CASTLING FREE DIVISORS

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Abstract. Let $\varphi : X \to S$ be a morphism between smooth complex analytic spaces, and let $f = 0$ define a free divisor on S. We prove that if the deformation space $T^1_{X/S}$ of φ is a Cohen-Macaulay $\mathcal{O}_X$–module of codimension 2, and all of the logarithmic vector fields for $f = 0$ lift via φ, then $f \circ \varphi = 0$ defines a free divisor on X.

We investigate several applications. When X is a representation of a reductive complex algebraic group G and φ is the quotient $X \to X/\!/G$ with $X/\!/G$ smooth, we describe sufficient conditions for $T^1_{X/S}$ to be Cohen–Macaulay of codimension 2. In particular, a free divisor on $\mathbb{C}^{n+1}$ lifts under the operation of "castling" to a free divisor on $\mathbb{C}^{n(n+1)}$, partially generalizing work of Granger–Mond–Schulze on linear free divisors. We give several other examples of such representations. Other applications include lifting free divisors via $\varphi : \mathbb{C}^{n+1} \to \mathbb{C}^n$ with critical set of codimension 2, and a generalization of a construction due to Buchweitz–Conca.

Contents

1. Introduction

1.1. Assume that $D \equiv \{f = 0\}$ is the germ of a free divisor in the smooth complex analytic germ $S = (\mathbb{C}^m, 0)$ and let $\varphi : X \to S$ be a holomorphic map from another smooth germ $X = (\mathbb{C}^n, 0)$ to S.

When is $E = \varphi^{-1}(D) \equiv \{f\varphi = 0\} \subseteq X$ again a free divisor?

1.2. If $\varphi : X \to S$ is any morphism of complex analytic germs, we write $\varphi^\flat : \mathcal{O}_S \to \mathcal{O}_X, f \mapsto f\varphi$ for the corresponding morphism of local analytic algebras.

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Furthermore, we throughout denote by $\mathfrak{m}_*$ the maximal ideal in $\mathcal{O}_*$ of germs of functions that vanish at 0, the distinguished point of the germ.

1.3. If $\varphi : X \to S$ is still any morphism of analytic germs and $\mathcal{M}$ an $\mathcal{O}_X$–module, we denote $T^i_{X/S}(\mathcal{M}) = H^i(\mathrm{Hom}_{\mathcal{O}_X}(\mathbb{L}_{X/S}, \mathcal{M}))$, the i^{th} tangent cohomology of X over S with values in $\mathcal{M}$. Here $\mathbb{L}_{X/S}$ is a cotangent complex for φ, well defined up to isomorphism in the derived category of coherent $\mathcal{O}_X$–modules (e.g., [GLS07, App. C]).

As usual, we abbreviate $T^i_{X/S} = T^i_{X/S}(\mathcal{O}_X)$, and write simply T^i_X if φ is (induced by) the structure map of the analytic $\mathbb{C}$–algebra $\mathcal{O}_X$.

1.4. Note that $T^0_{X/S}(\mathcal{M}) = \mathrm{Der}_S(\mathcal{O}_X, \mathcal{M})$ is the $\mathcal{O}_X$–module of $\varphi^{-1}(\mathcal{O}_S)$–linear vector fields on X with values in $\mathcal{M}$, or, shorter, the $\mathcal{O}_X$–module of *vertical vector fields* along φ with values in $\mathcal{M}$. If $\mathcal{M} = \mathcal{O}_X$, we simply speak of the module of vertical vector fields along φ.

1.5. If φ is *smooth*, then $T^i_{X/S}(\mathcal{M}) = 0$ for all $i \neq 0$, and all $\mathcal{M}$. As tangent cohomology localizes on X, the $\mathcal{O}_X$–modules $T^i_{X/S}(\mathcal{M})$, for $i > 0$, are thus (topologically) supported on the *critical locus* of φ, the (reduced) closed subgerm $C(\varphi) \subseteq X$, where φ fails to be smooth.

sit:ZJ **1.6.** If $\varphi : X \to S$ is any morphism of analytic germs, it induces the (dual) *Zariski–Jacobi sequence* in tangent cohomology, the long exact sequence of $\mathcal{O}_X$–modules

$$0 \longrightarrow T^0_{X/S} \longrightarrow T^0_X \xrightarrow{\mathrm{Jac}(\varphi)} T^0_S(\mathcal{O}_X) \xrightarrow{\delta} T^1_{X/S} \longrightarrow T^1_X \longrightarrow \cdots,$$

where $\mathrm{Jac}(\varphi)$ is the $\mathcal{O}_X$–dual to $d\varphi : \varphi^*\Omega^1_S \to \Omega^1_X$ that in turn sends $1 \otimes_{\mathcal{O}_S} ds$ to $d(s\varphi)$ for any function germ $s \in \mathcal{O}_S$.

If $x_1, ..., x_n$ are local coordinates on X and $s_1, ..., s_m$ are local coordinates on S, then a vector field

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$$Z = \sum_{i=1}^{n} g_i \frac{\partial}{\partial x_i} \in\ T^0_X\ \subseteq\ \bigoplus_{i=1}^{n} \mathcal{O}_X \frac{\partial}{\partial x_i}\,, \tag{1}$$

with coefficients $g_i \in \mathcal{O}_X$, maps to the vector field

al:vfformt0sox
$$\mathrm{Jac}(\varphi)(Z) = \sum_{j=1}^{m}\sum_{i=1}^{n} g_i \frac{\partial(s_j \circ \varphi)}{\partial x_i}\frac{\partial}{\partial s_j} \in\ T^0_S(\mathcal{O}_X)\ \subseteq\ \bigoplus_{j=1}^{m} \mathcal{O}_X \frac{\partial}{\partial s_j}\,. \tag{2}$$

1.7. Of particular importance is the $\mathcal{O}_S$–linear *Kodaira–Spencer map* defined by φ. It is the composition

$$\delta_{KS} = \delta^{\varphi}_{KS} = \delta \circ T^0_S(\varphi^\flat) : T^0_S \xrightarrow{T^0_S(\varphi^\flat)} \varphi_* T^0_S(\mathcal{O}_X) \xrightarrow{\varphi_*(\delta)} \varphi_* T^1_{X/S}\,.$$

that sends a vector field $D = \sum_{j=1}^{m} f_j \frac{\partial}{\partial s_j} \in T^0_S$ to the class

$$\delta_{KS}(D) = \delta\left(\sum_{j=1}^{m} f_j \varphi \frac{\partial}{\partial s_j}\right)$$

in $T^1_{X/S}$. Thus we have a commutative diagram

$$\begin{tikzcd} & & & T^0_S \arrow[d, "T^0_S(\varphi^\flat)"'] \arrow[dr, "\delta_{KS}"] & & & \\ 0 \arrow[r] & T^0_{X/S} \arrow[r] & T^0_X \arrow[r, "\mathrm{Jac}(\varphi)"] & T^0_S(\mathcal{O}_X) \arrow[r, "\delta"] & T^1_{X/S} \arrow[r] & T^1_X \arrow[r] & \cdots . \end{tikzcd}$$

1.8**.** The significance of the Kodaira–Spencer map is twofold: a vector field $D \in T^0_S$ is *liftable* to X, if, and only if, $\delta_{KS}(D) = 0$. Indeed, the exactness of the Zariski–Jacobi tangent cohomology sequence shows that the image $T^0_S(\varphi^\flat)(D) = \sum_{j=1}^m f_j \varphi \frac{\partial}{\partial s_j}$ in $T^0_S(\mathcal{O}_X)$ of the vector field D is induced from a vector field E on X, in that $T^0_S(\varphi^\flat)(D) = \mathrm{Jac}(\varphi)(E)$, for some E if, and only if, $\delta_{KS}(D) = 0$. One therefore calls the kernel of the Kodaira–Spencer map also the $\mathcal{O}_S$–submodule of *liftable vector fields* in T^0_S.

A deformation-theoretic interpretation is that such a lift trivializes the infinitesimal first-order deformation of X/S along D, whence we also say that X/S is (infinitesimally) *trivial along* D as soon as $\delta_{KS}(D) = 0$.

1.9**.** On the other hand, if φ is a *flat* morphism, then the Kodaira–Spencer map is *surjective*, if, and only if, φ represents a *versal deformation* of the fibre $X_0 = \varphi^{-1}(0) \subseteq X$ of φ over the origin ([Fle81]).

1.10**.** If X is smooth, then $T^1_X = 0$ and the inclusions of (1) and (2) are equalities. In particular, the dual Zariski-Jacobi sequence truncates to a resolution of $T^1_{X/S}$, with $T^0_{X/S}$ as the second syzygy module. When X and S are smooth, then in the language of the Thom–Mather theory of the singularities of differentiable maps, $T^1_{X/S}$ is isomorphic to the extended normal space of φ under right equivalence, while the cokernel of $\delta_{KS} : T^0_S \to T^1_{X/S}$ is isomorphic to the extended normal space of φ under left-right equivalence.

After this short excursion into the general theory of the (co-)tangent complex and its cohomology, we recall the pertinent facts about free divisors.

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1.11**.** Let $f \in \mathcal{O}_S$ be the germ of a function on S with zero locus the *divisor*, or *hypersurface* germ $V(f) \equiv \{f = 0\} \subseteq S$. Differentiating f yields the commutative diagram in Figure 1 of $\mathcal{O}_S$–modules with exact rows and exact columns, the rows exhibiting, one may say, defining, the *singular locus* Σ of $V(f)$ as well as the $\mathcal{O}_S$–module $\mathrm{Der}_S(-\log f)$ of *logarithmic vector fields* on S along $V(f)$, as cokernel, respectively kernel, of the $\mathcal{O}_S$–linear maps in the middle.

Here $\mathrm{Jac}(f)(D) = D(f)$ for any vector field or derivation $D \in T^0_S$, and $\mathrm{jac}(f)(D)$ is the class of $D(f)$ modulo f.

1.12. **Definition.** Recall that f defines a free divisor in S, if f is *reduced* and $\mathrm{Der}_S(-\log f)$ is a *free* $\mathcal{O}_S$–module, necessarily of rank $m = \dim S$ (see [Sai80]).

For example, the discriminants of versal deformations of isolated complete intersection singularities are free divisors. Another class of examples are the *linear free divisors* for which $\mathrm{Der}_S(-\log f)$ has a free basis of *linear* vector fields, which have degree 1 homogeneous coefficients.

We recall the basic notions of the theory.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
 & & \mathcal{O}_S \arrow[r, equal] \arrow[d, "in_2"'] & \mathcal{O}_S \arrow[d, "\cdot f"] & & \\
0 \arrow[r] & \mathrm{Der}_S(-\log f) \arrow[r, "\tilde\sigma_f"] \arrow[d, equal] & T^0_S \oplus \mathcal{O}_S \arrow[r, "(\mathrm{Jac}(f),f)"] \arrow[d, "pr_1"] & \mathcal{O}_S \arrow[r] \arrow[d] & \mathcal{O}_\Sigma \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Der}_S(-\log f) \arrow[r, "\sigma_f"] & T^0_S \arrow[r, "\mathrm{jac}(f)"] \arrow[d] & \mathcal{O}_S/(f) \arrow[r] \arrow[d] & \mathcal{O}_\Sigma \arrow[r] & 0 \\
 & & 0 & 0 & &
\end{tikzcd} \qquad (\dagger)$$

FIGURE 1. The commutative diagram exhibiting Σ and $\mathrm{Der}_S(-\log f)$, described in 1.11–1.13. fddiag

sit:saitomatrix 1.13. If $f \in \mathcal{O}_S$ defines a free divisor, $\{e_j\}_{j=1}^m$ is a choice of an $\mathcal{O}_S$–basis of $\mathrm{Der}_S(-\log f)$, and $\{\partial/\partial s_j\}_{j=1}^m$ is the canonical basis of T^0_S determined by local coordinates s_j on S (as in 1.6), then the matrix of the inclusion σ_f in Figure 1 with respect to these bases is a *Saito* or *discriminant matrix* for f.

The matrix of $\tilde\sigma_f$, where we extend the basis $\{\partial/\partial s_j\}_{j=1}^m$ of T^0_S by the canonical basis $1 \in \mathcal{O}_S$ of that free $\mathcal{O}_S$–module of rank 1, yields then the *extended* Saito or discriminant matrix for f, in that

$$\tilde\sigma_f(e_j) = \left(\sum_{i=1}^m a_{ij}\frac{\partial}{\partial s_i}, -h_j\right)$$

records that the vector field $D_j = \sum_{i=1}^m a_{ij}\frac{\partial}{\partial s_i}$ is logarithmic along f, with

$$D_j(\log f) = \frac{D_j(f)}{f} = h_j\,.$$

Moreover, the minor Δ_j obtained by removing the column corresponding to $\partial/\partial s_j$ and taking the determinant of the remaining square matrix equals $\partial f/\partial s_j$ up to multiplication by a unit, while the matrix of σ_f with respect to the chosen bases returns f times a unit.

In these terms, the vector fields $D_1, ..., D_m$ form a basis of the logarithmic vector fields as a submodule of T^0_S.

