MECHANICAL VIBRATIONS

Subject Code : 10ME72 **IA Marks : 25** **Hours/Week : 04**

Exam Hours : 03 **Total Hours : 52** **Exam Marks : 100**

PART- A

UNIT - 1

Introduction: Types of vibrations, Definitions, Simple Harmonic Motion (S.H.M.), Work done by harmonic force, Principle of super position applied to SHM, Beats, Fourier theorem and problems. **06 Hours**

UNIT -2

Undamped (Single Degree of Freedom) Free Vibrations: Derivations for spring mass systems, Methods of Analysis, Natural frequencies of simple systems, Springs in series and parallel, Torsional and transverse vibrations, Effect of mass of spring and Problems.

07 Hours

UNIT - 3

Damped free vibrations (1DOF): Types of damping**,** Analysis with viscous damping - Derivations for over, critical and under damped systems, Logarithmic decrement and Problems. **06 Hours**

UNIT - 4

Forced Vibrations (1DOF): Introduction**,** Analysis of forced vibration with constant harmonic excitation - magnification factor, rotating and reciprocating unbalances, excitation of support (relative & absolute amplitudes), force and motion Transmissibility, Energy dissipated due to damping and Problems. **07 Hours**

PART – B

UNIT – 5

Vibration Measuring Instruments and Whirling of shafts: Seismic Instruments – Vibrometers, Accelerometer, Frequency measuring instruments and

Problems. Whirling of shafts with and without damping, discussion of speeds above and below critical speeds and Problems. **06 Hours**

UNIT – 6

Systems with two degrees of Freedom: Principle modes of vibrations, Normal mode and natural frequencies of systems (without damping) – Simple spring mass systems, masses on tightly stretched strings, double pendulum, torsional systems, combined rectilinear and angular systems, geared systems and Problems. Undamped dynamic vibration absorber and Problems. **06 Hours**

UNIT - 7

Numerical Methods for multi degree freedom of systems: Introduction, Maxwell‟ s reciprocal theorem, Influence coefficients, Rayleigh‟ s method, Dunkerley‟ s method, Stodola method, Holzer‟ s method, Orthogonality of principal modes, method of matrix iteration and Problems. **09 Hours**

UNIT – 8

Modal analysis and Condition Monitoring: Signal analysis, dynamic testing of machines and structures, Experimental modal analysis, Machine condition monitoring and diagnosis**. 05 Hours**

TEXT BOOKS:

1. **Mechanical Vibrations,** S. S. Rao, Pearson Education Inc, 4th edition, 2003.

2. **Mechanical Vibrations,** V. P. Singh, Dhanpat Rai & Company, 3rd edition, 2006.

REFERENCE BOOKS:

1. **Theory of Vibration with Applications,** W. T. Thomson, M. D. Dahleh and C. Padmanabhan, Pearson Education Inc, 5th edition, 2008.

2. Mechanical Vibrations: S. Graham Kelly, Schaum‟ s outline Series, Tata McGraw Hill, Special Indian Edition, 2007.

3. Theory and Practice of Mechanical Vibrations: J. S. Rao & K. Gupta, New Age International Publications, New Delhi, 2001.

4. **Mechanical Vibrations,** G. K.Grover, Nem Chand and Bros, 6th edition, 1996.

TABLE OF CONTENTS

UNIT-1

INTRODUCTION

1.1 The study of vibration

A body is said to vibrate if it has periodic motion. Mechanical vibration is the study of oscillatory motions of bodies. Vibrations are harmful for engineering systems. Some

times vibrations can be useful. For example, vibratory compactors are used for compacting concrete during construction work. Excessive vibration causes discomfort to human beings, damage to machines and buildings and wear of machine parts such as bearings and gears. The study of vibrations is important to aeronautical, mechanical and civil engineers. It is necessary for a design engineer to have a sound knowledge of vibrations. The object of the sixth semester course on mechanical vibrations is to discuss the basic concepts of vibration with their applications. The syllabus covers fundamentals of vibration, undamped and damped single degree of freedom systems, multidegrees of freedom systems and continuous systems.

1.2 Examples of vibration

1.Beating of heart

2. Lungs oscillate in the process of breathing
3. Walking- Oscillation of legs and hands
4. Shivering- Oscillation of body in extreme cold
5. Speaking - Ear receives Vibrations to transmit message to brain
6. Vibration of atoms
7. Mechanical Vibrations

1.3 Classification of vibrations

One method of classifying mechanical vibrations is based on degrees of freedom. The number of degrees of freedom for a system is the number of kinematically independent variables necessary to completely descibe the motion of every particle in the system. Based on degrees of freedom, we can classify mechanical vibrations as follows:

1.Single Degree of freedom Systems
2.Two Degrees of freedom Systems
3.Multidegree of freedom Systems

4.Continuous Systems or systems with infinite degrees of freedom

A system is linear if its motion is governed by linear differential equations. A system is nonlinear if its motion is governed by nonlinear differential equations. If the excitation force is known at all times, the excitation is said to be deterministic. If the excitation force is unknown, but averages and standard derivations are known, the excitation is said

to be random. In this case the resulting vibrations are also random. Sometimes systems are subjected to short duration nonperiodic forces. The resulting vibrations are called transient vibrations. One example of a nonperiodic short duration excitation is the ground motion in an earthquake

The main causes of vibrations are:

1. Bad design
2. Unbalanced inertia forces
3. Poor quality of manufacture
4. Improper bearings (Due to wear & tear or bad quality)
5. Worn out gear teeth
6. External excitation applied on the system

The effects of vibrations are as follows:

1. Unwanted noise

2. Early failure due to cyclical stress(fatigue failure)
3. Increased wear
4. Poor quality product
5. Difficult to sell a product
6. Vibrations in machine tools can lead to improper machining of parts

1.4 Basic terms associated with vibrations

FREE VIBRATIONS

Vibrations under free or natural conditions. No disturbing forces.

Example: - Simple Pendulum

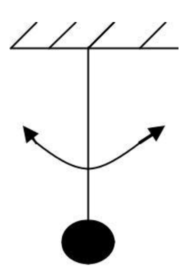


Fig 1.1 (a) Simple pendul

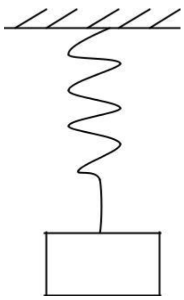
FORCED VIBRATIONS

Vibration due to impressed disturbing force

Examples

1.Electric bell-clipper oscillation under electromagnetic force.
2.I.C Engines-vibrations due to unbalanced inertia forces

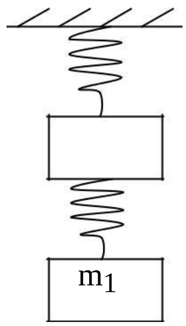
DEGREES OF FREEDOM



m_1

Single D.O.F

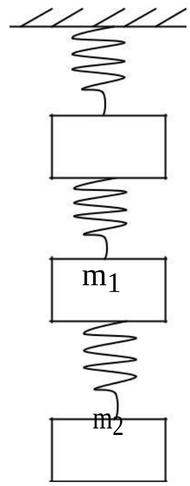
Fig 1.1 (a)



m_2

Two D.O.F

Fig 1.1(b)



m_3

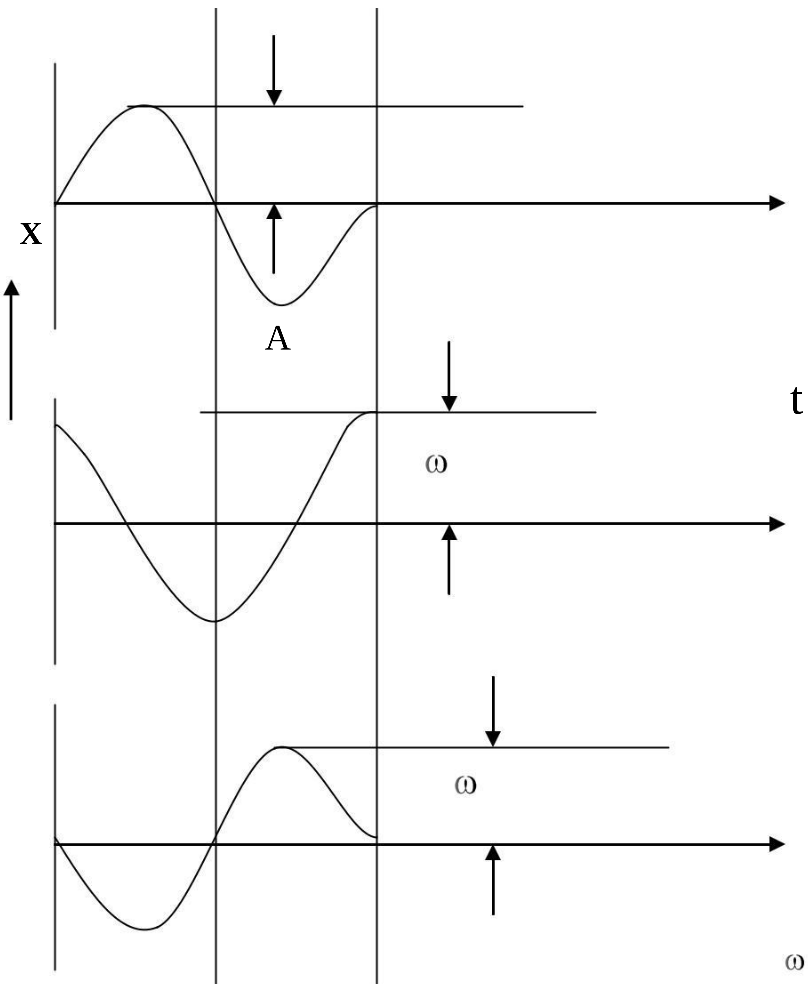
Three D.O.F

Fig 1.1(c)

Cantilever Beam

SIMPLE HARMONIC MOTION (S H M)

The oscillations of the mass shown in fig 1.1 (a**)** are described as simple harmonic motion. . Simple harmonic motion is represented graphically in fig 1.2



X-Displacement
A-amplitude T-Periodic Time f-Frequency
f=1/T

ω =Frequency in radians per second

t= time

•

x

••

X

A

²A

t

Fig 1.2 SHM

Simple harmonic motion is characterized by periodic oscillation about the equilibrium position. Each oscillation is one cycle. For S.H.M the time taken to execute one cycle, the period, is constant. The frequency of motion is the number of cycles executed in a fixed period of time, usually 1 second. The amplitude, the maximum displacement from equilibrium position, is also constant in S.H.M.

$\dot{X}$=Velocity

$= A\omega \cos \omega t$

$= A\omega \sin (\omega t + \pi/2)$

$\ddot{X}$=Acceleration

$= -\omega^2 A \sin \omega t$

$$= \omega^2 A \sin(\omega t + \pi)$$

$$= -\omega^2 x$$

From subject point of view the following notations and definitions are very important:

Periodic Motion:

It is a motion which repeats itself after equal intervals of time, e.g., the oscillations of simple pendulum

Time Period (T) :

It is the time required for one complete cycle or to and fro motion. The unit is seconds.

Frequency (f or ω) :

It is the number of cycles per unit time. The unit are radians/sec. or Hz.

Amplitude (X or A) :

It is the displacement of a vibrating body from its equilibrium position. It has units of length in general

.

Natural Frequency (f_n):

It is the frequency with which a body vibrates when subjected to an initial external disturbance and allowed to vibrate without external force being applied subsequently.

Fundamental Mode of Vibration:

A vibrating body may have more than one natural frequency and when it vibrates with the lowest natural frequency ,it is the Fundamental mode of vibration.

Degrees of Freedom:

It is the minimum number of coordinates required to describe the motion of system. Typically in our discussions 1DOF system will have one mass, e.g., a spring attached with one mass , 2 DOF system will have two masses and likewise we have 3DOFsystem. A continuous system like a beam or plate consisting of infinite number of particles with mass, are systems with infinite number of DOF.

