Data structures and Algorithms

Lecture 5: Priority Queues
[GT 5]

André van Renssen
School of Computer Science

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THE UNIVERSITY OF SYDNEY

Priority Queue ADT

Special type of ADT map to store a collection of key-value items where we can only remove smallest key:

- insert(k, v): insert item with key k and value v
- remove_min(): remove and return the item with smallest key
- min(): return item with smallest key
- size(): return how many items are stored
- is_empty(): test if queue is empty

We can also have a max version of this min version, but we cannot use both versions at once.

Example

A sequence of priority queue methods:

Method	Return value	Priority queue
insert(5,A)		{(5,A)}
insert(9,C)		{(5,A),(9,C)}
insert(3,B)		{(3,B),(5,A),(9,C)}
min()	(3,B)	{(3,B),(5,A),(9,C)}
remove_min()	(3,B)	{(5,A),(9,C)}
insert(7,D)		{(5,A),(7,D),(9,C)}
remove_min()	(5,A)	{(7,D),(9,C)}
remove_min()	(7,D)	{(9,C)}
remove_min()	(9,C)	{}
is_empty()	true	{}

Application: Stock Matching Engines

At the heart of modern stock trading systems are highly reliable systems known as **matching engines,** which match the stock trades of buyers and sellers.

Buyers post bids to buy a number of shares of a given stock at or below a specified price

Sellers post offers (asks) to sell a number of shares of a given stock at or above a specified price.

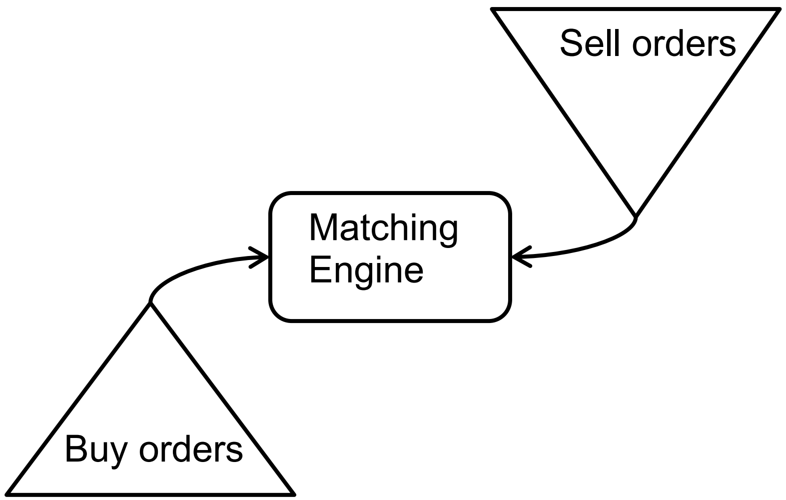
STOCK: EXAMPLE.COM

Buy Orders			Sell Orders		
Shares	**Price**	**Time**	**Shares**	**Price**	**Time**
1000	4.05	20 s	500	4.06	13 s
100	4.05	6 s	2000	4.07	46 s
2100	4.03	20 s	400	4.07	22 s
1000	4.02	3 s	3000	4.10	54 s
2500	4.01	81 s	500	4.12	2 s
			3000	4.20	58 s
			800	4.25	33 s
			100	4.50	92 s

Application: Stock Matching Engines

Buy and sell orders are organized according to a price-time priority, where price has highest priority and time is used to break ties

When a new order is entered, the matching engine determines if a trade can be immediately executed and if so, then it performs the appropriate matches according to price-time priority.



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Application: Stock Matching Engines

A matching engine can be implemented with two **priority queues**, one for buy orders and one for sell orders.

This data structure performs element removals based on priorities assigned to elements when they are inserted.

```
while True:
    bid ← buy_orders.remove_max()
    ask ← sell_orders.remove_min()
    if bid.price ≥ ask.price then
        carry out trade (bid, ask)
    else
        buy_orders.insert(bid)
        sell_orders.insert(ask)
```

STOCK: EXAMPLE.COM

Buy Orders			Sell Orders		
Shares	**Price**	**Time**	**Shares**	**Price**	**Time**
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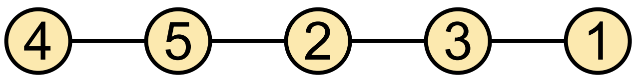
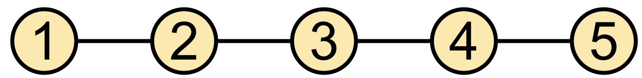
Sequence-based Priority Queue

Unsorted list implementation



- insert in O(1) time since we can insert the item at the beginning or end of the sequence
- remove_min and min in O(*n*) time since we have to traverse the entire list to find the smallest key

Sorted list implementation



- insert in O(n) time since we have to find the place where to insert the item
- remove_min and min in O(1) time since the smallest key is at the beginning

Method	Unsorted List	Sorted List
size, isEmpty	$O(1)$	$O(1)$
insert	$O(1)$	$O(n)$
min, removeMin	$O(n)$	$O(1)$

Priority Queue Sorting

We can use a priority queue to sort a list of keys:

1. iteratively insert keys into an empty priority queue
2. iteratively remove_min to get the keys in sorted order

Complexity analysis:

- n insert operations
- n remove_min operations

Either sequence-based implementation take $O(n^2)$

```
def priority_queue_sorting(A):
    pq ← new priority queue
    n ← size(A)
    for i in [0:n] do
        pq.insert(A[i])
    for i in [0:n] do
        A[i] = pq.remove_min()
```

