SOLVABLE GROUPS, FREE DIVISORS AND NONISOLATED MATRIX SINGULARITIES II: VANISHING TOPOLOGY

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Abstract. In this paper we use the results from the first part to compute the vanishing topology for matrix singularities based on certain spaces of matrices. We place the variety of singular matrices in a geometric configuration of free divisors which are the "exceptional orbit varieties" for repesentations of solvable groups. Because there are towers of representations for towers of solvable groups, the free divisors actually form a tower of free divisors $\mathcal{E}_n$, and we give an inductive procedure for computing the vanishing topology of the matrix singularities. The inductive procedure we use is an extension of that introduced by Lê-Greuel for computing the Milnor number of an ICIS. Instead of linear subspaces, we use free divisors arising from the geometric configuration and which correspond to subgroups of the solvable groups.

Here the vanishing topology involves a singular version of the Milnor fiber; however, it still has the good connectivity properties and is homotopy equivalent to a bouquet of spheres, whose number is called the singular Milnor number. We give formulas for this singular Milnor number in terms of singular Milnor numbers of various free divisors on smooth subspaces, which can be computed as lengths of determinantal modules. In addition to being applied to symmetric, general and skew-symmetric matrix singularities, the results are also applied to compute the Milnor number of isolated Cohen-Macaulay surface singularities in $\mathbb{C}^4$ and the difference of Betti numbers of Milnor fibers for isolated Cohen-Macaulay 3-fold singularities in $\mathbb{C}^5$.

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Introduction

In this paper we make use of the results from the first part of the paper [DP1] to introduce a method for computing the "vanishing topology" of nonisolated complex matrix singularities. A complex matrix singularity arises from a holomorphic germ $f_0 : \mathbb{C}^n, 0 \to M, 0$, where M denotes the space of $m \times m$ complex matrices, which may be either symmetric or skew-symmetric (and then m is even), or more general $m \times p$ complex matrices. If $\mathcal{V}$ denotes the "determinantal variety" of singular matrices, then $\mathcal{V}_0 = f_0^{-1}(\mathcal{V})$ is the corresponding matrix singularity. We shall also refer to the mapping f_0 as defining a matrix singularity; it can also be viewed as a "nonlinear section of $\mathcal{V}$" (although we also allow $n \geq \dim(M)$). In part I, we indicated many examples of matrix singularities for the classification of various types of singularities.

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For $m \times m$ matrices, if $n \leq \operatorname{codim}(\operatorname{sing}(\mathcal{V}))$ and f_0 is transverse to $\mathcal{V}$ off the origin, then $\mathcal{V}_0$ has an isolated singularity, defined by $H \circ f_0$, where $H : M \to \mathbb{C}$ denotes the determinant, or the Pfaffian in the skew-symmetric case (m even). Using algebraic resolutions, Goryunov-Mond [GM] showed that for isolated matrix singularities in all three cases, the Milnor number equaled τ, which is a $\mathcal{K}_H$–deformation theoretic codimension, with a correction term given by a two term Euler characteristic for an appropriate Tor complex.

$$\mu(H \circ f_0) \;=\; \tau + (\beta_0 - \beta_1)\,.$$

This explained an observed result of Bruce [Br] for simple symmetric matrix singularities for $n = 2 = \operatorname{codim}(\operatorname{sing}(\mathcal{V})) - 1$,

Although the Milnor number in the isolated case can be computed from Milnor's formula, the relation between it and the deformation theoretic codimension suggests there may exist such a relation in the nonisolated case, where there are no known general results about the topology of the Milnor fiber. However, the difficulty in determining the vanishing topology of matrix singularities in general is due to their highly singular structure. Hence, by the Kato-Matsumoto Theorem, its Milnor fiber will have very low connectivity and can have homology in many dimensions.

We overcome this problem by viewing $f_0 : \mathbb{C}^n, 0 \to M, 0$ as a nonlinear section of $\mathcal{V}$ and consider instead the "singular Milnor fiber". It is obtained as a "stabilization of f_0"and is homotopy equivalent to a bouquet of spheres of real dimension $n-1$. The number of such spheres $\mu_{\mathcal{V}}(f_0)$ is called the "singular Milnor number"of f_0, and it can be computed for free divisors $\mathcal{V}$ (in the sense of Saito [Sa]) by a Milnor -type formula as the length of of a determinantal module, [DM] and [D2]. In the case when $n < \dim(\operatorname{sing}(\mathcal{V}))$, then $\mathcal{V}_0$ is an isolated singularity and these are the usual Milnor fiber and Milnor number. That matrix singularities $\mathcal{V}$ are essentially never free divisors explains the need for a correction term in [GM] for the isolated case.

Instead we shall introduce an inductive method which extends that introduced by Lê-Greuel [LGr] for computing the Milnor number of an ICIS. Their method uses a geometric configuration formed from a flag of linear subspaces transverse to the map germ which we replace with a tower of linear free divisors constructed in Part I [DP1]. These arise from a tower of (modified) Cholesky–type representations of solvable linear algebraic groups. This allows us to adjoin a linear free divisor to the determinantal variety $\mathcal{V}$ to obtain another a linear free divisor, providing a "free completion"of $\mathcal{V}$.

The general form of the formula which we give expresses $\mu_{\mathcal{V}}(f_0)$ as a linear combination with integer coefficients

$$\mu_{\mathcal{V}}(f_0) \;=\; \sum_i a_i \mu_{\mathcal{W}_i}(f_0) \tag{0.1}$$

where the $\mathcal{W}_i$ are free divisors on linear subspaces of M. Thus, we can express $\mu_{\mathcal{V}}(f_0)$ as a linear combination of singular Milnor numbers, each of which can be computed using results from [D2] as lengths of determinantal modules.

If we view these singular Milnor numbers as functions on the space of germs f_0 transverse to the varieties off 0, then (0.1) can be written more simply as

$$\mu_{\mathcal{V}} \;=\; \sum_i a_i \mu_{\mathcal{W}_i}\,. \tag{0.2}$$

Furthermore, the method allows us to compute more generally the singular Milnor numbers for nonisolated matrix singularities on an ICIS X. There is a metatheorem which states that if X is defined by $\varphi : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$, and the formula (0.2) for $\mu_{\mathcal{V}}$ is obtained by the inductive process then the process also yields the formula

$$\mu_{\varphi, \mathcal{V}} \; = \; \sum_i a_i \mu_{\varphi, \mathcal{W}_i} \tag{0.3}$$

where $\mu_{\varphi, \mathcal{V}}(f_0)$, respectively $\mu_{\varphi, \mathcal{W}_i}(f_0)$, are the singular Milnor numbers for $f_0|X$ as nonlinear sections of $\mathcal{V}$, resp. $\mathcal{W}_i$, and can again be computed in terms of lengths of determinantal modules using a generalization of the Lê-Greuel theorem given in [D2].

These formulas are applied in § 6, 7, and 9 to obtain explicit formulas for symmetric and general 2×2 and 3×3 matrices and 4×4 skew-symmetric matrices.

Furthermore, general 2×3 matrix singularities are not complete intersection singularities; however they are Cohen–Macaulay singularities by the Hilbert-Burch Theorem [Hi], [Bh]. We next apply these methods in §8 to obtain the singular vanishing Euler characteristic $\tilde{\chi}_{\mathcal{V}}$ as a linear combination as in (0.2). We then deduce a formula for the Milnor number of isolated Cohen-Macaulay surface singularities in $\mathbb{C}^4$ as an alternating sum of lengths of determinantal modules (Theorem 8.3). Furthermore, for isolated 3–fold Cohen-Macaulay singularities, we give an analogous formula for the difference between the second and third Betti numbers $b_3 - b_2$ of the Milnor fiber (Theorem 8.4). This formula is also valid for Cohen–Macaulay singularities defined as matrix singularities defined on an ICIS.

This formula has been programmed in Macaulay2 by the second author [P2] and has been used to compute for the simple Cohen-Macaulay singularities, classified by Frühbis–Krüger–Neumer [FN], the Milnor numbers for those in $\mathbb{C}^4$ and the difference of Betti-numbers for the Milnor fiber for the 3–fold singularities in $\mathbb{C}^5$. In §11, these computer calculations are applied to verify a conjecture relating μ and τ for the surface case, and discover unexpected behavior of $b_3 - b_2$ and τ for the 3–fold singularities.

Besides obtaining general formulas as in (0.2) for the various cases, we also introduce two methods of reduction. In the case of 2×2 symmetric matrices, the terms in the linear combination represent the lengths of determinantal modules and the algebraic relations between these modules then allow us to combine them into a "Jacobian formula". This is a first step to finding more general reduction formulas to simplify (0.2).

The second method of "generic reduction" can be applied to all cases and uses the "defining codimensions" of the $\mathcal{W}_i$ in M. We may rewrite (0.2) in the form

$$\mu_{\mathcal{V}} \; = \; \lambda_0 \, + \, \lambda_1 \, + \cdots + \, \lambda_{N-1} \qquad\qquad (N = \dim(M)) \tag{0.4}$$

where λ_j denotes the sum of the terms in (0.2) for which the defining codimension of $\mathcal{W}_i$ is j. If $\mathrm{codim}(Im(df_0(0))) = k$ and we may apply a generic matrix transformation to f_0 so that $Im(df_0(0))$ projects submersively onto all of the defining linear subspaces of codimension $\geq k$ associated to the $\mathcal{W}_i$, then $\lambda_i(f_0) = 0$ for $i \geq k$, and the formula (0.4) can be reduced to

$$\mu_{\mathcal{V}}(f_0) \; = \; \lambda_0(f_0) \, + \, \lambda_1(f_0) \, + \cdots + \, \lambda_{k-1}(f_0)\,. \tag{0.5}$$

In essence the remaining terms are "higher order terms" which do not contribute in the generic case. We deduce a number of consequences of this reduction for the different types of matrices, and obtain $\mu = \tau$ type results for generic corank 1 mappings defining matrix singularities of the various types (Theorem 11.2).

Because the method applies quite generally to the exceptional orbit varieties for representations of solvable linear algebraic groups which form "block representations", these results will extend to many other representations of solvable linear algebraic groups.

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1. Outline of the method

We begin by outlining how we extend the Lê–Greuel method to apply to matrix singularities, and then illustrate the calculation for the simplest case of 2×2 symmetric matrices.

Let M be the space of $m \times m$ complex matrices which are symmetric or skew-symmetric, or $m \times p$ general matrices. We also let $\mathcal{V}$ denote the subvariety of singular matrices in M (by which we mean more singular than the generic matrix in M).

Definition 1.1. A matrix singularity is defined by a holomorphic germ

$$f_0 : \mathbb{C}^n, 0 \longrightarrow M, 0 \tag{1.1}$$

(or more generally, $f_0 : X, 0 \to M, 0$ for an analytic germ $X, 0$). The pull-back variety $\mathcal{V}_0 = f_0^{-1}(\mathcal{V})$ is the "matrix singularity" defined by f_0.

$$\begin{array}{ccccc} & & \mathbb{C}^n, 0 & \xrightarrow{f_0} & M, 0 \\ & & \uparrow & & \uparrow \\ f_0^{-1}(\mathcal{V}) & = & \mathcal{V}_0, 0 & \longrightarrow & \mathcal{V}, 0 \end{array} \tag{1.2}$$

For these singularities we require that f_0 is transverse to $\mathcal{V}$ off $0 \in \mathbb{C}^n$. The determinantal varieties $\mathcal{V}$ are highly singular. The singular set of the determinantal varieties has codimension in M equal to 3 (symmetric case), 4 general $m \times m$ case, or 6 for the skew symmetric case (m even); and by the Kato-Matsumoto Theorem [KM], the Milnor fiber of $\mathcal{V}_0$ will only be guaranteed to be 1-connected (symmetric case), 2-connected (general case), or 4-connected (skew-symmetric case).

To describe their vanishing topology, we initially replace the Milnor fiber by the "singular Milnor fiber". As $f_0 : \mathbb{C}^n, 0 \to M, 0$ is transverse to $\mathcal{V}$ off 0, we may use instead a stabilization $f_t : B_\varepsilon \to M$ of f_0. This means that for $t \neq 0$, f_t is transverse to $\mathcal{V}$ on B_ε. The *singular Milnor fiber* is then the fiber $\mathcal{V}_t = f_t^{-1}(\mathcal{V})$. By results in [DM] and [D2] (using a result of Lê), $\mathcal{V}_t$ is homotopy equivalent to a bouquet of spheres of real dimension $n-1$, whose number we denote by $\mu_{\mathcal{V}}(f_0)$ and which we call the "singular Milnor number". This continues to hold if $\mathcal{V}$ is a complete intersection, or if $f_0 : X, 0 \to M, 0$ for an ICIS $X, 0$ [D2]. If $\mathcal{V}$ is not a complete intersection, then we consider instead the *singular vanishing Euler characteristic* $\tilde{\chi}_{\mathcal{V}}(f_0) = \chi(\mathcal{V}_t) - 1$. These numbers have Milnor-type formulas if $\mathcal{V}$ is a free divisor or a free divisor on a smooth subspace (see §3).

As the determinantal varieties consisting of singular matrices are neither of these types, we will modify the method of Lê–Greuel to compute them inductively using free divisors. We recall how the Lê–Greuel formula is used to compute the Milnor number of an ICIS.

Computing Milnor Numbers of ICIS via Geometric Configurations. Unlike the case of isolated hypersurface singularities, except in the weighted homogeneous case there is no simple algebraic formula such as Milnor's formula for computing the Milnor number of an ICIS $f : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$. For the general case, the Lê–Greuel formula provides an inductive method as follows.

We choose a geometric configuration which consists of a full flag of subspaces $0 \subset \mathbb{C} \subset \mathbb{C}^2 \subset \cdots \subset \mathbb{C}^p$ transverse to f off 0. If $(y_1, \dots, y_p)$ denote coordinates defining these subspaces, we let $\mu_{y_1,\dots,y_k}(f) = \mu(\pi_k \circ f)$, where π_k denote projection onto the subspace $\mathbb{C}^k \times \{0\}$. Then, the Milnor number $\mu(f)$ is given by

$$\mu(f) \;=\; (\mu_{y_1,\dots,y_p}(f) + \mu_{y_1,\dots,y_{p-1}}(f)) \;-\; (\mu_{y_1,\dots,y_{p-1}}(f) \;+\; \mu_{y_1,\dots,y_{p-2}}(f)) \\ +\dots \pm ((\mu_{y_1,y_2}(f) \;+\; \mu_{y_1}(f)) - \mu_{y_1}(f))\,, \tag{1.3}$$

and each 2–term sum in parentheses represents the Milnor number of an isolated singularity on an ICIS and can be computed using the Lê–Greuel Theorem.

Theorem 1.2 (Lê–Greuel)**.** *For an ICIS $f = (f_1, f_2) : \mathbb{C}^n, 0 \to \mathbb{C}^{k+1}, 0$, with $f_2 : \mathbb{C}^n, 0 \to \mathbb{C}^k, 0$ also an ICIS,*

$$\mu(f) + \mu(f_2) \quad = \quad \dim_{\mathbb{C}}(\mathcal{O}_n/(f_2^* m_k + Jac(f)))\,.$$

Thus, $\mu(f)$ is not computed directly, but rather as an alternating sum of lengths of algebras, determined by a configuration of subspaces in $\mathbb{C}^p$.

Inductive Procedure for Computing Singular Milnor Numbers via Free Completions. We will use an analogous approach for computing the singular Milnor number of a matrix singularity. We give an inductive approach, for which the geometric configuration is given by a free divisor $\mathcal{E}_m$ appearing in one of the towers of free divisors from Part I [DP1] (see Table 2). This provides a "free completion"of the determinantal variety $\mathcal{D}_m$ of singular matrices, $\mathcal{E}_m = \pi^*\mathcal{E}_{m-1} \cup \mathcal{D}_m$.

Quite generally we define

Definition 1.3. A hypersurface singularity $\mathcal{W}, 0 \subset \mathbb{C}^N, 0$ has a *free completion* if there is a free divisor $\mathcal{V}, 0 \subset \mathbb{C}^N, 0$ such that $\mathcal{V} \cup \mathcal{W}, 0$ is again a free divisor.

Then, we may apply (3.4) of Lemma 3.5 to obtain

$$\mu_{\mathcal{D}_m}(f_0) \;=\; \mu_{\mathcal{E}_m}(f_0) \;-\; \mu_{\pi^*\mathcal{E}_{m-1}}(f_0) \;+\; (-1)^{n-1}\,\tilde{\chi}_{\pi^*\mathcal{E}_{m-1}\cap\mathcal{D}_m}(f_0)\,. \tag{1.4}$$

Because all of the $\pi^*\mathcal{E}_m$ are H-holonomic, the $\mu_{\pi^*\mathcal{E}_m}$ can be computed as lengths of determinantal modules by Theorem 3.1. This reduces the calculation of $\mu_{\mathcal{D}_m}(f_0)$ to computing $\tilde{\chi}_{\pi^*\mathcal{E}_{m-1}\cap\mathcal{D}_m}(f_0)$.

We proceed inductively to decompose $\pi^*\mathcal{E}_{m-1} \cap \mathcal{D}_m$ into a union of components each of which can be represented as divisors on ICIS. We then use either free completions for these divisors or completions by divisors which themselves have free completions. We may again inductively apply Lemma 3.5 to further reduce to computing the vanishing Euler characteristics for divisors on ICIS, where we repeat the inductive process. Eventually we are reduced to computing the singular Milnor numbers of almost free divisors on ICIS, which we can compute using either Theorem 3.1 or Theorem 3.2.

In analogy with the notation used to explain the case of ICIS, to represent the singular Milnor number of f_0 for a variety defined by $(g_1, \dots, g_r)$, we use the notation $\mu_{g_1,\dots,g_r}(f_0)$. The final form the formula will take is that of (0.2), where each $\mu_{\mathcal{W}_i}$ is given in the form just described.

If instead we consider matrix singularities $f_0 : X, 0 \to M, 0$ on an ICIS $X, 0$ defined by $\varphi : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$, then the same arguments may be repeated to obtain a formula of the form (0.3).

2×2 **Symmetric Matrix Singularities.**

As an initial example to illustrate these ideas, we consider the 2×2 symmetric matrices, denoted Sym_2 and use coordinates $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$. The variety of singular matrices is $\mathcal{D}_2^{sy}$ defined by $ac - b^2 = 0$. Then, by Theorem 6.2 of [DP1], it has a free completion $\mathcal{E}_2^{sy} = \pi^*\mathcal{E}_1^{sy} \cup \mathcal{D}_2^{sy}$, where $\mathcal{E}_2^{sy}$ is defined by $a\,(ac - b^2) = 0$ and $\pi^*\mathcal{E}_1^{sy}$ by $a = 0$.

