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Approximation to Impulsive Optimal Control Problems and Minimum Time Problem on Stratified Domains

Tese de Doutorado
Pós Graduação em Matemática

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15 de Março de 2016

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Tese apresentada como parte dos requisitos para obtenção do título de Doutor em Matemática, junto ao Programa de Pós Graduação em Matemática, do Instituto de Biociência, Letras e Ciências Exatas da Universidade Estadual Paulista "Júlio de Mesquita Filho", Campus de São José do Rio Preto.

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São José do Rio Preto, 15 de Março de 2016.

I dedicate this to my mother, father and sister,
my boyfriend Wanderson,
and my good friends.

ACKNOWLEDGMENTS

Thanks God for all the opportunity you gave to me during all my life. All the moments I felt insecure, afraid or desperate you showed me that I wasn't alone. Even in the most difficult moments I had I could see something better was coming because you showed me we need to have such difficult moments to learn something. If I am finishing my Phd today is thank you God and all the good people you put in my way.

My sincere thanks goes out to my adviser, Dr. Geraldo Nunes Silva, who has been my adviser since Master. He is a very good Professor and he was very patience with me during these six years. I also would like to thank Dr. Peter Robert Wolenski from Louisiana State University from the United States. He was a great adviser during my year abroad. I learnt a lot and had really good classes with him. When I was in the United States I made a lot of friends and I also would like to thank them. Because of them I had social life during that year and a lot of fun. In special, thanks Jean, Eddy, Rob, Lidia, Daniel, Tiffany, Elizabeth, Corina, Mengya, Denny, Francis, Yingzhan Wang, Sima, Xalia, Sarah, Christi. I miss everybody and I hope I can go to Baton Rouge again. Thanks The Chapel, bible study and English class.

Through my years of graduate school my parents and my sister, José Edivan and Maria José and Déborah, always supported me and were there for everything. These kind of things made me feel strong and keeping in the right direction. I can say the same about Wanderson. He was very kind all the time, patience, comprehensive and he was always by my side. I decided to go to the United States because he encouraged me and supported me all the time and he said that he would be with me in every moment. He didn't lie.

Thanks Simone and Luísa. Simone is a great friend and I had the opportunity to live with her when she was pregnant and when her baby was one year old. It was a great experience and now that they are so far from me I am missing them but I now they are much better now and I hope to see you soon.

Grant 2011/14121-9, 2014/05558-2 and 13/07375-0, São Paulo Research Foundation (FAPESP).

"*We are guided by the beauty of our weapons.*"

Leonard Cohen, First We Take Manhattan

RESUMO

Consideramos dois tipos de problemas de controle ótimo: a) Problemas de controle impulsivo e b) problemas de controle ótimo sobre domínios estratificados. Organizamos o trabalho em duas partes distintas. A primeira parte é dedicada ao estudo de um problema de controle impulsivo onde a técnica de reparametrização usual do problema impulsivo é usado para obter um problema regular. Então nós damos resultados de aproximações consistentes via discretização de Euler em que uma sequência de problemas aproximados é obtida com a propriedade que se existe uma subsequência de processos que são ótimos para o correspondente problema discreto que converge para algum processo limite, então o último é ótimo para o problema reparametrizado original. A partir da solução ótima reparametrizada somos capazes de fornecer a solução do problema impulsivo original. A segunda parte considera o problema de tempo mínimo definido sobre domínios estratificados. Definimos o problema e estabelecemos desigualdades de Hamilton Jacobi. Então, damos alguma motivações via Lei de Snell e o problema do Elvis e finalmente fornecemos condições de otimalidade necessárias e suficientes.

Palavras-chave: Aproximações Consistentes, problema de controle ótimo impulsivo, domínios estratificados e problema de tempo mínimo.

