (3.10), for  $v \in V(f_1)$  and  $g \in G$  we have

$$\alpha(\rho(g)(v)) \cdot f_2(\rho(g)(v)) = \chi_1(g)\chi_2(g)\alpha(v)f_2(v).$$

Then  $\alpha(\rho(g)(v)) = \chi_1(g)\alpha(v)$  on  $G \times (V(f_1) \setminus V(f_2))$ , and hence on the Zariski closure  $G \times V(f_1)$ . By Proposition 3.9,  $\frac{\alpha}{f_1}$  is an additive relative invariant, and hence so is  $\frac{\beta}{f_2}$ . If  $\frac{\alpha}{f_1}$  or  $\frac{\beta}{f_2}$  is not reduced, then by (3.10) neither is  $\frac{h_1}{f_1f_2}$ .

#### 4. Examples

We now give some examples of prehomogeneous vector spaces and their additive relative invariants.

**Example 4.1.** Let  $G = \left\{ \begin{pmatrix} a & 0 & 0 \\ b & a & 0 \\ c & 0 & a \end{pmatrix} \in \operatorname{GL}_3(\mathbb{C}) \right\}$  act on  $\mathbb{C}\{x,y,z\} = \mathbb{C}^3$  by multiplication, where this notation means that a,b,c may take any value that gives an invertible matrix. Then G has D = V(x) as the exceptional orbit variety, with  $f_1 = x \stackrel{\text{m}}{\longleftrightarrow} \chi_1 = a$ . Since G is abelian and the isotropy subgroup at  $(x,y,z) = (1,0,0) \in \Omega$  is trivial, G has 2 linearly independent additive functions, namely,  $h_1 = \frac{y}{x} \stackrel{\text{d}}{\longleftrightarrow} \Phi_1 = \frac{b}{a}$  and  $h_2 = \frac{z}{x} \stackrel{\text{d}}{\longleftrightarrow} \Phi_2 = \frac{c}{a}$ . On V(x), radial subsets of the form  $\alpha y + \beta z = 0$  are invariant, and the G-action multiplies both coordinates by  $\chi_1 = a$  as expected by Corollary 3.8.

More generally, fix coordinates  $x_1, \ldots, x_n$  on  $\mathbb{C}^n$  and let  $G \subseteq \mathrm{GL}(\mathbb{C}^n)$  consist of those matrices with all diagonals equal, the first column unrestricted, and all other entries equal to zero. Then G is abelian and has  $f_1 = x_1$  and n-1 linearly independent additive functions corresponding to the additive relative invariants  $\Phi_i = \frac{x_{i+1}}{x_1}, i = 1, \ldots, n-1$ .

**Example 4.2.** Let  $G = \left\{ \begin{pmatrix} a & 0 & 0 \\ b & a & 0 \\ 0 & 0 & c \end{pmatrix} \in \operatorname{GL}_3(\mathbb{C}) \right\}$  act on  $\mathbb{C}\{x,y,z\} = \mathbb{C}^3$  by multiplication. Then G has  $f_1 = x \stackrel{\longleftrightarrow}{\longleftrightarrow} \chi_1 = a$  and  $f_2 = z \stackrel{\longleftrightarrow}{\longleftrightarrow} \chi_2 = c$ . Since G is abelian and the isotropy subgroup at  $(x,y,z) = (1,1,0) \in \Omega$  is trivial,  $\dim(\mathscr{A}_1(G)) = 1$ ; a generator is  $h_1 = \frac{y}{x} \stackrel{\mathsf{a}}{\longleftrightarrow} \Phi_1 = \frac{b}{a}$ . On  $V(x), g \in G$  sends  $(0,y,z)^T$  to  $(0,\chi_1(g)y,\chi_2(g)z)^T$ ; hence, both V(x,y) and V(x,z) are G-invariant, but only the former has the behavior predicted by Corollary 3.8. Both V(x) and V(x) contain open orbits.

More generally, we may take direct products of groups of the type in Example 4.1, realized as block diagonal matrices. For  $n \in \mathbb{N}$  and  $k_1, \ldots, k_n \in \mathbb{Z}_{\geq 0}$ , this construction can produce a prehomogeneous vector space with  $\Omega^c$  consisting of n hyperplanes  $H_1, \ldots, H_n$ , such that for each i there are  $k_i$  independent additive relative invariants having poles on  $H_i$ .

**Example 4.3.** Let  $G = \left\{ \begin{pmatrix} a & 0 & 0 \\ b & \frac{a}{c} & 0 \\ 0 & 0 & c \end{pmatrix} \in \operatorname{GL}_3(\mathbb{C}) \right\}$  act on  $\mathbb{C}\{x,y,z\} = \mathbb{C}^3$  by multiplication. Then G has  $f_1 = x \stackrel{\text{m}}{\longleftrightarrow} \chi_1 = a$  and  $f_2 = z \stackrel{\text{m}}{\longleftrightarrow} \chi_2 = c$ . The generic isotropy subgroup is trivial and  $\dim([G,G]) = 1$ . Thus  $G/[G,G] \cdot G_{v_0}$  has no nontrivial additive functions, and there are no nontrivial additive relative invariants. The orbit structure of G agrees with that of the group in Example 4.2, but here neither V(x,y) nor V(x,z) exhibit the behavior described in Corollary 3.8.

**Example 4.4.** For  $n \geq 2$ , let  $G \subseteq GL_n(\mathbb{C})$  consist of all invertible lower-triangular matrices A such that each  $(A)_{ij}$  depends only on i-j, that is,  $(A)_{i+1,j+1} = (A)_{i,j}$  whenever this makes sense. Then G is a connected abelian linear algebraic group, and its action on  $\mathbb{C}^n = \mathbb{C}\{x_1,\ldots,x_n\}$  has an open orbit with exceptional orbit variety  $V(x_1)$  and a trivial generic isotropy subgroup. It follows that G has n-1

linearly independent additive functions. For  $1 \leq i < n$  define  $\Phi_i$  by

$$\Phi_{i} \begin{pmatrix} \begin{pmatrix} a_{1} & 0 & 0 & \cdots & 0 \\ a_{2} & a_{1} & 0 & \ddots & \vdots \\ a_{3} & a_{2} & a_{1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{n} & a_{n-1} & a_{n-2} & \cdots & a_{1} \end{pmatrix} = \frac{1}{(a_{1})^{i}} \begin{vmatrix} \frac{1}{i}a_{2} & a_{1} & 0 & \cdots & 0 \\ \frac{2}{i}a_{3} & a_{2} & a_{1} & \ddots & \vdots \\ \frac{3}{i}a_{4} & a_{3} & a_{2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \frac{i}{i}a_{i+1} & a_{i} & \cdots & \cdots & a_{2} \end{vmatrix}.$$

For instance, when n = 4 we have

$$\Phi_1 = \frac{a_2}{a_1}, \quad \Phi_2 = \frac{1}{a_1^2} \left( \frac{1}{2} a_2^2 - a_1 a_3 \right), \quad \text{and } \Phi_3 = \frac{1}{a_1^3} \left( \frac{1}{3} (a_2)^3 - a_1 a_2 a_3 + a_1^2 a_4 \right).$$

For at least  $n \leq 20$ , computer calculations show that each  $\Phi_i : G \to \mathbb{G}_a$  is a homomorphism, and hence  $\Phi_1, \ldots, \Phi_{n-1}$  is a basis of the space of additive functions. By (3.1), the corresponding additive relative invariants are of a form similar to  $\Phi_i$ , and for i > 1 have nonreduced denominators.

Example 4.5. Fix a positive integer n and let  $m \in \{n, n+1\}$ . Let  $L \subseteq \operatorname{GL}_n(\mathbb{C})$  (respectively,  $U \subseteq \operatorname{GL}_m(\mathbb{C})$ ) consist of invertible lower triangular matrices (resp., upper triangular unipotent matrices). Let  $G = L \times U$  act on the space  $M(n, m, \mathbb{C})$  of  $n \times m$  complex matrices by  $(A, B) \cdot M = AMB^{-1}$ . The classical LU factorization of a complex matrix asserts that this is a prehomogeneous vector space (see [DP12b]). For any matrix M, let  $M^{(k)}$  denote the upper-leftmost  $k \times k$  submatrix of M. By [DP12b, §6], the basic relative invariants are of the form  $f_i(M) = \det(M^{(i)})$ ,  $i = 1, \ldots, n$ , and  $f_i \stackrel{\longrightarrow}{\longleftrightarrow} \chi_i(A, B) = \det(A^{(i)})$ . The identity matrix  $v_0 = I$  (when m = n) or  $v_0 = \begin{pmatrix} I & 0 \end{pmatrix}$  (when m = n + 1) is in  $\Omega$  and has a trivial isotropy subgroup  $G_{v_0}$ .