By the preceding, it is sufficient to determine $\tilde{\chi}_{\pi^*\mathcal{E}_1^{sy} \cap \mathcal{D}_2^{sy}}(f_0)$. Then, set–theoretically,

$$\pi^*\mathcal{E}_1^{sy} \cap \mathcal{D}_2^{sy} \;=\; V(a, ad - b^2) \;=\; V(a, b)\,.$$

Hence,

$$\tilde{\chi}_{\pi^*\mathcal{E}_1^{sy} \cap \mathcal{D}_2^{sy}} \;=\; (-1)^{n-2}\mu_{a,b}\,.$$

Since $\mu_{\pi^*\mathcal{E}_1^{sy}}(f_0) = \mu_a(f_0)$, by substituting into (1.4) we obtain

$$\mu_{\mathcal{D}_2^{sy}}(f_0) \;=\; \mu_{\mathcal{E}_2^{sy}}(f_0) \,-\, (\mu_a(f_0) \,+\, \mu_{a,b}(f_0)) \tag{1.5}$$

where $\mu_{\mathcal{E}_2^{sy}}(f_0)$ can be computed via Theorem 3.1 as the length of a determinantal module and $\mu_a(f_0) + \mu_{a,b}(f_0)$, by the Lê–Greuel formula (Theorem 1.2). A complete statement is given in Theorem 6.1.

This example is especially simple as $\pi^*\mathcal{E}_1^{sy} \cap \mathcal{D}_2^{sy}$ is set–theoretically a complete intersection. In general it will require a number of inductive steps to decompose $\pi^*\mathcal{E}_{m-1} \cap \mathcal{D}_m$ and use auxiliary solvable group representations to constuct additonal free completions for the components.

Remark 1.4. In order to apply the inductive method, we must have the germ $f_0 : \mathbb{C}^n, 0 \to M, 0$ transverse off 0 to each of the free divisors on the subspaces and their intersections. We use the terminology that f_0 is *transverse to the associated varieties* to indicate that it is transverse to all of these associated free divisors and their intersections.

For matrix singularities, we only assume initially that f_0 is transverse off 0 to the determinantal variety $\mathcal{D}$. To ensure that f_0 is also transverse to the associated varieties, we may apply to f_0 an element of the larger groups GL_m or $\mathrm{GL}_m \times \mathrm{GL}_p$ which preserve the determinantal variety of singular matrices. The actions of the groups GL_m or $\mathrm{GL}_m \times \mathrm{GL}_p$ are transitive on the strata of the determinantal variety $\mathcal{D}$ (by the classification of complex bilinear forms and echelon form for linear transformations). The complement of $\mathcal{D}$ consists of matrices of maximal rank, and again by the classification, they belong to a single orbit of these groups. Hence, by the parametrized transversality theorem, for almost all elements g of the appropriate group, the composition of the action of g with f_0, denoted $g \cdot f_0$, is transverse to the associated varieties. Hence, these will preserve $\mathcal{D}$ and move f_0 into general position off 0 relative to the associated varieties.

There are three essential ingredients which allow the general computations to be carried out for the various matrix types in the later sections:

- First, the singular Milnor numbers are computed in terms of a certain deformation theoretic codimension for $\mathcal{K}_H$-equivalence. In §2 we relate this to the equivalence $\mathcal{K}_M$ for matrix singularities and a related equivalence $\mathcal{K}_{\mathcal{V}}$

for viewing germs as nonlinear sections of the variety $\mathcal{V}$ of singular matrices. We also recall the formulas for codimensions as lengths of modules.

- Second, we recall in §3 the formulas for computing the singular Milnor numbers and formulas involving them and singular vanishing Euler characteristics.
- Third, in §4 we summarize the results from part I which construct the towers of free divisors and certain auxiliary free divisors needed for the various types of matrix singularities.

2. Equivalence Groups for Matrix Singularities

There are several different equivalences that we shall consider for matrix singularities $f_0 : \mathbb{C}^n, 0 \to M, 0$ with $\mathcal{V}$ denoting the subvariety of singular matrices in M. The one used in classifications is $\mathcal{K}_M$*–equivalence:* We suppose that we are given an action of a group of matrices G on M. For symmetric or skew symmetric matrices, it is the action of $\mathrm{GL}_m(\mathbb{C})$ by $B \cdot A = B\, A\, B^T$. For general $m \times p$ matrices, it is the action of $\mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_p(\mathbb{C})$ by $(B, C) \cdot A = B\, A\, C^{-1}$. Given such an action, then the group $\mathcal{K}_M$ consists of pairs ψ, a germ of a diffeomorphism of $\mathbb{C}^n, 0$, and a holomorphic germ $B : \mathbb{C}^n, 0 \to G, I$. The action is given by

$$f_0(x) \mapsto f_1(x) \;=\; B(x) \cdot (f_0 \circ \varphi(x))\,.$$

For one space M and group G, we use the generic notation $\mathcal{K}_M$ for any of these groups of equivalence (Gervais had earlier considered this type of equivalence, referring to it as G–equivalence [Ge1], [Ge2]).

In addition to $\mathcal{K}_M$, there are two other commonly used groups.

$\mathcal{K}_{\mathcal{V}}$ and $\mathcal{K}_H$–equivalence for Matrix Singularities.

If we view f_0 as a “nonlinear section of $\mathcal{V}$”(even for a more general germ $\mathcal{V}, 0$), $\mathcal{K}_{\mathcal{V}}$–equivalence is defined by the actions of pairs of diffeomorphisms (Φ, φ), preserving $\mathbb{C}^n \times \mathcal{V}$ (see [D1]).

$$\begin{array}{ccccc} \mathbb{C}^n \times \mathbb{C}^N, 0 & \xrightarrow{\;\Phi\;} & \mathbb{C}^n \times \mathbb{C}^N, 0 & \xleftarrow{\;i\;} & \mathbb{C}^n \times \mathcal{V}, 0 \\ \pi \big\downarrow & & \pi \big\downarrow & & \\ \mathbb{C}^n, 0 & \xrightarrow{\;\varphi\;} & \mathbb{C}^n, 0 & & \end{array} \tag{2.1}$$

For $\mathcal{V}_0 = f_0^{-1}(\mathcal{V})$, it gives an ambient equivalence of $\mathcal{V}_0, 0 \subset \mathbb{C}^n, 0$.

There is a third equivalence, $\mathcal{K}_H$–equivalence, introduce in [DM], which requires moreover that Φ given above preserves all of the level sets of H. Here H is chosen to be a “good defining equation”for $\mathcal{V}$, which means there is an “Euler–like vector field”η such that $\eta(H) = H$. In the weighted homogeneous case such as for determinantal varieties, we use the Euler vector field (for general $\mathcal{V}$ we may always replace $\mathcal{V}$ by $\mathcal{V} \times \mathbb{C}$ and $\dfrac{\partial}{\partial t}$ is such a vector field for the defining equation $e^t \cdot H$).

All of these equivalence groups have corresponding unfolding groups and belong to the class of geometric subgroups of $\mathcal{A}$ or $\mathcal{K}$, so all of the basic theorems of singularity theory in the Thom-Mather sense are valid for them (see [D1], [D3] and [D6]). In particular, germs which have finite codimension for one of these groups have versal unfoldings, and the deformation theoretic spaces for these groups play an important role.

We let θ_N denote the module of germs of vector fields on $\mathbb{C}^N, 0$, and $I(\mathcal{V})$ the ideal of germs vanishing on $\mathcal{V}$, and define, after Saito [Sa] the module of *logarithmic vector fields*

$$\mathrm{Derlog}(\mathcal{V}) = \{\zeta \in \theta_N : \zeta(I(\mathcal{V})) \subseteq I(\mathcal{V})\}.$$

For good defining equation H, we also define

$$\mathrm{Derlog}(H) = \{\zeta \in \theta_N : \zeta(H) = 0\}.$$

If H is a good defining equation,

$$\mathrm{Derlog}(\mathcal{V}) \quad = \quad \mathrm{Derlog}(H) \oplus \mathcal{O}_{\mathbb{C}^N,0}\{\eta\}\,.$$

These modules both appear in infinitesimal calculations for the groups.

If $\mathrm{Derlog}(\mathcal{V})$ is generated by $\zeta_0, \ldots, \zeta_r$, then the extended tangent space is given by

$$T\mathcal{K}_{\mathcal{V},e} \cdot f_0 \; = \; \mathcal{O}_{\mathbb{C}^n,0}\{\frac{\partial f_0}{\partial x_1}, \ldots, \frac{\partial f_0}{\partial x_n}, \zeta_0 \circ f_0, \ldots, \zeta_r \circ f_0\}\,. \tag{2.2}$$

The analogue of the deformation tangent space T^1 is the extended $\mathcal{K}_{\mathcal{V}}$ normal space

$$N\mathcal{K}_{\mathcal{V},e} \cdot f_0 \quad = \quad \theta(f_0)/T\mathcal{K}_{V,e} \cdot f_0 \quad \simeq \quad \mathcal{O}_{\mathbb{C}^n,0}^{(p)}/T\mathcal{K}_{V,e} \cdot f_0$$

where as usual $\theta(f_0)$, the module of germs of holomorphic vector fields along f_0, is the free $\mathcal{O}_{\mathbb{C}^n,0}$ module generated by $\{\frac{\partial}{\partial x_i}\}$, $1 \leq i \leq n$. Likewise, if ζ_0 denotes the Euler-like vector field with the remaining ζ_i generating $\mathrm{Derlog}(H)$, then $T\mathcal{K}_{H,e}$ is obtained by deleting $\zeta_0 \circ f_0$ in (2.2), with $N\mathcal{K}_{H,e}$ denoting the corresponding quotient. As usual, the dimensions of these extended normal spaces are the extended codimensions $\mathcal{K}_{\mathcal{V},e}$-codim$(f_0)$, resp. $\mathcal{K}_{H,e}$-codim(f_0).

There is a direct relation between these groups and $\mathcal{K}_M$. The extended tangent space for $\mathcal{K}_M$ is obtained by an analogous formula to (2.2) except the generators of $\mathrm{Derlog}(\mathcal{V})$ are replaced by vector fields for the matrix equivalence group G acting on $M \simeq \mathbb{C}^N$. They are of the form $\xi_{v_i}(x) = \frac{\partial}{\partial t}(\exp(tv_i) \cdot x)_{|t=0}$, for $\{v_i\}$ a basis for the Lie algebra $\mathfrak{g}$ of G. In the terminology of part I, we refer to these as the "representation vector fields".

The reason these are so closely related for matrix singularities is due to a collection of results due to Józefiak [J], Józefiak-Pragacz [JP], and Gulliksen-Negård[GN]. Goryunov-Mond [GM] recognized that these results prove that for the three types of $m \times m$ matrices (symmetric, skew-symmeric (with m even), or general matrices) that the modules of vector fields generated by the representation vector fields are exactly $\mathrm{Derlog}(\mathcal{V})$, for $\mathcal{V}$ the determinantal variety of singular matrices. It then follows that $\mathcal{K}_M$ and $\mathcal{K}_{\mathcal{V}}$ have the same tangent spaces; and when using the standard methods for studying equivalence of singularities, they give the same equivalence.

In addition, as noted in [DM], if f_0 is weighted homogeneous for the same set of weights as $\mathcal{V}$, then the extended tangent spaces of f_0 for $\mathcal{K}_{\mathcal{V}}$ and $\mathcal{K}_H$ are the same. Hence,

$$\mathcal{K}_{M,e}\text{-codim}(f_0) \; = \; \mathcal{K}_{\mathcal{V},e}\text{-codim}(f_0) \; = \; \mathcal{K}_{H,e}\text{-codim}(f_0)\,. \tag{2.3}$$

Thus, Bruce's observed result [Br] about simple symmetric matrix singularites and the result of Goryunov-Mond [GM] both concern the relation between the Milnor number $\mu(H \circ f_0)$ and $\mathcal{K}_{H,e}$-codim(f_0). We next consider how this relates to the case of nonisolated matrix singularities.

3. Singular Milnor Fibers and Singular Milnor Numbers

The singular Milnor numbers can be explicitly computed in the case $\mathcal{V}$ is a *free divisor*. This term was introduced by Saito [Sa] for hypersurface germs $\mathcal{V}, 0 \subset \mathbb{C}^N, 0$ for which Derlog($\mathcal{V}$) is a free $\mathcal{O}_{\mathbb{C}^N}$–module, necessarily of rank N. In this case, if $f_0 : \mathbb{C}^n \to M, 0$ is transverse to $\mathcal{V}$ off 0 ($\in \mathbb{C}^n$), we refer to $\mathcal{V}_0 = f_0^{-1}(\mathcal{V})$ as an *almost free divisor* (AFD).

A free divisor $\mathcal{V}$ is called *holonomic* by Saito if at any point $z \in \mathcal{V}$ the generators of Derlog(V) evaluated at z span the tangent space of the stratum containing z of the canonical Whitney stratification of $\mathcal{V}$. If this still holds true using Derlog(H) instead then we say it is *H–holonomic* [D2].

Then the results in [DM] and [D2] combine to give the following formula for the singular Milnor number.

Theorem 3.1. *If $\mathcal{V} \subset \mathbb{C}^N$ is an H-holonomic free divisor, and $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^N, 0$ is transverse to $\mathcal{V}$ off 0, then*

$$\mu_{\mathcal{V}}(f_0) \quad = \quad \mathcal{K}_{H,e}\text{-codim}(f_0) \tag{3.1}$$

where the RHS is computed as the length of a determinantal module.

Almost Free Divisor (AFD) on an ICIS.

This formula further extends to the case $f_0 : X, 0 \to \mathbb{C}^N, 0$ where $X, 0 \subset \mathbb{C}^n, 0$ is an ICIS defined by $\varphi : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$. In our situation, we consider the case where $f_0|X$ is transverse to a H-holonomic free divisor $\mathcal{V}$ off 0. Then, as in §1, we consider a stabilization $f_t : B_\varepsilon \to M$ of f_0, for which $f_t|X \cap B_\varepsilon$ is transverse to $\mathcal{V}$ for $t \neq 0$. For $\mathcal{V}_t = f_t^{-1}(\mathcal{V})$, $\mathcal{V}_t \cap X \cap B_\varepsilon$ is homotopy equivalent to a bouquet of spheres of real dimension $n - p - 1$ [D2, §7]. We denote by $\mu_{\varphi,\mathcal{V}}(f_0)$ the number of such spheres and refer to this number as the *singular Milnor number of $f_0|X$*. Then, the singular Milnor number can be computed by the following generalization of the Lê-Greuel formula, see [D2, §9] or [D3, §4].

Theorem 3.2 (AFD on an ICIS)**.** *Let $\mathcal{V}, 0 \subset \mathbb{C}^N, 0$ be an H–holonomic free divisor as above. Suppose $X, 0 \subset \mathbb{C}^n, 0$ is an ICIS defined by $\varphi : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$, and that $f_0|X$ is transverse to $\mathcal{V}$ off 0. Let $F = (\varphi, f_0) : \mathbb{C}^n, 0 \to \mathbb{C}^{p+N}, 0$. Then,*

$$\mu_{\varphi,\mathcal{V}}(f_0) \,+\, \mu(\varphi) \quad = \quad \dim_{\mathbb{C}} \left(\mathcal{O}_{X,0}^{p+N} / \mathcal{O}_{X,0}\{ \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n}, \zeta_1 \circ f_0, \dots, \zeta_r \circ f_0 \} \right) . \tag{3.2}$$

With $\mu(\varphi)$ computed by the Lê-Greuel formula, (3.2) then yields the singular Milnor number $\mu_{\varphi,\mathcal{V}}(f_0)$. We also note that if $\mathcal{V} = \{0\}$ then (3.2) yields a module version of the Lê–Greuel formula. We next see that (3.2) can also be viewed as computing the singular Milnor number of F for a free divisor on a smooth subspace $\mathbb{C}^N \subset \mathbb{C}^{p+N}$. This is the form that many terms on the RHS of (0.2) will take in the formulas we obtain.

Proposition 3.3. *Let $\mathcal{V}, 0 \subset \mathbb{C}^N, 0$ be an H–holonomic free divisor.*

(1) *Let $\mathcal{V}' = \mathcal{V} \times \mathbb{C}^p, 0 \subset \mathbb{C}^{N+p}, 0$, and suppose $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^{N+p}, 0$ is algebraically transverse to $\mathcal{V}'$ off 0. Then for π denoting the projection $\mathbb{C}^{N+p} \to \mathbb{C}^N$*

$$\mu_{\mathcal{V}'}(f_0) \;=\; \mu_{\mathcal{V}}(\pi \circ f_0) \,.$$

(2) *Let $\mathcal{V}'', 0 = \mathcal{V} \times \{0\} \subset \mathbb{C}^{N+p}, 0$ be the image of $\mathcal{V}, 0$ via the inclusion $\mathbb{C}^N, 0 \subset \mathbb{C}^{N+p}, 0$ (so that $\mathcal{V}''$ is a free divisor in a linear subspace of $\mathbb{C}^{N+p}$). Suppose $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^{N+p}, 0$ is algebraically transverse to $\mathcal{V}''$ off 0 and for π' denoting the projection $\mathbb{C}^{N+p} \to \mathbb{C}^p$, $\varphi = \pi' \circ f_0 : \mathbb{C}^n, 0 \subset \mathbb{C}^p, 0$ is an ICIS. Then*

$$\mu_{\mathcal{V}''}(f_0) \; = \; \mu_{\varphi, \mathcal{V}}(\pi \circ f_0) \, .$$

Proof of Proposition 3.3. For 1), we first note that $\mathcal{V}'$ is also H–holonomic. If $\{S_i\}$ are the strata of the canonical Whitney stratification of $\mathcal{V}$, then $\{S_i \times \mathbb{C}^p\}$ are the strata for $\mathcal{V}' = \mathcal{V} \times \mathbb{C}^p$. Also, if Derlog$(\mathcal{V})$ has the set of free generators $\eta_1, \dots \eta_{N-1}$ and we use coordinates $(w_1, \dots, w_p)$ for $\mathbb{C}^p$, then we can trivially extend the η_i to $\mathbb{C}^{N+p}$ and adjoin $\{\frac{\partial}{\partial w_1}, \dots \frac{\partial}{\partial w_p}\}$ to obtain a set of free generators for Derlog$(\mathcal{V}')$. Thus, $\mathcal{V}'$ is also H–holonomic.

By a calculation similar to that for $\mathcal{K}_{V,e}$ in [D3], it follows that for any germ $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^{N+p}$, with $\pi : \mathbb{C}^{N+p} \to \mathbb{C}^N$ the projection, and $\mathcal{V}$ defined by H

$$\mathcal{K}_{H\circ\pi,e}\text{-codim}(f_0) \; = \; \mathcal{K}_{H,e}\text{-codim}(\pi \circ f_0) \, .$$

Then, by Theorem 3.1 we have 1).