ABSTRACT

We consider two types of optimal control problems: a) Impulsive control problems and b) optimal control problems in stratified domains. So we organize this work in two distinct parts. The first part is dedicated to the study of an impulsive optimal control problem where the usual reparametrization technique of the impulsive problem is used to obtain a regular problem. Then we provide consistent approximation results via Euler discretization in which a sequence of related approximated problems is obtained with the property that if there is a subsequence of processes which are optimal for the corresponding discrete problems which converge to some limit process, then the latter is optimal to the original reparametrized problem. From the reparametrized optimal solution we are able to provide the solution to the original impulsive problem.

The second part is regarding the minimal time problem defined on stratified domains. We sate the problem and establish Hamilton Jacobi inequalities. Then we give some motivation via SnellÂťs law and the Elvis problem and finally we provide necessary and sufficient conditions of optimality.

Keywords: Consistent approximations, impulsive optimal control problem, stratified domains, minimal time problem.

CONTENTS

NOTATION

bdry(B) - boundary of a set B;

$L_p^m([a,b])$ - space of all functions $f:[a,b]\to\mathbb{R}^m$ such that $f\in L_p$;

$L_{\infty,2}^m([a,b])$ - $L_\infty^m([a,b])$ endowed with the $L_2^m([a,b])$ inner product and norm;

r-int$B$ - relative interior of a set B;

r-bdry$B$ - relative boundary of a set B;

$|\cdot|_B$ - norm defined over B;

$\mathbb{R}_+^q$ - space of all $x\in\mathbb{R}^q$ so that $x\geq 0$;

$B(0,1)$ - open ball of center 0 and radius 1;

$B[0,1]$ - closed ball of center 0 and radius 1;

l.s.c - lower semicontinuous;

$\mathcal{M}_{n\times q}$ - space of the matrices $n\times q$;

$|\cdot|$ - norm in $\mathbb{R}^m$, for each $m\in\mathbb{N}$ or variation of a measure;

$\|\cdot\|$ - norm in a space that is not $\mathbb{R}^m$, for each $m\in\mathbb{N}$ or total variation of a measure;

$d(a,b)$ - distance between the points a and b;

$d_B(a)$ - distance between the point a and the set B;

$d_H(A,B)$ - Hausdorff distance between the sets A and B;

$\overline{\mathbb{R}}=\mathbb{R}\cup\{-\infty,+\infty\}$;

$\mathcal{B}_K([0,T])$ - set of all vectorial measures defined on $[0,T] \subset \mathbb{R}$ with values in K;

H-J - Hamilton-Jacobi;

$\text{proj}_A(y)$ - projection of a point y in a set A;

epi$f$ - Epigraph of a function f;

A^c - The complement of A in the space it is included;

INTRODUCTION

This work is divided in two parts. In the first one we study impulsive optimal control problem that we apply a theory called consistent approximations. It was introduced in [1, 5]. This theory uses approximated problems with finite dimension. From an infinite dimension problem (P), we can build a sequence of problems (P_N), where these problems have finite dimension and epi-converge to (P). This convergence ensures that all sequence of global or local minimum of (P_N) that converge, will converge to a global or local minimum of (P), respectively. It is necessary to use optimality functions to represent the first-order necessary conditions because for optimal control with state and control constraints, that are complex, it is easier to work with optimality functions than with classical forms as first-order necessary condition. In [1] is given an application of this theory by using an optimal control problem.

There exist many papers that study impulsive control systems, for example [2, 6]. The article [6] shows that the solution set of the impulsive system, given by differential inclusion, is weakly* closed and the article [2] builds a numerical approximation for the impulsive system, also given by differential inclusion, using the Euler's discretization. It is shown that there is a subsequence of the solution sequence obtained by this discretization that graph-converge to a solution of the original system. In this thesis is done an approximation by absolutely continuous measure, using the convergence of graph-measure.

There exist a lot of papers that talk about impulsive optimal control problems where the control systems involve measures, and that discuss theoretical conditions for optimality of control process. We can cite, for example, [7, 8, 9, 10, 11, 12, 13, 14]. On the other hand, the literature about numerical methods for impulsive optimal control problems is rather scarce.

