Markov Chains

Introduction: We suppose that whenever the process is in state i, there is a fixed probability P_{ij} that it will next be in state j. That is,

$$P(X_{n+1}=j \mid X_n=i, X_{n-1}=i_{n-1},\cdots,X_1=i_1,X_0=i_0\}=P_{ij}$$

for all states $i_0,i_1,\cdots,i_{n-1},i,j$ and all $n\geq 0$. Such a stochastic process is known as a *Markov chain.*

In Simple words, if the next state X_{n+1} depends only on current state X_n and NOT on any other prior state (i.e., X_{n+1} is NOT dependant on X_{n-1}, X_{n-2}, ... X_0) then the process can be modelled as a Markov Chain

The value P_{ij} represents the probability that the process will, when in state i, next make a transition into state j. Since probabilities are non-negative and since the process must make a transition into some state, we have that

$$P_{ij}\geq 0,\quad i,j\geq 0;\quad \sum_{j=0}^{\infty}P_{ij}=1,\quad i=0,1,\cdots$$

Let P denote the matrix of one-step transition probabilities P_{ij}, so that

Transition Probability Matrix

$$P=\begin{Vmatrix} P_{00} & P_{01} & P_{02}\cdots \\ P_{10} & P_{11} & P_{12}\cdots \\ \vdots & \vdots & \vdots \\ P_{i0} & P_{i1} & P_{i2}\cdots \\ \vdots & \vdots & \vdots \end{Vmatrix}$$

Example 1: Suppose that the chance of rain tomorrow depends on previous conditions only through whether or not it is raining today and not on past weather conditions. Suppose also that if it rains today, then it will rain tomorrow with probability α; and if it does not rain today, then it will rain tomorrow with probability β.

We assume that, the process has two states.
State 0: It rains
State 1: It does not rain
The above is a two-state Markov chain whose transition probabilities are given below

$$P=\begin{Vmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{Vmatrix}$$

Example 2: Consider a communication system which transmits the digit 0 and 1. Each digit transmitted must pass through several stages, at each of which there is a probability p that the digit entered will be unchanged when it leaves. Letting X_n denote the digit entering the n-th stage, then $\{X_n, n=0,1,\cdots\}$ is a two stage Markov chain having a transition probability matrix

$$P=\begin{Vmatrix} p & 1-p \\ 1-p & p \end{Vmatrix}$$

P00 = Bit entered stage as 0 and left stage as 0 (no error)
P01 = Bit entered stage as 0 and left stage as 1 (error!!!)
P11 = Bit entered stage as 1 and left stage as 1 (no error)
P10 = Bit entered stage as 1 and left stage as 0 (error!!!)

Example 3: On any given day Gary is either cheerful (C), so-so (S) or glum (G). If he is cheerful today, then he will be C, S, or G tomorrow with respective probabilities 0.5, 0.4, 0.1. If he is felling so-so today, then he will be C, S or G tomorrow with probabilities 0.3, 0.4, 0.3 respectively. If he is glum today, then he will be C, S or G tomorrow with probabilities 0.2, 0.3, 0.5.

Let, X_n denote Gary's mood on the n-th day, then $\{X_n, n \geq 0\}$ is a three state Markov chain.

(Also, Build the matrix if data given in this manner (columnwise): If he is cheerful, so-so or gloom today, then tomorrow he will be cheerful with probability 0.5, 0.3, 0.2, so-so with probability 0.4, 0.4, 0.3 and gloom with probability 0.1, 0.3, 0.5, respectively.)

State 0: Cheerful (C)
State 1: So-so (S)
State 2: Glum (G)

The transition probability matrix for this three state Markov chain is given below

$$P = \begin{Vmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{Vmatrix}$$

In each row, one value can be missing, which you can easily compute (because row sum always = 1.0 column sum NOT necessary to be 1.0)

Transforming Process into Markov Chain: Suppose that, whether or not, it rains today depends on previous weather conditions through the last two days. Specially suppose that if it has rained for the past two days, then it will rain tomorrow with probability 0.7; if it rained today but not yesterday, then it will rain tomorrow with probability 0.5; if it rained yesterday but not today, then it will rain tomorrow with probability 0.4; if it has not rained in the past two days, then it will rain tomorrow with probability 0.2.