The commutative diagram in Figure 1 yields as well Aleksandrov's characterization of free divisors.

prop:aleks 1.14. **Proposition** ([Ale90])**.** *A singular hypersurface germ $V(f) \subset S$ is a free divisor, if, and only if, the singular locus Σ is Cohen–Macaulay of codimension* 2 *in S.*

Proof. Indeed, the codimension of Σ in S is at least 2 if, and only if, f is squarefree, that is, $V(f)$ is reduced. On the other hand, $\mathrm{Der}_S(-\log f)$ is free if, and only if, $\mathcal{O}_\Sigma$ is of projective dimension, and thus, of codimension at most 2. $\square$

To prepare for our main result, we record how Figure 1 behaves with respect to base change.

lem:pb 1.15. **Lemma.** *Assume $V(f) \subset S$ is a free divisor and let $\varphi : X \to S$ be a morphism from an analytic germ X to S. If X is Cohen–Macaulay and the inverse image $\varphi^{-1}(\Sigma)$ of the singular locus Σ of $V(f)$ is still*[1] *of codimension 2, then the exact row (†) in Figure 1 pulls back to an exact sequence*

$$0 \longrightarrow \varphi^* \operatorname{Der}_S(-\log f) \xrightarrow{\varphi^\flat(\tilde{\sigma}_f)} T^0_S(\mathcal{O}_X) \oplus \mathcal{O}_X \xrightarrow{\alpha} \mathcal{O}_X \longrightarrow \mathcal{O}_{\varphi^{-1}(\Sigma)} \longrightarrow 0\,,$$

where $\alpha = \varphi^\flat(\operatorname{Jac}(f), f)$.

If furthermore $f\varphi$ remains a non-zero-divisor in $\mathcal{O}_X$, then the pull back of Figure 1 by φ gives a diagram with exact rows and columns.

Proof. The first assertion follows from the Hilbert–Burch Theorem, the second is then obvious. □

As we use the convention that the empty set has all codimensions, Lemma 1.15 also applies when $\Sigma = \emptyset$, with $\mathcal{O}_{\varphi^{-1}(\Sigma)} = 0$.

2. The Main Result

Fix as before a smooth germ S and let $f \in \mathcal{O}_S$ define a free divisor $V(f) \subset S$ with singular locus $\Sigma \subset V(f)$. By our definition, f is reduced. In fact, reducedness does not matter when computing the module of logarithmic vector fields.

lemma:notreduced 2.1. **Lemma** ([HM93, p. 313], [GS06, Lemma 3.4])**.** *If $f_1, f_2 \in \mathcal{O}_X$ define the same sets in X, then $\operatorname{Der}_X(-\log f_1) = \operatorname{Der}_X(-\log f_2)$.*

Proof. Let $g \in \mathcal{O}_X$ factor into distinct irreducible components as $g = g_1^{k_1} \cdots g_\ell^{k_\ell}$. By an easy argument using the product rule and the fact that $\mathcal{O}_X$ is a unique factorization domain, $\operatorname{Der}_X(-\log g) = \cap_i \operatorname{Der}_X(-\log g_i)$. The result follows. □

We now give our main result, a sufficient condition for the reduction of $f\varphi$ to define a free divisor in X.

altmthm 2.2. **Theorem.** *Let $\varphi : X \to S$ be a morphism of smooth germs and let $f \in \mathcal{O}_S$ define a free divisor $V(f) \subset S$ with singular locus $\Sigma \subset V(f)$. Assume that $\operatorname{Image}(\varphi) \nsubseteq V(f)$, i.e., $f\varphi$ is not zero. Let g be a reduction of $f\varphi$, a reduced function defining the same set as $f\varphi$. If*

mthm.t0xsiscm (a) *$T^0_{X/S}$ is free,*

mthm.ks (b) *the Kodaira–Spencer map $\delta_{KS} : T^0_S \to T^1_{X/S}$ vanishes on $\operatorname{Der}_S(-\log f)$, that is, $\delta_{KS} \circ \sigma_f = 0$, and*

mthm.singloc (c) *the inverse image $\varphi^{-1}(\Sigma)$ of the singular locus is still of codimension 2 in X,*

then g defines a free divisor *in X and its $\mathcal{O}_X$–module of logarithmic vector fields satisfies*

al:altderxeqn (3) $$\operatorname{Der}_X(-\log f\varphi) \cong T^0_{X/S} \oplus \varphi^* \operatorname{Der}_S(-\log f)\,.$$

By our convention on the codimension of the empty set, if $\Sigma = \emptyset$ then (c) is automatic.

Proof. The three $\mathcal{O}_X$–linear maps:

[1] Note that the codimension cannot go up under pullback.

- $\alpha = \varphi^\flat(\mathrm{Jac}(f), f) = (\mathrm{Jac}(f)\varphi, f\varphi) : T^0_S(\mathcal{O}_X) \oplus \mathcal{O}_X \to \mathcal{O}_X$ as above,
- $\beta = \mathrm{Jac}(\varphi) \oplus \mathrm{id}_{\mathcal{O}_X} : T^0_X \oplus \mathcal{O}_X \to T^0_S(\mathcal{O}_X) \oplus \mathcal{O}_X$, and
- $\gamma = (\mathrm{Jac}(f\varphi), f\varphi) : T^0_X \oplus \mathcal{O}_X \to \mathcal{O}_X$

satisfy $\gamma = \alpha\beta$ and give rise to the following diagram relating kernels and cokernels of these maps, where I is the ideal generated by $f\varphi$ and its partial derivatives.

$$
\begin{tikzcd}
 & & & & 0 \arrow[d] & & 0 \\
 & & & & \varphi^* \operatorname{Der}_S(-\log f) \arrow[r, "\omega"] \arrow[d, "\varphi^\flat \tilde{\sigma}_f"] & T^1_{X/S} \arrow[ur] \arrow[dd] & \\
 & & & & T^0_S(\mathcal{O}_X) \oplus \mathcal{O}_X \arrow[ur, "\delta+0"'] \arrow[d, "\alpha"] & & \\
0 \arrow[r] & \operatorname{Der}_X(-\log f\varphi) \arrow[r, "\tilde{\sigma}_{f\varphi}"] \arrow[uurrr] & T^0_X \oplus \mathcal{O}_X \arrow[rr, "\gamma"] \arrow[urr, "\beta"] & & \mathcal{O}_X \arrow[r] \arrow[d] & \mathcal{O}_X/I \arrow[r] \arrow[dl] & 0 \\
 & T^0_{X/S} \arrow[u] \arrow[ur] & & & \mathcal{O}_{\varphi^{-1}(\Sigma)} \arrow[dl] \arrow[d] & & \\
0 \arrow[ur] & 0 \arrow[u] & & 0 & 0 & &
\end{tikzcd}
$$

The horizontal sequence involving γ is the one described in 1.11 and 1.13 that defines $\operatorname{Der}_X(-\log f\varphi)$ and the singular locus of $V(f\varphi)$ (as a scheme). It is easily verified to be exact.

The vertical sequence involving α is exact by (c) as explained in Lemma 1.15 above. As X is smooth, $T^1_X = 0$, and the diagonal exact sequence including β is the direct sum of the identity on $\mathcal{O}_X$ and (the initial segment of) the exact dual Zariski–Jacobi sequence for φ as recalled in 1.6 above.

Now observe that $\omega(1\otimes D) = \delta\circ\varphi^\flat(\tilde{\sigma}_f)(1\otimes D) = \delta_{KS}(D)$, where $D \in \operatorname{Der}_S(-\log f) \subseteq T^0_S$ and $1\otimes D \in \mathcal{O}_X \otimes_{\mathcal{O}_S} \operatorname{Der}_S(-\log f)$ is the pulled-back vector field in $\varphi^* \operatorname{Der}_S(-\log f)$. As we assume in (b) that $\delta_{KS}(D) = 0$ for each logarithmic vector field along f, the Ker–Coker exact sequence defined by $\gamma = \alpha\beta$ splits into two short exact sequences for the kernels, respectively cokernels,

$$0 \longrightarrow T^0_{X/S} \longrightarrow \operatorname{Der}_X(-\log f\varphi) \longrightarrow \varphi^* \operatorname{Der}_S(-\log f) \longrightarrow 0$$

and

$$0 \longrightarrow T^1_{X/S} \longrightarrow \mathcal{O}_X/I \longrightarrow \mathcal{O}_{\varphi^{-1}(\Sigma)} \longrightarrow 0\ .$$

Since $\operatorname{Der}_S(-\log f)$ is a free $\mathcal{O}_S$–module by assumption, and thus $\varphi^* \operatorname{Der}_S(-\log f)$ is a free $\mathcal{O}_X$–module, it follows that the first exact sequence splits, giving the decomposition of $\operatorname{Der}_X(-\log f\varphi)$ in (3). By (a) and Lemma 2.1, $\operatorname{Der}_X(-\log g) = \operatorname{Der}_X(-\log f\varphi)$ is a free $\mathcal{O}_X$–module and hence g defines a free divisor. $\square$

As condition (a) can be difficult to prove directly, almost all of our examples use the following strengthening of the hypotheses of Theorem 2.2.

mthm **2.3. Theorem.** *Let $\varphi : X \to S$ be a morphism of smooth germs and let $f \in \mathcal{O}_S$ define a free divisor $V(f) \subset S$ with singular locus $\Sigma \subset V(f)$. If (b) and*

mthm.t1xs (d) *$T^1_{X/S}$ is Cohen–Macaulay of codimension 2,*

then $f\varphi$ is reduced and defines a free divisor, and $\mathrm{Der}_X(-\log f\varphi)$ has the decomposition of (3).

Proof. We check the conditions of Theorem 2.2. Condition (b) is assumed.

(d) implies (a), that $T^0_{X/S}$ is free, as it is a second syzygy module of $T^1_{X/S}$ via the dual Zariski–Jacobi sequence for φ.

Since $T^1_{X/S}$ is supported on the critical locus $C(\varphi)$ of φ, φ is smooth off a set of codimension 2. In particular, this implies that $f\varphi$ is nonzero: if $f\varphi = 0$, so $\mathrm{Image}(\varphi) \subseteq V(f)$, then φ is nowhere smooth.

For (c), first note that the codimension of $\varphi^{-1}(\Sigma)$ is ≤ 2, as the codimension cannot go up under pullback. Let φ' and φ'' be the restriction of φ to $C(\varphi)$ and its complement in X. Then $\varphi^{-1}(\Sigma) = (\varphi')^{-1}(\Sigma) \cup (\varphi'')^{-1}(\Sigma)$, both of which have codimension ≥ 2 in X: the first is contained in $C(\varphi)$, and the second because Σ has codimension 2 in S and φ'' is smooth. Thus we have (c).

By Theorem 2.2 and its proof, $\mathrm{Der}_X(-\log f\varphi)$ is free, with the decomposition of (3) and the exact sequence

$$0 \longrightarrow T^1_{X/S} \longrightarrow \mathcal{O}_X/I \longrightarrow \mathcal{O}_{\varphi^{-1}(\Sigma)} \longrightarrow 0\ ,$$

where I is generated by $f\varphi$ and its partial derivatives. The outer terms $T^1_{X/S}$ and $\mathcal{O}_{\varphi^{-1}(\Sigma)}$ are Cohen–Macaulay $\mathcal{O}_X$–modules of codimension 2 by assumption (d) and (c), whence $\mathcal{O}_X/I$ is also a Cohen–Macaulay $\mathcal{O}_X$–module of codimension 2. Since $\mathrm{codim}(\mathcal{O}_X/I) = 2$, $f\varphi$ is reduced and hence defines a free divisor. □

3. Applications

As a first application we obtain a result originally observed by Mond and van Straten [MvS01, Remark 1.5].

fibres **3.1. Theorem.** *Let C be the germ of an isolated complete intersection curve singularity. If $\varphi : X \to S$ is any versal deformation of C, then the union of the singular fibres of φ, that is, the pullback along φ of the discriminant $\Delta \subset S$ in the base, is a free divisor.*

More generally, if $f = 0$ defines a free divisor in S that contains the discriminant as a component, then its pre-image $f \circ \varphi = 0$ defines a free divisor in X.

Proof. It is well known (see [Loo84, 6.13, 6.12]) that Δ is a free divisor in S and that $T^1_{X/S}$ is a Cohen–Macaulay $\mathcal{O}_X$–module of codimension two. Finally, in this case the kernel of the Kodaira–Spencer map $\delta_{KS} : T^0_S \to T^1_{X/S}$ consists precisely of the logarithmic vector fields along Δ (see [BEGvB09]) and so all the assumptions of Theorem 2.3 are satisfied for Δ itself and then also for any free divisor in S that contains Δ as a component. □

3.2. Remark. Note that this result can only hold for versal deformations of isolated complete intersection singularities on *curves*. Indeed, for a versal deformation of any isolated complete intersection singularities the corresponding module $T^1_{X/S}$ is Cohen–Macaulay, but of codimension equal to the dimension of the singularity plus one ([Loo84, 6.12]).

3.3**.** Our second application pertains to the classical castling of group actions. Let $G = \mathrm{SL}(n, \mathbb{C})$ act on the affine space $V = M_{n,n+1}$ of $n \times (n+1)$ matrices over $\mathbb{C}$ by left multiplication. Use coordinates $\{x_{ij} : 1 \leq i \leq n, 1 \leq j \leq n+1\}$ for V, and let Δ_i be $(-1)^i$ times the $n \times n$ minor obtained by deleting the ith column of the generic matrix (x_{ij}). The quotient space $V//G$ is then again smooth and the corresponding invariant ring $R = \mathbb{C}[V]^G$ is the polynomial ring on the $n \times n$ minors $\{\Delta_i\}_{i=1,\ldots,n+1}$. In particular, $\dim R = n+1$, and the quotient map $\varphi : V \to V//G$ is smooth outside the null cone $\varphi^{-1}(0)$ that in turn is the determinantal variety defined by the vanishing of the maximal minors of the generic matrix, thus, Cohen–Macaulay of codimension 2.

thm:maxmin

3.4. **Theorem.** *Let $f \in \mathcal{O}_S$ define a free divisor in $S = \mathbb{C}^{n+1}$ that is not suspended, equivalently* [GS06]*, $\mathrm{Der}_S(-\log f) \subseteq \mathfrak{m}_S T^0_S$. Then $f(\Delta_1, \ldots, \Delta_{n+1})$ defines a free divisor on $\mathbb{C}^{n(n+1)}$.*

Proof. Let $X = V \cong \mathbb{C}^{n(n+1)}$, $S = V//G \cong \mathbb{C}^{n+1}$ and let $\varphi : X \to S$ be the natural morphism, smooth off the codimension 2 null cone $\varphi^{-1}(0)$.