Method	Unsorted List	Sorted List
size, isEmpty	$O(1)$	$O(1)$
insert	$O(1)$	$O(n)$
min, removeMin	$O(n)$	$O(1)$

Selection-Sort

Variant of pq-sort using unsorted sequence implementation:

1. inserting elements with n insert operations takes O(n) time
2. removing elements with n remove_min operations takes O(n^2)

Can be done in place
(no need for extra space)

Top level loop invariant:

- A[0:i] is sorted
- A[i:n] is the priority queue and all ⩾ A[i-1]

```
def selection_sort(A):
    n ← size(A)
    for i in [0:n] do
        # find s ⩾ i minimizing A[s]
        s ← i
        for j in [i:n] do
            if A[j] < A[s] then
                s ← j
        # swap A[i] and A[s]
        A[i], A[s] ← A[s], A[i]
```

Selection-Sort Example

i	A	s
0	<u>7</u>, 4, 8, <u>2</u>, 5, 3, 9	3
1	2, <u>4</u>, 8, 7, 5, <u>3</u>, 9	5
2	2, 3, <u>8</u>, 7, 5, <u>4</u>, 9	5
3	2, 3, 4, <u>7</u>, <u>5</u>, 8, 9	4
4	2, 3, 4, 5, <u>7</u>, 8, 9	4
5	2, 3, 4, 5, 7, <u>8</u>, 9	5
6	2, 3, 4, 5, 7, 8, <u>9</u>	6

```
def selection_sort(A):
    n ← size(A)
    for i in [0:n] do
        # find s ⩾ i minimizing A[s]
        s ← i
        for j in [i:n] do
            if A[j] < A[s] then
                s ← j
        # swap A[i] and A[s]
        A[i], A[s] ← A[s], A[i]
```

Insertion-Sort

Variant of pq-sort using sorted sequence implementation:

1. inserting elements with n insert operations takes $O(n^2)$ time
2. removing elements with n remove_min operations takes $O(n)$

Can be done in place
(no need for extra space)

Top level loop invariant:

- A[0:i] is the priority queue (and thus sorted)
- A[i:n] is yet-to-be-inserted

```
def insertion_sort(A):
    n ← size(A)
    for i in [1:n] do
        x ← A[i]
        # move forward entries > x
        j ← i
        while j > 0 and x < A[j-1] do
            A[j] ← A[j-1]
            j ← j - 1
        # if j>0 ⇒ x ≥ A[j-1]
        # if j<i ⇒ x < A[j+1]
        A[j] ← x
```

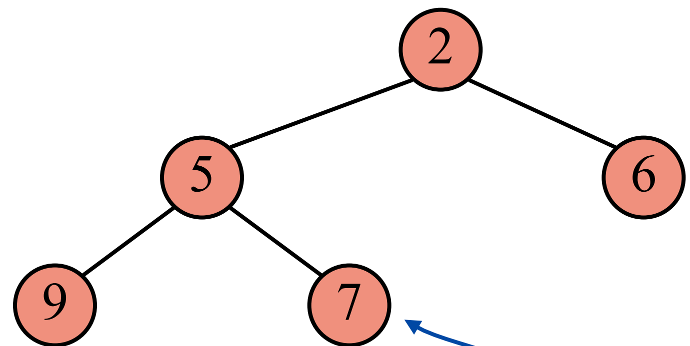
Insertion-Sort Example

i	A	j
1	7, 4, 8, 2, 5, 3, 9	0
2	4, 7, 8, 2, 5, 3, 9	2
3	4, 7, 8, 2, 5, 3, 9	0
4	2, 4, 7, 8, 5, 3, 9	2
5	2, 4, 5, 7, 8, 3, 9	1
6	2, 3, 4, 5, 7, 8, 9	6

```
def insertion_sort(A):
    n ← size(A)
    for i in [1:n] do
        x ← A[i]
        # move forward entries > x
        j ← i
        while j > 0 and x < A[j-1] do
            A[j] ← A[j-1]
            j ← j - 1
        # if j>0 ⇒ x ≥ A[j-1]
        # if j<i ⇒ x < A[j+1]
        A[j] ← x
```

Heap data structure (min-heap)

A heap is a binary tree storing (key, value) items at its nodes, satisfying the following properties:



