ON FITTING IDEALS OF LOGARITHMIC VECTOR FIELDS AND SAITO'S CRITERION

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Abstract. The germ of an analytic set (X, p) in $\mathbb{C}^n$ has an associated $\mathscr{O}_{\mathbb{C}^n,p}$–module $\mathrm{Der}(-\log X)$ of *logarithmic vector fields*, the ambient germs of holomorphic vector fields tangent to the smooth locus of X. For a module $L \subseteq \mathrm{Der}(-\log X)$ let $I_k(L)$ be the ideal generated by the $k \times k$ minors of a matrix of generators for L; these are the Fitting ideals of $\mathrm{Der}_{\mathbb{C}^n,p}/L$. We aim to: (i) find sufficient conditions on $\{I_k(L)\}$ to prove $L = \mathrm{Der}(-\log X)$; (ii) identify $\{I_k(\mathrm{Der}(-\log X))\}$, to provide a necessary condition for equality; and (iii) provide a geometric interpretation of these ideals.

Even for (X, p) smooth, an example shows that Fitting ideals alone are insufficient to prove equality, although we give a different criterion. Using (ii) and (iii) in the smooth case, we give partial answers to (ii) and (iii) for arbitrary (X, p). When (X, p) is a hypersurface, we give sufficient algebraic or geometric conditions for the reflexive hull of L to equal $\mathrm{Der}(-\log X)$; for L reflexive, this answers (i) and generalizes criteria of Saito for free divisors and Brion for linear free divisors.

Contents

Introduction

In [Sai80], Kyoji Saito introduced the notion of a *free divisor*, a complex hypersurface germ $(X, p) \subset (\mathbb{C}^n, p)$ for which the associated $\mathscr{O}_{\mathbb{C}^n,p}$–module $\mathrm{Der}(-\log X)_p$ of *logarithmic vector fields* is a free module, necessarily of rank n; geometrically, these are the vector fields tangent to (X, p). Although many classes of free divisors have been found, free divisors remain somewhat mysterious; for instance, it is not completely understood which hyperplane arrangements are free divisors.

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To determine when a set of n elements of $\operatorname{Der}(-\log X)_p$ forms a generating set, Saito proved a criterion (see Corollary 5.7) in terms of the determinant of a presentation matrix of these elements being reduced in $\mathscr{O}_{\mathbb{C}^n,p}$. Saito's criterion can only be satisfied when (X,p) is a free divisor. Several questions arise:

(1) For arbitrary (X,p), is there a necessary and sufficient condition on a submodule $L \subseteq \operatorname{Der}(-\log X)_p$ to have equality?
(2) What does Saito's criterion mean geometrically?

If X is not empty, $\mathbb{C}^n$, or a free divisor, then $\operatorname{Der}(-\log X)_p$ requires $> n$ generators. For (1), it is natural to mimic Saito's criterion by considering some condition on the ideals $I_k(L)$, $1 \leq k \leq n$, where $I_k(L)$ is generated by the $k \times k$ minors of a matrix of a generating set of L; these are the Fitting ideals of $\operatorname{Der}_{\mathbb{C}^n,p}/L$, but we shall abuse terminology and call them the *Fitting ideals of* L. An understanding of the geometric content of these ideals may also answer (2).

To this end, in this article we study these ideals of logarithmic vector fields. The Lie algebra structure of logarithmic vector fields has been studied, in particular in [HM93], but these ideals have not. The only results we are aware of ([HM93, Proposition 2.2]) address the radicals of $I_k(\operatorname{Der}(-\log X)_p)$ for $k \geq \dim(X)$.

In §1, we define these Fitting ideals, and demonstrate in Example 1.1 that for non-hypersurfaces, there is no general condition on the Fitting ideals of L sufficient to prove $L = \operatorname{Der}(-\log X)_p$, even when (X,p) is smooth. Nevertheless, an understanding of $\{I_k(\operatorname{Der}(-\log X)_p)\}$ can provide necessary conditions for L to equal $\operatorname{Der}(-\log X)_p$. The radicals of these ideals also contain information on various stratifications of (X,p).

In §2 we study smooth germs. Theorem 2.3 answers (1) completely for smooth (X,p), although our criterion is not in terms of Fitting ideals. In Propositions 2.7 and 2.8, we describe the corresponding Fitting ideals and their geometric content.

In §3 we study arbitrary analytic germs. Using our understanding of the smooth points of X and a generalization of the Nagata–Zariski Theorem due to [EH79] (see Theorem 3.2), we give upper bounds for $I_k(\operatorname{Der}(-\log X)_p)$ in Theorem 3.6 and Remark 3.7. (For instance, it follows that if J is a prime ideal defining an irreducible component of (X,p), then a certain Fitting ideal is contained in J, but not J^2; see Remark 3.9.) We describe the geometric meaning of certain Fitting ideals in Proposition 3.15.

In §4, we follow Saito and recall a useful duality between modules of vector fields and modules of meromorphic 1-forms. With this, under mild conditions the double dual of a module L of vector fields can be identified with a larger module of vector fields that we call the *reflexive hull* of L. For a hypersurface (X,p), $\operatorname{Der}(-\log X)_p$ is reflexive and hence equals its reflexive hull.

In §5, we apply our earlier work to a hypersurface (X,p). Proposition 5.1 addresses the geometric content of $I_n(\operatorname{Der}(-\log X)_p)$, and hence answers (2). Theorem 5.4 gives sufficient (algebraic or geometric) conditions for the reflexive hull of a module L of logarithmic vector fields to equal $\operatorname{Der}(-\log X)_p$. When L is already reflexive, as is the case for L free, this gives a condition for $L = \operatorname{Der}(-\log X)_p$. In particular, this recovers Saito's criterion for free divisors (Corollary 5.7), and a criterion of Michel Brion (Corollary 5.10) for linear free divisors associated to representations of linear algebraic groups. It was an attempt to understand and generalize Brion's result that originally motivated this work.

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1. Fitting ideals

We begin with some notation that we maintain for the whole paper.

1.1. Notation. For an open subset $U \subseteq \mathbb{C}^n$, let $\mathscr{O}_U$ denote the sheaf of holomorphic functions on U and let Der_U denote the sheaf of holomorphic vector fields on U. For a sheaf $\mathscr{S}$ (respectively, $\mathscr{S}_U$), let $\mathscr{S}_p$ (resp., $\mathscr{S}_{U,p}$) denote the stalk at $p \in U$. For an ideal I, let $V(I)$ denote the common zero set of $f \in I$. For a set or germ S and point $p \in U$, let $I(S)$ be the ideal of functions vanishing on S, and let $\mathscr{M}_p$ be the maximal ideal of functions vanishing at p; the ambient ring is given by context. For a germ g, a specific choice of representative will always be denoted by g' or g''.

All of our analytic sets and analytic germs will be reduced. Except for §5.3, by abuse of terminology a *Zariski closed* (respectively, *Zariski open*) subset shall mean an analytic subset (resp., the complement of an analytic subset).

For an analytic set $X \subseteq U$ the coherent sheaf $\mathrm{Der}(-\log X)$ of *logarithmic vector fields of X* is defined by

$$\mathrm{Der}(-\log X)(V) = \{\eta \in \mathrm{Der}_U(V) : \eta(I(X)) \subseteq I(X)\}, \tag{1.1}$$

where V is an arbitrary open subset of U and $I(X) \subseteq \mathscr{O}_U(V)$ is the ideal of functions on V vanishing on X. Each $\mathrm{Der}(-\log X)(V)$ is an $\mathscr{O}_U(V)$–module closed under the Lie bracket of vector fields, consisting of those vector fields on V tangent to (the smooth points of) X. For an analytic germ (X,p), $\mathrm{Der}(-\log X)_p$ is defined as $\mathrm{Der}(-\log X')_p$ for any representative X' of (X,p); it is an $\mathscr{O}_{\mathbb{C}^n,p}$–module closed under the Lie bracket. See [Sai80, HM93] for an introduction to logarithmic vector fields.

If $f : M \to N$ is a holomorphic map between complex manifolds, let $df_{(p)} : T_pM \to T_{f(p)}N$ denote the derivative at $p \in M$. For a complex vector space L of vector fields defined at p, define a subspace of $T_p\mathbb{C}^n$ by

$$\langle L\rangle_p = \{\eta(p) : \eta \in L\}. \tag{1.2}$$

1.2. Fitting ideals. Let U be an open subset of $\mathbb{C}^n$ and let $L \subseteq \mathrm{Der}_U(U)$ be a $\mathscr{O}_U(U)$–module of holomorphic vector fields on U. Let $(z_1,\ldots,z_n)$ be holomorphic coordinates on U. Choose a set of generators $\eta_1,\ldots,\eta_m$ of L. Writing $\eta_j = \sum_{i=1}^n a_{ij}\frac{\partial}{\partial z_i}$, form a $n \times m$ matrix $A = (a_{ij})$ with entries in $\mathscr{O}_U(U)$. We call A a *Saito matrix* for L. For any $1 \leq k \leq n$, consider the $\mathscr{O}_U(U)$ ideal $I_k(L)$ generated by $k \times k$ minors of a Saito matrix of L.

To see that $I_k(L)$ is well-defined, we give an equivalent definition. Identifying $\mathrm{Der}_U(U)$ with $\mathscr{O}_U(U)\{\frac{\partial}{\partial z_i}\}_{i=1..n}$ gives an exact sequence

$$\mathscr{O}_U^m \xrightarrow{A} \mathrm{Der}_U(U) \longrightarrow \mathrm{Der}_U(U)/L \longrightarrow 0.$$

Thus, $I_k(L)$ is also the $(n-k)$th Fitting ideal of $\mathrm{Der}_U(U)/L$, and hence well-defined. By abuse of terminology, we shall refer to $\{I_k(L)\}$ as the *Fitting ideals of L*.

For a submodule $L \subseteq \mathrm{Der}_{\mathbb{C}^n,p}$, the ideals $I_k(L) \subseteq \mathscr{O}_{\mathbb{C}^n,p}$ may be defined similarly. Of particular interest are the ideals $I_k(\mathrm{Der}(-\log X)_p)$, $k = 1,\ldots,n$, when

(X,p) is an analytic germ in $\mathbb{C}^n$. For instance, the radicals of these ideals encode the *logarithmic stratification* of (X,p), a (not necessarily finite) decomposition of (X,p) into smooth strata along which the germ is biholomorphically trivial (see [Sai80, §3]). Note that when this stratification is finite, it is equal to the canonical Whitney stratification of (X,p) ([DP14, Prop. 4.5]). Since this stratification is a rather subtle property of (X,p), we do not expect to be able to describe $I_k(\mathrm{Der}(-\log X)_p)$ explicitly. However, if $L \subseteq \mathrm{Der}(-\log X)_p$, then any property known about $I_k(\mathrm{Der}(-\log X)_p)$ gives a necessary condition to have $L = \mathrm{Der}(-\log X)_p$.

1.3. **Inadequacy of Fitting ideals.** An easy example shows that even when (X,p) is smooth, for $L \subseteq \mathrm{Der}(-\log X)_p$ the ideals $\{I_k(L)\}_k$ cannot detect whether $L = \mathrm{Der}(-\log X)_p$.

Example 1.1. Let $(X,0)$ be the origin in $\mathbb{C}^2$, defined by coordinates $x = y = 0$. The following logarithmic vector fields generate $\mathrm{Der}(-\log X)_p$:

$$\eta_1 = x\frac{\partial}{\partial x}, \quad \eta_2 = y\frac{\partial}{\partial x}, \quad \eta_3 = x\frac{\partial}{\partial y}, \quad \eta_4 = y\frac{\partial}{\partial y}.$$

Let $L = \mathscr{O}_{\mathbb{C}^2,0}\{\eta_2, \eta_3, \eta_1 - \eta_4\}$. Then $L \subsetneq \mathrm{Der}(-\log X)_p$, even though $I_k(L) = I_k(\mathrm{Der}(-\log X)_p)$ for $k = 1, 2$ and L is closed under the Lie bracket.