That the Kodaira–Spencer map restricted to the logarithmic vector fields along f vanishes is due to our assumption that $\mathrm{Der}_S(-\log f) \subseteq \mathfrak{m}_S T^0_S$ and the fact that we can exhibit lifts of a generating set of $\mathfrak{m}_S T^0_S$. Indeed, a computation shows that for $1 \leq p, q \leq n+1$ with $p \neq q$ and any $1 \leq r \leq n$,

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$$\begin{aligned}
\mathrm{Jac}(\varphi)\left(-\sum_{i=1}^{n} x_{iq}\frac{\partial}{\partial x_{ip}}\right) &= \Delta_p \frac{\partial}{\partial s_q} = T^0_S(\varphi^\flat)\left(s_p\frac{\partial}{\partial s_q}\right) \\
\mathrm{Jac}(\varphi)\left(\sum_{j=1}^{n+1} x_{rj}\frac{\partial}{\partial x_{rj}} - \sum_{i=1}^{n} x_{iq}\frac{\partial}{\partial x_{iq}}\right) &= \Delta_q \frac{\partial}{\partial s_q} = T^0_S(\varphi^\flat)\left(s_q\frac{\partial}{\partial s_q}\right).
\end{aligned} \tag{4}$$

(In each case, $\mathrm{Jac}(\varphi)$ applied to the sum over i gives a sum where the coefficient of $\frac{\partial}{\partial s_k}$ is of the form $\sum_{i=1}^{n} x_{iq}\frac{\partial \Delta_k}{\partial x_{ip}}$, which simplifies to $\pm\Delta_p$, $\pm\Delta_k$, or 0, depending on p, q, k. $\mathrm{Jac}(\varphi)$ applied to the sum over j gives $\sum_{k=1}^{n+1} \Delta_k \frac{\partial}{\partial s_k}$, as each minor is linear in row r. Or, see 4.11.) This shows that condition (b) of Theorem 2.3 is satisfied.

It remains to establish condition (d). This will follow from the dual Zariski–Jacobi sequence, once we show that the $\mathcal{O}_X$–module $T^0_{X/S}$ of vertical vector fields along the map φ is free. However, the Lie algebra $\mathfrak{sl}_n$ acts through derivations on $\mathcal{O}_X$, defining a $\mathcal{O}_X$–linear map $\mathfrak{sl}_n \otimes \mathcal{O}_X \to T^0_{X/S}$. This map is an isomorphism outside the null cone, as the smooth fibres there are regular orbits for the $\mathrm{SL}(n, \mathbb{C})$–action. Now both source and target of the exhibited map are reflexive $\mathcal{O}_X$–modules and the map is an isomorphism outside the null cone of codimension 2, whence it must be an isomorphism everywhere. □

3.5. **Remark.** Two types of vector fields on $M_{n,n+1}$ generate $\mathrm{Der}(-\log f\varphi)$. The first are lifts of a generating set of $\mathrm{Der}(-\log f)$, which may be found using (4). The second are the linear vector fields arising from the $\mathrm{SL}(n, \mathbb{C})$ action on $M_{n,n+1}$; these generate the module $T^0_{X/S}$ of vertical vector fields. Note that this is a minimal generating set, and that if the generators of $\mathrm{Der}(-\log f)$ are linear vector fields then $\mathrm{Der}(-\log f\varphi)$ is also generated by linear vector fields.

3.6. **Example.** The normal crossings divisor in $S = \mathbb{C}^{n+1}$ is the linear free divisor defined by $f = s_1 \cdots s_{n+1} = 0$. By Theorem 3.4, this pulls back to the linear free divisor $f\varphi = \Delta_1 \cdots \Delta_{n+1} = 0$, previously seen in [BM06, 7.4]. A generating set of $\mathrm{Der}(-\log f\varphi)$ consists of the n^2-1 vector fields arising from the $\mathrm{SL}(n,\mathbb{C})$ action on $M_{n,n+1}$, and lifts (as in (4)) of the $n+1$ generators $\left\{s_i \frac{\partial}{\partial s_i}\right\}_{i=1}^{n+1}$ of $\mathrm{Der}_S(-\log f)$.

3.7. **Example.** Let $f = 0$ be a reduced defining equation of a free surface in $\mathbb{C}^3$ which is not suspended. Pulling back f via $\varphi : M_{2,3} \to M_{1,3} \simeq \mathbb{C}^3$ produces the free divisor

$$f\left(-(x_{12}x_{23} - x_{13}x_{22}), (x_{11}x_{23} - x_{13}x_{21}), -(x_{11}x_{22} - x_{12}x_{21})\right) = 0$$

in $M_{2,3}$. For instance, $f = s_1(s_1s_3 - s_2^2)$ pulls back to the linear free divisor

$$(x_{12}x_{23} - x_{13}x_{22}) \cdot (-x_{12}x_{23}x_{11}x_{22} + x_{12}^2x_{23}x_{21} + x_{13}x_{22}^2x_{11} - x_{13}x_{22}x_{12}x_{21} + x_{11}^2x_{23}^2 - 2x_{11}x_{23}x_{13}x_{21} + x_{13}^2x_{21}^2) = 0.$$

3.8**.** The classical castling construction relates a representation ρ of a group G on $M_{n,m}$, $m < n$, to a representation ρ' of some G' on $M_{n,n-m}$, and vice versa. Then ρ has a Zariski open orbit if and only if ρ' has a Zariski open orbit, and the hypersurface component of the complement of each is defined by a homogeneous polynomial (H, respectively, H') in the respective generic maximal minors (§2.3 of [GMS11]). There is a bijection between the maximal minors of $M_{n,m}$ and $M_{n,n-m}$ defined by replacing a $m \times m$ minor Δ_I on $M_{n,m}$ with the $(n-m)\times(n-m)$ minor Δ_I' on $M_{n,n-m}$ formed by using the complementary set of rows and an appropriate sign. As polynomials in the minors, via this correspondence H and H' are the same up to multiplication by a unit.

Castling sends linear free divisors to linear free divisors by Proposition 2.10(4) of [GMS11]. For arbitrary free divisors, our Theorem 3.4 addresses the $n = m+1$ situation (in one direction), and it is reasonable to ask whether it holds more generally for arbitrary (n,m). One difficulty is that there is generally no morphism between $M_{n,m}$ and $M_{n,n-m}$ which sends Δ_I to Δ_I', or vice-versa, and hence it is unclear how to lift vector fields, or even what this means. In the classical situation, an underlying representation θ of a group H on a n-dimensional space is used in the construction of both ρ and ρ', and so gives a correspondence between the vector fields generated by the action of θ on the two spaces.

The general situation remains mysterious:

3.9. **Example.** For $(n,m) = (5,2)$, let Δ_{ij} denote the minor on $M_{5,2}$ obtained by using only rows i and j. A calculation using the software Macaulay2 or Singular shows that $\Delta_{14}\Delta_{15}(\Delta_{14}\Delta_{25} - \Delta_{15}\Delta_{24})(\Delta_{34}\Delta_{45} - \Delta_{35}^2) = 0$ defines a (non-linear) free divisor on $M_{5,2}$. Another computation shows that the corresponding divisor on $M_{5,3}$ is not free. It is unclear what additional hypotheses are necessary to generalize Theorem 3.4.

3.10**.** Note that Theorems 3.1 and 3.4 are somehow at opposite ends: in the first, the support of $T^1_{X/S}$, the critical locus of the versal deformation, is *finite* over its image, the discriminant, while in the second, the support of $T^1_{X/S}$ is collapsed by φ into a single point. Are there intermediate cases?

4. Coregular and Cofree Group Actions

We now generalize the ideas behind Theorem 3.4 to the case when $\varphi : X \to S$ is given by the quotient of X under a group action. Recall the following definitions.

4.1. **Definition.** If G is any reductive complex algebraic group, then a finite dimensional linear representation V is

(a) *coregular* if the quotient space $V//G$ is *smooth*;
(b) *cofree*, if further the natural projection $\varphi : V \to V//G$ is *flat*, equivalently (see [VP94, 8.1]), $\varphi : V \to V//G$ is coregular and *equidimensional* in that all fibres have the same dimension;
(c) *coreduced*, if the *null cone* $\varphi^{-1}(0)$ is reduced.

In algebraic terms, with $\mathbb{C}[V]$ the ring of polynomial functions, coregularity means that the ring of invariants $R = \mathbb{C}[V]^G$ is again a polynomial ring, while cofreeness means that further $\mathbb{C}[V]$ is free as an R–module. If $R = \mathbb{C}[f_1, ..., f_d]$ is the polynomial ring over the indicated invariant functions $f_j \in \mathbb{C}[V]$, then in the cofree case these functions form a regular sequence in $\mathbb{C}[V]$.

4.2. **Remark.** A famous conjecture by Popov suggests that equidimensionality of (the fibres of) the projection $\varphi : V \to V//G$ already implies coregularity and then automatically cofreeness for G connected semi-simple.

There are many examples of cofree representations, and even more that are coregular. We just mention Kempf's basic result that a representation is automatically cofree whenever $\dim V//G \leqslant 2$; see [VP94, Thm.8.6] or [Kem80]. For further lists of such representations see [Sch79, Lit89, Weh93].

4.3. **Remark.** In the case of Theorem 3.4 above, the action of $\mathrm{SL}(n, \mathbb{C})$ on $M_{n,n+1}$ is coregular, but not cofree.

4.4**.** To apply our main theorems to the quotient $X \to S$ of a coregular representation, $T^0_{X/S}$ must be free. There is a straightforward sufficient criterion for the stronger condition that $T^1_{X/S}$ is Cohen–Macaulay of codimension 2.

prop:coregulart1cm 4.5. **Proposition.** *Let $X = V$ be a coregular representation of the reductive complex algebraic group G with Lie algebra $\mathfrak{g}$ and quotient $S = V//G$. If the generic stabilizer of G on X is of dimension 0 and the natural morphism $\varphi : X \to S$ is smooth outside a set of codimension 2 in X, then*

conc:altgoxiso (i) *The natural $\mathcal{O}_X$–homomorphism $\mathfrak{g} \otimes \mathcal{O}_X \to T^0_{X/S}$ is an isomorphism;*
conc:altiscmcodim2 (ii) *$T^1_{X/S}$ is a Cohen–Macaulay $\mathcal{O}_X$–module of codimension 2*

Proof. φ is smooth outside of a set of codimension 2 in X and $T^1_{X/S}$ is supported on the critical locus of φ, so $\operatorname{codim}(\operatorname{supp} T^1_{X/S}) \geq 2$, or $\dim(T^1_{X/S}) \leq \dim(X) - 2$.

Since the generic stabilizer of G on X is of dimension zero, thus, a finite group, the $\mathcal{O}_X$–homomorphism $\rho : \mathfrak{g} \otimes \mathcal{O}_X \to T^0_{X/S}$ is an inclusion. On the set in X where φ is smooth, ρ is also locally surjective. As $\mathfrak{g} \otimes \mathcal{O}_X$ is free and $T^0_{X/S}$ is a second syzygy module (by the dual Zariski–Jacobi sequence), ρ is a homomorphism between reflexive modules which is an isomorphism off a set of codimension ≥ 2, and hence ρ is an isomorphism. This proves (i).

By (i) and the dual Zariski–Jacobi sequence, $\operatorname{projdim}_{\mathcal{O}_X} T^1_{X/S} \leq 2$. By the Auslander–Buchsbaum formula and the usual relation between depth and dimension,

$$\dim(X) - 2 \leq \operatorname{depth}(T^1_{X/S}) \leq \dim(T^1_{X/S}).$$

As $\dim(T^1_{X/S}) \leq \dim(X) - 2$, $T^1_{X/S}$ is Cohen–Macaulay of codimension 2. $\square$

There are coregular representations for which $T^0_{X/S}$ of the quotient is free, but $T^1_{X/S}$ is not Cohen–Macaulay of codimension 2 (e.g., Example 5.10). Our next result gives some insight into these cases, and also gives a necessary numerical condition for Proposition 4.5 to apply.

4.6. **Proposition.** *Let $X = V$ be a coregular representation of the reductive complex algebraic group G with Lie algebra $\mathfrak{g}$ and quotient $S = V//G$. Let $N = \dim(X)$, $d = \dim(S)$, and let $\delta_1, \ldots, \delta_d \geqslant 1$ be the degrees of the generating invariants. If the natural $\mathcal{O}_X$–homomorphism $\mathfrak{g} \otimes \mathcal{O}_X \to T^0_{X/S}$ is an isomorphism, then either*

- *$N = \sum_{\nu=1}^d \delta_\nu$ and $\dim(T^1_{X/S}) = N - 2$, or*
- *$N \neq \sum_{\nu=1}^d \delta_\nu$ and $\dim(T^1_{X/S}) = N - 1$.*

Proof. If $T^0_{X/S}$ is free and generated by the group action, then the dual Zariski–Jacobi sequence provides a graded free resolution of the graded $\mathcal{O}_X$–module $T^1_{X/S}$ of the form TODO: why right-to-left?

eqn:grres (5)

$$0 \longleftarrow T^1_{X/S} \longleftarrow \oplus_{\nu=1}^d \mathcal{O}_X(\delta_\nu) \longleftarrow \oplus_{\nu=1}^N \mathcal{O}_X(1) \longleftarrow \mathcal{O}_X^{\oplus(N-d)} \longleftarrow 0\,.$$

First, (5) implies $\operatorname{projdim}_{\mathcal{O}_X}(T^1_{X/S}) \leq 2$, and then the Auslander–Buchsbaum formula shows $\dim(T^1_{X/S}) \geq \dim(X) - 2$. Also by (5), the Hilbert–Poincaré series of $T^1_{X/S}$ satisfies

$$\begin{aligned}\mathbb{H}_{T^1_{X/S}}(t) &= \frac{1}{(1-t)^N}\left(\sum_{\nu=1}^d t^{-\delta_\nu} - Nt^{-1} + N - d\right)\\ &= \frac{\left(N - \sum_{\nu=1}^d \delta_\nu\right) + (1-t)p(t,t^{-1})}{(1-t)^{N-1}}\end{aligned}$$

for some Laurent polynomial $p(t,t^{-1}) \in \mathbb{Z}[t,t^{-1}]$. In particular, $\mathbb{H}_{T^1_{X/S}}$ has a pole at $t = 1$ of order $N - 1$ if and only if $N \neq \sum_{\nu=1}^d \delta_\nu$. Finally, the order of this pole equals $\dim(T^1_{X/S})$. $\square$

4.7. **Remark.** If we inspect the tables of cofree irreducible representations of simple groups in [VP94], we check readily that when the generic stabilizer is finite, the equation $\dim(X) = \sum_{\nu=1}^d \delta_\nu$ is satisfied. However, the tables of cofree irreducible representations of semisimple groups in [Lit89] show that this is not automatic; for instance (in the notation there), the representation $\omega_5 + \omega_1'$ of $B_5 + A_1$ has $\dim(X) = 64$ and $(\delta_i) = (2, 4, 6, 8, 8, 12)$, which falls $64 - 40 = 24$ short.

