Notes on the Elvis Problem, Part II

5 Minimal time functions

A minimal time problem uses the data of a controlled dynamical system and a target set. The problem consists of finding the fastest way to go from a given initial point to some point on the target by following a trajectory of the dynamical system. An optimal trajectory is one that achieves this aim. The Elvis problem has velocity sets that are fixed (that is, they do not depend on an individual point in a region, but only on the region itself. As such, an optimal trajectory, while it remains in a region, will use only one velocity, and is therefore a straight line. This allows for a considerable simplification of the analysis, and allows for the concepts introduced in Part I to be used instead of a more general and complicated dynamical systems theory.

5.1 One region with Constant Dynamics

We first consider the case where the velocity set is the same throughout $\mathbb{R}^n$. Let $F \in \mathcal{C}$.

Definition 5.1. *A trajectory of F is a piece-wise differentiable function $x(\cdot) : [0, T] \to \mathbb{R}^n$ with $\dot{x}(t) \in F$. For $X \in \mathbb{R}^n$ and $T > 0$, the Reachable Set from X at time T is defined by*

$$R^{(T)}(X) := \big\{ x(T) : \ x(\cdot) \text{ is a trajectory of } F \big\}.$$

Exercise 5.1. *Let $X_1 \in \mathbb{R}^n$ and $T > 0$.*

(a) Show that $R^{(T)}(X_1) = X_1 + TF := \big\{ X_1 + Tv : \ v \in F \big\}$.

(b) Characterize those $X_2 \in \mathbb{R}^n$ for which $X_2 \in R^{(T)}(X_1)$ for some $T > 0$.

(c) Show that

$$\gamma_F\,(X_2 - X_1) = \inf\Big\{ T : \ \exists \text{ trajectory } x(\cdot) \text{ of } F \text{ with } x(0) = X_1,\, x(T) = X_2 \Big\}$$

(Recall $\inf \emptyset = +\infty$.)

(d Let $v := \dfrac{X_2 - X_1}{\gamma_F(X_2 - X_1)}$ and $T := \gamma_F(X_2 - X_1)$. Show that the optimal trajectory from X_1 to X_2 is given by $x(t) = X_1 + tv$ for $t \in [0, T]$.

Suppose $S \subseteq \mathbb{R}^n$ is any closed set (the "target") and $F \in \mathcal{C}$ (the "admissible velocities"). The so-called minimal time function $T_S(\cdot)$ determined by S and F is defined by

$$T_S(x) = \min\Big\{ t : [x + tF] \cap S \neq \emptyset \Big\}.$$

Exercise 5.2. *Show that $T_S(x) = \inf\big\{ T : R^{(T)}(X) \cap S \neq \emptyset \big\} = \inf\big\{ \gamma_F(y - x) : \ y \in S \big\}$.*

5.2 Two regions with different velocity sets

Of course our Elvis problem does not have one single velocity set on the whole space, but rather different velocity sets on different regions with an interface separating the regions. We eventually will want more general configurations, but for now we begin with only two open half spaces $\mathcal{M}_1$ and $\mathcal{M}_2$. In this case there is a vector $\vec{n} \neq \mathbf{0}$ and a number $r \in \mathbb{R}$ so that

$$\mathcal{M}_1 = \big\{x : \langle \vec{n}, x\rangle < r\big\} \quad \text{and} \quad \mathcal{M}_2 = \big\{x : \langle \vec{n}, x\rangle > r\big\},$$

and the interface is

$$\Sigma := \operatorname{bdry}(\mathcal{M}_1) \cap \operatorname{bdry}(\mathcal{M}_2) = \big\{x : \langle \vec{n}, x\rangle = r\big\}.$$

Each $\mathcal{M}_i$, $i = 1, 2$, has an associated velocity set $F_i \in \mathcal{C}$. First, consider the case where $X_1 \in \mathcal{M}_1$ and $X_2 \in \mathcal{M}_2$, and we want to travel from X_1 to X_2 in the least time. We elaborate on what we mean by "travel." Any function $x(\cdot) : [0, T] \to \mathbb{R}^n$ that is differentiable at every $t \in [0, T]$ except possibly finite many points $\{t_1, t_2, \ldots, t_m\}$ is called an *arc*. If an arc $x(\cdot)$ satisfies

$$x(t) \in \mathcal{M}_i,\ t \in [0, T] \backslash \{t_1, t_2, \ldots, t_m\} \quad \Longrightarrow \quad \dot{x}(t) \in F_i,$$

then $x(\cdot)$ is called a trajectory (of the system). Thus "to travel from X_1 to X_2 in the least time" means to find a trajectory that satisfies $x(0) = X_1$, $x(T) = X_2$, and where T is as small as possible. Formally, the *minimal time problem* is stated as

$$\min\ T \qquad \text{such that } \exists \text{ trajectory } x(\cdot) : [0, T] \to \mathbb{R}^n \text{ with } x(0) = X_1, x(T) = X_2. \tag{$\mathcal{P}1$}$$

The reachable set $R^{(T)}(X_1)$ is defined as before in that it consists of all points X_2 for which there exists a trajectory $x(\cdot)$ for which $x(0) = X_1$ and $x(T) = X_2$.