Similar examples hold for the origin in higher dimensions, as the action of $\mathrm{SL}(n,\mathbb{C})$ and $\mathrm{GL}(n,\mathbb{C})$ on $\mathbb{C}^n$ have the same orbit structure, and a simple argument shows that the two modules of vector fields generated by these actions have the same Fitting ideals. This example may then be extended to apply to all smooth germs of codimension > 1.

2. Logarithmic vector fields of smooth germs

We first study a smooth analytic germ (X,p) in $\mathbb{C}^n$. Because of the coherence of the sheaf of logarithmic vector fields and the genericity of smooth points, we will later use this to study non-smooth germs.

The (only) example is:

Example 2.1. Let (X,p) in $\mathbb{C}^n$ be a smooth germ of dimension $d < n$. Choose local coordinates $y_1, \ldots, y_n$ for $\mathbb{C}^n$ near p so that the ideal of germs vanishing on X is $I(X) = (y_1, \ldots, y_{n-d})$ in $\mathscr{O}_{\mathbb{C}^n,p}$. Then the vector fields

$$\left\{\frac{\partial}{\partial y_k}\right\}_{k=n-d+1,\ldots,n} \quad \text{and} \quad \left\{y_j \frac{\partial}{\partial y_i}\right\}_{i,j\in\{1,\ldots,n-d\}} \tag{2.1}$$

are certainly in $\mathrm{Der}(-\log X)_p$.

We begin by describing a criterion, Theorem 2.3, for having a complete set of generators for $\mathrm{Der}(-\log X)_p$. It will follow that (2.1) is a complete set of generators for Example 2.1.

2.1. **A derivative for vector fields.** First, we make a definition used in our criterion. For $\eta \in \mathrm{Der}_{\mathbb{C}^n,p}$ with $\eta(p) = 0$, we shall define a directional derivative

$d(\widehat{\eta})_{(p)} : T_p\mathbb{C}^n \to T_p\mathbb{C}^n$ of η at p by choosing local coordinates $x_1, \ldots, x_n$ near p, writing $\eta = \sum_{i=1}^n g_i \frac{\partial}{\partial x_i}$, and then defining

$$d(\widehat{\eta})_{(p)}(v) = \sum_{i=1}^{n} d(g_i)_{(p)}(v) \frac{\partial}{\partial x_i}(p), \tag{2.2}$$

where we use the canonical coordinate on $\mathbb{C}$ to identify $T_0\mathbb{C}$ with $\mathbb{C}$ so that $d(g_i)_{(p)}(v) \in \mathbb{C}$ (or, interpret $d(g_i)_{(p)}(v)$ as a directional derivative). This operation does not depend on a choice of coordinates, as shown by applying the following lemma to $\{x_i - x_i(p)\}_i$:

Lemma 2.2. *If $\eta(p) = 0$ and $f(p) = 0$, then $d(\eta(f))_{(p)} = df_{(p)} \circ d(\widehat{\eta})_{(p)}$.*

Proof. Choose coordinates $(x_1, \ldots, x_n)$ near p. Since both sides are linear in η, it suffices to prove the claim when $\eta = g_i \frac{\partial}{\partial x_i}$, with $g_i(p) = 0$. In this case,

$$\begin{aligned} d(\eta(f))_{(p)}(v) &= d(g_i)_{(p)}(v) \cdot \frac{\partial f}{\partial x_i}(p) + g_i(p) \cdot d\left(\frac{\partial f}{\partial x_i}\right)_{(p)}(v) \\ &= df_{(p)}\left(d(g_i)_{(p)}(v)\frac{\partial}{\partial x_i}(p)\right) + 0 \\ &= df_{(p)}\left(d(\widehat{\eta})_{(p)}(v)\right). \end{aligned}$$

□

2.2. **Criterion for a generating set.** We are now able to state our criterion.

Theorem 2.3. *Let (X, p) be an analytic germ in $\mathbb{C}^n$ which is of pure dimension $d < n$. Let $\eta_1, \ldots, \eta_m \in \operatorname{Der}(-\log X)_p$, with $L_{\mathbb{C}} = \mathbb{C}\{\eta_i\}_{i=1,\ldots,m}$ and $L = \mathscr{O}_{\mathbb{C}^n,p}\{\eta_i\}_{i=1,\ldots,m}$. Let $L_{\mathbb{C},0}$ (respectively, L_0) denote the set of $\xi \in L_{\mathbb{C}}$ (respectively, $\xi \in L$) with $\xi(p) = 0$. In $\mathscr{O}_{\mathbb{C}^n,p}$, let $I = I(X)$ be the ideal of functions vanishing on X, and let $\mathscr{M}_p$ be the maximal ideal. The following are equivalent:*

1. *(X, p) is smooth and $L = \operatorname{Der}(-\log X)_p$;*
2. *$\dim(\langle L\rangle_p) = d$ and the map*
$$\alpha : L_{\mathbb{C},0} \to \operatorname{End}_{\mathbb{C}}(I/\mathscr{M}_p \cdot I)$$
defined by $\alpha(\xi) = (f \mapsto \xi(f))$ is surjective;
3. *$\dim(\langle L\rangle_p) = d$ and the map*
$$\beta : L_{\mathbb{C},0} \to \operatorname{End}_{\mathbb{C}}(T_p\mathbb{C}^n/\langle L\rangle_p)$$
defined by $\beta(\xi) = d(\widehat{\xi})_{(p)}$ is surjective.

In (2) or (3), we could equivalently have used L_0 instead of $L_{\mathbb{C},0}$.

The smoothness of (X, p) will follow from this lemma.

Lemma 2.4. *If (X, p) is an analytic germ in $\mathbb{C}^n$ of dimension d, with $p \in X$, then $\dim(\left\langle \operatorname{Der}(-\log X)_p\right\rangle_p) \leq d$, with equality if and only if (X, p) is smooth.*

Proof. Let $k = \dim(\left\langle \operatorname{Der}(-\log X)_p\right\rangle_p)$. By a theorem of Rossi [HM93, Rossi's Theorem, (c)], (X, p) has dimension $\geq k$; hence, $k \leq d$. If $k = d$, then by (b) of Rossi's Theorem, there is a reduced analytic germ Y such that $(X, p) \cong (\mathbb{C}^d, 0) \times Y$; but then Y is of dimension 0, and hence smooth. Conversely, if (X, p) is smooth, then [HM93, Existence Lemma] shows that $k \geq d$, and thus $k = d$. □

Proof of Theorem 2.3. We first show that $\dim(\langle L\rangle_p) = d$ implies that (X,p) is smooth. Since $L \subseteq \operatorname{Der}(-\log X)_p$, $\langle L\rangle_p \subseteq \left\langle \operatorname{Der}(-\log X)_p \right\rangle_p$. Since by Lemma 2.4 $\left\langle \operatorname{Der}(-\log X)_p \right\rangle_p$ has dimension $\leq d$, we have $\langle L\rangle_p = \left\langle \operatorname{Der}(-\log X)_p \right\rangle_p$ and both spaces have dimension d. By Lemma 2.4 again, (X,p) is smooth.

Thus we may assume in all cases that (X,p) is smooth, and $\langle L\rangle_p = \left\langle \operatorname{Der}(-\log X)_p \right\rangle_p = T_pX$ have dimension d. Choose local coordinates $(y_1,\ldots,y_n)$ vanishing at p so that $I=(y_1,\ldots,y_{n-d})$, just as in Example 2.1.

To show that α is well-defined, let D be the submodule of $\operatorname{Der}(-\log X)_p$ consisting of vector fields vanishing at p, and let $\alpha' : D \to \operatorname{End}_{\mathbb{C}}(I/\mathscr{M}_p\cdot I)$ be defined by $\alpha'(\xi)(f)=\xi(f)$. Let $\xi\in D$. Since $\xi \in \operatorname{Der}(-\log X)_p$, by definition the application of ξ to functions gives a linear map $I \to I$. If $f\in I$ and $g \in \mathscr{M}_p$, then $\xi(g)\in\mathscr{M}_p$ (as $\xi(p)=0$) and $\xi(f)\in I$. By the product rule, $\xi(f\cdot g)\in \mathscr{M}_p\cdot I$ and hence $\xi(\mathscr{M}_p\cdot I)\subseteq \mathscr{M}_p\cdot I$. Thus the map $f\mapsto \xi(f)$ descends to an endomorphism of $I/\mathscr{M}_p\cdot I$, as claimed, and so α' is well-defined. Since $L_{\mathbb{C},0}\subseteq L_0 \subseteq D$, restricting α' to one of these subspaces gives a well-defined map, including $\alpha = \alpha'|_{L_{\mathbb{C},0}}$.

We will show that (1) implies (2). The vector space $I/\mathscr{M}_p I$ is generated by $\{y_1,\ldots,y_{n-d}\}$, and (as in Example 2.1) $y_j\frac{\partial}{\partial y_i}\in L_0$ has

$$\left(y_j\frac{\partial}{\partial y_i}\right)(y_k)=\delta_{ik}y_j.$$

Since it follows that $\left\{\alpha'\left(y_j\frac{\partial}{\partial y_i}\right)\right\}_{1\leq i,j\leq n-d}$ is a $\mathbb{C}$-basis of $\operatorname{End}_{\mathbb{C}}(I/\mathscr{M}_p\cdot I)$, $\alpha'|_{L_0}$ is surjective.

However, we must show that $\alpha=\alpha'|_{L_{\mathbb{C},0}}$ is surjective. If $\xi\in L$ and $g\in\mathscr{M}_p$, then $g\cdot\xi\in L_0$ and $g\cdot\xi(I)\subseteq \mathscr{M}_p\cdot I$; hence, $\mathscr{M}_p\cdot L\subseteq\ker(\alpha')$. Since $\dim(\langle L\rangle_p)=d$, we may choose a basis for $L_{\mathbb{C}}$ consisting of $\eta_1,\ldots,\eta_d$ (with $\{\eta_i(p)\}$ linearly independent) and elements of $L_{\mathbb{C},0}$. Let $\zeta\in L_0$, and write $\zeta=\sum_i g_i\eta_i+\sum_j h_j\xi_j$, where $\xi_j\in L_{\mathbb{C},0}$. Evaluating at p and using the linear independence of $\{\eta_i(p)\}$, we see that all $g_i\in\mathscr{M}_p$. Let $\zeta'=\sum_j h_j(p)\xi_j\in L_{\mathbb{C},0}$, and observe that

$$\zeta-\zeta'=\sum_i g_i\eta_i+\sum_j (h_j-h_j(p))\xi_j\in\mathscr{M}_p\cdot L,$$

and hence $\alpha'(\zeta)=\alpha'(\zeta')=\alpha(\zeta')$. Thus α is surjective as well, giving us (2).

To show that (2) implies (1), we will first show

$$\ker(\alpha')\subseteq\mathscr{M}_p\cdot N, \tag{2.3}$$

where N is the submodule of $\operatorname{Der}(-\log X)_p$ generated by the vector fields in (2.1). If $\xi\in\ker(\alpha')$, then $\xi\in D$, and hence $\xi(p)=0$ and $\xi=\sum_i a_i\frac{\partial}{\partial y_i}$ with each $a_i\in\mathscr{M}_p$. Since $\xi\in\ker(\alpha')$, $\xi(y_i)\in\mathscr{M}_p\cdot I$ for $i=1,\ldots,n-d$. Hence, we may write

$$\xi=\sum_{i=1}^{n-d}\left(\sum_{j=1}^{n-d}b_{ij}y_j\right)\frac{\partial}{\partial y_i}+\sum_{n-d<i\leq n}a_i\frac{\partial}{\partial y_i}, \tag{2.4}$$

where $b_{i,j},a_i\in\mathscr{M}_p$. Examining (2.4), it is clear that $\xi\in\mathscr{M}_p\cdot N$, proving (2.3).

Now let $\zeta\in\operatorname{Der}(-\log X)_p$. Since $\{\eta_i(p)\}_{i=1}^d$ is a basis for $\langle L\rangle_p=\langle\operatorname{Der}(-\log X)\rangle_p$, write $\zeta(p)=\sum_{i=1}^d\lambda_i\eta_i(p)$ for $\lambda_i\in\mathbb{C}$. Let $\zeta'=\zeta-\sum_{i=1}^d\lambda_i\eta_i$, so that $\zeta'(p)=0$.