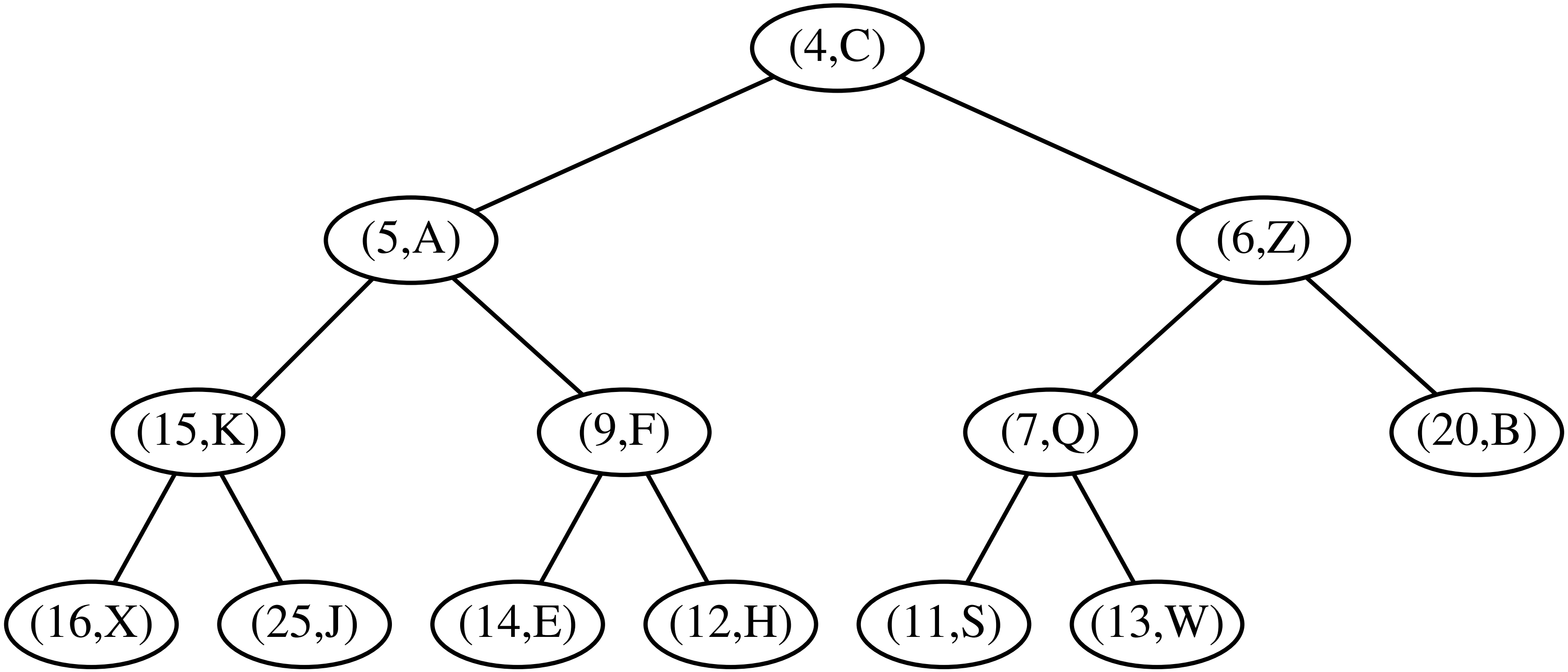
1. Heap-Order: for every node m ≠ root,

   key(m) ≥ key(parent(m))

2. Complete Binary Tree: let *h* be the height
   - every level i < h is full (i.e., there are 2^i nodes)
   - remaining nodes take leftmost positions of level h

The last node is the rightmost node of maximum depth

Example

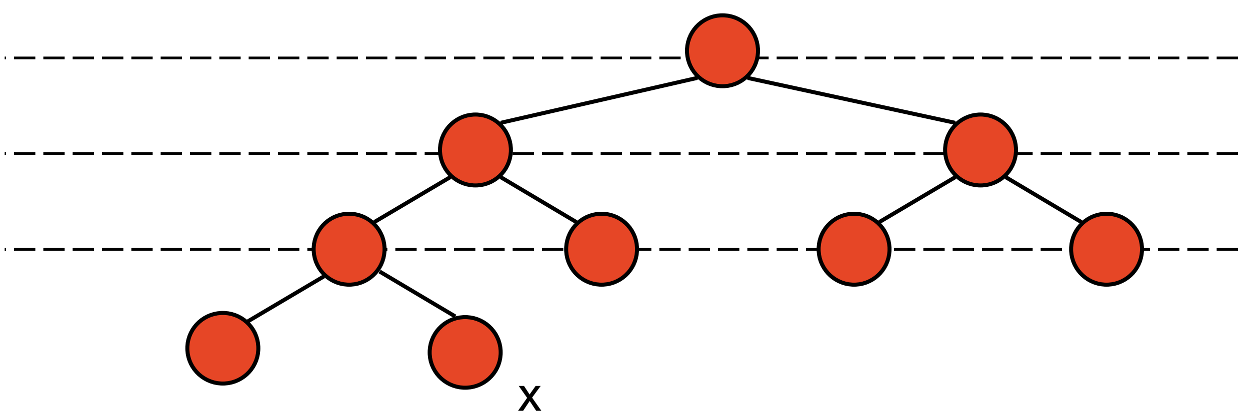


Minimum of a Heap

Fact: The root always holds the smallest key in the heap

Proof:

- Suppose the minimum key is at some internal node x
- Because of the heap property, as we move up the tree, the keys can only get smaller (assuming repeats, otherwise contradiction)
- If x is not the root, then its parent must also hold a smallest key
- Keep going until we reach the root