For 2), we observe that if we choose a stabilization f'_t of $\pi \circ f_0$ so that $0 \notin f_t'^{-1}(\mathcal{V})$ for $t \neq 0$, then $F_t = (\varphi, f'_t)$ is a stabilization of f_0 for $\mathcal{V}''$. Thus, the singular Milnor fiber of $\pi \circ f_0 | X$ for $\mathcal{V}$, where $X = \varphi^{-1}(0)$, is also the singular Milnor fiber of f_0 for $\mathcal{V}''$. This yields 2). $\square$

Remark 3.4. In the formula (0.1), if $\mathcal{W}_i \subset \mathbb{C}^N$ has codimension k, then if $n < k$, the corresponding singular Milnor fiber of $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^N, 0$ for $\mathcal{W}_i$ will be empty and hence have Euler characteristic 0. Likewise, if $n - p < k$ then for $X, 0 \subset \mathbb{C}^n, 0$ an ICIS defined by $\varphi : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$, the singular Milnor fiber of $f_0 : X, 0 \to \mathbb{C}^N, 0$ will be empty and hence have Euler characteristic 0. Thus, to make all of the formulas correct, we adopt the following convention.

Convention: If $n < k = \text{codim}(\mathcal{W}_i)$, then $\mu_{\mathcal{W}_i}(f_0) \overset{def}{=} (-1)^{n-k+1}$. Likewise if $n - p < k = \text{codim}(\mathcal{W}_i)$, then $\mu_{\varphi, \mathcal{W}_i}(f_0) \overset{def}{=} (-1)^{n-p-k+1}$.

Remark . The terms on the LHS of (3.2) can be viewed as computing the "relative singular Milnor number", which is given by rank$(H^{n-p-1}(X_t \cap B_\varepsilon, \mathcal{V}_t \cap X_t \cap B_\varepsilon; \mathbb{Z}))$, where X_t is the Milnor fiber of φ and $\mathcal{V}_t = f_t^{-1}(\mathcal{V})$. This follows because $\mathcal{V}_t \cap X_t \cap B_\varepsilon \simeq \mathcal{V}_t \cap X \cap B_\varepsilon$. Since each fiber is homotopy equivalent to a bouquet of spheres, the exact sequence for a pair yields the sum on the LHS of (3.2).

Singular Vanishing Euler Characteristic. In the case that $\mathcal{V}$ is not a complete intersection, we can still introduce a version of the vanishing Euler characteristic for the singular Milnor fiber (which is no longer homotopy equivalent to a bouquet of spheres). We suppose again that $f_0 : \mathbb{C}^n, 0 \to M, 0$ is transverse to $\mathcal{V}$ off 0, and consider a stabilization $f_t : B_\varepsilon \to M$ of f_0. We let the *singular vanishing Euler characteristic* be defined by

$$\tilde{\chi}_{\mathcal{V}}(f_0) \; \overset{def}{=} \; \tilde{\chi}(f_t^{-1}(\mathcal{V})) \; = \; \chi(f_t^{-1}(\mathcal{V})) - 1 \, .$$

As earlier, $\tilde{\chi}_{\mathcal{V}}(f_0)$ is independent of stabilization.

Similarly, if $X, 0$ is an ICIS defined by $\varphi : \mathbb{C}^n, 0 \to \mathbb{C}^p$ and $f_0 : X, 0 \to \mathbb{C}^N$ is transverse to $\mathcal{V}$ off 0, we define

$$\tilde{\chi}_{\varphi,\mathcal{V}}(f_0) \overset{def}{=} \tilde{\chi}(f_t^{-1}(\mathcal{V} \cap X)) = \chi(f_t^{-1}(\mathcal{V} \cap X)) - 1 .$$

This can be viewed as the singular vanishing Euler characteristic for the mapping $F_0 = (\varphi, f_0) : \mathbb{C}^n, 0 \to \mathbb{C}^p \times \mathbb{C}^N, 0$ since if $f_t|X : X \cap B_\varepsilon \to \mathbb{C}^N$ is transverse to $\mathcal{V}$, then $F_t = (\varphi, f_t) : B_\varepsilon \to \mathbb{C}^p \times \mathbb{C}^N$ is transverse to $\{0\} \times \mathcal{V}$. Thus, $\tilde{\chi}_{\varphi,\mathcal{V}}(f_0) = \tilde{\chi}_{\{0\}\times\mathcal{V}}(F_0)$.

We will compute singular Milnor numbers for nonlinear sections of hypersurface and complete intersection singularities. However, we will do so by using simple Euler characteristic arguments for the singular vanishing Euler characteristics combined with their calculation in terms of singular Milnor numbers. These, in turn, can be calculated algebraically using (3.1) and Theorem 3.2. The simplest version is for the case of subvarieties $\mathcal{V}, \mathcal{W} \subset \mathbb{C}^N$.

Lemma 3.5. *Suppose $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^N, 0$ is transverse to $\mathcal{V}$, $\mathcal{W}$ and $\mathcal{V} \cap \mathcal{W}$ off $0 \in \mathbb{C}^n$. Then,*

$$\tilde{\chi}_{\mathcal{W}\cup\mathcal{V}}(f_0) = \tilde{\chi}_{\mathcal{W}}(f_0) + \tilde{\chi}_{\mathcal{V}}(f_0) - \tilde{\chi}_{\mathcal{W}\cap\mathcal{V}}(f_0) . \tag{3.3}$$

In the case that $\mathcal{V}$ and $\mathcal{W}$ are both hypersurface singularities we obtain from (3.3)

$$\mu_{\mathcal{W}}(f_0) = \mu_{\mathcal{W}\cup\mathcal{V}}(f_0) - \mu_{\mathcal{V}}(f_0) + (-1)^{n-1}\tilde{\chi}_{\mathcal{W}\cap\mathcal{V}}(f_0) . \tag{3.4}$$

If instead $X, 0$ is an ICIS defined by $\varphi : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ and $f_0 : X, 0 \to \mathbb{C}^N, 0$ is transverse to $\mathcal{V}$ and $\mathcal{W}$ off 0, then there are the analogs for (3.3) and (3.4)

$$\tilde{\chi}_{\varphi,\mathcal{W}\cup\mathcal{V}}(f_0) = \tilde{\chi}_{\varphi,\mathcal{W}}(f_0) + \tilde{\chi}_{\varphi,\mathcal{V}}(f_0) - \tilde{\chi}_{\varphi,\mathcal{W}\cap\mathcal{V}}(f_0) \tag{3.5}$$

and

$$\mu_{\varphi,\mathcal{W}}(f_0) = \mu_{\varphi,\mathcal{W}\cup\mathcal{V}}(f_0) - \mu_{\varphi,\mathcal{V}}(f_0) + (-1)^{n-p-1}\tilde{\chi}_{\varphi,\mathcal{W}\cap\mathcal{V}}(f_0) . \tag{3.6}$$

Notation: To simplify formulas, we will view singular Milnor numbers and singular vanishing Euler characteristics as numerical functions on the space of germs transverse to the appropriate set of subvarieties off 0. Hence, a formula such as (3.4) will be written with evaluation on f_0 understood so it will take the form

$$\mu_{\mathcal{W}} = \mu_{\mathcal{W}\cup\mathcal{V}} - \mu_{\mathcal{V}} + (-1)^{n-1}\tilde{\chi}_{\mathcal{W}\cap\mathcal{V}} . \tag{3.7}$$

Also, we may apply Proposition 3.3 to obtain $\mu_{\pi^*\mathcal{E}}(f_0) = \mu_{\mathcal{E}}(\pi \circ f_0)$, so with this understanding, in all future formulas we will abbreviate $\mu_{\pi^*\mathcal{E}}$ to just $\mu_{\mathcal{E}}$.

Proof. The argument given in [D2, §8], although applied to Euler characteristics for hypersurfaces $\mathcal{W}$ and $\mathcal{V}$, applies equally well to reduced Euler characteristics ($\tilde{\chi} = \chi - 1$) for subvarieties to give (3.3). Then, for a hypersurface $\mathcal{W}$, we have $\tilde{\chi}_{\mathcal{W}}(f_0) = (-1)^{n-1}\mu_{\mathcal{W}}(f_0)$. Substituting for $\tilde{\chi}$ for all of the hypersurfaces in (3.3) and rearranging yields (3.4).

The same Euler characteristic argument used in verifying (3.3) also applies instead to $\{0\} \times \mathcal{Y} \subset \mathbb{C}^{p+N}$ for hypersurfaces $\mathcal{Y}$ and the map $F = (\varphi, f_0)$ yielding (3.5). Substituting $\tilde{\chi}_{\varphi,\mathcal{W}}(f_0) = (-1)^{n-p-1}\mu_{\varphi,\mathcal{W}}(f_0)$ for all of the hypersurfaces in (3.5) yields after rearranging (3.6). ∎

Intersections of Multiple Hypersurfaces. To compute $\tilde{\chi}_{\mathcal{V}\cap\mathcal{W}}$ we will use an inductive procedure which requires computing $\tilde{\chi}_{\cap_i \mathcal{W}_i}$ for a collection of hypersurfaces $\mathcal{W}_i$. We will use the following formula for k hypersurfaces $\mathcal{W}_i$:

$$\tilde{\chi}_{\cap_i \mathcal{W}_i} \;=\; \sum_{\mathbf{j}} (-1)^{|\mathbf{j}|+1} \tilde{\chi}_{\cup_{\mathbf{j}} \mathcal{W}_{j_i}} \tag{3.8}$$

for nonempty $\mathbf{j} = \{j_1, \dots, j_r\} \subset \{1, \dots, k\}$ with $|\mathbf{j}| = r$ (for a formula involving χ see [D2, Lemma 8.1], but an analogous argument works for $\tilde{\chi}$ using reduced homology).

Then, for mappings $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^N, 0$, substituting $\tilde{\chi}_{\cup_{\mathbf{j}} \mathcal{W}_{j_i}} = (-1)^{n-1} \mu_{\cup_{\mathbf{j}} \mathcal{W}_{j_i}}$ we obtain

Proposition 3.6. *For mappings $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^N, 0$ and a collection of hypersurfaces $\mathcal{W}_i, 0 \subset \mathbb{C}^N, 0$, $i = 1, \dots, k$, with $\cap_i \mathcal{W}_i$ not necessarily a complete intersection,*

$$\tilde{\chi}_{\cap_i \mathcal{W}_i} \;=\; (-1)^{n-k} \left(\sum_{\mathbf{j}} (-1)^{|\mathbf{j}|+k} \mu_{\cup_{\mathbf{j}} \mathcal{W}_{j_i}} \right) . \tag{3.9}$$

Remark 3.7. In the case that $\cap_i \mathcal{W}_i$ is a complete intersection, this formula reduces to Theorem 2 of [D2, §8].

4. Exceptional Orbit Varieties as Free Divisors

We recall the results from part I [DP1] which allow us to embed the varieties of singular matrices in a geometric configuration of divisors which form free divisors.

We use the notation from part I and let $M_{m,p}$ denote the space of $m \times p$ complex matrices, and Sym_m, respectively Sk_m, the subspaces of $M_{m,m}$ of symmetric, respectively skew–symmetric, complex matrices. Next, we let B_m denote the Borel subgroup of $\mathrm{GL}_m(\mathbb{C})$ consisting of lower triangular matrices and the group

$$C_m = \begin{pmatrix} 1 & 0 \\ 0 & B_{m-1}^T \end{pmatrix}$$

where B_{m-1}^T denote the group of upper triangular matrices of $\mathrm{GL}_{m-1}(\mathbb{C})$. Then, the (modified) *Cholesky-type representations* are given in Table 1, which is Table 1 of [DP1]. These representations give rise to *exceptional orbit varieties* which are the union of the positive codimension orbits of the representations. We denote these by : $\mathcal{E}_m^{sy}$ (for Sym_m); $\mathcal{E}_m$ (for $M_{m,m}$); $\mathcal{E}_{m-1,m}$ (for $M_{m-1,m}$); and $\mathcal{E}_m^{sk}$ (for Sk_m). Then, by Theorems 6.2, 7.1, and 8.1 in [DP1], these are linear free divisors. These are families of representations which, via natural inclusions of groups and spaces, together form towers of representations. Furthermore, the exceptional orbit varieties contain as components the corresponding "generalized determinant varieties", which we denote by: $\mathcal{D}_m^{sy}$, $\mathcal{D}_m$, $\mathcal{D}_{m-1,m}$, and $\mathcal{D}_m^{sk}$ respectively. The defining equations for the corresponding exceptional orbit varieties and generalized determinant varieties are given in Table 2. Because of the tower structure for the representations we have the inductive representation for the m–th exceptional orbit variety $\mathcal{E}_m$ and generalized determinant variety $\mathcal{D}_m$

$$\mathcal{E}_m \;=\; \mathcal{D}_m \cup \pi^* \mathcal{E}_{m-1} \, , \tag{4.1}$$

where π denotes a projection from the m–th representation V_m, $\pi : V_m \to V_{m-1}$. Then, by (4.1), in each case $\mathcal{D}_m$ has a free completion to $\mathcal{E}_m$ by $\pi^* \mathcal{E}_{m-1}$.

(Modified) Cholesky-Type Representations Yielding Free Divisors

(Modified) Cholesky-type Factorization	Matrix Space	Solvable Group	Representation
symmetric matrices	Sym_m	B_m	$B \cdot A = B\,A\,B^T$
general $m \times m$	$M_{m,m}$	$B_m \times C_m$	$(B, C) \cdot A = B\,A\,C^{-1}$
general $(m-1) \times m$	$M_{m-1,m}$	$B_{m-1} \times C_m$	$(B, C) \cdot A = B\,A\,C^{-1}$
Nonlinear Representation	Matrix Space	Solvable Lie algebra	Representation
skew-symmetric matrices	Sk_m	$\tilde{D}_m$	$Diff(Sk_m, 0)$

TABLE 1. Solvable group and solvable Lie algebra Block representations for (modified) Cholesky–type Factorizations, yielding free divisors.

$\mathcal{E}$	Defining Equation for $\mathcal{E}$	$\mathcal{D}$	Defining Equation for $\mathcal{D}$
$\mathcal{E}_m^{sy}$	$\prod_{k=1}^{m} \det(A^{(k)})$	$\mathcal{D}_m^{sy}$	$\det(A)$
$\mathcal{E}_m$	$\prod_{k=1}^{m} \det(A^{(k)}) \cdot \prod_{k=1}^{m-1} \det(\hat{A}^{(k)})$	$\mathcal{D}_m$	$\det(\hat{A}^{(m-1)}) \cdot \det(A)$
$\mathcal{E}_{m-1,m}$	$\prod_{k=1}^{m-1} \det(A^{(k)}) \cdot \prod_{k=1}^{m-1} \det(\hat{A}^{(k)})$	$\mathcal{D}_{m-1,m}$	$\det(A^{(m-1)}) \cdot \det(\hat{A}^{(m-1)})$
$\mathcal{E}_m^{sk}$	$\prod_{k=1}^{m-2} \det\left(\hat{\hat{A}}^{(k)}\right) \cdot \prod_{k=2}^{m} \mathrm{Pf}_{\{\epsilon(k),\ldots,k\}}(A)$	$\mathcal{D}_m^{sk}$	$\mathrm{Pf}_{\{\epsilon(m),\ldots,m\}}(A) \cdot \det(\hat{\hat{A}}^{(m-2)})$

TABLE 2. Defining Equations for the exceptional orbit varieties $\mathcal{E}$ and determinantal varieties $\mathcal{D}$ for the solvable group and solvable Lie algebra block representations. If $A = (a_{ij})$ denotes a general matrix, then $\hat{A}$ denotes the matrix obtained by deleting the first column of A, $\hat{\hat{A}}$, that obtained by deleting the first two columns of A, and $A^{(k)}$ denotes the $k \times k$ upper left–hand submatrix of a matrix A.

Remark 4.1. For Sk_m, in place of a solvable group, we have an infinite dimensional solvable Lie algebra $\tilde{D}_m$ which is an extension of the Lie algebra of the solvable Lie group

$$G_m = \begin{pmatrix} T_2 & 0_{2,m-2} \\ 0_{m-2,2} & B_{m-2} \end{pmatrix}$$

where T_2 is the group of 2×2 diagonal matrices. This extension is by a set of Pfaffian vector fields η_k for $2 \leq k \leq m-2$, see [DP1, §8] and [P, Chap. 5].

Remark 4.2. We may interleave the towers of general matrices so $M_{m-1,m-1} \subset M_{m-1,m} \subset M_{m,m}$. Then, the successive generalized determinantal varieties are defined by $\det(\hat{A}^{(m-1)})$ and then $\det(A)$.

Free Divisors arising from Restrictions of Block Representations. In addition to the free divisors arising from the representations in Table 1, we shall also use certain auxiliary free divisors arising from the restriction of representations. These are given in §9 of [DP1].

For Sym_3 we use coordinates given by

$$A = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}.$$

We define $\mathcal{Q}_f = \det(A_f)$ and $\mathcal{Q}_a = \det(A_a)$ where A_f and A_a are obtained from A by setting $f = 0$, respectively, $a = 0$. Interchanging the first and third coordinates in $\mathbb{C}^3$ will interchange $\mathcal{Q}_f$ and $\mathcal{Q}_a$ so any result for $\mathcal{Q}_f$ will have an analogous result for $\mathcal{Q}_a$. We let V_a denote the subspace where $a = 0$ and V_f, where $f = 0$. Then, we can summarize the appropriate results from Propositions 9.1 and 9.5 of [DP1].

Proposition 4.3. *The subvarieties of V_a defined by $b \cdot d \cdot \mathcal{Q}_a = 0$ and of V_f defined by $(ad - b^2) \cdot \mathcal{Q}_f = 0$ are linear free divisors.*

Hence, by Proposition 4.3, $V(\mathcal{Q}_a)$ has a free completion using the free divisor $V(bd)$, and we may complete $V(\mathcal{Q}_f)$ to a free divisor using $\mathcal{D}_2^{sy} = V(ad - b^2)$. Although $\mathcal{D}_2^{sy}$ is not a free divisor, it has a free completion $\mathcal{E}_2^{sy}$.

A Quiver Linear Free Divisor. A third special case of linear free divisors needed for our calculations occurs for the special case of 2×3 matrices. In [BM], Buchweitz and Mond proved that quivers of finite type give rise to free divisors. The quiver consisting of 3 arrows from vertices (representing $\mathbb{C}$) to a central vertex (representing $\mathbb{C}^2$) corresponds to the representation of $(\mathrm{GL}_2(\mathbb{C}) \times (\mathbb{C}^*)^3)/\mathbb{C}^*$ on $M_{2,3}$. If we use coordinates on $M_{2,3}$ given by $\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$, then the corresponding free divisor is defined by $(ae - bd)(af - cd)(bf - ce) = 0$.

Linear Free Divisors which are H–holonomic. Theorem 3.1 allows us to compute $\mu_{\mathcal{V}}(f_0)$ provided $\mathcal{V}$ is an H–holonomic free divisor. In this section we give two results establishing that free divisors are H–holonomic; one applies to towers of linear free divisors, and the other, to arbitrary low-dimensional linear free divisors.