Simple Harmonic Motion (SHM):

It is a periodic motion with acceleration always directed towards the equilibrium position. It can also be defined as projection of motion of a particle along a circle with uniform angular velocity on the diameter of circle.

Damping:

It is the resistance offered to the motion of a vibrating body by absorbing the energy of vibrations. Such vibrations are termed as damped vibrations

.

Forced Vibrations:

It is the vibration of a body when subjected to an external force which is periodic in nature and vibrations occur as long as external force is present.

Resonance:

It is said to occur in the system when the amplitude of vibrations are excessive leading to failure. This occurs in forced vibrations when the frequency of externally applied force is same as that of natural frequency of the body.

Linear and Non Linear Vibrations:

When the vibrations are represented by linear differential equations and laws of superposition are applicable for the system, we have Linear systems. Non linear vibrations

are experienced when large amplitudes are encountered and laws of superposition are not applicable.

Longitudinal, Transverse and Torsional Vibrations:

When the motion of mass of the system is parallel to the axis of the system, we have Longitudinal vibrations. When the motion of mass is perpendicular to the system axis the vibrations are Transverse vibrations and when the mass twists and untwists about the axis the vibrations are Torsional vibrations. Up and down motion of mass in a spring mass system represents Longitudinal vibrations. Vibration of a cantilever beam represents Transverse vibrations. The twisting and untwisting of a disc attached at the end of a shaft represents Torsional vibrations.

Vector representation of SHM:

Any SHM can be represented as by the equation , $x = A \sin\omega t$ ---(1) , where x is the displacement , A is the amplitude , ω is the circular frequency and t is the time.

Differentiating eqn.1 w.r.t. t we have velocity vector and differentiating eqn 1 twice we have the acceleration vector. If x1 and x2 are two displacement vectors with same frequencies then the phase difference between them is given by θ.

Principle of Superposition:

When two SHM of same frequencies are added the resulting motion is also a harmonic motion. Consider two harmonic motions $x1 = A1\sin\omega t$ and $x2 = A2 \sin(\omega t + \theta)$. Then if x is the resultant displacement , $x = x1 + x2$. The resultant amplitude $x = A \sin(\omega t +\theta)$, where A is the resultant amplitude and is acting at an angle θ w.r.t vector x1.

The above addition of SHMs can also be done graphically.

Problems:

1) Add the following harmonics analytically and check the solution graphically
$x1= 3\sin(\omega t + 30^\circ)$, $x2 = 4 \cos(\omega t +10^\circ)$

Solution:

Given : $x1= 3\sin(\omega t + 30^\circ)$, $x2 = 4 \cos(\omega t +10^\circ)$

Analytical method:

We know that, $x = x1 + x2 = A \sin(\omega t+ \theta)$

Make x1 and x2 to have same Sin terms always , i.e., $x2 = 4 \cos(\omega t +10^\circ + 90^\circ) = 4\sin(\omega t +100^\circ)$

Hence, $A \sin(\omega t+ \theta)= 3 \sin(\omega t + 30^\circ) + 4 \sin(\omega t +100^\circ)$ Expanding LHS and RHS

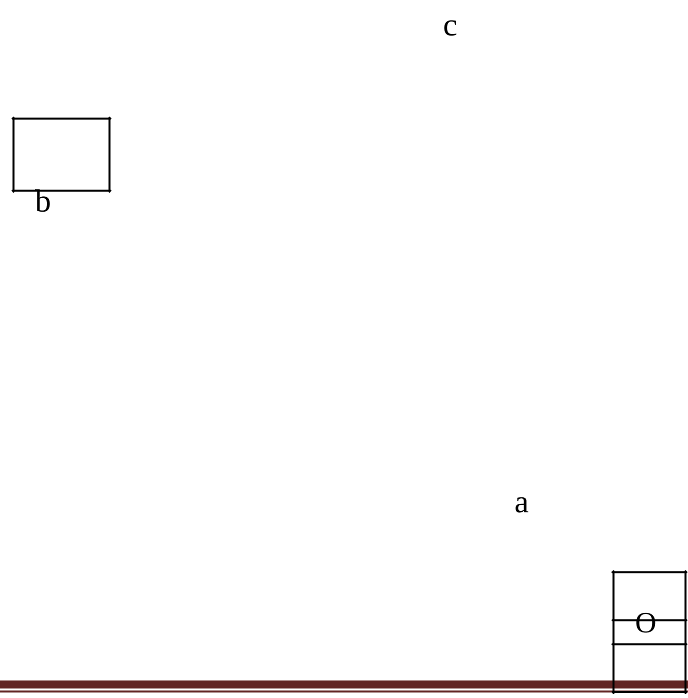
$A\sin\omega t \cos\theta + A \cos\omega t \sin\theta =3 \sin\omega t \cos 30^\circ + 3\cos\omega t\sin30^\circ + 4 \sin\omega t \cos 100^\circ + 4 \sin\omega t \sin 100^\circ$

$A \sin\omega t \cos\theta + A \cos\omega t \sin\theta = \sin\omega t(1.094) + \cos\omega t (5.44)$
Comparing the coefficients of $A \cos\theta$ and $A \sin\theta$ in the above equation $A\cos\theta = 1.094$, $A\sin\theta = 5.44$, $\tan\theta = A\sin\theta/ A\cos\theta = 5.44/1.904$ Therefore , $\theta = 70.7^\circ$ and $A = 1.094/\cos 70.7^\circ = 5.76$.

Graphical Method.:

Draw ox the reference line. With respect to ox, draw oa equal to 3 units in length at an angle of 30° to ox and ob equal to 4 units at an angle of 100° to ox. Complete the vector

polygon by drawing lines parallel to oa and ob to intersect at point c. Measure oc which should be equal to A and the angle oc makes with ox will be equal to θ. All angles measured in anticlockwise direction.



x
x

2) Repeat the above problem given , $x1 = 2Cos(\omega t + 0.5)$ and $x2 = 5Sin(\omega t + 1.0)$. The angles are in radians.

3) Add the following harmonic motions analytically or graphically.
$x1 = 10 Cos(\omega t + \pi/4)$ and $x2 = 8 Sin(\omega t + \pi/6)$.

4) A body is subjected to 2 harmonic motions

$x1 = 15sin(\omega t + \pi/6)$, $x2 = 8 cos(\omega t + \pi/6)$, what harmonic is to be given to the body to it to equilibrium.

Solution :

Let the harmonic to be given to the two harmonics to make it to be in equilibrium be $Asin (\omega t + \theta)$

Therefore, $Asin (\omega t + \theta)+ x1 + x2 = 0$

Hence, $Asin\omega t \cos \theta+A \cos\omega t \sin \theta+15sin\omega t \cos\pi/6+15cos\omega t \sin\pi/6+8 \cos\omega t \cos\pi/6+8 \sin\omega t \sin\pi/6 = 0$

$\sin\omega t (A \cos \theta+8.99038)+ \cos\omega t(A \sin \theta+14.4282) = 0$

Therefore, $A \cos \theta = -8.99038$
$A \sin \theta= - 14.4282$

Therefore, $\tan \theta = A \sin \theta / A \cos \theta = 14.4282/8.99038$, $\theta = 58.062^\circ$
From , $A \cos \theta = -8.99038$, substituting for $\theta = 58.062^\circ$, $A = 17.00$
Therefore, the motion is $x= 17sin(\omega t + 58.062^\circ)$

PROPERTIES OF OSCILLATORY MOTION

Peak value- Indicates space requirement.

An indication of maximum stress in the vibrating part

Average Value - Average value for complete sine wave is zero

For half sine wave $\bar{X} = 2A / \pi$

11

A-Amplitude

Mean square value - For sine wave $\overline{X^2} = 1/2\, A^2$

RMS Value = $A / \sqrt{2}$

Problem 1

The frequency of Vibrations of a machine is 150 Hz. Determine a) Its frequency in rad/sec. b)Time Period of oscillations. If the amplitude of vibrations is 0.8 mm, determine the acceleration

a) In m/s^2 b) In terms of g

Solution:

Given f = 150 Hz , A= 0.8 mm w= ? T=?

a = ? (in m/s^2) a = ? (in terms of g)

$w = 2\pi f = 2\pi(150) = 942$ Rad/sec

T= 1/f=1/150= 0.0066 sec = 6.66 milli seconds $x = A \sin(\omega t + \Phi)$

$= 0.8 \sin(942\, t + \Phi)$

$\dot{x} = 0.8\,(942)\cos(942\,t + \Phi)$

$\ddot{x} = -0.8\,(942)^2 \sin(942\,t + \Phi)$

$a = (\ddot{x})_{max} = 0.8\,(942)^2\ \text{mm/s}^2$

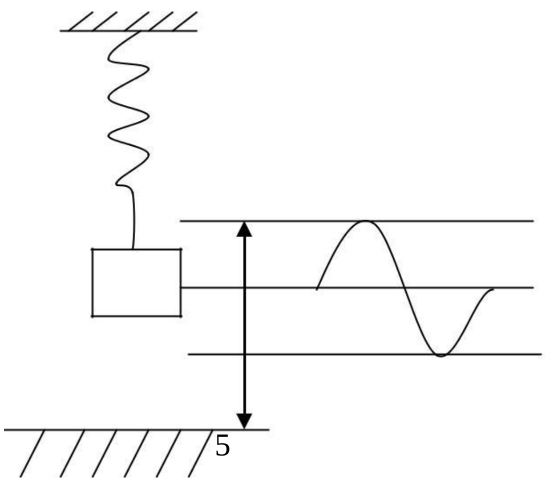
$= 710.61\ \text{m/s}^2$

$= 710.61/9.81 = 72.43\ g$

Problem 2.

A body suspended from a spring vibrates vertically up and down between two positions 3 and 5 cms above the ground. During each second it reaches the top position (5 cms above ground) 15 times. Find the time period, frequency, circular frequency and amplitude of motion.

Solution:



Amplitude = (5-3)/2 =1 cm.
f = Frequency =15 cps

T = Period = 1/15 Sec

ω

ω

= Circular Frequency

$= 2\pi f = 2\pi(15) = 30\pi$ rad/sec

3 3

1.6 Addition of harmonic motions of same frequency

x_1= A1 Sin t

x_2 = A2 Sin (t + Φ)

X = x_1+ x_2 = A1 Sin t + A2 Sin (t + Φ)

X = Sin t (A1+ A2 Cos Φ) + Cos t (A2 Sin Φ)

Let A1+ A2 Cos Φ = A Cos -------- 1

A2 Sin Φ = A Sin -------- 2

$X = \sin\omega t\,(A\cos\theta) + \cos\omega t\,(A\sin\theta)$

$X = A\sin(\omega t + \theta)$

$A^2(\sin^2\theta + \cos^2\theta) = (A_1 + A_2\cos\Phi)^2 + (A_2\sin\Phi)^2$

$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\Phi}$

From equations 1 and 2 we also get

$\tan\theta = A_2\sin\Phi/(A_1 + A_2\cos\Phi)$

Graphical Method for addition of two harmonic motions

1.7 BEATS

The phenomenon of beats occurs when two harmonic motions of slightly different frequencies and same amplitude are added. When the two harmonic motions are in the same phase, the resultant amplitude will be maximum. On the other hand, when the two motions are out of phase, they will provide minimum amplitude vibration.