Since  $\dim(G/[G,G]\cdot G_{v_0})=n+m-1$ , the quotient has m-1 linearly independent additive functions. In fact, for  $(A,B)\in G$  and  $1\leq i\leq m-1$ , let  $\Phi_i(A,B)=-(B)_{i,i+1}$ . A computation shows that these  $\Phi_i$  are linearly independent additive functions of G, with

$$h_i(M) = \frac{\det((M \text{ with column } i \text{ deleted})^{(i)})}{\det(M^{(i)})} \stackrel{\text{a}}{\longleftrightarrow} \Phi_i.$$

# 5. Linear free divisors

We now consider prehomogeneous vector spaces  $(G, \rho, V)$  for which the complement of the open orbit  $\Omega$  is a type of hypersurface called a linear free divisor. Our main theorem is that these have no nontrivial additive relative invariants, but this has significant consequences for their structure.

5.1. **Introduction.** Let  $\mathscr{O}_{\mathbb{C}^n,p}$  denote the ring of germs of holomorphic functions on  $\mathbb{C}^n$  at p, and  $\mathrm{Der}_{\mathbb{C}^n,p}$  the  $\mathscr{O}_{\mathbb{C}^n,p}$ -module of germs of holomorphic vector fields on  $\mathbb{C}^n$  at p. Associated to a germ (D,p) of a reduced analytic set in  $\mathbb{C}^n$  is the  $\mathscr{O}_{\mathbb{C}^n,p}$ -module of logarithmic vector fields defined by

$$\operatorname{Der}_{\mathbb{C}^n,p}(-\log D) := \{ \eta \in \operatorname{Der}_{\mathbb{C}^n,p} : \eta(I(D)) \subseteq I(D) \},$$

where I(D) is the ideal of germs vanishing on D. These are the vector fields tangent to (D, p), and form a Lie algebra using the Lie bracket of vector fields.

Let D be nonempty and  $(D, p) \neq (\mathbb{C}^n, p)$ . When  $\operatorname{Der}_{\mathbb{C}^n, p}(-\log D)$  is a free  $\mathscr{O}_{\mathbb{C}^n, p}(-\log D)$  is a free  $\mathscr{O}_{\mathbb{C}^n, p}(-\log D)$  is called a *free divisor*. Free divisors are always pure hypersurface germs that are either smooth or have singularities in codimension 1, and were first encountered as various types of discriminants.

Now let  $(G, \rho, V)$  be a prehomogeneous vector space, let  $\mathfrak{g}$  be the Lie algebra of G, and let  $D := V \setminus \Omega$  be the exceptional orbit variety. Let  $\mathrm{Der}_V(-\log D)_0$  denote the  $\eta \in \mathrm{Der}_V(-\log D)$  that are  $\mathit{linear}$ , that is, homogeneous of degree 0 (e.g.,  $(x+y)\frac{\partial}{\partial x}-2z\frac{\partial}{\partial y})$ . Differentiating the action of G gives a Lie algebra (anti-)homomorphism  $\tau:\mathfrak{g}\to \mathrm{Der}_V(-\log D)_0$ , where  $\xi_X:=\tau(X)$  is a vector field defined globally on V by

$$\tau(X)(v) = \frac{d}{dt} \left( \rho(\exp(tX))(v) \right) \Big|_{t=0}.$$

Thus,  $\tau(\mathfrak{g}) \subseteq \operatorname{Der}_V(-\log D)_0$  are finite-dimensional Lie subalgebras of  $\operatorname{Der}_V(-\log D)$ . As  $\tau(\mathfrak{g})(v) = T_v(G \cdot v)$ , the maximal minors of a matrix containing a basis of  $\tau(\mathfrak{g})$  are of degree  $\dim(V)$  and generate an ideal defining the set D.

When  $\mathrm{Der}_V(-\log D)$  has a free basis of linear vector fields, then D is called a linear free divisor. By [GMNRS09, Lemma 2.3], every linear free divisor D is the exceptional orbit variety of a prehomogeneous vector space with the following properties:

**Definition 5.1.** Let D be a linear free divisor in V. If  $(G, \rho, V)$  is a prehomogeneous vector space with open orbit  $V \setminus D$ , G connected, and  $\dim(G) = \dim(V)$ , then say that G defines the linear free divisor D.

It follows that  $\ker(\rho)$  is finite, and  $\tau(\mathfrak{g}) = \operatorname{Der}_V(-\log D)_0$ .

One such prehomogeneous vector space may be constructed in the following way. For a divisor  $D \subset V$  in a finite-dimensional complex vector space, let  $\operatorname{GL}(V)_D$  be the largest subgroup of  $\operatorname{GL}(V)$  that preserves D. Note that  $\operatorname{GL}(V)_D$  is algebraic. Then  $(\operatorname{GL}(V)_D)^0$  is a connected complex linear algebraic group with Lie algebra (anti-)isomorphic to  $\operatorname{Der}_V(-\log D)_0$ . For a linear free divisor  $D \subset V$ , the group  $(\operatorname{GL}(V)_D)^0 \subseteq \operatorname{GL}(V)$  with the inclusion representation is a prehomogeneous vector space that defines D ([GMNRS09, Lemma 2.3]). In fact, if  $(G, \rho, V)$  defines a linear free divisor D, then  $\rho(G) = (\operatorname{GL}(V)_D)^0$ .

5.2. **Brion's criterion.** Brion gave the following useful criterion for D to be a linear free divisor.

**Theorem 5.2** ([Bri06], [GMS11, Theorem 2.1]; see also [Pik]). For  $G = (GL(V)_D)^0$ , the following are equivalent:

- (1) D is a linear free divisor
- (2) Both of these conditions hold:
  - (a)  $V \setminus D$  is a unique G-orbit, and the corresponding isotropy groups are finite.
  - (b) Each irreducible component  $D_i$  of D contains an open G-orbit  $D_i^0$ , and the corresponding isotropy groups are extensions of finite groups by  $\mathbb{G}_m$ .

When these hold,  $\tau(\mathfrak{g})$  generates  $\operatorname{Der}_V(-\log D)$ , and each  $D_i^0 = \operatorname{smooth}(D) \cap D_i$ . The proof of Theorem 5.2 shows the following.

**Corollary 5.3.** Suppose that in the situation of Theorem 5.2, (2a) holds,  $D_i$  contains an open orbit  $D_i^0$ , and  $v_i \in D_i^0$ . Then  $G_{v_i}$  is an extension of a finite group by  $\mathbb{G}_{\mathrm{m}}$  if and only if the induced representation of  $G_{v_i}^0$  on the normal line to  $D_i$  at  $v_i$  is nontrivial.

The representation on the normal line is actually a quite familiar object.

**Lemma 5.4.** Let  $(G, \rho, V)$  be a prehomogeneous vector space with f as a basic relative invariant, and let v be a smooth point of D := V(f). If  $H \subseteq G_v$ , then the representation  $\rho_v : H \to \operatorname{GL}(L)$  on the normal line  $L = T_v V/T_v D$  to D at v is

$$\rho_v(h)(\ell) = \chi(h)\ell, \quad \text{for all } \ell \in L,$$

where  $f \stackrel{\text{m}}{\longleftrightarrow} \chi$ .

Geometrically,  $\rho_v$  acts on a normal slice to f=0 at v, and such a slice intersects all level sets of f. At the same time, the action of  $\rho(H)$  fixes v and translates between the level sets of f according to  $\chi$ .

Proof. The representation  $\rho|_H$  fixes v and hence induces a representation  $\rho'$  of H on the tangent space  $T_vV$ ; by silently identifying  $T_vV$  with V, we have  $\rho' = \rho|_H$ . Since  $\rho|_H$  leaves invariant D and fixes the smooth point v on D, the representation  $\rho'$  leaves invariant  $T_vD$ . Then the normal line is  $L = T_vV/T_vD$ , and  $\rho'$  produces the quotient representation  $\rho_v: H \to \operatorname{GL}(L)$  defined by  $\rho_v(h)(w+T_vD) = \rho'(h)(w) + T_vD$ .