Exercise 5.3. *This exercise refers to ($\mathcal{P}1$) where $X_1 \in \mathcal{M}_1$ and $X_2 \in \mathcal{M}_2$.*

(a) Show an optimal trajectory which begins at $X_1 \in \mathcal{M}_1$ will leave $\mathcal{M}_1$ and not return to any point in $\mathcal{M}_1$.

(b) Show that ($\mathcal{P}1$) is equivalent to

$$\min \Big\{\gamma_1(Q - X_1) + \gamma_2(X_2 - Q)\Big\} \quad \text{over } Q \in \Sigma$$

(c) Deduce that ($\mathcal{P}1$) is equivalent to

$$\min \Big\{\gamma_1(Q - X_1) + \gamma_2(X_2 - Q) + \mathcal{I}_\Sigma(Q)\Big\} \quad \text{over } Q \in \mathbb{R}^n \tag{$\mathcal{P}2$}$$

(Two problems are "equivalent" if a solution of one can lead to a solution of the other.)

We now introduce a further assumption.

$$\text{Each } F_i \text{ belongs to } \mathcal{C}_0. \tag{H1}$$

One of the consequences of this assumption is that for every X_1, $X_2 \in \mathbb{R}^n$, the problem $(\mathcal{P}1)$ has a feasible solution. This is because there are "admissible" velocities in every direction, and thus there is at least one trajectory $x(\cdot)$ defined on some interval $[0, T]$ with $x(0) = X_1$ and $x(T) = X_2$.

Exercise 5.4. *Assume (H1) holds.*

(a) For every X_1, $X_2 \in \mathbb{R}^n$, show there is at least one trajectory $x(\cdot)$ from X_1 to X_2 (consider different cases on whether the X_i's belong to the same half space).

(b) What are the necessary conditions for a trajectory $x(\cdot)$ to solve $(\mathcal{P}1)$?

(c) Assuming $X_1 \in \mathcal{M}_1$ and $X_2 \in \mathcal{M}_2$, what are the necessary and sufficient conditions for a point $Q \in \mathbb{R}^n$ to solve $(\mathcal{P}2)$?

Part (b) of the last exercise is a trick question, for it depends on whether X_1 and X_2 belong to the same manifold or not. Let us first assume $X_1 \in \mathcal{M}_1$ and $X_2 \in \mathcal{M}_2$ as in part (c), in which case we can rely on the results in Exercise 5.3. Indeed, via Rockafellar's Theorem, there exists $Q \in \Sigma$ and two vectors ζ_1, $\zeta_2 \in \mathbb{R}^n$ satisfying

$$\begin{aligned} \zeta_1 &\in \partial\gamma_{F_1}(Q - X_1), && (1)\\ -\zeta_2 &\in \partial\gamma_{F_2}(X_2 - Q), \quad \text{and} && (2)\\ \zeta_1 - \zeta_2 &\in N_\Sigma(Q). && (3) \end{aligned}$$

Using Exercise 4.2, we have for each $i = 1$, 2 that

$$\gamma_{F_i^\circ}(\zeta_i) = 1. \tag{4}$$

We know $v_1 := \dfrac{Q - X_1}{\gamma_{F_1}(Q - X_1)} \in F_1$ and $v_2 = \dfrac{X_2 - Q}{\gamma_{F_2}(X_2 - Q)} \in F_2$ are the two velocities used by the optimal trajectory, and by Exercise 4.2 again, that

$$\begin{aligned} v &\mapsto \langle \zeta_1, v\rangle \quad \text{is maximized over } v \in F_1 \text{ at } v = v_1 && (5)\\ v &\mapsto \langle -\zeta_2, v\rangle \quad \text{is maximized over } v \in F_2 \text{ at } v = v_2. && (6) \end{aligned}$$

Example 1. Let us now specialize to the case $n = 2$, $\mathcal{M}_1$ (resp. $\mathcal{M}_2$) the upper (resp. lower) half-plane, Σ the x-axis, $F_1 = r_1\overline{\mathbb{B}}$, $F_2 = r_2\overline{\mathbb{B}}$, where r_1, $r_2 > 0$. Recall $F_i^\circ = \frac{1}{r_1}\overline{\mathbb{B}}$ and $F_2^\circ = \frac{1}{r_2}\overline{\mathbb{B}}$. So for $i = 1$, 2, the ζ_i's in (1), (2) can be represented by an angle $\theta_i \in [0, 2\pi)$ with

$$\zeta_1 = \frac{1}{r_1}\begin{pmatrix}\sin(\theta_1)\\ \cos(\theta_1)\end{pmatrix} \quad \text{and} \quad -\zeta_2 = \frac{1}{r_2}\begin{pmatrix}\sin(\theta_2)\\ \cos(\theta_2)\end{pmatrix}.$$