Let$X_1 = A \sin \omega_1 t$

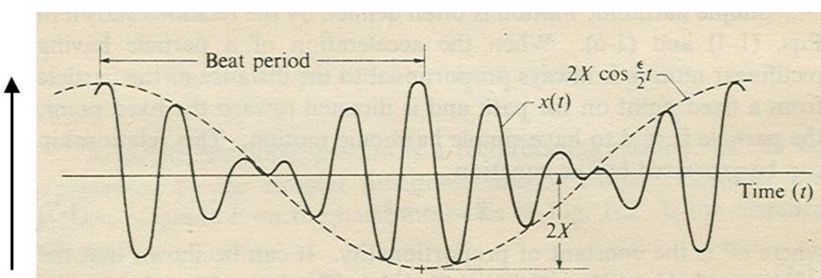
$X_2 = A \sin \omega_2 t$

$X = X_1 + X_2 = A \sin \omega_1 t + A \sin \omega_2 t$

$= 2 A \sin (\omega_1 + \omega_2)t \cos (\omega_1 - \omega_2)t$

$X = \sin [(\omega_1 + \omega_2)t]/2$ When $B = 2 A \cos [(\omega_1 - \omega_2)t]/2$

The Frequency of beats is $(\omega_1 - \omega_2)/2\pi$ Hz



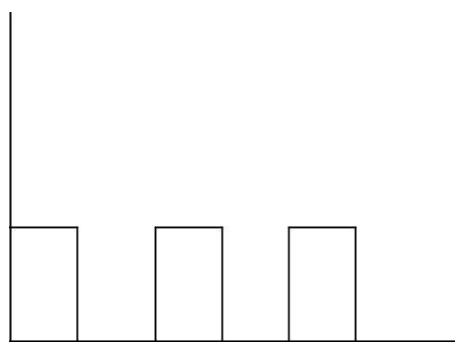
Amplitude

Graphical representation of Beats

1.8 Fourier series analysis

Forces acting on machines are generally periodic but this may not be harmonic for example the excitation force in a punching machine is periodic and it can be represented as shown in figure 1.3. Vibration analysis of system subjected to periodic but nonharmonic forces can be done with the help of Fourier series. The problem becomes a multifrequency excitation problem. The principle of linear superposition is applied and the total response is the sum of the response due to each of the individual frequency term.

Example :- Excitation force is periodic



Force Developed during punching operation

With the help of Fourier series vibration analysis of such problems can be done

Fourier Series

$$X(t) = a_o/2 + \sum_{n=1}^{\infty} (a_n \cos n\omega t) + b_n \sin n\omega t)$$

$\omega = 2\pi / T$ = Fundamental frequency

ao, a1,a2,……b1,b2….. are coefficients of infinite series

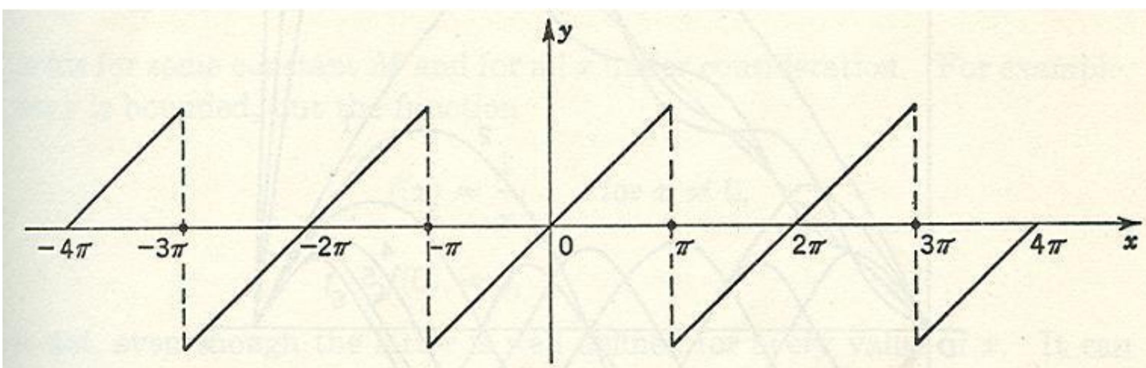
$(a_1\cos \omega t + b_1\sin \omega t)$ is First Harmonic $2\pi/\omega$

$$a_o = \omega/\pi \int_o^{2\pi/\omega} x(t)\,dt, \quad a_n = \omega/\pi \int_o^{2\pi/\omega} x(t)\cos(n\omega t)\,dt$$

$$bn= {}^{\omega}/\pi \int_{o} x(t)\ \sin(n\ {}^{\omega} t)dt$$

Problem 1.

Develop the Fourier Series for the curve shown in figure



π π

The function is defined as y=x (t) - < t <

$X(t)= a_o/2+ a_1\cos t+ a_2 \cos 2 t + \ldots\ldots + b_1 \sin t+ b_2 \sin 2 t + \ldots\ldots$

The equation for the curve for one cycle

for AB , ω X(t)= tω - <t< ω ω

= 2 / T = 2 / 2 = 1 π π

ω π π π

$a_{o}= 1/ \pi \int_{-\pi}^{\pi} t\,dt =0$

π

∫

π

-

$a_n= 1/ p$ t cos nt dt = 0

The graph is symmetrical about the origin and the function is odd

$a_o= a_n = 0$

π

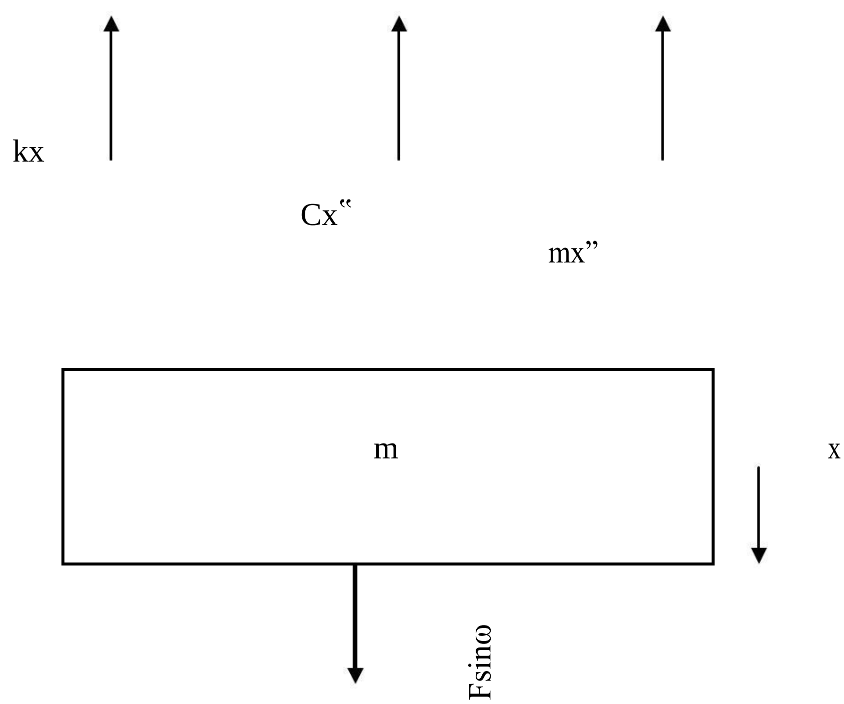
UNIT-04

FORCED VIBRATIONS

Forced vibrations are those whose amplitudes are maintained by application of external forces.Ringing of electric bell or machine tool vibrations are examples of forced vibrations.The external force maintaining the vibrations are called external excitation and are random, periodic or impulsive in nature.

Basic sources of excitation are external or inherent to the system. Machine subsystems are heated unevenly during operation and give rise to uneven deformation leading to generation of unbalanced force,Resonance of system produces large amplitudes leading to unbalanced forces.Similarly, defective assembly , bending and distortion of components, bearing defects leading to misalignment, uneven distribution of mass in rotating components lead to creation of unbalanced forces causing a system to vibrate forcibly.

Forced vibration of damped single degree of freedom system



kx is the spring force, cx‟ is the damping force and mx” is the inertia force and Fsinωt is the external excitation . x is the displacement of mass in the direction shown. The equation of motion is written as

mx”+cx‟ + kx = Fsinωt ----(i)

The solution of above equation is in 2 parts,

i) Complimentary function (cf) and

ii) particular integral(pi).

The total solution x = x(cf) + x(pi).

The x(cf) is the solution of equation mx”+cx‟ + kx = 0, which is written as $Ae^{-\xi\omega nt}$ $\sin(\omega_d t + \theta)$.

The particular integral x(pi) is assumed to be in the form $x = X \sin(\omega t - \theta)$, thus we have $dx/dt = x‟$ as $\omega X\sin(\omega t-\theta+\pi/2)$ and $(dx/dt)^2 = x” = \omega^2 X\sin(\omega t-\theta+\pi)$, substituting the values of x‟ amd x” in eqn. (i), we have,

$$m(\omega^2 X\sin(\omega t-\theta+\pi) + c(\omega X\sin(\omega t-\theta+\pi/2)) + k(X \sin(\omega t - \theta)) = F\sin\omega t$$

Rearranging ,

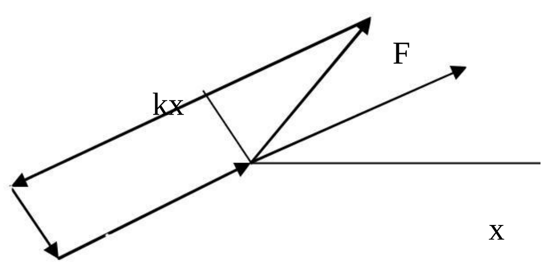
$$F\sin\omega t - kX \sin(\omega t - \theta) - c\omega X\sin(\omega t-\theta+\pi/2) - m\omega^2 X\sin(\omega t-\theta+\pi) = 0$$

Where,

$F\sin\omega t$ is the external force
Diplacement x lags the external force
kX is the spring force lagging F by θ

$c\omega X$ is the damping force lagging F by $(\theta+\pi/2)$
$m\omega^2 X$ is the inertia force, lagging F by $(\theta+\pi)$,

The vector diagram of these is as shown below:



From the geometry of diagram, we have,

$F^2 = (kx - m\omega^2 X)^2 + (c\omega X)^2$, simplifying

$X = F/((k - m\omega^2)^2 + (c\omega)^2)$

Therefore, the total solution can be written as,

$x = x(cf) + x(pi)$

$= Ae^{-\xi\omega nt} \sin(\omega_d t + \theta) + F \sin(\omega t - \theta) (/((k - m\omega^2)^2 + (c\omega)^2)$ ----(ii)

The eqn(ii) is total response which consists of two parts, first being the transient part, the first term in RHS , which dies out with time and the second part the x(pi), is the steady state vibration which does not die with time.

The expressions for amplitude X in dimensionless form and phase angle are as follows:

$X = (F/k)/\sqrt{(1-(\omega^2/\omega_n^2)^2 + (2\xi\omega/\omega_n)^2}$, (F/k) is called the Zero frequency deflection which is the deflection of spring mass under a steady force.

The phase angle, $\theta = \tan^{-1} ((2\xi\omega/\omega_n)/ 1-(\omega^2/\omega_n^2))$

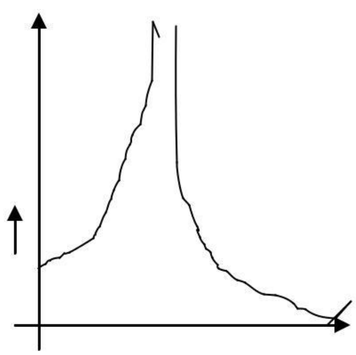
Magnification Factor:

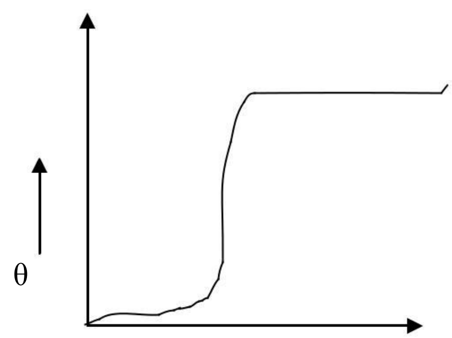
In a vibrating system the transient vibrations die out after passage of time and the steady state vibration continues with constant amplitude as long as the external excitations exsist, and this makes the study of steady state vibrations to be important for study and analysis.