Let  $h \in H$ , and  $w \in T_vV$ . For  $\lambda \in \mathbb{C}$  we have by (1.1),

(5.1) 
$$f(v + \lambda \cdot \rho(h)(w)) = f(\rho(h)(v + \lambda w)) = \chi(h)f(v + \lambda w).$$

Differentiating (5.1) with respect to  $\lambda$  and evaluating at  $\lambda = 0$  gives  $df_{(v)}(\rho'(h)(w)) = \chi(h)df_{(v)}(w)$ , or  $\rho'(h)(w) - \chi(h)w \in \ker(df_{(v)})$ . Since f is reduced and v is a smooth point,  $T_vD = \ker(df_{(v)}) : T_vV \to T_0\mathbb{C}$ . The definition of  $\rho_v$  then implies the result.

**Remark 5.5.** If  $(G, \rho, V)$  defines a linear free divisor D, then all of the results of this §5.2 hold for G as well as  $\rho(G) = (GL(V)_D)^0$ .

5.3. **The main theorem.** Let  $(G, \rho, V)$  define the linear free divisor  $D \subset V$  in the sense of Definition 5.1. Let  $f_1, \ldots, f_r$  be the basic relative invariants, so that  $\bigcup_{i=1}^r V(f_i)$  is the irreducible decomposition of  $D = V \setminus \Omega$ . Let  $f_i \stackrel{\text{m}}{\longleftrightarrow} \chi_i$ , and choose  $v_0 \in \Omega$ . For each i let  $v_i$  be a generic point on  $V(f_i)$ , an element of the open orbit of G in  $V(f_i)$ .

**Theorem 5.6.** Let G define the linear free divisor D, with the notation above. Then

- (1) The homomorphism  $(\chi_1, \ldots, \chi_r) : G \to (\mathbb{G}_m)^r$  is surjective and has kernel  $[G, G] \cdot G_{v_0}$ .
- (2)  $G/[G,G] \cdot G_{v_0}$  has only the trivial additive function and constant additive relative invariants.
- (3) For  $i \neq j$ , we have  $G_{v_i}^0 \subseteq \ker(\chi_i)$ .
- (4) The representation  $\chi_i|_{G_{v_i}^0}: G_{v_i}^0 \to \mathbb{G}_{\mathrm{m}}$  is surjective, with finite kernel.
- (5)  $\ker(\chi_i|_{G_{v_i}^0}) = G_{v_i}^0 \cap ([G, G] \cdot G_{v_0})$  is finite.
- (6) For  $i \neq j$ , the subgroup  $G_{v_i}^0 \cap G_{v_i}^0$  is a finite subset of  $[G, G] \cdot G_{v_0}$ .

*Proof.* Let  $H = [G, G] \cdot G_{v_0}$ . Claim (3) is just Lemma 3.11(1).

By Theorem 5.2, Corollary 5.3, and Lemma 5.4, each  $\chi_i|_{G_{v_i}^0}$  is nontrivial, thus surjective. As  $\dim(G_{v_i}^0) = 1 = \dim(\operatorname{GL}(\mathbb{C}))$ , the kernel is finite, giving (4).

If  $\Phi \in \mathscr{A}(G/H)$  is nontrivial, then by Proposition 3.3 there exists a nonzero h with  $h \stackrel{\mathrm{a}}{\longleftrightarrow} \Phi$ . By Proposition 3.5, write  $h = \frac{h_1}{g_1}$  as a reduced fraction, with  $g_1$  a polynomial relative invariant and  $g_1 \stackrel{\mathrm{m}}{\longleftrightarrow} \chi$ . Since  $\Phi$  is nontrivial,  $g_1$  cannot be constant; let  $f_i$  be an irreducible factor of  $g_1$ . If  $h_1(v_i) = 0$ , then by Proposition 3.7,  $h_1$  vanishes on  $G \cdot v_i$  and hence on  $\overline{G \cdot v_i} = V(f_i)$ ; but then  $f_i$  divides  $\gcd(h_1, g_1)$ , giving a contradiction. Hence  $h_1(v_i) \neq 0$ . By Lemma 3.11(2a) we have  $G_{v_i}^0 \subset \ker(\chi_i)$ , but this contradicts (4). Thus  $\Phi = 0$  is trivial, proving (2).

Since G/H is connected and abelian, by Proposition 2.1,  $G/H \cong (\mathbb{G}_{\mathrm{m}})^k \times (\mathbb{G}_{\mathrm{a}})^\ell$ . Choosing a free basis  $\epsilon_1, \ldots, \epsilon_k$  of X(G/H) and a basis  $\Phi_1, \ldots, \Phi_\ell$  of  $\mathscr{A}(G/H)$  gives an explicit isomorphism  $\theta = (\epsilon_1, \ldots, \epsilon_k, \Phi_1, \ldots, \Phi_\ell) : G/H \to (\mathbb{G}_{\mathrm{m}})^k \times (\mathbb{G}_{\mathrm{a}})^\ell$ . Theorem 1.1 and Proposition 1.3 show that we may let each  $\epsilon_i = \chi_i$ . Then, (1) follows from (2).

For (5), such elements are contained in the kernel by (1), and conversely by (3) and (1). By (4), the kernel is finite.

For (6), observe that by (3), such elements are in the kernel of the homomorphism of (1). For finiteness, use (5).

**Remark 5.7.** For each i > 0, the group  $G_{v_i}^0$  is a 1-dimensional connected complex linear algebraic group with a surjective homomorphism to  $\mathbb{G}_{\mathrm{m}}$ , and hence  $G_{v_i}^0 \cong \mathbb{G}_{\mathrm{m}}$ .

Theorem 5.6 has a number of immediate consequences. In particular, we may easily compute the number of irreducible components.

Corollary 5.8. Let G define the linear free divisor D, with the notation above. Then

(1) The hypersurface D has

$$r = \dim_{\mathbb{C}}(G/[G,G] \cdot G_{v_0}) = \dim_{\mathbb{C}}(G/[G,G]) = \dim_{\mathbb{C}}(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])$$

irreducible components, where  $\mathfrak{g}$  is the Lie algebra of G.

- (2) Every element of G may be written in a finite number of ways as a product of elements from the subgroups  $[G,G] \cdot G_{v_0}, G_{v_1}^0, \ldots, G_{v_r}^0$ . Each term of such a product is unique modulo  $[G,G] \cdot G_{v_0}$ .
- (3) Let  $S \subseteq \{1, ..., r\}$ . The subgroup of G that leaves invariant all level sets of  $f_i$  for  $i \in S$  is the product of the subgroups  $[G, G] \cdot G_{v_0}$ , and  $G_{v_i}^0$  for  $j \notin S$ .
- (4) As algebraic groups,  $G/[G,G] \cong (\mathbb{G}_m)^r$ .
- (5) The subgroup [G, G] contains all unipotent elements of G.

*Proof.* For (1), combine Theorem 5.6(2) and Corollary 2.3. Note also that  $G_{v_0}$  is finite by Theorem 5.2.

Let  $g \in G$ . By Theorem 5.6(4), for i > 0 there exists  $g_i \in G_{v_i}^0$  such that  $\chi_i(g) = \chi_i(g_i)$ . By Theorem 5.6(3) and 5.6(1),  $g(g_1 \cdots g_r)^{-1}$  is in the kernel of each  $\chi_i$ , and hence lies in  $[G, G] \cdot G_{v_0}$ . This proves existence for (2).

To address uniqueness, let  $g \in G$  and suppose  $g = g_0^1 \cdot g_1^1 \cdots g_r^1 = g_0^2 \cdot g_1^2 \cdots g_r^2$ , with  $g_0^j \in [G,G] \cdot G_{v_0}$  and  $g_i^j \in G_{v_i}^0$  for i > 0. By Theorem 5.6(1) and 5.6(3), for i > 0 we have  $\chi_i(g) = \chi_i(g_i^1) = \chi_i(g_i^2)$ . Since each  $\chi_i|_{G_{v_i}^0}$  has a finite kernel contained in  $[G,G] \cdot G_{v_0}$  by Theorem 5.6(5), for i > 0 we have  $g_i^1 = g_i^2$  modulo

 $\ker(\chi_i|_{G_{v_i}^0})$ . Since  $g_0^j$  is uniquely determined by  $g_1^j,\ldots,g_r^j$ , there are precisely

$$\prod_{i=1}^r \#(\ker(\chi_i|_{G_{v_i}^0}))$$

ways to write g in this way. This proves (2).

By (1.1),  $g \in G$  leaves invariant all level sets of  $f_i$  if and only if  $g \in \ker(\chi_i)$ . Then (2) and Theorems 5.6(3) and 5.6(5) prove (3).