By the surjectivity of α, there exists a $\xi \in L_{\mathbb{C},0}$ such that $\alpha'(\zeta') = \alpha(\xi)$. Thus,

$$\zeta - \left(\sum_{i=1}^{d} \lambda_i \eta_i + \xi\right) \in \ker(\alpha'),$$

and so by (2.3), $\zeta \in L + \mathscr{M}_p \cdot N$. Consequently,

$$\operatorname{Der}(-\log X)_p \subseteq L + \mathscr{M}_p \cdot N \subseteq L + \mathscr{M}_p \cdot \operatorname{Der}(-\log X)_p \subseteq \operatorname{Der}(-\log X)_p,$$

so that by Nakayama's Lemma, $L = \operatorname{Der}(-\log X)_p$. This proves (2).

To show that β in (3) is well-defined, observe that by the smoothness of (X, p), $\langle L\rangle_p = T_pX = \cap_{f\in I} \ker(df_{(p)})$. Let $\xi \in \operatorname{Der}(-\log X)_p$ vanish at p. Since for any $f \in I$ we have $\xi(f) \in I$, by Lemma 2.2, $d(\widehat{\xi})_{(p)}(T_pX) \subseteq \ker(df_{(p)})$. Since this is true of all $f \in I$, we have $d(\widehat{\xi})_{(p)}(T_pX) \subseteq T_pX$, and so $d(\widehat{\xi})_{(p)} : T_p\mathbb{C}^n \to T_p\mathbb{C}^n$ factors through the quotient $T_p\mathbb{C}^n \to T_p\mathbb{C}^n/T_pX$ to give a well-defined element $\beta(\xi)$.

We now show the equivalence of (2) and (3). Define the $\mathbb{C}$-linear map $\overline{\rho} : I \to \operatorname{Hom}_{\mathbb{C}}(T_p\mathbb{C}^n/T_pX, T_0\mathbb{C})$ by $\overline{\rho}(f) = df_{(p)}$; this is well-defined since for $f \in I$, $T_pX \subseteq \ker(df_{(p)})$. Since $y_1, \ldots, y_{n-d} \in I$, we know $\overline{\rho}$ is surjective. If $g \in \mathscr{M}_p$ and $f \in I$, then $d(f \cdot g)_{(p)} = 0$, and hence $\mathscr{M}_p \cdot I \subseteq \ker(\overline{\rho})$. Conversely, if $f = \sum_{i=1}^{n-d} a_i y_i \in I$ is in $\ker(\overline{\rho})$, then $0 = df_{(p)} = \sum_{i=1}^{n-d} a_i(p) d(y_i)_{(p)}$; by the linear independence of $d(y_i)_{(p)}$, all $a_i \in \mathscr{M}_p$. Hence $\mathscr{M}_p \cdot I = \ker(\overline{\rho})$, and so $\overline{\rho}$ factors through to a vector space isomorphism $\rho : I/\mathscr{M}_p \cdot I \to \operatorname{Hom}_{\mathbb{C}}(T_p\mathbb{C}^n/T_pX, T_0\mathbb{C})$.

For any $\xi \in \operatorname{Der}(-\log X)_p$ vanishing at p, Lemma 2.2 shows that we have the following commutative diagram.

$$\begin{tikzcd} I/\mathscr{M}_p \cdot I \arrow[r, "\alpha(\xi)"] \arrow[d, "\rho"'] & I/\mathscr{M}_p \cdot I \arrow[d, "\rho"'] \\ \operatorname{Hom}_{\mathbb{C}}(T_p\mathbb{C}^n/T_pX, T_0\mathbb{C}) \arrow[r, "(\beta(\xi))^*"] & \operatorname{Hom}_{\mathbb{C}}(T_p\mathbb{C}^n/T_pX, T_0\mathbb{C}) \end{tikzcd} \tag{2.5}$$

Since ρ is an isomorphism, it follows from (2.5) that α is surjective if and only if β is surjective, and hence (2) is equivalent to (3). $\square$

Remark 2.5. Since $\langle L\rangle_p = T_pX$, the map β in Theorem 2.3 describes how vector fields behave with respect to the normal space to X at p in $\mathbb{C}^n$, $T_p\mathbb{C}^n/T_pX$. So, too, does α: algebraically, the quotient of the Zariski tangent spaces of $\mathbb{C}^n$ and X at p is the dual of

$$(I + \mathscr{M}_p^2)/\mathscr{M}_p^2 \cong I/(I \cap \mathscr{M}_p^2) = I/\mathscr{M}_p \cdot I,$$

where the equality is because in this case, $I \cap \mathscr{M}_p^2 = \mathscr{M}_p \cdot I$. The map ρ constructed in the proof of Theorem 2.3 is essentially an identification between the dual of this "Zariski normal space" to X at p and the dual of the usual normal space to X at p.

Remark 2.6. If (X, p) is an arbitrary analytic germ with an irreducible component of dimension d in $\mathbb{C}^n$, then by the coherence of $\operatorname{Der}(-\log X)$ and Theorem 2.3, $\operatorname{Der}(-\log X)_p$ requires at least $d + (n - d)^2$ generators; the component of smallest dimension gives the strongest bound. Are germs for which $\operatorname{Der}(-\log X)_p$ is minimally generated in some way special, as is the case for hypersurfaces (see §5.2)?

2.3. **Fitting ideals for smooth germs.** We now compute the Fitting ideals associated to $\mathrm{Der}(-\log X)_p$ for (X,p) smooth.

Proposition 2.7. *Let (X,p) be a smooth germ of dimension d in $\mathbb{C}^n$, with $d<n$. In $\mathscr{O}_{\mathbb{C}^n,p}$, we have*

$$I_k(\mathrm{Der}(-\log X)_p) = \begin{cases} (I(X))^{k-d} & \text{if } k>d \\ (1) & \text{otherwise} \end{cases}. \tag{2.6}$$

Proof. Choose coordinates as in Example 2.1, with $I(X)=(y_1,\ldots,y_{n-d})$. Any Saito matrices have rows corresponding to the coefficients of $\frac{\partial}{\partial y_1},\ldots,\frac{\partial}{\partial y_n}$. By Theorem 2.3, the vector fields of (2.1) generate $\mathrm{Der}(-\log X)_p$. A Saito matrix M of these generators might be, in block form,

$$M = \begin{pmatrix} Y & & & & 0 \\ & Y & & & \\ & & \ddots & & \\ & & & Y & \\ 0 & & & & J_d \end{pmatrix}, \text{ where } Y = \begin{pmatrix} y_1 & y_2 & \cdots & y_{n-d} \end{pmatrix},$$

and for any ℓ, J_ℓ denotes the $\ell\times\ell$ identity matrix. Then $I_k(\mathrm{Der}(-\log X)_p)$ is generated by the determinants of the $k\times k$ submatrices of M. If $k\leq d$, then a $k\times k$ submatrix of J_d is nonsingular, and hence $I_k(\mathrm{Der}(-\log X)_p)$ contains 1. Now let $k>d$. Any submatrix which uses more than one column from the same Y block will have determinant zero, as there is an obvious relation between the columns. Also, any submatrix which uses a non-symmetric choice of rows and columns of J_d will have a zero row or column. Hence, the only nonzero generators of $I_k(\mathrm{Der}(-\log X)_p)$ will be

$$\det\begin{pmatrix} y_{i_1} & & & \\ & \ddots & & \\ & & y_{i_\ell} & \\ & & & J_{k-\ell} \end{pmatrix} = \prod_{j=1}^{\ell} y_{i_j},$$

where each $i_j\in\{1,\ldots,n-d\}$, and (since $0\leq k-\ell\leq d$) $k-d\leq \ell\leq k$. Thus $I_k(\mathrm{Der}(-\log X)_p)$ is generated by all monomials of degree $k-d$ in $\{y_1,\ldots,y_{n-d}\}$, which is exactly as claimed. $\square$

2.4. **Geometry of the Fitting ideals for smooth germs.** We may give a geometric interpretations to one of these ideals. Let α and β be defined as in Theorem 2.3.

Proposition 2.8. *Let (X,p) be a smooth germ of dimension d in $\mathbb{C}^n$, with $d<n$. Let $L\subseteq \mathrm{Der}(-\log X)_p$ be a submodule, and let $L_0\subseteq L$ be the submodule of vector fields vanishing at p. Then the following are equivalent:*

1. *in $\mathscr{O}_{\mathbb{C}^n,p}$, we have $I_{d+1}(L)\nsubseteq \mathscr{M}_p^2$;*
2. *$\dim(\langle L\rangle_p)=d$ and the map $\alpha|_{L_0}$ is nonzero;*
3. *$\dim(\langle L\rangle_p)=d$ and the map $\beta|_{L_0}$ is nonzero.*

We need a lemma for the proof. Let $M(p,q,\mathbb{C})$ be the space of $p\times q$ matrices with complex entries. Since $M(p,q,\mathbb{C})$ is a vector space, there is a canonical identification $T_AM(p,q,\mathbb{C})\simeq M(p,q,\mathbb{C})$ for all A.

Lemma 2.9. *Let $\gamma : (-\epsilon, \epsilon) \to M(m, m, \mathbb{C})$ be a differentiable curve. Define $A = \gamma(0)$, $B = \gamma'(0)$, and $\delta(t) = \det(\gamma(t))$. Then the following are equivalent:*

(i) $\delta(0) = 0$ *and* $\delta'(0) \neq 0$*;*
(ii) A *has rank* $m-1$ *and* $B \cdot \ker(A) \nsubseteq \operatorname{im}(A)$.

Proof. Since the statement is obvious when $m = 1$, assume $m > 1$. Let $\operatorname{adj}(A)$ denote the adjugate of $A \in M(p, p, \mathbb{C})$. By Jacobi's formula,

$$\delta'(0) = \operatorname{tr}(\operatorname{adj}(\gamma(0)) \cdot \gamma'(0)) = \operatorname{tr}(\operatorname{adj}(A) \cdot B). \tag{2.7}$$

By hypothesis and (2.7), in both cases we must have $\operatorname{rank}(A) = m - 1$, so assume this. Since $\det(A) = 0$, we have $A \cdot \operatorname{adj}(A) = \operatorname{adj}(A) \cdot A = \det(A) \cdot I = 0$, and thus

$$\operatorname{im}(\operatorname{adj}(A)) \subseteq \ker(A) \qquad \text{and} \qquad \operatorname{im}(A) \subseteq \ker(\operatorname{adj}(A)). \tag{2.8}$$

By the first inclusion of (2.8), $\operatorname{rank}(\operatorname{adj}(A)) \leq 1$; as also $\operatorname{adj}(A) \neq 0$, we have $\operatorname{rank}(\operatorname{adj}(A)) = 1$ and may write $\operatorname{adj}(A) = \lambda \cdot \omega$ for some nonzero $\lambda \in M(m, 1, \mathbb{C})$ and $\omega \in M(1, m, \mathbb{C})$. By dimensional considerations, the inclusions of (2.8) are equalities, and we may identify these spaces using the structure of $\operatorname{adj}(A)$:

$$\begin{aligned} &\operatorname{im}(\operatorname{adj}(A)) = \ker(A) = \mathbb{C} \cdot \lambda \\ \text{and} \qquad &\operatorname{im}(A) = \ker(\operatorname{adj}(A)) = \{z \in M(m, 1, \mathbb{C}) : \omega \cdot z = 0\}. \end{aligned} \tag{2.9}$$

By (2.7),

$$\delta'(0) = \operatorname{tr}(\lambda \cdot (\omega \cdot B)) = (\omega \cdot B) \cdot \lambda,$$

and in particular, $\mathbb{C} \cdot \delta'(0) = \omega \cdot B \cdot \ker(A)$. The equivalence follows from this equation and (2.9). ☐

Proof of 2.8. By (2.5), (2) is equivalent to (3). Let M be a Saito matrix of a generating set of L, and N be a $(d+1) \times (d+1)$ submatrix of M. By Proposition 2.7, $\det(N) \in \mathscr{M}_p$. Observe that if $\dim(\langle L \rangle_p) < d$, then $\operatorname{rank}(N(p)) < d$ and so by Lemma 2.9 we have $\det(N) \in \mathscr{M}_p^2$.