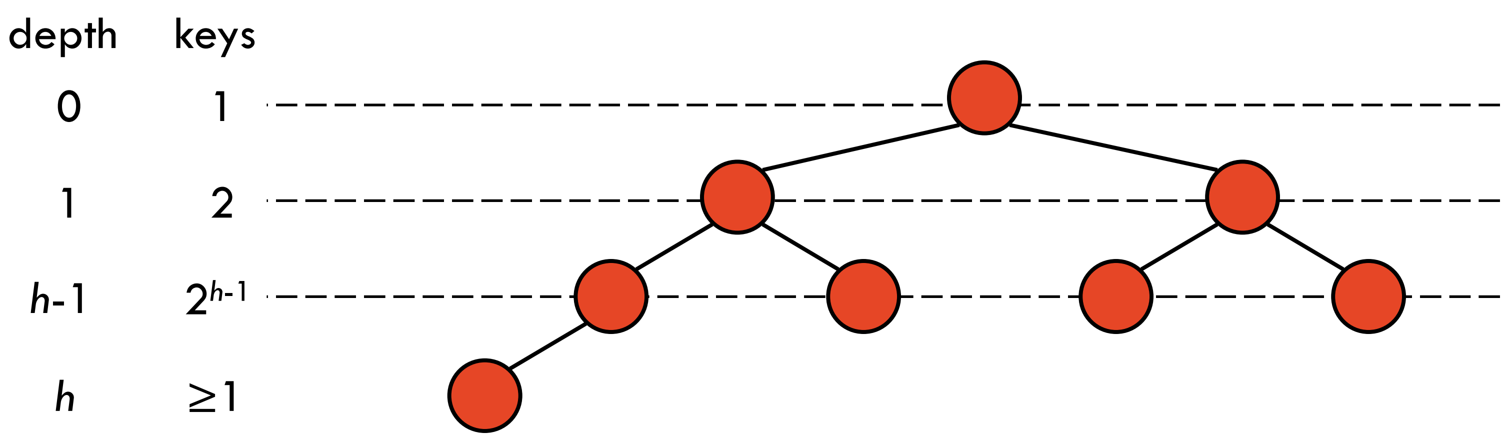


Height of a Heap

Fact: A heap storing n keys has height $\log n$

Proof:

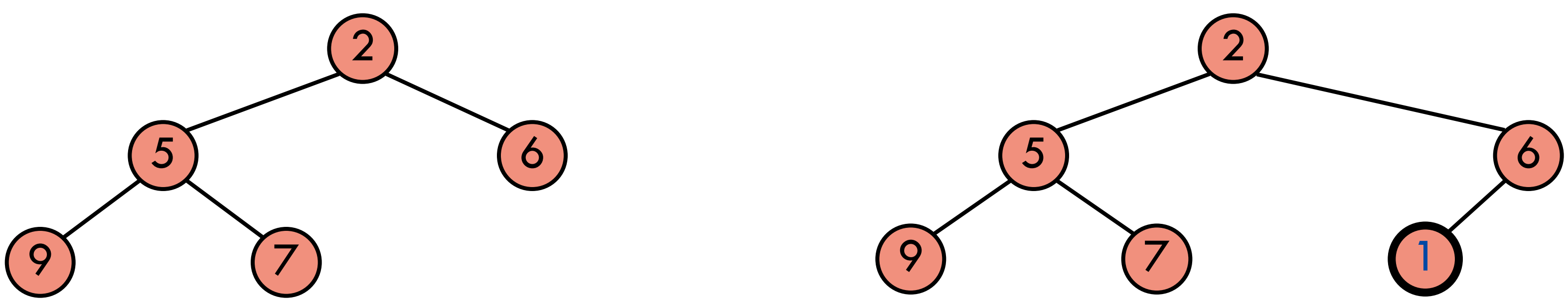
- Let h be the height of a heap storing n keys
- Since there are 2^i keys at depth $i = 0, \ldots, h - 1$ and at least one key at depth h, we have $n \geq 1 + 2 + 4 + \ldots + 2^{h-1} + 1$
- Thus, $n \geq 2^h$, applying $\log_2$ on both sides, $\log_2 n \geq h$



Insertion into a Heap

- Create a new node with given key
- Find location for new node
- Restore the heap-order property

`insert(1)`



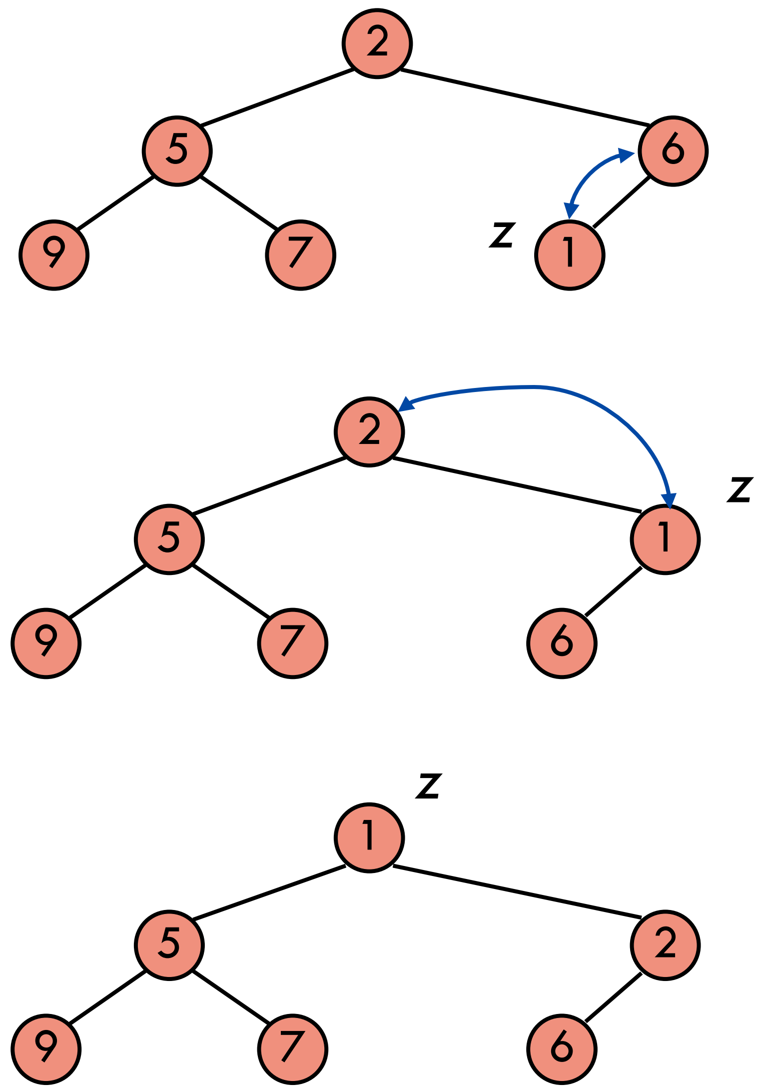
Upheap

Restore heap-order property by swapping keys along upward path from insertion point

```
def up_heap(z):
    while z ≠ root and
          key(parent(z)) > key(z) do
        swap key of z and parent(z)
        z ← parent(z)
```