TODO: have replaced 'cofree' with 'coregular'. Is this actually useful? It's really just a restatement of the main theorem.

For coregular representations our main result takes the following form.

thm:cofree 4.8. **Theorem.** *Let $X = V$ be (the germ of) a coregular representation of G and set $S = V//G$, (the germ of) the space of closed orbits, with $\varphi : X \to S$ the associated flat morphism between smooth germs. Assume that $T^1_{X/S}$ is a Cohen–Macaulay $\mathcal{O}_X$–module of codimension 2. If $f \in \mathcal{O}_S$ defines a free divisor such that the restriction of the Kodaira–Spencer map to the logarithmic vector fields along f vanishes, then $f\varphi \in \mathcal{O}_X$ defines a free divisor in X.*

Proof. Condition (b) and (d) of Theorem 2.3 are assumed, so this follows directly from that theorem. □

4.9. To prove that vector fields lift across $\varphi : X \to X//G$, the following technique is sometimes useful.

lemma:grouplift

4.10. **Lemma.** *Let $X = V$ be a coregular representation of the algebraic group G with quotient $S = V//G$. Let ρ_X and ρ_S be representations of an algebraic group H on X, respectively, S. If $\varphi : X \to S$ is equivariant with respect to the action of H, then all vector fields on S obtained by differentiating ρ_S lift across φ.*

Proof. Differentiating gives representations $d\rho_X$ and $d\rho_S$ of $\mathfrak{h}$, the Lie algebra of H, as Lie algebras of vector fields on X, respectively, S. Since φ is equivariant, for each $Y \in \mathfrak{h}$, $d\rho_X(Y)$ is φ-related to $d\rho_S(Y)$, and hence $d\rho_S(Y)$ lifts to $d\rho_X(Y)$. □

sit:explainslvfs

4.11. **Example.** This argument may be used in the castling situation of Theorem 3.4. There, $\mathrm{GL}(n+1, \mathbb{C})$ has representations ρ_X and ρ_S on $X = M_{n,n+1}$ and $S = M_{1,n+1}$ defined by

$$\rho_X(A)(X) = XA^T \qquad \rho_S(A)(Y) = Y \operatorname{adj}(A) = Y \det(A) A^{-1},$$

where $\operatorname{adj}(A)$ is the adjugate of A. (If $M_{n,n+1} \simeq V \otimes W$, with $\dim(V) = n$, $\dim(W) = n+1$, and $M_{1,n+1} \simeq \mathbb{C} \otimes W^*$, then ρ_X is a representation on W and ρ_S is the contragredient representation of ρ_X.) A calculation shows that φ is equivariant with respect to ρ_X and ρ_S. Since $d\rho_S$ produces a generating set of $\mathrm{Der}_S(-\log\{0\})$, any $\eta \in \mathrm{Der}_S(-\log\{0\})$ will lift. (Note that the lifts in (4) have been simplified.)

4.12. We now investigate a method for determining the generating invariants of lowest degree. First we observe that if $\mathrm{Jac}(\varphi)$ was known, then it would be easy to determine a generating set of invariants.

4.13. **Proposition.** *Let $X = V$ be a coregular representation of G, and let $\varphi : X \to S = V//G$ with $N = \dim(X)$ and $d = \dim(S)$. If $E = \sum_{i=1}^N x_i \frac{\partial}{\partial x_i} \in T_X^0$ is the Euler vector field and we write*

$$\mathrm{Jac}(\varphi)(E) = \sum_{j=1}^d \tilde{f}_j \frac{\partial}{\partial s_j} \in T_S^0(\mathcal{O}_X),$$

then the coefficient functions $\tilde{f}_j$ form a generating set of the invariants in $\mathbb{C}[V]^G \subseteq \mathbb{C}[V]$.

Proof. Observe that $\varphi^\flat : \mathcal{O}_S = \mathbb{C}[s_1, \ldots, s_d] \to \mathbb{C}[f_1, \ldots, f_d] = \mathbb{C}[V]^G \subseteq \mathbb{C}[V]$ is the canonical inclusion, that is, $\varphi^\flat(s_j) = f_j$. Since each f_j is homogeneous, we have

$$\mathrm{Jac}(\varphi)(E) = \sum_{j=1}^d \left(\sum_{i=1}^N x_i \frac{\partial (s_j \circ \varphi)}{\partial x_i} \right) \frac{\partial}{\partial s_j} = \sum_{j=1}^d \deg(f_j) f_j \frac{\partial}{\partial s_j}.$$

Now each $\deg(f_j) > 0$, and we are in characteristic zero, so that the functions $\tilde{f}_j = \deg(f_j) f_j$ also form a generating set of invariants. □

4.14. We now describe a way to compute the $\mathcal{O}_X$–ideal generated by the invariants, even without knowledge of the invariants. From this ideal we may recover the invariants of lowest degree.

4.15. **Proposition.** *Let $X = V$ be a coregular representation of G with finite generic stabilizer, and let $\varphi : X \to S = X//G$, with $N = \dim(X)$. Let $f_1, \ldots, f_d$ be the generating invariants, and let $J = (f_1, \ldots, f_d)\mathcal{O}_X$. Let $K \subseteq (\mathcal{O}_X)^N$ be the $\mathcal{O}_X$–module of $b = (b_i)$ such that $\sum_{i=1}^N b_i a_i = 0$ for any linear vector field $\sum_{i=1}^N a_i \frac{\partial}{\partial x_i}$ on X arising from the action of the Lie algebra $\mathfrak{g}$ of G. Let I be the $\mathcal{O}_X$–ideal consisting of $\sum_{i=1}^N b_i x_i$, where $(b_i) \in K$. If $T^1_{X/S}$ is a Cohen–Macaulay $\mathcal{O}_X$–module of codimension 2, then $J = I$.*

Proof. For a homogeneous invariant $g \in \mathcal{O}_X$, $(\frac{\partial g}{\partial x_i}) \in K$, and hence $g \in I$. It follows that $J \subseteq I$.

Since $\dim(T^1_{X/S})$ is the dimension of the critical locus, φ is smooth off a set of codimension 2. By Proposition 4.5, $T^0_{X/S}$ is generated by the action $\theta : \mathfrak{g} \to \mathfrak{gl}(V) \cong V \otimes V^\vee$ of $\mathfrak{g}$. Tensoring with $\mathbb{C}[V] \cong \mathcal{O}_X$ gives a map $\rho : \mathfrak{g} \otimes \mathbb{C}[V] \to V \otimes V^\vee \otimes \mathbb{C}[V] \cong V \otimes \mathbb{C}[V](1) \cong T^0_X$, which may be identified with the first map in the dual Zariski–Jacobi sequence

$$0 \longrightarrow \mathfrak{g} \otimes \mathcal{O}_X \xrightarrow{\rho} T^0_X \xrightarrow{\mathrm{Jac}(\varphi)} T^0_S(\mathcal{O}_X) \longrightarrow T^1_{X/S} \longrightarrow 0.$$

Split this into short exact sequences and take $\mathcal{O}_X$–duals to get the exact sequence

xy:ses (6)
$$0 \longrightarrow N^* \xrightarrow{\psi} \Omega^1_X \xrightarrow{\rho^*} \mathfrak{g}^* \otimes \mathcal{O}_X, \tag{6}$$

where $N = \mathrm{Image}(\mathrm{Jac}(\varphi))$, ψ is the dual of $\mathrm{Jac}(\varphi)$, and $\Omega^1_X \cong (T^0_X)^* = (\Omega^1_X)^{**}$ as the smoothness of X implies the reflexivity of Ω^1_X. By this identification, the Euler derivation $E \in T^0_X$ gives a map $\tilde{E} : \Omega^1_X \to \mathcal{O}_X$ defined by $\tilde{E}\left(\sum a_i dx_i\right) = \sum a_i x_i$.

Observe that under the obvious identification of $(\mathcal{O}_X)^N$ with Ω^1_X, $K \cong \ker(\rho^*)$, and $I = \tilde{E}(\ker(\rho^*))$. Let $b \in \ker(\rho^*)$. By the exactness of (6), there exists an $n \in N^*$ such that $b = \psi(n)$. Then by the form of ψ and the homogeneity of $f_1, \ldots, f_d$, we have

$$\left(\tilde{E}(b)\right) = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix} \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_d}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n} & \cdots & \frac{\partial f_d}{\partial x_n} \end{pmatrix} \begin{pmatrix} n_1 \\ \vdots \\ n_d \end{pmatrix} \in J.$$

□

5. Examples of group actions

5.1. We now discuss a few examples of coregular and cofree group actions, and sufficient conditions for a free divisor to pull back via $\varphi : X \to S = X//G$ to another free divisor. These examples come from classifications which provide the number and degrees of the generating invariants.

To check our hypotheses for φ, however, it is necessary to choose specific generating invariants. For many of the examples below, we have used Macaulay2 to find all invariants of the given degrees[2], make a choice of generating invariants to find an explicit form for φ, compute the dimension of the critical locus of φ, and find the module of liftable vector fields. A different choice of generating invariants

[2] The vector space of degree d invariants of a linear representation of a connected group is just the space of degree d polynomials annihilated by the linear vector fields corresponding to the Lie algebra action.

gives a different presentation of $\mathbb{C}[X]^G$ as a polynomial ring, a new φ' (equal to φ composed with a diffeomorphism in S), and a different module of liftable vector fields.

Note also that there are many other examples in, e.g., [Lit89].

Special linear group.

5.2. **Example.** Let $\rho : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{GL}(V)$, $V = \mathbb{C}x \oplus \mathbb{C}y$, be the standard representation of $G = \mathrm{SL}(2,\mathbb{C})$. Differentiating this representation gives the vector fields

eqn:sl2diffops

$$d\rho(e) = x\partial_y, \quad d\rho(f) = y\partial_x, \quad d\rho(h) = x\partial_x - y\partial_y \tag{7}$$

on V, where $\mathfrak{sl}_2 = \mathbb{C}\{e, f, h\}$.

Consider the nth symmetric power $X = \mathbb{S}\mathrm{ym}^n(V)$ of ρ, where $\mathbb{S}\mathrm{ym}^n(V)$ has the $\mathbb{C}$-basis $z_i = x^{n-i}y^i$ for $i = 0, \ldots, n$. Differentiating this G–representation shows that e, f, and h act on each $x^{n-i}y^i$ by the corresponding differential operator in (7). Let $\varphi : X \to X/\!/G = S$.

For $1 \leq n \leq 4$, the resulting representation appears in the list of cofree representations of [Lit89], along with the dimension g of the generic isotropy subgroup, and the number ($= \dim(S)$) and degrees of the generating invariants. For $n = 1, 2$, Proposition 4.5 does not apply because $g = 1$.

When $n = 3$ (the "binary cubics"), $g = 0$, $\dim(S) = 1$, and the sole generating invariant is

$$f_1 = -3z_1^2z_2^2 + 4z_0z_2^3 + 4z_1^3z_3 - 6z_0z_1z_2z_3 + z_0^2z_3^2.$$

Since it is readily checked that $\varphi = (f_1)$ is smooth off a set of codimension 2, by Proposition 4.5 $T^1_{X/S}$ is Cohen-Macaulay of codimension 2. We compute that any $\eta \in \mathrm{Der}_S(-\log s_1)$ will lift, so by Theorem 4.8 f_1 itself (i.e., the lift of $\{0\} \subseteq S$) defines a free divisor. (A linear change of coordinates takes f_1 to [GMS11, 2.11(2)].)

When $n = 4$ (the "binary quartics"), $g = 0$, $\dim(S) = 2$, and the generating invariants are

$$f_1 = 3z_2^2 - 4z_1z_3 + z_0z_4 \qquad \text{and} \qquad f_2 = z_2^3 - 2z_1z_2z_3 + z_0z_3^2 + z_1^2z_4 - z_0z_2z_4.$$

$\varphi = (f_1, f_2)$ is smooth off a set of codimension 2, so by Proposition 4.5 $T^1_{X/S}$ is Cohen-Macaulay of codimension 2. The liftable vector fields are $\mathrm{Der}_S(-\log(s_1^3 - 27s_2^2))$. Since all reduced plane curve singularities are free divisors, by Theorem 4.8 any reduced plane curve which contains $s_1^3 - 27s_2^2$ as a factor lifts to a free divisor in $\mathbb{S}\mathrm{ym}^4(V)$.