When we look for the usual optimal control problems, there are some works that aim to solve them using discrete approximation by Euler [15, 16, 17] or Runge Kutta [18, 19, 20, 21]. The scheme used is 1) discretize the optimal control problem and 2) solve the resulting nonlinear optimization problem. The choice of the method of resolution depends on the structure of the optimal control problem and personal preferences. Among the several proposals for solutions of nonlinear optimization problems arising from discretization, we cite some more recent [17], [21].

This work aims to contribute with the presentation of the Euler's method application for impulsive optimal control problems. We show that an impulsive optimal control problem can be reparametrized and discretized by Euler's method to generate a subsequence of optimal trajectories of Euler that converge to an optimal trajectory of the reparametrized problem, using an appropriate metric. From that we can find the optimal solution to the continuous problem. We are given a generalization of valid results for non impulsive optimal control problems [16].

This part is organized as follow. In Chapter 1 we summarize all the definitions and results that we need to get our desired results. We define the impulsive optimal control problem and introduce the theory of consistent approximations in Chapter 2. In Chapter 3 we get the approximated problems to our impulsive optimal control problem and finally the consistent approximations. We also show the convergence between a sequence of global or local minimum to the approximated problems and the local or global minimum to the original problem.

The second part is about a minimum time problem defined on stratified domains. The minimal time problem has been studied in many works, [22], [23], [28], [29], [41]. This problem consists in achieving the target in the shortest time as possible along a trajectory of the system. In [22] is defined the minimal time function. They show that the minimal time function is a proximal solution to the Hamilton-Jacobi equality. In [23] is defined a constant multifunction over $\mathbb{R}^n$, and it is gotten a characterization of the minimal time function in terms of the *gauge function* that is totally related to the Hamilton-Jacoby equation in this case. In our work, $\mathbb{R}^n$ is been written as an union of embedded manifolds in $\mathbb{R}^n$. We call such collection of manifolds by stratified domain; They were defined in [4]. Each manifold must satisfy some properties and over each one is defined a multifunction. Each multifunction gives the velocities that can be used over such manifold. A trajectory can travel between different manifolds and have different velocities in different places. Each multifunction is Lipschitz over their domains. This case is different of the cases cited above because their multifunction is defined over $\mathbb{R}^n$ and when we define another multifunction over $\mathbb{R}^n$, depending of the ones we have over each manifold, such multifunction is not necessarily Lipschitz.

A good example of this kind of problem was studied in [24]. Professor Timothy of Hope College in Holland, Michigan used to play in a lake with his dog, called Elvis. He throw a ball into the lake and Elvis, that was on the shoreline, needed to fetch it. He noted that his dog was not taking the path of shortest distance to get to the ball instead Elvis was taking the path of smallest amount of time. He considered the velocities that Elvis could achieve on the shoreline and into the water, that are different, and made all the calculations to discover what the path of smallest time is. In this particular example there are two different manifolds, water and shoreline, and over each manifold the velocity was considered constant. Another case about Elvis was studied in [25].

In [26] was studied a system with a stratified differential inclusion problem, and proved results on weak and strong invariance. This technique allows the value function to be lower semicontinuous, which opens up many potential applications. They also demonstrated that versions of the Compactness of Trajectories and Filippov Theorems can be applied to these systems, which are tools that have great importance in the theory of standard differential inclusions. Stratified systems were introduced by [4], where they considered an optimal control formulation (instead of differential inclusions). They provided conditions that guarantee the existence of solution and also some sufficient conditions for optimality. There are other works with stratified domains, [27], and one that is not published yet but the stratification is a little bit different from ours.