H–holonomic free divisors which appear in towers. Let $\mathcal{E}$ be a free divisor arising as the exceptional orbit variety of a representation $G \to \mathrm{GL}(W)$, which itself is one step of a tower of representations as defined in part I ([DP1]). For example, $\mathcal{E}$ could be any of the hypersurfaces in the following Theorem, which is proven in detail in §6.3 of [P] using the technique we will describe.

Theorem 4.4 (Theorem 6.2.2 in [P])**.** *The linear free divisors $\mathcal{E}_m^{sy}$, $\mathcal{E}_m$, and $\mathcal{E}_{m-1,m}$ listed in Table 2 are H–holonomic.*

Outline of Proof. We outline what is a fairly lengthy argument which is proven in detail in §6.3 of [P]. Readers are encouraged to refer there for the full details.

First, it is proven that there are only a finite number of orbits of G in W by classifying them, giving normal forms for representatives of each orbit. The tower structure makes this step significantly easier, because the classification at a lower level of the tower can be combined with the inclusion of the group action and vector spaces to put an arbitrary $w \in W$ into a "partial normal form" $g_1 \cdot w$ (for example, a certain submatrix of $g_1 \cdot w$ contains only zeros and ones in a certain pattern). Then, another element of G is applied to put $g_1 \cdot w$ into a normal form. As the resulting list of normal forms is finite, there are a finite number of G–orbits in W (and thus in the exceptional orbit variety $\mathcal{E}$), and so $\mathcal{E}$ is holonomic.

Second, we let $G_H \subset G$ be the connected codimension 1 Lie subgroup whose Lie algebra of vector fields generates $\mathrm{Derlog}\,(H)$. To show $\mathcal{E}$ is H–holonomic, it is sufficient to prove that G_H acts transitively on all non-open G–orbits (or, the G–orbits in $\mathcal{E}$ are the G_H–orbits in $\mathcal{E}$). Thus we consider each normal form n (representing a non-open orbit) with an arbitrary $g \in G$, and show that there exists an $h \in G$ in the isotropy subgroup of n with $hg \in G_H$. Thus, if $n = g \cdot v$ then $n = hg \cdot v$ with $hg \in G_H$. It follows that $G \cdot n = G_H \cdot n$. □

H–holonomic free divisors in small dimensions. Since we use other linear free divisors described above, we also provide the following sufficient condition for a hypersurface to be H–holonomic. In low dimensions, the criterion can be checked by a computer using a computer algebra system such as Macaulay2 or Singular.

Let $\mathcal{V}, 0 \subset \mathbb{C}^n, 0$ be a reduced hypersurface with good defining equation H. Let M be an $\mathcal{O}_{\mathbb{C}^n,0}$-module of vector fields on $\mathbb{C}^n, 0$. We let for $z \in \mathbb{C}^n$,

$$< M >_{(z)} \; = \; \{\eta(z) \,|\, \eta \in M\}$$

be the linear subspace of $T_z\mathbb{C}^n$. The logarithmic and H–logarithmic tangent spaces are defined to be

$$T_{log}\mathcal{V}_z \; = < \mathrm{Derlog}\,(V) >_{(z)} \qquad \text{and} \qquad T_{log}H_z \; = < \mathrm{Derlog}\,(H) >_{(z)} \; .$$

For $0 \leq k \leq n$, define the varieties $D_k = \{z \in \mathcal{V} \,|\, \dim (T_{log}\mathcal{V}_z) \leq k\}$ and $H_k = \{z \in \mathcal{V} \,|\, \dim (T_{log}H_z) \leq k\}$.

Proposition 4.5. *With the preceding notation, if, for all $0 \leq k < n$,*

(i) *all irreducible components of $(D_k, 0)$ have dimension $\leq k$ at 0, and*
(ii) *$(D_k, 0) = (H_k, 0)$ as germs,*

then $(\mathcal{V}, 0)$ is H–holonomic.

Proof. For $z \in \mathcal{V}$, let S_z denote the stratum of the canonical Whitney stratification of $\mathcal{V}$ containing z. Then, $\mathcal{V}$ is holonomic if and only if $T_{log}\mathcal{V}_z = T_z S_z$ for all $z \in \mathcal{V}$, and it is H-holonomic if and only if $T_{log}H_z = T_z S_z$ for all $z \in \mathcal{V}$.

First, we observe that the conditions imply $\mathcal{V}$ is holonomic for if not, then there is a stratum S of highest dimension, say k, on which it fails. Then, there is a Zariski open set U of S consisting of those $z \in S$ with $T_{log}\mathcal{V}_z \subsetneq T_z S_z$. Then, $U \subset D_{k-1}$, and $\dim D_{k-1} \geq k$, contradicting i). A similar argument using $T_{log}H_z$ shows if $\mathcal{V}$ is not H–holonomic, then $\dim D_{k-1} \geq k$, contradicting ii) given that i) holds. □

Computer algebra systems such as Macaulay2 and Singular have built-in functions to perform each of the steps necessary to use Proposition 4.5 to show that a hypersurface is H–holonomic, including: finding generators of $\mathrm{Derlog}\,(V)$ and $\mathrm{Derlog}\,(H)$ (as certain syzygies), determining the ideals defining each D_k and H_k, computing the radicals and primary decompositions of these ideals, computing the dimensions of the irreducible components of D_k, and testing pairs of ideals for equality. In particular, the linear free divisors in Proposition 4.3 and the quiver linear free divisor in $M_{2,3}$ are H–holonomic. When we assert that a hypersurface is an H–holonomic free divisor and give no reference, it will be understood that we have used an implementation ([P2]) of this approach in Macaulay2 ([M2]) to check Saito's Criterion and the conditions of Proposition 4.5.

5. A Metatheorem and Generic Reduction

In this section we introduce two ideas which both extend and simplify the formulas for singular Milnor numbers which we will obtain.

Metatheorem. The results on matrix singularities for $f_0 : \mathbb{C}^n, 0 \to M, 0$ can be extended to the case of matrix singularities on an ICIS X. In fact given a formula (0.2) for $\mu_{\mathcal{V}}$, the following metatheorem asserts that there is a corresponding formula for the singular Milnor number of $f_0|X, 0 \to M, 0$.

5.1 (Metatheorem). *If X is an ICIS defined by $\varphi : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$, and the formula (0.2) for $\mu_{\mathcal{V}}$ is obtained by the inductive procedure, then the same procedure also yields the formula (with the same coefficients a_i)*

$$\mu_{\varphi, \mathcal{V}} \;=\; \sum_i a_i \mu_{\varphi, \mathcal{W}_i} \tag{5.1}$$

where $\mu_{\varphi, \mathcal{V}}(f_0)$, respectively $\mu_{\varphi, \mathcal{W}_i}(f_0)$, are the singular Milnor numbers for $f_0|X$ as nonlinear sections of $\mathcal{V}$, resp. $\mathcal{W}_i$, and can be computed as lengths of determinantal modules.

Likewise, if instead we have a formula for the vanishing Euler characteristic $\tilde{\chi}_{\mathcal{V}}$ having the same form as in (0.2)

$$\tilde{\chi}_{\mathcal{V}} \;=\; \sum_i b_i \mu_{\mathcal{W}_i} \tag{5.2}$$

and obtained by the inductive process, then there is an analogous formula

$$\tilde{\chi}_{\varphi, \mathcal{V}} \;=\; (-1)^p \left(\sum_i b_i \mu_{\varphi, \mathcal{W}_i} \right) . \tag{5.3}$$

Proof. This result follows because at each inductive step, the decomposition into the associated varieties will be the same. Then, in place of using the formulas in Lemma 3.5 and Theorem 3.1 for germs f_0 on $\mathbb{C}^n$, we use the versions of Lemma 3.5 for $f_0|X$ on an ICIS X and Theorem 3.2. Also, for a variety in M defined by $(g_1, \dots, g_r)$, in place of $\mu_{g_1, \dots, g_r}(f_0)$ we use $\mu_{(g_1, \dots, g_r) \circ \pi}((\varphi, f_0))$, with $\pi : \mathbb{C}^{r+p} \to \mathbb{C}^r$ denoting the projection. This we denote by $\mu_{\varphi, g_1, \dots, g_r}(f_0)$. This can be seen by observing that in terms of singular vanishing Euler characteristics, we repeatedly use (3.3) from Lemma 3.5. However, for $f_0|X$ we repeated use instead (3.5). Thus the formulas in terms of singular vanishing Euler characteristics will have the same form. However, in writing the formulas in terms of singular Milnor numbers, $\tilde{\chi}_{\mathcal{W}_i} = (-1)^{n-k} \mu_{\mathcal{W}_i}$ where k is the codimension of $\mathcal{W}_i$; while $\tilde{\chi}_{\varphi, \mathcal{W}_i} = (-1)^{n-p-k} \mu_{\varphi, \mathcal{W}_i}$. Since the extra factor of $(-1)^p$ will occur for every term on each side, it will cancel yielding (5.1). However, for $\tilde{\chi}_{\varphi, \mathcal{V}}$ versus $\tilde{\chi}_{\mathcal{V}}$, there is an extra factor of $(-1)^p$ for each term on the RHS, resulting in the desired formula (5.3). □

Generic Reduction. Given a matrix singularity defined by f_0, we may apply an element g of the group G which acts on the space of matrices M to obtain $f_1 = g \cdot f_0$ which is $\mathcal{K}_M$–equivalent to f_0 and has same singular Milnor number. By Remark 1.4 we can apply g so that f_1 is transverse to the associated varieties, allowing us to compute $\mu_{\mathcal{D}}(f_0)$ using formulas of the form (0.2). However, we can do more and this leads to the idea of *generic reduction.*

We can simplify the form which the formulas take if we can choose f_1 so as many of the terms in (0.2) vanish. We can achieve this by considering $df_0(0)$ and the effect of applying g to it to obtain $df_1(0)$.

Suppose $M_i \subset M$ is a linear subspace and that $\mathcal{W}_i, 0 \subset M_i, 0$, can be defined as the pullback of a free divisor by the projection $\pi_i : M_i \to \mathbb{C}^{m_i}$, where $\dim M_i$ and m_i are the minimal such. Then, the *defining dimension* of $\mathcal{W}_i$ is $\operatorname{codim} M_i + m_i$ and the *defining codimension* of $\mathcal{W}_i$ is $\dim M_i - m_i$. We let λ_ℓ denote the sum of the terms in (0.2) for the $\mathcal{W}_i$ of defining codimension ℓ. Then, by generic reduction we mean that an element g of G is applied so that $df_1(0)$ projects submersively via π_i onto each M_i of codimension $\geq \operatorname{codim}(Im(df_1(0)))$. Then, all of the terms $\lambda_\ell(f_1)$ will be 0 for $\ell \geq \operatorname{codim}(Im(df_1(0)))$.

In certain cases, the classification of linear matrix singularities may prevent us from obtaining an f_1 with the full generic reduction; however, we will still apply g to obtain as many terms vanishing as possible. The results obtained in the later sections will indicate how generic reduction simplifies the formulas. In §11 we deduce specific consequences of generic reduction for all of the matrix types for generic corank 1 matrix mappings and for the computations for Cohen–Macaulay singularities.

6. Symmetric Matrix Singularities

By the results of [DP1] summarized in §4, the exceptional orbit variety $\mathcal{E}_m^{sy}$ of the representation of B_m on Sym_m is a linear free divisor and the determinantal variety $\mathcal{D}_m^{sy}$ has a free completion given by

$$\mathcal{E}_m^{sy} \; = \pi^* \mathcal{E}_{m-1}^{sy} \cup \mathcal{D}_m^{sy} \tag{6.1}$$

for the projection $\pi : Sym_m \to Sym_{m-1}$.

Furthermore, by Theorem 4.4, $\mathcal{E}_m^{sy}$ is H–holonomic; hence by Theorem 3.1, for a nonlinear section $f_0 : \mathbb{C}^n, 0 \to Sym_m$, transverse to $\mathcal{E}_m^{sy}$ off 0, the singular Milnor number $\mu_{\mathcal{E}_m^{sy}}$ is the length of the determinantal module

$$N\mathcal{K}_{H,e}\,(f_0) \;\simeq\; N\mathcal{K}_{\tilde{B}_m,e}\,(f_0)$$

where $\tilde{B}_m$ is the subgroup of B_m which preserves the defining equation H of $\mathcal{E}_m^{sy}$. The corresponding Lie algebra of representation vector fields is $\operatorname{Derlog}(H)$.

Hence, by Lemma 3.5 and (6.1), we have quite generally

$$\mu_{\mathcal{D}_m^{sy}} \;=\; \mu_{\mathcal{E}_m^{sy}} \,-\, \mu_{\mathcal{E}_{m-1}^{sy}} \,+\, (-1)^{n-1}\, \tilde{\chi}_{\pi^* \mathcal{E}_{m-1}^{sy} \cap \mathcal{D}_m^{sy}}\,. \tag{6.2}$$

Thus, we are reduced to inductively computing $\tilde{\chi}_{\pi^* \mathcal{E}_{m-1}^{sy} \cap \mathcal{D}_m^{sy}}$. We note that the simplest case of $\mathcal{D}_1^{sy} = \{0\} \subset Sym_1 \simeq \mathbb{C}$ just yields isolated hypersurface singularities and $\mu_{\mathcal{D}_1^{sy}} = \mu$ when applied to $f_0 : \mathbb{C}^n, 0 \to Sym_1, 0 \simeq \mathbb{C}, 0$. We have already carried out the calculation for 2×2 symmetric matrices in §1 which leads to the following theorem.

Theorem 6.1. *For the space of germs transverse to the associated varieties for $\mathcal{E}_2^{sy}$ off 0,*

$$\mu_{\mathcal{D}_2^{sy}} \;=\; \mu_{\mathcal{E}_2^{sy}} \,-\, (\mu_a \,+\, \mu_{a,b}) \tag{6.3}$$

where $\mu_{\mathcal{E}_2^{sy}} = \mathcal{K}_{\tilde{B}_2,e}$-codim and $\mu_a + \mu_{a,b}$ is the length of a determinantal module by the Lê-Greuel formula (Theorem 1.2).

By the Metatheorem 5.1 there is an analogue of (6.3) for the Milnor number $\mu_{\varphi,\mathcal{D}_2^{sy}}$ on the ICIS $X = \varphi^{-1}(0)$ defined by $\varphi : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$.

Proof. We have already obtained (6.3), and the metaversion follows from the Metatheorem. □

We observe that for germs $f_0 : \mathbb{C}^2, 0 \to Sym_2, 0$ transverse to $\mathcal{D}_2^{sy}$ off 0, $\det \circ f_0$ defines an isolated hypersurface singularity and the Milnor number $\mu(\det \circ f_0) = \dim \mathcal{O}_{\mathbb{C}^2,0}/Jac(\det \circ f_0)$. The Milnor fiber of $\det \circ f_0$ equals the singular Milnor fiber of f_0, and hence the Milnor number and singular Milnor number agree. For $n > 3$ and $f_0 : \mathbb{C}^n, 0 \to Sym_2, 0$ (transverse to $\mathcal{D}_2^{sy}$ off 0), $\det \circ f_0$ no longer has an isolated singularity; however, the singular Milnor number is still defined. We can collapse the formula in Theorem 6.1 to yield a Jacobian–type formula for the singular Milnor number.

Corollary 6.2 (Jacobian Formula)**.** *If $n \geq 2$ and $f_0 : \mathbb{C}^n, 0 \to Sym_2, 0$ has rank ≥ 1 (and is transverse to $\mathcal{D}_2^{sy}$ off 0), then*

$$\mu_{\mathcal{D}_2^{sy}}(f_0) \;=\; \dim_{\mathbb{C}} \left(\mathcal{O}_{\mathbb{C}^n,0}/(Jac(\det \circ f_0) + Jac(f_0))\right) \tag{6.4}$$

where $Jac(f_0)$ is the ideal generated by the 3×3 minors of df_0.

We note in the corollary, if $n = 2$ then there are no 3×3 minors, so the formula reduces to Milnor's formula. It is not clear that the restriction to rank ≥ 1 is required, although the proof we give uses it.

Proof of Corollary 6.2.
As $f_0 : \mathbb{C}^n, 0 \to Sym_2, 0$ has $\operatorname{rank}(df_0(0)) \geq 1$, we may apply a matrix transformation on Sym_2 so that $df_0(0)$ has non-zero upper-left entry nonzero. Furthermore, we may assume that under the transformation, f_0 is transverse off zero to the line $a = b = 0$, so the composition of f_0 with projection onto the (a, b)–subspace has an isolated singularity at 0. Thus, after applying the transformation, we may apply a change of coordinates in $\mathbb{C}^n, 0$ so that for $y = (y_1, \dots, y_{n-1})$, f_0 has the form

$$f_0(x, y) \;=\; \begin{pmatrix} x & g(x, y) \\ g(x, y) & h(x, y) \end{pmatrix}.$$

Then, by assumption $(x, y) \mapsto (x, g(x, y))$ has an isolated singularity at 0. Hence, if $g_0(y) = g(0, y)$, then g_0 has an isolated singularity at 0 and $\mu_a(f_0) + \mu_{a,b}(f_0) = \mu(g_0)$. By Theorem 3.1 $\mu_{\mathcal{E}_2^{sy}}(f_0) = \dim_{\mathbb{C}} N\mathcal{K}_{H,e} f_0$. We will show that there is a surjective projection $N\mathcal{K}_{H,e} f_0 \to \mathcal{O}_{\mathbb{C}^{n-1},0}/Jac(g_0)$ with kernel the RHS of (6.4). Then, by Theorem 6.1 and the above remark, the result follows.

For H the defining equation for $\mathcal{E}_2^{sy}$, $\operatorname{Derlog}(H)$ is generated by $\zeta_1 = a\dfrac{\partial}{\partial b} + 2b\dfrac{\partial}{\partial c}$ and $\zeta_2 = a\dfrac{\partial}{\partial a} - c\dfrac{\partial}{\partial c}$. Then, if instead we write $f_0(x, y) = (x, g(x, y), h(x, y))$, we obtain the generators for $T\mathcal{K}_{H,e} f_0$ as an $\mathcal{O}_{\mathbb{C}^n,0}$–module

$$\frac{\partial f_0}{\partial x} = (1, g_x, h_x) \qquad \text{and} \qquad \frac{\partial f_0}{\partial y_i} = (0, g_{y_i}, h_{y_i})$$

and

$$\zeta_1 \circ f_0 = (0, x, 2g) \qquad \text{and} \qquad \zeta_2 \circ f_0 = (x, 0, -h)\,.$$

We may choose for generators for $\theta(f_0)$: $\varepsilon_1' = (1, g_x, h_x)$, $\varepsilon_2 = (0,1,0)$, and $\varepsilon_3 = (0,0,1)$. By the above, $\varepsilon_1' \in T\mathcal{K}_{H,e}f_0$; hence the projection of $\theta(f_0)$ to $\mathcal{O}_{\mathbb{C}^n,0}\{\varepsilon_2, \varepsilon_3\}$ maps $T\mathcal{K}_{H,e}f_0$ onto

$$L = \mathcal{O}_{\mathbb{C}^n,0}\{(x, 2g), (0, h + xh_x - 2gg_x), (g_{y_i}, h_{y_i}), i = 1, \dots, n-1\}$$

with kernel $\mathcal{O}_{\mathbb{C}^n,0}\{\varepsilon_1'\}$. Thus, $N\mathcal{K}_{H,e}f_0$ is mapped isomorphically to $\mathcal{O}_{\mathbb{C}^n,0}\{\varepsilon_2, \varepsilon_3\}/L$.

