Do Dogs Know Angles?

Jonathan W Gardner, Kaleb Gardner

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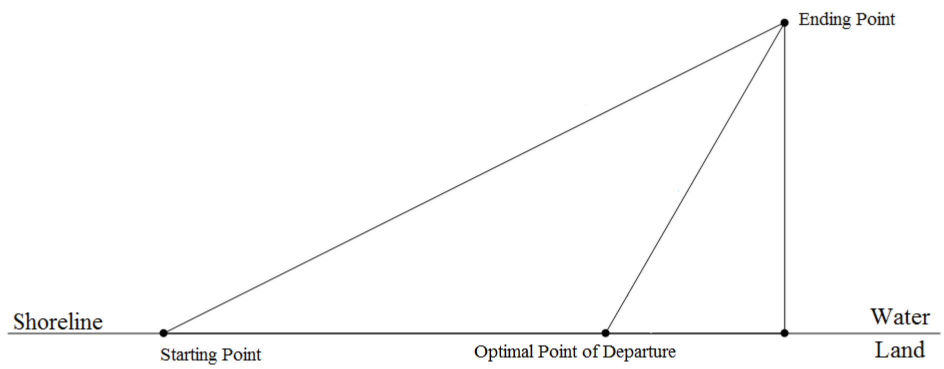
There once was a small dog named Elvis who was capable of doing calculus problems. At least, that what Hope College Mathematics Professor Timothy Pennings would want you to believe, and his statistics are pretty convincing. He noted in the article titled “Do Dogs Know Calculus ?” that while playing fetch on the shores of Lake Michigan, Elvis managed to get pretty close to the path which took the least amount of time.

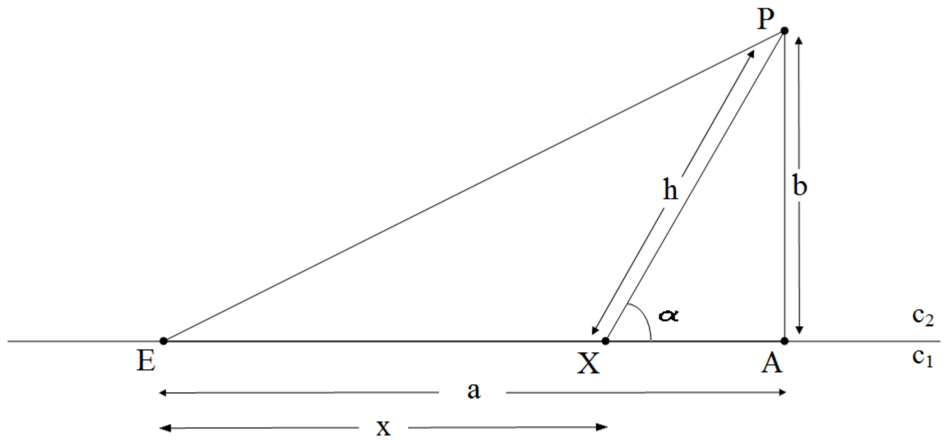


Now, most agree that dogs do not know calculus and so the question still remains. How does Elvis manage to always find the optimal path ? In this paper we will suggest a possible answer to this question : maybe dogs know angles. To get there, we :

(a) Use calculus to construct the optimal path and
(b) Analyze the geometry behind the optimal solution.

First we decided to see how difficult of a problem this would be and so we began by defining the path mathematically.

The calculus method to solve this problem would first involve deriving a formula that would give the total amount of time needed to travel any given path.



Now say that x is the point of departure while c_1 is the speed of Elvis on land and c_2 is his speed in water.

$$Time = \frac{Distance}{Velocity} = \frac{x}{c_1} + \frac{\sqrt{b^2+(a-x)^2}}{c_2}$$

Now to determine the optimal path, we take the derivative of the formula and solve for x when the time derivative of the time is set equal to zero (0).

$$T'(x) = \frac{1}{c_1} - \frac{a-x}{c_2\sqrt{b^2+(a-x)^2}}$$

We find that :

$$x = a - \frac{bc_2}{\sqrt{(c_1)^2-(c_2)^2}}$$

How is it that a simple Welsh Corgi could figure out something as complicated as this? This is cause for more investigation. Considering that dogs generally act on instinct and what they see, we chose to search for a geometric solution to this problem. We first looked at the angle at which Elvis leaves the shore line, or the angle of departure, to see if we could find some consistency. We named this angle "Angle Alpha" (α) and went on to define it.

$$Tan(\alpha) = \frac{b}{a-x} = \frac{\sqrt{(c_1)^2+(c_2)^2}}{c_2}$$

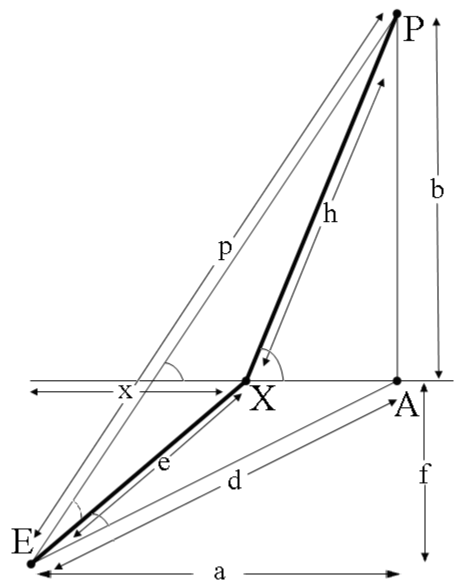
This was the solution we were looking for! The angle, α was solely dependent on the speed of Elvis, and nothing else. The angle is always constant assuming that Elvis doesn't change his speed. For example, if the speed in water is $\frac{1}{2}$ ($c_2 = \frac{1}{2}$) and the speed on land is 1 ($c_1 = 1$), then the angle of departure will be the arctangent of the $\sqrt{3}$, making $\alpha{=}\frac{\pi}{6}$. Therefore we conclude that Elvis, after many years of playing fetch, had developed a sense of his own speed and learned the optimal angle of departure. This angle would always lead him to the optimal point X.