For (4), by Proposition 2.1 there is an isomorphism  $\phi: G/[G,G] \to \mathbb{G}_{\mathrm{m}}^{k} \times \mathbb{G}_{\mathrm{a}}^{\ell}$  with  $r = k + \ell$ . By Theorem 5.6(1), there is a quotient homomorphism  $\psi: \mathbb{G}_{\mathrm{m}}^{k} \times \mathbb{G}_{\mathrm{a}}^{\ell} \to \mathbb{G}_{\mathrm{m}}^{r}$  with finite kernel  $\phi([G,G] \cdot G_{v_{0}})$ . The Jordan decomposition implies that  $\{1\} \times \mathbb{G}_{\mathrm{a}}^{\ell} \subseteq \ker(\phi)$ , and for this kernel to be finite requires  $\ell = 0, k = r$ .

Finally, (5) is a consequence of (4) and the Jordan decomposition.

**Example 5.9.** On  $\mathbb{C}^3$ , fix coordinates x, y, z. The linear free divisor  $x(xz-y^2)=0$  is defined by the solvable group

$$G = \left\{ \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ \frac{b^2}{a} & \frac{2bc}{a} & \frac{c^2}{a} \end{pmatrix} \in \operatorname{GL}(\mathbb{C}^3) \right\}.$$

We have  $f_1 = x \stackrel{\text{m}}{\longleftrightarrow} \chi_1 = a$  and  $f_2 = xz - y^2 \stackrel{\text{m}}{\longleftrightarrow} \chi_2 = c^2$ . At  $v_0 = (1,0,1) \in \Omega$ ,  $v_1 = (0,1,0) \in V(f_1)$ , and  $v_2 = (1,0,0) \in V(f_2)$ , the (generic) isotropy subgroups are defined by, respectively,  $(a,b,c) = (1,0,\pm 1)$ , (b,c) = (0,1), and (a,b) = (1,0). As [G,G] is defined by a = c = 1, we see  $[G,G] \cdot G_{v_0}$  is defined by a = 1,  $c = \pm 1$ . Theorem 5.6 and Corollary 5.8 are easy to verify.

Recall that L(K) denotes the Lie algebra of an algebraic group K. On the level of Lie algebras, Theorem 5.6 implies the following.

Corollary 5.10. As vector spaces,

$$\mathfrak{g} = [\mathfrak{g},\mathfrak{g}] \oplus \bigoplus_{i=1}^r L(G_{v_i}).$$

For i = 1, ..., r there exist unique  $X_i \in L(G_{v_i})$  such that  $L(G_{v_i}) = \mathbb{C}X_i$ , and such that for all j we have  $d(\chi_i)_{(e)}(X_j) = \delta_{ij}$  and  $\xi_{X_j}(f_i) = \delta_{ij}f_i$ . For  $X \in [\mathfrak{g}, \mathfrak{g}]$  and any j,  $\xi_X(f_j) = 0$ .

Proof. Differentiating the homomorphism of Theorem 5.6(1) gives a homomorphism  $\lambda: \mathfrak{g} \to L((\mathbb{G}_{\mathrm{m}})^r) = \bigoplus_{i=1}^r L(\mathbb{G}_{\mathrm{m}})$  with kernel  $[\mathfrak{g},\mathfrak{g}]$ . By Theorem 5.6(3) and (4), under  $\lambda$  each  $L(G_{v_i})$  surjects onto the *i*th copy of  $L(\mathbb{G}_{\mathrm{m}})$ , and is zero on the rest. This gives the vector space decomposition, and proves that for  $i \neq j$ ,  $d(\chi_i)_{(e)}(L(G_{v_j})) = 0$ . Now choose the unique  $X_i \in L(G_{v_i})$  such that  $d(\chi_i)_{(e)}(X_i) = 1$ .

The rest of the statement follows by differentiating  $f_i(\rho(\exp(tX))(v)) = \chi_i(\exp(tX))f_i(v)$  to show that for all  $X \in \mathfrak{g}$  and i,

(5.2) 
$$\xi_X(f_i) = d(\chi_i)_{(e)}(X) \cdot f_i. \qquad \Box$$

Although each  $X_i$  depends on the choice of  $v_i \in V(f_i)$ , modulo  $[\mathfrak{g}, \mathfrak{g}]$  each  $X_i$  is independent of the choice of  $v_i$ .

**Remark 5.11.** More generally, let  $(G, \rho, V)$  be a prehomogeneous vector space with no nontrivial additive relative invariants. (By Remark 3.13, this happens, for instance, if the ideal I of Remark 1.2 is not contained in  $(f_i)^2$  for every basic relative invariant  $f_i$ .) Then by the same argument in Theorem 5.6 above,  $G/[G, G] \cdot G_{v_0}$ 

is an algebraic torus, of dimension equal to the number of irreducible hypersurface components of the exceptional orbit variety  $V \setminus \Omega$ .

5.4. The structure of G. We now use Theorem 5.6 and Corollary 5.8 to study the structure of algebraic groups defining linear free divisors.

Let G be a connected complex algebraic group. Let Rad(G) denote the radical of G, the maximal connected normal solvable subgroup. The (algebraic) Levi decomposition of G writes

$$G = \operatorname{Rad}_u(G) \rtimes L,$$

where  $\operatorname{Rad}_u(G)$  is the unipotent radical of G, the largest connected unipotent normal subgroup of G, consisting of all unipotent elements of  $\operatorname{Rad}(G)$ ; and L is a Levi subgroup, a maximal connected reductive algebraic subgroup of G, unique up to conjugation [Bor91, 11.22]. Moreover,  $L = \operatorname{Z}(L)^0 \cdot [L, L]$  for  $\operatorname{Z}(L)$  the center of L,  $\operatorname{Z}(L) \cap [L, L]$  is finite, [L, L] is semisimple, and  $(\operatorname{Z}(L))^0 = L \cap \operatorname{Rad}(G)$  is a maximal torus of  $\operatorname{Rad}(G) = \operatorname{Rad}_u(G) \rtimes \operatorname{Z}(L)^0$  [Bor91, 14.2, 11.23].

Groups defining linear free divisors have the following structure.

Corollary 5.12. Let G define the linear free divisor D and have the Levi decomposition above. Then

- (1) The number of irreducible components of D is  $r = \dim(Z(L))$ .
- (2)  $[G, G] = \operatorname{Rad}_u(G) \rtimes [L, L].$
- (3)  $G = [G, G] \cdot (Z(L))^0$ , with  $[G, G] \cap (Z(L))^0$  finite.
- (4)  $(\chi_1, \ldots, \chi_r)|_{Z(L)^0} : Z(L)^0 \to (\mathbb{G}_m)^r$  is surjective, with a finite kernel.

*Proof.* Let  $R = \operatorname{Rad}_u(G)$ . Since G is the semidirect product of R and L, a straightforward calculation shows that  $[G, G] = [R, R] \cdot [R, L] \cdot [L, L]$ . By Corollary 5.8(5) and connectedness,  $R \subseteq [G, G]$ . Since  $R \subseteq G$ , we have  $[R, R], [R, L] \subseteq R$  and hence

$$[G,G] = R \cdot [G,G] = R \cdot [R,R] \cdot [R,L] \cdot [L,L] = R \cdot [L,L].$$

As R consists of unipotent elements and  $\operatorname{Rad}(G) \cap L$  consists of semisimple elements,  $R \cap L = \{e\}$  and hence  $R \cap [L, L] = \{e\}$ . Since  $R \subseteq [G, G]$  are normal subgroups of G, we have  $R \subseteq [G, G]$ . This proves the decomposition (2) as abstract groups, and hence as complex algebraic groups ([OV90, §3.3–3.4]).

From this, we conclude

(5.3) 
$$G = R \cdot L = R \cdot [L, L] \cdot (Z(L))^{0} = [G, G] \cdot (Z(L))^{0}.$$

Clearly,  $[L, L] \subseteq [G, G] \cap L$ . If  $g \in [G, G] \cap L$ , then by (2) we have  $g = k \cdot \ell$  for  $k \in R$  and  $\ell \in [L, L] \subseteq L$ , and hence  $k = g\ell^{-1} \in L \cap R = \{e\}$ ; thus  $[G, G] \cap L = [L, L]$ . In particular,  $[G, G] \cap (Z(L))^0 = [L, L] \cap (Z(L))^0$  is finite. This proves (3).

By (3) and an isomorphism theorem,

$$G/[G,G] \cong (\operatorname{Z}(L))^0/([G,G] \cap \operatorname{Z}(L)^0),$$

and hence  $\dim(Z(L)) = \dim(G/[G,G])$ . Corollary 5.8(1) then proves (1).