Conditions (5), (6) imply the optimal velocities are

$$v_1 = r_1 \begin{pmatrix} \sin(\theta_1) \\ \cos(\theta_1) \end{pmatrix} \quad \text{and} \quad v_2 = r_2 \begin{pmatrix} \sin(\theta_2) \\ \cos(\theta_2) \end{pmatrix},$$

which geometrically explains the angles θ_1, θ_2 — see Figure 1. Now $N_\Sigma(Q)$ equals the cone that is the y-axis, and so (3) implies

$$\zeta_1 - \zeta_2 \in \left\{ \begin{pmatrix} 0 \\ y \end{pmatrix} : y \in \mathbb{R} \right\}.$$

Therefore the first component of $\zeta_1 - \zeta_2$ is equal to 0, or that

$$\frac{\sin(\theta_1)}{r_1} = \frac{\sin(\theta_2)}{r_2}. \tag{7}$$

This is the classical version of Snell's Law. One might observe (7) imposes an implicit restriction on θ_1 if $r_1 > r_2$, for it must satisfy $\left|\sin(\theta_2)\right| \leq \frac{r_2}{r_1}$. This says optimal trajectories do not enter $\mathcal{M}_2$ at too large an angle. $\square$

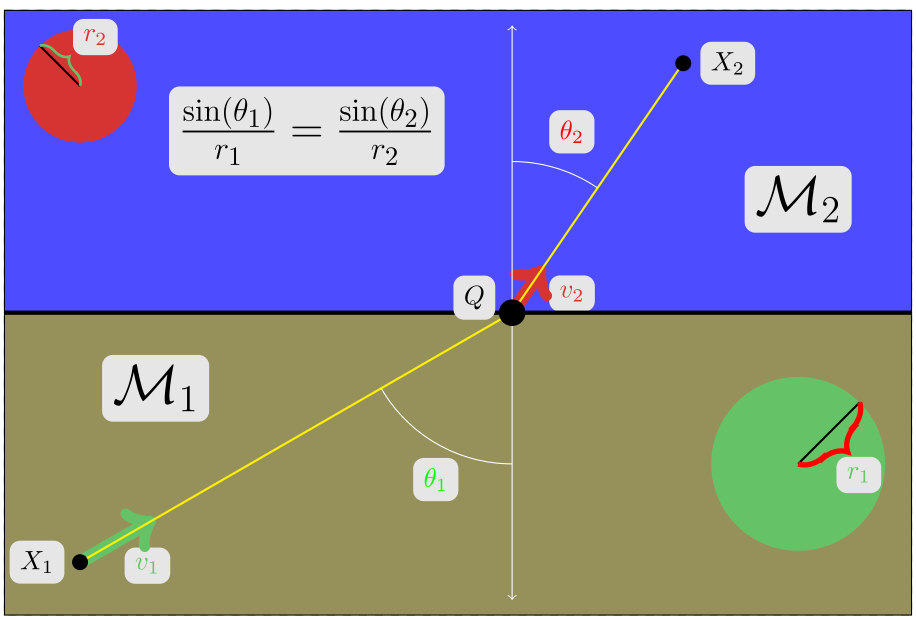

Figure 1: Snell's Law

We emphasize that our derivation of Snell's Law (7) is a very special case of what we have proven (modulo Rockafellar's Theorem, which we have not proven). The conditions (1)-(3) are valid for any choices of $F_i \in \mathcal{C}_0$, and these imply (4)-(6).

Exercise 5.5. *Let $\mathcal{M}_1$, $\mathcal{M}_2$, and Σ as in Example 1. We write $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2$ as a typical velocity vector. Find the optimal solution(s) for the following collections of data* *(a)* $F_1 = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} : \ u^2 + v^2 \leq 1 \right\}$; $F_2 = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} : \ u^2 + v^2 \leq 4 \right\}$; $X_1 = \begin{pmatrix} -4 \\ -4 \end{pmatrix}$; $X_2 = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$

(b) $F_1 = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} : \ \frac{u^2}{4} + v^2 \leq 1 \right\}$; $F_2 = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} : \ 4u^2 + \frac{v^2}{9} \leq 1 \right\}$; $X_1 = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$; $X_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

(c) $F_1 = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} : |u| + |v| \leq 1 \right\}$; $F_2 = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} : u^2 + v^2 \leq 1 \right\}$; $X_1 = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$; $X_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

(d) $F_1 = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} : \max\{|u|, |v|\} \leq 1 \right\}$; $F_2 = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} : \frac{u^2}{4} + v^2 \leq 1 \right\}$; $X_1 = \begin{pmatrix} -2 \\ -4 \end{pmatrix}$; $X_2 = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$

5.3 Calculating the reachable set for two regions

Unlike the case of just one region, finding the reachable set explicitly is somewhat more difficult for two or more regions.

Exercise 5.6. *Assume (H1), and suppose $X_1 \in \mathbb{R}^n$. Show the following.*

(a) If $T_1 < T_2$, then $R^{(T_1)}(X_1) \subsetneq R^{(T_2)}(X_1)$.

