Warm-up

Problem 1. Sort the following array using merge-sort: $A = [5, 8, 2, 0, 23, 786, -2, 65]$. Give all arrays on which recursive calls are made and show how they are merged back together.

Problem 2. Consider the following algorithm.

```
function REVERSE(A)
    if |A| = 1 then
        return A
    else
        B ← first half of A
        C ← second half of A
        return concatenate REVERSE(C) with REVERSE(B)
```

Let $T(n)$ be the running time of the algorithm on an instance of size n. Write down the recurrence relation for $T(n)$ and solve it by unrolling it.

Problem solving

Problem 3. Given an array A holding n objects, we want to test whether there is a *majority* element; that is, we want to know whether there is an object that appears in more than $n/2$ positions of A.

Assume we can test equality of two objects in $O(1)$ time, but we cannot use a dictionary indexed by the objects. Your task is to design an $O(n \log n)$ time algorithm for solving the majority problem.

a) Show that if x is a majority element in the array then x is a majority element in the first half of the array or the second half of the array

b) Show how to check in $O(n)$ time if a candidate element x is indeed a majority element.

c) Put these observation together to design a divide an conquer algorithm whose running time obeys the recurrence $T(n) = 2T(n/2) + O(n)$

d) Solve the recurrence by unrolling it.

Problem 4. Let A be an array with n distinct numbers. We say that two indices $0 \leq i < j < n$ form an inversion if $A[i] > A[j]$. Modify merge sort so that it computes the number of inversions of A.

Problem 5. Given a sorted array A containing distinct non-negative integers, find the smallest non-negative integer that isn't stored in the array. For simplicity, you can assume there is such an integer, i.e., $A[n-1] > n-1$

Example: $A = [0, 1, 3, 5, 7]$, result: 2

Problem 6. Design an $O(n)$ time algorithm for the majority problem.

Problem 1. Sort the following array using merge-sort: $A = [5, 8, 2, 0, 23, 786, -2, 65]$. Give all arrays on which recursive calls are made and show how they are merged back together.

