ADDITIVE RELATIVE INVARIANTS AND THE COMPONENTS OF A LINEAR FREE DIVISOR

BRIAN PIKE

Abstract. A *prehomogeneous vector space* is a rational representation $\rho : G \to \mathrm{GL}(V)$ of a connected complex linear algebraic group G that has a Zariski open orbit $\Omega \subset V$. Mikio Sato showed that the hypersurface components of $D := V \setminus \Omega$ are related to the rational characters $H \to \mathrm{GL}(\mathbb{C})$ of H, an algebraic abelian quotient of G. Mimicking this work, we investigate the *additive functions* of H, the homomorphisms $\Phi : H \to (\mathbb{C}, +)$. Each such Φ is related to an *additive relative invariant*, a rational function h on V such that $h \circ \rho(g) - h = \Phi(g)$ on Ω for all $g \in G$. Such an h is homogeneous of degree 0, and helps describe the behavior of certain subsets of D under the G–action.

For those prehomogeneous vector spaces with D a type of hypersurface called a linear free divisor, we prove there are no nontrivial additive functions of H, and hence H is an algebraic torus. From this we gain insight into the structure of such representations and prove that the number of irreducible components of D equals the dimension of the abelianization of G. For some special cases (G abelian, reductive, or solvable, or D irreducible) we simplify proofs of existing results. We also examine the homotopy groups of $V \setminus D$.

Contents

Introduction

A *prehomogeneous vector space* is a rational representation $\rho : G \to \mathrm{GL}(V)$ of a connected complex linear algebraic group G that has a (unique) Zariski open orbit Ω in V. These representations have been much studied from the viewpoint of number theory and harmonic analysis (e.g., [Kim03]). A particularly useful tool is to consider the rational *characters* $\chi : H \to \mathbb{G}_{\mathrm{m}} := \mathrm{GL}(\mathbb{C})$ of the abelian quotient $H := G/[G,G] \cdot G_{v_0}$, where G_{v_0} is the isotropy subgroup at any $v_0 \in \Omega$. There is a correspondence between these characters and 'relative invariants'; a *relative*

Date: December 16, 2014.

2010 *Mathematics Subject Classification.* 11S90 (Primary); 20G20, 32S25, 17B66 (Secondary).

Key words and phrases. prehomogeneous vector space, relative invariant, additive relative invariant, linear algebraic group, linear free divisor, free divisor, homotopy group.

invariant h is a rational function on V such that on Ω, h is holomorphic and $h \circ \rho$ is merely h multiplied by a character of G (see (1.1)). The relative invariants are constructed from the irreducible polynomials defining the hypersurface components of the algebraic set $V \setminus \Omega$. Thus, for example, the number of hypersurface components of $V \setminus \Omega$ equals the rank of the free abelian group of characters of H.

In this paper we study the *additive functions* of H, the rational homomorphisms $\Phi : H \to \mathbb{G}_{\mathrm{a}} := (\mathbb{C}, +)$. This complements the classical study of the characters, as up to isomorphism $\mathbb{G}_{\mathrm{m}}$ and $\mathbb{G}_{\mathrm{a}}$ are the only 1-dimensional connected complex linear algebraic groups. Since H is connected and abelian, $H \cong {\mathbb{G}_{\mathrm{m}}}^k \times {\mathbb{G}_{\mathrm{a}}}^\ell$ with k the rank of the character group of H and ℓ the dimension of the vector space of additive functions of H; the characters and additive functions thus describe H completely. Of particular interest for us, the number of irreducible hypersurface components of $V \setminus \Omega$ equals

$$k = \dim(H) - \ell \tag{0.1}$$

(see Corollary 2.3), where $\dim(H)$ is easily computable.

The content of the paper is as follows. After reviewing the basic properties of prehomogeneous vector spaces in §1 and abelian linear algebraic groups in §2, we study the additive functions of H in §3. Just as the characters of H are related to relative invariants, we show that the additive functions of H are related to 'additive relative invariants' (see Definition 3.1). By Proposition 3.4, a rational function k on V is an *additive relative invariant* if and only if $k(\rho(g)(v)) - k(v)$ is independent of $v \in \Omega$ for all $g \in G$, and then this difference gives the additive function $G \to \mathbb{G}_{\mathrm{a}}$ associated to k. These additive relative invariants are homogeneous of degree 0 and of the form $\frac{h}{f}$ for a polynomial relative invariant f (Proposition 3.5). Geometrically, an additive relative invariant $\frac{h}{f}$ describes a G–invariant subset $f = h = 0$ of $f = 0$, with G permuting the sets

$$V_\epsilon = \{x : f(x) = h(x) - \epsilon = 0\}, \qquad \epsilon \in \mathbb{C},$$

by $\rho(g)(V_\epsilon) = V_{\chi(g)\cdot\epsilon}$ for χ the character corresponding to the relative invariant f (see Proposition 3.8); the converse also holds if $\deg(\frac{h}{f}) = 0$ and f is reduced. In §3.6 we investigate which characters and additive functions vanish on which isotropy subgroups, a key ingredient for §5. In §3.7 we establish the algebraic independence of a generating set of relative invariants and the numerators of a basis of the additive relative invariants. Finally, in §3.8 we conjecture that every additive relative invariant may be written as a sum of additive relative invariants of the form $\frac{h_i}{f_i}$, where f_i is a power of an irreducible polynomial relative invariant. We prove this splitting behavior when the additive relative invariant may be written as a sum of appropriate fractions.

In §4 we describe some examples of prehomogeneous vector spaces and their additive relative invariants.

In §5, we study prehomogeneous vector spaces with the property that $D := V \setminus \Omega$ is a type of hypersurface called a *linear free divisor*. Such objects have been of much interest recently (e.g., [BM06, GMNRS09, GMS11, DP12a, DP12b]), and were our original motivation. Using a criterion due to Brion and the results of §3.6, we show in Theorem 5.6 that such prehomogeneous vector spaces have no nontrivial additive relative invariants or nontrivial additive functions. Then by (0.1) the number of irreducible components of D equals $\dim(H)$, and this may be easily computed using only the Lie algebra $\mathfrak{g}$ of G (Theorem 5.8; see also Remark 5.11). We also gain insight into the structure of G and the behavior of the G–action (§§5.3, 5.4, 5.5). In §5.6, we study the special cases where G is abelian, reductive, or solvable, or where D is irreducible, simplifying proofs of several known results. In §5.7 we observe that, although the number of components of a linear free divisor is computable from the abstract Lie algebra structure $\mathfrak{g}$ of G, the degrees of the polynomials defining these components are not, and seem to require the representation $\mathfrak{g} \to \mathfrak{gl}(V)$. Lemma 5.27 can compute these degrees, but requires data that are themselves difficult to compute. Finally, in §5.8 we use our earlier results to study the homotopy groups of the complement of a linear free divisor.

Acknowledgements: We would like to thank Mathias Schulze for very helpful discussions that led to this work. He deserves much of the credit for Corollary 5.24. We also thank Christian Barz for finding an error in an example.

1. Prehomogeneous vector spaces

We begin by briefly reviewing prehomogeneous vector spaces. Our reference is [Kim03], although the results we describe were developed by M. Sato in the 1960's ([Sat90]).

In the whole article, we shall study only complex linear algebraic groups. For such a K, let K^0 denote the connected component of K containing the identity, $L(K)$ the Lie algebra of K, and K_v the isotropy subgroup at v of a particular K–action.

Let V be a finite-dimensional complex vector space, and G a connected complex linear algebraic group. Let $\rho : G \to \mathrm{GL}(V)$ be a rational representation of G, i.e., a homomorphism of linear algebraic groups. When G has an open orbit Ω in V, then we call (G, ρ, V) a *prehomogeneous vector space*. Then Ω is unique and Zariski open, so that the complement $\Omega^c = V \setminus \Omega$ is an algebraic set in V. We call Ω^c the *exceptional orbit variety* as it is the union of the non-open orbits of G; others use *discriminant* or *singular set*.

One of the basic theorems of prehomogeneous vector spaces is that the hypersurface components of Ω^c may be detected from certain multiplicative characters of G. More precisely, for any complex linear algebraic group K let $X(K)$ denote the set of rational (multiplicative) *characters*, that is, the homomorphisms $K \to \mathbb{G}_{\mathrm{m}} := \mathrm{GL}(\mathbb{C})$ of linear algebraic groups. Then a rational function f on V that is not identically 0 is a *relative invariant* if there exists a $\chi \in X(G)$ such that

$$f(\rho(g)(v)) = \chi(g) \cdot f(v) \tag{1.1}$$

for all $v \in \Omega$ and $g \in G$, in which case we[1] write $f \xleftrightarrow{\mathrm{m}} \chi$. By (1.1), the zeros and poles of f may occur only on Ω^c.

Actually, f and χ provide almost the same information when $f \xleftrightarrow{\mathrm{m}} \chi$. Using f, we may choose any $v_0 \in \Omega$ and then recover χ by defining

$$\chi(g) = \frac{f(\rho(g)(v_0))}{f(v_0)} \in (\mathbb{C}^*, \cdot) \cong \mathbb{G}_{\mathrm{m}}, \qquad g \in G.$$

Conversely, using χ we may recover a nonzero constant multiple h of f by choosing any $v_0 \in \Omega$ and any nonzero value $h(v_0) \in \mathbb{C}$, defining h on Ω by $h(\rho(g)(v_0)) =$

[1] Usually this is written $f \longleftrightarrow \chi$, but we shall use similar notation for the relationship between additive functions and additive relative invariants.

$\chi(g)h(v_0)$, and then using the density of Ω to extend h uniquely to a rational function on V.

Let $X_1(G)$ be the set of $\chi \in X(G)$ for which there exists an f with $f \overset{\mathrm{m}}{\longleftrightarrow} \chi$. Then $X_1(G)$ is an abelian group: if $f_i \overset{\mathrm{m}}{\longleftrightarrow} \chi_i$ and $n_i \in \mathbb{Z}$ for $i = 1, 2$, then $(f_1)^{n_1}(f_2)^{n_2} \overset{\mathrm{m}}{\longleftrightarrow} (\chi_1)^{n_1}(\chi_2)^{n_2} \in X_1(G)$. The group $X_1(G)$ contains information about the hypersurface components of Ω^c.

Theorem 1.1 (e.g., [Kim03, Theorem 2.9])**.** *Let (G, ρ, V) be a prehomogeneous vector space with exceptional orbit variety Ω^c, and $S_i = \{x \in V : f_i(x) = 0\}$, $i = 1, \ldots, r$ the distinct irreducible hypersurface components of Ω^c. Then the irreducible polynomials $f_1, \ldots, f_r$ are relative invariants which are algebraically independent, and any relative invariant is of the form $c \cdot (f_1)^{m_1} \cdots (f_r)^{m_r}$ for nonzero $c \in \mathbb{C}$, and $m_i \in \mathbb{Z}$. Moreover, if each $f_i \overset{\mathrm{m}}{\longleftrightarrow} \chi_i$, then $X_1(G)$ is a free abelian group of rank r generated by $\chi_1, \ldots, \chi_r$.*

The (homogeneous) irreducible polynomials $f_1, \ldots, f_r$ of the Theorem are called the *basic relative invariants* of (G, ρ, V).

Remark 1.2. Since $v \in \Omega$ if and only if the corresponding orbit map $g \mapsto \rho(g)(v)$ is a submersion at $e \in G$, and this derivative depends linearly on $v \in V$, the set Ω^c may be defined by an ideal generated by homogeneous polynomials of degree $\dim(V)$ (see §5.1).

There exists another description of $X_1(G)$. Fix an element $v_0 \in \Omega$ with isotropy subgroup G_{v_0}. Since the orbit of v_0 is open, we have $\dim(G_{v_0}) = \dim(G) - \dim(V)$. Define the algebraic groups

$$G_1 = [G, G] \cdot G_{v_0} \qquad \text{and} \qquad H = G/G_1.$$

The group G_1 does not depend on the choice of $v_0 \in \Omega$ as all such isotropy subgroups are conjugate, and $[G, G] \subseteq G_1$. By (1.1), every $\chi \in X_1(G)$ has $G_1 \subseteq \ker(\chi)$, and thus factors through the quotient to give a corresponding $\widetilde{\chi} \in X(H)$. In fact, the map $\chi \mapsto \widetilde{\chi}$ is an isomorphism:

Proposition 1.3 (e.g., [Kim03, Proposition 2.12])**.** *Let (G, ρ, V) be a prehomogeneous vector space with open orbit Ω. Then for any $v_0 \in \Omega$,*

$$X_1(G) \cong X(G/[G, G] \cdot G_{v_0}).$$

Consequently, the rank of the character group $X(H)$ equals the number of irreducible hypersurface components of $V \setminus \Omega$.

2. Abelian complex linear algebraic groups

Let (G, ρ, V) be a prehomogeneous vector space, and use the notation of the previous section with $H = G/[G, G] \cdot G_{v_0}$ for some $v_0 \in \Omega$. To understand the rank of $X(H)$ and thus the number of irreducible hypersurface components of the exceptional orbit variety Ω^c, we must understand H. Since H is abelian and connected, its structure is very simple. Recall that over $\mathbb{C}$ there are exactly two distinct 1-dimensional connected linear algebraic groups, $\mathbb{G}_{\mathrm{m}} = \mathrm{GL}(\mathbb{C}^1) \cong (\mathbb{C}^*, \cdot)$ and $\mathbb{G}_{\mathrm{a}} = (\mathbb{C}, +)$.

Proposition 2.1 (e.g., [OV90, §3.2.5])**.** *An abelian connected complex linear algebraic group K is isomorphic to $(\mathbb{G}_{\mathrm{m}})^k \times (\mathbb{G}_{\mathrm{a}})^\ell$ for nonnegative integers k and ℓ, where the exponents denote a repeated direct product.*

Note that this Proposition is false for some other fields.