(b) For any $T_0 > 0$, a state X_2 belongs to $\mathrm{bdry} R^{(T_0)}(X_1)$ if and only if T_0 solves the problem (P1).

(c) For any T_1, $T_2 \geq 0$, show that $R^{(T_1+T_2)}(X_1) = R^{(T_1)}\big(R^{(T_2)}(X_1)\big)$. (The notation here is that $R^{(T)}(S) := \bigcup\big\{R^{(T)}(s) : \ s \in S\big\}$.)

Consider $\mathcal{M}_1$, $\mathcal{M}_2$, Σ as in Example 1, and $F_1 = r_1\overline{\mathbb{B}}$, $F_2 = r_2\overline{\mathbb{B}}$.

Case 1: To be even more specific, let $X_1 = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$, $r_1 = 2$, $r_2 = 1$ (see Figure 2. We can use Snell's Law to determine which points are on the boundary of $R^{(T)}(X)$. Since $r_1 > r_2$, any point $X_2 \in R^{(T)}(X_1) \bigcap \mathcal{M}_1$ can be obtained with a velocity vector of the form $v = r_1 \frac{X_2 - X_1}{\|X_2 - X_1\|}$. That is, no additional points in $\mathcal{M}_1$ can be reached with a trajectory that leaves $\mathcal{M}_1$, which is not necessarily the case if $r_1 < r_2$ as will be discussed below. For example, suppose a trajectory starts at X_1 and uses velocity $v = \sqrt{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \in F_1$. The point $X_1 + Tv$ is on the boundary of $R^{(T)}(X)$ until it hits hits Σ at $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$, and this happens when $-3 + T_1\sqrt{2} = 0$, or at time $T_1 = \frac{3}{\sqrt{2}}$. Of course the incidental angle is $\frac{\pi}{4}$. Thus the trajectory enters $\mathcal{M}_2$ "optimally" if it leaves at an angle θ_2 that satisfies $\frac{\sin(\theta_2)}{r_2} = \sin(\theta_2) = \frac{\sqrt{2}/2}{r_1} = \frac{\sqrt{2}}{4}$ ($\theta_2 \approx .36137$). Thus the point $X_2 := \begin{pmatrix} 3 \\ 0 \end{pmatrix} + (T - T_1)\begin{pmatrix} \sin(\theta_2) \\ \cos(\theta_2) \end{pmatrix}$ belongs to the boundary of $R^{(T)}(X_1)$ when $T > T_1$. The time T_1 of course depends on θ_1, but the analysis is the same for each angle θ_1 where $|\theta| < \pi$. Also, similar analysis applies for any $X_1 \in \mathcal{M}_1$ and any $0 < r_2 < r_1$. Some special cases are pictured in Figure 2, where the reachable set $R^{(3)}(X_1)$ is the region enclosed in red.

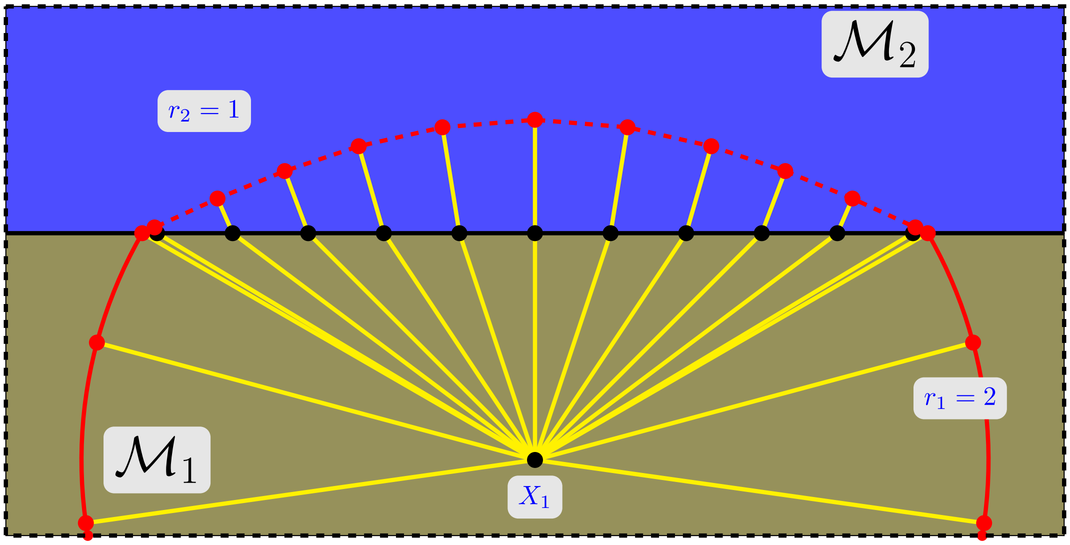

Figure 2: Examples of "optimal" trajectories with $T = 3$ in Case 1

$$X_1 = \begin{pmatrix} 0 \\ -3 \end{pmatrix},\ r_1 = 2,\ r_2 = 1$$

$$Q = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

$$T_1 = \gamma_{F_1}(Q - X_1) = \frac{\|Q-X_1\|}{2}$$

$$\sin(\theta_1) = \frac{x}{\|Q-X_1\|}$$

$$\sin(\theta_2) = \frac{r_2\,\sin(\theta_1)}{r_1} = \frac{x}{2\,\|Q-X_1\|}$$

$$\cos(\theta_2) = \frac{\sqrt{4\|Q-X_1\|^2-(x)^2}}{2\,\|Q-X_1\|}$$

$$X_2 = Q + (3 - T_1)\begin{pmatrix} \sin(\theta_2) \\ \cos(\theta_2) \end{pmatrix}$$