It remains only to show that if $\dim(\langle L \rangle_p) = d$, then (1) is equivalent to (2). Assume, then, that $\dim(\langle L \rangle_p) = d$, and hence $T_pX = \langle L \rangle_p$. Choose coordinates as in Example 2.1, with $I(X) = (y_1, \ldots, y_{n-d})$. Choose $\eta_1, \ldots, \eta_d \in L$ spanning $\langle L \rangle_p$ at p such that $L = \mathscr{O}_{\mathbb{C}^n,p}\{\eta_i\}_{i=1}^d + L_0$. Let M be a Saito matrix of L with columns corresponding to $\eta_1, \ldots, \eta_d$ and a set of generators of L_0. It is enough to consider determinants of submatrices of M. Observe that $M(p)$ is zero, except for the $d \times d$ submatrix A of rank d corresponding to $\eta_1, \ldots, \eta_d$ and $\frac{\partial}{\partial y_{n-d+1}}, \ldots, \frac{\partial}{\partial y_n}$. By Lemma 2.9, it is necessary for any $(d+1) \times (d+1)$ submatrix N of M to contain A in order to have $\det(N) \notin \mathscr{M}_p^2$.

Thus, consider a submatrix N of M containing A, as well as a column corresponding to $\xi \in L_0$ and a row corresponding to $\frac{\partial}{\partial y_j}$, with $j < n-d+1$. For any $v \in T_p\mathbb{C}^n$, let $\gamma_v(t) = N(p + tv)$, and observe that $\gamma_v(0)$ has rank d, $\operatorname{im}(\gamma_v(0))$ corresponds to a projection of T_pX onto $V = \mathbb{C}\{\frac{\partial}{\partial y_j}(p), \frac{\partial}{\partial y_{n-d+1}}(p), \ldots, \frac{\partial}{\partial y_n}(p)\}$, $\ker(\gamma_v(0))$ corresponds to $\mathbb{C} \cdot \xi$, and $\gamma_v'(0) \cdot \ker(\alpha_v(0))$ corresponds to a projection of $\mathbb{C} \cdot d(\widehat{\xi})_{(p)}(v)$ onto V. By Lemma 2.9, $(\det \circ \gamma_v)'(0) \neq 0$ if and only if the $\frac{\partial}{\partial y_j}$–component of $d(\widehat{\xi})_{(p)}(v)$ is nonzero; moreover, such a $v \notin T_pX$ since $d(\widehat{\xi})_{(p)}(T_pX) \subseteq T_pX$.

If $\det(N) \in \mathscr{M}_p \backslash \mathscr{M}_p^2$, then there exists a v so that $\frac{d}{dt}(\det(N(p+tv)))|_{t=0} \neq 0$, and by the above $d(\widehat{\xi})_{(p)}(v)$ has a nonzero $\frac{\partial}{\partial y_j}$–coefficient, so that $\beta(\xi) \neq 0$. Conversely, if $\beta(\xi) \neq 0$, then there exists a $v \notin T_pX$ and a $j < n-d+1$ so that $d(\widehat{\xi})_{(p)}(v)$ has a nonzero $\frac{\partial}{\partial y_j}$–coefficient, and by the above argument the corresponding submatrix N has $\det(N) \notin \mathscr{M}_p^2$. □

3. Logarithmic vector fields of arbitrary germs

In this section, we study the logarithmic vector fields for an arbitrary analytic germ (X,p) in $\mathbb{C}^n$ by using the properties of logarithmic vector fields at nearby smooth points. We shall study the associated ideals and their geometric meaning.

3.1. **Choice of representatives.** We shall often choose representatives of an analytic germ (X,p) and a collection of vector fields in $\mathrm{Der}(-\log X)_p$, and then study the behavior of these representatives at points near to p. It is convenient to choose representatives on an open U containing p so that the behavior of the representatives on U reflect only the behavior of the original germs. For any germ f, a specific choice of representative will be denoted by f' or f''.

Lemma 3.1. *Let (X,p) be an analytic germ in $\mathbb{C}^n$ and let $\eta_1,\ldots,\eta_m \in \mathrm{Der}(-\log X)_p$. For any choice of representatives of these germs on open neighborhoods of p, there exists an open neighborhood U of p such that:*

(1) all representatives are defined on U;
(2) if $X = \cup_{s\in S} X_s$ is an irreducible decomposition of germs, then there is a corresponding irreducible decomposition $X' = \cup_{s\in S} X'_s$ on U where $(X'_s,p) = (X_s,p)$;
(3) each $\eta'_i \in \mathrm{Der}(-\log X')(U)$;
(4) if $\mathscr{O}_{\mathbb{C}^n,p}\cdot\{\eta_i\}_{i=1}^m = \mathrm{Der}(-\log X)_p$, then $\{\eta'_i\}_{i=1}^m$ generates the sheaf $\mathrm{Der}(-\log X')$ on U;
(5) U is, e.g., a polydisc, so that U is convex (using real line segments).

Proof. Without loss of generality, assume that the hypothesis of (4) is satisfied (by, e.g., increasing m). Without mentioning it explicitly, each U_i will be an open neighborhood of p. Let X'' and $\eta''_1,\ldots,\eta''_m$ be representatives, and let U_1 be an open set on which these representatives are all defined.

Decompose (X,p) into irreducible components as in (2), and find $U_2 \subseteq U_1$ containing representatives X''_s of each (X_s,p). Since each (X_s,p) is irreducible, there exists a $U_3 \subseteq U_2$ with the property that for any open neighborhood $V \subseteq U_3$ containing p, $V \cap X''_s$ is irreducible in V, and hence $\cup_{s\in S}(V\cap X''_s)$ is an irreducible decomposition of $V \cap (\cup_{s\in S} X''_s)$ in V. Since $\cup_{s\in S} X''_s$ is a representative of (X,p), it and X'' are equal on some $U_4 \subseteq U_3$.

By the coherence of $\mathrm{Der}(-\log(U_4 \cap X''))$ and the hypothesis of (4), there is a $U_5 \subseteq U_4$ on which representatives of $\eta_1,\ldots,\eta_m$ can be found which generate the sheaf $\mathrm{Der}(-\log(U_5\cap X''))$; by the definition of germ, there is a $U_6 \subseteq U_5$ on which $\eta''_1,\ldots,\eta''_m$ generate $\mathrm{Der}(-\log(U_6 \cap X''))$.

Finally, let $U \subseteq U_6$ be a polydisc containing p, and set $X' = U\cap X''$, $X'_s = U \cap X''_s$, and $\eta'_i = \eta''_i|_U$. It is easily checked that all conditions are satisfied. □

Observe that by (2), X'_s is irreducible in U, and hence $I(X'_s) \subseteq \mathscr{O}_U(U)$ is prime and X'_s is pure-dimensional. By (5), if $f \in \mathscr{O}_U(U)$ has $f(q) = 0$ for some $q \in U$, then by Hadamard's Lemma we may write $f = \sum_{i=1}^n f_i \cdot (x_i - x_i(q))$, with $f_i \in \mathscr{O}_U(U)$. More generally, if $f \in \mathscr{O}_U(U)$ has $f \in \mathscr{M}_q^k \subseteq \mathscr{O}_{U,q}$, then $f \in \mathscr{M}_q^k \subseteq \mathscr{O}_U(U)$.

3.2. **Symbolic powers of ideals.** Before proceeding, we recall the concept of the symbolic power of an ideal. Let R be a commutative Noetherian ring with identity. For a prime ideal $\mathfrak{p} \subseteq R$, it is not necessarily true that $\mathfrak{p}^\ell$ is $\mathfrak{p}$-primary (e.g., [Eis95, §3.9.1]); the $\mathfrak{p}$-primary component of $\mathfrak{p}^\ell$ is called the *ℓth symbolic power of* $\mathfrak{p}$ and is denoted by $\mathfrak{p}^{(\ell)}$.

A result originally due to Zariski and Nagata says that $\mathfrak{p}^{(\ell)}$ is the ideal of functions which vanish to order $\geq \ell$ on $V(\mathfrak{p})$.

Theorem 3.2 ([EH79, Corollary 1])**.** *Let $\mathfrak{p}$ be a prime ideal of a ring R, and let S be a set of maximal ideals $\mathscr{M}$ containing $\mathfrak{p}$ such that $R_{\mathscr{M}}/\mathfrak{p}_{\mathscr{M}}$ is a regular local ring, and $\cap_{\mathscr{M}\in S}\mathscr{M} = \mathfrak{p}$. Then*

$$\mathfrak{p}^{(\ell)} \supseteq \bigcap_{\mathscr{M}\in S} \mathscr{M}^\ell \supseteq \bigcap_{\substack{\mathscr{M}\supseteq\mathfrak{p} \\ \mathscr{M} \text{ maximal}}} \mathscr{M}^\ell,$$

with equalities if R is regular.

In particular, we have

Corollary 3.3. *If $\mathfrak{p}$ is a prime ideal of $\mathscr{O}_U(U)$, and $N \subseteq \mathrm{Smooth}(V(\mathfrak{p}))$ is such that the closure $\overline{N} = V(\mathfrak{p})$, then*

$$\mathfrak{p}^{(\ell)} = \bigcap_{p\in N} \mathscr{M}_p^\ell = \bigcap_{p\in V(\mathfrak{p})} \mathscr{M}_p^\ell.$$

Proof. $\mathscr{O}_U$ is a regular ring. Since $\overline{N} = V(\mathfrak{p})$, we have

$$\cap_{p\in N}\mathscr{M}_p = I(N) = I(\overline{N}) = \mathfrak{p}.$$

Then apply Theorem 3.2. □

Remark 3.4. Since primary decomposition commutes with localization, so does the symbolic power.

It is also useful to know the following characterization of $\mathfrak{p}^{(\ell)}$ when $\mathfrak{p}$ defines a complete intersection.

Lemma 3.5. *If $\mathfrak{p}$ is a prime ideal in $\mathscr{O}_{U,p}$ generated by a regular sequence, then $\mathfrak{p}^{(\ell)} = \mathfrak{p}^\ell$ for all $\ell \geq 1$.*

Proof. $\mathscr{O}_{U,p}$ is Cohen-Macaulay so this follows from, e.g., Proposition 3.76 of [Vas98]. □

3.3. **Fitting ideals.** We can now prove an upper bound for the Fitting ideals of logarithmic vector fields.

Theorem 3.6. *Let (X,p) be an analytic germ in $\mathbb{C}^n$, with $X = \cup_{s\in S}X_s$ the decomposition into irreducible components. Let $1 \leq k \leq n$. Then in $\mathscr{O}_{\mathbb{C}^n,p}$, we have*

$$I_k(\mathrm{Der}(-\log X)_p) \subseteq \bigcap_{\substack{s\in S \\ k>\dim(X_s)}} (I(X_s))^{(k-\dim(X_s))}, \tag{3.1}$$

where the exponents denote symbolic powers and are sharp*, that is, changing the RHS of (3.1) by either increasing the exponents or intersecting with any nontrivial symbolic power of any $I(X_s)$ with $k \leq \dim(X_s)$ would make the statement false. Moreover, the difference between the two sides of (3.1) is supported on* $\mathrm{Sing}(X)$*.*

For example, $I_n(\mathrm{Der}(-\log X)_p)$ reflects both the irreducible components of (X,p) and the dimensions of these components.

Proof. Let $\eta_1,\ldots,\eta_m$ generate $\mathrm{Der}(-\log X)_p$. Find representatives of (X,p) and each η_i on an open neighborhood U of p, as in Lemma 3.1. Let $M=\mathrm{Der}(-\log X')(U)$, and observe that it is generated by $\eta'_1,\ldots,\eta'_m$.

For $s\in S$, let $N_s=\mathrm{Smooth}(X')\cap X'_s=\mathrm{Smooth}(X'_s)\setminus\left(\cup_{r\in S\setminus\{s\}}X'_r\right)$. Since N_s is a dense open subset of X'_s, $\overline{N_s}=X'_s$.