Magnification factor M.F. is one parameter in study of forced vibrations which is defined as the ratio of amplitude of steady state response X to X_{st} the zero frequency deflection or the static response under steady load F.

The M.F . is given by, M.F. = $X/X_{st} = 1/(\sqrt{(1-(\omega^2/\omega_n^2))^2 + (2\xi\omega/\omega_n)^2})$

This M.F. depends upon the frequency ratio ω/ω_n and the damping factor ξ . From the plots of M.F. versus frequency ratio and phase angle, θ versus frequency ratio also called frequency response curves following observations can be made: (Refer any standard text for detailed curves)





MF

ω/ω_n ω/ω_n

i) Phase angle is 90º at resonance
ii) M.F. is infinity at resonance and $\xi = 0$

iii) For all frequencies th MF reduces with damping

iv) Maximum amplitude occurs at left of resonance

v) For small values of frequency ratio, the inertia and damping forces are small resulting in small phase angles.Impressed force is nearly equal to spring force.

vi) For frequency ratio of 1, the inertia force is balanced by the spring force.The impressed force balances the damping force.

vii) For large values of frequency ratio, inertia force increases to a large value and damping and spring forces are small.

viii) The frequency at which the maximum amplitude occurs is obtained by using the relation $\omega_p = \omega_n \sqrt{(1 - 2\xi^2)}$, where ω_p is the frequency at which maximum amplitude occurs.

ix) Condition for resonance , $(MF)_{resonance} = (X_r / X_{st}) = (1/2\xi)$

Solution by complex algebra:

Let the equation of motion be written as

$$mx'' + cx' + kx = Fe^{i\omega t}$$

the response of which is $x = X e^{i(\omega t - \theta)}$. substituting the expressions for x" and x" into the equation of motion and simplifying , we have $(-m\omega^2 + ic\omega + k) X e^{i(\omega t - \theta)} = Fe^{i\omega t}$, from which

,

$Xe^{-i\theta} = (F/(k - m\omega^2) + ic\omega)$, from which using $x = X e^{i(\omega t - \theta)}$, the real part of x is given by $Re(Fe^{i\omega t}/(k - m\omega^2) + ic\omega)$.

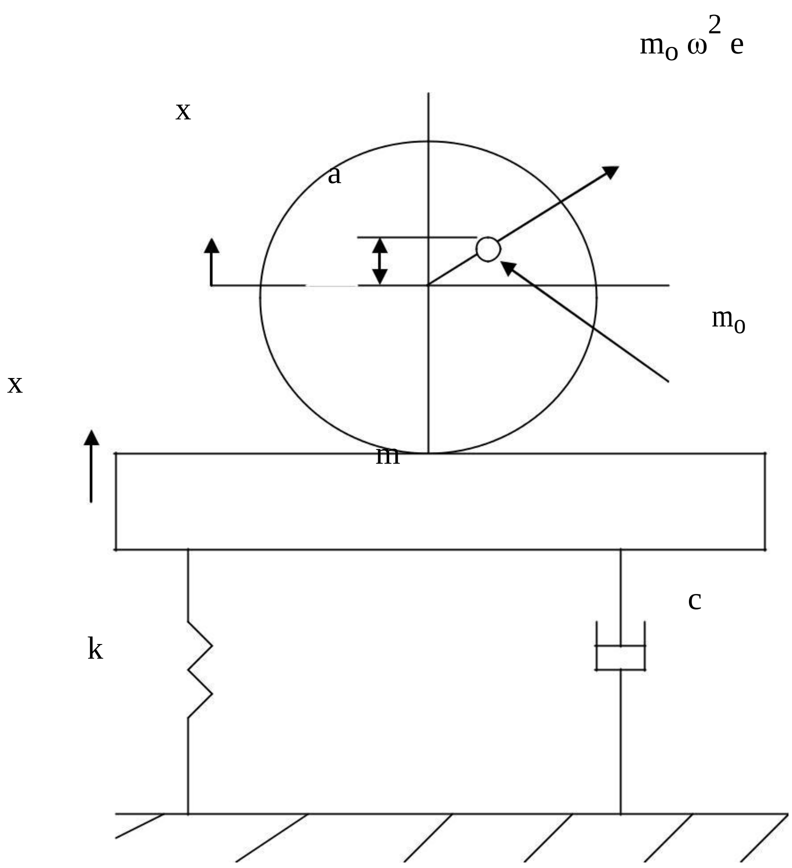
Introducing the complexfrquency response H(ω) as ratio of output $Xe^{-i\theta}$ to input F i.e, $H(\omega) = Xe^{-i\theta} / F = (1/(k - m\omega^2) + ic\omega)$

$= X/X_{st} = 1/\sqrt{((k - m\omega^2)^2 + (c\omega)^2)}$ The phase angle, $\theta = \tan^{-1}(c\omega/(k - m\omega^2))$

Rotating and Reciprocating unbalance

a = esinωt

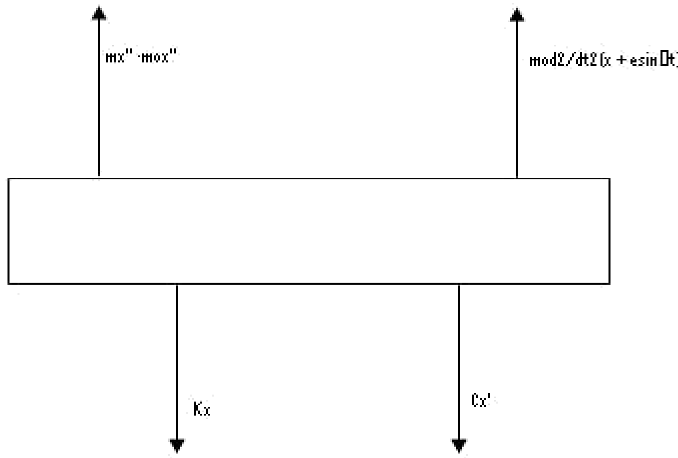
e = eccentricity



The figure shows a rotating equipment rotating at a speed of ω rad./sec. Let m_o be the unbalance mass rotating with its CG at a distance of e from centre.This unbalanced mass gives rise to a centrifugal force , equal to $m_o \omega^2 e$ .Let m be the total mass of equipment inclusive of m_o and at any instant of time m_o make an angle of ωt.

The equation of motion for this system can be written considering the effective mass „$m-m_o$‟ and the unbalanced mass „m_o‟ .

Referring figure as shown below, we have the effective displacement of m_o is sum of „x‟ and „esinωt‟ . Hence we can write the equation of motion in the vertical direction as $(m-m_o)x'' + (m_o)d^2(x + e\sin\omega t)/dt^2 = -Kx - Cx'$



Ie, $mx'' - m_o x'' + m_o x'' + m_o d\{\omega_e \cos\omega t\}/dt = -Kx -$

Cx' $mx'' - m_o \omega^2 e \sin\omega t = -Kx - Cx'$

$mx'' + Cx' + Kx = m_o \omega^2 \sin\omega t$

The above equation is similar

to $mx'' + Cx' + Kx = F\sin\omega t$

Hence for an under damped system, we get the expression for steady state amplitude as

$$X = \frac{m_o\omega^2 e/K}{\sqrt{(1-(w/w_n)^2)^2 + (2\xi w/w_n)^2}}$$

Therefore

$$\frac{X}{(m_o e/m)} = \frac{(w/w_n)^2}{\sqrt{(1-(w/w_n)^2)^2 + (2\xi w/w_n)^2}}$$

$$\Phi = \tan^{-1}\{2\xi(w/w_n) / (1 - (w/w_n)^2)\}$$

Same analysis is extended to reciprocating masses where exciting force becomes $m_o e w^2 \sin wt$ where m_o = Unbalanced mass of reciprocating masses.

The complete solution for the unbalanced system is

$$x = A_2 e^{-\xi\omega_n t}(\sin\omega_d t + \Phi_2) + (m_o e w^2/k)/(\sqrt{(1-(\omega^2/\omega_n^2)^2 + (2\xi\omega/\omega_n)^2})$$

The following points are concluded for unbalanced system:

- Damping factor plays an important role in controlling the amplitudes during resonance. For low values of frequency ratio, X tends to 0.
- For low values of frequency ratio (w/w_n), X tends to 0.
- At high speeds of operation, damping effects are negligible.
- The peak amplitudes occur to right of resonance unlike for balanced systems. At resonance, $w = w_n$ ie: $X / m_o e/m = 1/2\xi$

Also, $(X)_{resonance} = m_o e / 2m\xi$

From the plot of ($X / em_o\ m$) v/s ω/ω_n , it iserved that at low speeds, because the inertia force is small, all the curves start from zero and at resonance ($X / m_o e/m$)= $1/2\xi$ and the amplitude of such vibrations can be controlled by the damping provided in the system. For very large frequency ratio, ($X / m_o e/m$) tends to one.

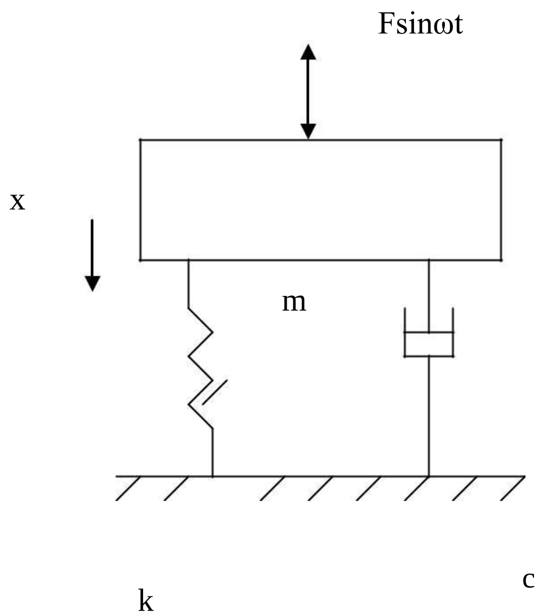
VIBRATION ISOLATION AND TRANSMISSION:

Vibration Isolation:

High speed machines and engines due to unbalance give rise to vibrations of excessive amplitudes and due to the unbalance forces being setup, the foundations can be damaged. Hence there is a need to eliminate or reduce the vibrations being transmitted to the foundations, springs, dampers, etc. are placed between the machines and the foundations to reduce the vibrations or minimize then. These elements isolate the vibrations by absorbing the vibration energy. This isolation of vibrations is expressed in terms of force or motion transmitted to the foundation. The requirements of these isolating elements are that there should be no connection between the vibrating system & the foundation & it is to be ensured that in case of failure of isolators the system is still in of position on the foundation. Rubber acts effectively as an isolator during shear loading. The sound transmitted by it is also low. Heat and oil affect the rubber and it is usually preferred for light loads & high frequency oscillation. Felt pals are used for low frequency ratios. Many small sized felt pads are used instead of a single large pad. Cork can be used for compressive loads.

Helical & leaf springs of metal are used as isolators for high frequency ratios. They are not affected by air, water or oil. The sound transmitted by them can be reduced by covering them with pads of felt, rubber or cork.