Finally, (4) may be checked on the level of Lie algebras. By (3), we have  $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}] \oplus \mathfrak{z}(\mathfrak{l})$ , where  $\mathfrak{g}$  and  $\mathfrak{l}$  are the Lie algebras of G and L, and  $\mathfrak{z}(\mathfrak{l})$  is the center of  $\mathfrak{l}$ . For  $\chi = (\chi_1, \ldots, \chi_r)$ , we have  $\ker(d\chi_{(e)}) = [\mathfrak{g},\mathfrak{g}]$  by Theorem 5.6(1), and hence  $d\chi_{(e)}|_{\mathfrak{z}(\mathfrak{l})}$  is an isomorphism. In fact, by Theorem 5.6(1) the kernel in (4) is  $(Z(L))^0 \cap ([G,G] \cdot G_{v_0})$ .

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**Remark 5.13.** An arbitrary connected complex linear algebraic group with Levi decomposition  $G = \operatorname{Rad}_u(G) \rtimes L$  has

$$[G,G] = [\operatorname{Rad}_u(G), \operatorname{Rad}_u(G)] \cdot [\operatorname{Rad}_u(G), L] \cdot [L,L] \subseteq \operatorname{Rad}_u(G) \times [L,L],$$

and hence  $[\operatorname{Rad}_u(G), \operatorname{Rad}_u(G)] \cdot [\operatorname{Rad}_u(G), L] \subseteq \operatorname{Rad}_u(G)$ ; if G defines a linear free divisor, then by Corollary 5.12(2) these are both equalities.

**Remark 5.14.** If B is a Borel subgroup of [L, L] and  $T \subseteq B$  is a maximal torus of [L, L], then it follows from [Bor91, 11.14] that  $\operatorname{Rad}_u(G) \cdot B \cdot \operatorname{Z}(L)^0$  is a Borel subgroup of G and  $T \cdot \operatorname{Z}(L)^0$  is a maximal torus of G. In particular, the number of irreducible components is at most the dimension of the maximal torus.

Consider the following examples of linear free divisors.

**Example 5.15.** We continue Example 5.9. Since G is solvable, [L, L] is trivial and hence  $\operatorname{Rad}_u(G)$  is defined by a=c=1, and a maximal torus  $L=\operatorname{Z}(L)$  is defined by b=0. Corollary 5.12 is easy to check; in particular, the 2-element subgroup  $L\cap G_{v_0}$  lies in the kernel of  $(\chi_1,\chi_2):G\to \mathbb{G}_{\operatorname{m}}^2$  restricted to  $(\operatorname{Z}(L))^0$ .

The following example is neither reductive nor solvable.

Example 5.16 ([DP12b, Example 9.4]). Define the algebraic group

$$G = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & d & e & 0 \\ f & a & b & i \end{pmatrix} \in \operatorname{GL}(\mathbb{C}^4) \right\}.$$

Let S be the space of  $4 \times 4$  symmetric matrices with the usual coordinates  $x_{ij}$ ,  $1 \le i \le j \le 4$ . Let  $V \subset S$  be the subspace where  $x_{11} = 0$ . Let  $\rho : G \to \operatorname{GL}(V)$  be defined by  $\rho(A)(M) = AMA^T$ . Note that  $\ker(\rho) = \{\pm I\}$ . A Levi decomposition of G has L defined by f = g = h = 0 and  $\operatorname{Rad}(G)$  defined by c = d = b - e = 0; the Borel subgroup of lower-triangular  $g \in G$  is not normal in G. Then  $\operatorname{Rad}_u(G)$  is defined by a = b = e = i = 1 and c = d = 0, [L, L] is defined by f = g = h = 0 and be - cd = a = i = 1, and  $L \cap \operatorname{Rad}(G) = \operatorname{Z}(L)^0$  is defined by b = e and c = d = f = g = h = 0. Finally, [G, G] is defined by a = be - cd = i = 1.

The exceptional orbit variety is the linear free divisor defined by  $f_1 \cdot f_2 \cdot f_3$ , where

$$f_1 = \begin{vmatrix} x_{22} & x_{23} \\ x_{23} & x_{33} \end{vmatrix}, \qquad f_2 = \begin{vmatrix} 0 & x_{12} & x_{13} \\ x_{12} & x_{22} & x_{23} \\ x_{13} & x_{23} & x_{33} \end{vmatrix}, \qquad f_3 = \begin{vmatrix} 0 & x_{12} & x_{13} & x_{14} \\ x_{12} & x_{22} & x_{23} & x_{24} \\ x_{13} & x_{23} & x_{33} & x_{34} \\ x_{14} & x_{24} & x_{34} & x_{44} \end{vmatrix},$$

corresponding to the characters  $\chi_1 = (cd - be)^2$ ,  $\chi_2 = a^2(cd - be)^2$ ,  $\chi_3 = a^2(cd - be)^2i^2$ , respectively. Let

$$v_0 = \begin{pmatrix} \begin{smallmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad v_1 = \begin{pmatrix} \begin{smallmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} \begin{smallmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} \begin{smallmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in V$$

be generic points in  $\Omega$  and on each  $V(f_i)$ . Then  $G_{v_0}$ ,  $G_{v_1}$ ,  $G_{v_2}$ ,  $G_{v_3}$  respectively consist of all elements of G of the form

$$\begin{pmatrix} \gamma & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix}, \qquad \begin{pmatrix} \frac{1}{a} & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & \epsilon \end{pmatrix}, \qquad \begin{pmatrix} \frac{1}{a} & 0 & 0 & 0 \\ 0 & \frac{1}{a} & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \qquad \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & a \end{pmatrix},$$

where  $\gamma^2 = \delta^2 = \epsilon^2 = 1$  and  $a \in \mathbb{C}$ . With these calculations, it is straightforward to check the conclusions of Theorem 5.6 and Corollaries 5.8 and 5.12.

5.5. The structure of the Lie algebra. We now summarize our results for the structure of the Lie algebra  $\mathfrak g$  of G. In contrast to the terminology above, the usual Levi decomposition of  $\mathfrak g$  expresses  $\mathfrak g$  as the semidirect sum of the radical  $\mathfrak{rad}$   $\mathfrak g$ , defined as the maximal solvable ideal, and a semisimple Levi subalgebra; these correspond to the Lie algebras of  $\operatorname{Rad}(G)$  and [L, L], respectively.

**Proposition 5.17.** Let  $(G, \rho, V)$  define a linear free divisor  $D \subset V$ . Let G have a Levi decomposition as above, and let  $\mathfrak{g}$ ,  $\mathfrak{r}$ , and  $\mathfrak{l}$  be the Lie algebras of G,  $\mathrm{Rad}_u(G)$ , and L, respectively, and let  $\mathfrak{z}(\mathfrak{l})$  denote the center of  $\mathfrak{l}$ . Then as vector spaces  $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{z}(\mathfrak{l}) \oplus [\mathfrak{l}, \mathfrak{l}]$  where the ideal  $\mathfrak{r}$  consists of nilpotent elements,  $\mathfrak{z}(\mathfrak{l})$  is abelian and consists of semisimple elements,  $[\mathfrak{l}, \mathfrak{l}]$  is semisimple, the ideal  $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{r} \oplus [\mathfrak{l}, \mathfrak{l}]$ , the ideal  $\mathfrak{r} \oplus \mathfrak{z}(\mathfrak{l})$  equals the radical of  $\mathfrak{g}$ , and  $\mathfrak{l} = \mathfrak{z}(\mathfrak{l}) \oplus [\mathfrak{l}, \mathfrak{l}]$ . We thus have

$$\begin{aligned} [\mathfrak{r},\mathfrak{r}] \subseteq \mathfrak{r} & \quad [\mathfrak{r},[\mathfrak{l},\mathfrak{l}]] \subseteq \mathfrak{r} & \quad [\mathfrak{r},\mathfrak{z}(\mathfrak{l})] \subseteq \mathfrak{r} \\ [[\mathfrak{l},\mathfrak{l}],[\mathfrak{l},\mathfrak{l}]] = [\mathfrak{l},\mathfrak{l}] & \quad [[\mathfrak{l},\mathfrak{l}],\mathfrak{z}(\mathfrak{l})] = 0 \\ [\mathfrak{z}(\mathfrak{l}),\mathfrak{z}(\mathfrak{l})] = 0. \end{aligned}$$

*Proof.* Most of this follows from the algebraic Levi decomposition of G. Since [L, L] is semisimple and  $\operatorname{Rad}_u(G)$  is unipotent, their Lie algebras are semisimple and nilpotent, respectively. Since  $\mathfrak{r}$  is nilpotent, all of its elements are nilpotent. Then apply Corollary 5.12(2).