5.3. **Example.** Let V be the standard representation of $G = \mathrm{SL}(3,\mathbb{C})$. Then $X = \mathbb{S}\mathrm{ym}^3(V)$, the "ternary cubics," has dimension 10, finite isotropy subgroup, $S = X/\!/G$ has dimension 2, and the invariants g_S, g_T have degree 4 and 6 (e.g., [Stu08, 4.4.7, 4.5.3]). Then $\varphi = (g_S, g_T)$ is smooth off a set of codimension 2, and the liftable vector fields are exactly $\mathrm{Der}_S(-\log(64s_1^3 - s_2^2))$. By Proposition 4.5 and Theorem 4.8, any reduced isolated plane curve singularity which contains $64s_1^3 - s_2^2$ as a factor lifts via φ to a free divisor in X.

5.4. **Example.** Let V be the standard representation of $\mathrm{SL}(2,\mathbb{C})$, and let $X = \mathbb{S}\mathrm{ym}^2(V) \otimes \mathbb{S}\mathrm{ym}^2(V)$, a representation of $G = \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$. Use the basis $y_{ij} = x_1^ix_2^{2-i} \otimes x_1^jx_2^{2-j}$, $0 \leq i, j \leq 2$, for X. By [Lit89], this cofree representation has finite generic isotropy subgroup, $S = X/\!/G$ of dimension 3, with invariants g_2, g_3, and g_4, $\deg(g_i) = i$. We compute the invariants as $g_2 = 2y_{11}^2 - 2y_{12}y_{10} + y_{20}y_{02} -$

$2y_{21}y_{01} + y_{22}y_{00}$, $g_3 = y_{20}y_{11}y_{02} - y_{21}y_{10}y_{02} - y_{20}y_{12}y_{01} + y_{22}y_{10}y_{01} + y_{21}y_{12}y_{00} - y_{22}y_{11}y_{00}$, and $g_4 = -4y_{11}^4 + 8y_{12}y_{11}^2y_{10} - 4y_{12}^2y_{10}^2 + 2y_{20}y_{12}y_{10}y_{02} - 4y_{21}y_{11}y_{10}y_{02} + 2y_{22}y_{10}^2y_{02} - (1/2)y_{20}^2y_{02}^2 - 4y_{20}y_{12}y_{11}y_{01} + 8y_{21}y_{11}^2y_{01} - 4y_{22}y_{11}y_{10}y_{01} + 2y_{21}y_{20}y_{02}y_{01} - 4y_{21}^2y_{01}^2 + 2y_{22}y_{20}y_{01}^2 + 2y_{20}y_{12}^2y_{00} - 4y_{21}y_{12}y_{11}y_{00} + 2y_{22}y_{12}y_{10}y_{00} + 2y_{21}^2y_{02}y_{00} - 3y_{22}y_{20}y_{02}y_{00} + 2y_{22}y_{21}y_{01}y_{00} - (1/2)y_{22}^2y_{00}^2$. For $\varphi = (g_2, g_3, g_4)$, φ is smooth off a set of codimension 2 and the liftable vector fields are $\mathrm{Der}(-\log\Delta)$ for the free divisor defined by $\Delta = s_1^6 - 10s_1^3s_2^2 + 4s_1^4s_3 + 27s_2^4 - 18s_1s_2^2s_3 + 5s_1^2s_3^2 + 2s_3^3$. By Proposition 4.5 and Theorem 4.8, any free divisor containing Δ as a factor lifts to a free divisor in X.

5.5. **Example.** Let V be the standard representation of $\mathrm{SL}(2,\mathbb{C})$, and let $X = \mathbb{S}\mathrm{ym}^3(V) \otimes V$, a representation of $G = \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$. On X, use the basis $z_{ij} = x_1^i x_2^{3-i} \otimes x_j$, where $0 \le i \le 3$, $1 \le j \le 2$. By [Lit89], this cofree representation has finite generic isotropy subgroup, $S = X/\!/G$ of dimension 2, with invariants g_2 and g_6, $\deg(g_i) = i$. We compute the invariants as $g_2 = 3z_{22}z_{11} - 3z_{21}z_{12} - z_{32}z_{01} + z_{31}z_{02}$ and $g_6 = 27z_{22}^3z_{11}^3 - 81z_{21}z_{22}^2z_{11}^2z_{12} + 81z_{21}^2z_{22}z_{11}z_{12}^2 - 27z_{21}^3z_{12}^3 - 27z_{32}z_{22}^2z_{11}^2z_{01} + 27z_{32}z_{21}z_{22}z_{11}z_{12}z_{01} + 27z_{31}z_{22}^2z_{11}z_{12}z_{01} + 9z_{32}^2z_{11}^2z_{12}z_{01} - 27z_{31}z_{21}z_{22}z_{12}^2z_{01} - 18z_{31}z_{32}z_{11}z_{12}^2z_{01} + 9z_{31}^2z_{12}^3z_{01} + 9z_{32}z_{21}z_{22}^2z_{01}^2 - 9z_{31}z_{22}^3z_{01}^2 + 6z_{32}^2z_{22}z_{11}z_{01}^2 - 15z_{32}^2z_{21}z_{12}z_{01}^2 + 9z_{31}z_{32}z_{22}z_{12}z_{01}^2 - (2/3)z_{32}^3z_{01}^3 + 27z_{32}z_{21}z_{22}z_{11}^2z_{02} - 9z_{32}^2z_{11}^3z_{02} - 27z_{32}z_{21}^2z_{11}z_{12}z_{02} - 27z_{31}z_{21}z_{22}z_{11}z_{12}z_{02} + 18z_{31}z_{32}z_{11}^2z_{12}z_{02} + 27z_{31}z_{21}^2z_{12}^2z_{02} - 9z_{31}^2z_{11}z_{12}^2z_{02} - 18z_{32}z_{21}^2z_{22}z_{01}z_{02} + 18z_{31}z_{21}z_{22}^2z_{01}z_{02} + 9z_{32}^2z_{21}z_{11}z_{01}z_{02} - 21z_{31}z_{32}z_{22}z_{11}z_{01}z_{02} + 21z_{31}z_{32}z_{21}z_{12}z_{01}z_{02} - 9z_{31}^2z_{22}z_{12}z_{01}z_{02} + 2z_{31}z_{32}^2z_{01}^2z_{02} + 9z_{32}z_{21}^3z_{02}^2 - 9z_{31}z_{21}^2z_{22}z_{02}^2 - 9z_{31}z_{32}z_{21}z_{11}z_{02}^2 + 15z_{31}^2z_{22}z_{11}z_{02}^2 - 6z_{31}^2z_{21}z_{12}z_{02}^2 - 2z_{31}^2z_{32}z_{01}z_{02}^2 + (2/3)z_{31}^3z_{02}^3$. Then $\varphi = (g_2, g_6)$ is smooth off a set of codimension 2, and the liftable vector fields are $\mathrm{Der}(-\log\Delta)$, for the plane curve $\Delta = (s_1^3 - s_2)(2s_1^3 - 3s_2)$. By Proposition 4.5 and Theorem 4.8, any reduced plane curve containing Δ as a factor lifts to a free divisor in X.

Special orthogonal group.

5.6. **Example.** Let V be the standard representation of $G = \mathrm{SO}(n,\mathbb{C})$. Consider the representation $\mathbb{S}\mathrm{ym}^2(V)$, which we identify with the action of G on the space $X = \mathrm{Symm}_n(\mathbb{C})$ of $n \times n$ symmetric matrices by $A \cdot M = AMA^T$. Since multiples of the identity are fixed by G, X decomposes as the direct sum of the trivial 1-dimensional representation (on $\mathbb{C} \cdot I$ for the identity I) and a representation (on the traceless matrices) which appears on the lists of [VP94] and [Lit89] of irreducible representations. As a result, we know that the generic stabilizer is finite, the generating invariants are $g_1, \ldots, g_n$, with $\deg(g_i) = i$, and $S = X/\!/G$ has dimension n.

Since G acts by conjugation, it preserves the characteristic polynomial $\det(t \cdot I - M) = t^n + h_1t^{n-1} + \cdots + h_n$ of M. When restricted to the subspace D of diagonal matrices, $h_i = (-1)^i\sigma_i$, where σ_i is the ith elementary symmetric polynomial in the diagonal entries; it follows that each $h_{k+1} \notin \mathbb{C}[h_1, \ldots, h_k]$, and hence $g_i = h_i$ are generating invariants for $i = 1, \ldots, n$. Let $\varphi = (g_n, \ldots, g_1)$; under the identification of $(s_1, \ldots, s_n) \in S$ with the monic degree n polynomial $t^n + s_nt^{n-1} + \cdots + s_1 \in \mathbb{C}[t]$, $\varphi(M) = \det(t \cdot I - M)$.

The derivative of φ may be computed at points having a symmetric "Jordan form" described in [Gan59, I, §2–3]; as each $A \in \mathrm{Symm}_n(\mathbb{C})$ is in a G-orbit of such a normal form and φ is invariant under the G action, this calculation shows that the critical locus of φ is the (codim ≥ 2) set of symmetric matrices for which the Jordan canonical form has at least two Jordan blocks with the same eigenvalue. Thus, the discriminant is the locus of monic polynomials of degree n having a repeated root, i.e., the free divisor defined by the (classical) discriminant Δ of the polynomial $t^n + \sum_{k=0}^{n-1} s_{k+1} t^k$.

Observe that $\theta = \varphi|_D = (\theta_1, \ldots, \theta_n)$ is a finite map with the same discriminant, which may be understood as the quotient $\mathbb{C}^n \to \mathbb{C}^n /\!/ S_n$ under the action of the symmetric group. Using [Zak83, §1.7], generators for $\mathrm{Der}_S(-\log \Delta)$ are of the form $\eta_k = \sum_\ell \alpha_{k\ell} \frac{\partial}{\partial s_\ell}$, where $\alpha_{k\ell} \circ \theta = (\nabla\theta_k, \nabla\theta_\ell)$, ∇ is the gradient, and $(\cdot,\cdot)$ is the dot product. Each η_k lifts across $\varphi = (\varphi_1, \ldots, \varphi_n)$ to what is almost $\nabla\varphi_k$, except with off-diagonal coefficients scaled by $\frac{1}{2}$ when the coordinates $\{x_{ij}\}_{1 \leq i \leq j \leq n}$ on $\mathrm{Symm}_n(\mathbb{C})$ are obtained by restricting the usual coordinates on $M_{n,n}$. By Proposition 4.5 and Theorem 4.8, using φ to pull back any free divisor containing Δ as a factor produces another free divisor.

For instance, when $n = 2$ we have $g_1 = -x_{11} - x_{22}$, $g_2 = x_{11}x_{22} - x_{12}^2$, and $\Delta = s_2^2 - 4s_1$. For $n = 3$, $\Delta = s_2^2 s_3^2 - 4s_1 s_3^3 - 4s_2^3 + 18 s_1 s_2 s_3 - 27 s_1^2$. For $n = 4$, $\Delta = s_2^2 s_3^2 s_4^2 - 4 s_1 s_3^3 s_4^2 - 4 s_2^3 s_4^3 + 18 s_1 s_2 s_3 s_4^3 - 27 s_1^2 s_4^4 - 4 s_2^2 s_3^3 + 16 s_1 s_3^4 + 18 s_2^3 s_3 s_4 - 80 s_1 s_2 s_3^2 s_4 - 6 s_1 s_2^2 s_4^2 + 144 s_1^2 s_3 s_4^2 - 27 s_2^4 + 144 s_1 s_2^2 s_3 - 128 s_1^2 s_3^2 - 192 s_1^2 s_2 s_4 + 256 s_1^3$.

Now let $T \subseteq \mathrm{Symm}_n(\mathbb{C})$ consist of the traceless matrices. Under φ, T maps to the subspace $Z \subset \mathbb{C}[t]$ consisting of monic polynomials of degree n with the coefficient of t^{n-1} equal to zero. Let $\varphi' : T \to Z$ be defined by restricting φ. By our calculations of $d\varphi$, it follows fairly easily that φ' is a submersion at $A \in T$ if and only if φ is a submersion at A. Accordingly, $C(\varphi') = C(\varphi) \cap T$, and the discriminant of φ' is defined by $\Delta' = \Delta|_Z$. By the same argument used for φ, φ' is smooth off a set of codimension ≥ 2. $\mathrm{Der}_S(-\log \Delta)$ contains an element of the form $\cdots + n\frac{\partial}{\partial s_n}$, so it is easy to find $\alpha_1, \ldots, \alpha_{n-1} \in \mathrm{Der}_S(-\log \Delta)$ which are also tangent to Z, and hence restrict to elements of $\mathrm{Der}_Z(-\log \Delta')$. Note that $\varphi'|_{D \cap T}$ is a finite map with the same discriminant; by [Arn76], the restrictions of $\alpha_1, \ldots, \alpha_{n-1}$ are a free basis for $\mathrm{Der}_Z(-\log \Delta')$. Since each α_i lifts via φ to a vector field which is tangent to T, the restriction of each α_i lifts via φ' to a vector field on T. Thus, by Proposition 4.5 and Theorem 4.8, using φ' to pull back any free divisor having Δ' as a factor produces another free divisor.

Orthogonal group.

5.7. Let the orthogonal group $G = \mathrm{O}(n, \mathbb{C})$ act on the space $V = M_{n,m}$ of complex $n \times m$ matrices by multiplication on the left. By [VP94, §9.3], the ring of invariants is generated by the $\binom{m+1}{2}$ inner products of pairs of columns, allowing repetition. If $\mathrm{Symm}_m(\mathbb{C})$ denotes the space of $m \times m$ complex symmetric matrices, these are equivalently the degree 2 polynomials given by the entries of $\varphi : X = M_{n,m} \to S = \mathrm{Symm}_m(\mathbb{C})$ defined by $\varphi(A) = A^T \cdot A$. By [VP94, §9.4], there are relations between these generators precisely when $n < m$.