For any linear algebraic group K, let $\mathscr{A}(K)$ be the complex vector space of rational homomorphisms $\Phi : K \to \mathbb{G}_{\mathrm{a}}$, sometimes called *additive functions of K* ([Spr98, §3.3]). When K is connected, $\Phi \in \mathscr{A}(K)$ is determined by $d\Phi_{(e)}$, and hence $\mathscr{A}(K)$ is finite-dimensional. When K has the decomposition of Proposition 2.1, the rank of $X(K)$ and the dimension of $\mathscr{A}(K)$ are related to k and ℓ.

Lemma 2.2. *If $K = (\mathbb{G}_{\mathrm{m}})^k \times (\mathbb{G}_{\mathrm{a}})^\ell$, then $X(K)$ is free of rank k and $\dim_{\mathbb{C}}(\mathscr{A}(K)) = \ell$.*

Proof. By the Jordan decomposition, any $\chi \in X(K)$ will have $\{1\}^k \times (\mathbb{G}_{\mathrm{a}})^\ell \subseteq \ker(\chi)$, so χ factors through to an element of $X((\mathbb{G}_{\mathrm{m}})^k)$. Thus $X(K) \cong X((\mathbb{G}_{\mathrm{m}})^k)$, and similarly $\mathscr{A}(K) \cong \mathscr{A}((\mathbb{G}_{\mathrm{a}})^\ell)$. Finally, an easy exercise shows that $\mathrm{rank}(X((\mathbb{G}_{\mathrm{m}})^k)) = k$, and $\dim(\mathscr{A}((\mathbb{G}_{\mathrm{a}})^\ell)) = \ell$. □

Thus, for a prehomogeneous vector space we have:

Corollary 2.3. *Let (G, ρ, V) be a prehomogeneous vector space with open orbit Ω. Let $v_0 \in \Omega$ and $H = G/[G,G] \cdot G_{v_0}$. Then the number of irreducible hypersurface components of Ω^c equals $\dim(H) - \dim(\mathscr{A}(H))$.*

Proof. By Theorem 1.1, Proposition 1.3, Proposition 2.1, and Lemma 2.2, the number of irreducible components equals $\mathrm{rank}(X_1(G)) = \mathrm{rank}(X(H)) = \dim(H) - \dim(\mathscr{A}(H))$. □

It is natural, then, to study these additive functions and their geometric meaning.

3. Additive relative invariants

In this section, we develop for additive functions the analogue of the multiplicative theory from §1. We encourage the reader to consult the examples of §4 as needed. Let (G, ρ, V) be a prehomogeneous vector space with open orbit Ω. Let $v_0 \in \Omega$ and $H = G/[G,G] \cdot G_{v_0}$.

3.1. Definition.

We define additive relative invariants similarly to (multiplicative) relative invariants.

Definition 3.1. A rational function h on V is an *additive relative invariant* of (G, ρ, V) if there exists a $\Phi \in \mathscr{A}(G)$ so that

$$h(\rho(g)(v)) - h(v) = \Phi(g) \tag{3.1}$$

for all $v \in \Omega$ and $g \in G$. In this situation, we write $h \overset{\mathrm{a}}{\longleftrightarrow} \Phi$.

By (3.1), the poles of such an h may occur only on Ω^c, and h is constant on the orbits of $\ker(\Phi)$.

3.2. Basic properties.

We now establish some basic facts about additive relative invariants. First we investigate the uniqueness of the relationship $h \overset{\mathrm{a}}{\longleftrightarrow} \Phi$.

Proposition 3.2. *Let (G, ρ, V) be a prehomogeneous vector space.*

1. *If $h_1 \overset{\mathrm{a}}{\longleftrightarrow} \Phi$ and $h_2 \overset{\mathrm{a}}{\longleftrightarrow} \Phi$, then there exists an $\alpha \in \mathbb{C}$ with $h_1 = \alpha + h_2$.*
2. *If $h \overset{\mathrm{a}}{\longleftrightarrow} \Phi_1$ and $h \overset{\mathrm{a}}{\longleftrightarrow} \Phi_2$, then $\Phi_1 = \Phi_2$.*

Proof. For (1), fix $v_0 \in \Omega$ and let $\alpha = h_1(v_0) - h_2(v_0)$, so $\Phi(g) + h_1(v_0) = \alpha + \Phi(g) + h_2(v_0)$ for all $g \in G$. Applying (3.1) shows that $h_1 = \alpha + h_2$ on Ω, and thus on V.

(2) is immediate from (3.1). $\square$

Let $\mathscr{A}_1(G)$ be the set of $\Phi \in \mathscr{A}(G)$ for which there exists a rational function h on V with $h \xleftrightarrow{\mathrm{a}} \Phi$. We now identify the additive functions in $\mathscr{A}_1(G)$, analogous to Proposition 1.3.

Proposition 3.3. *As vector spaces, $\mathscr{A}_1(G) \cong \mathscr{A}(H)$, where $H = G/[G,G] \cdot G_{v_0}$ and $v_0 \in \Omega$.*

Proof. Let $\Phi \in \mathscr{A}_1(G)$ with $h \xleftrightarrow{\mathrm{a}} \Phi$. Evaluating (3.1) at $v_0 \in \Omega$ and $g \in G_{v_0}$ shows that $G_{v_0} \subseteq \ker(\Phi)$. Since $\mathbb{G}_\mathrm{a}$ is abelian, $[G,G] \subseteq \ker(\Phi)$. Thus any $\Phi \in \mathscr{A}_1(G)$ factors through the quotient $\pi : G \to H$ to a unique $\overline{\Phi} \in \mathscr{A}(H)$, with $\Phi = \overline{\Phi} \circ \pi$. The map $\rho : \mathscr{A}_1(G) \to \mathscr{A}(H)$ defined by $\rho(\Phi) = \overline{\Phi}$ is $\mathbb{C}$-linear.

Conversely, define the $\mathbb{C}$-linear map $\sigma : \mathscr{A}(H) \to \mathscr{A}(G)$ by $\sigma(\overline{\Phi}) = \overline{\Phi} \circ \pi$. For $\overline{\Phi} \in \mathscr{A}(H)$, we have $G_{v_0} \subseteq \ker(\overline{\Phi} \circ \pi)$, and hence we may define a function $\overline{h} : \Omega \to \mathbb{C}$ by $\overline{h}(\rho(g)(v_0)) = (\overline{\Phi} \circ \pi)(g)$. By the argument[2] of [Kim03, Proposition 2.11], $\overline{h}$ is a regular function on Ω that may be extended to a rational function h on V. By construction, $h(\rho(g)(v)) - h(v) = (\overline{\Phi} \circ \pi)(g)$ for $v = v_0$ and all $g \in G$, but this implies that this equation holds for all $v \in \Omega$ and all $g \in G$. Thus $h \xleftrightarrow{\mathrm{a}} \overline{\Phi} \circ \pi$, and $\sigma : \mathscr{A}(H) \to \mathscr{A}_1(G)$.

Finally, check that ρ and σ are inverses using the uniqueness of the factorization through π. $\square$

Additive relative invariants are the rational functions on V having the following property.

Proposition 3.4. *Let $h : V \to \mathbb{C}$ be a rational function on V, holomorphic on Ω. Then h is an additive relative invariant if, and only if, for all $g \in G$,*

$$h(\rho(g)(v)) - h(v) \text{ is independent of } v \in \Omega.$$

Proof. By (3.1), additive relative invariants have this property.

Conversely, define $\Phi : G \to \mathbb{C}$ by $\Phi(g) = h(\rho(g)(v)) - h(v)$ for any $v \in \Omega$. As a composition of regular functions, Φ is regular on G. Let $g_1, g_2 \in G$, and $v \in \Omega$. Then

$$\begin{aligned}
\Phi(g_1 g_2^{-1}) &= h\big(\rho(g_1 g_2^{-1})(v)\big) - h(v) \\
&= h\big(\rho(g_1)(\rho(g_2^{-1})(v))\big) - h\big(\rho(g_2^{-1})(v)\big) + h\big(\rho(g_2^{-1})(v)\big) - h(v) \\
&= \Big(h(\rho(g_1)(v_1)) - h(v_1)\Big) + \Big(h(v_1) - h(\rho(g_2)(v_1))\Big),
\end{aligned}$$

where $v_1 = \rho(g_2^{-1})(v) \in \Omega$. By our hypothesis, $\Phi(g_1 g_2^{-1}) = \Phi(g_1) - \Phi(g_2)$. Thus $\Phi : G \to \mathbb{G}_\mathrm{a}$ is a homomorphism of algebraic groups, and since $h \xleftrightarrow{\mathrm{a}} \Phi$ we have $\Phi \in \mathscr{A}_1(G)$. $\square$

[2]This uses the fact that $\mathbb{C}$ has characteristic 0.

3.3. **Homogeneity.** Relative invariants are always homogeneous rational functions on V because ρ is a linear representation. Similarly, additive relative invariants are homogeneous of degree 0.

Proposition 3.5. *If $h \overset{\mathrm{a}}{\longleftrightarrow} \Phi$ and h is not the zero function, then h may be written as $h = \frac{h_1}{f_1}$, where h_1 and f_1 are homogeneous polynomials with $\deg(h_1) = \deg(f_1)$, h_1 and f_1 have no common factors, and f_1 is a relative invariant.*

Proof. First note that by Remark 1.2, for any $t \in \mathbb{C}^*$, we have $v \in \Omega$ if and only if $t \cdot v \in \Omega$; we use this fact implicitly in the rest of the proof since Definition 3.1 describes the behavior of h only on Ω.

Let $t \in \mathbb{C}^*$, and define the rational function h_t on V by $h_t(v) = h(t \cdot v)$. Since ρ is a linear representation,

$$h_t(\rho(g)(v)) = h(t \cdot \rho(g)(v)) = h(\rho(g)(t \cdot v)). \tag{3.2}$$

Applying (3.1) to (3.2) gives

$$h_t(\rho(g)(v)) = \Phi(g) + h(t \cdot v) = \Phi(g) + h_t(v). \tag{3.3}$$

Since (3.3) holds for all $g \in G$ and $v \in \Omega$, we have $h_t \overset{\mathrm{a}}{\longleftrightarrow} \Phi$.

By Proposition 3.2(1), there exists a function $\alpha : \mathbb{C}^* \to \mathbb{C}$ such that

$$h(t \cdot v) = h(v) + \alpha(t) \tag{3.4}$$

for all $t \in \mathbb{C}^*$ and $v \in \Omega$. If $s, t \in \mathbb{C}^*$ and $v \in \Omega$, then using (3.4) repeatedly shows

$$h(v) + \alpha(st) = h(s(t \cdot v)) = h(t \cdot v) + \alpha(s) = h(v) + \alpha(t) + \alpha(s),$$

or $\alpha(st) = \alpha(s) + \alpha(t)$. By (3.4), we have $\alpha(1) = 0$, and hence $\alpha : (\mathbb{C}^*, \cdot) \to (\mathbb{C}, +)$ is a group homomorphism.

Fixing some $v \in \Omega$ and instead using (3.4) to define α shows that $\alpha : \mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{a}}$ is regular, hence a homomorphism of complex linear algebraic groups. By the Jordan decomposition, $\alpha = 0$. Then $h(t \cdot v) = h(v)$ for all nonzero $t \in \mathbb{C}$ and $v \in \Omega$; by density, h is homogeneous of degree 0.

Thus we may write $h = \frac{h_1}{f_1}$, with h_1 and f_1 homogeneous polynomials of equal degree, and without common factors. Since h may only have poles on Ω^c, f_1 defines a hypersurface in Ω^c or is a nonzero constant. By Theorem 1.1, f_1 is a relative invariant. ☐

This gives an apparently nontrivial result about the structure of prehomogeneous vector spaces for which the exceptional orbit variety has no hypersurface components.

Corollary 3.6. *Let (G, ρ, V) be a prehomogeneous vector space, let $v_0 \in \Omega$, and let r be the number of irreducible hypersurface components of $V \setminus \Omega$.*

1. *If $r = 0$, then $\mathscr{A}_1(G) = \{0\}$ and $G = [G, G] \cdot G_{v_0}$.*
2. *If $d := \dim(G/[G, G] \cdot G_{v_0}) \leq 1$, then $\mathscr{A}_1(G) = \{0\}$ and $r = d$.*

In particular, $V \setminus \Omega$ has no hypersurface components if and only if $G = [G, G] \cdot G_{v_0}$ for $v_0 \in \Omega$.

Proof. Suppose $r = 0$, so $X_1(G) = \{1\}$. Let $\Phi \in \mathscr{A}_1(G)$, and choose h with $h \overset{\mathrm{a}}{\longleftrightarrow} \Phi$. Assume that Φ is nonzero and hence h is not constant. By Proposition 3.5, write $h = \frac{h_1}{f_1}$ for polynomials f_1 and h_1 with $\deg(h_1) = \deg(f_1) > 0$, and f_1 a relative invariant. If $f_1 \overset{\mathrm{m}}{\longleftrightarrow} \chi_1$, then by our hypothesis $\chi_1 = 1$. By (1.1), f_1 is a nonzero constant on Ω, contradicting the fact that $\deg(f_1) > 0$. Thus $\Phi = 0$ and so $\mathscr{A}_1(G) = \{0\}$. Since there are no nontrivial characters or additive functions, by Proposition 2.1 and Lemma 2.2 we find that $H = G/[G,G] \cdot G_{v_0}$ consists of a single point, hence $G = [G,G] \cdot G_{v_0}$, proving (1).

Now suppose $d \leq 1$. If $d = 0$, then by Corollary 2.3 and Proposition 3.3, $r = 0 - \dim(\mathscr{A}_1(G)) \geq 0$, implying the statement. If $d = 1$, then by the same reasoning either $r = 0$ or $r = 1$, but the former is impossible by (1). Now apply Corollary 2.3 to show $\mathscr{A}_1(G) = \{0\}$. □

3.4. **Global equation.** Our definition of an additive relative invariant only involves the behavior of the function on Ω. We may describe the behavior on all of V.