Next we further project $\mathcal{O}_{\mathbb{C}^n,0}\{\varepsilon_2, \varepsilon_3\}$ onto $\mathcal{O}_{\mathbb{C}^n,0}\{\varepsilon_2\}$. Under this projection L maps to $\mathcal{O}_{\mathbb{C}^n,0}\{(x, g_{y_i}, i = 1, \dots, n-1\}$. Thus,

$$\mathcal{O}_{\mathbb{C}^n,0}\{\varepsilon_2, \varepsilon_3\}/L \;\to\; \mathcal{O}_{\mathbb{C}^n,0}/(x, g_{y_i}, i = 1, \dots, n-1) \;\simeq\; \mathcal{O}_{\mathbb{C}^{n-1},0}/Jac(g_0)$$

is a surjective homomorphism onto the Jacobian algebra of g_0, which has length $\mu(g_0)$.

Hence, it is enough to show that the kernel of this projection has the required form. Since $\{(x, g_{y_i}, i = 1, \dots, n-1\}$ is a regular sequence, the only relations between these elements are the trivial ones. Thus, the kernel of the projection is generated by

(6.5)

$$(0, h + xh_x - 2gg_x),\ (0, xh_{y_i} - 2gg_{y_i}), \text{ and } (0, g_{y_i}h_{y_j} - g_{y_j}h_{y_i}), \quad 1 \le i, j \le n-1\,.$$

Then, $\det \circ f_0 = xh - g^2$ and, provided $n \ge 3$, the 3×3 minors of df_0 are the 2×2 determinants $g_{y_i}h_{y_j} - g_{y_j}h_{y_i}$. Thus, under the isomorphism $\mathcal{O}_{\mathbb{C}^n,0}\{\varepsilon_3\} \simeq \mathcal{O}_{\mathbb{C}^n,0}$, the generators in (6.5) are mapped to the the generators of $Jac(\det \circ f_0) + Jac(df_0)$. Thus, the kernel of the projection is isomorphic to the RHS of (6.4). $\square$

As a second application of Theorem 6.1, in §11 we will obtain a "$\mu = \tau$"- type formula for generic corank 1 maps defining 2×2 symmetric matrix singularities.

3×3 **Symmetric Matrices:** Next, we consider $\mu_{\mathcal{D}_3^{sy}}$ and use coordinates for Sym_3 given by $A = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$.

By our earlier discussion, $\mathcal{D}_3^{sy} \subset Sym_3$ has a free completion $\mathcal{E}_3^{sy} = \pi^*\mathcal{E}_2^{sy} \cup \mathcal{D}_3^{sy}$, with $\mathcal{E}_3^{sy}$ defined by $a\,(ad - b^2) \cdot \det(A) = 0$. Then, by (6.2), it is sufficient to determine $\tilde{\chi}_{\pi^*\mathcal{E}_2^{sy} \cap \mathcal{D}_3^{sy}}$. To apply the inductive procedure, we will use the auxiliary linear free divisors given by Proposition 4.3 (which arise from subgroups of B_3). We obtain the following formulas for singular Milnor numbers.

Proposition 6.3. *On the space of germs transverse off* 0 *to the associated varieties for* $V(\mathcal{Q}_a)$,

(1)

$$\mu_{\mathcal{Q}_a} \;=\; \mu_{bd\cdot\mathcal{Q}_a} \,-\, (\mu_{d,bc(bf-2ce)} + \mu_d) \,+\, (\mu_{d,c,bf} + \mu_{d,c}) \,-\, (\mu_{b,cd} + \mu_b). \tag{6.6}$$

(2) *There is an analogous formula for* $\mu_{\mathcal{Q}_f}$ *obtained from 1) by composing* f_0 *with the permutation* $(a, b, c, d, e, f) \mapsto (f, e, c, d, b, a)$.

By the Metatheorem 5.1 there is an analog of (6.6) for the Milnor number $\mu_{\varphi,\mathcal{Q}_a}$ *on the ICIS* $X = \varphi^{-1}(0)$ *defined by* $\varphi : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$.

Remark 6.4. The RHS of (6.6) is computed as the alternating sum of lengths of four determinantal modules using Theorems 3.1 and 3.2 (by Proposition 4.3, $V(b \cdot d \cdot \mathcal{Q}_a)$ is an H–holonomic linear free divisor and $V(bc(bf - 2ce))$ is an H–holonomic linear free divisor for 2×2 general matrices).

Proof. As we just noted, by Proposition 4.3, $V(b \cdot d \cdot \mathcal{Q}_a)$ is an H–holonomic linear free divisor. Hence, $V(\mathcal{Q}_a)$ has a free completion, so we may apply Lemma 3.5 to obtain

$$\mu_{\mathcal{Q}_a} \;=\; \mu_{bd\cdot\mathcal{Q}_a} \,-\, \mu_{bd} \,+\, (-1)^{n-1}\tilde{\chi}_{bd,\mathcal{Q}_a}\,. \tag{6.7}$$

Then, it is sufficent to compute $\tilde{\chi}_{bd,\mathcal{Q}_a}$. Then,

$$V(bd,\mathcal{Q}_a) \;=\; V(b,\mathcal{Q}_a) \cup V(d,\mathcal{Q}_a) \;=\; V(b,cd) \cup V(d, b(bf-2ce))\,.$$

Also, $V(b,cd) \cap V(d,b(bf-2ce)) = V(b,d)$. Hence, applying Lemma 3.5, we obtain

$$\tilde{\chi}_{bd,\mathcal{Q}_a} \;=\; (-1)^{n-2}\big(\mu_{b,cd} \,+\, \mu_{d,b(bf-2ce)} - \mu_{b,d}\big)\,. \tag{6.8}$$

Now, $V(bc(bf-2ce))$ is a linear free divisor for the 2×2 general matrices. Thus, by the metaversion of Lemma 3.5

$$\mu_{d,b(bf-2ce)} \;=\; \mu_{d,bc(bf-2ce)} \,-\, \mu_{d,c} \,-\, \mu_{d,c,bf}\,. \tag{6.9}$$

Substituting (6.9) and (6.8) into (6.7) and replacing

$$\mu_{bd} \,-\, \mu_{b,d} = \mu_b \,+\, \mu_d$$

yields (6.6). □

Then, using the formula given in Proposition 6.3, together with the fact that $\mathcal{E}_3^{sy}$ and $\mathcal{D}_2^{sy} \cup V(\mathcal{Q}_f)$ are H–holonomic free divisors by Proposition 4.3, we may compute the singular Milnor number $\mu_{\mathcal{D}_3^{sy}}$ using the following theorem.

Theorem 6.5. *For the space of germs transverse to the associated varieties for $\mathcal{E}_3^{sy}$ off 0, the singular Milnor number can be computed by*

$$\mu_{\mathcal{D}_3^{sy}} \;=\; \mu_{\mathcal{E}_3^{sy}} - \mu_{\mathcal{D}_2^{sy}\cup\mathcal{Q}_f} \,+\, \mu_{\mathcal{Q}_f} \,-\, \big((\mu_{a,\mathcal{Q}_a} + \mu_a) \,+\, (\mu_{a,b,c\cdot d} + \mu_{a,b})\big) \tag{6.10}$$

*where $\mu_{\mathcal{E}_3^{sy}} = \mathcal{K}_{\tilde{B}_3,e}$-*codim*, where $\tilde{B}_3$ is the subgroup of B_3 preserving the defining equation for $\mathcal{E}_3^{sy}$.*

By the Metatheorem 5.1 there is an analog of (6.10) for the Milnor number $\mu_{\varphi,\mathcal{D}_3^{sy}}$ on the ICIS $X = \varphi^{-1}(0)$ defined by $\varphi : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$.

Remark 6.6. In the RHS of (6.10), the first two terms are lengths of determinantal modules, $\mu_{\mathcal{Q}_f}$ is computed by Proposition 6.3, and of the last two groups of pairs of terms, the first pair is computed using the meta-version of (6.6) and Theorem 3.2, and the second is the length of a determinantal module by Theorem 3.2.

Proof. We may apply Lemma 3.5 to (6.1) to obtain

$$\mu_{\mathcal{D}_3^{sy}} \;= \mu_{\mathcal{E}_3^{sy}} \,-\, \mu_{\mathcal{E}_2^{sy}} \,+\, \tilde{\chi}_{\pi^*\mathcal{E}_2^{sy}\cap\mathcal{D}_3^{sy}}\,, \tag{6.11}$$

provided we can compute $\tilde{\chi}_{\pi^*\mathcal{E}_2^{sy}\cap\mathcal{D}_3^{sy}}$. Then, as $\mathcal{E}_2^{sy}$ is defined by $a\,(ad-b^2) = 0$,

$$\begin{aligned}\pi^*\mathcal{E}_2^{sy} \cap \mathcal{D}_3^{sy} \;&=\; (V(a)\cap\mathcal{D}_3^{sy}) \,\cup\, (V(ad-b^2)\cap\mathcal{D}_3^{sy}) \\ &=\; V(a,\mathcal{Q}_a) \,\cup\, V(ad-b^2,\mathcal{Q}_f)\,. \end{aligned}\tag{6.12}$$

Also, $V(a,\mathcal{Q}_a) \,\cap\, V(ad-b^2,\mathcal{Q}_f) = V(a,b,c\cdot d)$. Thus, applying Lemma 3.5, we obtain

$$\tilde{\chi}_{\pi^*\mathcal{E}_2^{sy}\cap\mathcal{D}_3^{sy}} \;=\; \tilde{\chi}_{\mathcal{Q}_a} \,+\, \tilde{\chi}_{ad-b^2,\mathcal{Q}_f} - \tilde{\chi}_{a,b,c\cdot d}\,. \tag{6.13}$$

Also, by Lemma 3.5

$$\mu_{\mathcal{Q}_f} \;=\; \mu_{(ad-b^2)\cdot\mathcal{Q}_f} \,-\, \mu_{ad-b^2} + (-1)^{n-1}\tilde{\chi}_{ad-b^2,\mathcal{Q}_f}\,. \tag{6.14}$$

Then, for (6.11), we can use (6.14) to substitute for $\tilde{\chi}_{ad-b^2,\mathcal{Q}_f}$ in (6.13), then evaluate the vanishing singular Euler characteristics in terms of singular Milnor numbers, and by Theorem 6.1 replace

$$\mu_{\mathcal{E}_2^{sy}} \; - \; \mu_{ad-b^2} \;=\; \mu_a \;+\; \mu_{a,b}\; . \tag{6.15}$$

This yields (6.10). □

In §11 we will also obtain a "$\mu = \tau$"- type formula for generic corank 1 maps defining 3×3 symmetric matrix singularities.

7. General Matrix Singularities

By the results [DP1, Theorem 7.1] for general matrices, summarized in §4, together with Theorem 4.4, both $\mathcal{E}_m$ in $M_{m,m}$, and $\mathcal{E}_{m-1,m}$ in $M_{m-1,m}$ are $H-$holonomic linear free divisors. Moreover, the determinant variety $\mathcal{D}_m$ in $M_{m,m}$ and the generalized determinant variety $\mathcal{D}_{m-1,m}$ in $M_{m-1,m}$, which has defining equation $\det(\hat{A}^{(m-1)}) = 0$, have free completions given by

$$\begin{aligned} \mathcal{E}_m \;&=\; \pi^*\mathcal{E}_{m-1,m} \cup \mathcal{D}_m \qquad \text{and} \\ \mathcal{E}_{m-1,m} \;&=\; \pi'^*\mathcal{E}_{m-1} \cup \mathcal{D}_{m-1,m}\, , \end{aligned} \tag{7.1}$$

for the projections $\pi : M_{m,m} \to M_{m-1,m}$ and $\pi' : M_{m-1,m} \to M_{m-1,m-1}$.

We first use these free completions to compute the singular Milnor number $\mu_{\mathcal{D}_2}$ for $\mathcal{D}_2 \subset M_{2,2}$.

2×2 **Matrices:**

We use coordinates $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on $M_{2,2}$ and consider the modified Cholesky–type representation. Then, by [DP1, Theorem 7.1], the exceptional orbit variety $\mathcal{E}_2$ is defined by $a\,b \cdot (ad - bc) = 0$. We then have the following

Theorem 7.1. *On the space of germs transverse off* 0 *to the associated varieties for* $\mathcal{E}_2$*,*

$$\mu_{\mathcal{D}_2} = \mu_{\mathcal{E}_2} - \left((\mu_a + \mu_{a,cb}) \,+\, (\mu_b + \mu_{b,ad})\right). \tag{7.2}$$

Here $\mu_{\mathcal{E}_2} = \mathcal{K}_{\tilde{G}_2,e}$-codim *where* $\tilde{G}_2$ *is the subgroup of* $B_2 \times C_2$ *preserving the defining equation* $a\,b \cdot (ad - bc) = 0$*. By the Metatheorem 5.1 there is an analogue of (7.2) for singular Milnor number* μ_{φ, D_2} *on an ICIS* $X = \varphi^{-1}(0)$ *defined by* $\varphi : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$.

Remark 7.2. Each pair $\mu_a + \mu_{a,cb}$ and $\mu_b + \mu_{b,ad}$ is computed as the length of a determinantal module by Theorem 3.2.

As a corollary of the proof we obtain the following which will be used in the calculations for the skew-symmetric case.

Corollary 7.3. *With the assumptions of Theorem 7.1,*

$$\mu_{a(ad-bc)} = \mu_{\mathcal{E}_2} - (\mu_b + \mu_{b,ad}) \tag{7.3}$$

and

$$\mu_{ad\,(ad-bc)} = \mu_{\mathcal{E}_2} + \left((\mu_d + \mu_{d,abc}) \,-\, (\mu_b + \mu_{b,ad})\right). \tag{7.4}$$

There are also corresponding meta-versions of these formulas.

Proof of Theorem 7.1 and Corollary 7.3. First, $\mathcal{D}_2$ has the H–holonomic free completion $\mathcal{E}_2$ defined by $ab \cdot (ad - bc)$. Thus,

$$\mu_{\mathcal{D}_2} \; = \mu_{\mathcal{E}_2} \, - \, \mu_{ab} \, + \, (-1)^{n-1} \tilde{\chi}_{ab,(ad-bc)} \, . \tag{7.5}$$

Since $V(ab, ad - bc) = V(a, bc) \cup V(b, ad)$ with $V(a, bc) \cap V(b, ad) = V(a, b)$, by Lemma 3.5

$$\tilde{\chi}_{ab,(ad-bc)} \; = (-1)^{n-2} \big(\mu_{a,bc} \, + \, \mu_{b,ad} \, - \, \mu_{a,b} \big) \, . \tag{7.6}$$

Then, substituting (7.6) into (7.5) and replacing

$$\mu_{ab} \, - \, \mu_{a,b} \; = \; \mu_a \, + \, \mu_b$$

yields (7.2).

For Corollary 7.3, the argument for (7.3) is similar using instead that $\mathcal{E}_2$ is a free completion of $V(a(ad - bc))$. While for (7.4) we use

$$V(ad\,(ad-bc)) = V(a(ad-bc)) \cup V(d) \quad \text{ with } \quad V(a(ad-bc)) \cap V(d) = V(d, abc) \, .$$

By Lemma 3.5

$$\mu_{ad(ad-bc)} \; = \mu_{a(ad-bc)} \, + \, \mu_d \, + \, \mu_{d,abc} \tag{7.7}$$

and then we substitute (7.3) for $\mu_{a(ad-bc)}$. $\square$

As for symmetric matrices, we deduce in §11 a "$\mu = \tau$"–type formula for generic corank 1 germs for 2×2 general matrices.

2×3 **Matrices:**

We use coordinates $\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$ on $M_{2,3}$ and consider the modified Cholesky–type representation. Again by [DP1, Theorem 7.1], the exceptional orbit variety $\mathcal{E}_{2,3}$ is a free divisor and is defined by $a\, b \cdot (ae - bd) \cdot (bf - ce) = 0$.

We use this free divisor to compute μ_V where $V = V((ae - bd) \cdot (bf - ce))$. To simplify notation, we let V_j denote the subvariety of $M_{2,3}$ defined by the determinant of the submatrix obtained by deleting the j–th column. Also, we denote the union $V_i \cup V_j$ by $V_{i\,j}$. Then, $V((ae - bd) \cdot (bf - ce)) = V_{1\,3}$. Once we have computed μ_V for $V = V_{1\,3}$, then we may compute μ_V for $V = V_{i\,j}$ by permuting the coordinates corresponding to the permutation of the columns sending $(1, 3)$ to (i, j).

Theorem 7.4. *For the space of germs transverse to the associated varieties for $\mathcal{E}_{2,3}$ off* 0*,*

$$\mu_{V_{1\,3}} \; = \; \mu_{\mathcal{E}_{2,3}} \, - \, (\mu_{a,bde(bf-ce)} + \mu_a) \, + \, (\mu_{a,e,bdf} + \mu_{a,e}) \, - \, (\mu_{b,ace} + \mu_b) \, . \tag{7.8}$$

Here $\mu_{\mathcal{E}_{2,3}} = \mathcal{K}_{\tilde{G}_3,e}$-codim *where $\tilde{G}_3$ is the subgroup of $B_2 \times C_3$ which preserves the defining equation* $ab \cdot (ae - bd) \cdot (bf - ce) = 0$.

By the Metatheorem 5.1, there is an analogue of (7.8) for singular Milnor number $\mu_{\varphi,V_{1\,3}}$ on the ICIS $X = \varphi^{-1}(0)$ defined by $\varphi : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$.

Remark 7.5. Each grouped pair on the RHS of (7.8) can be computed using Theorem 3.2 for AFD's on an ICIS and the first term by Theorem 3.1. Thus, the RHS of (7.8) is computed as the alternating sum of the lengths of four determinantal modules.

We can obtain the corresponding formulas for $\mu_{V_{1\,2}}$, resp. $\mu_{V_{2\,3}}$ by applying (7.8) after first composing f_0 with the permutation $(a, b, c, d, e, f) \mapsto (a, c, b, d, f, e)$, respectively, $(a, b, c, d, e, f) \mapsto (b, a, c, e, d, f)$.