Correctness: after swapping the subtree rooted at z has the property

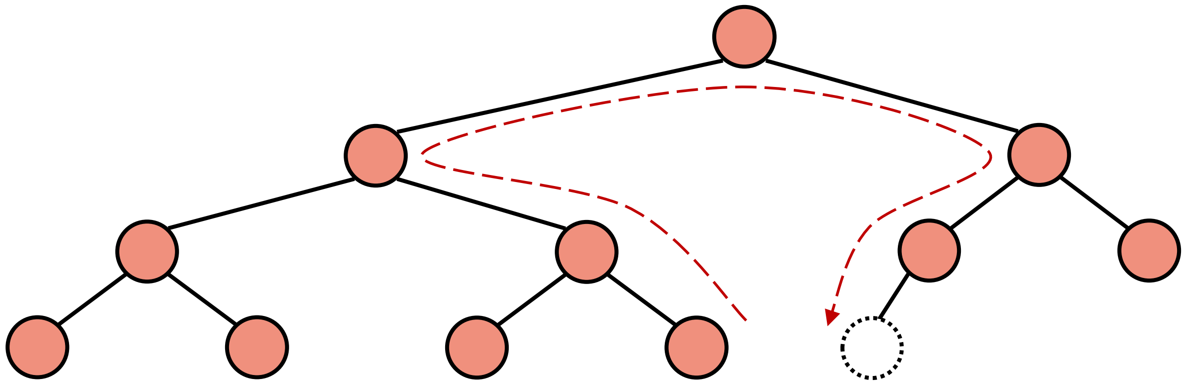
Complexity: O(log n) time because the height of the heap is log n