Let $q\in N_s$, and let $k>\dim(X_s)$. Since $\eta'_1,\ldots,\eta'_m$ generate $\mathrm{Der}(-\log X')_q$ as a $\mathscr{O}_{U,q}$ module, the localization of M at q is equal to $\mathrm{Der}(-\log X')_q$. Observe that localizing M commutes with taking minors of a presentation matrix of M. Thus by Proposition 2.7, as $\mathscr{O}_{U,q}$ modules,

$$\mathscr{O}_{U,q}\cdot I_k(M)=I_k(\mathscr{O}_{U,q}\cdot M)=I_k(\mathrm{Der}(-\log X')_q)=I(X'_s)^{k-\dim(X_s)}. \tag{3.2}$$

Since $q\in X_s$, by (3.2), $\mathscr{O}_{U,q}\cdot I_k(M)\subseteq\mathscr{M}_q^{k-\dim(X_s)}$. By a version of Hadamard's lemma for holomorphic functions, it follows that in $\mathscr{O}_U$, $I_k(M)\subseteq\mathscr{M}_q^{k-\dim(X_s)}$. As this is true for all $q\in N_s$, by Corollary 3.3 it follows that $I_k(M)\subseteq(I(X'_s))^{(k-\dim(X_s))}$ whenever $k>\dim(X_s)$.

This proves that as $\mathscr{O}_U(U)$ ideals,

$$I_k(M)\subseteq\bigcap_{\substack{s\in S\\ k>\dim(X_s)}}(I(X'_s))^{(k-\dim(X_s))}. \tag{3.3}$$

Since localization commutes with taking minors, M localized at p is $\mathrm{Der}(-\log X)_p$, $I(X'_s)$ localized at p is $I(X_s)$, and as symbolic powers commute with localization, (3.1) follows from (3.3). By (3.2) and Lemma 3.5, the two sides of (3.1) agree at smooth points of X.

To show that the exponents are sharp, fix $s\in S$ and $k>\dim(X_s)$. Suppose that in $\mathscr{O}_{\mathbb{C}^n,p}$,

$$I_k(\mathrm{Der}(-\log X)_p)\subseteq(I(X_s))^{(k-\dim(X_s)+1)}. \tag{3.4}$$

As above, choose representatives on an open set U containing p. Each side of (3.4) is the stalk at p of the coherent ideal sheaf $\mathscr{J}$ and $\mathscr{K}$ on U generated by $I_k(M)$ and $(I(X'_s))^{(k-\dim(X_s)+1)}$, respectively. It thus follows from (3.4) that there exists some open $V\subseteq U$ containing p on which $\mathscr{J}|_V\subseteq\mathscr{K}|_V$. Let $q\in V\cap\mathrm{Smooth}(X')\cap X'_s$, take the stalks at q, and apply Proposition 2.7 and Lemma 3.5 to find that in $\mathscr{O}_{\mathbb{C}^n,q}$,

$$\mathscr{J}_q=(I(X'_s))^{k-\dim(X_s)}\subseteq\mathscr{K}_q=(I(X'_s))^{k-\dim(X_s)+1};$$

this is a contradiction.

If $s\in S$ and $k\leq\dim(X_s)$, then no symbolic power of $I(X_s)$ can appear on the right side of (3.1), either by the same argument, or by the existence of certain vector fields constructed in [HM93, Existence Lemma]. $\square$

Remark 3.7. Theorem 3.6 is improved significantly by the following observation. If $\mathrm{Der}(-\log X)_p\subseteq\mathrm{Der}(-\log Y)_p$ for some (Y,p), then $I_k(\mathrm{Der}(-\log X)_p))\subseteq I_k(\mathrm{Der}(-\log Y)_p)$, and hence applying Theorem 3.6 to (Y,p) can reduce the upper bound for $I_k(\mathrm{Der}(-\log X)_p)$. In particular, this applies to $Y=\mathrm{Sing}(X)$, $Y=\mathrm{Sing}(\mathrm{Sing}(X))$, etc.

Remark 3.8. The RHS of (3.1) is part of the primary decomposition of the LHS, containing those primary ideals corresponding to isolated primes.

Remark 3.9. Let (X_s, p) be an irreducible component of (X, p) and let $d = \dim(X_s)$. By Theorem 3.6, $I_{d+1}(\mathrm{Der}(-\log X)_p)$ is contained in $I(X_s)$, but not contained in $I(X_s)^{(2)}$.

Example 3.10. Consider the hypersurface X in $\mathbb{C}^4$ defined by $xw - yz = 0$, with $\mathrm{Sing}(X)$ the origin. Then $M = \mathrm{Der}(-\log X)_0$ is generated by seven vector fields, and by Theorem 3.6 and Remark 3.7, we know

$$\begin{aligned} I_4(M) &\subseteq (xw - yz) \cap (x, y, z, w)^4, \\ I_3(M) &\subseteq (x, y, z, w)^3, \\ I_2(M) &\subseteq (x, y, z, w)^2, \\ \text{and } I_1(M) &\subseteq (x, y, z, w); \end{aligned}$$

in fact, a Macaulay2 computation shows that these are all equalities.

Example 3.11. Consider the hypersurface X_3 in $\mathbb{C}^4$ defined by $xy(x-y)(xz - yw) = 0$. Inductively let $X_i = \mathrm{Sing}(X_{i+1})$, so that $X_0 \subseteq X_1 \subseteq X_2 \subseteq X_3$, with each X_i a union of irreducible complete intersections of dimension i. Then $M = \mathrm{Der}(-\log X_3)_0$ is generated by 4 vector fields, and applying Theorem 3.6 to all X_i computes $I_4(M)$ and $I_1(M)$ exactly, while an upper bound with the correct radical is produced for $I_3(M)$. For $I_2(M)$, Theorem 3.6 does not detect that the logarithmic stratification of X_3 is not finite, as the vector fields of M have rank ≤ 1 on $x = y = 0$. This corresponds to X_3 not being biholomorphically trivial along $x = y = 0$.

Example 3.12. Consider the Whitney umbrella X_2 in $\mathbb{C}^3$ defined by $x^2 - y^2 z = 0$. Then $X_1 = \mathrm{Sing}(X_2)$ is the smooth set $x = y = 0$ and $M = \mathrm{Der}(-\log X_2)_0$ is generated by 4 vector fields. Applying Theorem 3.6 to X_1 and X_2 does not compute any $I_k(M)$ exactly. Although the bounds for $I_2(M)$ and $I_3(M)$ have the correct radical, the upper bound for $I_1(M)$ is (1). If we recognize that each $\eta \in M$ must be tangent to the origin, then Theorem 3.6 and Remark 3.7 compute $I_1(M)$ and $I_3(M)$ exactly.

Example 3.13. Consider the variety X defined by the ideal I generated by the 2×2 minors of a generic 3×3 symmetric matrix. Then X has dimension 3, and $\mathrm{Sing}(X)$ is the origin. Here, the symbolic powers of I differ from the usual powers of I. The module $M = \mathrm{Der}(-\log X)_0$ is generated by 24 vector fields. Theorem 3.6 and Remark 3.7 compute $I_1(M), \ldots, I_5(M)$ exactly, while $I_6(M)$ differs from the computed upper bound.

The sharpness described in Theorem 3.6 provides a necessary condition on a submodule of logarithmic vector fields to be a complete generating set. However, it is far from sufficient.

Example 3.14. Let $f \in \mathscr{O}_{\mathbb{C}^n,p}$ define a reduced hypersurface (X, p). Then the vector fields

$$\left\{ \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j} \right\}_{1 \leq i < j \leq n} \quad \text{and} \quad \left\{ f \frac{\partial}{\partial x_i} \right\}_{1 \leq i \leq n} \tag{3.5}$$

generate a module $L \subseteq \mathrm{Der}(-\log X)_p$ of vector fields which vanish on $\mathrm{Sing}(X)$; nevertheless, at any nearby smooth point q of X, $L = \mathrm{Der}(-\log X)_q$ as $\mathscr{O}_{\mathbb{C}^n,q}$–modules (by, e.g., Theorem 2.3). Since the proof of Theorem 3.6 relied entirely on behavior at the smooth points, each $I_k(L)$ should satisfy (3.1) and the sharpness conditions of Theorem 3.6, although in general $L \neq \mathrm{Der}(-\log X)_p$.

3.4. **Geometry of the Fitting ideals.** Let $L \subseteq \mathrm{Der}(-\log X)_p$ be a submodule. We can give a geometric interpretation to a certain Fitting ideal of L satisfying the sharpness condition of Theorem 3.6 with respect to a certain component of (X, p).

Proposition 3.15. *Let (X, p) be an analytic germ in $\mathbb{C}^n$, and let (X_0, p) be an irreducible component of (X, p) of dimension d. Let $L = \mathscr{O}_{\mathbb{C}^n,p}\{\eta_1, \ldots, \eta_m\} \subseteq \mathrm{Der}(-\log X)_p$. Choose representatives of (X, p), (X_0, p), and each η_i, and let U be an open neighborhood of p on which these representatives satisfy the conditions of Lemma 3.1. Let $L' = \mathscr{O}_U\{\eta'_1, \ldots, \eta'_m\}$. Then the following are equivalent:*

(1) $I_{d+1}(L) \nsubseteq (I(X_0))^{(2)}$ in $\mathscr{O}_{\mathbb{C}^n,p}$;
(2) for every open neighborhood $V \subseteq U$ containing p, there exists a $q \in \mathrm{Smooth}(X') \cap X'_0 \cap V$ such that at q, L' and $(X', q) = (X'_0, q)$ satisfy the equivalent conditions of Proposition 2.8;
(3) for every open neighborhood $V \subseteq U$ containing p, there exists a Zariski open, dense subset W of $X'_0 \cap V$ such that at every $q \in W$, L' and $(X', q) = (X'_0, q)$ satisfy the equivalent conditions of Proposition 2.8.

Proof. The set $A = \{q \in U : \mathscr{O}_{U,q} \cdot I_{d+1}(L') \subseteq \mathscr{M}_q^2 \text{ in } \mathscr{O}_{U,q}\}$ is defined by the vanishing of f and the partials of f, for all $f \in I_{d+1}(L')$, and hence is a closed analytic subset of U. Since $\mathrm{Sing}(X')$ and X'_0 are closed analytic sets, $B = (A \cup \mathrm{Sing}(X')) \cap X'_0$ is a closed analytic subset of X'_0. Thus $G = X'_0 \setminus B$ is a Zariski open subset of X'_0, and is the set of $q \in N = \mathrm{Smooth}(X') \cap X'_0$ where L' and $(X', q) = (X'_0, q)$ satisfy one of the equivalent conditions of Proposition 2.8. Note that $G = N \setminus A$.

Since G is a Zariski open subset of the irreducible X'_0, either $\overline{G} = X'_0$ (and B is nowhere dense in X'_0) or $G = \emptyset$ (and $B = X'_0$).

If $G = \emptyset$, then at every $q \in N$, $q \in A$ and hence (by, e.g., Hadamard's Lemma) $I_{d+1}(L') \subseteq \mathscr{M}_q^2$ in $\mathscr{O}_U(U)$. Since $\overline{N} = X'_0$, Corollary 3.3 shows that $I_{d+1}(L') \subseteq (I(X_0))^{(2)}$, and localizing at p shows that (1) is false. However, (2) and (3) are also false.

If $\overline{G} = X'_0$, then G is dense in N, and (2) and (3) are true. Suppose that (1) were false. Then on some open $V \subseteq U$ containing p, $\mathscr{O}_U(V) \cdot I_{d+1}(L') \subseteq (I(X'_0))^{(2)}$ in $\mathscr{O}_U(V)$. Let $q \in G \cap V$. By Corollary 3.3, $I_{d+1}(L') \subseteq \mathscr{M}_q^2$ in $\mathscr{O}_{U,q}$, and hence $q \in A$. But since $A \cap G = \emptyset$, this is a contradiction. □

4. Reflexive modules

Before discussing hypersurfaces in §5, we recall some useful background on reflexive modules.