Proposition 3.7. *Let $h \overset{\mathrm{a}}{\longleftrightarrow} \Phi$ with h nonzero. As in Proposition 3.5, write $h = \frac{h_1}{f_1}$ for polynomials h_1 and f_1, with $f_1 \overset{\mathrm{m}}{\longleftrightarrow} \chi$. Then for all $g \in G$ and $v \in V$,*

$$h_1(\rho(g)(v)) = \chi(g) \cdot (f_1(v) \cdot \Phi(g) + h_1(v)) \,. \tag{3.5}$$

Proof. Define the regular functions $L, R : G \times V \to \mathbb{C}$ using the left, respectively, right sides of (3.5). By (3.1) and (1.1), these functions agree on $G \times \Omega$, and thus on $G \times V$. □

Of course, (3.5) also holds for $h = 0 \overset{\mathrm{a}}{\longleftrightarrow} \Phi = 0$ if $h_1 = 0$ and f_1 is a nonzero constant.

3.5. **Geometric interpretation.** We may now provide a geometric interpretation of additive relative invariants. For polynomials h_1, h_2, f on V with f irreducible and nonzero, we will say that h_1 *and* h_2 *agree to order* k *on* $V(f)$ if $h_1 - h_2 \in (f)^{k+1}$, or equivalently, if in coordinates $\frac{\partial^I}{\partial x^I}(h_1 - h_2)(v) = 0$ for all $v \in V(f)$ and all multi-indices I with $|I| \leq k$.

Proposition 3.8. *Let $f_1, \ldots, f_r$ be the basic relative invariants, with $f_i \overset{\mathrm{m}}{\longleftrightarrow} \chi_i$. Let $h = \frac{h_1}{f}$ for homogeneous polynomials h_1 and f with $\deg(h_1) = \deg(f)$, h written in lowest terms, and $f = f_1^{k_1} \cdots f_r^{k_r}$ with $k_i \geq 0$. Let $\chi = \chi_1^{k_1} \cdots \chi_r^{k_r}$. Then the following are equivalent:*

(1) h is an additive relative invariant.
(2) For all $i = 1, \ldots, r$ with $k_i > 0$, and all $g \in G$, $h_1 \circ \rho(g)$ and $\chi(g) \cdot h_1$ agree to order $k_i - 1$ on $V(f_i)$.
(3) For all $g \in G$, $h_1 \circ \rho(g) - \chi(g) \cdot h_1 \in \mathbb{C}[V]$ is divisible by f.

In particular, when f is reduced and we let $V_\epsilon = \{x : f(x) = h_1(x) - \epsilon = 0\}$ for all $\epsilon \in \mathbb{C}$, then h is an additive relative invariant if and only if $\rho(g)(V_\epsilon) = V_{\chi(g) \cdot \epsilon}$ for all $g \in G$ and $\epsilon \in \mathbb{C}$.

Proof. If (1), then by (3.5), for all $g \in G$ and all $i = 1, \ldots, r$ we have

$$(h_1 \circ \rho(g)) - \chi(g) \cdot h_1 \in (f_i)^{k_i},$$

implying (2).

(2) implies (3) because $f_1, \ldots, f_r$ all define distinct irreducible hypersurfaces.

If (3), then for all g, v we have

$$h_1(\rho(g)(v)) - \chi(g) \cdot h_1(v) = f(v) \cdot \beta(g, v) \tag{3.6}$$

for some $\beta \in \mathbb{C}[G \times V]$. For each $g \in G$, both sides of (3.6) should be zero or homogeneous of degree $\deg(h_1) = \deg(f)$ as functions on V; hence, $\beta(g, -) \in \mathbb{C}[V]$ is zero or homogeneous of degree 0, i.e., constant, and so $\beta \in \mathbb{C}[G]$. Finally, rearrange (3.6) and apply (1.1) and Proposition 3.4.

For the last part of the claim, if h is an additive relative invariant then $\rho(g)(V_\epsilon) = V_{\chi(g)\cdot\epsilon}$ by (3.5). Conversely, $\rho(g)(V_\epsilon) = V_{\chi(g)\cdot\epsilon}$ for all $\epsilon \in \mathbb{C}$ implies that $h_1 \circ \rho(g) - \chi(g) \cdot h_1$ vanishes on $V(f)$, and hence is divisible by f, i.e., (3). $\square$

3.6. **Vanishing lemma.** We now prove that certain characters and additive functions vanish on certain isotropy subgroups. This is a key observation used in §5.

Lemma 3.9. *Let $v \in V$.*

(i) Let f_1 be a relative invariant, $f_1 \xleftrightarrow{\mathrm{m}} \chi$. If f_1 is defined at v and $f_1(v) \neq 0$, then $G_v \subseteq \ker(\chi)$.

(ii) Let $\frac{h_1}{f_1}$ be an additive relative invariant written as in Proposition 3.5, with h_1 nonzero, $\frac{h_1}{f_1} \xleftrightarrow{\mathrm{a}} \Phi$, and $f_1 \xleftrightarrow{\mathrm{m}} \chi$. Then the following conditions are equivalent:
- *$h_1(v) = 0$ or $G_v \subseteq \ker(\chi)$,*
- *$f_1(v) = 0$ or $G_v \subseteq \ker(\Phi)$.*

Proof. Let $g \in G_v$. For (i), by (1.1) we have

$$f_1(v) = f_1(\rho(g)(v)) = \chi(g) f_1(v),$$

and (i) follows. For (ii), by Proposition 3.7 we have

$$h_1(v) = h_1(\rho(g)(v)) = \chi(g)\left(f_1(v)\Phi(g) + h_1(v)\right),$$

or

$$h_1(v) \cdot (1 - \chi(g)) = \chi(g) \cdot f_1(v) \cdot \Phi(g). \tag{3.7}$$

Now consider the circumstances when one side of (3.7) vanishes for all $g \in G_v$; the other side vanishes, too. $\square$

Lemma 3.9 is best understood when we consider generic points on the hypersurface components of Ω^c.

Lemma 3.10. *Let $f_1, \ldots, f_r$ be the basic relative invariants of (G, ρ, V), with $f_i \xleftrightarrow{\mathrm{m}} \chi_i$. Let $v_i \in V(f_i)$, with $v_i \notin V(f_j)$ for $i \neq j$.*

(1) If $i \neq j$, then $G_{v_i} \subseteq \ker(\chi_j)$.

(2) Suppose $h \xleftrightarrow{\mathrm{a}} \Phi$, with h nonzero and $h = \frac{h_1}{g_1}$ as in Proposition 3.5, and $g_1 = f_1^{k_1} \cdots f_r^{k_r}$ the factorization of g_1. Then:

(a) If $k_i > 0$ and $h_1(v_i) \neq 0$, then $G_{v_i}^0 \subseteq \ker(\chi_i)$.

(b) If $k_i = 0$, then $G_{v_i} \subseteq \ker(\Phi)$.

Proof. (1) follows from Lemma 3.9(i). For (2a), apply Lemma 3.9(ii) to show that $G_{v_i} \subseteq \ker(\chi_1^{k_1} \cdots \chi_r^{k_r})$. Then by (1), $G_{v_i} \subseteq \ker(\chi_i^{k_i})$, and this implies (2a). In the situation of (2b), $g_1(v) \neq 0$ and by (1) we have $G_{v_i} \subseteq \ker(\chi_1^{k_1} \cdots \chi_r^{k_r})$. Applying Lemma 3.9(ii) then shows (2b). $\square$

Remark 3.11. Since h is written in lowest terms in Lemma 3.10(2), nearly any choice of v_i in $V(f_i)$ has $h_1(v_i) \neq 0$. Thus, if v_i is generic and f_i appears nontrivially in the denominator of an additive relative invariant, then by Lemma 3.10(1), Lemma 3.10(2a), and Theorem 1.1, the subgroup $G_{v_i}^0$ lies in the kernel of every $\chi \in X_1(G)$.

Remark 3.12. In the situation of Lemma 3.10(2a), the results of [Pik, §5] imply that the ideal I of Remark 1.2 is contained in $(f_i)^2$. In particular, if $I \nsubseteq (f_i)^2$ then there are no additive relative invariants with f_i appearing in the denominator.

Remark 3.13. Lemma 3.10 says much about the dimensions of the orbits of certain algebraic subgroups H of G. In particular, if $G_v^0 \subseteq H$ then $H_v^0 = G_v^0$ and it follows that $\dim(H \cdot v) = \dim(G \cdot v) - \operatorname{codim}(H)$.

3.7. **Algebraic independence.** By Theorem 1.1, a set of basic relative invariants are algebraically independent polynomials; in fact, we may add to this set the numerators of a basis of the additive relative invariants.

Proposition 3.14. *Let $f_1, \ldots, f_r$ be a set of basic relative invariants, with $f_i \overset{\mathrm{m}}{\longleftrightarrow} \chi_i$. Let $\Phi_1, \ldots, \Phi_s$ be a basis of $\mathscr{A}_1(G)$, and for $1 \leq i \leq s$ let $\frac{h_i}{g_i}$ be an additive relative invariant written in lowest terms with $\frac{h_i}{g_i} \overset{\mathrm{a}}{\longleftrightarrow} \Phi_i$. Let $v_i \in V(f_i)$ with $v_i \notin V(f_j)$ for $i \neq j$. Then:*

(1) Each f_i is invariant under the action of $[G,G] \cdot G_{v_0}$, and G_{v_j} for $j \neq i$.
(2) Each h_i is invariant under the action of $[G,G] \cdot G_{v_0}$, and G_{v_j} for j with $g_i(v_j) \neq 0$.
(3) The polynomials $f_1, \ldots, f_r, h_1, \ldots, h_s \in \mathbb{C}[V]$ are algebraically independent over $\mathbb{C}$.

Proof. Claim (1) follows from (1.1), Proposition 1.3, and Lemma 3.10(1). Claim (2) follows from Propositions 3.7 and 3.3, and Lemma 3.10(2b).

A criterion of Jacobi (e.g., [Hum90, §3.10]) states that a set of m polynomials in $\mathbb{C}[V]$, $m \leq \dim(V)$, is algebraically independent over $\mathbb{C}$ if and only if some $m \times m$ minor of their Jacobian matrix is not the zero polynomial. We shall equivalently show that the wedge product of their exterior derivatives is not identically zero.

Let $\mathfrak{g}$ be the Lie algebra of G, and for $X \in \mathfrak{g}$ let ξ_X be the vector field on V defined by $\xi_X(v) = \frac{d}{dt}\left(\rho(\exp(tX))(v)\right)|_{t=0}$ (see also §5.1). If $f \overset{\mathrm{m}}{\longleftrightarrow} \chi$ and $X \in \mathfrak{g}$, then differentiating (1.1) shows

$$(\xi_X(f))(v) = df_{(v)}(\xi_X(v)) = d\chi_{(e)}(X) \cdot f(v). \tag{3.8}$$

Similarly, if $X \in \mathfrak{g}$ and $g_i \overset{\mathrm{m}}{\longleftrightarrow} \chi_i'$ then differentiating (3.5) shows

$$d(h_i)_{(v)}(\xi_X(v)) = d(\chi_i')_{(e)}(X) \cdot h_i(v) + g_i(v) \cdot d(\Phi_i)_{(e)}(X). \tag{3.9}$$

Since $(\chi_1, \ldots, \chi_r, \Phi_1, \ldots, \Phi_s)(G) = (\mathbb{G}_\mathrm{m})^r \times (\mathbb{G}_\mathrm{a})^s$, choose $X_1, \ldots, X_r, Y_1, \ldots, Y_s \in \mathfrak{g}$ such that $d(\chi_i)_{(e)}(X_j) = \delta_{ij}$, $d(\chi_i)_{(e)}(Y_j) = 0$, $d(\Phi_i)_{(e)}(X_j) = 0$, and $d(\Phi_i)_{(e)}(Y_j) = \delta_{ij}$, where δ_{ij} denotes the Kronecker delta function. Because each χ_i' is some product of the χ_j, we have $d(\chi_i')_{(e)}(Y_j) = 0$. Then by (3.8) and (3.9), evaluating $df_1 \wedge \cdots \wedge df_r \wedge dh_1 \wedge \cdots \wedge dh_s$ at $(\xi_{X_1}, \cdots, \xi_{X_r}, \xi_{Y_1}, \cdots, \xi_{Y_s})(v)$ for any $v \in \Omega$ equals

$$\begin{vmatrix} f_1(v) & & 0 & & & \\ & \ddots & & & 0 & \\ 0 & & f_r(v) & & & \\ & & & g_1(v) & & 0 \\ & * & & & \ddots & \\ & & & 0 & & g_r(v) \end{vmatrix} \neq 0. \qquad \square$$

Remark 3.15. By Remark 3.13 the dimension of the $[G,G]\cdot G_{v_0}$–orbit of any $v \in \Omega$ is $\dim(G \cdot v) - \operatorname{codim}([G,G] \cdot G_{v_0}) = \dim(V) - (r+s)$. (Note that in Ω the orbits of $[G, G]$ and $[G, G] \cdot G_{v_0}$ agree.) Although the level set of $(f_1, \ldots, f_r, h_1, \ldots, h_s)$ containing v and the orbit of such a v will thus agree locally, it is doubtful that $f_1, \ldots, f_r, h_1, \ldots, h_s$ always generate the subring of $\mathbb{C}[V]$ of all $[G, G] \cdot G_{v_0}$–invariant polynomials, as invariant subrings are not always finitely generated.

3.8. **Decomposition.** For prehomogeneous vector spaces, any nontrivial factor of a relative invariant is itself a relative invariant; a crucial fact used in the proof is the unique factorization of a rational function on V.

An analogous statement for additive relative invariants might be that any additive relative invariant may be expressed as a sum of additive relative invariants with 'simpler' denominators. Example 4.4 shows that these denominators may be powers of basic relative invariants, and hence we conjecture:

Conjecture 3.16. *There exists a basis $\Phi_1, \ldots, \Phi_s$ for $\mathscr{A}_1(G)$ such that if we choose reduced rational functions h_i with $h_i \xleftrightarrow{\mathrm{a}} \Phi_i$, then for each i the denominator of h_i is a positive power of some basic relative invariant.*

It would be natural to use a partial fraction expansion to prove this, but such an expansion does not always exist for rational functions of several variables. The question of the existence of a partial fraction decomposition seems to be the main obstacle in proving this conjecture.