We thus restrict ourselves to $n \geq m$, so that $V /\!/ G \cong \mathrm{Symm}_m(\mathbb{C})$ is coregular and the quotient is given by φ. We prove some basic properties.

lemma:onproperties 5.8. **Lemma.** *Let φ be as above, with $n \geq m$. Then*

cond:submersion (i) *For $A \in X$, φ is a submersion at A if and only if* $\mathrm{rank}(A) = m$.

cond:crit (ii) $C(\varphi) = \{A \in X : \mathrm{rank}(A) < m\}$ *has codimension $n - m + 1$. The discriminant $\Delta \subset \mathrm{Symm}_m(\mathbb{C})$ is the set of singular matrices.*

cond:stab (iii) *The generic stabilizer of G is trivial when $n = m$ and otherwise isomorphic to $\mathrm{O}(n - m, \mathbb{C})$.*

cond:lift (iv) *All* $\eta \in \mathrm{Der}_S(-\log \Delta)$ *lift.*

Proof. Differentiating $t \mapsto \varphi(A+tB)$ shows that $d\varphi_{(A)}(B) = A^T B + B^T A$. If $\mathrm{rank}(A) = m$ and $C \in \mathrm{Symm}_m(\mathbb{C}) \simeq T_{\varphi(A)} \mathrm{Symm}_m(\mathbb{C})$, then there exists a $K \in M(n,m,\mathbb{C})$ such that $K^T A = \frac{1}{2}C$, and so $d\varphi_{(A)}(K) = C$. If $\mathrm{rank}(A) < m$, then let v be a nonzero column vector in $\ker(A)$. For any B, $v^T(A^T B + B^T A)v = 0$, but it is easy to produce some $C \in \mathrm{Symm}_m(\mathbb{C})$ with $v^T C v \neq 0$. This proves (i).

The first part of (ii) follows from (i) and linear algebra. Since $\mathrm{rank}(\varphi(A)) \leq \mathrm{rank}(A)$, $\Delta \subseteq V(\det)$. If $B \in V(\det)$, then let $D = G^T B G$ be the diagonalization of B as the matrix of a symmetric bilinear form. If $H = G^{-1}$, then since D has diagonal entries in $\{0,1\}$, $D = D^2$ and $B = H^T D H = (DH)^T(DH)$. Appending zeros to the bottom of the $m \times m$ DH produces an $A \in X$ with $\varphi(A) = B$ and $\mathrm{rank}(A) = \mathrm{rank}(B)$. Thus $V(\det) \subseteq \Delta$.

(iii) follows from computing the stabilizer at $\begin{pmatrix} I \\ 0 \end{pmatrix} \in X$.

For (iv), observe that $\mathrm{GL}(m,\mathbb{C})$ has representations ρ_X and ρ_S on X and S defined by

$$\rho_X(A)(B) = BA^T \qquad \text{and} \qquad \rho_S(A)(C) = ACA^T.$$

Since φ is equivariant with respect to these representations, the vector fields from ρ_S lift by Lemma 4.10. A particular free $\mathcal{O}_{\mathrm{Symm}_m(\mathbb{C})}$–resolution of $\mathcal{O}_{\mathrm{Symm}_m(\mathbb{C})}/(\det)$ constructed by Józefiak may be interpreted as saying that the vector fields from ρ_S generate the module $\mathrm{Der}_S(-\log\det)$ (see [GM05, §3.2]). It follows that all elements of $\mathrm{Der}_S(-\log\det)$ are liftable. □

For the purposes of applying Proposition 4.5 and Theorem 2.3, the case where $n = m+1$ is particularly nice.

prop:on 5.9. **Proposition.** *Let $G = \mathrm{O}(m+1,\mathbb{C})$ act on $X = M_{m+1,m}$ by multiplication on the left, and let $\varphi : X \to S = X//G \cong \mathrm{Symm}_m(\mathbb{C})$ be the quotient map. If f defines a free divisor in S which contains the hypersurface of singular matrices in $\mathrm{Symm}_m(\mathbb{C})$, then $f \circ \varphi$ defines a free divisor in X.*

Proof. We check the hypotheses of Theorem 2.3.

By Lemma 5.8(ii) and (iii) $C(\varphi)$ has codimension 2 and the generic stabilizer has dimension 0. Hence by Proposition 4.5, $T^1_{X/S}$ is Cohen-Macaulay of codimension 2, giving us (d).

Since $f = 0$ contains the singular matrices, $\mathrm{Der}_S(-\log f) \subseteq \mathrm{Der}_S(-\log\det)$, and all of these vector fields lift by Lemma 5.8(iv). Thus we have (b). □

Theorem 2.2 may also produce free divisors from the square case ($n = m$), provided we can prove $T^0_{X/S}$ is free.

ex:square 5.10. **Example.** Let $G = O(m,\mathbb{C})$ act on $X = M_{m,m}$ by multiplication on the left, with coregular quotient $\varphi : X \to S \cong \mathrm{Symm}_m(\mathbb{C})$. By Lemma 5.8(ii), $\dim(T^1_{X/S}) = \dim(X) - 1$. Let $m \leq 8$, as Macaulay2 calculations then show that $T^0_{X/S}$ is free. Lemma 5.8(iv) identifies the liftable vector fields. Thus, for a free divisor in S containing the singular matrices in $\mathrm{Symm}_m(\mathbb{C})$, conditions (a) and (b) of Theorem 2.2 are satisfied, and (c) is easy to check.

For instance, use coordinates $\begin{pmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix}$ for $\mathrm{Symm}_2(\mathbb{C})$ and $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ for $M_{2,2}$. Then the free divisor defined by $f = s_{11}(s_{11}s_{22} - s_{12}^2)$ satisfies (c), and so by

Theorem 2.2 the reduction of $f\varphi = (x_{11}^2 + x_{21}^2)(x_{11}x_{22} - x_{12}x_{21})^2$ defines a free divisor in X. Indeed, a change of coordinates on X takes $\frac{f\varphi}{\det}$ to the well-known example $x_{11}x_{12}(x_{11}x_{22} - x_{12}x_{21})$. Similar examples for higher m have been shown by David Mond.

Symplectic group.

5.11**.** Let n be even and let $G = \mathrm{Sp}(n,\mathbb{C}) \subseteq \mathrm{GL}(n,\mathbb{C})$ be the symplectic group acting on the space $V = M_{n,m}$ by multiplication on the left. Let $\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, for I the identity, and let $(u,v) = u^T\Omega v$ denote the skew-symmetric bilinear form preserved by G. By [VP94, §9.3], the ring of invariants is generated by the $\binom{m}{2}$ degree 2 polynomials of the form (u,v), where u and v are columns. If $\mathrm{Sk}_m(\mathbb{C})$ denotes the space of $m \times m$ complex skew-symmetric matrices, these are equivalently the polynomials given by the entries of $\varphi : X = M_{n,m} \to S = \mathrm{Sk}_m(\mathbb{C})$ defined by $\varphi(A) = A^T\Omega A$. By [VP94, §9.4], there are relations between these generators precisely when $m \geq n+2$.

Thus assume that $1 < m \leq n+1$, so that $V/\!/G \cong \mathrm{Sk}_m(\mathbb{C})$ is coregular and the quotient is given by φ. Before we prove some basic properties, recall that the rank of any $C \in \mathrm{Sk}_m(\mathbb{C})$ is always even, and that the square of the Pfaffian $\mathrm{Pf} : \mathrm{Sk}_m(\mathbb{C}) \to \mathbb{C}$ is equal to the determinant function. We prove some basic properties.

`lemma:spproperties` 5.12. **Lemma.** *Let φ be as above, with $1 < m \leq n+1$.*

`cond:spsubmersion` (i) *For $A \in X$, φ is a submersion at A if and only if $\mathrm{rank}(A) \geq m-1$.*

`cond:spcrit` (ii) *$C(\varphi) = \{A \in X : \mathrm{rank}(A) < m-1\}$ has codimension $2(n-m+2)$. The discriminant $\Delta = \{C \in \mathrm{Sk}_m(\mathbb{C}) : \mathrm{rank}(C) < m-1\}$.*

`cond:spvfs` (iii) *$\mathrm{Der}_S(-\log\Delta)$ is generated by the linear vector fields coming from the $\mathrm{GL}(m,\mathbb{C})$ action $A \cdot C = ACA^T$.*

`cond:splift` (iv) *All $\mathrm{Der}_S(-\log\Delta)$ lift across φ.*

`cond:spstab` (v) *The dimension of the generic stabilizer is $\frac{1}{2}\left((n-m)^2 + m\right)$ when $1 \leq m \leq \frac{n}{2}$, and $\frac{1}{2}\left((n-m)^2 + n - m\right)$ when $\frac{n}{2} \leq m \leq n+1$.*

First we prove a lemma.

`lemma:rankconstruction` 5.13. **Lemma.** *Let $B \in M_{m,n}$ have rank $\geq m-1$. Then for any $C \in \mathrm{Sk}_m(\mathbb{C})$ there exists a $D \in M_{n,m}$ and $E \in \mathrm{Symm}_m(\mathbb{C})$ such that $C = BD + E$.*

Proof. The $\mathrm{rank}(B) = m$ case is clear, with $E = 0$. Let $\mathrm{rank}(B) = m-1$, and let $v \notin \mathrm{Image}(B)$. By our rank assumption, every $z \in M_{m,1}$ is the sum of an element of $\mathrm{Image}(B)$ and a multiple of v. Applying this to each column of C, there exists an $A \in M_{n,m}$ and a $w \in M_{m,1}$ such that $C = BA + vw^T$. Now write $w = Bu + \lambda v$, for $u \in M_{n,1}$ and $\lambda \in \mathbb{C}$. Then as required,

$$C = B(A - uv^T) + \left((Buv^T) + (Buv^T)^T + \lambda vv^T\right). \quad \square$$

Proof of 5.12. Differentiating $t \mapsto A + sD$, shows that $d\varphi_{(A)}(D) = A^T\Omega D + D^T\Omega A$.

If $\mathrm{rank}(A) \geq m-1$, then $\mathrm{rank}(A^T\Omega) \geq m-1$. Let $C \in \mathrm{Sk}_m(\mathbb{C})$, and by Lemma 5.13 write $\frac{1}{2}C = A^T\Omega D + E$, where $E \in \mathrm{Symm}_m(\mathbb{C})$. Then $d\varphi_{(A)}(D) = C$. If $\mathrm{rank}(A) < m-1$, then let v, w be two linearly independent vectors in $\ker(A)$. Although $v^T(d\varphi_{(A)}(D))w = 0$ for any D, there exists $C \in \mathrm{Sk}_m(\mathbb{C})$ such that $v^TCw \neq 0$: if $v = \sum v_ie_i$ and $w = \sum w_ie_i$ are expressed in terms of a basis and E_{ij} is an elementary matrix (with one nonzero entry), then $v^T(E_{ij} - E_{ij}^T)w = v_iw_j - v_jw_i$. Since v, w are linearly independent, this proves (i).

The first part of (ii) follows from (i) and linear algebra. If $A \in C(\varphi)$, then $\mathrm{rank}(A^T\Omega A) < m-1$, and hence the discriminant Δ is contained in the claimed set.

The converse will follow by showing that if $C \in \mathrm{Sk}_m(\mathbb{C})$ has rank $2r$, then there exists an $A \in M_{n,m}$ of rank $2r$ with $\varphi(A) = C$. By the standard form of skew-symmetric bilinear forms, there exists a $K \in \mathrm{GL}(m,\mathbb{C})$ such that K^TCK is block diagonal, with r blocks of the form $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and the remainder zero. There exists a permutation matrix P such that $P^T\Omega P$ is block diagonal, with $\frac{n}{2}$ copies of J. Let $B \in M_{n,m}$ be zero, except with a copy of the identity in the upper left $2r \times 2r$ submatrix (it fits when $m = n+1$ because m is then odd and hence $2r \leq n$, and also fits when $m \leq n$). A calculation shows that $B^TP^T\Omega PB = K^TCK$, and hence $\varphi(PBK^{-1}) = C$ with $\mathrm{rank}(PBK^{-1}) = 2r$.

For (iii), since this action preserves all rank varieties in S, such vector fields are in $\mathrm{Der}_S(-\log\Delta)$. When m is even, then Δ is defined by the Pfaffian Pf. By a free resolution due to Józefiak–Pragacz (see [GM05, §3.3]), the vector fields from the action generate $\mathrm{Der}_S(-\log \mathrm{Pf})$.

When m is odd, then Δ is defined by the ideal $I = (P_1, \ldots, P_m)$, where P_i is the Pfaffian after deleting row i and column i. Let $\eta' \in \mathrm{Der}_S(-\log\Delta)$, so that $\eta'(P_i) \in I$ for all i. Calculations show that for all i, j, there exists a linear vector field ξ_{ij} coming from the action such that $\xi_{ij}(P_k)$ is 0 if $i \neq k$ and P_j if $i = k$. An appropriate linear combination added to η' will produce an η which annihilates each of $P_1, \ldots, P_m$. Let $\pi : \mathrm{Sk}_{m+1}(\mathbb{C}) \to \mathrm{Sk}_m(\mathbb{C})$ be the projection which deletes the last row and column. If $\eta = \sum_{1\leq i<j\leq m} \alpha_{ij}\frac{\partial}{\partial x_{ij}}$, then let $\tilde{\eta} = \sum_{1\leq i<j\leq m} \alpha_{ij} \circ \pi \frac{\partial}{\partial x_{ij}}$ be a vector field on $\mathrm{Sk}_{m+1}(\mathbb{C})$. A calculation shows that $\tilde{\eta}$ must annihilate Pf on $\mathrm{Sk}_{m+1}(\mathbb{C})$. By the even case, $\tilde{\eta}$ may be written in terms of the linear vector fields coming from the $\mathrm{GL}(m+1,\mathbb{C})$ action on $\mathrm{Sk}_{m+1}(\mathbb{C})$. Since $\tilde{\eta}$ does not depend on the last column, the coefficients of the linear vector fields may be restricted to functions on $\mathrm{Sk}_m(\mathbb{C})$ and the corresponding linear vector fields on $\mathrm{Sk}_m(\mathbb{C})$ used, to express η in terms of the linear vector fields coming from the $\mathrm{GL}(m,\mathbb{C})$ action. This proves (iii), and (iv) follows just as for Lemma 5.8(iv).