Remark 5.18. Theorem 6.1 of [GMNRS09] describes a normal form for a basis of the logarithmic vector fields of a linear free divisor. A key ingredient is a maximal subspace of simultaneously diagonalizable linear logarithmic vector fields, i.e., the vector fields corresponding to the Lie algebra t of a maximal torus. By Remark 5.14, t may always be chosen to include 3(1).

- 5.6. Consequences in a few special cases. We now apply Theorem 5.6 and its corollaries to several special types of linear free divisors.
- 5.6.1. Abelian groups. The normal crossings divisor in a vector space V is given by the union of all coordinate hyperplanes for some choice of vector space coordinates. This is a linear free divisor, and is the only linear free divisor defined by an abelian group.

**Corollary 5.19** (Theorem 2.12 of [GMS11]). Let V be a finite-dimensional complex vector space and suppose that a connected complex linear algebraic group  $G \subseteq \operatorname{GL}(V)$  defines a linear free divisor D in V. Then G is abelian if and only if D is equivalent, under a change of coordinates in V, to the normal crossings divisor.

*Proof.* Let 
$$n = \dim(G) = \dim(V)$$
.

If D is the normal crossings divisor, then after choosing a basis of V, G is the diagonal group in GL(V), isomorphic to  $(\mathbb{G}_{\mathrm{m}})^n$ . Thus G is abelian.

If G is abelian, then by Corollary 5.8(1), D has  $\dim(G) = n$  irreducible components. By Corollary 5.8(4), G is isomorphic to  $\mathbb{G}_m^n$ , and hence is a maximal torus in  $\mathrm{GL}(V)$ . As all such tori are conjugate in  $\mathrm{GL}(V)$ , we may choose coordinates on V so that G is the group of diagonal matrices. A calculation then shows that D is the normal crossings divisor.

5.6.2. Irreducible linear free divisors. We now examine the groups that produce irreducible linear free divisors. Observe that if  $G \subseteq \operatorname{GL}(V)$  defines a linear free divisor  $D \subset V$ , then by the construction of §5.2 we must have all  $\lambda \cdot I \in G$  for I the identity element and  $\lambda \in \mathbb{C}^*$ .

Recall that an algebraic group H is called *perfect* if [H, H] = H. For instance, semisimple groups are perfect.

**Corollary 5.20.** Let  $D \subset V$  be a linear free divisor defined by the group  $G \subseteq \operatorname{GL}(V)$ . Let  $H = G \cap \operatorname{SL}(V)$ , and let  $K = (\mathbb{C}^*) \cdot I \subseteq G$ . The following are equivalent:

- (1) D has 1 irreducible component;
- (2)  $H^0 = [G, G];$
- (3)  $G = K \cdot [G, G];$
- (4)  $H^0$  is perfect;
- (5) There exists a perfect connected codimension 1 algebraic subgroup J of G. When these hold,  $J = [G, G] = H^0$ .

*Proof.* Clearly  $G = K \cdot H$ , and there are a finite number of ways to write  $g \in G$  as a product of elements of K and H. It follows that  $\dim(H) = n-1$ . The multiplication morphism  $K \times H^0 \to G$  has a connected image of dimension  $\dim(G)$ , and hence  $G = K \cdot H^0$ .

If (1), then by Corollary 5.8(1),  $\dim([G,G]) = n-1$ . Since  $[G,G] \subseteq H$  are of the same dimension and [G,G] is connected, we have  $[G,G] = H^0$  and (2).

If (2), then by the above work,  $G = K \cdot [G, G]$ , giving (3).

Suppose (3). Since  $[G,G] \subseteq H$  we have  $\dim([G,G]) \le n-1$ . Also,  $\dim(G) \le \dim(K) + \dim([G,G])$  shows  $\dim([G,G]) \ge n-1$ , and hence  $\dim([G,G]) = n-1$ . Since they are connected and of the same dimension,  $[G,G] = H^0$ . Since K is in the center of G, we have [KN,KM] = [N,M] for any subgroups N and M of G. In particular,

$$H^0 = [G, G] = [K[G, G], K[G, G]] = [[G, G], [G, G]] = [H^0, H^0],$$

giving (4).

If (4), then since  $\dim(H^0) = n - 1$ , we have (5).

If (5), then since  $J = [J, J] \subseteq [G, G] \subseteq H$  and  $\dim(H) = n - 1$ , we have  $\dim([G, G]) = n - 1$ . By Corollary 5.8(1), D has 1 irreducible component, proving (1). Finally, note that  $J \subseteq [G, G] \subseteq H^0$  are all connected algebraic groups of the same dimension, and hence equal.

**Remark 5.21.** The case when H is semisimple was thoroughly explored in [GMS11]; by the Levi decomposition of G, this is equivalent to  $\dim(\mathbf{Z}(L)) = 1$  and  $\mathrm{Rad}_u(G) = \{e\}$ . Are there linear free divisors with H perfect and  $\mathrm{Rad}_u(G) \neq \{e\}$ ? Since an irreducible representation that is a prehomogeneous vector space has H semisimple by [Kim03, Theorem 7.21], in such an example the representation must be reducible.

5.6.3. Reductive groups. For reductive groups, we have the following.

**Corollary 5.22** ([GMS11, Lemma 2.6]). For a linear free divisor D defined by a reductive group G, the number of irreducible components of D equals the dimension of the center of G. Let H be the subgroup of G leaving invariant the level sets of the product  $f_1 \cdots f_r$ . Then D is irreducible if and only if  $H^0$  is semisimple.

*Proof.* Let G have a Levi decomposition. Since  $\operatorname{Rad}_u(G)$  is trivial and G is itself a Levi subgroup, apply Corollary 5.12(1) to get the first statement.

If D is irreducible, then r=1 and so by Corollary 5.8(3),  $H=[G,G]\cdot G_{v_0}$ . In particular,  $H^0=[G,G]$ , and this is semisimple by the structure theory.

Conversely, suppose  $H^0$  is semisimple. Since  $H = \ker(\chi_1 \cdots \chi_r)$ , by Theorem 5.6(1),  $H^0$  has codimension 1 in G. Since  $H^0$  is perfect, by Corollary 5.20, D is irreducible.

Granger–Mond–Schulze also show [GMS11, Theorem 2.7] that for a linear free divisor  $D \subset V$  defined by a reductive group G, the number of irreducible hypersurface components of D equals the number of irreducible G-modules in V.

Example 5.23. Let  $D \subset V$  be a linear free divisor constructed from a quiver Q, a finite connected oriented graph with vertex set  $Q_0$ , edge set  $Q_1$ , and a dimension vector  $d: Q_0 \to \mathbb{N}$  (see [GMNRS09, BM06, GMS11]). Here, the group G is a product over  $Q_0$  of general linear groups, so G is reductive with  $\dim(\mathbb{Z}(G)) = |Q_0|$ , and V is the space of representations of (Q, d). When d is a real Schur root and Q has no oriented cycles, then G has an open orbit and a theorem of Kac states that the complement has  $|Q_0| - 1$  irreducible hypersurface components ([GMNRS09, §4]). The apparent disagreement with Corollary 5.22 is resolved by observing that G does not define D (in the sense of Definition 5.1) as the representation  $\rho$  of G has a 1-dimensional kernel contained in  $\mathbb{Z}(G)$ , whereas  $\rho(G)$  defines D and is reductive with center of dimension  $|Q_0| - 1$ .

5.6.4. Solvable groups. Recall that by the Lie–Kolchin Theorem, any solvable linear algebraic group  $G \subset GL(V)$  has a basis of V in which G is lower triangular.

**Corollary 5.24.** Let  $D \subset V$  be a linear free divisor defined by a solvable group  $G \subseteq \operatorname{GL}(V)$ . Fix any basis which makes G lower triangular, and let  $\phi : G \to (\mathbb{G}_m)^{\dim(V)}$  send g to the diagonal entries of g. Then

- (1) The number of components of D equals the dimension of the maximal torus of G, and also  $\dim(\phi(G))$ .
- (2) D has a hyperplane component.
- (3) Let  $G_u$  be the subgroup consisting of unipotent elements of G. Then  $[G, G] = G_u = \ker(\phi)$ .
- (4) Every  $\chi \in X_1(G)$  factors through  $\phi$ .