Proposition 3.17. *Let $\frac{h_1}{f_1 f_2}$ be an additive relative invariant written in lowest terms, with f_1 and f_2 polynomial relative invariants having no common factors. If*

$$\frac{h_1}{f_1 f_2} = \frac{\alpha}{f_1} + \frac{\beta}{f_2}$$

for α, β homogeneous polynomials of the same degree as f_1 and f_2, respectively, then $\frac{\alpha}{f_1}$ and $\frac{\beta}{f_2}$ are additive relative invariants written in lowest terms.

Proof. The statement is true when $h_1 = 0$ and f_1, f_2 are constants, so assume h_1 is nonzero.

Let $f_i \xleftrightarrow{\mathrm{m}} \chi_i$ and $\frac{h_1}{f_1 f_2} \xleftrightarrow{\mathrm{a}} \Phi$. By algebra we have

$$h_1 = \alpha \cdot f_2 + \beta \cdot f_1 \tag{3.10}$$

and so by Proposition 3.7 we have for all $(g, v) \in G \times V$,

$$\begin{aligned} &\alpha(\rho(g)(v)) \cdot f_2(\rho(g)(v)) + \beta(\rho(g)(v)) \cdot f_1(\rho(g)(v)) \\ &\qquad = \chi_1(g)\chi_2(g)\left(f_1(v)f_2(v)\Phi(g) + \alpha(v)f_2(v) + \beta(v)f_1(v)\right). \end{aligned} \tag{3.11}$$

Rearranging (3.11) and using (1.1) gives

$$\begin{aligned} &(\alpha(\rho(g)(v)) - \chi_1(g)\alpha(v)) \cdot \chi_2(g)f_2(v) \\ &\qquad = \chi_1(g)f_1(v)\left(\chi_2(g)f_2(v)\Phi(g) + \chi_2(g)\beta(v) - \beta(\rho(g)(v))\right). \end{aligned} \tag{3.12}$$

For each $g \in G$, the left side of (3.12) is divisible by f_1, and since f_1 and f_2 have no common factors, $\alpha \circ \rho(g) - \chi_1(g) \cdot \alpha$ is divisible by f_1. By Proposition 3.8, $\frac{\alpha}{f_1}$ is an additive relative invariant, and thus so is $\frac{\beta}{f_2}$. If $\frac{\alpha}{f_1}$ or $\frac{\beta}{f_2}$ is not in lowest terms, then by (3.10) neither is $\frac{h_1}{f_1 f_2}$. $\square$

Remark 3.18. If in Proposition 3.17 we omit the assumption that α and β are homogeneous, then we may take homogeneous parts of (3.10) to find an α' and β' that are homogeneous of the correct degree.

4. Examples

We now give some examples of additive relative invariants. Our first two examples show there may be arbitrary numbers of linearly independent additive relative invariants associated to a single hypersurface, and arbitrary numbers of hypersurfaces.

Example 4.1. Let $G = \left\{ \left(\begin{smallmatrix} a & 0 & 0 \\ b & a & 0 \\ c & 0 & a \end{smallmatrix} \right) \in \mathrm{GL}_3(\mathbb{C}) \right\}$ act on $\mathbb{C}\{x, y, z\} = \mathbb{C}^3$ by multiplication, where this notation means that a, b, c may take any value that gives an invertible matrix. Then G has $D = V(x)$ as the exceptional orbit variety, with $f_1 = x \xleftrightarrow{\mathrm{m}} \chi_1 = a$. Since G is abelian and the isotropy subgroup at $(x, y, z) = (1, 0, 0) \in \Omega$ is trivial, G has 2 linearly independent additive functions, namely, $h_1 = \frac{y}{x} \xleftrightarrow{\mathrm{a}} \Phi_1 = \frac{b}{a}$ and $h_2 = \frac{z}{x} \xleftrightarrow{\mathrm{a}} \Phi_2 = \frac{c}{a}$. On $V(x)$, radial subsets of the form $\alpha y + \beta z = 0$ are invariant, and the G–action multiplies both coordinates by $\chi_1 = a$ as expected by Proposition 3.8.

More generally, fix coordinates $x_1, \ldots, x_n$ on $\mathbb{C}^n$ and let $G \subseteq \mathrm{GL}(\mathbb{C}^n)$ consist of those matrices with all diagonals equal, the first column unrestricted, and all other entries equal to zero. Then G is abelian and has $f_1 = x_1$ and $n - 1$ linearly independent additive functions corresponding to the additive relative invariants $h_i = \frac{x_{i+1}}{x_1}$, $i = 1, \ldots, n - 1$.

Example 4.2. Let $G = \left\{ \left(\begin{smallmatrix} a & 0 & 0 \\ b & a & 0 \\ 0 & 0 & c \end{smallmatrix} \right) \in \mathrm{GL}_3(\mathbb{C}) \right\}$ act on $\mathbb{C}\{x, y, z\} = \mathbb{C}^3$ by multiplication. Then G has $f_1 = x \xleftrightarrow{\mathrm{m}} \chi_1 = a$ and $f_2 = z \xleftrightarrow{\mathrm{m}} \chi_2 = c$. Since G is abelian and the isotropy subgroup at $(x, y, z) = (1, 0, 1) \in \Omega$ is trivial, $\dim(\mathscr{A}_1(G)) = 1$; a generator is $h_1 = \frac{y}{x} \xleftrightarrow{\mathrm{a}} \Phi_1 = \frac{b}{a}$. On $V(x)$, $g \in G$ sends $(0, y, z)^T$ to $(0, \chi_1(g)y, \chi_2(g)z)^T$; hence, both $V(x, y)$ and $V(x, z)$ are G–invariant, but only the former has the behavior predicted by Proposition 3.8. Both $V(x)$ and $V(z)$ contain open orbits.

More generally, we may take direct products of groups of the type in Example 4.1, realized as block diagonal matrices. For $n \in \mathbb{N}$ and $k_1, \ldots, k_n \in \mathbb{Z}_{\geq 0}$, this construction can produce a prehomogeneous vector space with Ω^c consisting of n hyperplanes $H_1, \ldots, H_n$, such that for each i there are k_i linearly independent additive relative invariants having poles on H_i.

Next, we observe that the existence of an additive relative invariant does not depend only on the orbit structure of the action.

Example 4.3. Let $G = \left\{ \left(\begin{smallmatrix} a & 0 & 0 \\ b & \frac{a}{c} & 0 \\ 0 & 0 & c \end{smallmatrix} \right) \in \mathrm{GL}_3(\mathbb{C}) \right\}$ act on $\mathbb{C}\{x, y, z\} = \mathbb{C}^3$ by multiplication. Then G has $f_1 = x \xleftrightarrow{\mathrm{m}} \chi_1 = a$ and $f_2 = z \xleftrightarrow{\mathrm{m}} \chi_2 = c$. The generic isotropy subgroup is trivial and $\dim([G, G]) = 1$. Thus $G/[G, G] \cdot G_{v_0}$ has no nontrivial additive functions, and there are no nontrivial additive relative invariants. The orbit structure of G agrees with that of the group in Example 4.2, but here neither $V(x, y)$ nor $V(x, z)$ exhibit the behavior described in Proposition 3.8.

An additive relative invariant may unavoidably have a nontrivial power of a basic relative invariant in its denominator.

Example 4.4. For $n \geq 2$, let $G \subseteq \mathrm{GL}_n(\mathbb{C})$ consist of all invertible lower-triangular matrices A such that each $(A)_{ij}$ depends only on $i - j$, that is, $(A)_{i+1,j+1} = (A)_{i,j}$ whenever this makes sense. Then G is a connected abelian linear algebraic group, and its action on $\mathbb{C}^n = \mathbb{C}\{x_1, \ldots, x_n\}$ has an open orbit with exceptional orbit variety $V(x_1)$ and a trivial generic isotropy subgroup. It follows that G has $n-1$ linearly independent additive functions. For $1 \leq i \leq n-1$ define Φ_i by

$$\Phi_i \left(\begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 \\ a_2 & a_1 & 0 & \ddots & \vdots \\ a_3 & a_2 & a_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_1 \end{pmatrix} \right) = \frac{1}{(a_1)^i} \begin{vmatrix} \frac{1}{i}a_2 & a_1 & 0 & \cdots & 0 \\ \frac{2}{i}a_3 & a_2 & a_1 & \ddots & \vdots \\ \frac{3}{i}a_4 & a_3 & a_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \frac{i}{i}a_{i+1} & a_i & \cdots & \cdots & a_2 \end{vmatrix}.$$

For instance, when $n = 4$ we have

$$\Phi_1 = \frac{a_2}{a_1}, \quad \Phi_2 = \frac{1}{a_1^2}\left(\frac{1}{2}a_2^2 - a_1a_3\right), \quad \text{and } \Phi_3 = \frac{1}{a_1^3}\left(\frac{1}{3}(a_2)^3 - a_1a_2a_3 + a_1^2a_4\right).$$

For at least $n \leq 20$, computer calculations show that each $\Phi_i : G \to \mathbb{G}_{\mathrm{a}}$ is a homomorphism, and hence $\Phi_1, \ldots, \Phi_{n-1}$ is a basis of the space of additive functions. By (3.1), the corresponding additive relative invariants are of a form similar to Φ_i (e.g., substitute x_i for a_i), and for $i > 1$ have nonreduced denominators.

Example 4.5. Fix a positive integer n and let $m \in \{n, n+1\}$. Let $L \subseteq \mathrm{GL}_n(\mathbb{C})$ (respectively, $U \subseteq \mathrm{GL}_m(\mathbb{C})$) consist of invertible lower triangular matrices (resp., upper triangular unipotent matrices). Let $G = L \times U$ act on the space $M(n, m, \mathbb{C})$ of $n \times m$ complex matrices by $(A, B) \cdot M = AMB^{-1}$. The classical *LU factorization* of a complex matrix asserts that this is a prehomogeneous vector space (see [DP12b]). For any matrix M, let $M^{(k)}$ denote the upper-leftmost $k \times k$ submatrix of M. By [DP12b, §6], the basic relative invariants are of the form $f_i(M) = \det(M^{(i)})$, $i = 1, \ldots, n$, and $f_i \xleftrightarrow{\mathrm{m}} \chi_i(A, B) = \det(A^{(i)})$. For I the identity matrix, $v_0 = I$ when $m = n$, or $v_0 = \begin{pmatrix} I & 0 \end{pmatrix}$ when $m = n + 1$, is in Ω and has a trivial isotropy subgroup G_{v_0}.

Since $\dim(G/[G,G]\cdot G_{v_0}) = n+m-1$, the quotient has $m-1$ linearly independent additive functions. In fact, for $(A, B) \in G$ and $1 \leq i \leq m-1$, let $\Phi_i(A, B) = -(B)_{i,i+1}$. A computation shows that these Φ_i are linearly independent additive functions of G, with

$$h_i(M) = \frac{\det((M \text{ with column } i \text{ deleted})^{(i)})}{\det(M^{(i)})} \xleftrightarrow{\mathrm{a}} \Phi_i.$$

5. Linear free divisors

We now consider prehomogeneous vector spaces (G, ρ, V) for which the complement of the open orbit Ω is a type of hypersurface called a linear free divisor. Our main theorem is that these have no nontrivial additive relative invariants, but this has significant consequences for their structure.

5.1. **Introduction.** Let $\mathscr{O}_{\mathbb{C}^n,p}$ denote the ring of germs of holomorphic functions on $\mathbb{C}^n$ at p, and $\mathrm{Der}_{\mathbb{C}^n,p}$ the $\mathscr{O}_{\mathbb{C}^n,p}$–module of germs of holomorphic vector fields on $\mathbb{C}^n$ at p. Associated to a germ (D, p) of a reduced analytic set in $\mathbb{C}^n$ is the $\mathscr{O}_{\mathbb{C}^n,p}$–module of *logarithmic vector fields* defined by

$$\mathrm{Der}_{\mathbb{C}^n,p}(-\log D) := \{\eta \in \mathrm{Der}_{\mathbb{C}^n,p} : \eta(I(D)) \subseteq I(D)\},$$

where $I(D)$ is the ideal of germs vanishing on D. These are the vector fields tangent to (D, p), and form a Lie algebra using the Lie bracket of vector fields.

Let D be nonempty and $(D,p) \neq (\mathbb{C}^n, p)$. When $\mathrm{Der}_{\mathbb{C}^n,p}(-\log D)$ is a free $\mathscr{O}_{\mathbb{C}^n,p}$–module, necessarily of rank n, then (D,p) is called a *free divisor*. Free divisors are always pure hypersurface germs that are either smooth or have singularities in codimension 1, and were first encountered as various types of discriminants.

Now let (G,ρ,V) be a prehomogeneous vector space, let $\mathfrak{g}$ be the Lie algebra of G, and let $D := V \setminus \Omega$ be the exceptional orbit variety. Let $\mathrm{Der}_V(-\log D)_0$ denote the $\eta \in \mathrm{Der}_V(-\log D)$ that are *linear*, that is, homogeneous of degree 0 (e.g., $(x+y)\frac{\partial}{\partial x} - 2z\frac{\partial}{\partial y}$). Because D is G–invariant, differentiating the action of G gives a Lie algebra (anti-)homomorphism $\tau : \mathfrak{g} \to \mathrm{Der}_V(-\log D)_0$, where $\xi_X := \tau(X)$ is a vector field defined globally on V by

$$\xi_X(v) = \tau(X)(v) = \frac{d}{dt}\left(\rho(\exp(tX))(v)\right)\Big|_{t=0}.$$

Thus, $\tau(\mathfrak{g}) \subseteq \mathrm{Der}_V(-\log D)_0$ are finite-dimensional Lie subalgebras of the module $\mathrm{Der}_V(-\log D)$. As $\tau(\mathfrak{g})(v) = T_v(G\cdot v)$, the maximal minors of a matrix containing the coefficients of a basis of $\tau(\mathfrak{g})$ are of degree $\dim(V)$ and generate an ideal defining the set D.