TRANSMISSIBILITY:



In a spring mass dashpot system subjected to harmonically varying external force, the spring and dashpot become the vibration isolators and the spring force and damping force are the forces between the mass and foundation. Thus the force transmitted to the foundation (F_{tr}) is vector sum of the spring force (kX) and damping force (cωX). We can write,

$F_{tr} = X\,\sqrt{(K^2 + c^2\omega^2)}$, substituting for X as $X = F/((k - m\omega^2)^2 + (c\omega)^2)$, we have F_{tr} equal to, $F_{tr} = F\,(\sqrt{(K^2 + c^2\omega^2)} / ((k - m\omega^2)^2 + (c\omega)^2)$

Transmissibility is defined as the ratio of force transmitted to the foundation to the force impressed on the system i.e.,

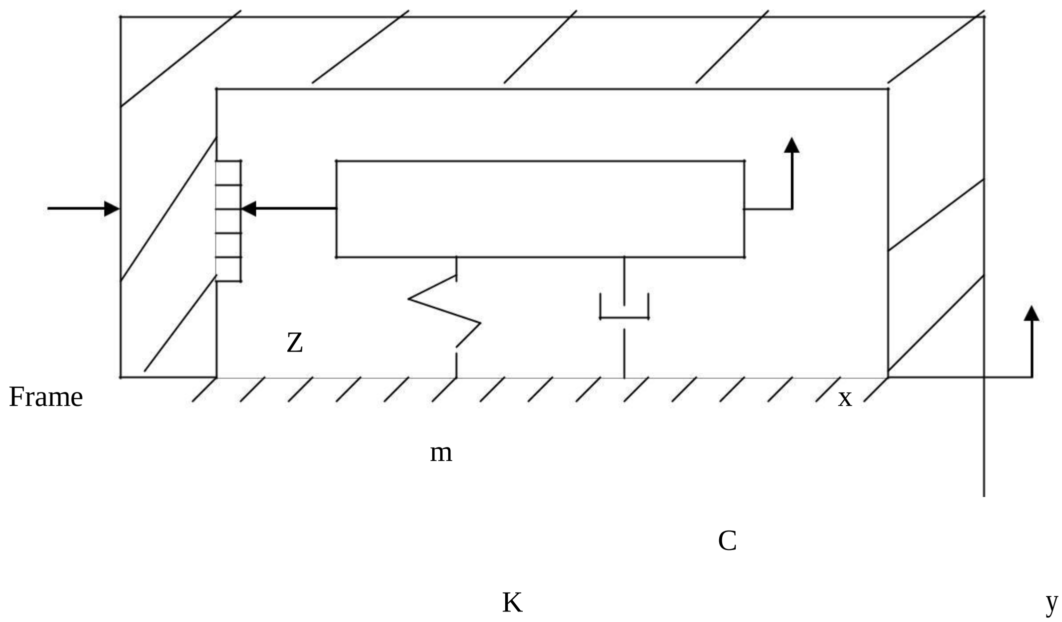
$$T_r = \varepsilon = F_{tr} / F = \sqrt{(1 + (c\omega/k)^2} / (\sqrt{(1-(\omega^2/\omega_n^2)^2 + (2\xi\omega/\omega_n)^2})$$

The angle of lag of the transmitted force is ,

$$(\theta - \alpha) = \tan^{-1}((2\xi\omega/\omega_n) / 1-(\omega^2/\omega_n^2)) - \tan^{-1}(2\xi\omega/\omega_n)$$

Plot of T_r versus ω/ω_n (refer a text book) for various values of ξ , is called the transmissibility curve . From the plot it is seen that all curves start from 1 and transmissibility T_r is always desired to be less than 1, as it ensures that transmitted force to the foundation is minimum and better isolation is achieved. The operating values of frequency ratio to achieve this effect should be greater than √2 and the region beyond this value of frequency ratio is called mass control zone where isolation is most effective. In the plot the frequency ratio values upto 0.6 are spring control zone and from 0.6 to √2 is damping control zone and beyond that is mass control zone.

FORCED VIBRATION DUE TO EXCITATION OF SUPPORT:



VIBRATING BODY/

Figure shows a basic sesmic instrument used for measuring vibrations. When the system is excited by the vibrations of the base, the mass „m‟ is subjected to a displacement „x‟ . If we consider „y‟ be the motion of the base, then the absolute amplitude of mass „m‟ is the displacement „x‟ . If „Z‟ is considered as the displacement of mass „m‟ w.r.t the frame, then we have a relative motion of „m‟ w.r.t the frame.

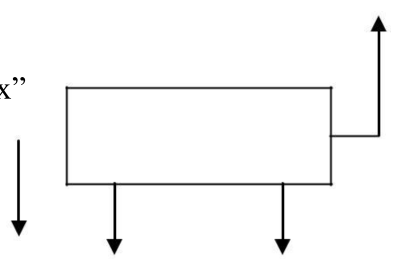
Absolute amplitude: (neglect z)

Let the displacement of base be „y‟ viz: a sinusoidal motion, given by $y = Y\sin\omega t$

For such a system the equation for motion can be written as

x

mx”



K(x-y) C(x‟ -y‟)

mx” + K(x-y) + C(x‟ -y‟) = 0 i.e, mx” + Cx‟ + Kx – Ky - Cy‟ = 0 Substituting for y and y‟ , we get

$mx'' + Cx' + Kx – Ky\sin\omega t – C\omega y\cos\omega t = 0$

$mx'' + Cx' + Kx = y \{Ky\sin\omega t + C\omega\cos\omega t = 0\}$

$= y(\sqrt{K^2 + (c\omega)^2}).\sin(\omega t + \alpha)$ → (1)

Where $\alpha = \tan^{-1} \{c\omega/K\} = \tan^{-1} \{2\xi\omega/\omega_n\}$

The solution of (1) consists of CF and PI.

The PI is $x = X \sin(\omega t + \alpha - \theta$ → (a)

(a) is similar to $x = X \sin (\omega t – \theta)$, where X is the steady state amplitude.

$$X = \frac{y (\sqrt{K^2 + (c\omega)^2})}{(\sqrt{(K - (c\omega^2)^2 + (c\omega)^2})}$$

Therefore $X/y = (\sqrt{1+ (2\xi\omega/\omega_n)^2}) / (\sqrt{(1 – (\omega/\omega_n)^2)^2 + (2\xi\omega/\omega_n)^2})$ → (b)

$\Phi = (\tan^{-1} (\underline{2\xi\omega/\omega_n}^2 \;/ 1 – (\underline{\omega/\omega_n})^2))$

$\alpha = (\tan^{-1} \{2\xi\omega/\omega_n\})$

$(\Phi - \alpha) = (\tan^{-1} (\underline{2\xi\omega/\omega_n}^2 \;/ 1 – (\underline{\omega/\omega_n})^2)) - (\tan^{-1} \{2\xi\omega/\omega_n\})$ ---- ©

Equations (a), (b) and (c) completely define the motion of the mass due to the support or base excitation. The ratio X/y is called the displacement transmissibility

Relative Amplitude:

If the displacement of the mass is considered relative to the frame and if this relative

displacement is called z , then we have,

$z = x – y$

or, $x = y + z$

substituting this value of x in the equation of motion, $m(y'' + z'')+c(y' +z')+K(y+z-y) = 0$

$my''+mz''+cz' +kz = - my''$

$mz'' + cz' + kz = -m(-\omega^2 y \sin\omega t)$ i.e., $mz'' + cz' + kz = m\omega^2 y \sin\omega t$ similar to eqn b, we have

$z/y = (\omega/\omega_n)^2 / (\sqrt{(1 – (\omega/\omega_n)^2)^2 + (2\xi\omega/\omega_n)^2})$

The expressions for $\Phi – \alpha$ is same as given above for absolute amplitude.

Energy dissipated by Damping.

When a syatem undergoes steady state forced vibration swith viscous damping, energy gets absorbed by the dashpot . The energy dissipated or workdone per cycle is given by,

Energy dissipated/ cycle = $\pi c \omega x^2$, where x is the amplitude of steady state vibrations.

The power required for vibrating the system can be obtained by the relation

Power = Energy dissipated/ cycle / Sec. , Watts.

Sharpness of Resonance:

In forced vibration, quantity Q is related to damping which becomes a measure of the sharpness of resonance. It also gives the side band of frequencies ω_1 and ω_2 on either side of the resonance by which resonance can be avoided during operation. The expression of Q is given as follows:

$Q = \omega_n / (\omega_2 - \omega_1) = 1/2\xi$

UNIT-03

Damped Free Vibrations

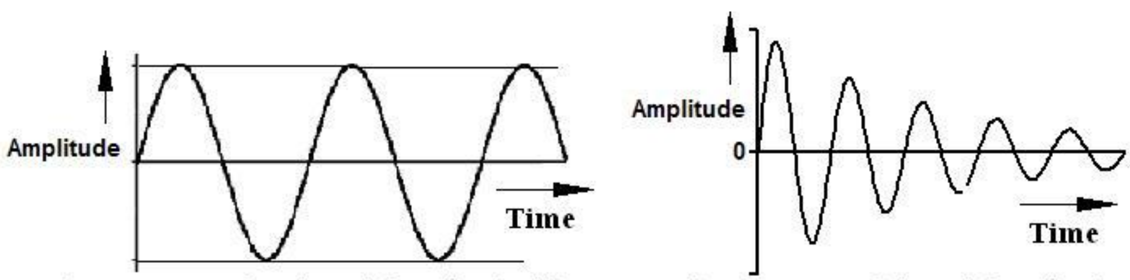
Single Degree of Freedom Systems

Introduction:

Damping – dissipation of energy.

For a system to vibrate, it requires energy. During vibration of the system, there will be continuous transformation of energy. Energy will be transformed from potential/strain to kinetic and vice versa.

In case of undamped vibrations, there will not be any dissipation of energy and the system vibrates at constant amplitude. Ie, once excited, the system vibrates at constant amplitude for infinite period of time. But this is a purely hypothetical case. But in an actual vibrating system, energy gets dissipated from the system in different forms and hence the amplitude of vibration gradually dies down. Fig.1 shows typical response curves of undamped and damped free vibrations.



Response curve of undamped free vibration Time Response curve of damped free vibration

Fig 1

Types o damping:

(i) Viscous damping

In this type of damping, the damping resistance is proportional to the relative velocity between the vibrating system and the surroundings. For this type of damping, the differential equation of the system becomes linear and hence the analysis becomes easier. A schematic representation of viscous damper is shown in Fig.2.

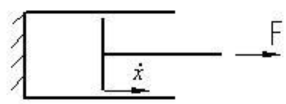


Fig 2

Here, $F \propto \dot{x}$ or $F = c\dot{x}$, where, F is damping resistance, $\dot{x}$ is relative velocity and c is the damping coefficient.

1

(ii) Dry friction or Coulomb damping

In this type of damping, the damping resistance is independent of rubbing velocity and is practically constant.

(iii) Structural damping

This type of damping is due to the internal friction within the structure of the material, when it is deformed.

Spring-mass-damper system:

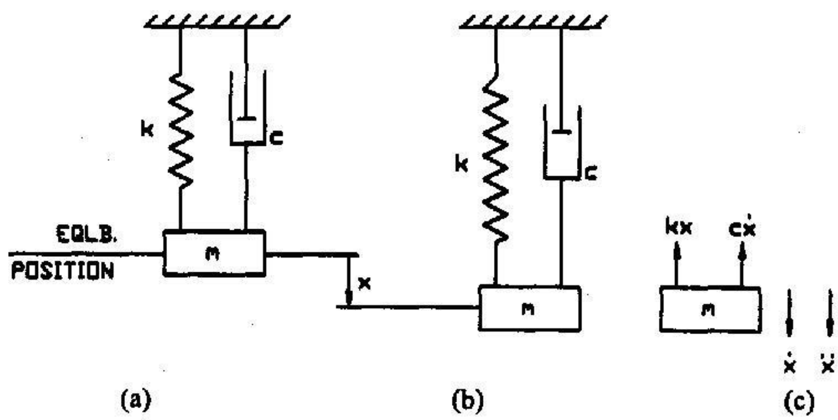


Fig 3

Fig.3 shows the schematic of a simple spring-mass-damper system, where, m is the mass of the system, k is the stiffness of the system and c is the damping coefficient.