For (v), the Lie algebra of the isotropy subgroup at $P = \begin{pmatrix} I_m \\ 0 \end{pmatrix}$ when $m \leq n$, or at $P = \begin{pmatrix} I_n & 0 \end{pmatrix}$ when $m = n+1$, is straightforward to compute by considering the cases $1 \leq m \leq \frac{n}{2}$, $\frac{n}{2} \leq m \leq n$, and $m = n+1$. □

When $m = n+1$, by Lemma 5.12, Proposition 4.5, and Theorem 2.3, we have

5.14. **Proposition.** *Let n be even. Let $G = \mathrm{Sp}(n,\mathbb{C}) \subseteq \mathrm{GL}(n,\mathbb{C})$ act on $X = M_{n,n+1}$ by multiplication on the left, and let $\varphi : X \to S/\!/G \cong \mathrm{Sk}_{n+1}(\mathbb{C})$ be the quotient map. Let Δ be as in Lemma 5.12. If f defines a free divisor in S for which $\mathrm{Der}_S(-\log f) \subseteq \mathrm{Der}_S(-\log\Delta)$, then $f \circ \varphi$ defines a free divisor in $M_{n,n+1}$.*

6. Maps $\varphi : \mathbb{C}^{n+1} \to \mathbb{C}^n$

Another class of maps for which $T^1_{X/S}$ is often Cohen–Macaulay of codimension 2 are germs $\varphi : \mathbb{C}^{n+1} \to \mathbb{C}^n$ for which the critical set has codimension 2. In fact, this is the idea behind Theorem 3.1, the versal deformations of isolated complete intersection curve singularities.

prop:nnmo 6.1. **Proposition.** *Let $X = \mathbb{C}^{n+1}$, $S = \mathbb{C}^n$, and let $\varphi : \mathbb{C}^{n+1} \to \mathbb{C}^n$ be holomorphic with critical set $C(\varphi) \subseteq \mathbb{C}^{n+1}$. If $C(\varphi)$ is nonempty and has codimension 2, then $T^1_{X/S}$ is a Cohen-Macaulay $\mathcal{O}_X$-module of codimension 2. The vertical vector fields are the $\mathcal{O}_X$-module generated by $\eta = \sum_{i=1}^{n+1} (-1)^i d_i \frac{\partial}{\partial x_i}$, where d_i is the determinant of* $\operatorname{Jac}(\varphi)$ *with column i deleted.*

Proof. [Loo84, Proposition 6.12] uses the Buchsbaum–Rim complex to prove that for $g : \mathbb{C}^n \to \mathbb{C}^p$, $n \geq p$, if $C(g)$ has dimension $p-1$ (the expected dimension), then $\operatorname{coker}(\operatorname{Jac}(g))$ is a Cohen–Macaulay $\mathcal{O}_X$–module of dimension $p-1$. Thus, $T^1_{X/S} \cong \operatorname{coker}(\operatorname{Jac}(\varphi))$ is Cohen–Macaulay of codimension 2, and the Buchsbaum–Rim complex for $\bigwedge^1 \operatorname{Jac}(\varphi) = \operatorname{Jac}(\varphi)$ is exact and of the form

xy:cnnmo
$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\cdot \epsilon \eta} (\mathcal{O}_X)^{n+1} \xrightarrow{\operatorname{Jac}(\varphi)} (\mathcal{O}_X)^n \longrightarrow T^1_{X/S} \longrightarrow 0\ , \tag{8}$$

where $\epsilon = (-1)^{\binom{n+2}{2}}$. Hence $T^0_{X/S}$ is the free module generated by η. $\square$

6.2. **Example.** Let $\varphi : \mathbb{C}^3 \to \mathbb{C}^2$ be defined by $\varphi(x_1, x_2, x_3) = (x_1^2 + x_2^3, x_2^2 + x_1 x_3)$. The critical locus $V(x_1, x_2 x_3)$ has codimension 2, and the discriminant is the plane curve defined by $\Delta = s_1^2 - s_2^3$. A Macaulay2 computation shows that the generators of $\operatorname{Der}_S(-\log \Delta)$ are exactly the liftable vector fields. By Proposition 6.1 and Theorem 2.3, we conclude that $\varphi^{-1}(\Delta)$ is a free divisor defined by

$$\Delta\varphi = x_1(-3x_2^4 x_3 - 3x_1 x_2^2 x_3^2 - x_1^2 x_3^3 + 2x_1 x_2^3 + x_1^3).$$

A generating set of $\operatorname{Der}_X(-\log \Delta\varphi)$ consists of lifts of a generating set of $\operatorname{Der}_S(-\log \Delta)$, and the vertical vector field $-3x_1 x_2^2 \frac{\partial}{\partial x_1} + 2x_1^2 \frac{\partial}{\partial x_2} - (4x_1 x_2 - 3x_2^2 x_3) \frac{\partial}{\partial x_3}$.

Since adding components to a hypersurface reduces the set of logarithmic vector fields, pulling back any isolated plane curve singularity containing $\Delta = 0$ yields a free divisor which contains $\Delta\varphi = 0$. For instance, it is easy to find a parameterized family of reduced plane curves, e.g.,

$$F_\lambda : (s_1^2 - s_2^3)(\lambda_1 s_1 + \lambda_2 s_2) = 0, \qquad (\lambda_1, \lambda_2) \neq (0, 0),$$

which then lift via φ to a parameterized family of free divisors in $\mathbb{C}^3$.

6.3. **Example.** Let $\varphi : \mathbb{C}^4 \to \mathbb{C}^3$ be defined by $\varphi(x_1, x_2, x_3, x_4) = (x_1 x_3, x_2^2 - x_3^3, x_2 x_4)$. Since $C(\varphi)$ has codimension 2, $T^1_{X/S}$ is Cohen-Macaulay of codimension 2. Although the module of liftable vector fields is not associated with a free divisor, each $s_i \frac{\partial}{\partial s_i}$, $i = 1, 2, 3$, is liftable. Hence, any free divisor in $\mathbb{C}^3$ containing the normal crossings divisor $s_1 s_2 s_3 = 0$ will lift via φ to a free divisor in $\mathbb{C}^4$.

7. Adding components and dimensions

We shall now examine a way to add components to a free divisor on $\mathbb{C}^m$ to produce a free divisor on $\mathbb{C}^m \times \mathbb{C}^n$. Use coordinates $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_n)$ on $\mathbb{C}^m$ and $\mathbb{C}^n$.

prop:ffstar 7.1. **Proposition.** *Let $I = (g_1, \ldots, g_n)$ be a $\mathcal{O}_{\mathbb{C}^m}$–ideal such that $\mathcal{O}_{\mathbb{C}^m}/I$ is Cohen-Macaulay of codimension 2. If $h \in \mathcal{O}_{\mathbb{C}^m}$ defines a free divisor on $\mathbb{C}^m$ with*

eqn:containderlog
$$\operatorname{Der}_{\mathbb{C}^m}(-\log h) \subseteq \operatorname{Der}_{\mathbb{C}^m}(-\log V(g_1, \ldots, g_n)), \tag{9}$$

then $h\cdot(\sum_{i=1}^n g_i y_i)$ *defines a free divisor on* $X=\mathbb{C}^m\times\mathbb{C}^n$.

Proof. Let $S=\mathbb{C}^m\times\mathbb{C}$ have coordinates $(z_1,\dots,z_m,t)$. Define $\varphi: X\to S$ by $\varphi(x,y)=(x,\sum_{i=1}^n g_i(x)\cdot y_i)$. Let $f(z,t)=h(z)\cdot t$ define the free divisor in S which is the "product-union" of $V(h)$ and $\{0\}\subset\mathbb{C}$. The statement will then follow from Theorem 2.3 by lifting f via φ.

To check condition (d) of the Theorem, observe that with respect to the coordinates given, the matrix form of $\mathrm{Jac}(\varphi)$ is

$$\begin{pmatrix}\mathrm{I}_{m,m} & & 0_{m,n} & \\ * & g_1 & \cdots & g_n\end{pmatrix},$$

where the subscripts on I and 0 denote the sizes of identity and zero blocks. In particular, $T^1_{X/S}\cong\operatorname{coker}\mathrm{Jac}(\varphi)$ is isomorphic to $\mathcal{O}_X/(I\otimes_{\mathcal{O}_{\mathbb{C}^m}}\mathcal{O}_X)\cong(\mathcal{O}_{\mathbb{C}^m}/I)\otimes_{\mathcal{O}_{\mathbb{C}^m}}\mathcal{O}_X$. Since $\mathcal{O}_{\mathbb{C}^m}/I$ is a Cohen-Macaulay $\mathcal{O}_{\mathbb{C}^m}$-module of codimension 2, it follows that $T^1_{X/S}$ is a Cohen-Macaulay $\mathcal{O}_X$-module of codimension 2.

For (b), $\mathrm{Der}(-\log f)$ is generated by elements of $\mathrm{Der}(-\log h)$ extended to S with 0 as the coefficient of $\frac{\partial}{\partial t}$, and $t\frac{\partial}{\partial t}$. The latter lifts:

$$\mathrm{Jac}(\varphi)\left(\sum_{i=1}^n y_i\frac{\partial}{\partial y_i}\right)=\left(\sum_{i=1}^n g_i y_i\right)\frac{\partial}{\partial t}=T^0_S(\varphi^\flat)\left(t\frac{\partial}{\partial t}\right).$$

Now, if $\eta=\sum_{i=1}^m a_i\frac{\partial}{\partial z_i}$ is logarithmic for $V(g_1,\dots,g_n)$, then there exists $\gamma_{j,k}\in\mathcal{O}_{\mathbb{C}^n}$ such that $\eta(g_j)=\sum_{k=1}^n\gamma_{j,k}\cdot g_k$ for all j. Then η lifts as well:

$$\begin{aligned}
&\mathrm{Jac}(\varphi)\left(\sum_{i=1}^m a_i\frac{\partial}{\partial x_i}-\sum_{j,k=1}^n y_j\gamma_{j,k}\frac{\partial}{\partial y_k}\right)\\
&=\sum_{i=1}^m a_i\left(\frac{\partial}{\partial z_i}+\left(\sum_{j=1}^n\frac{\partial g_j}{\partial z_i}y_j\right)\frac{\partial}{\partial t}\right)-\left(\sum_{j,k=1}^n y_j\gamma_{j,k}g_k\right)\frac{\partial}{\partial t}\\
&=\sum_{i=1}^m a_i\frac{\partial}{\partial z_i}+\left(\sum_{i=1}^m\sum_{j=1}^n a_i\frac{\partial g_j}{\partial z_i}y_j\right)\frac{\partial}{\partial t}-\left(\sum_{j=1}^n y_j\eta(g_j)\right)\frac{\partial}{\partial t}\\
&=\sum_{i=1}^m a_i\frac{\partial}{\partial z_i}+\left(\sum_{j=1}^n y_j\eta(g_j)\right)\frac{\partial}{\partial t}-\left(\sum_{j=1}^n y_j\eta(g_j)\right)\frac{\partial}{\partial t}\\
&=T^0_S(\varphi^\flat)(\eta).
\end{aligned}$$

By assumption (9), all generators of $\mathrm{Der}(-\log f)$ lift. □

7.2. **Remark.** In the proof, the vertical vector fields of φ are generated by the $\mathcal{O}_{\mathbb{C}^m}$–syzygies of $(g_1,\dots,g_n)$.

7.3. **Remark.** There is no need for $(g_1,\dots,g_n)$ to be a minimal generating set.

TODO: is this used?

rem:smoothffstar 7.4. **Remark.** The conclusion of Proposition 7.1 also holds if $I=(1)$. Then some g_i is a unit in the local ring, and so a local change of coordinates of X takes $h\cdot(\sum_{i=1}^n g_i y_i)$ to $h\cdot y_1$.

7.5**.** To find an h and I which satisfy (9), a natural approach is to use the ideal (J_h, h) defining the singular locus Σ of $V(h)$. In particular, we have the following generalization of the "ff^*" construction of Buchweitz–Conca ([BC12, Theorem 8.1]), where we have removed the hypothesis that h be weighted homogeneous.

cor:ffstar

7.6. **Corollary.** *If $h \in \mathcal{O}_{\mathbb{C}^m}$ defines a free divisor on $\mathbb{C}^m$ and $g_1, \ldots, g_n$ generate the $\mathcal{O}_{\mathbb{C}^m}$–ideal $I = (J_h, h)$, then $h \cdot (\sum_{i=1}^n g_i y_i)$ defines a free divisor on $\mathbb{C}^m \times \mathbb{C}^n$. In particular, $h \cdot \left(h y_{m+1} + \sum_{i=1}^m \frac{\partial h}{\partial x_i} y_i\right)$ always defines a free divisor on $\mathbb{C}^m \times \mathbb{C}^{m+1}$, and if $h \in J_h$ then $h \cdot \left(\sum_{i=1}^m \frac{\partial h}{\partial x_i} y_i\right)$ defines a free divisor on $\mathbb{C}^m \times \mathbb{C}^m$.*

Proof. It is enough to prove the first sentence, as the rest follows. Let Σ be the singular locus of $V(h)$, defined by I. If $V(h)$ is smooth, then $I = (1)$ and we may apply Remark 7.4. Otherwise, $\mathcal{O}_{\mathbb{C}^m}/I$ is Cohen–Macaulay of codimension 2 by Proposition 1.14. Any vector field which is logarithmic to $V(h)$ is also logarithmic to Σ. Now apply Proposition 7.1. $\square$

However, this is not the only way to find a satisfactory h and I.