x	T_1	$\sin(\theta_1)$	$\sin(\theta_2)$	$\cos(\theta_2)$	X_2
-5	$\frac{\sqrt{34}}{2}$	$\frac{-5}{\sqrt{34}}$	$\frac{-5}{2\sqrt{34}}$	$\frac{1}{2}\sqrt{\frac{111}{34}}$	$\begin{pmatrix} -5.03264 \\ .0764 \end{pmatrix}$
-4	$\frac{5}{2}$	$\frac{-4}{5}$	$\frac{-2}{5}$	$\frac{\sqrt{21}}{5}$	$\begin{pmatrix} -4.2 \\ .4583 \end{pmatrix}$
-3	$\frac{\sqrt{18}}{2}$	$\frac{-3}{\sqrt{18}}$	$\frac{-3}{2\sqrt{18}}$	$\sqrt{\frac{63}{72}}$	$\begin{pmatrix} -3.3107 \\ .8219 \end{pmatrix}$
-2	$\frac{\sqrt{13}}{2}$	$\frac{-2}{\sqrt{13}}$	$\frac{-1}{\sqrt{13}}$	$\sqrt{\frac{48}{52}}$	$\begin{pmatrix} -2.3321 \\ 1.1503 \end{pmatrix}$
-1	$\frac{\sqrt{10}}{2}$	$\frac{-1}{\sqrt{10}}$	$\frac{-1}{2\sqrt{10}}$	$\sqrt{\frac{39}{40}}$	$\begin{pmatrix} -1.2243 \\ 1.401 \end{pmatrix}$
0	$\frac{3}{2}$	0	0	1	$\begin{pmatrix} 0 \\ 3/2 \end{pmatrix}$

The optimal velocity in the first part of the trajectory is $v_1 = \frac{2(Q-X_1)}{\|Q-X_1\|}$, and in the second part is $v_2 = \frac{X_2-Q}{(Q-X_1)}$.

Case 2: Now let us specify $F_1 = \overline{\mathbb{B}}$, $F_2 = 2\overline{\mathbb{B}}$, and $X_1 = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$; similar considerations apply for any $0 < r_1 < r_2$ and $X_1 \in \mathcal{M}_1$. When $0 \le T \le 3$ the reachable set is just $X + T\overline{\mathbb{B}}$. When $T > 3$, it becomes more interesting because there are now three separate types of an optical arc.

Subcase 2a: A trajectory never leaves $\mathcal{M}_1$. In particular, every point in $\mathcal{M}_1 \bigcap \{X_1 + T\overline{I\!B}\}$ belongs to $R^{(T)}(X_1)$, although as we'll see, not all of the boundary points belong to the boundary of $R^{(T)}(X_1)$.

Subcase 2b: A trajectory leaves $\mathcal{M}_1$ and spends the remaining time in $\mathcal{M}_2$; this is the analyzed in a manner similar to Case 1, and produces those points in $\mathcal{M}_2 \bigcap R^{(T)}(X_1)$. Examples are shown in Figure 3 with $T = 6$.