If x is the displacement of the system, from Newton's second law of motion, it can be written

$$m\ddot{x} = -c\dot{x} - kx$$

Ie $$m\ddot{x} + c\dot{x} + kx = 0 \quad (1)$$

This is a linear differential equation of the second order and its solution can be written as

$$x = e^{st} \qquad (2)$$

Differentiating (2), $$\frac{dx}{dt} = \dot{x} = se^{st}$$

$$\frac{d^2x}{dt^2} = \ddot{x} = s^2e^{st}$$

Substituting in (1), $$ms^2e^{st} + cse^{st} + ke^{st} = 0$$

$$\left(ms^2 + cs + k\right)e^{st} = 0$$

Or $$ms^2 + cs + k = 0 \qquad (3)$$

Equation (3) is called the characteristic equation of the system, which is quadratic in s. The two values of s are given by

$$s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \qquad (4)$$

The general solution for (1) may be written as

$$x = C_1 e^{s_1 t} + C_2 e^{s_2 t} \qquad (5)$$

Where, C_1 and C_2 are arbitrary constants, which can be determined from the initial conditions.

In equation (4), the values of $s_1 = s_2$, when $\left(\frac{c}{2m}\right)^2 = \frac{k}{m}$

Or, $$\frac{c}{2m} = \sqrt{\frac{k}{m}} = \omega_n \qquad (6)$$

Or $c = 2m\omega_n$, which is the property of the system and is called critical damping coefficient and is represented by c_c.

Ie, critical damping coefficient = $c_c = 2m\omega_n$

The ratio of actual damping coefficient c and critical damping coefficient c_c is called damping factor or damping ratio and is represented by ζ.

$$\text{Ie, } \zeta = \frac{c}{c_c} \qquad (7)$$

In equation (4), $\frac{c}{2m}$ can be written as $\frac{c}{2m} = \frac{c}{c_c} \cdot \frac{c_c}{2m} = \zeta.\omega_n$

Therefore, $$s_{1,2} = -\zeta.\omega_n \pm \sqrt{(\zeta.\omega_n)^2 - \omega_n^2} = \left[-\zeta \pm \sqrt{\zeta^2 - 1}\right]\omega_n \qquad (8)$$

The system can be analyzed for three conditions.

(i) $\zeta > 1$, ie, $c > c_c$, which is called over damped system.

(ii) $\zeta = 1$. ie, $c = c_c$, which is called critically damped system.

(iii) $\zeta < 1$, ie, $c < c_c$, which is called under damped system.

Depending upon the value of ζ, value of s in equation (8), will be real and unequal, real and equal and complex conjugate respectively.

(i) Analysis of over-damped system ($\zeta > 1$).

In this case, values of s are real and are given by

$s_1 = \left[-\zeta + \sqrt{\zeta^2 - 1}\right]\omega_n$ and $s_2 = \left[-\zeta - \sqrt{\zeta^2 - 1}\right]\omega_n$

Then, the solution of the differential equation becomes

$$x = C_1 e^{\left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t} + C_2 e^{\left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t} \qquad (9)$$

This is the final solution for an over damped system and the constants C_1 and C_2 are obtained by applying initial conditions. Typical response curve of an over damped system is shown in fig.4. The amplitude decreases exponentially with time and becomes zero at t = ∞.

3

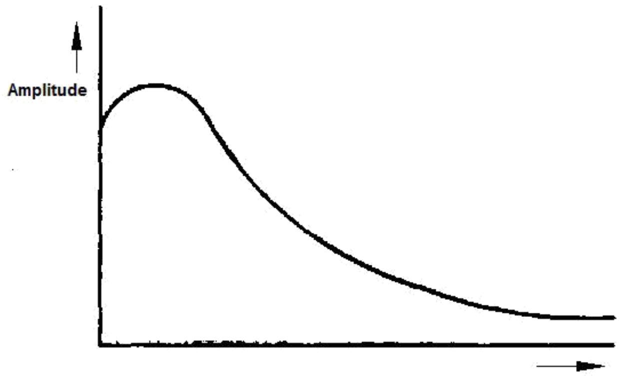


Fig 4 Typical response curve of an over damped system

(ii) Analysis of critically damped system ($\zeta = 1$).

In this case, based on equation (8), $s_1 = s_2 = -\omega_n$

The solution of the differential equation becomes

$$x = C_1 e^{s_1 t} + C_2 t e^{s_2 t}$$

Ie, $x = C_1 e^{-\omega_n t} + C_2 t e^{-\omega_n t}$

Or, $x = (C_1 + C_2 t) e^{-\omega_n t}$ (10)

This is the final solution for the critically damped system and the constants C_1 and C_2 are obtained by applying initial conditions. Typical response curve of the critically damped system is shown in fig.5. In this case, the amplitude decreases at much faster rate compared to over damped system.

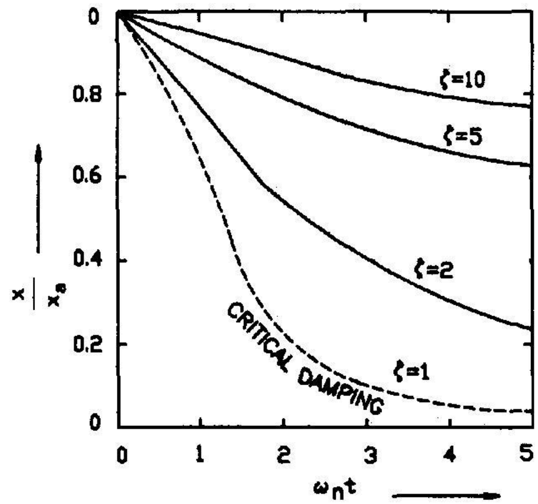


Fig 5 **Displacement-time plots of over-damped and critically damped systems with zero starting velocity**

4

(iii) Analysis of under damped system (ζ < 1).

In this case, the roots are complex conjugates and are given by

$$s_1 = \left[-\zeta + j\sqrt{1-\zeta^2}\right]\omega_n$$

$$s_2 = \left[-\zeta - j\sqrt{1-\zeta^2}\right]\omega_n$$

The solution of the differential equation becomes

$$x = C_1 e^{\left(-\zeta + j\sqrt{1-\zeta^2}\right)\omega_n t} + C_2 e^{\left(-\zeta - j\sqrt{1-\zeta^2}\right)\omega_n t}$$

This equation can be rewritten as

$$x = e^{-\zeta\omega_n t}\left[C_1 e^{j\sqrt{1-\zeta^2}\,\omega_n t} + C_2 e^{-j\sqrt{1-\zeta^2}\,\omega_n t}\right] \qquad (11)$$

Using the relationships

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

Equation (11) can be written as

$$x = e^{-\zeta\omega_n t}\left[C_1\left\{\cos\sqrt{1-\zeta^2}\,\omega_n t + j\sin\sqrt{1-\zeta^2}\,\omega_n t\right\} + C_2\left\{\cos\sqrt{1-\zeta^2}\,\omega_n t - j\sin\sqrt{1-\zeta^2}\,\omega_n t\right\}\right]$$

$$\sqrt{1-\zeta^2} \qquad \sqrt{1-\zeta^2} \tag{12}$$

In equation (12), constants (C_1+C_2) and j(C_1-C_2) are real quantities and hence, the equation can also be written as

$$x = e^{-\zeta\omega_n t}\left[A\left\{\cos\sqrt{1-\zeta^2}\,\omega_n t\right\} + B\left\{\sin\sqrt{1-\zeta^2}\,\omega_n t\right\}\right]$$

Or,

$$x = A_1 e^{-\zeta\omega_n t}\left[\left\{\sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \varphi_1\right)\right\}\right] \tag{13}$$

The above equations represent oscillatory motion and the frequency of this motion is represented by

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n \tag{14}$$

Where, ω_d is the damped natural frequency of the system. Constants A_1 and Φ_1 are determined by applying initial conditions. The typical response curve of an under damped system is shown in Fig.6.

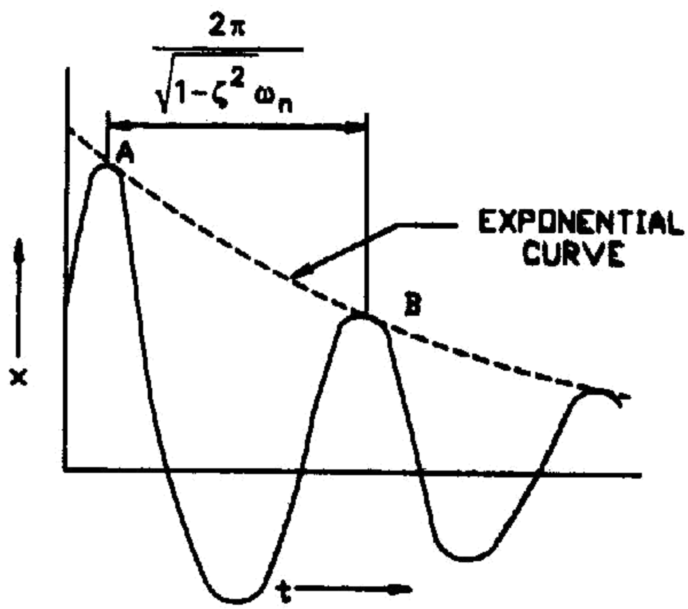


Fig 6 Typical response curve of an under damped system

Applying initial conditions,

$\underline{x = X_o}$ at t = 0; and $\dot{x} = 0$ at t = 0, and finding constants A_1 and Φ_1, equation (13) becomes

$$x = \frac{X_o}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \tag{15}$$

The term $\frac{X_o}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}$ represents the amplitude of vibration, which is observed to decay exponentially with time.

6

LOGARITHMIC DECREMENT

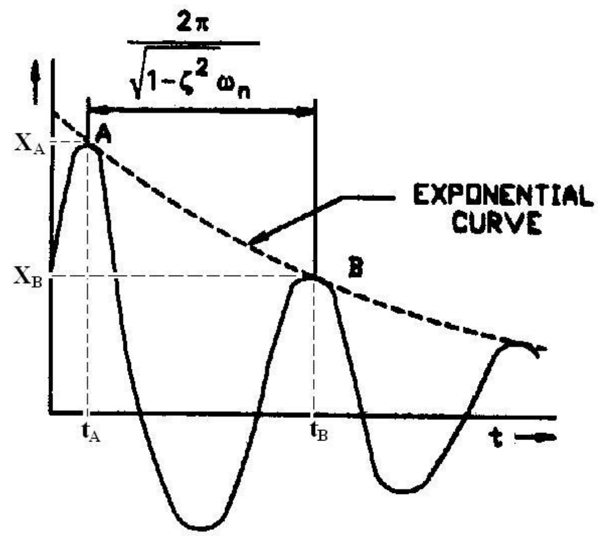


Fig.7 Logarithmic decrement

Referring to Fig.7, points A & B represent two successive peak points on the response curve of an under damped system. X_A and X_B represent the amplitude corresponding to points A & B and t_A & t_B represents the corresponding time.

We know that the natural frequency of damped vibration = $\omega_d = \sqrt{1-\zeta^2}\,\omega_n$ rad/sec.

Therefore, $f_d = \dfrac{\omega_d}{2\pi}$ cycles/sec

Hence, time period of oscillation = $t_B - t_A = \dfrac{1}{f_d} = \dfrac{2\pi}{\omega_d} = \dfrac{2\pi}{\sqrt{1-\zeta^2}\,\omega_n}$ sec (16)

From equation (15), amplitude of vibration

$$X_A = \frac{X_o}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t_A}$$

$$X_B = \frac{X_o}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t_B}$$

Or, $$\frac{X_A}{X_B} = e^{-\zeta\omega_n(t_A - t_B)} = e^{\zeta\omega_n(t_B - t_A)}$$

Using eqn. (16), $$\frac{X_A}{X_B} = e^{\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}}$$

Or, $$\log_e \frac{X_A}{X_B} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

7

This is called logarithmic decrement. It is defined as the logarithmic value of the ratio of two successive amplitudes of an under damped oscillation. It is normally denoted by δ.

$$\text{Therefore, } \delta = \log_e \frac{X_A}{X_B} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \qquad (17)$$

This indicates that the ratio of any two successive amplitudes of an under damped system is constant and is a function of damping ratio of the system.