We adopt the following notation for this section. Let U be an open subset of $\mathbb{C}^n$. Let $\mathscr{O} = \mathscr{O}_{\mathbb{C}^n}|_U$ (respectively, $\widetilde{\mathscr{O}}$) be the sheaf of holomorphic functions (resp., meromorphic functions) on U. Let $\Omega^1 = \Omega^1_U$ (respectively, $\widetilde{\Omega}^1 = \widetilde{\Omega}^1_U$) be the $\mathscr{O}$–module of holomorphic (resp., meromorphic) 1-forms on U, and let $\mathrm{Der} = \mathrm{Der}_U$ be the $\mathscr{O}$–module of holomorphic vector fields on U. For a $\mathscr{O}$–module $\mathscr{N}$, denote its $\mathscr{O}$–dual by $\mathscr{N}^* = \mathscr{H}\mathrm{om}_{\mathscr{O}}(\mathscr{N}, \mathscr{O})$; for an open $V \subseteq U$, $\mathscr{N}^*(V) = \mathrm{Hom}_{\mathscr{O}|_V}(\mathscr{N}|_V, \mathscr{O}|_V)$ is the $\mathscr{O}(V)$–module of $\mathscr{O}|_V$–module morphisms $\mathscr{N}|_V \to \mathscr{O}|_V$.

Let $\theta : \mathrm{Der} \times \widetilde{\Omega}^1 \to \widetilde{\mathscr{O}}$, defined by $\theta(V)(\eta, \omega) = (x \mapsto \omega(\eta(x))) \in \widetilde{\mathscr{O}}(V)$, be the standard pairing (or "inner product") between vector fields and 1-forms, extended to meromorphic forms. Just as for pairings between vector spaces, such a pairing may sometimes be used to identify the $\mathscr{O}$–dual of a submodule $\mathscr{N} \subseteq \mathrm{Der}$ with a submodule of $\widetilde{\Omega}^1$, and vice-versa. Following [Sai80, (1.6)], we have

Lemma 4.1. *Let $f \in \mathscr{O}(U)$, $f \neq 0$.*

(1) Let $\mathscr{D}$ be a $\mathscr{O}$–submodule of Der, with $f \cdot \mathrm{Der} \subseteq \mathscr{D} \subseteq \mathrm{Der}$. Then $\mathscr{D}^*$ is canonically isomorphic to a $\mathscr{O}$–submodule $\mathscr{M} \subseteq \widetilde{\Omega}^1$, with $\Omega^1 \subseteq \mathscr{M} \subseteq \frac{1}{f} \cdot \Omega^1$, by the pairing $\theta|_{\mathscr{D} \times \mathscr{M}} : \mathscr{D} \times \mathscr{M} \to \mathscr{O}$.

(2) Let $\mathscr{M}$ be a $\mathscr{O}$–submodule of $\widetilde{\Omega}^1$, with $\Omega^1 \subseteq \mathscr{M} \subseteq \frac{1}{f} \cdot \Omega^1$. Then $\mathscr{M}^*$ is canonically isomorphic to a $\mathscr{O}$–submodule $\mathscr{D} \subseteq \mathrm{Der}$, with $f \cdot \mathrm{Der} \subseteq \mathscr{D} \subseteq \mathrm{Der}$, by the pairing $\theta|_{\mathscr{D} \times \mathscr{M}} : \mathscr{D} \times \mathscr{M} \to \mathscr{O}$.

The modules and maps are independent of f.

Proof. Fix holomorphic coordinates $x_1, \dots, x_n$ on U. Throughout, V and W will denote arbitrary open subsets with $W \subseteq V \subseteq U$.

For (1), define the $\mathscr{O}$–submodule $\mathscr{M} \subseteq \widetilde{\Omega}^1$ by $\mathscr{M}(V) = \{\omega \in \widetilde{\Omega}^1(V) : \theta(V)(\mathscr{D}(V), \omega) \subseteq \mathscr{O}(V)\}$, and define the maps

$$\begin{aligned} &\mathscr{D}^* \to \mathscr{M} && \text{on } V, \quad \varphi \mapsto \frac{1}{f} \sum_{i=1}^{n} \varphi(V) \left(f \frac{\partial}{\partial x_i} \right) dx_i \in \mathscr{M}(V), \\ \text{and} \quad &\mathscr{M} \to \mathscr{D}^* && \text{on } V, \quad \omega \mapsto (W \mapsto (\eta \mapsto \theta(W)(\eta, \omega))) \in \mathscr{D}^*(V). \end{aligned}$$

Check that the images lie in the claimed spaces, that the maps are morphisms of $\mathscr{O}$–modules, and that composition in either order gives the identity. By definition and by the surjectivity of the first map, $\Omega^1 \subseteq \mathscr{M} \subseteq \frac{1}{f}\Omega^1$. Since the second map is independent of f and is the inverse of the first, both maps are independent of f.

For (2), define the $\mathscr{O}$–submodule $\mathscr{D} \subseteq \mathrm{Der}$ by $\mathscr{D}(V) = \{\eta \in \mathrm{Der}(V) : \theta(V)(\eta, \mathscr{M}(V)) \subseteq \mathscr{O}(V)\}$, and define the maps

$$\begin{aligned} &\mathscr{M}^* \to \mathscr{D} && \text{on } V, \quad \varphi \mapsto \sum_{i=1}^{n} \varphi(V)(dx_i) \frac{\partial}{\partial x_i} \in \mathscr{D}(V), \\ \text{and} \quad &\mathscr{D} \to \mathscr{M}^* && \text{on } V, \quad \eta \mapsto (W \mapsto (\omega \mapsto \theta(W)(\eta, \omega))) \in \mathscr{M}^*(V). \end{aligned}$$

Check the same conditions as for (1). By definition, $f \cdot \mathrm{Der} \subseteq \mathscr{D} \subseteq \mathrm{Der}$. □

Remark 4.2. The hypothesis that $f \cdot \mathrm{Der} \subseteq \mathscr{D}$ for some nonzero $f \in \mathscr{O}(U)$ is equivalent to $I_n(\mathscr{D}(U)) \neq (0)$.

Definition 4.3. For a $\mathscr{O}$–module $\mathscr{N}$ of vector fields (respectively, 1-forms) as in Lemma 4.1, call the module constructed in (1) (resp., (2)) the *realization* of $\mathscr{N}^*$ as a module of 1-forms (resp., vector fields) and denote it by $R(N)$.

Example 4.4. Let X be the origin in $\mathbb{C}^2$. If $\mathscr{M} = \mathrm{Der}_{\mathbb{C}^2}(-\log X)$, then applying Lemma 4.1(1) gives $\mathscr{M}^* \cong R(\mathscr{M}) = \Omega^1_{\mathbb{C}^2}$, and (2) gives $\mathscr{M}^{**} \cong R(R(\mathscr{M})) = \mathrm{Der}_{\mathbb{C}^2}$.

Remark 4.5. For a coherent $\mathscr{O}$–submodule $\mathscr{N}$ of Der or $\widetilde{\Omega}^1$ as in Lemma 4.1, and $p \in U$, $R(\mathscr{N})_p$ depends only on $\mathscr{N}_p$. For, there is a canonical isomorphism $(\mathscr{N}^*)_p \cong \mathrm{Hom}_{\mathscr{O}_p}(\mathscr{N}_p, \mathscr{O}_p)$ ([GR84, A.4.4]), and by construction $R(\mathscr{N})_p$ depends only on $(\mathscr{N}^*)_p$. Since $\mathscr{N}^*$ is coherent, it is enough to understand the realization operation at stalks.

Example 4.6. For a hypersurface X in $U \subseteq \mathbb{C}^n$ defined by a reduced $f \in \mathscr{O}(U)$, the sheaf of *logarithmic* 1*-forms* $\Omega^1_U(\log X)$ consists of the meromorphic 1-forms ω such that $f \cdot \omega$ and $f \cdot d\omega$ are holomorphic ([Sai80, (1.1)]). In [Sai80, (1.6)], Saito shows that at each $p \in U$, $\Omega^1_{U,p}(\log X)$ and $\mathrm{Der}(-\log X)_p$ are each the $\mathscr{O}_{U,p}$–dual of the other by the pairing θ_p. By coherence, $R(\mathrm{Der}(-\log X)) = \Omega^1_U(\log X)$ and $R(\Omega^1_U(\log X)) = \mathrm{Der}(-\log X)$.

Recall that for a $\mathscr{O}$–module $\mathscr{N}$, there is a natural morphism $\mathscr{N} \to \mathscr{N}^{**}$, and that $\mathscr{N}$ is *reflexive* when this morphism is an isomorphism[1]. If $\mathscr{D} \subseteq$ Der and Lemma 4.1 is used to construct the module $R(R(\mathscr{D})) \subseteq$ Der and an isomorphism $\mathscr{D}^{**} \to R(R(\mathscr{D}))$, then composition with the natural map $\mathscr{D} \to \mathscr{D}^{**}$ gives an interesting homomorphism between two submodules of Der.

Corollary 4.7. *Let $f \in \mathscr{O}(U)$ with $f \neq 0$, let $\mathscr{D} \subseteq$ Der be a $\mathscr{O}$–submodule containing $f \cdot$ Der, and let $i : \mathscr{D} \to \mathscr{D}^{**}$ be the canonical map. If the identifications in the proof of Lemma 4.1 are used to construct an isomorphism $j : \mathscr{D}^{**} \to R(R(\mathscr{D})) \subseteq$ Der, then $j \circ i$ is the inclusion map. In particular, $\mathscr{D} \subseteq R(R(\mathscr{D}))$, and $\mathscr{D}$ is reflexive if and only if $\mathscr{D} = R(R(\mathscr{D}))$.*

Proof. By the proof of Lemma 4.1(1), we have a $\mathscr{O}$–module $\mathscr{M} = R(\mathscr{D}) \subseteq \widetilde{\Omega}^1$ and an isomorphism $\rho : \mathscr{M} \to \mathscr{D}^*$. By the proof of Lemma 4.1(2), we have a module $R(\mathscr{M}) \subseteq$ Der and an isomorphism $\sigma : \mathscr{M}^* \to R(\mathscr{M}) = R(R(\mathscr{D}))$. Let $\kappa : \mathscr{D}^{**} \to \mathscr{M}^*$ be defined by $\kappa(V)(\varphi) = (W \mapsto \varphi(W) \circ \rho(W)) \in \mathscr{M}^*(V)$ for open $W \subseteq V \subseteq U$. Then $j : \mathscr{D}^{**} \to R(R(\mathscr{D}))$ in the statement is defined by $j = \sigma \circ \kappa$.

For open $W \subseteq V \subseteq U$ and $\eta \in \mathscr{D}(V)$, $(\kappa \circ i)(V)(\eta) \in \mathrm{Hom}_{\mathscr{O}|_V}(\mathscr{M}|_V, \mathscr{O}|_V)$ is defined by $W \mapsto (\omega \mapsto \theta(W)(\eta, \omega))$. Writing η in coordinates and applying $\sigma(V)$ shows that $(\sigma \circ \kappa \circ i)(V)$ is the identity on $\mathscr{D}$, so $j \circ i$ is the inclusion.

Since j is an isomorphism and i must be injective, $\mathscr{D}$ is reflexive if and only if i is surjective, if and only if $j \circ i$ is surjective, that is, $\mathscr{D} = (j \circ i)(\mathscr{D}) = R(R(\mathscr{D}))$. $\square$

Remark 4.8. An analogous result, proved similarly, holds for a module $\mathscr{M}$ of forms with $\Omega^1 \subseteq \mathscr{M} \subseteq \frac{1}{f} \cdot \Omega^1$.

We shall need the following property of reflexive sheaves.