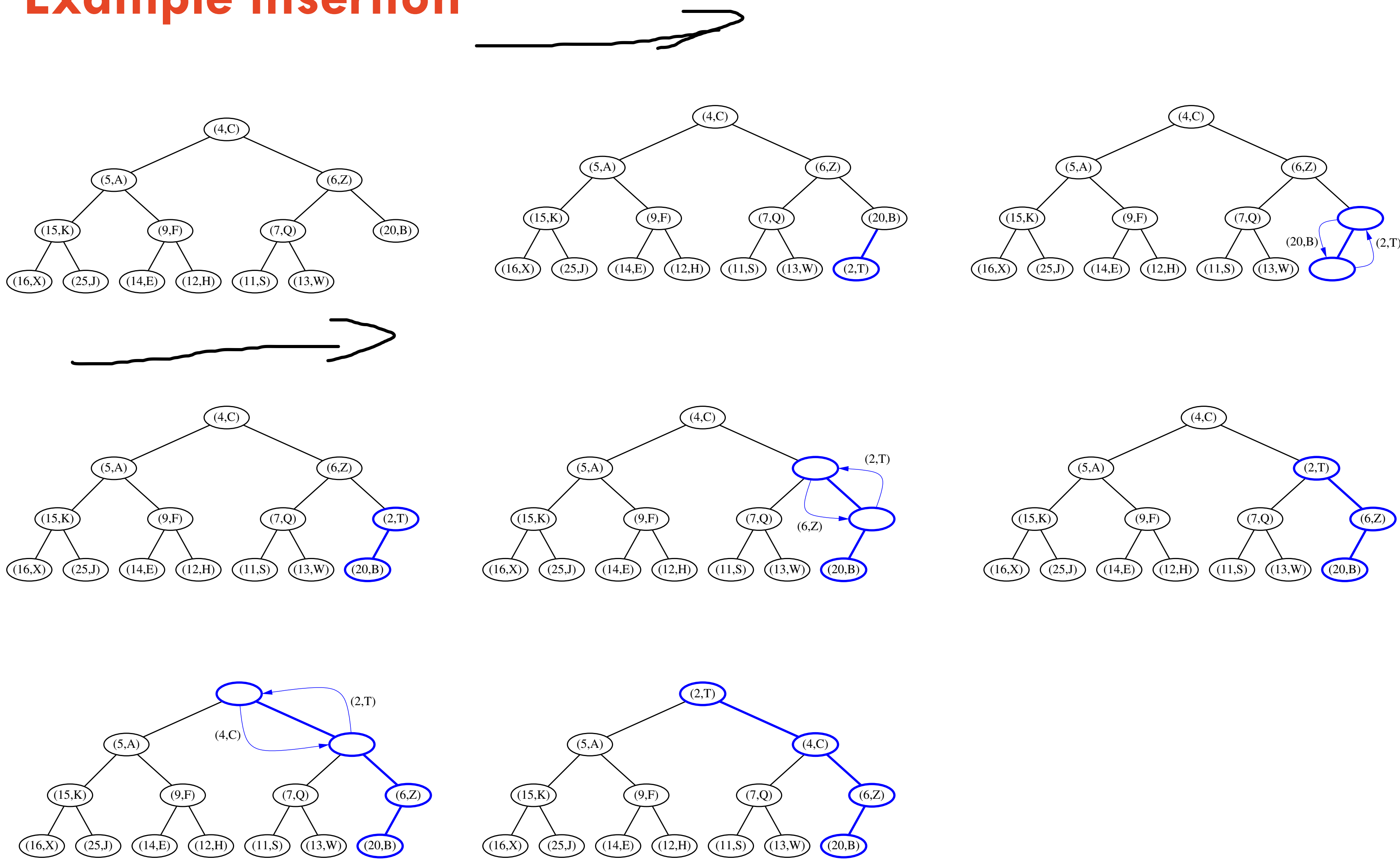
Finding the position for insertion

- start from the last node
- go up until a left child or the root is reached
- if we reach the root then need to open a new level
- otherwise, go to the sibling (right child of parent)
- go down left until a leaf is reached

Complexity of this search is O(log n) because the height is log n.
Thus, overall complexity of insertion is O(log n) time



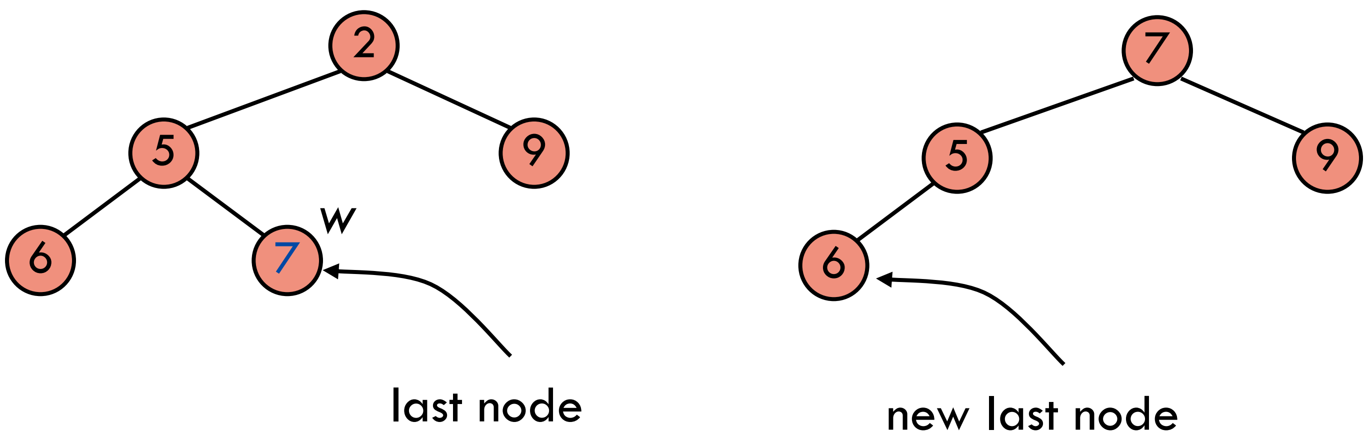
Example insertion



Removal from a Heap

- Replace the root key with the key of the last node *w*
- Delete *w*
- Restore the heap-order property