For small values of ζ, $\delta \approx 2\pi\zeta$

If X_0 represents the amplitude at a particular peak and X_n represents the amplitude after 'n' cycles, then, logarithmic decrement = $\delta = \log_e \frac{X_0}{X_1} = \log_e \frac{X_1}{X_2} = \ldots\ldots = \log_e \frac{X_{n-1}}{X_n}$

Adding all the terms, $n\delta = \log_e \frac{X_0}{X_1} \cdot \frac{X_1}{X_2} \ldots\ldots \frac{X_{n-1}}{X_n}$

$$\text{Or, } \delta = \frac{1}{n} \log_e \frac{X_0}{X_n} \qquad (18)$$

MECHANICAL VIBRATION

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Solved problems

1) The mass of a spring-mass-dashpot system is given an initial velocity $5\omega_n$, where ω_n is the undamped natural frequency of the system. Find the equation of motion for the system, when (i) $\zeta = 2.0$, (ii) $\zeta = 1.0$, (i) $\zeta = 0.2$.

Solution:

Case (i) For $\zeta = 2.0$ – Over damped system

For over damped system, the response equation is given by

$$x = C_1 e^{\left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t} + C_2 e^{\left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t}$$

Substituting $\zeta = 2.0$, $x = C_1 e^{[-0.27]\omega_n t} + C_2 e^{[-3.73]\omega_n t}$ (a)

Differentiating, $\dot{x} = -0.27\omega_n C_1 e^{-0.27\omega_n t} - 3.73\omega_n C_2 e^{-3.73\omega_n t}$ (b)

Substituting the initial conditions

$x = 0$ at $t = 0$; and $\dot{x} = 5\omega_n$ at $t = 0$ in (a) & (b),

$0 = C_1 + C_2$ (c)

$5\omega_n = -0.27\,\omega_n C_1 - 3.73\,\omega_n C_2$ (d)

Solving (c) & (d), $C_1 = 1.44$ and $C_2 = -1.44$.

Therefore, the response equation becomes

$$x = 1.44\left(e^{[-0.27]\omega_n t} - e^{[-3.73]\omega_n t}\right) \quad \text{(e)}$$

Case (ii) For $\zeta = 1.0$ – Critically damped system

For critically damped system, the response equation is given by

$$x = (C_1 + C_2 t)e^{-\omega_n t} \quad \text{(f)}$$

Differentiating, $$\dot{x} = -(C_1 + C_2 t)\omega_n e^{-\omega_n t} + C_2 e^{-\omega_n t} \quad \text{(g)}$$

Substituting the initial conditions

x = 0 at t = 0; and $\dot{x} = 5\omega_n$ at t = 0 in (f) & (g),

$C_1 = 0$ and $C_2 = 5\omega_n$

Substituting in (f), the response equation becomes

$$x = (5\omega_n t)e^{-\omega_n t} \quad \text{(h)}$$

Case (iii) For $\zeta = 0.2$ – under damped system

For under damped system, the response equation is given by

$$x = A_1 e^{-\zeta\omega_n t}\left[\left\{\sin\sqrt{1-\zeta^2}\,\omega_n t + \varphi_1\right\}\right]$$

Substituting $\zeta = 0.2$, $$x = A_1 e^{-0.2\omega_n t}\left[\left\{\sin\left(0.98\omega_n t + \varphi_1\right)\right\}\right] \quad \text{(p)}$$

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Differentiating,

$$\dot{x} = -0.2\omega_n A_1 e^{-0.2\omega_n t}\left[\left\{\sin\left(0.98\omega_n t + \varphi_1\right)\right\}\right] + 0.98\omega_n A_1 e^{-0.2\omega_n t}\cos\left(0.98\omega_n t + \varphi_1\right) \quad \text{(q)}$$

Substituting the initial conditions

x = 0 at t = 0; and $\dot{x} = 5\omega_n$ at t = 0 in (p) & (q),

$A_1\sin\Phi_1 = 0$ and $A_1\cos\Phi_1 = 5.1$

Solving, $A_1 = 5.1$ and $\Phi_1 = 0$

Substituting in (p), the response equation becomes

$$x = 5.1e^{-0.2\omega_n t}\left[\left\{\sin\left(0.98\omega_n t\right)\right\}\right] \quad \text{(r)}$$

2) A mass of 20kg is supported on two isolators as shown in fig.Q.2. Determine the undamped and damped natural frequencies of the system, neglecting the mass of the isolators.

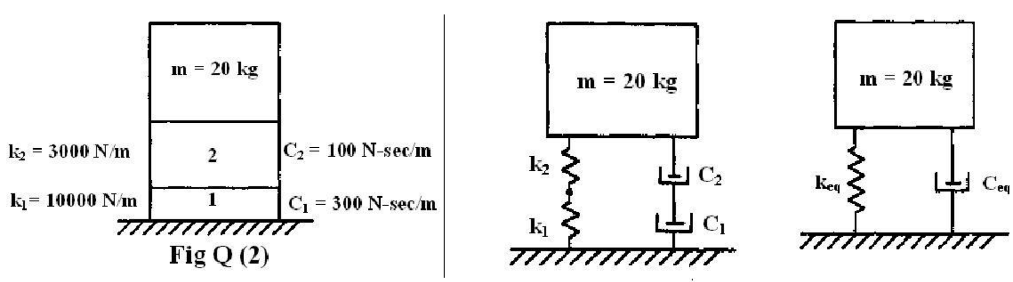

Fig Q (2)

Solution:

Equivalent stiffness and equivalent damping coefficient are calculated as

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} = \frac{1}{10000} + \frac{1}{3000} = \frac{13}{30000}$$

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{300} + \frac{1}{100} = \frac{4}{300}$$

Undamped natural frequency = $\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{30000/13}{20}} = 10.74\,rad/\sec$

$$f_n = \frac{10.74}{2\pi} = 1.71\,cps$$

Damped natural frequency = $\omega_d = \sqrt{1-\zeta^2}\,\omega_n$

$$\zeta = \frac{C_{eq}}{2\sqrt{k_{eq}m}} = \frac{300/4}{2\cdot\sqrt{30000/13\cdot 20}} = 0.1745$$

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$$\therefore \omega_d = \sqrt{1 - 0.1745^2} \cdot 10.74 = 10.57 rad / \sec$$

Or, $$f_d = \frac{10.57}{2 \cdot \pi} = 1.68 cps$$

3) A gun barrel of mass 500kg has a recoil spring of stiffness 3,00,000 N/m. If the barrel recoils 1.2 meters on firing, determine,

(a) initial velocity of the barrel

(b) critical damping coefficient of the dashpot which is engaged at the end of the recoil stroke

(c) time required for the barrel to return to a position 50mm from the initial position.

Solution:

(a) Strain energy stored in the spring at the end of recoil:

$$P = \frac{1}{2} kx^2 = \frac{1}{2} \cdot 300000 \cdot 1.2^2 = 216000 N - m$$

Kinetic energy lost by the gun barrel:

$$T = \frac{1}{2} mv^2 = \frac{1}{2} \cdot 500 \cdot v^2 = 250 v^2$$, where v = initial velocity of the barrel

Equating kinetic energy lost to strain energy gained, ie T = P, $250v^2 = 216000$

v = 29.39m/s

(b) Critical damping coefficient = $C_c = 2\sqrt{km} = 2\sqrt{300000 \cdot 500} = 24495 N-\text{sec}/m$

(c) Time for recoiling of the gun (undamped motion):

Undamped natural frequency = $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{300000}{500}} = 24.49 r/s$

Time period = $\tau = \dfrac{2\pi}{\omega_n} = \dfrac{2\pi}{24.29} = 0.259$ sec

Time of recoil = $\dfrac{\tau}{4} = \dfrac{0.259}{4} = 0.065$ sec

Time taken during return stroke:

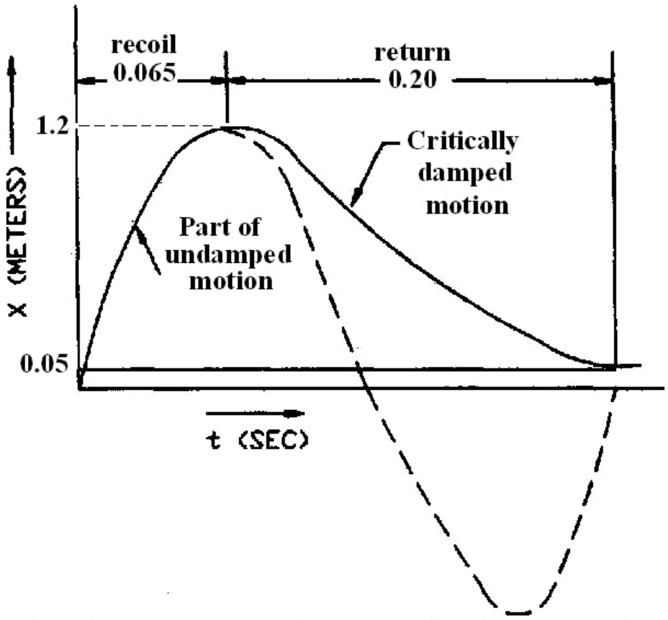
Response equation for critically damped system = $x = (C_1 + C_2 t)e^{-\omega_n t}$

Differentiating, $\dot{x} = C_2 e^{-\omega_n t} - (C_1 + C_2 t)\omega_n e^{-\omega_n t}$

Applying initial conditions, x = 1.2, at t = 0 and $\dot{x} = 0$ at t = 0, $C_1 = 1.2$, & $C_2 = 29.39$

Therefore, the response equation = $x = (1.2 + 29.39t)e^{-24.49t}$ When x = 0.05m, by trial and error, t = 0.20 sec

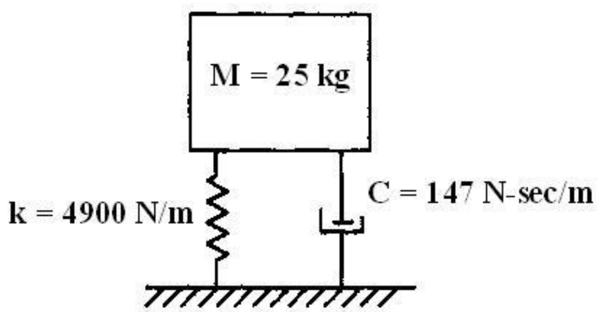
Therefore, total time taken = time for recoil + time for return = 0.065 + 0.20 = 0.265 sec
The displacement – time plot is shown in the following figure.



Displacement – time plot of recoil and return stroke

4) A 25 kg mass is resting on a spring of 4900 N/m and dashpot of 147 N-se/m in parallel. If a velocity of 0.10 m/sec is applied to the mass at the rest position, what will be its displacement from the equilibrium position at the end of first second?

Solution:



The above figure shows the arrangement of the system.