5 8 2 0   23 786 -2 65

5 8   2 0   23 786   -2 65   Base Case

5 8   0 2   23 786   -2 65

0 2 5 8   -2 23 65 786

-2 0 2 23 58 65 786

Problem 2. Consider the following algorithm.

```
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```
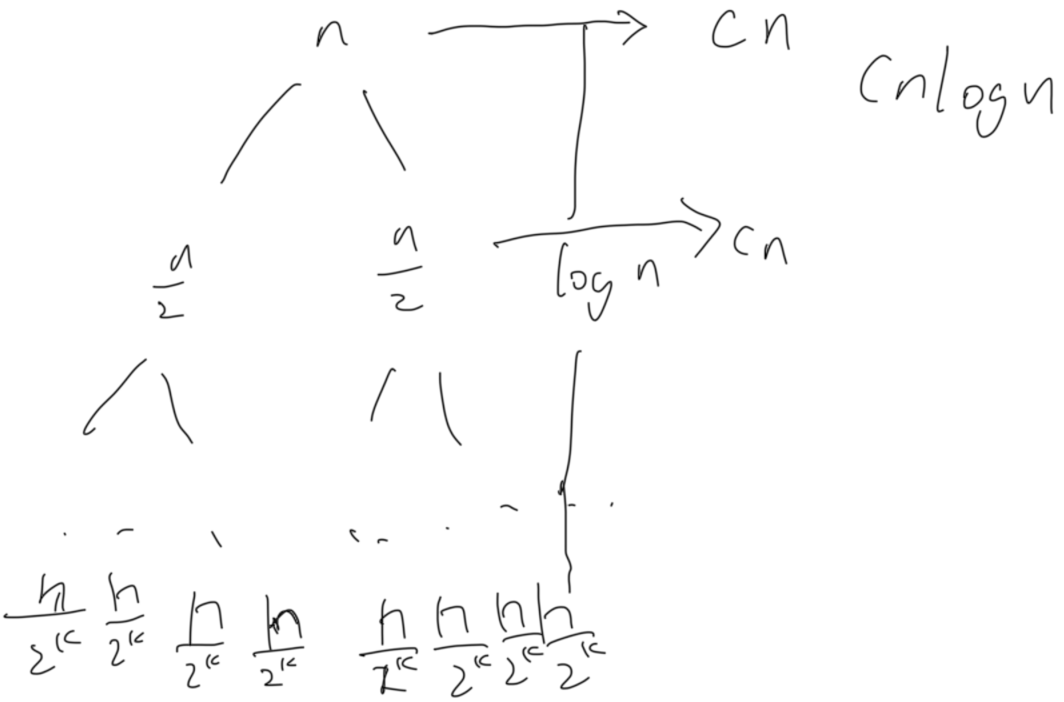

2 Recu

Let $T(n)$ be the running time of the algorithm on an instance of size n. Write down the recurrence relation for $T(n)$ and solve it by unrolling it.

$$T(n) = \begin{cases} 2T(\frac{n}{2}) + O(n) & n > 1 \\ O(1) & n = 1 \end{cases}$$

Concatenate takes O(n) because we need to create new Array with size n



Problem 3. Given an array A holding n objects, we want to test whether there is a *majority element*; that is, we want to know whether there is an object that appears in more than $n/2$ positions of A.

Assume we can test equality of two objects in $O(1)$ time, but we cannot use a dictionary indexed by the objects. Your task is to design an $O(n \log n)$ time algorithm for solving the majority problem.

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c) Put these observation together to design a divide an conquer algorithm whose running time obeys the recurrence $T(n) = 2T(n/2) + O(n)$

d) Solve the recurrence by unrolling it.

(a). Since there are more than $\frac{n}{2}$ A in the Array

· the number of A in the first half could be k, where $0 \le k \le \frac{n}{2}$

· then the number of A in the second half could be $1+\frac{n}{2}-k$

· If $k \le \frac{n}{4}$, then $\frac{n}{2}-k \geq \frac{n}{4}$

If $k \geq \frac{n}{4}$, then the requirement is also satisfied

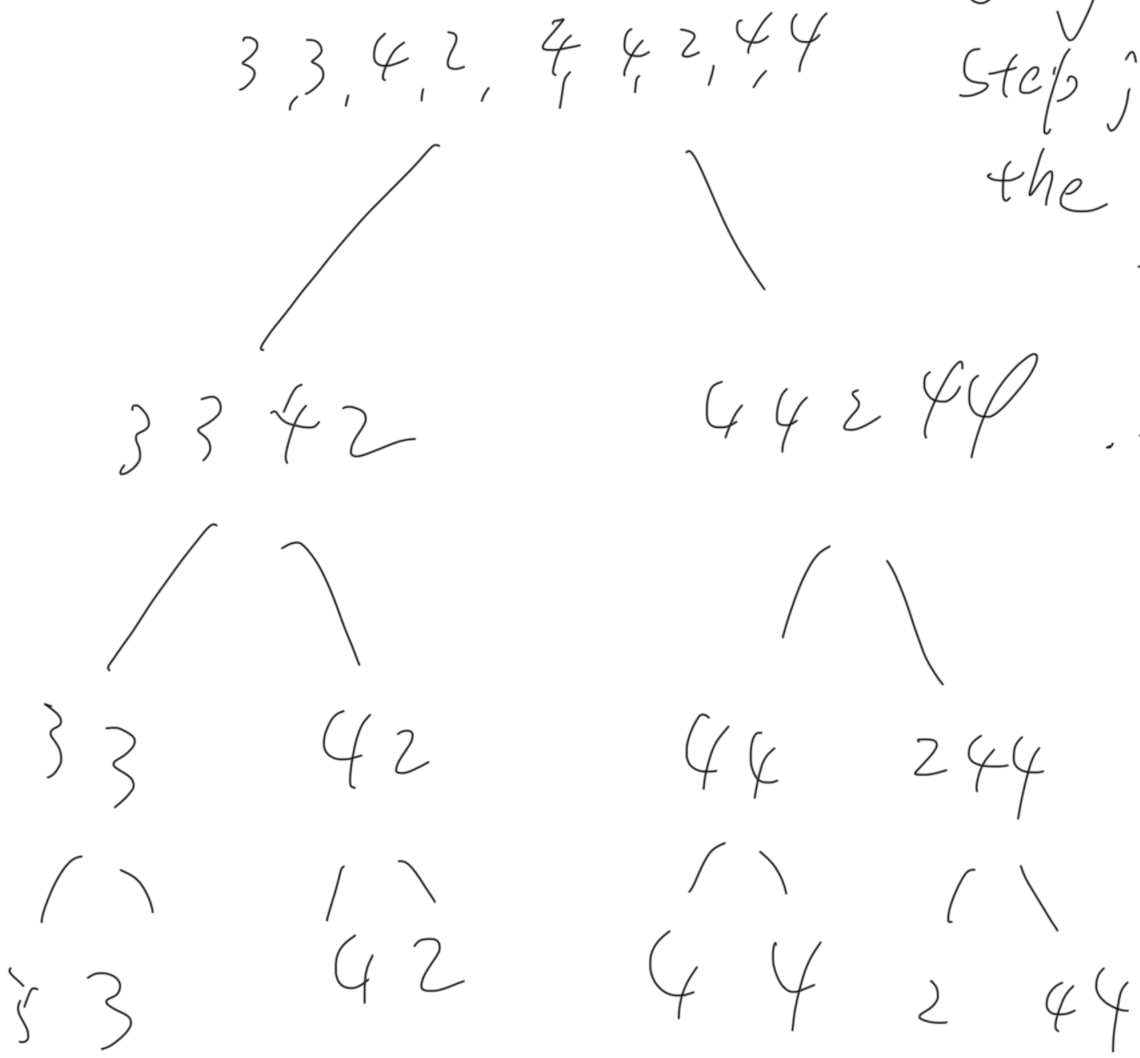
(b) just go through the array

(c) Similar to Q2, but we need not to concatenate

$$T(n)=\begin{cases} 2T(n)+O(n) & n>1 \\ O(1) & n=1 \end{cases}$$

- We ~~still need to merge array~~

- every recursive step just give the candidate to previous level



- then each level need to do a linear check (go through the array)

① dividing $O(1)$

② Conquer $O(n)$

同时计数传回 candidate 的数量，选出新的candidate

Problem 4. Let A be an array with n distinct numbers. We say that two indices $0 \le i < j < n$ form an inversion if $A[i] > A[j]$. Modify merge sort so that it computes the number of inversions of A.

$A = [0, 2, 1, 3]$ $\quad A[i] > A[j]$

$\quad\quad\quad\quad i \;\; j$

(1) 每次 merge 的时候，每当 put right array element into sorted list

# of current level inversion += # of elements currently in left side

$L[l] > R[r]$

Problem 5. Given a sorted array A containing distinct non-negative integers, find the smallest non-negative integer that isn't stored in the array. For simplicity, you can assume there is such an integer, i.e., $A[n-1] > n-1$

Example: $A = [0, 1, 3, 5, 7]$, result: 2

- We can do binary search since sorted list
- from $0 \to n$, do binary search respectively