Proof of Theorem 7.4. First, $V((ae - bd)(bf - ce))$ has as a free completion $\mathcal{E}_{2,3} = V(ab(ae - bd)(bf - ce))$. By Lemma 3.5,

$$\mu_{V_{1\,3}} \;=\; \mu_{\hat{\mathcal{E}}_{(1,3)}} \;-\; \mu_{ab} \;+\; (-1)^{n-1}\tilde{\chi}_{ab,(ae-bd)(bf-ce)}. \tag{7.9}$$

Since $V(ab, (ae - bd)(bf - ce)) = V(a, bd(bf - ce)) \cup V(b, ace)$ and $V(a, bd(bf - ce)) \cap V(b, ace) = V(a, b)$, we have by Lemma 3.5 (by evaluating the $\tilde{\chi}$ as singular Milnor numbers),

$$\tilde{\chi}_{ab,(ae-bd)(bf-ce)} \;=\; (-1)^{n-2}\big(\mu_{a,bd(bf-ce)} \;+\; \mu_{b,ace} \;-\; \mu_{a,b}\big)\,. \tag{7.10}$$

Then, $V(bd(bf - ce))$ has a free completion $V(ebd(bf - ce))$. Thus by the meta-version of Lemma 3.5,

$$\mu_{a,bd(bf-ce)} \;=\; \mu_{a,bde(bf-ce)} \;-\; \mu_{a,e} \;-\; \mu_{a,e,bdf}\,. \tag{7.11}$$

Then, by substituting (7.11) for $\mu_{a,bd(bf-ce)}$ into (7.10), then substituting the resulting expression into (7.9), and lastly replacing

$$\mu_{ab} \;-\; \mu_{a,b} \;=\; \mu_a \;+\; \mu_b\,,$$

we obtain the result. □

Remark 7.6. We have also obtained a formula for 3×3 general matrix singularities; however, we are not including it in this paper.

8. Vanishing Topology for Cohen–Macaulay Singularities in $\mathbb{C}^n$

In this section we apply the preceding results in reverse to obtain a formula for the singular vanishing Euler characteristic for Cohen–Macaulay singularities in $\mathbb{C}^n$ defined as $\mathcal{V}_0 = f_0^{-1}(\mathcal{V})$. Here $\mathcal{V}$ is the variety of singular matrices of rank ≤ 1 in $M_{2,3}$ and $f_0 : \mathbb{C}^n, 0 \to M_{2,3}, 0$ is transverse to $\mathcal{V}$ off 0. We then apply this formula in several different ways. First, if $n = 4, 5$ or 6, then $\mathcal{V}_0$ will be an isolated surface, resp. 3-fold, resp. 4-fold, singularity. In the case of $n = 4$, we obtain a formula for the Milnor number for isolated Cohen–Macaulay surface singularities as the sum of lengths of determinantal modules. Furthermore in the case of the Cohen–Macaulay 3-fold singularities, we obtain a formula for the difference of the second and third Betti numbers $b_3 - b_2$ of the Milnor fiber. We furthermore deduce bounds on these Betti numbers. In §11, we shall implement these formulas using the results of §7, with a software package developed for Macaulay2, to compute the Milnor number for simple Cohen–Macaulay surface singularities and $b_3 - b_2$ for 3–fold singularities.

In addition, if we consider instead Cohen–Macaulay singularities on an ICIS X defined by φ, then we obtain analogous results in each case using the corresponding meta-versions of the results. Finally, we also use these results to obtain formulas for the Milnor numbers of functions defining ICIS on isolated Cohen–Macaulay singularities.

Singular Vanishing Euler Characteristic for Nonisolated Cohen–Macaulay Singularities in $\mathbb{C}^n$.

Let $M_{2,3}$ denote the space of 2×3 matrices with $\mathcal{V}$ the variety of singular matrices of rank ≤ 1. Consider $f_0 : \mathbb{C}^n, 0 \to M_{2,3}, 0$. Because $\mathcal{V}$ is not a complete intersection, f_0 does not have a singular Milnor number $\mu_{\mathcal{V}}(f_0)$. However, we can compute $\tilde{\chi}_{\mathcal{V}}(f_0)$.

Theorem 8.1. *For a germ $f_0 : \mathbb{C}^n, 0 \to M_{2,3}, 0$ which is transverse to the associated varieties off 0, let $\mathcal{V}_0 = f_0^{-1}(\mathcal{V})$ be the nonisolated Cohen–Macaulay singularity. Then, the singular vanishing Euler characteristic is computed by*

$$\tilde{\chi}_{\mathcal{V}}(f_0) \;=\; (-1)^{n-1}\left(\mu_{V_{1\,2\,3}}(f_0) \;-\; \sum \mu_{V_{i\,j}}(f_0) + \sum_{i=1}^{3} \mu_{V_i}(f_0)\right) \tag{8.1}$$

where the first sum is over $\{i, j\} = \{1, 2\}, \{1, 3\}, \{2, 3\}$ and $V_{1\,2\,3} = V_1 \cup V_2 \cup V_3$.

By the Metatheorem 5.1 there is an analog of (8.1) for vanishing Euler characteristic $\tilde{\chi}_{\varphi,D_2}$ on the ICIS $X = \varphi^{-1}(0)$ defined by $\varphi : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$.

Remark 8.2. Here we are using the notation of §7. The $\mu_{V_{i\,j}}$ are computed by Theorem 7.4, and the μ_{V_i} are computed by Theorem 7.1. Also, as explained in §4, the variety $V_{1\,2\,3}$ is an H–holonomic linear free divisor corresponding to a quiver representation by Buchweitz-Mond [BM]. Hence, $\mu_{V_{1\,2\,3}}$ can be computed as the length of a determinantal module by Theorem 3.1.

As we will see in §11, we can frequently apply generic reduction by applying an element of $\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_3(\mathbb{C})$ to f_0 so that, depending on rank of $df_0(0)$, the terms in (8.1) either vanish or their computation considerably simplifies.

Milnor Numbers for Isolated Cohen–Macaulay Surface Singularities in $\mathbb{C}^4$.

We now consider the special case of $f_0 : \mathbb{C}^4, 0 \to M_{2,3}, 0$ which is transverse to $\mathcal{V}$ off 0. By the Hilbert-Burch Theorem, $\mathcal{V}_0 = f_0^{-1}(\mathcal{V})$ is an isolated Cohen Macaulay surface singularity. By results of Wahl [Wa] (in the weighted homogeneous case) and Greuel-Steenbrink [GS], its Milnor fiber has first Betti number $b_1 = 0$. By convention, the second Betti number is referred to as the Milnor number $\mu(\mathcal{V}_0)$.

In this case, the versal unfolding of $\mathcal{V}_0$ in the sense of algebraic geometry is obtained by a deformation of the mapping f_0, see [Sh]. Thus, what we call the singular Milnor fiber is actually the Milnor fiber of $\mathcal{V}_0$ since a stabilization of f_0 will only (transversely) intersect the smooth part of $\mathcal{V}$. Hence, we may compute $\mu(\mathcal{V}_0) = \tilde{\chi}_{\mathcal{V}}(f_0)$. By applying an element of $\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_3(\mathbb{C})$ to f_0 we may assume that f_0 is transverse to all of the associated varieties for each V_i and $V_{i\,j}$. Then, the preceding results yield the following formula for $\mu(\mathcal{V}_0)$.

Theorem 8.3. *For a germ $f_0 : \mathbb{C}^4, 0 \to M_{2,3}, 0$ which is transverse to the associated varieties off 0, let $\mathcal{V}_0 = f_0^{-1}(\mathcal{V})$ be the isolated Cohen–Macaulay surface singularity. Then, the Milnor number is computed by*

$$\mu(\mathcal{V}_0) \;=\; \sum \mu_{V_{i\,j}}(f_0) - \sum_{i=1}^{3} \mu_{V_i}(f_0) \;-\; \mu_{V_{1\,2\,3}}(f_0) \tag{8.2}$$

where the first sum is over $\{i, j\} = \{1, 2\}, \{1, 3\}, \{2, 3\}$*. By the Metatheorem 5.1 there is an analogue of (8.2) for the Milnor number* $\mu(\mathcal{V}_0)$ *on the ICIS* $X = \varphi^{-1}(0)$ *defined by* $\varphi : \mathbb{C}^n, 0 \to \mathbb{C}^{n-4}, 0$.

All of Remark 8.2 applies equally well to Theorem 8.3.

Betti Numbers of Milnor Fibers for Isolated Cohen–Macaulay 3–fold Singularities in $\mathbb{C}^5$.

We consider the case $f_0 : \mathbb{C}^5, 0 \to M_{2,3}, 0$ which is transverse to $\mathcal{V}$ off 0. Now $\mathcal{V}_0 = f_0^{-1}(\mathcal{V})$ is an isolated Cohen–Macaulay 3–fold singularity. A stabilization of f_0 will miss the isolated singular point $0 \in \mathcal{V}$; hence the singular Milnor fiber for f_0 is the Milnor fiber of $\mathcal{V}_0$. Thus, the singular vanishing Euler characteristic of f_0 is the vanishing Euler characteristic of $\mathcal{V}_0$. The results of Greuel–Steenbrink still apply; and so the first Betti number $b_1(\mathcal{V}_0) = 0$ (in fact, they show that the Milnor fiber of $\mathcal{V}_0$ is simply connected). Thus, $\tilde{\chi}_{\mathcal{V}}(f_0) = b_2(\mathcal{V}_0) - b_3(\mathcal{V}_0)$. Then, we may compute this difference

Theorem 8.4. *For a germ* $f_0 : \mathbb{C}^5, 0 \to M_{2,3}, 0$ *which is transverse to the associated varieties off* 0*, let* $\mathcal{V}_0 = f_0^{-1}(\mathcal{V})$ *be the nonisolated Cohen–Macaulay* 3*–fold singularity. Then,*

$$b_3(\mathcal{V}_0) - b_2(\mathcal{V}_0) \;=\; \sum \mu_{V_{i\,j}}(f_0) - \sum_{i=1}^{3} \mu_{V_i}(f_0) \;-\; \mu_{V_{1\,2\,3}}(f_0) \tag{8.3}$$

where the first sum is over $\{i, j\} = \{1, 2\}, \{1, 3\}, \{2, 3\}$.

By the Metatheorem 5.1 there is an analogue of (8.3) for the difference $b_2(\mathcal{V}_0 \cap X) - b_3(\mathcal{V}_0 \cap X)$ *on the ICIS* $X = \varphi^{-1}(0)$ *defined by* $\varphi : \mathbb{C}^n, 0 \to \mathbb{C}^{n-5}, 0$.

There are analogous remarks as earlier regarding the computation of the RHS of (8.3). Depending on the sign of the RHS of (8.3), it gives either a crude lower bound on $b_2(\mathcal{V}_0)$ if the RHS is positive, or on $b_3(\mathcal{V}_0)$ if the RHS is negative.

Milnor Numbers for Isolated ICIS singularities on Isolated Cohen–Macaulay Singularities.

As a final consequence of the meta-versions of the preceding results, we consider $\mathcal{V}_0$ an isolated Cohen-Macaulay surface or 3-fold singularity defined by $f_0 : \mathbb{C}^n, 0 \to M_{2,3}, 0$ for $n = 4, 5$. Also, let $\varphi : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ be an ICIS germ, with $n - p \leq \dim \mathcal{V}_0$, and so that $\varphi|\mathcal{V}_0$ has an isolated singularity. We let $X_0 = \varphi^{-1}(0) \cap \mathcal{V}_0$ and consider the Milnor fiber X_t of $\varphi|\mathcal{V}_0$. Then, X_0 is again an isolated Cohen-Macaulay (point, curve or surface) singularity. We can use the preceding results to compute the Milnor number.

Corollary 8.5. *In the preceding situation, the Milnor number of the restriction* $\mu(X_0) = \chi_{\varphi, \mathcal{V}}(f_0)$*, which can be computed using the meta-version of (8.1) which becomes the meta-versions of either (8.2) or (8.3).*

Proof. We may construct of stabilization of $f = (\varphi, f_0) : \mathbb{C}^n, 0 \to \mathbb{C}^p \times M_{2,3}$ which is the transverse intersection of a Milnor fiber of φ with a singular Milnor fiber of f_0 (as a nonlinear section of $\mathcal{V}$, which is the Milnor fiber of $\mathcal{V}_0$. However, another stabilization of f is obtained as the intersection of the Milnor fiber of φ with $\mathcal{V}_0$, which is the Milnor fiber of X_0. Both of these are stabilizations of $\{0\} \times \mathcal{V} \subset \mathbb{C}^p \times M_{2,3}$. Hence, they are diffeomorphic. Thus, they have the same Euler characteristic. For the second, we obtain the Milnor number $\mu(X_0)$. For the first, we have $\chi_{\varphi,\mathcal{V}}(f_0)$, and the meta-version of (8.1) allows us to compute it. This becomes the meta-version of either (8.2) or (8.3). $\square$

9. Skew–Symmetric Matrix Singularities

We use coordinates for Sk_4 given by

$$A \;=\; \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix}.$$

The determinantal variety $\mathcal{D}_4^{sk}$ has reduced defining equation the Pfaffian $\mathrm{Pf}(A)$, which we shall denote simply as Pf. Then, by [DP1, Theorem 8.1] and also [P, Theorem 5.2.21], the nonlinear solvable Lie algebra $\mathcal{L}_4$ determines a free divisor $\mathcal{E}_4^{sk}$, which is defined by $a\,b\,d\,(be-dc)\cdot \mathrm{Pf}(A) = 0$. Also $a\,b\,d\,(be-dc) = 0$ defines a free divisor $\mathcal{E}_2'$ (the product union of $\{0\} \subset \mathbb{C}$ defined by $a = 0$ with $\mathcal{E}_2$ for the 2×2 upper right-hand submatrix of A). Hence, the Pfaffian hypersurface $\mathcal{D}_4^{sk}$ has a free completion by this free divisor

$$\mathcal{E}_4^{sk} = \pi^*\mathcal{E}_2' \cup \mathcal{D}_4^{sk}\,.$$

We denote $\pi^*\mathcal{E}_2'$ simply by $\mathcal{E}_2'$. We can also use this to give a free completion of $V((be-dc)\cdot \mathrm{Pf}(A))$. We next use this free completion to compute the singular Milnor number $\mu_{\mathcal{D}_4^{sk}}$ via the following theorem.

Theorem 9.1. *For the space of germs transverse to the associated varieties for $\mathcal{E}_4^{sk}$ off 0, the singular Milnor number can be computed by*

$$\mu_{\mathcal{D}_4^{sk}} \;=\; \mu_{\mathcal{E}_4^{sk}} \,-\, \mu_{a,f,(be-cd)} \,+\, \lambda_1 \,+\, \lambda_2 \,+\, \lambda_3\,, \tag{9.1}$$

where again λ_k is a sum of terms of defining codimension k and are given by

$$\begin{aligned} \lambda_1 \;&=\; -\big(\mu_{b,cd\,(af+cd)} + \mu_{d,be\,(af-be)} + 2\mu_{a,(be-cd)} \,+\, \mu_{f,(be-cd)}\big) \\ \lambda_2 \;&=\; -\,\big(\mu_{be-cd} \,+\, \mu_{a,b,c\cdot d} \,+\, \mu_{a,d,b\cdot e}\big) \\ \lambda_3 \;&=\; (\mu_{a,b,d} + \mu_{b,d}) - \mu_{abd}\,. \end{aligned} \tag{9.2}$$

Here $\mu_{\mathcal{E}_4^{sk}} = \mathcal{K}_{\tilde{\mathcal{L}}_4,e}$-codim, where $\tilde{\mathcal{L}}_4$, is the Lie subalgebra of $\mathcal{L}_4$, preserving the defining equation for $\mathcal{E}_4^{sk}$.

By the Metatheorem 5.1 there is an analogue of (9.1) (and (9.2)) for the Milnor number $\mu_{\varphi,\mathcal{D}_4^{sk}}$ on the ICIS $X = \varphi^{-1}(0)$ defined by $\varphi : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$.

Also, the terms in the λ_i can be computed using the meta-versions of Theorem 7.1 and Corollary 7.3.

Proof. We first consider $V((be-cd)\cdot \mathrm{Pf})$. By Lemma 3.5

$$\mu_{\mathrm{Pf}} \;=\; \mu_{(be-cd)\cdot\mathrm{Pf}} \,-\, \mu_{be-cd} \,+\, (-1)^{n-1}\tilde{\chi}_{be-cd,\mathrm{Pf}}\,. \tag{9.3}$$

As $\mathcal{E}_4^{sk}$ as a free completion of $V((be-cd)\cdot \mathrm{Pf})$, by Lemma 3.5

$$\mu_{(be-cd)\cdot\mathrm{Pf}} \;=\; \mu_{\mathcal{E}_4^{sk}} \,-\, \mu_{abd} \,+\, (-1)^{n-1}\tilde{\chi}_{abd,(be-cd)\cdot\mathrm{Pf}}\,. \tag{9.4}$$

Next, to compute $\tilde{\chi}_{be-cd,\mathrm{Pf}}$ we observe

$$V(be-cd, \mathrm{Pf}) \;=\; V(be-cd, af) \;=\; V(a, be-cd) \cup V(f, be-cd)$$

and $V(a, be-cd) \cap V(f, be-cd) = V(a, f, be-cd)$. Hence, by Lemma 3.5

$$\begin{aligned} \tilde{\chi}_{be-cd,\mathrm{Pf}} &= \tilde{\chi}_{a,be-cd} + \tilde{\chi}_{f,be-cd} - \tilde{\chi}_{a,f,be-cd} \\ &= (-1)^{n-2}\big(\mu_{a,be-cd} + \mu_{f,be-cd} + \mu_{a,f,be-cd}\big)\,. \end{aligned} \tag{9.5}$$

Lastly, we consider $\tilde{\chi}_{abd,(be-cd)\cdot \mathrm{Pf}}$.

$$V(abd, (be-cd)\cdot \mathrm{Pf}) = V(a, (be-cd)) \cup V(b, cd(af+cd)) \cup V(d, be(af-be))\,.$$

In addition,

$$\begin{aligned} V(a, (be-cd)) \cap V(b, cd(af+cd)) &= V(a, b, cd) \\ V(a, (be-cd)) \cap V(d, be(af-be)) &= V(a, d, be) \\ V(b, cd(af+cd)) \cap V(d, be(af-be)) &= V(b, d)\,; \end{aligned} \tag{9.6}$$

and

$$V(a, (be-cd)) \cap V(b, cd(af+cd)) \cap V(d, be(af-be)) = V(a, b, d)\,. \tag{9.7}$$

Thus, since all of the terms on the RHS of (9.6) and (9.7) will define AFD's on ICIS, we may apply (3.8) and evaluate each $\tilde{\chi}$ as a singular Milnor number to obtain

$$\begin{aligned} \tilde{\chi}_{abd,(be-cd)\cdot \mathrm{Pf}} &= (-1)^{n-2}\big(\mu_{a,be-cd} + \mu_{b,cd(af+cd)} + \mu_{d,be(af-be)}\big) \\ &\quad - (-1)^{n-3}\big(\mu_{a,b,cd} + \mu_{a,d,be} - \mu_{b,d}\big) + (-1)^{n-3}\mu_{a,b,d}\,. \end{aligned} \tag{9.8}$$

Finally, we substitute (9.8) into (9.4), and substitute the resulting (9.4) and (9.5) into (9.3). After rearranging terms and simplifying coefficients we obtain (9.1). □

Remark 9.2. Because there are several ways to give a free completion for $\mathcal{D}_4^{sk}$, there are several variations on the formulas given in Theorem 9.1. We have given a version which is conceptually shortest in terms of having to compute the fewest number of singular Milnor numbers in (9.1).