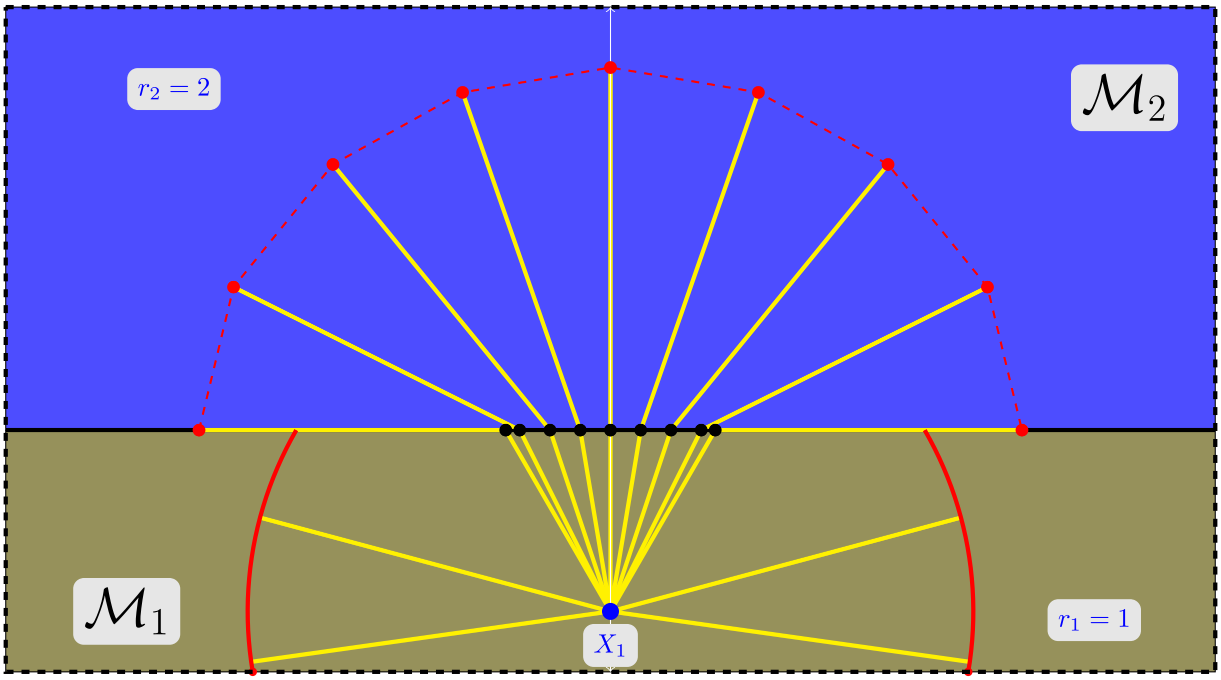

Figure 3: Examples of optimal trajectories of Subcases 2a and 2b

Subcase 2c: An optimal trajectory may hit Σ at some time $T_1 < T$, and stay on Σ until time T_2 and then re-enter $\mathcal{M}_1$.

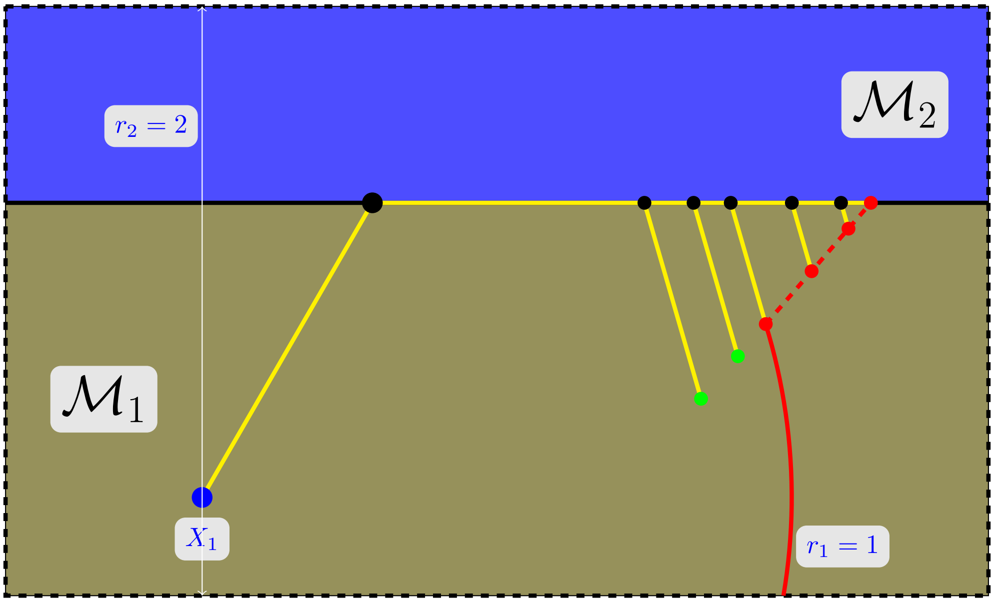

Figure 4: Examples of trajectories in Subcase 2c. Points at the end of optimal arcs are red, and those in the interior of $R^{(6)}(X_1)$ are green.

$$X_1 = \begin{pmatrix} 0 \\ -3 \end{pmatrix},\ r_1 = 1,\ r_2 = 2$$

$$Q_1 = \begin{pmatrix} \sqrt{3} \\ 0 \end{pmatrix}; \quad Q_2 = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

$$T_1 = \sqrt{12}\,; \quad T_2 = T_1 + \tfrac{1}{2}(x - \sqrt{3})$$

$$X_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = Q_2 + \tfrac{1}{2}(6 - T_2)\begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$$

x	T_2	X_2	$\lVert X_2 - X_1 \rVert$
4.5	4.848	$\begin{pmatrix} 5.076 \\ -1.995 \end{pmatrix}$	5.1745
5	5.098	$\begin{pmatrix} 5.451 \\ -1.5622 \end{pmatrix}$	5.6374
5.3781	5.2871	$\begin{pmatrix} 5.7345 \\ -1.2347 \end{pmatrix}$	6
6	5.598	$\begin{pmatrix} 6.201 \\ -.69615 \end{pmatrix}$	6.6151
6.5	5.848	$\begin{pmatrix} 6.576 \\ -.26314 \end{pmatrix}$	7.123
6.8038	6	$\begin{pmatrix} 6.8038 \\ 0 \end{pmatrix}$	7.4359

The full reachable set is the union of all the trajectories coming from one of the subcases. The boundary of the reachable set would typically consist of all endpoints of one of the optimal trajectories, but his is not necessarily the case as seen by the green points in Figure 4.

