# Exercises on the Elvis Problem - Part I

## 3 Gauge functions

Suppose $F \in \mathcal{C}$ is given. The gauge function $\gamma_F : X \to [0, \infty]$ is defined by

$$\gamma_F(x) = \inf\left\{t > 0 : \frac{1}{t}\, x \in F\right\}.$$

By convention, if $rx \notin F$ for all $r > 0$, then $\gamma_F(x) = +\infty$. We mainly will be interested in only the case where $F \in \mathcal{C}_0$.

**Exercise 3.1.** *Let $F \in \mathcal{C}$ with $\mathbf{0} \in F$. Show the following:*

*(a) $v \in F$ if and only if $\gamma_F(v) \leq 1$.*

(a) *Proof.*

**Claim 3.1.** *If $F \in \mathcal{C}$ with $\mathbf{0} \in F$ then for $t \in \mathbb{R}, |\frac{1}{t}| \leq 1, \frac{1}{t}F \subset F$ (where $\frac{1}{t}F := \{\frac{1}{t}x : x \in F\}$).*

*Proof.* Let $x \in F$ and $|\frac{1}{t}| \leq 1$. Since $F$ is convex, $\frac{1}{t}x \in F$. □

($\Rightarrow$) Let $v \in F$. Then $1 \in \{t \geq 0 : \frac{1}{t}\, x \in F\}$ so that $\gamma_F(v) \leq 1$.
($\Leftarrow$) Let $v \notin F$. If $|\frac{1}{t}| \leq 1$ then $\frac{1}{t}v \notin F$ by Claim 3.1. Accordingly, if $\frac{1}{t}v \in F$ then $|\frac{1}{t}| > 1$. This implies that $\gamma_F(v) > 1$. □