Ellipses

1 Setup

Consider two open half spaces $\mathcal{M}_1, \mathcal{M}_2 \in \mathbb{R}^2$ with an interface separating the two regions, Σ, defined by the x-axis..

Each $\mathcal{M}_i$, $i = 1, 2$, has an associated velocity set F_i. For positive constants a_i, b_i, the velocity sets are defined by

$$F_i = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : \frac{x^2}{a_i^2} + \frac{y^2}{b_i^2} \leq 1 \right\} \quad \text{then} \quad F_i^\circ = \left\{ \begin{pmatrix} \zeta \\ \xi \end{pmatrix} : a_i^2\zeta^2 + b_i^2\xi^2 \leq 1 \right\}.$$

From conditions (1) and (2), for a fixed angle $0 \leq \theta \leq \pi$, find ζ_i using velocities in F_i°. Namely,

$$\zeta_1 = \begin{pmatrix} \frac{\sin(\theta_1)}{a_1} \\ \frac{\cos(\theta_1)}{b_1} \end{pmatrix} \quad \text{and} \quad -\zeta_2 = \begin{pmatrix} \frac{\sin(\theta_2)}{a_2} \\ \frac{\cos(\theta_2)}{b_2} \end{pmatrix}.$$

From conditions (5) and (6), the velocities given by θ are

$$v_i = \begin{pmatrix} a_i \sin(\theta_i) \\ b_i \cos(\theta_i) \end{pmatrix}.$$

2 Snell's Law for Ellipses

Now, since $N_\Sigma(Q)$ equals the cone that is the y-axis, condition (3) implies

$$\zeta_1 - \zeta_2 \in \left\{ \begin{pmatrix} 0 \\ y \end{pmatrix} : y \in \mathbb{R} \right\}.$$

Therefore the first component of $\zeta_1 - \zeta_2$ is equal to 0, or that

$$\frac{\sin(\theta_1)}{a_1} = \frac{\sin(\theta_2)}{a_2}. \tag{7}$$

3 Critical Angle

But (7) imposes an implicit restriction on θ_1 if $a_2 > a_1$. Therefore, θ_1 must satisfy $\left|\sin(\theta_1)\right| \leq \frac{a_1}{a_2}$. In other words, optimal trajectories do not enter $\mathcal{M}_2$ at too large an angle.

Figure 1: Example velocity set F (dark blue), $2F$ (gray), and F° (green) with $a = 2, b = 1$.