*Proof.* At first we proceed without the hypothesis that G defines a linear free divisor. Since G is connected and solvable, the Levi decomposition of G has  $\operatorname{Rad}(G) = G$ ,  $\operatorname{Rad}_u(G) = G_u$ , L is a maximal torus of G, and in this case  $[G,G] \subseteq G_u$ . Note that  $\phi$  is a homomorphism of linear algebraic groups, and by the definition of unipotent,  $G_u = \ker(\phi)$ . Then since  $\dim(G) = \dim(G_u) + \dim(L) = \dim(\ker(\phi)) + \dim(L)$ , we have

(5.4) 
$$\dim(L) = \dim(\phi(G)).$$

Now let  $\chi \in X(G)$ . By the Jordan decomposition,  $G_u \subseteq \ker(\chi)$ , and hence  $\ker(\phi) \subseteq \ker(\chi)$ . If  $\overline{\chi}$  and  $\overline{\phi}$  are the induced homomorphisms on  $G/G_u$ , then the homomorphism  $\lambda = \overline{\chi} \circ (\overline{\phi})^{-1} : \phi(G) \to \mathbb{G}_m$  satisfies  $\chi = \lambda \circ \phi$ . Since  $\lambda$  is a character on a subtorus of  $(\mathbb{G}_m)^{\dim(V)}$ , by [Bor91, 8.2]  $\lambda$  extends to a character  $\psi : (\mathbb{G}_m)^{\dim(V)} \to \mathbb{G}_m$  with  $\chi = \psi \circ \phi$ . This proves (4).

Now assume that G defines a linear free divisor D. By Corollary 5.12(1), the number of components of D equals  $\dim(\mathbf{Z}(L))$ . Then the observation that  $L = \mathbf{Z}(L)$  and (5.4) implies (1). By Corollary 5.8(5) we have  $G_u \subseteq [G, G]$ , and hence  $[G, G] = G_u = \ker(\phi)$ , proving (3).

Finally, the Lie–Kolchin Theorem guarantees an invariant complete flag in V, hence an invariant hyperplane H. As H cannot intersect  $\Omega$ , H is a component of D, proving (2). In these lower-triangular coordinates, this H is defined by the "first coordinate".

**Remark 5.25.** By (4), the characters corresponding to the basic relative invariants are functions of the diagonal entries.

5.7. **Degrees of the components.** Let G define a linear free divisor D with basic relative invariants  $f_1, \ldots, f_r$ . We have seen that r may be computed from the Lie algebra  $\mathfrak{g}$  of G; may the degrees of the  $f_i$  be computed from  $\mathfrak{g}$ ?

If we only use the abstract Lie algebra structure of  $\mathfrak{g}$ , then the answer is no:

**Example 5.26.** Consider the following two linear free divisors in  $\mathbb{C}^5$ :

$$D_1: \quad (x_3x_5 - x_4^2) \begin{vmatrix} 0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{vmatrix} = 0,$$
 and  $D_2: \quad (x_2^2x_3^2 - 4x_1x_3^3 - 4x_2^3x_4 + 18x_1x_2x_3x_4 - 27x_4^2x_1^2)x_5 = 0.$ 

This  $D_1$  is [DP12b, Example 9.4], while  $D_2$  is the product-union of a hyperplane with [GMS11, Theorem 2.11(2)]. The degrees of the polynomials defining the irreducible components of  $D_1$  and  $D_2$  differ. However, the groups which define  $D_1$  and  $D_2$  have the same abstract Lie algebra structure:  $\mathfrak{gl}_2(\mathbb{C}) \oplus \mathfrak{gl}_1(\mathbb{C})$ . Thus,  $D_1$  and  $D_2$  are constructed from inequivalent representations of the same abstract Lie algebra.

Of course, since the representation of the Lie algebra determines the divisor, the representation undoubtedly contains the information necessary to compute these degrees. How may we do so effectively?

**Lemma 5.27.** Let  $G \subseteq GL(V)$  define the linear free divisor D with basic relative invariants  $f_1, \ldots, f_r$ . Choose  $X_i \in \mathfrak{g} \subseteq \mathfrak{gl}(V)$  such that  $d(\chi_i)_{(e)}(X_j) = \delta_{ij}$ , or equivalently,  $\xi_{X_i}(f_j) = \delta_{ij}f_j$ . Let I denote the identity endomorphism in both  $G \subseteq GL(V)$  and  $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ , with  $\xi_I$  the Euler vector field. Let the module  $A \subseteq Der_V$  consist of the vector fields annihilating all  $f_i$ . Then

(5.5) 
$$\xi_I(f_i) = \deg(f_i)f_i,$$

$$(5.6) d(\chi_i)_{(e)}(I) = \deg(f_i),$$

(5.7) 
$$\chi_i(\lambda \cdot I) = \lambda^{\deg(f_i)} \quad \text{for all } \lambda \in \mathbb{C}^*,$$

(5.8) 
$$I = \sum \deg(f_j) X_j \bmod [\mathfrak{g}, \mathfrak{g}],$$

(5.9) 
$$and \xi_I = \sum \deg(f_j) \xi_{X_j} \bmod A.$$

*Proof.* Corollary 5.10 shows that such  $X_i$  exist and its proof includes the useful identity (5.2). By (5.2), the  $\xi_{X_i}$  have the claimed effects on  $f_j$  for all i, j. Then (5.5) is just the Euler relation, and applying (5.2) shows (5.6). Integrating (5.6) shows (5.7). By (5.6), for each i we have

$$d(\chi_i)_{(e)} \left( I - \sum_j \deg(f_j) X_j \right) = \deg(f_i) - \deg(f_i) = 0,$$

and since  $\cap_{i=1}^r \ker(d(\chi_i)_{(e)}) = [\mathfrak{g}, \mathfrak{g}]$  by Theorem 5.6(1), we have (5.8). A similar argument using (5.5) shows that  $\xi_I - \sum \deg(f_j)\xi_{X_j}$  annihilates each  $f_i$ , giving (5.9).

It is unclear whether the embedding  $\mathfrak{g} \subseteq \mathfrak{gl}(V)$  may be used to find the degrees without first finding either the  $\chi_i$ ,  $d(\chi_i)_{(e)}$ , or  $f_i$ .

**Remark 5.28.** For  $d \in \mathbb{N}$  let  $R_d \subseteq \mathbb{G}_m$  denote the group of dth roots of unity. Then in the situation of Lemma 5.27, (5.7) implies that

$$\ker(\chi_i|_{\mathbb{G}_{\mathrm{m}}\cdot I}) = \ker(\chi_i) \cap \mathbb{G}_{\mathrm{m}} \cdot I = R_{\deg(f_i)} \cdot I.$$

Then by Theorem 5.6(1) and number theory,  $([G, G] \cdot G_{v_0}) \cap \mathbb{G}_{\mathrm{m}} \cdot I = R_d \cdot I$  for d the greatest common divisor of  $\{\deg(f_1), \ldots, \deg(f_r)\}$ .

5.8. Homotopy groups of  $V \setminus D$ . The results above can give some insight into the topology of the complement of a linear free divisor D and two types of (global) Milnor fiber associated to D.

**Proposition 5.29.** Let  $G \subseteq \operatorname{GL}(V)$  define a linear free divisor  $D \subset V$  with irreducible components defined by  $f_1, \ldots, f_r$ , and  $f_i \stackrel{\operatorname{m}}{\longleftrightarrow} \chi_i$ . Let L be a Levi factor of G. Let  $v_0 \in \Omega$ , let  $F = (f_1, \ldots, f_r) : V \to \mathbb{C}^r$ , let K be the fiber of  $f_1 \cdots f_r$  containing  $v_0$ , and let  $P \subset K$  be the fiber of F containing  $v_0$ . Use  $v_0$  or  $e \in G$  as the base point for all homotopy groups. Then for n > 1,

$$\pi_n(L) \cong \pi_n([L,L]) \cong \pi_n(G) \cong \pi_n([G,G]) \cong \pi_n(P) \cong \pi_n(\Omega) \cong \pi_n(K),$$
  
and for  $n=1$  we have the exact diagrams in Figure 1.

*Proof.* If  $G_1 \subset G_2 \subset G_3$  are Lie groups, then by [Ste51, §7.4–7.5] the map of cosets  $G_3/G_1 \to G_3/G_2$  is a fiber bundle with fiber  $G_2/G_1$ . Thus, we have the following fiber bundles:

$$p_{1}: G \to G/G_{v_{0}} \cong \Omega$$
 
$$p_{2}: [G, G] \to [G, G]/([G, G] \cap G_{v_{0}})$$
 
$$p_{3}: G \to G/\operatorname{Rad}_{u}(G) \cong L$$
 
$$p_{4}: [G, G] \to [G, G]/\operatorname{Rad}_{u}(G) \cong [L, L]$$
 
$$p_{5}: G \to G/[G, G] \cong \mathbb{G}_{m}^{r}$$
 
$$p_{6}: L \to L/[L, L] \cong \mathbb{G}_{m}^{r}$$
 
$$q: G/G_{v_{0}} \to G/[G, G] \cdot G_{v_{0}}$$

The identifications of the codomains of  $p_1$ ,  $p_3$ ,  $p_4$ ,  $p_5$ , and  $p_6$  are by, respectively, the orbit map  $\alpha_{v_0}$  of  $v_0$ , the definition of the Levi decomposition, Corollary 5.12(2), Corollary 5.8(4), and the isomorphism

$$L/[L, L] \cong (G/\operatorname{Rad}_u(G))/([G, G]/\operatorname{Rad}_u(G)) \cong G/[G, G]$$

that follows from the Levi decomposition and Corollary 5.12(2).