For generic corank 1 skew–symmetric matrix singularities, it will follow by generic reduction that all of the λ_i for $i > 0$ in (9.1) vanish. In §11 we further compute the two remaining terms and will obtain a "$\mu = \tau$"–type result.

10. Higher Multiplicities of Linear Free Divisors

We will begin computing the general formulas in the special cases of mappings f_0 within restricted classes with a goal of relating $\mu_{\mathcal{D}}(f_0)$ for $\mathcal{D}$ a determinantal variety and $\tau = \mathcal{K}_{M,e}$-codim(f_0). For this we must first compute $\mu_{\mathcal{E}}(f_0)$ for various H–holonomic free divisors $\mathcal{E}$ and then apply the results of the previous sections.

We begin with the simplest case where f_0 is a generic linear section. Then, we are really computing the higher multiplicities for (H–holonomic) linear free divisors. We recall that for a hypersurface (or more generally a complete intersection) $\mathcal{V}, 0 \subset \mathbb{C}^N, 0$ we may define for $0 < k < N$ the k–th higher multiplicity, denoted $\mu_k(\mathcal{V})$, as the singular Milnor number $\mu_{\mathcal{V}}(i)$ for a generic linear section $i : \mathbb{C}^k, 0 \to \mathbb{C}^N, 0$. This is analogous to the definition of Teissier's μ_* sequence for isolated hypersurface singularities [Te] and [LeT]. To be consistent with our earlier notation, if $k < \ell =$ codim$\mathcal{V}$, then we let $\mu_k(\mathcal{V}) \overset{def}{=} (-1)^{k-\ell+1}$. If $\mathcal{V}$ is a hypersurface then $\mu_0(\mathcal{V}) = 1$.

Very surprisingly, in the case of H–holonomic linear free divisors, these higher multiplicities can be computed independent of the specific linear free divisor $\mathcal{V}$.

Free Divisor	$\mathcal{E}_m^{sy}$	$\mathcal{E}_m$	$\mathcal{E}_{m-1,m}$	$\mathcal{E}_m^{sk}$
μ_k	$\binom{\binom{m+1}{2}-1}{k}$	$\binom{m^2-1}{k}$	$\binom{m(m-1)-1}{k}$	$\sigma_k(1^{\binom{m}{2}-(m-2)}, 2, 2, \ldots, [(m+1)/2])$

TABLE 3. Higher Multiplicities for the exceptional orbit varieties $\mathcal{E}$ for the solvable group and solvable Lie algebra block representations in Table 1.

Proposition 10.1. *If $\mathcal{V}, 0 \subset \mathbb{C}^N, 0$ is an H–holonomic linear free divisor, then*

$$\mu_k(\mathcal{V}) \;=\; \binom{N-1}{k} \qquad 0 < k < N\,. \tag{10.1}$$

Hence, for any H–holonomic linear free divisor in $\mathbb{C}^N$, there is the duality relation

$$\mu_k(\mathcal{V}) \;=\; \mu_{N-1-k}(\mathcal{V}) \qquad 0 \leq k \leq N-1.$$

Before proving the proposition, we point out as a consequence that any two H–holonomic linear free divisors in $\mathbb{C}^N$ will always have the same higher multiplicities.

Example 10.2. There are three exceptional orbit varieties in $M_{2,3}$: that for the action of the solvable group $B_2 \times C_3$ given by modified Cholesky factorization; the "quiver discriminant" arising from the reductive group $(\mathrm{GL}_3 \times (\mathbb{C}^*)^3)/\mathbb{C}^*$ for the quiver representation just mentioned; and that for $(\mathbb{C}^*)^6$ given by the coordinate hyperplane arrangement. These are quite distinct H–holonomic linear free divisors in $M_{2,3}$. However, by Proposition 10.1, the k–th higher multiplicities for them all equal $\binom{5}{k}$.

We thus obtain the higher multiplicities for the linear free divisors listed in Table 2.

Proposition 10.3. *For the free divisors given in Table 2, the higher multiplicities are given by Table 3.*

In the table, σ_k denotes the k–th elementary symmetric function, and 1^ℓ denotes 1 being repeated ℓ times and $2, 2, \ldots, [(m+1)/2]$ denotes the sequence of $m-3$ integers $2, 2, 3, 3, \ldots,$ truncated at $[(m+1)/2]$.

Remark 10.4. We note that in the table $\mathcal{E}_3^{sy}$, $\mathcal{E}_{2,3}$ and $\mathcal{E}_4^{sk}$ are linear free divisors in $\mathbb{C}^6$; but $\mathcal{E}_4^{sk}$ will have different higher multiplicities because it is not a linear free divisor. In fact the values $\sigma_k(1^4, 2) = 6, 14, 16, 9, 2$ for $k = 1, \ldots, 5$ also do not satisfy the duality property in Proposition 10.1. Surprisingly, the higher multiplicities $\mu_k(\mathcal{D}_2^{sy})$, $\mu_k(\mathcal{D}_3^{sy})$, $\mu_k(\mathcal{D}_2)$, and $\mu_k(\mathcal{D}_4^{sk})$ do satisfy the duality property. This follows by the calculations in §6, 7 and 9. For $\mathcal{D}_2^{sy}$, $\mathcal{D}_2$ and $\mathcal{D}_4^{sk}$ it also follows because their defining equations have Morse singularities at 0, and the restrictions to a generic section are again Morse singularities and their Milnor fiber is the singular Milnor fiber of the generic section. Thus, all of the nonzero higher multiplicities equal 1. By contrast the higher multiplicities $\mu_k(\mathcal{D}_3^{sy}) = 1, 2, 4, 4, 2, 1$ for $k = 0, 1, \ldots, 5$ still satisfy the duality property. This leads to the

Question/Conjecture: The higher multiplicities for the determinantal varieties $\mathcal{D}_n^{sy}$ and $\mathcal{D}_n$ satisfy the duality property.

Because duality does not hold for $\mathcal{E}_4^{sk}$, it suggests that the result for $\mathcal{D}_4^{sk}$ may only be a low dimension phenomenon.

Proof. Both propositions are a consequence of the fact that for all such free divisors $\mathcal{V}$, the module $N\mathcal{K}_{\mathcal{V},e} \cdot i$ is (weighted) homogeneous in the sense of [D5]; hence by Theorem 1 of [D5] its length is given by a formula in terms of its weights. This will yield the result.

The weighted homogeneous case for $N\mathcal{K}_{\mathcal{V},e} \cdot f_0$, concerns $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^N, 0$ with $\mathcal{V}$ a free divisor such that we can choose weights for $\mathbb{C}^n$ and $\mathbb{C}^N$ so that: i) both f_0 and $\mathcal{V}$ are weighted homogeneous for the same weights; and ii) the generators of Derlog(H) may also be chosen to be weighted homogeneous for these weights. In our cases, we use weights 0 for the coordinates of $\mathbb{C}^N$ and 1 for the weights of the coordinates x_j for $\mathbb{C}^n$. Then, as the section i is linear, $\dfrac{\partial i}{\partial x_j}$ has weight 0 and for linear free divisors, $\zeta_j \circ i$ has weight 1, while for $\mathcal{E}_m^{sk}$ the last $m-3$ generators will have weights $2, 2, 3, 3, \dots$ as in the statement. Then, by Theorem 1 of [D5], $\mu_k(\mathcal{E}) = \mu_{\mathcal{E}}(i)$ will equal $\sigma_k(1, \dots, 1)$ with $(N-1)$ 1's $(= \binom{N-1}{k})$ for a linear free divisor $\mathcal{E}$, or $\sigma_k(1, \dots, 1, 2, 2, \dots, [(m+1)/2])$ with $(\binom{m}{2} - (m-2))$ 1's in the case of $\mathcal{E} = \mathcal{E}_m^{sk}$ (and $N = \binom{m}{2}$). ☐

We use the preceding propositions in conjunction with two other properties of higher multiplicities which follow from Proposition 3.3.

Proposition 10.5. *Let $\mathcal{V}, 0 \subset \mathbb{C}^N, 0$ be an H–holonomic free divisor.*

(1) *If $\mathcal{V}' = \mathcal{V} \times \mathbb{C}^p, 0 \subset \mathbb{C}^{N+p}, 0$, then*
$$\mu_k(\mathcal{V}') \;=\; \mu_k(\mathcal{V}) \qquad \text{for } 0 \leq k < N\,.$$
(2) *If $\mathcal{V}'', 0 = \mathcal{V} \times \{0\} \subset \mathbb{C}^{N+p}, 0$ is the image of $\mathcal{V}, 0$ via the inclusion $\mathbb{C}^N, 0 \subset \mathbb{C}^{N+p}, 0$ (so that $\mathcal{V}''$ is a free divisor in a linear subspace of $\mathbb{C}^{N+p}$), then*
$$\mu_k(\mathcal{V}'') \;=\; \mu_{k-p}(\mathcal{V}) \quad \text{if } k \geq p, \text{ and } = (-1)^{p-k} \text{ if } k < p\,.$$

Proof. For 1), we can choose a generic linear section $i : \mathbb{C}^k, 0 \to \mathbb{C}^{N+p}$ of $\mathcal{V}'$ so that $\pi \circ i$ is also a generic linear section of $\mathcal{V}$ and the result follows from 1) of Proposition 3.3.

For 2), provided $k \geq p$, we may choose a generic linear section $i : \mathbb{C}^k, 0 \to \mathbb{C}^{N+p}$ so that i is transverse to $\mathbb{C}^p$ and if $W = i^{-1}(0) \times \mathbb{C}^N$ then $\pi \circ i|W$ is a generic linear section of $\mathcal{V}$. Then, 2) follows by applying 2) of Proposition 3.3. ☐

11. $\mu = \tau - \gamma$ Type Results for Matrix Singularities

In this section we consider the relation between μ and τ for singularities defined by f_0. Here μ will denote a singular Milnor number $\mu_{\mathcal{V}}(f_0)$ or possibly the Milnor number of a Cohen–Macaulay isolated surface singularity, and τ will denote an appropriate $\mathcal{K}_{H,e}$–codim of f_0. We will be concerned with how much μ differs from τ or equivalently consider the difference $\gamma = \tau - \mu$. We recall that results of Greuel and Looijenga for (see [L, Chap. 7]) ICIS $X, 0$ with μ the usual Milnor number and τ the Tjurina number (which is also the $\mathcal{K}_e$–codim). They show that $\mu \geq \tau$ with equality if X is weighted homogeneous. Thus, for ICIS, $\gamma \leq 0$. An analogous result was shown to hold for the "discriminant Milnor number" in [DM]. For matrix singularities, we consider what form such a result takes. We will show for matrix singularities which are hypersurfaces defined by corank 1 mappings that

$\gamma = 0$. However, when we consider Cohen–Macaulay singularities defined from 2×3 matrices there are some fundamental changes which occur and γ becomes positive.

Corank 1 mappings and $\mu = \tau$–type Results. We begin by considering matrix singularities defined by corank 1 mappings $f_0 : \mathbb{C}^n, 0 \to M, 0$ for various spaces of matrices M (with $\dim M = N$). Here corank refers to the corank of $df_0(0)$ and not that of the specific matrices $f_0(x)$.

As a prelude, we first consider germs $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^N, 0$ with $n \geq N$ and $\mathcal{V} \subset \mathbb{C}^N$ an H–holonomic linear free divisor. We consider such corank 1 mappings which are generic, in the sense that $W = df_0(0)(\mathbb{C}^n)$ is a generic linear section of $\mathcal{V}$. We suppose $w_0 \notin W$. Then, by the inverse function theorem, we may change coordinates in $\mathbb{C}^n, 0$ so that f_0 has the form

$$f_0(x, y) \;=\; \sum_{i=1}^{N-1} x_i w_i \;+\; g(x, y)\, w_0$$

where $(x, y) = (x_1, \dots, x_{N-1}, y_1, \dots, y_{n-N+1})$, $\{w_1, \dots w_{N-1}\}$ is a basis for W, and $dg(0) = 0$.

Then, W being generic means that $f_1(x) = \sum_{i=1}^{N-1} x_i w_i$ is a generic linear section. Hence, by Proposition 10.1 $\mu_{\mathcal{V}}(f_1) = \mu_1(\mathcal{V}) = 1$. Then, let $\zeta_1, \dots, \zeta_{N-1}$ be the generators for $\mathrm{Derlog}(H)$ for H a good defining equation for $\mathcal{V}$. We write $\zeta_j = a_0^{(j)} w_0 + w'_j$. Then, the projection of $\mathcal{O}_{\mathbb{C}^{N-1},0}\{w_0, w_1, \dots, w_{N-1}\}$ onto $\mathcal{O}_{\mathbb{C}^{N-1},0}\{w_0\} \simeq \mathcal{O}_{\mathbb{C}^{N-1},0}$ along $\mathcal{O}_{\mathbb{C}^{N-1},0}\{w_1, \dots, w_{N-1}\}$ induces an isomorphism

$$N\mathcal{K}_{H,e} \cdot f_1 \;\simeq\; \mathcal{O}_{\mathbb{C}^{N-1},0}/(a_0^{(1)} \circ f_1, \dots, a_0^{(N-1)} \circ f_1)\,. \tag{11.1}$$

However, by Theorem 3.1 and the above, this equals 1. Hence, $(a_0^{(1)} \circ f_1, \dots, a_0^{(N-1)} \circ f_1)$ provides a system of local coordinates for $\mathbb{C}^{N-1}, 0$.

For the linear free divisors corresponding to the matrix singularities we are considering, we may further apply Mather's Lemma to conclude that the generic corank 1 germs are $\mathcal{K}_M$–equivalent to a germ of the form

$$f_0(x, y) \;=\; \sum_{i=1}^{N-1} x_i w_i \;+\; g(y)\, w_0 \tag{11.2}$$

with $g(y)$ defining an isolated singularity on $\mathbb{C}^{n-N+1}, 0$. We can then compute the singular Milnor number for generic corank 1 germs as follows.

Proposition 11.1. *Let $\mathcal{V} \subset \mathbb{C}^N, 0$ an H–holonomic linear free divisor, and $f_0(x, y)$ be a generic corank 1 mapping for $\mathcal{V}$ given by (11.2). Then,*

$$\mu_{\mathcal{V}}(f_0) \;=\; \mu(g)\,.$$

Proof. We note that $\dfrac{\partial f_0}{\partial x_j} = w_j$, and $\dfrac{\partial f_0}{\partial y_i} = \dfrac{\partial g}{\partial y_i}$. In addition, by the above discussion,

$$(a_0^{(1)} \circ f_0, \dots, a_0^{(N-1)} \circ f_0) \;=\; (a_0^{(1)} \circ f_1, \dots, a_0^{(N-1)} \circ f_1).$$

Hence, as earlier, projecting $\mathcal{O}_{\mathbb{C}^n,0}\{w_0, w_1, \dots, w_{N-1}\}$ onto $\mathcal{O}_{\mathbb{C}^n,0}\{w_0\} \simeq \mathcal{O}_{\mathbb{C}^n,0}$ along $\mathcal{O}_{\mathbb{C}^n,0}\{w_1, \dots, w_{N-1}\}$ induces an isomorphism

$$
\begin{aligned}
N\mathcal{K}_{H,e} \cdot f_0 \;&\simeq\; \mathcal{O}_{\mathbb{C}^n,0}/(a_0^{(1)} \circ f_0, \dots, a_0^{(N-1)} \circ f_0, \frac{\partial g}{\partial y_1}, \dots, \frac{\partial g}{\partial y_{n-N+i}}) \\
&\simeq\; \mathcal{O}_{\mathbb{C}^{n-N+1},0}/(\frac{\partial g}{\partial y_1}, \dots, \frac{\partial g}{\partial y_{n-N+i}}). \qquad (11.3)
\end{aligned}
$$

Then, by Theorem 3.1 and (11.3),

$$\mu_{\mathcal{V}}(f_0) \;=\; \dim_{\mathbb{C}} N\mathcal{K}_{H,e} \cdot f_0 \;=\; \mu(g).$$

□

11.1. A "$\mu = \tau$"- type Formula for Matrix singularities.

We now consider a generic corank 1 germ $f_0 : \mathbb{C}^{n+N-1}, 0 \to M, 0$ where M is any of the spaces of $m \times m$ matrices with ($\dim M = N$). In the case $M = Sym_m$, Bruce [Br] shows that f_0 is $\mathcal{K}_M$–equivalent to germs of one of two types. The first of which is generic in our sense

$$f_0(x_1, \dots, x_{N-1}, y_1, \dots, y_n) \;=\; \begin{pmatrix} g_0(x,y) & x_1 & x_2 & \cdots & x_{m-1} \\ x_1 & x_m & x_{m+1} & \cdots & x_{2m-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{m-1} & x_{2m-3} & \cdots & \cdots & x_{N-1} \end{pmatrix},$$

where $g_0(x,y) = \sum \varepsilon_i\, x_i + g(y_1, \dots, y_n)$ for generic tuples $(\varepsilon_1, \dots, \varepsilon_{N-1})$, and g defines an isolated hypersurface singularity on $\mathbb{C}^n$. In fact, further normalization allows many $\varepsilon_i = 0$ (see [Br]). For general and skew–symmetric cases there are analogous normal forms. For example, for 2×2 general and 4×4 skew–symmetric cases they take the form

$$\begin{pmatrix} g_0(x,y) & x_1 \\ x_2 & x_3 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & g_0(x,y) & x_1 & x_2 \\ -g_0(x,y) & 0 & x_3 & x_4 \\ -x_1 & -x_3 & 0 & x_5 \\ -x_2 & -x_4 & -x_5 & 0 \end{pmatrix}.$$

with $g_0(x,y)$ of the same form as above.

Then, for this class of germs for any of the matrix types we obtain a $\mu = \tau$–type result.