Lemma 4.9. *For $i = 1, 2$, let $\mathscr{D}_i \subseteq$ Der be a coherent $\mathscr{O}$–module, with some nonzero $f_i \in \mathscr{O}(U)$ such that $f_i \cdot \mathrm{Der} \subseteq \mathscr{D}_i \subseteq$ Der. If $\mathscr{D}_1$ and $\mathscr{D}_2$ are equal off a set of codimension ≥ 2, then $R(\mathscr{D}_1) = R(\mathscr{D}_2)$ are reflexive.*

Proof. By Remark 4.5, $R(\mathscr{D}_1)$ is equal to $R(\mathscr{D}_2)$ off the same set of codimension ≥ 2. Each $R(\mathscr{D}_i)$ is reflexive and coherent, as is the case for the $\mathscr{O}$–dual of any coherent sheaf ([GR84, A.4.4]). Since U is normal, coherent reflexive sheaves which are equal off a set of codimension ≥ 2 are equal ([Har80, Prop. 1.6]). $\square$

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[1]For $\mathscr{N} \subseteq$ Der and $\mathscr{N} \subseteq \frac{1}{f}\Omega^1$, this map is automatically injective because Der and $\frac{1}{f}\Omega^1$ (and thus $\mathscr{N}$) are torsion-free.

By Remark 4.5, realization is a well-defined operation on the stalk of a coherent $\mathscr{O}$–submodule $\mathscr{N}$ satisfying the hypotheses of Lemma 4.1, and so there is a clear definition for such a stalk being *reflexive*. By coherence, Corollary 4.7, and Remark 4.8, $\mathscr{N}_p$ is reflexive if and only if $\mathscr{N}_p = R(R(\mathscr{N}_p))$, and these conditions at stalks are equivalent to those for $\mathscr{N}$ restricted to a small enough open set containing p. We call $R(R(\mathscr{N}_p))$ the *reflexive hull* of $\mathscr{N}_p$.

There is the following characterization of reflexive modules of logarithmic vector fields.

Proposition 4.10. *Let (X, p) be an analytic germ in $\mathbb{C}^n$, and let (H, p) be the union of the hypersurface components of (X, p), setting $H = \emptyset$ if $\dim(X) \neq n - 1$. Then*

(1) $\mathrm{Der}(-\log X)_p$ is reflexive if and only if either (X, p) is empty, is $(\mathbb{C}^n, p)$, or is the hypersurface germ (H, p).
(2) $R(\mathrm{Der}(-\log X)_p) = R(\mathrm{Der}(-\log H)_p)$ is the module of logarithmic 1-forms for (H, p).
(3) $R(R(\mathrm{Der}(-\log X)_p)) = \mathrm{Der}(-\log H)_p$.

Proof. For one direction of (1), $\mathrm{Der}(-\log \emptyset)_p = \mathrm{Der}(-\log \mathbb{C}^n)_p = \mathrm{Der}_{\mathbb{C}^n,p}$ is free and thus reflexive. If (X, p) is a hypersurface germ, necessarily (H, p), then $\mathrm{Der}(-\log X)_p$ is reflexive by [Sai80, (1.7)].

For (2), choose representatives of (X, p) and (H, p) on an open set U chosen as in Lemma 3.1. Let Y' be the union of the codimension ≥ 2 components of X', so that $X' = H' \cup Y'$. Since $\mathrm{Der}(-\log X')$ equals $\mathrm{Der}(-\log H')$ off of Y', by Lemma 4.9, $R(\mathrm{Der}(-\log X')) = R(\mathrm{Der}(-\log H'))$. This gives the equality of (2), and the interpretation as logarithmic forms is due to Saito (see Example 4.6).

By (2) we have $R(R(\mathrm{Der}(-\log X)_p)) = R(R(\mathrm{Der}(-\log H)_p))$. Since $\mathrm{Der}(-\log H)_p$ is reflexive by (1), this proves (3).

To finish (1), if $\mathrm{Der}(-\log X)_p$ is reflexive then by (3), $\mathrm{Der}(-\log X)_p = \mathrm{Der}(-\log H)_p$. If $\mathrm{Der}(-\log X)_p = \mathrm{Der}_{\mathbb{C}^n,p}$, then either (X, p) is empty or is $(\mathbb{C}^n, p)$; if not, then $(X, p) = (H, p)$, which must be a hypersurface. □

Remark 4.11. A coherent $\mathscr{O}$–module $\mathscr{F}$ is reflexive if and only if (at least locally) it can be written in an exact sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{E} \longrightarrow \mathscr{G} \longrightarrow 0,$$

where $\mathscr{E}$ is locally free and $\mathscr{G}$ is torsion-free ([Har80, Prop 1.1]). For a reduced hypersurface X defined near p by f, we have

$$0 \longrightarrow \mathrm{Der}(-\log X) \xrightarrow{\alpha} \mathrm{Der} \oplus \mathscr{O} \xrightarrow{\beta} (f, \mathrm{jac}(f)) \longrightarrow 0,$$

where $\mathrm{jac}(f)$ is the Jacobian ideal generated by the partial derivatives of f, $\alpha(\eta) = (\eta, -\frac{\eta(f)}{f})$, and $\beta(\eta, g) = \eta(f) + f \cdot g$.

5. Hypersurfaces and free divisors

We now apply our earlier results to hypersurface germs. If L is a module of vector fields logarithmic to a hypersurface (X, p), we give a criterion for $R(R(L)) = \mathrm{Der}(-\log X)_p$. This generalizes criteria of Saito and Brion for free divisors.

5.1. **Hypersurfaces.** First, we summarize our earlier results for a hypersurface component of an analytic germ.

Proposition 5.1. *Let (X,p) be a germ of an analytic set in $\mathbb{C}^n$. Let $\eta_1,\dots,\eta_m \in \operatorname{Der}(-\log X)_p$, $L_{\mathbb{C}} = \mathbb{C}\{\eta_1,\dots,\eta_m\}$, and $L = \mathscr{O}_{\mathbb{C}^n,p}\cdot L_{\mathbb{C}}$. Let (X_0,p) be an irreducible hypersurface component of (X,p). Choose representatives of (X,p), (X_0,p), and each η_i, and let U be an open neighborhood of p on which these representatives satisfy the conditions of Lemma 3.1. Then the following conditions are equivalent:*

1. *$I_n(L) \nsubseteq (I(X_0))^2$ in $\mathscr{O}_{\mathbb{C}^n,p}$;*
2. *for every open neighborhood $V \subseteq U$ containing p, there exists a $q \in X_0'$ satisfying one of the following equivalent conditions:*
   - *(a) there exists $g \in I_n(L')$ such that $dg_{(q)} \neq 0$;*
   - *(b) there exists $g \in I_n(L')$ such that $g \notin \mathscr{M}_q^2$ in $\mathscr{O}_{\mathbb{C}^n,q}$;*
   - *(c) q is a smooth point of X' and $\mathscr{O}_{\mathbb{C}^n,q}\cdot L' = \operatorname{Der}(-\log X')_q$;*
   - *(d) $\dim(\langle L'\rangle_q) = n-1$ and there exists a nonzero $\xi \in L'_{\mathbb{C}}$ vanishing at q such that, if $f_0 \in \mathscr{O}_{\mathbb{C}^n,q}$ is a reduced defining equation for (X_0',q), then one of the following equivalent conditions holds:*
     - *(i) $\xi(f_0) = \gamma\cdot f_0$, with $\gamma(q)\neq 0$;*
     - *(ii) $\xi(f_0)$ has a nonzero derivative at q;*
     - *(iii) $\operatorname{im}(d(\widehat{\xi})_{(q)}) \nsubseteq \langle L'\rangle_q$;*
3. *for every open neighborhood $V \subseteq U$ containing p, there exists a closed analytic set $Y \subseteq V \cap X_0'$ of codimension ≥ 2 in V such that at every $q \in (\operatorname{Smooth}(X)\cap X_0)\setminus Y$, L' and $(X',q) = (X_0',q)$ satisfy the equivalent conditions of (2).*

Proof. First, observe that if any of the conditions of (2) hold for some $q \in X_0'$, then q is a smooth point of X': either g locally defines a smooth hypersurface (necessarily $(X',q) = (X_0',q)$), q is assumed smooth, or we use Lemma 2.4.

Since (X_0,p) is a hypersurface germ, by Lemma 3.5, $I(X_0)^2 = (I(X_0))^{(2)}$. Proposition 3.15 shows that (1) is equivalent to one of several equivalent conditions (listed in Proposition 2.8) which should be satisfied at some point of $\operatorname{Smooth}(X')\cap X_0'\cap V$ for every open neighborhood $V\subseteq U$ containing p, or equivalently, at all points in a Zariski open subset of $\operatorname{Smooth}(X')\cap X_0'\cap V$ for every open neighborhood $V\subseteq U$ containing p. One of these equivalent conditions, Proposition 2.8(1), is the same as (2b).

It remains only to check that the conditions of (2) are equivalent. By a simple argument, (2a) is equivalent to (2b). By Proposition 2.8, (2b) is equivalent to $\dim(\langle L\rangle_q) = n-1$ and the existence of a nonzero $\xi \in L'_{\mathbb{C}}$ vanishing at q with $\alpha(\xi)\neq 0$ (equivalently, $\beta(\xi)\neq 0$). Since $I(X_0') = (f_0)$ in $\mathscr{O}_{\mathbb{C}^n,q}$ and $\xi \in \operatorname{Der}(-\log X_0')_q$, $\xi(f_0) = \gamma\cdot f_0$ for some $\gamma \in \mathscr{O}_{\mathbb{C}^n,q}$. By the definition of α and β, (2(d)i) is the condition that $\alpha(\xi)\neq 0$, and (2(d)iii) is the condition the $\beta(\xi)\neq 0$. To see that (2(d)ii) is equivalent to (2(d)i), use the product rule on the equation $\xi(f_0) = \gamma\cdot f_0$ and the observation that $d(f_0)_{(q)} \neq 0$ as (X_0',q) is smooth and f_0 reduced.

It remains to prove that (2c) is equivalent to, e.g., (2(d)i). But α is a $\mathbb{C}$-linear map to a 1-dimensional vector space, and hence α is nonzero if and only if α is surjective. Thus, the equivalence follows by Theorem 2.3. □

Remark 5.2. Let $f \in \mathscr{O}_{\mathbb{C}^n,p}$ define a reduced hypersurface (X,p). In conditions (2(d)i) or (2(d)ii), f_0 could be any reduced defining equation for (X_0,q), including (a representative of) f. Since $(\frac{1}{\gamma}\xi)(f) = f$ in $\mathscr{O}_{\mathbb{C}^n,q}$, these conditions imply that $\frac{1}{\gamma}\xi$ is an *Euler-like vector field* for f in some neighborhood of q. This neighborhood may not include p, as there are hypersurfaces without Euler-like vector fields.

Remark 5.3. Let (X, p) be a hypersurface. If a free $\mathscr{O}_{\mathbb{C}^n,p}$–module $L \subseteq \mathrm{Der}(-\log X)_p$ of rank n has $\dim(\langle L\rangle_q) = n$ for $q \notin X$, then L is called a *free* structure for* (X, p) in [Dam03]. If $L \neq \mathrm{Der}(-\log X)_p$, then Proposition 5.1 gives a geometric interpretation of how L must differ from $\mathrm{Der}(-\log X)_p$.

Using the notation and results of §4, we have the following criterion for a set of logarithmic vector fields to generate all such vector fields for a hypersurface germ.

Theorem 5.4. *Let (X, p) be a hypersurface germ in $\mathbb{C}^n$ defined locally by a reduced $f \in \mathscr{O}_{\mathbb{C}^n,p}$. Let L be a submodule of $\mathrm{Der}(-\log X)_p$. If $I_n(L) \subseteq (h)$ for a reduced h implies that $h|f$ (equivalently, the hypersurface component of the analytic germ (Z, p) defined by $I_n(L)$ is (X, p)), and every irreducible hypersurface component (X_0, p) of (X, p) satisfies one of the equivalent conditions of Proposition 5.1, then $R(L)$ is the module of germs of logarithmic 1-forms of (X, p) and*

$$R(R(L)) = \mathrm{Der}(-\log X)_p.$$

If also L is reflexive, then $L = \mathrm{Der}(-\log X)_p$.

Proof. The equivalence of the two conditions is straightforward. Choose representatives of X and L, and choose an open set U containing p satisfying Lemma 3.1. Define Z' using $I_n(L')$. Let $\mathscr{L}'$ be the $\mathscr{O}_U$–module generated by L'.