Note: here 'not' add korle matrix er value gulo different hoye jabe!!!

If we let the state at time n depend only on whether or not it is raining at time n, then the above model is not a Markov chain. However, we can transform the above model into a Markov chain by saying that the state at any time is determined by the weather conditions during both that day and the previous day. That means, we can say that the process is in

State 0: if it rained both today and yesterday
State 1: if it rained today but not yesterday
State 2: if it rained yesterday but not today
State 3: if it did not rain either yesterday or today.

The preceding would then represent a four-state Markov chain having a transition probability matrix

Today's Today is Tomorrow's Yesterday
Today's "Today" is Tomorrow's "Yesterday"

y = yesterday
T = Today
t = Tomorrow

$$P = \begin{Vmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{Vmatrix}$$

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P01 ==>(yT=0) => (Tt=1)

Random Walk Model: A Markov chain whose state space is given by integers $i = 0, \pm 1, \pm 2, \cdots$ is said to be a random walk if, for some number $0 < p < 1$,

$$P_{i,i+1} = p = 1 - P_{i,i-1}, \qquad i = 0, \pm 1, \pm 2, \cdots$$

The preceding Markov chain is called a random walk for we may think of it as being a model for an individual walking on a straight line who at each point of time either takes one step to the right with probability p or one step to the left with probability $1 - p$.

A Gambling Model: Consider a gambler who, at each play of the game, either wins \$1 with probability p or loses \$1 with probability $1 - p$. If we suppose that our gambler quits playing either when he goes broke or he attains a fortune of \$*N*, then the gambler's fortune is a Markov chain having transition probabilities

$$P_{i,i+1} = p = 1 - P_{i,i-1}, \qquad i = 1, 2, \cdots, N-1$$

$$P_{0,0} = P_{N,N} = 1$$

Sate 0 and *N* are called *absorbing* state since once entered they are never left. Note that the above is a finite state random walk with absorbing barriers (Sate 0 and *N*).

Chapman-Kolmogorov Equation: We have already defined one-step transition probabilities P_{ij}. We now define the *n*-step transition probabilities P_{ij}^n to be the probability that a process in state *i* will be in state *j* after n additional transitions. That is,

$$P_{ij}^n = P\{X_{n+m} = j \mid X_m = i\}, \qquad n \ge 0, i, j \ge 0$$

The *Chapman-Kolmogorov equations* provide a method for computing these n-step transition probabilities. These equations are

$$\boxed{P_{ij}^{n+m} = \sum_{k=0}^{\infty} P_{ik}^n P_{kj}^m \qquad \text{for all } n, m \ge 0, \text{ all } i, j}$$

n-step transition matrix may be obtained by multiplying the matrix P by itself *n* times.

Example 4: Consider Example 1, in which the weather is considered as a two-state Markov chain. If $\alpha = 0.7$ and $\beta = 0.4$, then calculate the probability that it will rain four days from today given that it is raining today.

Solution: The one-step transition probability matrix is given by

$$P = \begin{Vmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{Vmatrix}$$

Hence,

$$P^2 = P \cdot P = \begin{Vmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{Vmatrix} \cdot \begin{Vmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{Vmatrix}$$

$$= \begin{Vmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{Vmatrix}$$

$$P^4 = P^2 \cdot P^2 = \begin{Vmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{Vmatrix} \cdot \begin{Vmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{Vmatrix}$$
$$= \begin{Vmatrix} 0.5749 & 0.4251 \\ 0.5668 & 0.4332 \end{Vmatrix}$$

It is NOT (P00)^4, Rather it is (P^4)00
So, Find the P^4 Matrix at first

and the desired probability P_{00}^4 equals 0.5749. (Ans.)