When $\mathrm{Der}_V(-\log D)$ has a free basis of linear vector fields, then D is called a *linear free divisor*. By [GMNRS09, Lemma 2.3], every linear free divisor D is the exceptional orbit variety of a prehomogeneous vector space with the following properties:

Definition 5.1. Let D be a linear free divisor in V. If (G,ρ,V) is a prehomogeneous vector space with open orbit $V \setminus D$, G connected, and $\dim(G) = \dim(V)$, then say that G *defines the linear free divisor* D.

It follows that $\ker(\rho)$ is finite, and $\tau(\mathfrak{g}) = \mathrm{Der}_V(-\log D)_0$.

One such prehomogeneous vector space may be constructed in the following way. For a divisor $D \subset V$ in a finite-dimensional complex vector space, let $\mathrm{GL}(V)_D$ be the largest subgroup of $\mathrm{GL}(V)$ that preserves D. Note that $\mathrm{GL}(V)_D$ is algebraic. Then $(\mathrm{GL}(V)_D)^0$ is a connected complex linear algebraic group with Lie algebra (anti-)isomorphic to $\mathrm{Der}_V(-\log D)_0$. For a linear free divisor $D \subset V$, the group $(\mathrm{GL}(V)_D)^0 \subseteq \mathrm{GL}(V)$ with the inclusion representation is a prehomogeneous vector space that defines D ([GMNRS09, Lemma 2.3]). In fact, if (G,ρ,V) defines a linear free divisor D, then $\rho(G) = (\mathrm{GL}(V)_D)^0$.

5.2. **Brion's criterion.** Brion gave the following useful criterion for D to be a linear free divisor.

Theorem 5.2 ([Bri06], [GMS11, Theorem 2.1]; see also [Pik]). *For $G = (\mathrm{GL}(V)_D)^0$, the following are equivalent:*

(1) D is a linear free divisor.

(2) Both of these conditions hold:

- *(a) $V \setminus D$ is a unique G-orbit, and the corresponding isotropy groups are finite.*
- *(b) Each irreducible component D_i of D contains an open G-orbit D_i^0, and the corresponding isotropy groups are extensions of finite groups by $\mathbb{G}_{\mathrm{m}}$.*

When these hold, $\tau(\mathfrak{g})$ generates $\mathrm{Der}_V(-\log D)$, and each $D_i^0 = \mathrm{smooth}(D) \cap D_i$.

The proof of Theorem 5.2 shows the following.

Corollary 5.3. *Suppose that in the situation of Theorem 5.2, (2a) holds, D_i contains an open orbit D_i^0, and $v_i \in D_i^0$. Then G_{v_i} is an extension of a finite group by $\mathbb{G}_{\mathrm{m}}$ if and only if the induced representation of $G_{v_i}^0$ on the normal line to D_i at v_i is nontrivial.*

The representation on the normal line is actually quite familiar.

Lemma 5.4. *Let (G, ρ, V) be a prehomogeneous vector space with f as a basic relative invariant, and let v be a smooth point of $D := V(f)$. If $H \subseteq G_v$, then the representation $\rho_v : H \to \mathrm{GL}(L)$ on the normal line $L = T_vV/T_vD$ to D at v is*

$$\rho_v(h)(\ell) = \chi(h)\ell, \qquad \textit{for all } \ell \in L,$$

where $f \xleftrightarrow{\mathrm{m}} \chi$.

Geometrically, ρ_v acts on a normal slice to $f = 0$ at v, and such a slice intersects all level sets of f. At the same time, the action of $\rho(H)$ fixes v and translates between the level sets of f according to χ.

Proof. The representation $\rho|_H$ fixes v and hence induces a representation ρ' of H on the tangent space T_vV; by silently identifying T_vV with V, we have $\rho' = \rho|_H$. Since $\rho|_H$ leaves invariant D and fixes the smooth point v on D, the representation ρ' leaves invariant T_vD. Then the normal line is $L = T_vV/T_vD$, and ρ' produces the quotient representation $\rho_v : H \to \mathrm{GL}(L)$ defined by $\rho_v(h)(w + T_vD) = \rho'(h)(w) + T_vD$.

Let $h \in H$ and $w \in T_vV$. For $\lambda \in \mathbb{C}$ we have by (1.1),

$$f(v + \lambda \cdot \rho(h)(w)) = f(\rho(h)(v + \lambda w)) = \chi(h)f(v + \lambda w). \tag{5.1}$$

Differentiating (5.1) with respect to λ and evaluating at $\lambda = 0$ gives $df_{(v)}(\rho'(h)(w)) = \chi(h)df_{(v)}(w)$, or $\rho'(h)(w) - \chi(h)w \in \ker(df_{(v)})$. Since f is reduced and v is a smooth point, $T_vD = \ker(df_{(v)} : T_vV \to T_0\mathbb{C})$. The definition of ρ_v then implies the result. ☐

Remark 5.5. If (G, ρ, V) defines a linear free divisor D in the sense of Definition 5.1, then all of the results of this §5.2 hold for G as well as $\rho(G) = (\mathrm{GL}(V)_D)^0$.

5.3. **The main theorems.** Let (G, ρ, V) define the linear free divisor $D \subset V$ in the sense of Definition 5.1. Let $f_1, \ldots, f_r$ be the basic relative invariants, so that $\cup_{i=1}^r V(f_i)$ is the irreducible decomposition of $D = V \setminus \Omega$. Let $f_i \xleftrightarrow{\mathrm{m}} \chi_i$, and choose $v_0 \in \Omega$. For $1 \leq i \leq r$ let v_i be a generic point on $V(f_i)$, an element of the open orbit of G in $V(f_i)$.

Theorem 5.6. *Let G define the linear free divisor D, with the notation above. Then:*

1. *The homomorphism $(\chi_1, \ldots, \chi_r) : G \to (\mathbb{G}_{\mathrm{m}})^r$ is surjective and has kernel $[G, G] \cdot G_{v_0}$.*
2. *$G/[G, G] \cdot G_{v_0}$ has only the trivial additive function and (G, ρ, V) has only constant additive relative invariants.*
3. *For $i \neq j$, we have $G_{v_j}^0 \subseteq \ker(\chi_i)$.*
4. *The representation $\chi_i|_{G_{v_i}^0} : G_{v_i}^0 \to \mathbb{G}_{\mathrm{m}}$ is surjective, with finite kernel.*
5. *$\ker(\chi_i|_{G_{v_i}^0}) = G_{v_i}^0 \cap ([G, G] \cdot G_{v_0})$ is finite.*
6. *For $i \neq j$, the subgroup $G_{v_i}^0 \cap G_{v_j}^0$ is a finite subset of $[G, G] \cdot G_{v_0}$.*

Proof. Let $G_1 = [G,G]\cdot G_{v_0}$. Claim (3) is just Lemma 3.10(1).

By Theorem 5.2, Corollary 5.3, and Lemma 5.4, each $\chi_i|_{G^0_{v_i}}$ is nontrivial, thus surjective. As $\dim(G^0_{v_i}) = 1 = \dim(\mathrm{GL}(\mathbb{C}))$, the kernel is finite, giving (4).

If $\Phi \in \mathscr{A}(G/G_1)$ is nontrivial, then by Proposition 3.3 there exists a non-constant h with $h \overset{\mathrm{a}}{\longleftrightarrow} \Phi$. By Proposition 3.5, write $h = \frac{h_1}{g_1}$ as a reduced fraction, with g_1 a polynomial relative invariant and $g_1 \overset{\mathrm{m}}{\longleftrightarrow} \chi$. Since Φ is nontrivial, g_1 cannot be constant and so by Theorem 1.1 is a product of basic relative invariants; let f_i be an irreducible factor of g_1. If $h_1(v_i) = 0$, then by Proposition 3.7, h_1 vanishes on $G\cdot v_i$ and hence on $\overline{G\cdot v_i} = V(f_i)$; but then f_i divides $\gcd(h_1, g_1)$, a contradiction of how we wrote h, hence $h_1(v_i) \neq 0$. By Lemma 3.10(2a) we have $G^0_{v_i} \subset \ker(\chi_i)$, but this contradicts (4). Thus $\Phi = 0$ is trivial, proving (2).

Since G/G_1 is connected and abelian, by Proposition 2.1, $G/G_1 \cong (\mathbb{G}_{\mathrm{m}})^k \times (\mathbb{G}_{\mathrm{a}})^\ell$. Choosing a free basis $\epsilon_1, \ldots, \epsilon_k$ of $X(G/G_1)$ and a basis $\Phi_1, \ldots, \Phi_\ell$ of $\mathscr{A}(G/G_1)$ gives an explicit isomorphism $\theta = (\epsilon_1, \ldots, \epsilon_k, \Phi_1, \ldots, \Phi_\ell) : G/G_1 \to (\mathbb{G}_{\mathrm{m}})^k \times (\mathbb{G}_{\mathrm{a}})^\ell$. Theorem 1.1 and Proposition 1.3 show that we may let each $\epsilon_i = \chi_i$. Then, (1) follows from (2).

For (5), such elements are contained in the kernel by (1), and conversely by (3) and (1). By (4), the kernel is finite.

For (6), observe that by (3), such elements are in the kernel of the homomorphism of (1). For finiteness, use (5). □

Remark 5.7. For each $i > 0$, the group $G^0_{v_i}$ is a 1–dimensional connected complex linear algebraic group with a surjective homomorphism to $\mathbb{G}_{\mathrm{m}}$, and hence $G^0_{v_i} \cong \mathbb{G}_{\mathrm{m}}$.

Theorem 5.6 has a number of immediate consequences. In particular, we may easily compute the number of irreducible components.

Theorem 5.8. *Let G define the linear free divisor D, with the notation above. Then:*

(1) The hypersurface D has
$$r = \dim_{\mathbb{C}}(G/[G,G]\cdot G_{v_0}) = \dim_{\mathbb{C}}(G/[G,G]) = \dim_{\mathbb{C}}(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])$$
irreducible components, where $\mathfrak{g}$ is the Lie algebra of G.

(2) Every element of G may be written in a finite number of ways as a product of elements from the subgroups $[G,G]\cdot G_{v_0}, G^0_{v_1}, \ldots, G^0_{v_r}$. Each term of such a product is unique modulo $[G,G]\cdot G_{v_0}$.

(3) Let $S \subseteq \{1, \ldots, r\}$. The subgroup of G that leaves invariant all level sets of f_i for $i \in S$ is normal in G, and is the product of the subgroups $[G,G]\cdot G_{v_0}$, and $G^0_{v_j}$ for $j \notin S$.

(4) As algebraic groups, $G/[G,G] \cong (\mathbb{G}_{\mathrm{m}})^r$.

(5) The subgroup $[G,G]$ contains all unipotent elements of G.

Proof. For (1), combine Theorem 5.6(2) and Corollary 2.3. Note also that G_{v_0} is finite by Theorem 5.2.

Let $g \in G$. By Theorem 5.6(4), for $i > 0$ there exists $g_i \in G^0_{v_i}$ such that $\chi_i(g) = \chi_i(g_i)$. By Theorem 5.6(3) and 5.6(1), $g(g_1 \cdots g_r)^{-1}$ is in the kernel of each χ_i, and hence lies in $[G,G]\cdot G_{v_0}$. This proves existence for (2).

To address uniqueness, let $g \in G$ and suppose that for $j = 1, 2$ we have $g = g_{0,j}\cdot g_{1,j}\cdots g_{r,j}$, with $g_{0,j} \in [G,G]\cdot G_{v_0}$ and $g_{i,j} \in G^0_{v_i}$ for $i > 0$. By Theorem 5.6(1) and 5.6(3), for $i > 0$ we have $\chi_i(g) = \chi_i(g_{i,1}) = \chi_i(g_{i,2})$. Then for $i > 0$

we have $g_{i,1} = g_{i,2}$ modulo $\ker(\chi_i|_{G^0_{v_i}})$, and this kernel is finite and contained in $[G,G] \cdot G_{v_0}$ by Theorem 5.6(5). Since $g_{0,j}$ is uniquely determined by $g_{1,j}, \ldots, g_{r,j}$, there are precisely

$$\prod_{i=1}^{r} \#(\ker(\chi_i|_{G^0_{v_i}}))$$

ways to write g in this way. This proves (2).

By (1.1), $g \in G$ leaves invariant all level sets of f_i if and only if $g \in \ker(\chi_i)$. Hence, the subset H leaving invariant all level sets of f_i for $i \in S$ is an intersection of kernels and thus a normal subgroup. By Theorems 5.6(1) and 5.6(3), $[G,G] \cdot G_{v_0}$ and $G^0_{v_j}$ for $j \notin S$ are in H. Conversely, by (2) and Theorems 5.6(3) and 5.6(5), and the normality of $[G,G] \cdot G_{v_0}$, any element of H may be written as a product of elements of these subgroups. This proves (3).

To prove (4), consider the diagram

$$\begin{tikzcd} G/[G,G] \arrow[r, "\phi"] \arrow[d, "\kappa"] & {\mathbb{G}_{\mathrm{m}}}^k \times {\mathbb{G}_{\mathrm{a}}}^\ell \arrow[d, "\psi"] \\ G/[G,G] \cdot G_{v_0} \arrow[r, "\chi"] & {\mathbb{G}_{\mathrm{m}}}^r \end{tikzcd}$$

where ϕ is an isomorphism that exists by Proposition 2.1, κ is the quotient map, χ is the isomorphism induced from $(\chi_1, \ldots, \chi_r)$ by Theorem 5.6(1), and ψ makes the diagram commutative. The Jordan decomposition implies that $\{1\} \times {\mathbb{G}_{\mathrm{a}}}^\ell \subseteq \ker(\psi)$, but this kernel is isomorphic to $\ker(\kappa)$, which is finite because G_{v_0} is finite. Hence $\ell = 0$ and $k \leq r$, and the surjectivity of ψ requires $k \geq r$, proving (4).