7.7. **Example.** Let $M = M_{n,n}$ be the space of $n \times n$ complex matrices with coordinates $\{x_{ij}\}$, let $N = M_{n-1,n}$, and let $\pi : M \to N$ be the projection which deletes the last row. Let f define a free divisor on N for which $\mathrm{Der}_N(\log f) \subseteq D$, where $D = \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{O}_N$ is generated by the finite-dimensional Lie algebra of vector fields $\mathfrak{g}$ obtained by differentiating the representation $\rho : \mathrm{GL}(n-1, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C}) \to \mathrm{GL}(N)$ defined by $\rho(A, B)(X) = AXB^{-1}$. (For instance, f could define a linear free divisor on N obtained by restricting ρ to an appropriate subgroup.)

Now ρ leaves invariant $N_0 = \{X : \mathrm{rank}(X) < n-1\} \subset N$, and hence all elements of $\mathfrak{g}$, D, and $\mathrm{Der}_N(\log f)$ are tangent to the variety N_0. N_0 is Cohen-Macaulay of codimension 2 and defined by $I = (g_1, \ldots, g_n)$, where $g_i : N \to \mathbb{C}$ deletes column i and takes the determinant. Then I is also generated by $(-1)^{n+i} g_i$, $i = 1, \ldots, n$, and since $\sum_{i=1}^n (-1)^{n+i} g_i x_{ni} = \det$ on M, by Proposition 7.1 $(f \circ \pi) \cdot \det$ defines a free divisor on M. By the lifts in the proof and the observation that the vertical vector fields are generated by linear vector fields (e.g., by Hilbert–Burch), we see that if f defines a linear free divisor on N then $(f \circ \pi) \cdot \det$ defines a linear free divisor on M. (This linear free divisor case partially recovers [Pik10, Prop. 5.3.7].)

As a concrete example, for the linear free divisor on $M_{2,3}$ defined by

$$f = x_{11} x_{12} \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix} \begin{vmatrix} x_{12} & x_{13} \\ x_{22} & x_{23} \end{vmatrix},$$

$(f \circ \pi) \cdot \det$ defines a linear free divisor D on $M_{3,3}$, part of the "modified LU" series of [DP12, Pik10]. In fact, D may be constructed from $\{x_{11} = 0\} \subset M_{1,1}$ by repeatedly applying Proposition 7.1, as, e.g., $\mathrm{Der}_{M_{2,2}}(-\log(x_{11} x_{12}(x_{11} x_{22} - x_{12} x_{21}))) \subseteq \mathrm{Der}_{M_{2,2}}(-\log(x_{12}, x_{22}))$.

This observation likely explains why adding generic determinantal components to a free divisor often produces another free divisor.

8. Lifting Euler vector fields

Theorem 2.3 requires that all elements of $\mathrm{Der}_S(-\log f)$ lift. We now examine a way to relax this hypothesis, at least for the Euler vector field of a weighted-homogeneous free divisor f. We first examine how general Theorem 2.3 is in a well-understood situation.

ex:fdalready 8.1. **Example.** Suppose that $\varphi : X = \mathbb{C}^n \to S = \mathbb{C}$ (and hence $f \circ \varphi$ for $f = s_1$) already defines a free divisor. What does Theorem 2.3 prove?

Here, $T^1_{X/S} \cong \operatorname{coker} \operatorname{Jac}(\varphi) \cong \mathcal{O}_X/J_\varphi$, where J_φ is the Jacobian ideal of partials of φ. If $\varphi \in J_\varphi$ (equivalently, there exists an "Euler-like" vector field η such that $\eta(\varphi) = \varphi$), then $T^1_{X/S}$ is Cohen-Macaulay of codimension 2 by Proposition 1.14. The vector field $s_1 \frac{\partial}{\partial s_1}$ which generates $\operatorname{Der}_S(-\log s_1)$ lifts if and only if $\varphi \in J_\varphi$. Hence, Theorem 2.3 recovers that φ defines a free divisor exactly when $\varphi \in J_\varphi$, in which case the conclusion says that $\operatorname{Der}_X(-\log \varphi)$ is the direct sum of $\mathcal{O}_X \cdot \eta$ and the (vertical) vector fields which annihilate φ.

This deficiency is easily remedied by a standard trick, to give the following corollary of Theorem 2.3.

cor:mthm 8.2. **Corollary.** *Let $\varphi : X \to S$ be a morphism of smooth germs with module $L = \ker(\delta_{KS}) \subseteq T^0_S$ of liftable vector fields. Let $f \in \mathcal{O}_S$ define a free divisor $V(f) \subset S$ with singular locus $\Sigma \subset V(f)$. Let $(w_1, \dots, w_m)$ be a set of nonnegative integral weights for the coordinates $(s_1, \dots, s_m)$ on S. Let $E = \sum_{i=1}^m w_i s_i \frac{\partial}{\partial s_i} \in T^0_S$ be the corresponding Euler vector field, so that $T^0_S(\varphi^\flat)(E) = \sum_{i=1}^m w_i (s_i \circ \varphi) \frac{\partial}{\partial s_i} \in T^0_S(\mathcal{O}_X)$. If f is weighted homogeneous of degree d with respect to these weights,*

$$N = T^0_S(\mathcal{O}_X)/(\operatorname{Image}(\operatorname{Jac}(\varphi)) + \mathcal{O}_X \cdot T^0_S(\varphi^\flat)(E))$$

is a Cohen-Macaulay $\mathcal{O}_X$–module of codimension 2, and $\operatorname{Der}_S(-\log f) \subseteq L + \mathcal{O}_S \cdot E$, then $f \circ \varphi$ defines a free divisor.

Proof. Let t be a coordinate on $\mathbb{C}$, and let $\varphi = (\varphi_1, \dots, \varphi_m)$. Define $\theta : Y = X \times \mathbb{C} \to S$ by $\theta(x,t) = (e^{w_1 t} \cdot \varphi_1(x), \dots, e^{w_m t} \cdot \varphi_m(x))$. Since

$$\begin{aligned} \theta^\flat(f)(x,t) &= f(e^{w_1 t} \cdot \varphi_1(x), \dots, e^{w_m t} \cdot \varphi_m(x)) \\ &= e^{dt} \cdot \varphi^\flat(f)(x), \end{aligned}$$

and e^{dt} is a unit in $\mathcal{O}_Y$, applying Theorem 2.3 to lift $V(f)$ via θ will show that $V(f \circ \varphi) \times \mathbb{C}$ is a free divisor in Y. It follows that $V(f \circ \varphi)$ is a free divisor in X. It remains only to check the hypotheses of the Theorem.

A matrix representation of $\operatorname{Jac}(\theta)$ is

eqn:jactheta (10)
$$\begin{pmatrix} e^{w_1 t} \frac{\partial \varphi_1}{\partial x_1} & \cdots & e^{w_1 t} \frac{\partial \varphi_1}{\partial x_n} & w_1 e^{w_1 t} \varphi_1 \\ \vdots & \ddots & \vdots & \vdots \\ e^{w_m t} \frac{\partial \varphi_m}{\partial x_1} & \cdots & e^{w_m t} \frac{\partial \varphi_m}{\partial x_n} & w_m e^{w_m t} \varphi_m \end{pmatrix}, \tag{10}$$

with values in $T^0_S(\mathcal{O}_Y)$. The isomorphism $\psi : T^0_S(\mathcal{O}_Y) \to T^0_S(\mathcal{O}_Y)$ with $\psi(\frac{\partial}{\partial s_i}) = e^{-w_i t} \frac{\partial}{\partial s_i}$ shows that deleting the exponential coefficients in (10) gives an isomorphic cokernel. Thus, $T^1_{Y/S} \cong \operatorname{coker} \operatorname{Jac}(\theta)$ is isomorphic to $N \otimes_{\mathcal{O}_X} \mathcal{O}_Y$, and hence a Cohen–Macaulay $\mathcal{O}_Y$–module of codimension 2. This shows (d).

Now let $\eta = \sum_{i=1}^m a_i \frac{\partial}{\partial s_i} \in T^0_S$ be homogeneous of degree λ, i.e., $\lambda = \deg(a_i) - w_i$ for $i = 1, \dots, m$. Suppose that η lifts under φ to some $\xi = \sum_{j=1}^n b_j \frac{\partial}{\partial x_j} \in T^0_X$, so that $a_i \circ \varphi = \sum_{j=1}^n b_j \frac{\partial \varphi_i}{\partial x_j}$ for $i = 1, \dots, m$. Let $\xi' \in T^0_Y$ have the same defining equation. Then

$$\begin{aligned}
\operatorname{Jac}(\theta)\left(e^{\lambda t}\cdot\xi'\right) &= \sum_{i=1}^{m} e^{(\lambda+w_i)t}\left(\sum_{j=1}^{n} b_j\frac{\partial\varphi_i}{\partial x_j}\right)\frac{\partial}{\partial s_i}\\
&= \sum_{i=1}^{m} e^{\deg(a_i)t}\cdot(a_i\circ\varphi)\frac{\partial}{\partial s_i}\\
&= \sum_{i=1}^{m} a_i\circ\left(e^{w_1 t}\cdot\varphi_1,\ldots,e^{w_m t}\cdot\varphi_m\right)\frac{\partial}{\partial s_i}\\
&= T^0_S(\theta^\flat)(\eta).
\end{aligned}$$

Thus, homogeneous elements of L lift via θ. E also lifts, as $\operatorname{Jac}(\theta)(\frac{\partial}{\partial t}) = T^0_S(\theta^\flat)(E)$. It follows that the module generated by homogeneous elements of $L+\mathcal{O}_S E$ lift via θ; since f is weighted homogeneous, $\operatorname{Der}_S(-\log f)$ has a homogeneous generating set, and hence elements of $\operatorname{Der}_S(-\log f)$ lift via θ, proving (b). □

8.3**.** This corollary may be applied to maps between spaces of the same dimension, and may create free divisors without an Euler–like vector field.

8.4. **Example.** Let $\varphi : X = \mathbb{C}^3 \to S = \mathbb{C}^2$ be defined by $\varphi(x_1,x_2,x_3) = (x_1^2 + x_2^3, x_2^2+x_1x_3)$, and let $f = s_1s_2(s_1+s_2)$. Let L be the module of vector fields liftable by φ. Although $T^1_{X/S}$ is Cohen-Macaulay of codimension 2, $\operatorname{Der}_S(-\log f) \nsubseteq L$. However, $\operatorname{Der}_S(-\log f) \subseteq L + \mathcal{O}_S\cdot E$, and the module N associated to φ is also Cohen-Macaulay of codimension 2. By Corollary 8.2, $f\circ\varphi$ defines a free divisor; it has no Euler-like vector field.

8.5. **Example.** Let $\varphi : X = \mathbb{C}^3 \to S = \mathbb{C}^3$ be defined by $\varphi(x_1,x_2,x_3) = (x_1x_3 + x_2^2, x_2, x_3)$. Although $T^1_{X/S}$ is not, the module N is Cohen-Macaulay of codimension 2. As $L+\mathcal{O}_S E$ contains $\operatorname{Der}_S(-\log f)$ for, e.g., $f = s_1s_2s_3$ or $f = s_1s_3(s_1s_3 - s_2^2)$, by Corollary 8.2 each such $f\varphi$ defines a free divisor in X.

8.6. **Remark.** If f is multi-weighted homogeneous, that is, weighted homogeneous of degree d_k with respect to weights $(w_{1k},\ldots,w_{mk})$ for $k=1,\ldots,p$ (or, $f=0$ is invariant under the action of an algebraic p-torus), then a version of Corollary 8.2 holds, with E replaced by the p Euler vector fields. To adapt the proof, let $\theta : X\times\mathbb{C}^p \to S$ be defined by

$$\theta(x,t) = \left(e^{\sum_{k=1}^{p} w_{1k}t_k}\cdot\varphi_1(x),\cdots,e^{\sum_{k=1}^{p} w_{mk}t_k}\cdot\varphi_m(x)\right),$$

for $\varphi = (\varphi_1,\ldots,\varphi_m)$, and assume that η is multi-weighted homogeneous.

For instance, if $f = s_1\cdots s_m$ is the normal crossings divisor in $S=\mathbb{C}^m$ with m weightings of the form $(0,\cdots,1,\cdots,0)$, then N is the cokernel of

$$A = \begin{pmatrix}
\frac{\partial\varphi_1}{\partial x_1} & \cdots & \frac{\partial\varphi_1}{\partial x_n} & \varphi_1 & 0 & \cdots & 0\\
\frac{\partial\varphi_2}{\partial x_1} & \cdots & \frac{\partial\varphi_2}{\partial x_n} & 0 & \varphi_2 & \cdots & 0\\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
\frac{\partial\varphi_m}{\partial x_1} & \cdots & \frac{\partial\varphi_m}{\partial x_n} & 0 & 0 & \cdots & \varphi_m
\end{pmatrix}.$$

When each φ_i is nonzero, then $\ker(A) \cong \cap_i \operatorname{Der}_X(-\log\varphi_i) = \operatorname{Der}_X(-\log\varphi_1\cdots\varphi_m) = \operatorname{Der}_X(-\log\varphi^\flat(f))$; the conditions on N ensure that $\ker(A)$ is free, and $\varphi_1\cdots\varphi_m$ is reduced and nonzero.

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