Write $Z' = X' \cup Y'$, where Y' consists of the irreducible components of Z' having codimension ≥ 2. At $q \notin Z'$, $\mathscr{L}'_q$, Der_q, and $\mathrm{Der}(-\log X')_q$ are equal. Let X'_i, $i = 1, \ldots, k$, be the irreducible hypersurface components of X'. By Proposition 5.1 there is an analytic set $B_i \subseteq X_i$ of codimension ≥ 2 in U such that $\mathscr{L}'$ and $\mathrm{Der}(-\log X')$ are equal at every $q \in (\mathrm{Smooth}(X') \cap X'_i) \setminus B_i$.

Thus, $\mathscr{L}'$ and $\mathrm{Der}(-\log X')$ are equal off $\mathrm{Sing}(X') \cup Y' \cup \bigcup_{i=1}^k B_i$, which is of codimension ≥ 2. By Lemma 4.9 and Proposition 4.10(2), $R(L) = R(\mathrm{Der}(-\log X)_p)$ is the module of logarithmic 1-forms. By Proposition 4.10(3), $R(R(L)) = R(R(\mathrm{Der}(-\log X)_p)) = \mathrm{Der}(-\log X)_p$. For the final statement, use the last claim of Corollary 4.7. $\square$

Remark 5.5. Conversely, if $L = \mathrm{Der}(-\log X)_p$ for a hypersurface (X, p), then L is reflexive and the hypotheses of Theorem 5.4 are satisfied by Proposition 4.10(1) and Remark 3.9.

Example 5.6. Let $f \in \mathscr{O}_{\mathbb{C}^n,p}$ define a reduced hypersurface germ (X, p). Let $L \subseteq \mathrm{Der}(-\log X)_p$ be the module generated by the vector fields of Example 3.14. At $q \notin X$, there exist n linearly independent elements of L. At every $q \in \mathrm{Smooth}(X)$, L will satisfy, e.g., Proposition 5.1(2(d)i). Thus by Theorem 5.4, $R(L)$ is the module of logarithmic 1-forms for X, and $R(R(L)) = \mathrm{Der}(-\log X)$.

That such a generic construction works may be surprising, but the algebraic conditions for a $\varphi \in \mathrm{Hom}_{\mathscr{O}_{\mathbb{C}^n,p}}(f \cdot \mathrm{Der}_{\mathbb{C}^n,p}, \mathscr{O}_{\mathbb{C}^n,p})$ to extend (uniquely) to L, and for a corresponding $\omega \in \frac{1}{f}\Omega^1_{\mathbb{C}^n,p}$ to be logarithmic to (X, p), are the same.

5.2. **Free divisors.** A hypersurface germ (X, p) in $\mathbb{C}^n$ is called a *free divisor* if $\mathrm{Der}(-\log X)_p$ is a free module, necessarily of rank n. Theorem 5.4 implies the following result, for which the equivalence of (1) and (2) is due to Saito.

Corollary 5.7 (Saito's criterion, [Sai80, (1.8)ii])**.** *Let (X,p) be a hypersurface germ defined locally by a reduced $f \in \mathscr{O}_{\mathbb{C}^n,p}$. The following are equivalent:*

(1) (X,p) is a free divisor;
(2) there exists $\eta_1,\dots,\eta_n \in \operatorname{Der}(-\log X)_p$, such that for $L = \mathscr{O}_{\mathbb{C}^n,p}\{\eta_1,\dots,\eta_n\}$, $I_n(L) = (f)$ in $\mathscr{O}_{\mathbb{C}^n,p}$;
(3) there exists $\eta_1,\dots,\eta_n \in \operatorname{Der}(-\log X)_p$, linearly independent off X, such that for every irreducible component of (X_0,p) of (X,p), $L = \mathscr{O}_{\mathbb{C}^n,p}\{\eta_1,\dots,\eta_n\}$ satisfies one of the equivalent conditions of Proposition 5.1.

Moreover, $L = \operatorname{Der}(-\log X)_p$.

Proof. If (1), then let $\eta_1,\dots,\eta_n$ be a free basis of $\operatorname{Der}(-\log X)_p = L$. Since the vector fields are linearly independent off (X,p), linearly dependent on (X,p), and the principal ideal $I_n(L) = (g)$ must satisfy the sharpness conditions of Theorem 3.6, we have $I_n(L) = (f)$, which is (2).

If (2), then $\eta_1,\dots,\eta_n$ are linearly independent off (X,p). Since f is reduced, for every (X_0,p), L satisfies condition Proposition 5.1(1). This proves (3).

If (3), then the free (thus reflexive) module L satisfies the hypotheses of Theorem 5.4, so $L = \operatorname{Der}(-\log X)_p$. This proves (1). □

Remark 5.8. For a free divisor (X,p), the equivalent conditions of Proposition 5.1(2) or (3) are satisfied at every smooth point of X (that is, $Y = \emptyset$). This follows from the coherence of $\operatorname{Der}(-\log X)$ and condition (2c).

Remark 5.9. There is a second "Saito's criterion" (see [Sai80, (1.9)]): if L is the module generated by $\eta_1,\dots,\eta_n \in \operatorname{Der}_{\mathbb{C}^n,p}$, L is closed under the Lie bracket of vector fields, (X,p) is defined as a set by $I_n(L) = (g)$, and $g \in \mathscr{O}_{\mathbb{C}^n,p}$ is reduced, then (X,p) is a free divisor and $L = \operatorname{Der}(-\log X)_p$. To prove this, use a generalization of the Frobenius Theorem (e.g., [Nag66]) to show that $L \subseteq \operatorname{Der}(-\log X)_p$, and then apply Corollary 5.7.

By the same argument, there are versions of Theorem 5.4 and Corollary 5.7 where L is a submodule of $\operatorname{Der}_{\mathbb{C}^n,p}$ closed under the Lie bracket of vector fields, and (X,p) is the hypersurface components of the set defined by $I_n(L)$,

5.3. **Linear free divisors.** We now show that Corollary 5.7 generalizes a theorem of Brion concerning 'linear' free divisors. Here, we work in the algebraic category.

Let V be a complex vector space of dimension n and let $D \subsetneq V$ be a reduced hypersurface. We say D (or $(D,0)$) is a *linear free divisor* if $\operatorname{Der}(-\log D)_0$ has a free basis of n vector fields which are *linear*, homogeneous of degree 0 (e.g., $(3x-2y)\partial_x - z\partial_y$). By Saito's criterion, a linear free divisor in V must be defined by a homogeneous polynomial of degree n. All linear free divisors arise from a rational representation $\rho : G \to \operatorname{GL}(V)$ of a connected complex linear algebraic group G with $n = \dim(G)$, a Zariski open orbit Ω, and with $D = V \setminus \Omega$ (see [GMNRS09, §2]).

For now, let $\rho : G \to \operatorname{GL}(V)$ be a rational representation of a connected complex linear algebraic group G with Lie algebra $\mathfrak{g}$ and a Zariski open orbit Ω. Differentiating ρ gives a Lie algebra homomorphism $d\rho_{(e)} : \mathfrak{g} \to \operatorname{End}(V)$. Since V is a vector space, we can give a canonical identification $\phi_v : V \to T_vV$ for each $v \in V$, and then define a Lie algebra (anti-)homomorphism $\tau : \mathfrak{g} \to \operatorname{Der}(-\log(V \setminus \Omega))$ by $\tau(X)(v) = \phi_v\left(d\rho_{(e)}(X)(v)\right)$ (see [DP12]). Thus $\tau(\mathfrak{g})$ is a finite-dimensional Lie algebra of linear vector fields, logarithmic to $V \setminus \Omega$. For a linear free divisor D, there is a representation so that $\tau(\mathfrak{g})$ generates the module $\operatorname{Der}(-\log D)_0$.

Michel Brion used his work on log-homogeneous varieties ([Bri07]) to prove the following necessary and sufficient condition for D to be a linear free divisor.

Corollary 5.10 ([Bri06], [GMS11, Theorem 2.1])**.** *Let V be a complex vector space of dimension n, and let $D \subseteq V$ be a reduced hypersurface. Let $G \subseteq \mathrm{GL}(V)$ be the largest connected subgroup which preserves D, with Lie algebra $\mathfrak{g}$. Let $\rho : G \to \mathrm{GL}(V)$ be the inclusion map. Then the following are equivalent:*

(1) $(D, 0)$ is a linear free divisor and $\tau(\mathfrak{g})$ generates $\operatorname{Der}(-\log D)_0$;

(2) Both:

(a) $V \setminus D$ is a unique G-orbit, and the corresponding isotropy groups are finite; and

(b) The smooth part in D of each irreducible component of D is a unique G-orbit, and the corresponding isotropy groups are extensions of finite groups by the multiplicative group $\mathbb{G}_{\mathrm{m}} = (\mathbb{C} \setminus \{0\}, \cdot)$.

Our proof differs from [GMS11] by using Corollary 5.7 instead of [Bri07].

Proof. Let $L = \mathscr{O}_{\mathbb{C}^n,p} \cdot \tau(\mathfrak{g})$.

For $v \in V$, let G_v denote the isotropy subgroup at v, and let G_v^0 be the identity component of G_v. The Lie algebra $\mathfrak{g}_v$ of G_v consists of those $Y \in \mathfrak{g}$ such that $\tau(Y)$ vanishes at v, and $T_v(G \cdot v) = \langle \tau(\mathfrak{g}) \rangle_v$. Thus (2a) implies that $n = \dim(G)$; as this is also true for (1), assume $n = \dim(G)$.

Suppose that $v \in V$ has $\dim(G \cdot v) = n - 1$, and hence $\overline{G \cdot v}$ is a hypersurface defined by a reduced, irreducible $f \in \mathbb{C}[V]$. Then ρ induces a representation $\rho_v : G_v \to \mathrm{GL}(N)$ on the normal space N to $G \cdot v$ at v, and by assumption $\dim(N) = \dim(G_v) = 1$. It follows that ρ_v acts on N by multiplication by a character $G_v \to \mathbb{G}_{\mathrm{m}}$. By a Lemma in [Pik], if ρ has an open orbit then this character is the restriction of some $\chi : G \to \mathbb{G}_{\mathrm{m}}$ with $f(\rho(g)(w)) = \chi(g) \cdot f(w)$ for all $g \in G$ and $w \in V$. Setting $g = \exp(t \cdot X)$ and differentiating, we see that for $X \in \mathfrak{g}_v$,

$$\tau(X)(f) = d\chi_{(e)}(X) \cdot f, \tag{5.1}$$

where $d\chi_{(e)}(X) \in \mathbb{C}$. By (5.1), the constant function $d\chi_{(e)}(X)$ plays the role of γ in condition Proposition 5.1(2(d)i). Thus, this condition is satisfied at such a v for L if and only if $\rho_v|_{G_v^0}$ is nontrivial (and hence is an isomorphism $G_v^0 \to \mathbb{G}_{\mathrm{m}}$).

If (1), then by the above observations and Corollary 5.7, (2a) and (2b) follow readily, except that we only know the tangent spaces to the claimed orbits. Mather's Lemma on Lie Group Actions, an understanding of the connectedness of the smooth locus of a complex analytic set, and Lemma 2.4 may be combined to show the orbits are as claimed.

If (2), then $\tau(\mathfrak{g})$ is a n-dimensional vector space of vector fields with $\dim(\langle \tau(\mathfrak{g}) \rangle_v) = n$ for all $v \notin D$, and $\dim(\langle \tau(\mathfrak{g}) \rangle_v) = n - 1$ for all $v \in \mathrm{Smooth}(D)$. All that remains before applying Corollary 5.7 is to show that for $v \in \mathrm{Smooth}(D)$, $\rho_v|_{G_v^0}$ is nontrivial. Since G_v^0 is reductive by assumption, $\rho|_{G_v^0}$ decomposes as a direct sum of representations. It follows that the normal line may be realized as an actual 1-dimensional subspace W of V, complementary to $(\phi_v)^{-1}(T_v(G \cdot v)) \subseteq V$. If $\rho_v|_{G_v^0}$ is trivial, then $\rho|_{G_v^0}$ fixes all points in W and hence fixes all points in $v + W$; as $v + W$ is a line transverse to D, it intersects $V \setminus D$. As a result, a trivial $\rho_v|_{G_v^0}$ contradicts (2a). $\square$

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