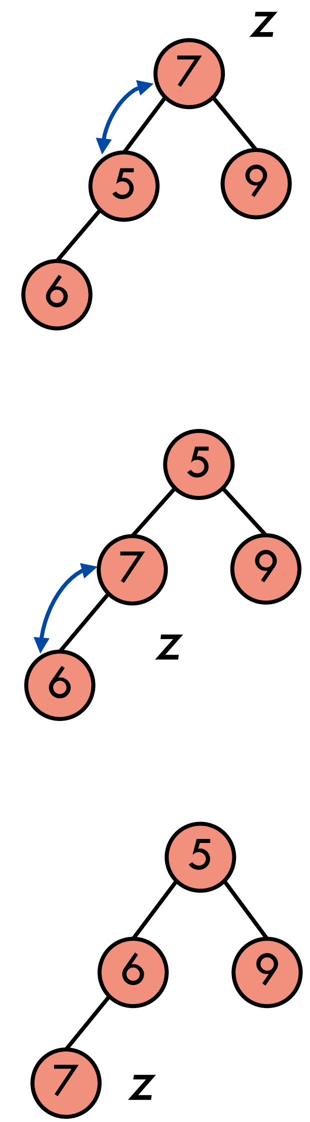
`remove_min()`



Downheap

Restore heap-order property by swapping keys along downward path from the root

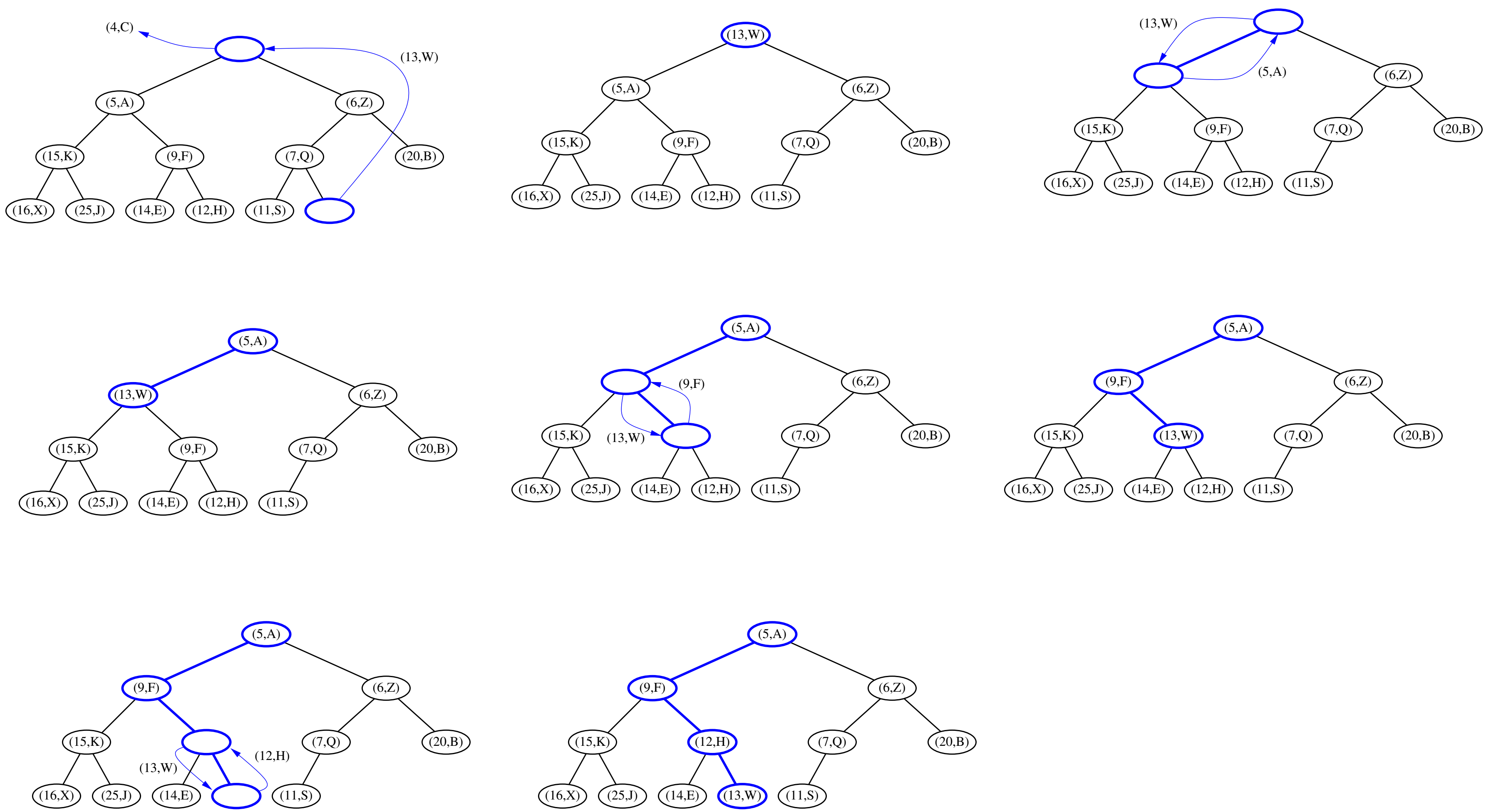
```
def down_heap(z):
    while z has child with
            key(child) < key(z) do
        x ← child of z with smallest key
        swap keys of x and z
        z ← x
```