6 Moving faster on Σ

We looked in Subcase 2c above at a trajectory moving on the interface Σ, but we assumed that that speed was the same as the subspace. We next consider the possibility that Σ is like a road or a boardwalk in which faster movement is possible. Suppose the interface Σ also has a velocity set G. The velocity set G restricts the motion on Σ, and G is assumed to be closed, convex, and satisfying $\langle \vec{n}, G\rangle := \big\{\langle \vec{n}, v\rangle : \ v \in G\big\} = \{0\}$. The latter requirement is needed so that for any trajectory that begins on Σ moving with a velocity in G stays on Σ. The assumption (H1) now needs a slight modification since obviously the interior of G in $\mathbb{R}^n$ is empty. The modification of (H1) is to require there exists $\delta > 0$ so that $\Sigma \cap \delta\overline{I\!B} \subseteq G$, or that 0 belongs to the *relative interior* of G. We will always assume

$$\big[F_1 \bigcup F_2\big] \bigcap \Sigma \subseteq G, \tag{H2}$$

which in effect says that any movement just off of Σ can be done at least as fast as being on Σ.

Exercise 6.1. *(a) Consider the case where $X_1 \in \mathcal{M}_1$ and $X_2 \in \mathcal{M}_2$. Show that (P1) is equivalent to the problem*

$$\min\ J\big(Q_1,\, Q_2\big) \quad \textit{over } Q_1,\, Q_2 \in \mathbb{R}^n, \tag{$\mathcal{P}3$}$$

where

$$J\big(Q_1,\, Q_2\big) := \Big\{\gamma_{F_1}(Q_1 - X_1) + \gamma_G(Q_2 - Q_1) + \gamma_{F_2}(X_2 - Q_2) + \mathcal{I}_\Sigma(Q_1) + \mathcal{I}_\Sigma(Q_2)\Big\}$$

(b) In the case where $X_1 \in \mathcal{M}_1$ and $X_2 \in \Sigma$, show that $(\mathcal{P}1)$ is equivalent to $(\mathcal{P}2)$.

(c) In the case where both X_1 and X_2 belong to $\mathcal{M}_1$, show there are at most two types of optimal solutions, and

$$\min(\mathcal{P}) = \min\big\{\min(\mathcal{P}3), \gamma_G(X_2 - X_1)\big\}.$$

We make some simple observations in the case where each velocity set is a ball $r\overline{I\!B}$ where $r > 0$. To fix notation, suppose $F_i = r_i\overline{I\!B}$, $i = 1,\, 2$, and on the interface Σ, $G = \Sigma \cap r_3\overline{I\!B}$. Assumption (H2) in this case is equivalent to assuming $r_3 \geq \max\{r_1,\, r_2\}$.

Exercise 6.2. *(a) If $r_3 \leq \min\{r_1,\, r_2\}$, show that for any $X_1 \in \mathcal{M}_1$ and $X_2 \in \mathcal{M}_2$, an optimal solution to ($\mathcal{P}3$) has $Q_1 = Q_2$. That is, any optimal trajectory spends no time on the interface Σ.*

(b) Give an example in $\mathbb{R}^2$ (i.e. produce $X_1,\, X_2 \in \mathbb{R}^2$ and $r_1,\, r_2,\, r_3 > 0$) where the solution to ($\mathcal{P}3$) has $Q_1 \neq Q_2$.

(c) Suppose $r_1,\, r_2,\, r_3 > 0$ with $r_3 > \max\{r_1, r_2\}$, characterize those points $X_1 \in \mathcal{M}_1$ and $X_2 \in \mathcal{M}_2$ for which $Q_1 = Q_2$.

The necessary (and sufficient) conditions for Q_1 and Q_2 to solve $(\mathcal{P}3)$ are the existence of $\zeta_i \in \mathbb{R}^n$, $i = 1, 2, 3, 4$, so that

$$\begin{array}{rclcrcl} \zeta_1 & \in & \partial\gamma_{F_1}(Q_1 - X_1) & \qquad & \zeta_3 & \in & \partial\gamma_G(Q_2 - Q_1) \\ -\zeta_2 & \in & \partial\gamma_G(Q_2 - Q_1) & & -\zeta_4 & \in & \partial\gamma_{F_2}(X_2 - Q_2) \\ \zeta_1 - \zeta_2 & \in & N_\Sigma(Q_1) & & \zeta_3 - \zeta_4 & \in & N_\Sigma(Q_2) \end{array}$$

which imply

$$\gamma_{F_1^\circ}(\zeta_1) = \gamma_{G^\circ}(-\zeta_2) = \gamma_{G^\circ}(\zeta_3) = \gamma_{F_2^\circ}(-\zeta_4) = 1. \tag{8}$$