Rule : Since $Tan(\alpha_{\text{optimal}}) = \frac{\sqrt{(c_1)^2-(c_2)^2}}{c_2}$, then :

(a) If $\angle(AXP) = \alpha \geq \alpha_{\text{optimal}}$, then Elvis leaves the shoreline and swims directly to point P.

(b) If $\angle(AXP) = \alpha \leq \alpha_{\text{optimal}}$, then Elvis runs until he reaches the point where $\alpha = \alpha_{\text{optimal}}$, and then swims to point P.

With that mystery solved, we moved to consider what would happen if Elvis had not started on the shore, but farther in land and had to complete the same task. Would there also be a simple geometric solution or would this be much more complicated? We began in much of the same way, by mapping out the path and deriving a formula for the total time. This problem is much more complicated, yet relatively simple when it comes to deriving the formula.



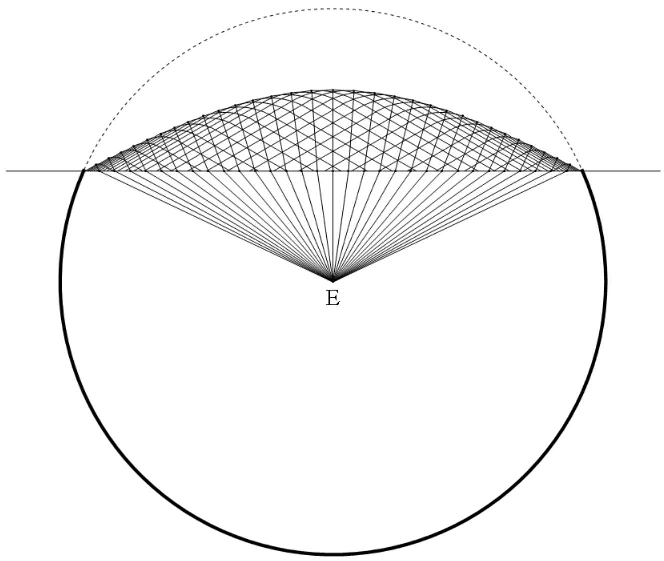
The figure is defined mathematically.

$$Time = \frac{Distance}{Velocity} = \frac{\sqrt{x^2+f^2}}{c_1} + \frac{\sqrt{(a-x)^2+b^2}}{c_2}$$

Again we found the derivative and set it equal to zero (0) in order to find the optimal path.

$$T'(x) = \frac{x}{c_1\sqrt{f^2+x^2}} - \frac{a-x}{c_2\sqrt{b^2+(a-x)^2}} = 0$$

Solving for x then becomes an extremely difficult task for any human. We know by the formula's complexity that Elvis could not find this path based on calculus methods. So again, we turn to a geometric solution. After analyzing every possible angle, we find absolutely no consistency. The only logical conclusion is that Elvis could not use an angle to discover the optimal path and so we believe that he could not do this problem at all.

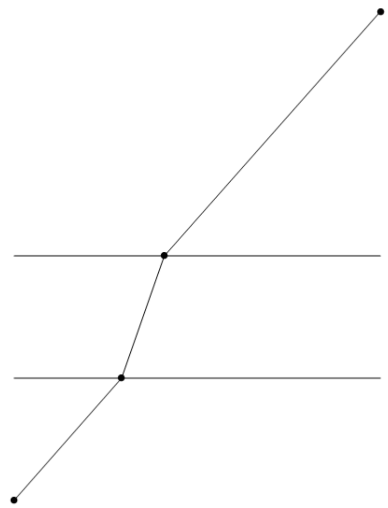


For a little more investigation, images were created to show exactly how much area Elvis could cover at a specific starting point and taking into account what we know about the angle of departure once he reaches the shoreline. When we add the optimal path calculated from calculus methods from the starting point to a specified point on the shoreline, we have what we call the "Elvis Circle".

Even after this, there is no visual evidence to suggest that Elvis could do this problem at all. However, we do notice that the optimal paths follow a path we have seen before. We began studying Snell's Law of Optics :

$$c_1 \sin(\theta) = c_2 \sin(\phi)$$

We returned to the angles we were looking at before to discover that Snell's Law always holds true. This was a big discovery because it connected the way in which light has always travelled and the path someone would take to get to a specific place as fast as possible. This was again, cause for further investigation. Immediately, we began looking into other cases in which this may hold true. A river would provide a similar problem. If someone needed to run to a river and cross it to get to something on the other side, would the optimal path be the exact same path a beam of light would take to pass through a thick lens or a glass of water?



After looking into it further, we have concluded that this is in fact true.

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