Correctness: after swap z heap-order property is restored up to level of z

Complexity: O(log n) time because the height of the heap is log n

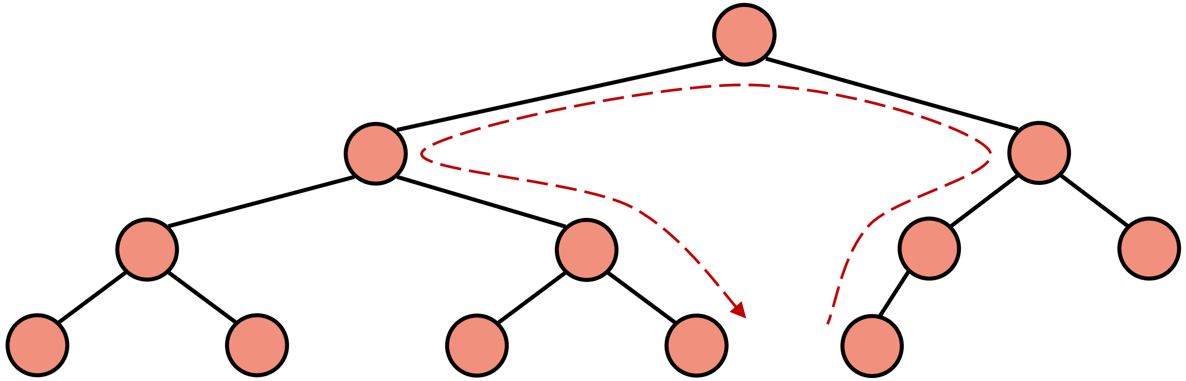
Example removal



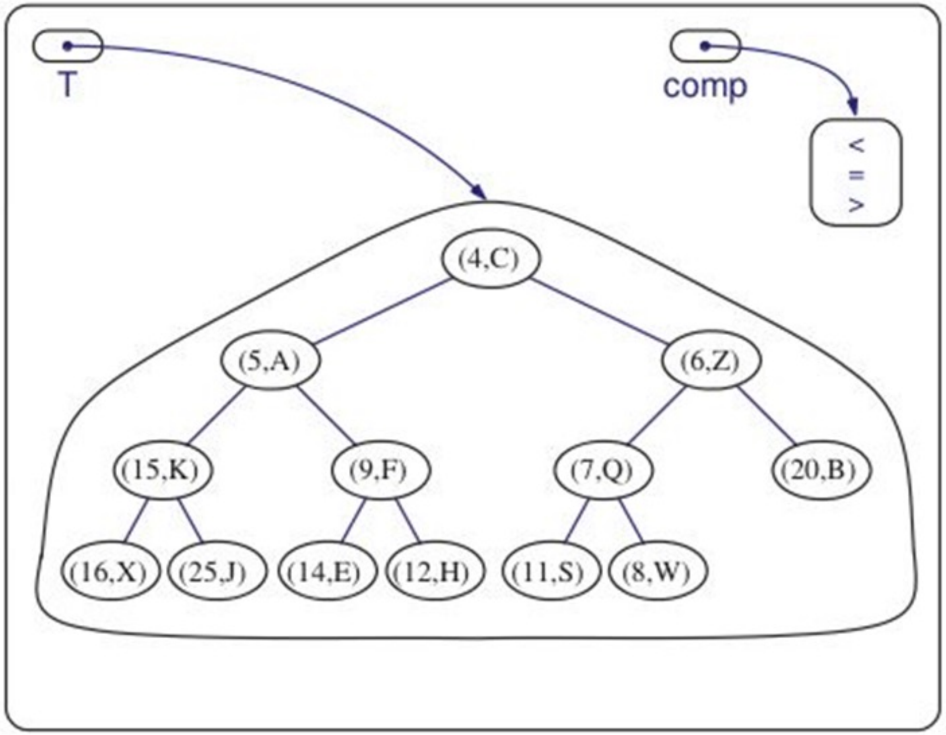
Finding next last node after deletion

- start from the (old) last node
- go up until a right child or the root is reached
- if we reach the root then need to close a level
- otherwise, go to the sibling (left child of parent)
- go down right until a leaf is reached

Complexity of this search is O(log n) because the height is log n.
Thus, overall complexity of deletion is O(log n) time



Heap-based implementation of a priority queue



Operation	Time
size, isEmpty	$O(1)$
min,	$O(1)$
insert	$O(\log n)$
removeMin	$O(\log n)$

Heap-Sort

Consider a priority queue with *n* items implemented with a heap:

- the space used is $O(n)$
- methods insert and remove_min take O(log n)

Recall that priority-queue sorting uses:

- n insert ops
- n remove_min ops

Heap-sort is the version of priority-queue sorting that implements the priority queue with a heap. It runs in O(n log n) time.

Heap-in-array implementation

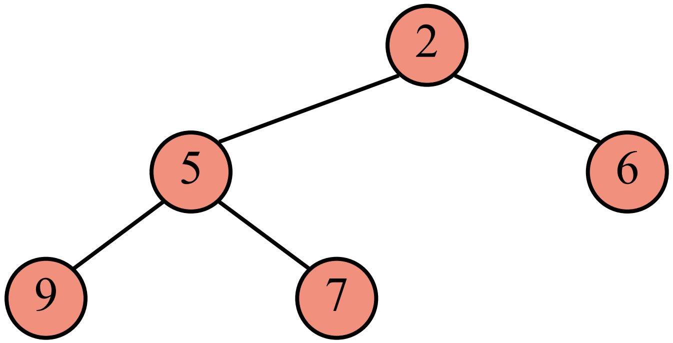
We can represent a heap with *n* keys by means of an array of length *n*

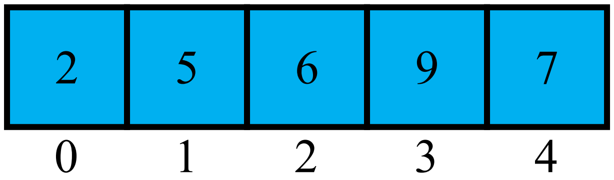
Special nodes:

- root is at 0
- last node is at n-1

For the node at index i:

- the left child is at index $2i+1$
- the right child is at index $2i+2$
- Parent is at index $\lfloor (i-1)/2 \rfloor$





Refinements and Generalization

Heap-sort can be arranged to work in place using part of the array for the output and part for the priority queue

A heap on n keys can be constructed in O(n) time. But the n remove_min still take O(n log n) time

Sometimes it is useful to support a few more operations (all are given a pointer to e):

- remove(e): Remove item e from the priority queue
- replace_key(e, k): update key of item e with k
- replace_value(e, v): update value of item e with v

Summary: Priority queue implementations

Method	Unsorted List	Sorted List	Heap
size, isEmpty	$O(1)$	$O(1)$	$O(1)$
insert	$O(1)$	$O(n)$	$O(\log n)$
min	$O(n)$	$O(1)$	$O(1)$
removeMin	$O(n)$	$O(1)$	$O(\log n)$
remove	$O(1)$	$O(1)$	$O(\log n)$
replaceKey	$O(1)$	$O(n)$	$O(\log n)$
replaceValue	$O(1)$	$O(1)$	$O(1)$

Implementing a Priority Queue

Entries: An object that keeps track of the associations between keys and values

Comparators: A function or an interface to compare entry objects

compare(a, b): returns an integer i such that

- $i < 0$ if $a \prec b$,
- $i = 0$ if $a = b$
- $i > 0$ if $a \succ b$

Warning: do not assume that compare(a,b) is always -1, 0, 1

Stock Application Revisited

Online trading system where orders are stored in two priority queues (one for sell orders and one for buy orders) as (p, t, s) entries:

- The key is (p, t), the price of the order p and the time t such that we first sort by p and break ties with t
- The value is s, the number of shares the order is for

How do we implement the following:

- What should we do when a new order is placed?
- What if someone wishes to cancel their order before it executes?
- What if someone wishes to update the price or number of shares for their order?