Finally, (5) is a consequence of (4) and the Jordan decomposition. $\square$

Example 5.9 ([BM06, Example 2.1],[GMNRS09, Example 5.1])**.** On $\mathbb{C}^3$, fix coordinates x, y, z. The linear free divisor $x(xz - y^2) = 0$ is defined by the solvable group

$$G = \left\{ \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ \frac{b^2}{a} & \frac{2bc}{a} & \frac{c^2}{a} \end{pmatrix} \in \mathrm{GL}(\mathbb{C}^3) \right\}.$$

We have $f_1 = x \overset{\mathrm{m}}{\longleftrightarrow} \chi_1 = a$ and $f_2 = xz - y^2 \overset{\mathrm{m}}{\longleftrightarrow} \chi_2 = c^2$. At $v_0 = (1,0,1) \in \Omega$, $v_1 = (0,1,0) \in V(f_1)$, and $v_2 = (1,0,0) \in V(f_2)$, the (generic) isotropy subgroups are defined by, respectively, $(a,b,c) = (1,0,\pm 1)$, $(b,c) = (0,1)$, and $(a,b) = (1,0)$. As $[G,G]$ is defined by $a = c = 1$, we see $[G,G] \cdot G_{v_0}$ is defined by $a = 1$, $c = \pm 1$. Theorems 5.6 and 5.8 are easy to verify.

Recall that $L(K)$ denotes the Lie algebra of an algebraic group K. Let δ_{ij} denote the Kronecker delta function. On the level of Lie algebras, Theorem 5.6 implies the following.

Corollary 5.10. *As vector spaces,*

$$\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \bigoplus_{i=1}^{r} L(G_{v_i}).$$

For $i = 1, \ldots, r$ there exist unique $X_i \in L(G_{v_i})$ such that $L(G_{v_i}) = \mathbb{C}X_i$, and such that for all j we have $d(\chi_i)_{(e)}(X_j) = \delta_{ij}$ and $\xi_{X_j}(f_i) = \delta_{ij} \cdot f_i$. For $X \in [\mathfrak{g}, \mathfrak{g}]$ and any j, $\xi_X(f_j) = 0$.

Proof. Differentiating the homomorphism of Theorem 5.6(1) gives a homomorphism $\lambda : \mathfrak{g} \to L((\mathbb{G}_{\mathrm{m}})^r) = \oplus_{i=1}^r L(\mathbb{G}_{\mathrm{m}})$ with kernel $[\mathfrak{g}, \mathfrak{g}]$. By Theorems 5.6(3) and 5.6(4), under λ each $L(G_{v_i})$ surjects onto the ith copy of $L(\mathbb{G}_{\mathrm{m}})$, and is zero on the rest. This gives the vector space decomposition, and proves that for $i \neq j$, $d(\chi_i)_{(e)}(L(G_{v_j})) = 0$. Now choose the unique $X_i \in L(G_{v_i})$ such that $d(\chi_i)_{(e)}(X_i) = 1$.

The rest of the statement follows from (3.8). □

Each X_i depends on the choice of $v_i \in V(f_i)$, but any two choices will differ by an element of $[\mathfrak{g}, \mathfrak{g}]$.

Remark 5.11. More generally, let (G, ρ, V) be a prehomogeneous vector space with no nontrivial additive relative invariants. (For instance, by Remark 3.12, this happens if the ideal of Remark 1.2 is not contained in $(f_i)^2$ for every basic relative invariant f_i.) Then by the same argument in Theorems 5.6 and 5.8, $G/[G,G] \cdot G_{v_0}$ is an algebraic torus, of dimension equal to the number of irreducible hypersurface components of the exceptional orbit variety $V \setminus \Omega$.

5.4. **The structure of G.** We now use Theorems 5.6 and 5.8 to study the structure of algebraic groups defining linear free divisors.

Let G be a connected complex algebraic group. Let $\mathrm{Rad}(G)$ denote the *radical* of G, the maximal connected normal solvable subgroup. The (algebraic) *Levi decomposition* of G writes

$$G = \mathrm{Rad}_u(G) \rtimes L,$$

where $\mathrm{Rad}_u(G)$ is the *unipotent radical* of G, the largest connected unipotent normal subgroup of G, consisting of all unipotent elements of $\mathrm{Rad}(G)$; and L is a *Levi subgroup*, a maximal connected reductive algebraic subgroup of G, unique up to conjugation ([Bor91, 11.22]). Moreover, $L = \mathrm{Z}(L)^0 \cdot [L, L]$ for $\mathrm{Z}(L)$ the center of L, $\mathrm{Z}(L) \cap [L, L]$ is finite, $[L, L]$ is semisimple, and $(\mathrm{Z}(L))^0 = L \cap \mathrm{Rad}(G)$ is a maximal torus of $\mathrm{Rad}(G) = \mathrm{Rad}_u(G) \rtimes \mathrm{Z}(L)^0$ ([Bor91, 14.2, 11.23]).

Groups defining linear free divisors have the following structure.

Corollary 5.12. *Let G define the linear free divisor D and have the Levi decomposition above. Then:*

1. *The number of irreducible components of D equals* $\dim(\mathrm{Z}(L))$.
2. $[G, G] = \mathrm{Rad}_u(G) \rtimes [L, L]$.
3. $G = [G, G] \cdot (\mathrm{Z}(L))^0$*, with* $[G, G] \cap (\mathrm{Z}(L))^0$ *finite.*
4. $(\chi_1, \ldots, \chi_r)|_{\mathrm{Z}(L)^0} : \mathrm{Z}(L)^0 \to (\mathbb{G}_{\mathrm{m}})^r$ *is surjective, with a finite kernel.*

Proof. Let $R = \mathrm{Rad}_u(G)$. Since G is the semidirect product of R and L, a straightforward calculation shows that $[G, G] = [R, R] \cdot [R, L] \cdot [L, L]$. By Theorem 5.8(5) and connectedness, $R \subseteq [G, G]$. Since $R \trianglelefteq G$, we have $[R, R], [R, L] \subseteq R$ and hence

$$[G, G] = R \cdot [G, G] = R \cdot [R, R] \cdot [R, L] \cdot [L, L] = R \cdot [L, L].$$

As R consists of unipotent elements and $\mathrm{Rad}(G) \cap L$ consists of semisimple elements, $R \cap L = \{e\}$ and hence $R \cap [L, L] = \{e\}$. Since $R \subseteq [G, G]$ are normal subgroups of G, we have $R \trianglelefteq [G, G]$. This proves the decomposition (2) as abstract groups, and hence as complex algebraic groups ([OV90, §3.3–3.4]).

From this, we conclude

$$G = R \cdot L = R \cdot [L, L] \cdot (\mathrm{Z}(L))^0 = [G, G] \cdot (\mathrm{Z}(L))^0.$$

Clearly, $[L,L] \subseteq [G,G] \cap L$. If $g \in [G,G] \cap L$, then by (2) we have $g = k \cdot \ell$ for $k \in R$ and $\ell \in [L,L] \subseteq L$, hence $k = g\ell^{-1} \in L \cap R = \{e\}$ and so $g = \ell \in [L,L]$; thus $[G,G] \cap L = [L,L]$. In particular, $[G,G] \cap (\mathrm{Z}(L))^0 = [L,L] \cap (\mathrm{Z}(L))^0$, which is finite. This proves (3).

By (3) and an isomorphism theorem,

$$G/[G,G] \cong (\mathrm{Z}(L))^0/([G,G] \cap \mathrm{Z}(L)^0),$$

and hence $\dim(\mathrm{Z}(L)) = \dim(G/[G,G])$. Theorem 5.8(1) then proves (1).

Finally, (4) may be checked on the level of Lie algebras. By (3), we have $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}] \oplus \mathfrak{z}(\mathfrak{l})$, where $\mathfrak{g}$ and $\mathfrak{l}$ are the Lie algebras of G and L, and $\mathfrak{z}(\mathfrak{l})$ is the center of $\mathfrak{l}$. For $\chi = (\chi_1, \ldots, \chi_r)$, we have $\ker(d\chi_{(e)}) = [\mathfrak{g},\mathfrak{g}]$ by Theorem 5.6(1), and hence $d\chi_{(e)}|_{\mathfrak{z}(\mathfrak{l})}$ is an isomorphism. In fact, by Theorem 5.6(1) the kernel in (4) is $(\mathrm{Z}(L))^0 \cap ([G,G] \cdot G_{v_0})$. □

Remark 5.13. An arbitrary connected complex linear algebraic group with Levi decomposition $G = \mathrm{Rad}_u(G) \rtimes L$ has

$$[G,G] = [\mathrm{Rad}_u(G), \mathrm{Rad}_u(G)] \cdot [\mathrm{Rad}_u(G), L] \cdot [L,L] \subseteq \mathrm{Rad}_u(G) \rtimes [L,L],$$

and hence $[\mathrm{Rad}_u(G), \mathrm{Rad}_u(G)] \cdot [\mathrm{Rad}_u(G), L] \subseteq \mathrm{Rad}_u(G)$; if G defines a linear free divisor, then by Corollary 5.12(2) these are both equalities.

Remark 5.14. If B is a *Borel subgroup* of $[L,L]$, a maximal connected solvable subgroup, and $T \subseteq B$ is a maximal torus of $[L,L]$, then it follows from [Bor91, 11.14] that $\mathrm{Rad}_u(G) \cdot B \cdot \mathrm{Z}(L)^0$ is a Borel subgroup of G and $T \cdot \mathrm{Z}(L)^0$ is a maximal torus of G. In particular, the number of irreducible components is at most the dimension of the maximal torus.

Consider the following examples of linear free divisors.

Example 5.15. We continue Example 5.9. Since G is solvable, $[L,L]$ is trivial and hence $\mathrm{Rad}_u(G)$ is defined by $a = c = 1$, and a maximal torus $L = \mathrm{Z}(L)$ is defined by $b = 0$. Corollary 5.12 is easy to check; in particular, the 2-element subgroup $L \cap G_{v_0}$ lies in the kernel of $(\chi_1, \chi_2) : G \to \mathbb{G}_{\mathrm{m}}{}^2$ restricted to $(\mathrm{Z}(L))^0$.

The following example is neither reductive nor solvable.

Example 5.16 ([DP12b, Example 9.4])**.** Define the algebraic group

$$G = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & d & e & 0 \\ f & g & h & i \end{pmatrix} \in \mathrm{GL}(\mathbb{C}^4) \right\}.$$

Let S be the space of 4×4 symmetric matrices with the usual coordinates x_{ij}, $1 \le i \le j \le 4$. Let $V \subset S$ be the subspace where $x_{11} = 0$. Let $\rho : G \to \mathrm{GL}(V)$ be defined by $\rho(A)(M) = AMA^T$. Note that $\ker(\rho) = \{\pm I\}$. A Levi decomposition of G has L defined by $f = g = h = 0$ and $\mathrm{Rad}(G)$ defined by $c = d = b - e = 0$ (the Borel subgroup of lower-triangular $g \in G$ is not normal in G). Then $\mathrm{Rad}_u(G)$ is defined by $a = b = e = i = 1$ and $c = d = 0$, $[L,L]$ is defined by $f = g = h = 0$ and $be - cd = a = i = 1$, and $L \cap \mathrm{Rad}(G) = \mathrm{Z}(L)^0$ is defined by $b = e$ and $c = d = f = g = h = 0$. Finally, $[G,G]$ is defined by $a = be - cd = i = 1$.

The exceptional orbit variety is the linear free divisor defined by $f_1 \cdot f_2 \cdot f_3$, where

$$f_1 = \begin{vmatrix} x_{22} & x_{23} \\ x_{23} & x_{33} \end{vmatrix}, \qquad f_2 = \begin{vmatrix} 0 & x_{12} & x_{13} \\ x_{12} & x_{22} & x_{23} \\ x_{13} & x_{23} & x_{33} \end{vmatrix}, \qquad f_3 = \begin{vmatrix} 0 & x_{12} & x_{13} & x_{14} \\ x_{12} & x_{22} & x_{23} & x_{24} \\ x_{13} & x_{23} & x_{33} & x_{34} \\ x_{14} & x_{24} & x_{34} & x_{44} \end{vmatrix},$$

corresponding to the characters $\chi_1 = (cd - be)^2$, $\chi_2 = a^2(cd - be)^2$, $\chi_3 = a^2(cd - be)^2 i^2$, respectively. Let

$$v_0 = \begin{pmatrix} 0&1&0&0\\ 1&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}, \qquad v_1 = \begin{pmatrix} 0&1&0&0\\ 1&0&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 0&0&1&1\\ 0&0&1&0\\ 1&1&0&0\\ 1&0&0&0 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 0&1&0&0\\ 1&1&0&0\\ 0&0&1&0\\ 0&0&0&0 \end{pmatrix} \in V$$

be generic points in Ω and on each $V(f_i)$. Then G_{v_0}, G_{v_1}, G_{v_2}, G_{v_3} respectively consist of all elements of G of the form

$$\begin{pmatrix} \gamma&0&0&0\\ 0&\gamma&0&0\\ 0&0&\delta&0\\ 0&0&0&\epsilon \end{pmatrix}, \qquad \begin{pmatrix} \frac{1}{a}&0&0&0\\ 0&a&0&0\\ 0&0&\delta&0\\ 0&0&0&\epsilon \end{pmatrix}, \qquad \begin{pmatrix} \frac{1}{a}&0&0&0\\ 0&\frac{1}{a}&0&0\\ 0&0&a&0\\ 0&0&0&a \end{pmatrix}, \qquad \begin{pmatrix} \epsilon&0&0&0\\ 0&\epsilon&0&0\\ 0&0&\delta&0\\ 0&0&0&a \end{pmatrix},$$

where $\gamma^2 = \delta^2 = \epsilon^2 = 1$ and $a \in \mathbb{C}$. With these calculations, it is straightforward to check the conclusions of Theorems 5.6 and 5.8 and Corollary 5.12.

5.5. **The structure of the Lie algebra.** We now summarize our results for the structure of the Lie algebra $\mathfrak{g}$ of G. In contrast to the terminology for groups, the usual *Levi decomposition* of $\mathfrak{g}$ expresses $\mathfrak{g}$ as the semidirect sum of the *radical* $\mathfrak{rad}\,\mathfrak{g}$, defined as the maximal solvable ideal, and a semisimple *Levi subalgebra*; these correspond to the Lie algebras of $\mathrm{Rad}(G)$ and $[L, L]$, respectively.