Probability of Rain after four days given today Not Raining = 0.5668
Probability of No Rain after four days given today Raining = 0.4251 ***Probability of NO Rain after four days given today Not Raining = 0.4332***

Example 5: Consider Example in Page 2. Given that it rained on Monday and Tuesday, what is the probability that it will rain on Thursday? **What is probability it will Rain on Thu, but not on Wed? (Ans: P^2 Matrix er value of index 01) OR, (P^2)01 = 0.12**

Solution: The two-step transition matrix is given by

$$P^2 = P \cdot P = \begin{Vmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{Vmatrix} \cdot \begin{Vmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{Vmatrix}$$
$$= \begin{Vmatrix} 0.49 & 0.12 & 0.21 & 0.18 \\ 0.35 & 0.20 & 0.15 & 0.30 \\ 0.20 & 0.12 & 0.20 & 0.48 \\ 0.10 & 0.16 & 0.10 & 0.64 \end{Vmatrix}$$

(P^2)ij
j: State 0 = Rained Today & Yesterday
State 1= Rained Today, but NOT yesterday
State 0 = Rained THU & Wed
State 1 = Rained THU, but Not Wed

Since rain on Thursday is equivalent to the process being in either state 0 or sate 1 on Thursday, the desired probability is given by $P_{00}^2 + P_{01}^2 = 0.49 + 0.12 = 0.61$ (Ans.)

(it does NOT matter whether it Rained or Not at the intermediate day = Wednesday)

☺Good Luck☺

What is the Probability that it Won't Rain of Thursday given it Rained on Mon & Tuesday?
Ans: (P^2)02 + (P^2)03

What is the Probability that it will Rain of Thursday given it Rained Neither on Monday Nor onTuesday?
Ans: (P^2)30 + (P^2)31

What is the Probability that GARY will be cheerful after FOUR days given that he is Cheerful today?
Ans: (P^4)00

What is the Probability that GARY will be cheerful after FOUR days given that he is Gloom today?
Ans: (P^4)20

What is the Probability that GARY will NOT be cheerful after FOUR days given that he is Gloom today?

Ans: 1 - (P^4)20 OR { (P^4)21 + (P^4)22 }

Suppose:

Yesterday: Sunday (S)

Today: Monday (M)

Tomorrow: Tuesday (T)

State 0: if it rained both today and yesterday
State 1: if it rained today but not yesterday
State 2: if it rained yesterday but not today
State 3: if it did not rain either yesterday or today.

~~$P_{00} \Rightarrow$ S M~~
yesten

✓	⇒	means rainy day
X	⇒	means no rain

$P_{00} \Rightarrow$ S✓ (yesterday) M✓ (~~yesten~~ Today) — state 0 $\Rightarrow$ M✓ (yesterday) T✓ (Today) — state 0 $\Rightarrow$ **it is** possible. probability = 0·7

Yesterday/Monday state = 0
(means Rain on Mon & Sun)
Today/Tuesday state = 0
(means Rain on Tue & Mon)

$P_{01} \Rightarrow$ S✓ (Yesterday) M✓ (Today) — state 0 $\Rightarrow$ M✗ (yesterday) T✓ (Today) — state 1 $\Rightarrow$ Not Possible because given M✓ conflict M✗

Yesterday/Monday state = 0
(means Rain on Mon & Sun)
Today/Tuesday's state = 1
(means Rain on Tue,
but NO rain on Mon)

~~$P_{10} \Rightarrow$ S✗ M✓ $\Rightarrow$ M✓ T✓ $\Rightarrow$ Possible~~
~~state 1~~ ~~state 0~~

$P_{02} \Rightarrow$ S✓ M✓ — state 0 $\Rightarrow$ M✓ T✗ — state 2 — possible probability = 1−0·7 = 0·3

Yesterday/Monday state = 0
(means Rain on Mon & Sun)
Today/Tuesday state = 2
(means NO Rain on Tue, but Rain on Mon)

$P_{03} \Rightarrow$ S✓ M✓ — state 0 $\Rightarrow$ M✗ T✗ — state 3 ⟶ not possible given M✓ conflict M✗

Yester./Mon=0 (Rain on Mon+Sun)
Today/Tues=3 (NO Rain Tue+Mon)