The Minimal Time Problem Overview

1 Overview

Consider two open half spaces $\mathcal{M}_1$ and $\mathcal{M}_2$ with an interface separating the two regions, Σ.

This means there is a vector $\vec{n} \neq \mathbf{0}$ and a number $r \in \mathbb{R}$ so that

$$\mathcal{M}_1 = \{x : \langle \vec{n}, x \rangle < r\} \quad \text{and} \quad \mathcal{M}_2 = \{x : \langle \vec{n}, x \rangle > r\},$$

and the interface is

$$\Sigma := \{x : \langle \vec{n}, x \rangle = r\}.$$

Each $\mathcal{M}_i$, $i = 1, 2$, has an associated velocity set F_i. In particular, let F_i be convex and bounded with $\mathbf{0} \in \text{int}(F)$. Now identify two points, $X_1 \in \mathcal{M}_1$ and $X_2 \in \mathcal{M}_2$. Then the minimal time problem should find the trajectory from X_1 to X_2 using the velocities in F_1 while in $\mathcal{M}_1$ and F_2 while in $\mathcal{M}_2$ which takes the least amount of time.

Another way of phrasing this problem is to find the point $Q \in \Sigma$ so that together the time it takes to travel from X_1 to Q and Q to X_2 is minimized. The gauge function, $\gamma_F : X \to [0, \infty]$ is defined by

$$\gamma_F(x) = \inf\left\{t \geq 0 : \frac{1}{t}\, x \in F\right\}.$$

Using this terminology, we can now say that our minimal time problem is

$$\min \left\{\gamma_{F_1}(Q - X_1) + \gamma_{F_1}(X_2 - Q)\right\} \quad \text{over } Q \in \Sigma.$$

Then using the indicator function, we can then say rephrase our problem as

$$\min \left\{\gamma_{F_1}(Q - X_1) + \gamma_{F_1}(X_2 - Q) + \mathcal{I}_\Sigma(Q)\right\} \quad \text{over } Q \in \mathbb{R}^n. \tag{$\mathcal{P}2$}$$

Now to solve for Q, consider the subgradient $\partial f(x)$ at a point $x \in \text{dom} f$ given by

$$\partial f(x) := \{\xi \in X : f(y) \geq f(x) + \langle \xi, y - x \rangle \ \forall y \in X\}. \tag{1}$$

Then from (1) we can say that $Q \in \mathbb{R}^n$ solves the problem ($\mathcal{P}2$) if and only if

$$\mathbf{0} \in \partial\left\{\gamma_{F_1}((\cdot) - X_1) + \gamma_{F_2}(X_2 - (\cdot)) + \mathcal{I}_\Sigma(Q)\right\}(Q). \tag{2}$$

Note that For $v \in F$, the normal cone $N_F(v)$ is given by

$$N_F(v) := \{\zeta : \langle \zeta, v' - v \rangle \leq 0 \quad \forall v' \in F\}.$$

This together with (1) gives

$$\partial \mathcal{I}_\Sigma(Q) = N_\Sigma(Q) := \{\zeta : \langle \zeta, y - Q \rangle \leq 0, \forall y \in \Sigma\}$$

Since $\gamma_{F_1}((\cdot)-X_1), \gamma_{F_2}(X_2-(\cdot)), \mathcal{I}_\Sigma \in \mathcal{F}$ apply Rockafellar's Theorem,

$$\begin{aligned}\partial\Big\{\gamma_{F_1}((\cdot)-X_1) \quad + \quad \gamma_{F_2}(X_2-(\cdot)) \quad + \quad \mathcal{I}_\Sigma(\cdot)\Big\}(Q)\\ = \quad \partial\gamma_{F_1}(Q-X_1) + \partial\gamma_{F_2}(X_2-Q) + N_\Sigma(Q)\end{aligned}$$

and equivalently there exist two vectors ζ_1, $\zeta_2 \in \mathbb{R}^n$ satisfying

$$\zeta_1 \in \partial\gamma_{F_1}(Q-X_1), \tag{3}$$
$$-\zeta_2 \in \partial\gamma_{F_2}(X_2-Q), \quad \text{and} \tag{4}$$
$$\zeta_1-\zeta_2 \in N_\Sigma(Q). \tag{5}$$

If Q solves the problem $(\mathcal{P}2)$, then

$$0 = \zeta_1 - \zeta_2 + \bar{\zeta} \tag{6}$$

where $\bar{\zeta} \in N_\Sigma$. One can then show with Exercise 4.2 that for each $i = 1$, 2 that

$$\gamma_{F_i^\circ}(\zeta_i) = 1. \tag{7}$$

We know if $v_1 := \dfrac{Q-X_1}{\gamma_{F_1}(Q-X_1)} \in F_1$ and $v_2 = \dfrac{X_2-Q}{\gamma_{F_2}(X_2-Q)} \in F_2$ are the two velocities used by the optimal trajectory, and by Exercise 4.2 again, that

$$v \mapsto \langle\zeta_1, v\rangle \quad \text{is maximized over } v \in F_1 \text{ at } v = v_1$$
$$v \mapsto \langle-\zeta_2, v\rangle \quad \text{is maximized over } v \in F_2 \text{ at } v = v_2.$$

2 Snell's Law

In the simple case, the two velocity sets are $F_i = r_i\overline{\mathbb{B}}$ and $r_i > 0$ and $i = 1, 2$ and the interface is the x-axis. Then $F_i^\circ = \frac{1}{r_i}\overline{\mathbb{B}}$ and N_Σ is the y-axis. Now designate all points that are distance 1 from the origin as

$$\begin{pmatrix}\sin\theta\\ \cos\theta\end{pmatrix}$$

Further, by (7)

$$(3) \quad \Rightarrow \quad \zeta_1 = \frac{1}{r_1}\begin{pmatrix}\sin\theta_1\\ \cos\theta_1\end{pmatrix}$$
$$(4) \quad \Rightarrow \quad \zeta_2 = \frac{1}{r_2}\begin{pmatrix}\sin\theta_2\\ \cos\theta_2\end{pmatrix}.$$

Then since (5) $\Rightarrow \zeta_1 - \zeta_2 \in$ y-axis we have

$$\frac{1}{r_1}\sin\theta_1 - \frac{1}{r_2}\sin\theta_2 = 0.$$

Which gives Snell's Law, which is stated in the form

$$\frac{\sin\theta_1}{r_1} = \frac{\sin\theta_2}{r_2}.$$