Proposition 5.17. *Let (G, ρ, V) define a linear free divisor $D \subset V$. Let G have a Levi decomposition as above, and let $\mathfrak{g}$, $\mathfrak{r}$, and $\mathfrak{l}$ be the Lie algebras of G, $\mathrm{Rad}_u(G)$, and L, respectively, and let $\mathfrak{z}(\mathfrak{l})$ denote the center of $\mathfrak{l}$. Then as vector spaces $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{z}(\mathfrak{l}) \oplus [\mathfrak{l}, \mathfrak{l}]$ where the ideal $\mathfrak{r}$ consists of nilpotent elements, $\mathfrak{z}(\mathfrak{l})$ is abelian and consists of semisimple elements, $[\mathfrak{l}, \mathfrak{l}]$ is semisimple, the ideal $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{r} \oplus [\mathfrak{l}, \mathfrak{l}]$, the ideal $\mathfrak{r} \oplus \mathfrak{z}(\mathfrak{l})$ equals the radical of $\mathfrak{g}$, and $\mathfrak{l} = \mathfrak{z}(\mathfrak{l}) \oplus [\mathfrak{l}, \mathfrak{l}]$. We thus have:*

$$\begin{array}{ccc} [\mathfrak{r}, \mathfrak{r}] \subseteq \mathfrak{r} & [\mathfrak{r}, [\mathfrak{l}, \mathfrak{l}]] \subseteq \mathfrak{r} & [\mathfrak{r}, \mathfrak{z}(\mathfrak{l})] \subseteq \mathfrak{r} \\ & [[\mathfrak{l}, \mathfrak{l}], [\mathfrak{l}, \mathfrak{l}]] = [\mathfrak{l}, \mathfrak{l}] & [[\mathfrak{l}, \mathfrak{l}], \mathfrak{z}(\mathfrak{l})] = 0 \\ & & [\mathfrak{z}(\mathfrak{l}), \mathfrak{z}(\mathfrak{l})] = 0. \end{array}$$

Proof. Most of this follows from the algebraic Levi decomposition of G. Since $[L, L]$ is semisimple and $\mathrm{Rad}_u(G)$ is unipotent, their Lie algebras are semisimple and nilpotent, respectively. Since $\mathfrak{r}$ is nilpotent, all of its elements are nilpotent. Then apply Corollary 5.12(2). $\square$

Remark 5.18. Theorem 6.1 of [GMNRS09] describes a normal form for a basis of the logarithmic vector fields of a linear free divisor. A key ingredient is a maximal subspace of simultaneously diagonalizable linear logarithmic vector fields, i.e., the vector fields corresponding to the Lie algebra $\mathfrak{t}$ of a maximal torus. By Remark 5.14, $\mathfrak{t}$ may always be chosen to contain $\mathfrak{z}(\mathfrak{l})$.

5.6. **Some special cases.** We now apply our results to several special types of linear free divisors.

5.6.1. *Abelian groups.* The *normal crossings divisor* in a vector space V is given by the union of all coordinate hyperplanes for some choice of vector space coordinates. It is a linear free divisor, and is the only linear free divisor defined by an abelian group.

Corollary 5.19 ([GMS11, Theorem 2.12])**.** *Let V be a finite-dimensional complex vector space and suppose that a connected complex linear algebraic group $G \subseteq \mathrm{GL}(V)$ defines a linear free divisor D in V. Then G is abelian if and only if D is equivalent, under a change of coordinates in V, to the normal crossings divisor.*

Proof. Let $n = \dim(G) = \dim(V)$.

If D is the normal crossings divisor, then after choosing a basis of V, G is the diagonal group in $\mathrm{GL}(V)$, isomorphic to $(\mathbb{G}_{\mathrm{m}})^n$. Thus G is abelian.

If G is abelian, then by Theorem 5.8(1), D has $\dim(G) = n$ irreducible components. By Theorem 5.8(4), G is isomorphic to $(\mathbb{G}_{\mathrm{m}})^n$, and hence is a maximal torus in $\mathrm{GL}(V)$. As all such tori are conjugate in $\mathrm{GL}(V)$, we may choose coordinates on V so that G is the group of diagonal matrices. A calculation then shows that D is the normal crossings divisor. □

5.6.2. *Irreducible linear free divisors.* We now examine the groups that produce irreducible linear free divisors. Observe that if $G \subseteq \mathrm{GL}(V)$ defines a linear free divisor $D \subset V$, then $\lambda \cdot I \in (\mathrm{GL}(V)_D)^0 = G$ for I the identity element and $\lambda \in \mathbb{C}^*$.

Recall that an algebraic group H is called *perfect* if $[H, H] = H$. For instance, semisimple groups are perfect.

Corollary 5.20. *Let $D \subset V$ be a linear free divisor defined by the group $G \subseteq \mathrm{GL}(V)$. Let $H = G \cap \mathrm{SL}(V)$, and let $K = (\mathbb{C}^*) \cdot I \subseteq G$. The following are equivalent:*

(1) D has 1 irreducible component.
(2) $H^0 = [G, G]$.
(3) $G = K \cdot [G, G]$.
(4) H^0 is perfect.
(5) There exists a perfect connected codimension 1 algebraic subgroup J of G.

When these hold, $J = [G, G] = H^0$.

Proof. Clearly $G = K \cdot H$, and there are a finite number of ways to write $g \in G$ as a product of elements of K and H. It follows that $\dim(H) = n-1$. The multiplication morphism $K \times H^0 \to G$ has a connected image of dimension $\dim(G)$, and hence $G = K \cdot H^0$.

If (1), then by Theorem 5.8(1), $\dim([G, G]) = n - 1$. Since $[G, G] \subseteq H$ are of the same dimension and $[G, G]$ is connected, we have $[G, G] = H^0$ and (2).

If (2), then by the above work, $G = K \cdot [G, G]$, giving (3).

Suppose (3). Since $[G, G] \subseteq H$ we have $\dim([G, G]) \leq n - 1$. Also, $\dim(G) \leq \dim(K) + \dim([G, G])$ shows $\dim([G, G]) \geq n - 1$, and hence $\dim([G, G]) = n - 1$. Since they are connected and of the same dimension, $[G, G] = H^0$. Since K is in the center of G, we have $[K \cdot N, K \cdot M] = [N, M]$ for any subgroups N and M of G. In particular,

$$H^0 = [G, G] = [K \cdot [G, G], K \cdot [G, G]] = [[G, G], [G, G]] = [H^0, H^0],$$

giving (4).

If (4), then since $\dim(H^0) = n - 1$, we have (5).

If (5), then since $J = [J, J] \subseteq [G, G] \subseteq H$ and $\dim(H) = n - 1$, we have $\dim([G, G]) = n - 1$. By Theorem 5.8(1), D has 1 irreducible component, proving (1). Finally, note that $J \subseteq [G, G] \subseteq H^0$ are all connected algebraic groups of the same dimension, hence equal. □

Remark 5.21. The case when H is semisimple was thoroughly explored in [GMS11]; by the Levi decomposition of G, this is equivalent to $\dim(\mathrm{Z}(L)) = 1$ and $\mathrm{Rad}_u(G) = \{e\}$. Are there other irreducible linear free divisors, with H^0 perfect and $\mathrm{Rad}_u(G) \neq \{e\}$? Since an irreducible representation that is a prehomogeneous vector space has

H semisimple by [Kim03, Theorem 7.21], in such an example the representation must be reducible.

5.6.3. *Reductive groups.* For reductive groups, we have the following.

Corollary 5.22 ([GMS11, Lemma 2.6])**.** *For a linear free divisor D defined by a reductive group G, the number of irreducible components of D equals the dimension of the center of G. Let H be the subgroup of G leaving invariant the level sets of the product $f_1 \cdots f_r$. Then D is irreducible if and only if H^0 is semisimple.*

Proof. Let G have a Levi decomposition. Since $\mathrm{Rad}_u(G)$ is trivial and G is itself a Levi subgroup, apply Corollary 5.12(1) to get the first statement.

If D is irreducible, then $r = 1$ and so by Theorem 5.8(3), $H = [G, G] \cdot G_{v_0}$. In particular, $H^0 = [G, G]$, and this is semisimple by the structure theory.

Conversely, suppose H^0 is semisimple. Since $H = \ker(\chi_1 \cdots \chi_r)$, by Theorem 5.6(1), H^0 has codimension 1 in G. Since H^0 is perfect, by Corollary 5.20, D is irreducible. □

Granger–Mond–Schulze also show ([GMS11, Theorem 2.7]) that for a linear free divisor $D \subset V$ defined by a reductive group G, the number of irreducible hypersurface components of D equals the number of irreducible G–modules in V.

Example 5.23. Let $D \subset V$ be a linear free divisor constructed from a *quiver* Q, a finite connected oriented graph with vertex set Q_0, edge set Q_1, and a dimension vector $d : Q_0 \to \mathbb{N}$ (see [GMNRS09, BM06, GMS11]). Here, the group G is a product over Q_0 of general linear groups, so G is reductive with $\dim(\mathrm{Z}(G)) = |Q_0|$, and V is the space of representations of (Q, d). When d is a real Schur root and Q has no oriented cycles, then G has an open orbit and a theorem of Kac states that the complement has $|Q_0| - 1$ irreducible hypersurface components ([GMNRS09, §4]). The apparent disagreement with Corollary 5.22 is resolved by observing that G does not define D in the sense of Definition 5.1 as the representation ρ of G has a 1-dimensional kernel contained in $\mathrm{Z}(G)$, whereas $\rho(G)$ defines D and is reductive with center of dimension $|Q_0| - 1$.

5.6.4. *Solvable groups.* Recall that by the Lie–Kolchin Theorem, any solvable linear algebraic group $G \subset \mathrm{GL}(V)$ has a basis of V in which G is lower triangular.

Corollary 5.24. *Let $D \subset V$ be a linear free divisor defined by a solvable group $G \subseteq \mathrm{GL}(V)$. Fix any basis which makes G lower triangular, and let $\phi : G \to (\mathbb{G}_{\mathrm{m}})^{\dim(V)}$ send g to the diagonal entries of g. Then:*

(1) The number of components of D equals the dimension of the maximal torus of G, and also $\dim(\phi(G))$.
(2) D has a hyperplane component.
(3) Let G_u be the subgroup consisting of unipotent elements of G. Then $[G, G] = G_u = \ker(\phi)$.
(4) Every $\chi \in X_1(G)$ factors through ϕ.

Proof. At first we proceed without the hypothesis that G defines a linear free divisor. Since G is connected and solvable, the Levi decomposition of G has $\mathrm{Rad}(G) = G$, $\mathrm{Rad}_u(G) = G_u$, L is a maximal torus of G, and in this case $[G, G] \subseteq G_u$. Note that ϕ is a homomorphism of linear algebraic groups, and by the definition of unipotent, $G_u = \ker(\phi)$. Then since $\dim(G) = \dim(G_u) + \dim(L) = \dim(\ker(\phi)) + \dim(L)$, we have

$$\dim(L) = \dim(\phi(G)). \tag{5.2}$$

Now let $\chi \in X(G)$. By the Jordan decomposition, $G_u \subseteq \ker(\chi)$, and hence $\ker(\phi) \subseteq \ker(\chi)$. If $\overline{\chi}$ and $\overline{\phi}$ are the induced homomorphisms on G/G_u, then the homomorphism $\lambda = \overline{\chi} \circ (\overline{\phi})^{-1} : \phi(G) \to \mathbb{G}_{\mathrm{m}}$ satisfies $\chi = \lambda \circ \phi$. Since λ is a character on a subtorus of $(\mathbb{G}_{\mathrm{m}})^{\dim(V)}$, by [Bor91, 8.2] λ extends to a character $\psi : (\mathbb{G}_{\mathrm{m}})^{\dim(V)} \to \mathbb{G}_{\mathrm{m}}$ with $\chi = \psi \circ \phi$. This proves (4).

Now assume that G defines a linear free divisor D. By Corollary 5.12(1), the number of components of D equals $\dim(\mathrm{Z}(L))$. Then the observation that $L = \mathrm{Z}(L)$ and (5.2) implies (1). By Theorem 5.8(5) we have $G_u \subseteq [G, G]$, and hence $[G, G] = G_u = \ker(\phi)$, proving (3).

Finally, the Lie–Kolchin Theorem guarantees an invariant complete flag in V, hence an invariant hyperplane H. As H cannot intersect Ω, H is a component of D, proving (2). In these lower-triangular coordinates, this H is defined by the "first coordinate". □

Remark 5.25. By Corollary 5.24(4), the characters corresponding to the basic relative invariants are functions of the diagonal entries.

5.7. **Degrees of the components.** Let G define a linear free divisor D with basic relative invariants $f_1, \ldots, f_r$. We have seen that r may be computed from the Lie algebra $\mathfrak{g}$ of G; may the degrees of the f_i be computed from $\mathfrak{g}$?

If we only use the abstract Lie algebra structure of $\mathfrak{g}$, then the answer is no:

Example 5.26. Consider the following two linear free divisors in $\mathbb{C}^5$:

$$\begin{aligned} D_1: &\quad (x_3x_5 - x_4^2)\begin{vmatrix} 0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{vmatrix} = 0, \\ \text{and } D_2: &\quad (x_2^2x_3^2 - 4x_1x_3^3 - 4x_2^3x_4 + 18x_1x_2x_3x_4 - 27x_4^2x_1^2)x_5 = 0. \end{aligned}$$

This D_1 is [DP12b, Example 9.4], while D_2 is the product-union of a hyperplane with [GMS11, Theorem 2.11(2)]. The degrees of the polynomials defining the irreducible components of D_1 and D_2 differ. However, the groups defining D_1 and D_2 have the same abstract Lie algebra structure: $\mathfrak{gl}_2(\mathbb{C}) \oplus \mathfrak{gl}_1(\mathbb{C})$. Thus, D_1 and D_2 are constructed from inequivalent representations of the same abstract Lie algebra.