Critical damping coefficient = $c_c = 2m\omega_n$

Where $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4900}{25}} = 14 r/s$

Therefore, $c_c = 2 \cdot 25 \cdot 14 = 700\ N - \sec/m$

Since C< C_c, the system is under damped and $\zeta = \frac{c}{c_c} = \frac{147}{700} = 0.21$

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Hence, the response equation is $x = A_1 e^{-\zeta\omega_n t}\left[\left\{\sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \varphi_1\right)\right\}\right]$

Substituting ζ and ω_n, $x = A_1 e^{-0.21\cdot 14t}\left[\left\{\sin\left(\sqrt{1-0.21^2}\,14t + \varphi_1\right)\right\}\right]$ $x = A_1 e^{-2.94t}\left[\left\{\sin\left(13.7t + \varphi_1\right)\right\}\right]$

Differentiating, $\dot{x} = -2.94\,A_1 e^{-2.94t}\left[\left\{\sin\left(13.7t + \varphi_1\right)\right\}\right] + 13.7\,A_1 e^{-2.94t}\cos\left(13.7t + \varphi_1\right)$

Applying the initial conditions, x = 0, at t = 0 and $\dot{x} = 0.10 m/s$ at t = 0

$\Phi_1 = 0$

$0.10 = -2.94\,A_1\left[\left\{\sin\left(\varphi_1\right)\right\}\right] + 13.7\,A_1\cos\left(\varphi_1\right)$

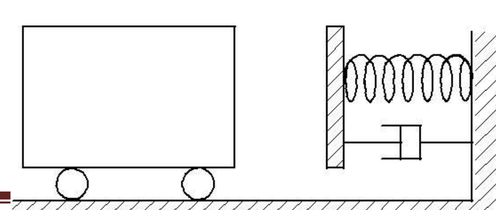
Since, $\Phi_1 = 0$, $0.10 = 13.7\,A_1$; $A_1 = 0.0073$

Displacement at the end of 1 second = $x = 0.0073e^{-2.94}\left[\left\{\sin\left(13.7\right)\right\}\right] = 3.5\cdot 10^{-4}\,m$

5) A rail road bumper is designed as a spring in parallel with a viscous damper. What is the bumper's damping coefficient such that the system has a damping ratio of 1.25, when the bumper is engaged by a rail car of 20000 kg mass. The stiffness of the spring is 2E5 N/m. If the rail car engages the bumper, while traveling at a speed of 20m/s, what is the maximum deflection of the bumper?

k

m



c

Solution: Data = m = 20000 kg; k = 200000 N/m; ζ = 1.25

Critical damping coefficient =

$$c_c = 2 \cdot \sqrt{m \cdot k} = 2 \cdot \sqrt{20000 \cdot 200000} = 1.24 \cdot 10^5 \, N - \sec/m$$

Damping coefficient $C = \zeta \cdot C_C = 1.25 \cdot 1.24 \cdot 10^5 = 1.58 \cdot 10^5 \, N - \sec/m$

Undamped natural frequency = $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{200000}{20000}} = 3.16 r/s$

Since ζ = 1.25 , the system is over damped.

For over damped system, the response equation is given by

$$x = C_1 e^{\left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t} + C_2 e^{\left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t}$$

Substituting ζ = 1.25, $x = C_1 e^{[-0.5]\omega_n t} + C_2 e^{[-2.0]\omega_n t}$ (a)

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Differentiating, $\dot{x} = -0.5\omega_n C_1 e^{-0.5\omega_n t} - 2.0\omega_n C_2 e^{-2.0\omega_n t}$ (b)

Substituting the initial conditions

x = 0 at t = 0; and $\dot{x} = 20 m/s$ at t = 0 in (a) & (b),

$0 = C_1 + C_2$ (c)

$20 = -0.5\,\omega_n C_1 - 2.0\,\omega_n C_2$ (d)

Solving (c) & (d), $C_1 = 4.21$ and $C_2 = -4.21$

Therefore, the response equation becomes

$$x = 4.21\left(e^{[-1.58 \cdot t]} - e^{[-6.32 \cdot t]}\right)m \qquad \text{(e)}$$

The time at which, maximum deflection occurs is obtained by equating velocity equation to zero.

Ie, $\dot{x} = -0.5\omega_n C_1 e^{-0.5\omega_n t} - 2.0\omega_n C_2 e^{-2.0\omega_n t} = 0$

Ie, $-6.65e^{-1.58t} + 26.61e^{-6.32t} = 0$

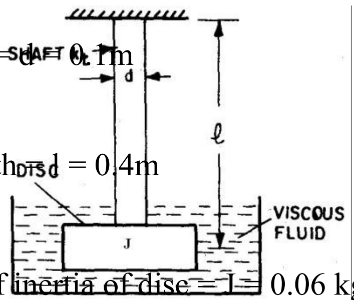
Solving the above equation, t = 0.292 secs.

Therefore, maximum deflection at t = 0.292secs,

Substituting in (e), $x = 4.21\left(e^{[-1.58 \cdot 0.292]} - e^{[-6.32 \cdot 0.292]}\right)m$, = 1.99m.

6) A disc of a torsional pendulum has a moment of inertia of 6E-2 kg-m^2 and is immersed in a viscous fluid. The shaft attached to it is 0.4m long and 0.1m in

diameter. When the pendulum is oscillating, the observed amplitudes on the same side of the mean position for successive cycles are 9^0, 6^0 and 4^0. Determine (i) logarithmic decrement (ii) damping torque per unit velocity and (iii) the periodic time of vibration. Assume G = 4.4E10 N/m^2, for the shaft material.



Shaft dia. = d = 0.1m

Shaft length = l = 0.4m

Moment of inertia of disc = J = 0.06 kg-m^2.

Modulus of rigidity = G = 4.4E10 N/m^2

Solution:

The above figure shows the arrangement of the system.

$$\delta = \log_e \frac{9}{6} = \log_e \frac{6}{4} = 0.405$$

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We know that logarithmic decrement $= \delta = \dfrac{2\pi\zeta}{\sqrt{1-\zeta^2}}$, rearranging which, we get

Damping factor $\zeta = \dfrac{\delta}{\sqrt{4\pi^2+\delta^2}} = \dfrac{0.405}{\sqrt{4\pi^2+0.405^2}} = 0.0645$

Also, $\zeta = \dfrac{C}{C_C}$, where, critical damping coefficient $= C_C = 2\sqrt{k_t J}$

Torsional stiffness $= k_t = \dfrac{GI_p}{l} = \dfrac{G}{l}\cdot\dfrac{\pi d^4}{32} = \dfrac{4.4\cdot 10^{10}}{0.4}\cdot\dfrac{\pi\cdot 0.1^4}{32} = 1.08\cdot 10^6\ N-m/rad$

Critical damping coefficient $= C_C = 2\sqrt{k_t J} = 2\sqrt{1.08\cdot 10^6\cdot 0.06} = 509 N-m/rad$

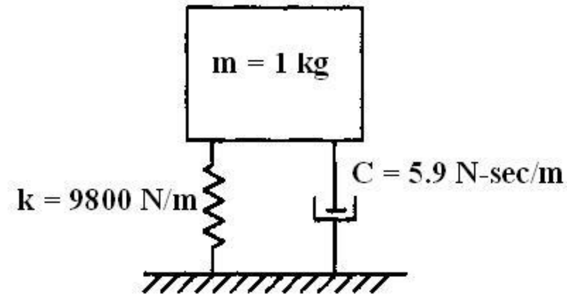
Damping coefficient of the system $= C = C_C\cdot\zeta = 509\cdot 0.0645 = 32.8 N-m/rad$

(iii) Periodic time of vibration $= \tau = \dfrac{1}{f_d} = \dfrac{2\pi}{\omega_d} = \dfrac{2\pi}{\omega_n\sqrt{1-\zeta^2}}$

Where, undamped natural frequency $= \omega_n = \sqrt{\dfrac{k_t}{J}} = \sqrt{\dfrac{1.08\cdot 10^6}{0.06}} = 4242.6 rad/\sec$

Therefore, $\tau = \dfrac{2\pi}{4242.6\cdot\sqrt{1-0.0645^2}} = 0.00148 \sec$

7) A mass of 1 kg is to be supported on a spring having a stiffness of 9800 N/m. The damping coefficient is 5.9 N-sec/m. Determine the natural frequency of the system. Find also the logarithmic decrement and the amplitude after three cycles if the initial displacement is 0.003m.



Solution:

Undamped natural frequency = $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{9800}{1}} = 99 r/s$

Damped natural frequency = $\omega_d = \sqrt{1 - \zeta^2}\,\omega_n$

Critical damping coefficient = $c_c = 2 \cdot m \cdot \omega_n = 2 \cdot 1 \cdot 99 = 198 N - \sec/m$

Damping factor = $\zeta = \dfrac{c}{c_c} = \dfrac{5.9}{198} = 0.03$

Hence damped natural frequency = $\omega_d = \sqrt{1 - \zeta^2}\,\omega_n = \sqrt{1 - 0.013^2} \cdot 99 = 98.99 rad/\sec$

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Logarithmic decrement = $\delta = \dfrac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \dfrac{2\cdot\pi\cdot 0.03}{\sqrt{1-0.03^2}} = 0.188$

Also, $\delta = \dfrac{1}{n}\log_e \dfrac{X_0}{X_n}$; if $x_0 = 0.003$,

then, after 3 cycles, $\delta = \dfrac{1}{n}\log_e \dfrac{X_0}{X_n}$ ie, $0.188 = \dfrac{1}{3}\cdot\log_e \dfrac{0.003}{X_3}$

ie, $X_3 = \dfrac{0.003}{e^{3\cdot 0.188}} = 1.71\cdot 10^{-3}\, m$

8) The damped vibration record of a spring-mass-dashpot system shows the following data.

Amplitude on second cycle = 0.012m; Amplitude on third cycle = 0.0105m;

Spring constant k = 7840 N/m; Mass m = 2kg. Determine the damping constant, assuming it to be viscous.

Solution:

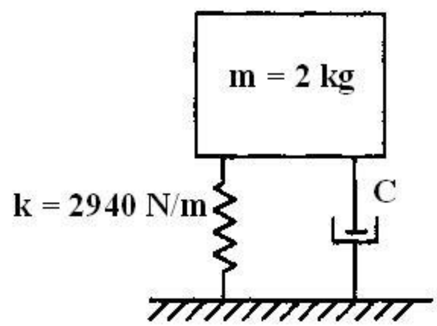
Here, $\delta = \log_e \dfrac{X_2}{X_3} = \log_e \dfrac{0.012}{0.0105} = 0.133$

Also, $\delta = \dfrac{2\pi\zeta}{\sqrt{1-\zeta^2}}$, rearranging, $\zeta = \dfrac{\delta}{\sqrt{4\pi^2+\delta^2}} = \dfrac{0.133}{\sqrt{4\pi^2+0.133^2}} = 0.021$

Critical damping coefficient = $c_c = 2\cdot\sqrt{m\cdot k} = 2\cdot\sqrt{2\cdot 7840} = 250.4 N-\sec/m$

Damping coefficient $C = \zeta \cdot C_C = 0.021 \cdot 250.4 = 5.26 N - sec/m$

9) A mass of 2kg is supported on an isolator having a spring scale of 2940 N/m and viscous damping. If the amplitude of free vibration of the mass falls to one half its original value in 1.5 seconds, determine the damping coefficient of the isolator.



Solution:

Undamped natural frequency = $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2940}{2}} = 38.34 r/s$

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Critical damping coefficient = $c_c = 2 \cdot m \cdot \omega_n = 2 \cdot 2 \cdot 38.34 = 153.4 N - \sec/m$

Response equation of under damped system = $x = A_1 e^{-\zeta\omega_n t}\left[\left\{\sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \varphi_1\right)\right\}\right]$

Here, amplitude of vibration = $A_1 e^{-\zeta\omega_n t}$

If amplitude = X_0 at t = 0, then, at t = 1.5 sec, amplitude = $\frac{X_0}{2}$ √

Ie, $A_1 e^{-\zeta\omega_n \cdot 0} = X_0$ or $A_1 = X_0$

Also, $A_1 e^{-\zeta\omega_n \cdot 1.5} = \frac{X_0}{2}$ or $X_0 \cdot e^{-\zeta \cdot 38.34 \cdot 1.5} = \frac{X_0}{2}$ or $e^{-\zeta \cdot 38.34 \cdot 1.5} = \frac{1}{2}$

Ie, $e^{\zeta \cdot 38.34 \cdot 1.5} = 2$, taking log, $\zeta \cdot 38.34 \cdot 1.5 = 0.69 \therefore \zeta = 0.012$

Damping coefficient $C = \zeta \cdot C_C = 0.012 \cdot 153.4 = 1.84 N - \sec/m$

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