The optimal velocities are

$$v_1 := \frac{Q_1 - X_1}{\gamma_{F_1}(Q_1 - X_1)} \in F_1 \qquad u = \frac{Q_2 - Q_1}{\gamma_G(Q_2 - Q_1)} \in G \qquad v_2 := \frac{X_2 - Q_2}{\gamma_{F_2}(X_2 - Q_2)} \in F_2$$

and the Max Principle says

$$v \mapsto \langle \zeta_1, v\rangle \quad \text{is maximized over } v \in F_1 \text{ at } v = v_1 \tag{9}$$
$$v \mapsto \langle -\zeta_2, v\rangle \quad \text{is maximized over } v \in G \text{ at } v = u. \tag{10}$$
$$v \mapsto \langle \zeta_3, v\rangle \quad \text{is maximized over } v \in G \text{ at } v = u \tag{11}$$
$$v \mapsto \langle -\zeta_4, v\rangle \quad \text{is maximized over } v \in F_2 \text{ at } v = v_2 \tag{12}$$

Example 2. Again consider the case with $n = 2$, $\mathcal{M}_1$ the lower half space with velocity set $F_1 = r_1\overline{\mathbb{B}}$, $\mathcal{M}_2$ the upper half space with velocity set $F_2 = r_2\overline{\mathbb{B}}$, and Σ the x-axis with velocity set $G = \Sigma \bigcap r_3\overline{\mathbb{B}}$. Consider $(\mathcal{P}1)$ with $X_1 \in \mathcal{M}_1$ and $X_2 \in \mathcal{M}_2$, and sufficiently far apart so that $Q_1 \neq Q_2$ — see Figure 6.

Snell's Law now operates at both switching points Q_1 and Q_2, and the incidental angles satisfy

$$\sin(\theta_1) = \frac{r_1}{r_3} \quad \text{and} \quad \sin(\theta_2) = \frac{r_2}{r_3}. \tag{13}$$

We now address the issue when $X_1 := \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $X_2 := \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ are such that the corresponding Q_i's are different. For definiteness, suppose the situation is similar to what is shown in Figure 6 where $y_1 < 0 < y_2$ and $x_1 \leq x_2$. We write $Q_i := \begin{pmatrix} q_i \\ 0 \end{pmatrix}$, and we want to characterize when $q_1 \leq q_2$. This happens if and only if (13) holds, and the values q_i can be directly calculated from this information. The conclusion is that $Q_1 \neq Q_2$ if and only if

$$x_1 + (-y_1)\tan\left(\sin^{-1}\left(\frac{r_1}{r_3}\right)\right) < x_2 + (y_2)\tan\left(\sin^{-1}\left(\frac{r_2}{r_3}\right)\right).$$

Exercise 6.3. *Consider the examples in Exercise 5.5 where we now add the velocity set* $G := \left\{\begin{pmatrix} u \\ 0 \end{pmatrix} : |u| \leq 3\right\}$ *on the interface* Σ*. Solve (P3) for each of (a)-(d).*

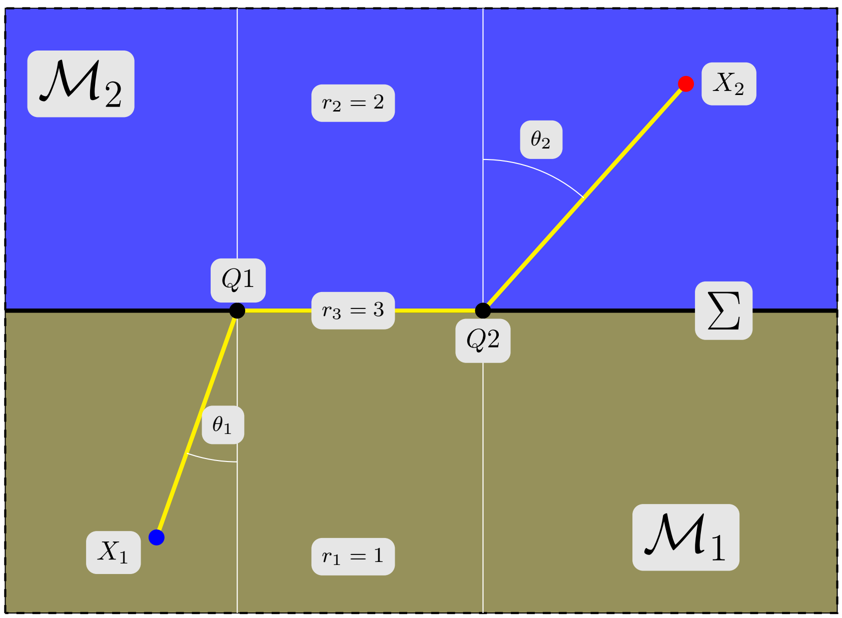

Figure 5: Case 1 where $Q_1 \neq Q_2$