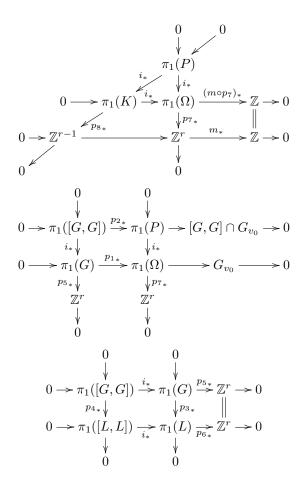
Since  $G_{v_0}$  and  $[G, G] \cap G_{v_0}$  are finite,  $p_1$  and  $p_2$  are covering spaces with deck transformation groups isomorphic to  $G_{v_0}$  and  $[G, G] \cap G_{v_0}$ , respectively [Hat02, Proposition 1.40].

For q, we use the orbit map to identify  $G/G_{v_0}$  with  $\Omega$  and then by Theorem 5.6(1) compose with the isomorphism  $G/[G,G]\cdot G_{v_0}\to (\mathbb{G}_{\mathrm{m}})^r$  induced by  $(\chi_1,\ldots,\chi_r):G\to (\mathbb{G}_{\mathrm{m}})^r$ . This gives a fiber bundle

$$p_7:\Omega\to(\mathbb{G}_{\mathrm{m}})^r$$

defined by  $p_7(\rho(g)(v_0)) = (\chi_1, \dots, \chi_r)(g)$  with fiber  $\ker(\chi_1, \dots, \chi_r)/G_{v_0} \cong [G, G] \cdot G_{v_0}/G_{v_0}$ . As the action of [G, G] on  $[G, G] \cdot G_{v_0}/G_{v_0}$  is smooth and transitive,

FIGURE 1. The exact diagrams describing the fundamental groups of the spaces in Proposition 5.29. For a map  $\phi$ , let  $\phi_*$  denote the associated map of fundamental groups. Each i is constructed from an inclusion map. We do not claim commutativity of these diagrams because of the many possible identifications of  $\pi_1(\mathbb{G_m}^k) \cong \mathbb{Z}^k$  and, e.g.,  $G/[G,G] \cong \mathbb{G_m}^r$ .



and the isotropy subgroup at  $eG_{v_0}$  is  $[G,G] \cap G_{v_0}$ , the fiber of  $p_7$  is isomorphic to  $[G,G]/([G,G] \cap G_{v_0})$ .

Let  $m: (\mathbb{G}_{\mathrm{m}})^r \to \mathbb{G}_{\mathrm{m}}$  and  $n: (\mathbb{C}^*)^r \to \mathbb{C}^*$  both be defined by  $(a_1, \ldots, a_r) \mapsto a_1 \cdots a_r$ . By (1.1) we have a commutative diagram

$$\begin{array}{ccc}
(\mathbb{G}_{\mathbf{m}})^r & \xrightarrow{m} \mathbb{G}_{\mathbf{m}} \\
\downarrow^{p_7} & & \downarrow^{\gamma} \\
& & \downarrow^{\gamma} \\
& & \downarrow^{\gamma} \\
& & & \downarrow^{\gamma} \\
& & & \downarrow^{\gamma} \\
& & & & \downarrow^{\gamma}
\end{array}$$

where  $\beta(a_1, \ldots, a_r) = (a_1 f_1(v_0), \ldots, a_r f_r(v_0))$  and  $\gamma(a) = a(f_1 \cdots f_r)(v_0)$  are homeomorphisms. By the diagram, P is homeomorphic to the fiber of  $p_7$ , that is,  $P \cong [G, G]/([G, G] \cap G_{v_0})$ , and hence P is the codomain of  $p_2$ . Also, K is homeomorphic to  $(m \circ p_7)^{-1}(1)$ ; restricting  $p_7$  gives a fiber bundle

$$p_8: K \to \ker(m) \cong (\mathbb{G}_{\mathrm{m}})^{r-1}$$

with fiber P.

By [Hat02, Proposition 4.48], a fiber bundle is a Serre fibration, that is, it possesses the homotopy lifting property for CW complexes, and these have the usual homotopy long exact sequence of a fibration [Hat02, Theorem 4.41]; apply this sequence to all  $p_i$  and  $m \circ p_7$ . Note that the connected unipotent group  $\operatorname{Rad}_u(G)$  is diffeomorphic to some  $\mathbb{C}^p$ , and hence is contractible [OV90, §3.3.6]. Also,  $G_{v_0}$  and  $[G,G] \cap G_{v_0}$  are finite,  $P \cong [G,G]/[G,G] \cap G_{v_0}$  is connected,  $\pi_1((\mathbb{G}_m)^k) \cong \mathbb{Z}^k$ , and  $\pi_i((\mathbb{G}_m)^k) \cong 0$  for i > 1.

The long exact sequences show that for n > 1 there is a diagram, not necessarily commutative,

$$\pi_n([L,L]) \stackrel{p_4}{\leftarrow} \pi_n([G,G]) \stackrel{p_2}{\rightarrow} \pi_n(P) \stackrel{p_8}{\rightarrow} \pi_n(K)$$

$$\downarrow^{p_6} \downarrow \qquad \qquad \downarrow^{p_5} \downarrow \qquad \qquad \downarrow^{p_7} \downarrow \qquad \downarrow^{m \circ p_7}$$

$$\pi_n(L) \stackrel{p_3}{\longleftarrow} \pi_n(G) \stackrel{p_1}{\longrightarrow} \pi_n(\Omega)$$

where an arrow labeled by  $\phi$  represents an isomorphism occurring in the long exact sequence of the fibration  $\phi$ , either  $\phi_*$  or  $i_*$  for i the inclusion of the fiber. The sequences also give the exact diagrams in Figure 1.

Thus, the homotopy groups may largely be computed from the homotopy groups of the semisimple part [L, L] of G. For instance, if G is solvable then  $[L, L] = \{e\}$  and hence by Proposition 5.29, the spaces  $\Omega$ , P, and K are  $K(\pi, 1)$ 's, as shown in [DP12a].

**Remark 5.30.** For the cohomology  $H^k(\Omega; F)$ , consider the covering space

$$\widetilde{G} = \operatorname{Rad}_u(G) \rtimes \left(\operatorname{Z}(L)^0 \times [L,L]\right) \xrightarrow{\quad \phi \quad} G \xrightarrow{\quad \alpha \quad} G/G_{v_0} \cong \Omega \ ,$$

where  $\phi$  is an  $n=|\mathbf{Z}(L)^0\cap[L,L]|$ -fold covering given by multiplication, and  $\alpha$  is an  $m=|G_{v_0}|$ -fold covering. If F is a field of characteristic 0 or a prime not dividing nm, then by [Hat02, Proposition 3G.1], each  $H^k(\Omega;F)$  is isomorphic to the  $\phi^{-1}(G_{v_0})$ -invariant classes in  $H^k(\widetilde{G};F)$ . Note that  $\widetilde{G}$  is a product topologically. See also [GMNRS09, Theorem 1.6] and [CJNMM96].

5.9. (Non-linear) free divisors. Let (D,p) be an arbitrary free divisor in  $(\mathbb{C}^n,p)$ . May the number of components of (D,p) be computed from the structure of  $M=\operatorname{Der}_{\mathbb{C}^n,p}(-\log D)$ ? One natural candidate,  $\dim_{\mathbb{C}}(M/\mathscr{O}_{\mathbb{C}^n,p}\cdot[M,M])$ , does not work. For example, if D is a hyperplane in  $\mathbb{C}^2$ , then the number computed is 0; in fact,  $M=\mathscr{O}_{\mathbb{C}^n,p}\cdot[M,M]$  whenever  $\operatorname{Der}_{\mathbb{C}^n,p}(-\log D)\nsubseteq\mathscr{M}_p\cdot\operatorname{Der}_{\mathbb{C}^n,p}$  for  $\mathscr{M}_p$  the maximal ideal. Other examples give answers too large: for the plane curve  $D=V((a^2-b^3)(a^7-b^{13}))$ , the number computed is 35.

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