Of course, since the representation of the Lie algebra determines the divisor, the representation undoubtedly contains the information necessary to compute these degrees. How may we do so effectively?

Lemma 5.27. *Let $G \subseteq \mathrm{GL}(V)$ define the linear free divisor D with basic relative invariants $f_1, \ldots, f_r$, with $f_i \overset{\mathrm{m}}{\longleftrightarrow} \chi_i$. Choose $X_i \in \mathfrak{g} \subseteq \mathfrak{gl}(V)$ such that $d(\chi_i)_{(e)}(X_j) = \delta_{ij}$, or equivalently, $\xi_{X_i}(f_j) = \delta_{ij} f_j$. Let I denote the identity endomorphism in both $G \subseteq \mathrm{GL}(V)$ and $\mathfrak{g} \subseteq \mathfrak{gl}(V)$, with ξ_I the Euler vector field. Let* *the module $A \subseteq \mathrm{Der}_V$ consist of the vector fields annihilating all f_i. Then:*

$$\xi_I(f_i) = \deg(f_i) f_i, \tag{5.3}$$

$$d(\chi_i)_{(e)}(I) = \deg(f_i), \tag{5.4}$$

$$\chi_i(\lambda \cdot I) = \lambda^{\deg(f_i)} \quad \textit{for all } \lambda \in \mathbb{C}^*, \tag{5.5}$$

$$I = \sum_{j=1}^{r} \deg(f_j) X_j \bmod [\mathfrak{g}, \mathfrak{g}], \tag{5.6}$$

$$\textit{and } \xi_I = \sum_{j=1}^{r} \deg(f_j) \xi_{X_j} \bmod A. \tag{5.7}$$

Proof. Corollary 5.10 shows that such X_i exist. By (3.8), the ξ_{X_i} have the claimed effects on f_j for all i, j. Then (5.3) is just the Euler relation, and applying (3.8) shows (5.4). Integrating (5.4) shows (5.5). By (5.4), for each i we have

$$d(\chi_i)_{(e)}\Big(I - \sum_{j=1}^{r} \deg(f_j) X_j\Big) = \deg(f_i) - \deg(f_i) = 0,$$

and since $\cap_{i=1}^{r} \ker(d(\chi_i)_{(e)}) = [\mathfrak{g}, \mathfrak{g}]$ by Theorem 5.6(1), we have (5.6). A similar argument using (5.3) shows that $\xi_I - \sum_{j=1}^{r} \deg(f_j) \xi_{X_j}$ annihilates each f_i, giving (5.7). $\square$

It is unclear whether the embedding $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ may be used to find the degrees without first finding either the χ_i, $d(\chi_i)_{(e)}$, or f_i.

Remark 5.28. For $d \in \mathbb{N}$ let $R_d \subseteq \mathbb{G}_{\mathrm{m}}$ denote the group of dth roots of unity. Then in the situation of Lemma 5.27 with $G \subseteq \mathrm{GL}(V)$, (5.5) implies that

$$\ker(\chi_i|_{\mathbb{G}_{\mathrm{m}} \cdot I}) = \ker(\chi_i) \cap (\mathbb{G}_{\mathrm{m}} \cdot I) = R_{\deg(f_i)} \cdot I.$$

Then by Theorem 5.6(1) and number theory, $([G, G] \cdot G_{v_0}) \cap (\mathbb{G}_{\mathrm{m}} \cdot I) = R_d \cdot I$ for d the greatest common divisor of $\{\deg(f_1), \ldots, \deg(f_r)\}$.

5.8. **Homotopy groups of $V \setminus D$.** The results above can give some insight into the topology of the complement of a linear free divisor D and two types of (global) Milnor fiber associated to D.

Proposition 5.29. *Let $G \subseteq \mathrm{GL}(V)$ define a linear free divisor $D \subset V$ with irreducible components defined by $f_1, \ldots, f_r$, and $f_i \xleftrightarrow{\mathrm{m}} \chi_i$. Let L be a Levi factor of G. Let $v_0 \in \Omega$, let $F = (f_1, \ldots, f_r) : V \to \mathbb{C}^r$, let K be the fiber of $f_1 \cdots f_r$ containing v_0, and let $P \subset K$ be the fiber of F containing v_0. Use v_0 or $e \in G$ as the base point for all homotopy groups. Then for $n > 1$,*

$$\pi_n(L) \cong \pi_n([L, L]) \cong \pi_n(G) \cong \pi_n([G, G]) \cong \pi_n(P) \cong \pi_n(\Omega) \cong \pi_n(K),$$

and for $n = 1$ we have the exact diagrams in Figure 1.

Proof. If $G_1 \subset G_2 \subset G_3$ are Lie groups, then by [Ste51, §7.4–7.5] the map of cosets $G_3/G_1 \to G_3/G_2$ is a fiber bundle with fiber G_2/G_1. Thus, we have the following

FIGURE 1. The exact diagrams describing the fundamental groups of the spaces in Proposition 5.29. For a map ϕ, let ϕ_* denote the associated map of fundamental groups. Each i is constructed from an inclusion map. We do not claim commutativity of these diagrams.

$$
\begin{tikzcd}
 & & & & 0 \arrow[d] & 0 \arrow[dl] & \\
 & & & & \pi_1(P) \arrow[dl, "i_*"'] \arrow[d, "i_*"] & & \\
 & & 0 \arrow[r] & \pi_1(K) \arrow[r, "i_*"] \arrow[dll, "p_{8*}"] & \pi_1(\Omega) \arrow[r, "(m\circ p_7)_*"] \arrow[d, "p_{7*}"] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbb{Z}^{r-1} \arrow[rrr] \arrow[dl] & & & \mathbb{Z}^r \arrow[r, "m_*"] \arrow[d] & \mathbb{Z} \arrow[r] & 0 \\
0 & & & & 0 & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \pi_1([G,G]) \arrow[r, "p_{2*}"] \arrow[d, "i_*"'] & \pi_1(P) \arrow[r] \arrow[d, "i_*"] & {[G,G]} \cap G_{v_0} \arrow[r] & 0 \\
0 \arrow[r] & \pi_1(G) \arrow[r, "p_{1*}"] \arrow[d, "p_{5*}"'] & \pi_1(\Omega) \arrow[r] \arrow[d, "p_{7*}"] & G_{v_0} \arrow[r] & 0 \\
 & \mathbb{Z}^r \arrow[d] & \mathbb{Z}^r \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \pi_1([G,G]) \arrow[r, "i_*"] \arrow[d, "p_{4*}"'] & \pi_1(G) \arrow[r, "p_{5*}"] \arrow[d, "p_{3*}"] & \mathbb{Z}^r \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \pi_1([L,L]) \arrow[r, "i_*"'] \arrow[d] & \pi_1(L) \arrow[r, "p_{6*}"'] \arrow[d] & \mathbb{Z}^r \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}
$$

fiber bundles:

$$
\begin{aligned}
p_1 &: G \to G/G_{v_0} \cong \Omega & p_2 &: [G,G] \to [G,G]/([G,G] \cap G_{v_0}) \\
p_3 &: G \to G/\mathrm{Rad}_u(G) \cong L & p_4 &: [G,G] \to [G,G]/\mathrm{Rad}_u(G) \cong [L,L] \\
p_5 &: G \to G/[G,G] \cong \mathbb{G}_{\mathrm{m}}{}^r & p_6 &: L \to L/[L,L] \cong \mathbb{G}_{\mathrm{m}}{}^r \\
q &: G/G_{v_0} \to G/[G,G] \cdot G_{v_0} & &
\end{aligned}
$$

The identifications of the codomains of p_1, p_3, p_4, p_5, and p_6 are by, respectively, the orbit map α_{v_0} of v_0, the definition of the Levi decomposition, Corollary 5.12(2), Theorem 5.8(4), and the isomorphism

$$L/[L,L] \cong (G/\mathrm{Rad}_u(G))/([G,G]/\mathrm{Rad}_u(G)) \cong G/[G,G]$$

that follows from the Levi decomposition and Corollary 5.12(2).

Since G_{v_0} and $[G,G] \cap G_{v_0}$ are finite, p_1 and p_2 are covering spaces with deck transformation groups isomorphic to G_{v_0} and $[G,G] \cap G_{v_0}$, respectively ([Hat02, Proposition 1.40]).

For q, we use the orbit map to identify G/G_{v_0} with Ω and then by Theorem 5.6(1) compose with the isomorphism $G/[G,G] \cdot G_{v_0} \to (\mathbb{G}_{\mathrm{m}})^r$ induced by $(\chi_1, \ldots, \chi_r) : G \to (\mathbb{G}_{\mathrm{m}})^r$. This gives a fiber bundle

$$p_7 : \Omega \to (\mathbb{G}_{\mathrm{m}})^r$$

defined by $p_7(\rho(g)(v_0)) = (\chi_1, \ldots, \chi_r)(g)$ with fiber homeomorphic to $\ker(\chi_1, \ldots, \chi_r)/G_{v_0} = [G,G] \cdot G_{v_0}/G_{v_0}$. As the action of $[G,G]$ on $[G,G] \cdot G_{v_0}/G_{v_0}$ is smooth and transitive, and the isotropy subgroup at eG_{v_0} is $[G,G] \cap G_{v_0}$, the fiber of p_7 is isomorphic to $[G,G]/([G,G] \cap G_{v_0})$.

Let $m : (\mathbb{G}_{\mathrm{m}})^r \to \mathbb{G}_{\mathrm{m}}$ and $n : (\mathbb{C}^*)^r \to \mathbb{C}^*$ both be defined by $(a_1, \ldots, a_r) \mapsto a_1 \cdots a_r$. By (1.1) we have a commutative diagram

(5.8)
$$\begin{tikzcd}
 & (\mathbb{G}_{\mathrm{m}})^r \arrow[r, "m"] \arrow[d, "\beta"] & \mathbb{G}_{\mathrm{m}} \arrow[d, "\gamma"] \\
\Omega \arrow[ur, "p_7"] \arrow[r, "F"] \arrow[rr, bend right, "f_1 \cdots f_r"'] & (\mathbb{C}^*)^r \arrow[r, "n"] & \mathbb{C}^*
\end{tikzcd}$$

where $\beta(a_1, \ldots, a_r) = (a_1 f_1(v_0), \ldots, a_r f_r(v_0))$ and $\gamma(a) = a(f_1 \cdots f_r)(v_0)$ are homeomorphisms. By (5.8), P is homeomorphic to the fiber of p_7, that is, $P \cong [G,G]/([G,G] \cap G_{v_0})$, and hence P is the codomain of p_2. Also, K is homeomorphic to $(m \circ p_7)^{-1}(1)$; restricting p_7 gives a fiber bundle

$$p_8 : K \to \ker(m) \cong (\mathbb{G}_{\mathrm{m}})^{r-1}$$

with fiber P.

By [Hat02, Proposition 4.48], a fiber bundle is a Serre fibration, that is, it possesses the homotopy lifting property for CW complexes, and these have the usual homotopy long exact sequence of a fibration ([Hat02, Theorem 4.41]); apply this sequence to all p_i and $m \circ p_7$. Note that the connected unipotent group $\mathrm{Rad}_u(G)$ is diffeomorphic to some $\mathbb{C}^p$, and hence is contractible ([OV90, §3.3.6]). Also, G_{v_0} and $[G,G] \cap G_{v_0}$ are finite, $P \cong [G,G]/[G,G] \cap G_{v_0}$ is connected, $\pi_1((\mathbb{G}_{\mathrm{m}})^k) \cong \mathbb{Z}^k$, and $\pi_i((\mathbb{G}_{\mathrm{m}})^k) \cong 0$ for $i > 1$.

The long exact sequences show that for $n > 1$ there is a diagram, not necessarily commutative,

$$\begin{tikzcd}
\pi_n([L,L]) \arrow[d, "p_6"'] & \pi_n([G,G]) \arrow[l, "p_4"'] \arrow[r, "p_2"] \arrow[d, "p_5"'] & \pi_n(P) \arrow[r, "p_8"] \arrow[d, "p_7"'] & \pi_n(K) \arrow[dl, "m \circ p_7"] \\
\pi_n(L) & \pi_n(G) \arrow[l, "p_3"'] \arrow[r, "p_1"] & \pi_n(\Omega) &
\end{tikzcd}$$

where an arrow labeled by ϕ represents an isomorphism occurring in the long exact sequence of the fibration ϕ, either ϕ_* or i_* for i the inclusion of the fiber. The sequences also give the exact diagrams in Figure 1. □

Thus, the homotopy groups may largely be computed from the homotopy groups of the semisimple part $[L,L]$ of G. For instance, if G is solvable then $[L,L] = \{e\}$ and hence by Proposition 5.29, Ω, P, and K are $K(\pi, 1)$ spaces, as shown in [DP12a].

5.9. **(Non-linear) free divisors.** Let (D, p) be an arbitrary free divisor in $(\mathbb{C}^n, p)$. May the number of components of (D, p) be computed from the structure of $M = \mathrm{Der}_{\mathbb{C}^n,p}(-\log D)$? One natural guess, $\dim_{\mathbb{C}}(M/\mathscr{O}_{\mathbb{C}^n,p} \cdot [M, M])$, does not work. For example, if D is a hyperplane in $\mathbb{C}^2$, then the number computed is 0; in fact, $M = \mathscr{O}_{\mathbb{C}^n,p} \cdot [M, M]$ whenever $\mathrm{Der}_{\mathbb{C}^n,p}(-\log D) \nsubseteq \mathscr{M}_p \cdot \mathrm{Der}_{\mathbb{C}^n,p}$ for $\mathscr{M}_p$ the maximal ideal. Other examples give answers too large: for the plane curve $D = V((a^2 - b^3)(a^7 - b^{13}))$, the number computed is 35.

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Dept. of Computer and Mathematical Sciences, University of Toronto Scarborough, 1265 Military Trail, Toronto, ON M1C 1A4, Canada

E-mail address: `bapike@gmail.com`