Theorem 11.2 ($\mu = \tau$ for generic corank 1 germs)**.** *We let $(\mathcal{D}, \mathcal{E})$ denote any of the pairs $(\mathcal{D}_2^{sy}, \mathcal{E}_2^{sy})$, $(\mathcal{D}_3^{sy}, \mathcal{E}_3^{sy})$, $(\mathcal{D}_2, \mathcal{E}_2)$, or $(\mathcal{D}_4^{sk}, \mathcal{E}_4^{sk})$ and f_0 any of the corresponding generic corank 1 germs as above. Then,*

$$\mu_{\mathcal{D}}(f_0) \;=\; \mu(g) \;=\; \mathcal{K}_{H,e}\text{-}codim(f_0)$$

where H is the defining equation for the free divisor $\mathcal{E}$.

If moreover g is weighted homogeneous, then

$$\mu_{\mathcal{D}}(f_0) \;=\; \mathcal{K}_{H',e}\text{-}codim(f_0) \;=\; \mathcal{K}_{M,e}\text{-}codim(f_0)$$

where H' is the defining equation for $\mathcal{D}$.

Proof. We first consider 2×2 symmetric matrices. By Theorem 6.1, Theorem 3.1 and generic reduction,

$$\mu_{\mathcal{D}_2^{sy}}(f_0) \;=\; \mu_{\mathcal{E}_2^{sy}}(f_0) \;=\; \mathcal{K}_{H,e}\text{-codim}(f_0)$$

where H is the defining equation for $\mathcal{E}_2^{sy}$. Then a direct calculation analogous to that in the proof of Corollary 6.2 shows $N\mathcal{K}_{H,e}(f_0) \simeq \mathcal{O}_{\mathbb{C}^n,0}/Jac(g)$, yielding the first equality. Lastly, if g is weighted homogeneous, with H' the defining equation for $\mathcal{D}_2^{sy}$, then Derlog(H') has linear generators. Hence, for $\xi \in$ Derlog(H'),

$$\xi \circ f_0 \ \in \ (x_1, x_2, g) \cdot \theta(f_0) \ \subset \ T\mathcal{K}_{H,e}(f_0)\,.$$

Hence, $\mathcal{K}_{H',e}$-codim$(f_0) = \mathcal{K}_{H,e}$-codim$(f_0)$, and by (2.3) these equal $\mathcal{K}_{M,e}$-codim(f_0), completing the proof .

The proof for 2×2 general matrices is virtually identical to that for 2×2 symmetric matrices using instead Theorem 7.1.

Next, for 3×3 symmetric matrices the argument is similar to that for the 2×2 case except for the first step. Instead, we first, apply Theorem 6.5 and generic reduction. Since $df_0(0)(\mathbb{C}^{n+5})$ projects submersively onto all subspaces of dimension ≤ 5, all terms of defining codimension ≥ 1 are zero so we obtain

$$\mu_{\mathcal{D}_3^{sy}}(f_0) \ = \ \mu_{\mathcal{E}_3^{sy}}(f_0) - \mu_{a,\mathcal{Q}a}(f_0)\,.$$

Then, by the meta-version of Proposition 6.3 and generic reduction,

$$\mu_{a,\mathcal{Q}a}(f_0) \ = \ \mu_{a,bd\cdot\mathcal{Q}a}(f_0) - \mu_{a,d,bc(bf-2ce)}(f_0)\,.$$

However, both $V(bd\cdot\mathcal{Q}a)$ and $V(bc(bf-2ce))$ are H–holonomic linear free divisors (by Theorem 4.4 and Proposition 4.3). By a change of coordinates in the source, we may assume that both a and d are coordinates for $\mathbb{C}^n$. Thus, by Proposition 11.1 applied to the restrictions of f_0 to the linear subspaces $V(a)$ and $V(a,d)$,

$$\mu_{a,bd\,\mathcal{Q}a}(f_0) \ = \ \mu_{a,d,bc(bf-2ce)}(f_0) \ = \ \mu(g)\,.$$

Thus, $\mu_{a,\mathcal{Q}a}(f_0) = 0$ and $\mu_{\mathcal{D}_3^{sy}}(f_0) = \mu_{\mathcal{E}_3^{sy}}(f_0)$. The remainder of the proof follows as for the 2×2 symmetric case.

Lastly, the proof for the 4×4 skew–symmetric case follows the proof for the 3×3 symmetric matrices, but with just one difference. By Theorem 9.1 and generic reduction, (9.1) simplifies to

$$\mu_{\mathcal{D}_4^{sk}}(f_0) \ = \ \mu_{\mathcal{E}_4^{sk}}(f_0) \ - \ \mu_{a,f,(be-cd)}(f_0)\,. \tag{11.4}$$

The homogeneous generators ζ_i for Derlog(H), with H the defining equation for $\mathcal{E}_4^{sk}$, consist of four linear vector fields and a quadratic vector field obtained from the Pfaffian vector field. Thus, the $\dfrac{\partial}{\partial x_{1,2}}$–components a_j of the $\zeta_i \circ f_0$ have degrees $1, 1, 1, 1, 2$ in the x_i. Thus, by a calculation similar to the above one for 3×3 symmetric matrices together with Theorem 3.1

$$\mu_{\mathcal{E}_4^{sk}}(f_0) \ = \ \mathcal{K}_{H,e}\text{-codim}(f_0) \ = \ 2\mu(g)\,.$$

However, by Theorem 7.1, generic reduction and Proposition 11.1 applied to the restriction of f_0 to $V(a,f)$,

$$\mu_{a,f,(be-cd)}(f_0) \ = \ \mu_{a,f,bc(be-cd)}(f_0) \ = \ \mu(g)\,.$$

Hence, we obtain from (9.7) and (11.4)

$$\mu_{\mathcal{D}_4^{sk}}(f_0) \ = \ \mu(g)\,.$$

The remainder of the proof is identical to that for 3×3 symmetric matrices. $\square$

Remark 11.3. What is surprising in all of these cases is that the number of singular vanishing cycles for the matrix singularities equals the number of vanishing cycles for the isolated singularity g, although there is at this point no known geometric reason for this agreement.

This leads to the conjecture

Conjecture: For all generic corank 1 matrix singularities for $m \times m$ symmetric, general, or skew–symmetric (for m even) matrices, there is a $\mu = \tau$ result, where $\mu = \mu_{\mathcal{D}}$ and $\tau = \mathcal{K}_{H,e}$-codim, for H the defining equation for the appropriate $\mathcal{E}$. If moreover g is weighted homogeneous, both of these equal $\mathcal{K}_{M,e}$-codim $= \mathcal{K}_{H',e}$-codim $= \mu(g)$, where H' is the defining equation for $\mathcal{D}$.

This result contrasts with the situation for generic corank 1 germs $f_0 : \mathbb{C}^n, 0 \to M_{2,3}, 0$ for the varieties $V_{i,j}$ in the space of 2×3 general matrices. Now by Theorem 7.4 and generic reduction the singular Milnor number is zero. Then, using generic reduction and Theorem 8.1 together with Proposition 11.1, we obtain the following for the variety of singular matrices $\mathcal{V}$ in $M_{2,3}$.

Corollary 11.4. *If $f_0 : \mathbb{C}^n, 0 \to M_{2,3}$ is a generic corank 1 germ as above with $n \geq 6$, then*

$$\tilde{\chi}_{\mathcal{V}}(f_0) \;=\; (-1)^{n-1}\mu_{V_{123}}(f_0) = (-1)^{n-1}\mu(g)\,.$$

If g is weighted homogeneous, these equal the $\mathcal{K}_{M,e}$–codimension of f_0.

Corollary 11.4 substitutes for the $\mu = \tau$ formula in this case. A simple example of this can be seen in the list in [FN] for codimension 2 Cohen-Macaulay singularities in $\mathbb{C}^6$. Example $A_0^\sharp$ in the list, has $g(u) = u^k$, an A_{k-1} singularity and the τ, which is the $\mathcal{K}_{M,e}$–codimension, equals $k-1$. Calculations of the singular vanishing Euler characteristic using the Macaulay2 package [P2] for computing the formula in Theorem 8.1 yields $-(k-1)$ as claimed above.

$\mu = \tau - 1$–type Results for Cohen-Macaulay Surface Singularities.

Having obtained above a number of $\mu = \tau$ results for hypersurfaces, we ask what form results take for Cohen-Macaulay singularities. If $f_0 : \mathbb{C}^4, 0 \to M_{2,3}, 0$ is a germ transverse off 0 to the variety $\mathcal{V}$ of singular matrices, then $\mathcal{V}_0 = f_0^{-1}(\mathcal{V})$ is an isolated Cohen-Macaulay surface singularity. We use the $\mathcal{K}_{M,e}$-codimension of f_0 for τ, and the Milnor number $\mu(\mathcal{V}_0)$ for μ.

Specifically the simple isolated Cohen-Macaulay surface singularities arise in this way and were classified by Frühbis-Krüger and Neumer (Theorem 3.3 of [FN]). These turn out to be precisely the rational triple points (c.f. [Tj]). They include both a number of infinite families and discrete cases. As well in [FN] are identified the singularities just outside the simple range.

Until recently the only method to compute the Milnor number involved using a partial resolution of $\mathcal{V}_0$. There are now two new ways to compute the Milnor number. In the recent thesis of Pereira ([Pe]), she applies a Lê-Greuel type method to a generic linear function on the surface. This method requires that the number of critical points of the linear function on the Milnor fiber be computed directly by hand. Also, Theorem 8.3 provides an effective formula for computing $\mu(\mathcal{V}_0)$, and this has been implemented by the second author as a package [P2] in Macaulay2. Taken together, these computations include all of the simple isolated Cohen-Macaulay surface singularities, as well as certain nonsimple cases.

Summary of the Results for Isolated Cohen-Macaulay Surface Singularities :

1) Pereira computes the Milnor number for many discrete cases and the entirety of many of the infinite families of simple singularities. Based on her results she has conjectured (6.3.1 of [Pe]) and verified for her cases that for $\mathcal{V}_0$ quasihomogeneous,

$$\mu(\mathcal{V}_0) = \tau(\mathcal{V}_0) - 1. \tag{11.5}$$

2) Using the package [P2], we have verified (11.5) for all of the discrete examples, for the first few examples of each infinite family, and for a number of cases just outside the simple region (e.g., Table 4 in the Appendix §12).

With further work, Theorem 8.3 should provide a method to prove (11.5) for large classes of singularities. One immediate consequence is that while for ICIS $\gamma = \tau - \mu \leq 0$, now for non-ICIS $\gamma = \tau - \mu$ becomes positive. The relation (11.5) would be a striking complement to a similar pattern found in listings of certain space curve singularities (see Tables 1, 2a, 2b of [Fr]).

$\mu = \tau - \gamma$ for Cohen-Macaulay 3–fold Singularities in $\mathbb{C}^5$.

We next consider isolated Cohen-Macauly 3-fold singularities $\mathcal{V}_0, 0 \subset \mathbb{C}^5, 0$ defined by $f_0 : \mathbb{C}^5, 0 \to M_{2,3}, 0$, with $\mathcal{V}_0 = f_0^{-1}(\mathcal{V})$. Again by the results of Greuel–Steenbrink [GS], the first (vanishing) Betti number of the Milnor fiber of $\mathcal{V}_0$, $b_1(\mathcal{V}_0) = 0$. As there are two possibly non-vanishing Betti numbers for the Milnor fiber, we replace the Milnor number by $b_3(\mathcal{V}_0) - b_2(\mathcal{V}_0)$. We can use Theorem 8.4 to compute $b_3(\mathcal{V}_0) - b_2(\mathcal{V}_0)$ and investigate whether an analogue of (11.5) holds.

We apply Theorem 8.4 to the classification of simple isolated Cohen-Macaulay 3-fold singularities in $\mathbb{C}^5$ (Theorem 3.5 of [FN]). We compute (8.3) using the package [P2], and summarize the results in Table 5 in the Appendix §12 .

We summarize the main observed conclusions from the calculations. These conclusions concern the values and behavior of $\gamma = \tau - (b_3 - b_2)$ (where $\tau = \mathcal{K}_{M,e}$-codim), and the behavior of γ and $b_3 - b_2$ in simple infinite families. We emphasize that although we state the expected form of these for infinite families, we have so far only verified them for a small range of values in each infinite family.

Summary of the Results for Isolated Cohen-Macaulay 3-fold Singularities :

a) $\gamma \geq 2$ and increases in value as we move higher in the classification.
b) $b_3 - b_2 \geq -1$, with equality for the generic linear section and one infinite family.
c) $b_3 - b_2$ is constant for certain infinite families with values -1 (one family), 0 (two families), and 1 (two families).
d) γ is constant in all other considered infinite families in Table 5 with only one exception where both $b_3 - b_2$ and γ increase with τ.
e) For singularities of the form $\begin{pmatrix} x & y & z \\ w & v & g(x,y) \end{pmatrix}$ with g a simple hypersurface singularity (cases 2-6 in Table 5), $\gamma = 3$ and $b_3 - b_2 = \mu(g) - 1$.

As each $b_i \geq 0$, knowing $b_3 - b_2$ gives lower bounds on b_3 when $b_3 - b_2 > 0$ and on b_2 when $b_3 - b_2 < 0$. In particular, the generic Cohen-Macaulay 3-fold singularity as well as one infinite family must have $b_2 > 0$. In fact, we expect that both b_2 and b_3 will increase with τ in families with $b_3 - b_2$ constant.

Remark 11.5. These results reveal that there are (at least) two quite different (and mutually exclusive) types of behavior occurring for infinite families of isolated

Cohen-Macaulay 3 fold singularities. One where $b_3 - b_2$ is constant in the family and one where γ is constant. A basic question is what different geometric properties are responsible for the two different types of behavior? Second, as γ increases within the classification, how can it be computed independently via other geometric properties of the singularities?

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12. Appendix: Computations for Cohen–Macaulay Singularities

Table 4: Some non-simple isolated Cohen-Macaulay surface singularities in $\mathbb{C}^4$

Presentation matrix	τ	μ
$\begin{pmatrix} z & y & x \\ x & w & z^2+y^4 \end{pmatrix}$	11	10
$\begin{pmatrix} z & y & x \\ x & w & y^3+z^3 \end{pmatrix}$	10	9
$\begin{pmatrix} z & y & x^2+y^2 \\ x & w & w^2+xw+z^2 \end{pmatrix}$	13	12
$\begin{pmatrix} x & y & z \\ w & zx+x^2 & w+yz \end{pmatrix}$	9	8
$\begin{pmatrix} z & y & x^2 \\ w^2 & x & y+w^2 \end{pmatrix}$	8	7
$\begin{pmatrix} z & y & x^2+z^2 \\ w^2 & x & y+w^2 \end{pmatrix}$	8	7

Table 5: The simple isolated Cohen-Macaulay 3-fold singularities in $\mathbb{C}^5$

Presentation matrix	Parameters tried	τ	$b_3 - b_2$
$\begin{pmatrix} x & y & z \\ w & v & x \end{pmatrix}$		1	-1
$\begin{pmatrix} x & y & z \\ w & v & x^{k+1}+y^2 \end{pmatrix}$	$1 \le k \le 4$	$k+2$	$k-1$
$\begin{pmatrix} x & y & z \\ w & v & xy^2+x^{k-1} \end{pmatrix}$	$4 \le k \le 6$	$k+2$	$k-1$
$\begin{pmatrix} x & y & z \\ w & v & x^3+y^4 \end{pmatrix}$		8	5
$\begin{pmatrix} x & y & z \\ w & v & x^3+xy^3 \end{pmatrix}$		9	6
$\begin{pmatrix} x & y & z \\ w & v & x^3+y^5 \end{pmatrix}$		10	7
$\begin{pmatrix} w & y & x \\ z & w & y+v^k \end{pmatrix}$	$2 \le k \le 5$	$2k-1$	-1
$\begin{pmatrix} w & y & x \\ z & w & y^k+v^2 \end{pmatrix}$	$2 \le k \le 5$	$k+2$	$k-2$
$\begin{pmatrix} w & y & x \\ z & w & yv+v^k \end{pmatrix}$	$2 \le k \le 5$	$2k$	0
$\begin{pmatrix} w+v^k & y & x \\ z & w & yv \end{pmatrix}$	$2 \le k \le 5$	$2k+1$	0
$\begin{pmatrix} w+v^2 & y & x \\ z & w & y^2+v^k \end{pmatrix}$	$2 \le k \le 5$	$2k$	$k-2$
$\begin{pmatrix} w & y & x \\ z & w & y^2+v^3 \end{pmatrix}$		7	1
$\begin{pmatrix} v^2+w^k & y & x \\ z & w & v^2+y^l \end{pmatrix}$	$(k,l) \in \{(2,2),(2,3),(2,4),(3,3)\}$	$k+l+1$	$k+l-3$
$\begin{pmatrix} v^2+w^k & y & x \\ z & w & yv \end{pmatrix}$	$2 \le k \le 4$	$k+4$	$k-1$
$\begin{pmatrix} v^2+w^k & y & x \\ z & w & y^2+v^l \end{pmatrix}$	$(k,l) \in \{(2,3),(2,4),(3,3)\}$	$k+l+2$	$k+l-3$
$\begin{pmatrix} wv+v^k & y & x \\ z & w & yv+v^k \end{pmatrix}$	$3 \le k \le 6$	$2k+1$	1
$\begin{pmatrix} wv+v^k & y & x \\ z & w & yv \end{pmatrix}$	$3 \le k \le 6$	$2k+2$	1
$\begin{pmatrix} wv+v^3 & y & x \\ z & w & y^2+v^3 \end{pmatrix}$		8	2
$\begin{pmatrix} wv & y & x \\ z & w & y^2+v^3 \end{pmatrix}$		9	2
$\begin{pmatrix} w^2+v^3 & y & x \\ z & w & y^2+v^3 \end{pmatrix}$		9	3
$\begin{pmatrix} z & y & x \\ x & w & v^2+y^2+z^k \end{pmatrix}$	$2 \le k \le 4$	$k+4$	k
$\begin{pmatrix} z & y & x \\ x & w & v^2+yz+y^kw \end{pmatrix}$		$2k+5$	

Table 5: The simple isolated Cohen-Macaulay 3-fold singularities in $\mathbb{C}^5$

Presentation matrix	Parameters tried	τ	$b_3 - b_2$
$\begin{pmatrix} z & y & x \\ x & w & v^2 + yz + y^{k+1} \end{pmatrix}$		$2k+4$	
$\begin{pmatrix} z & y & x \\ x & w & v^2 + yw + z^2 \end{pmatrix}$		8	4
$\begin{pmatrix} z & y & x \\ x & w & v^2 + y^3 + z^2 \end{pmatrix}$		9	5
$\begin{pmatrix} z & y & x + v^2 \\ x & w & vy + z^2 \end{pmatrix}$		7	2
$\begin{pmatrix} z & y & x + v^2 \\ x & w & vz + y^2 \end{pmatrix}$		8	3
$\begin{pmatrix} z & y & x + v^2 \\ x & w & y^2 + z^2 \end{pmatrix}$		9	4

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