In Chapter 4 we introduce the stratified domain that is consider in our work. We define the minimal time problem and show the Hamilton-Jacoby inequalities for it. To get that we need to prove weak and strong invariance. As we said before, weak and strong invariance was proved in [26], but as we are assuming that our dynamics is constant over each manifold we could provide better results than the ones provided in [26]. In Chapter 5 we give some motivation, we explain better the Elvis problem and show some pictures that can help the reader to understand better our point. We give necessary and sufficient conditions to the minimal time problem and we finally show when the sufficient conditions doesn't hold with an example.

CHAPTER 1

PRELIMINARIES

The main goal of this chapter is to introduce some important concepts that have been used in our analysis in this thesis. In Section 1.1 we give some results and definitions of nonsmooth analysis, for more information see the references [28], [29]. The measure theory is given in Section 1.2 following references [28], [30], [31]. We give a brief introduction about weak and strong invariance in Section 1.3. These concepts are pertinent to get the Hamilton-Jacobi inequalities for the minimum time problem. For a complete treatment, see for example [32], [33]. Finally, we put together some theorems and definitions that are very important in our analysis in Section 1.4, for more information see [30], [34].

1.1 Nonsmooth Analysis

Suppose C is a closed subset of $\mathbb{R}^n$ and $x \in C$. We say that $\xi \in \mathbb{R}^n$ belongs to the **proximal normal cone** to C at x, written $N_C^P(x)$, if there exists $\sigma > 0$ so that

$$\langle \xi, \bar{x} - x \rangle \leq \sigma |\bar{x} - x|^2 \;\; \forall \bar{x} \in C.$$

If C is convex then $\sigma = 0$.

We say that $\zeta \in \mathbb{R}^n$ belongs to the **limiting normal cone**, written $N_C(x)$, if there exist some sequence $\{x_N\}_{N \in \mathbb{N}}$ and $\{\zeta_N\}_{N \in \mathbb{N}}$ with $x_N \in C$ for all $N \in \mathbb{N}$, $x_N \to x$ and $\zeta_N \to \zeta$ such that $\zeta_N \in N_C^P(x_N)$ for all N.

Observation 1. *If C is convex $N_C^P(x) = N_C(x)$.*

Suppose $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is lower semicontinuous. We can define the **epigraph** of

f by

$$\text{epi}f = \{(x, r) : x \in \text{dom}f, r \geq f(x)\}.$$

Let $x \in \text{dom}f$. We say that $\xi \in \mathbb{R}^n$ belongs to the **proximal subdifferential** of f at x, written $\partial_P f(x)$ if

$$(\xi, -1) \in N^P_{\text{epi}f}(x, f(x)).$$

Analogously, we say that $\zeta \in \mathbb{R}^n$ belongs to the **limiting subdifferential** of f at x, written $\partial f(x)$ if it satisfies an inclusion like the last one but changing the proximal normal cone for the limiting cone.

Observation 2. *By Proposition 4.3.6, [28], if* $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ *is convex we have*

$$\partial_P f(x) = \partial f(x) = \{\xi : \langle \xi, \bar{x} - x\rangle \leq f(\bar{x}) - f(x) \ \ \forall x \in \mathbb{R}^n\}.$$

Let C be a closed subset of $\mathbb{R}^n$. Its **indicator function** $I_C : \mathbb{R}^n \to [0, +\infty]$ is given by

$$I_C(x) = \begin{cases} 0 & \text{if } x \in C \\ +\infty & \text{otherwise.} \end{cases}$$

As C is a closed set we have that $I_C(\cdot)$ is lower semicontinuous.

Proposition 1. *Let* $x \in C$*, where* C *is a closed subset of* $\mathbb{R}^n$*. Then,*

$$\partial_P I_C(x) = N^P_C(x).$$

Proof. Suppose $C \subseteq \mathbb{R}^n$ is closed and let $x \in C$ and $\xi \in \partial_P I_C(x)$, then $(\xi, -1) \in N^P_{\text{epi}I_C}(x, I_C(x))$, and by the definition of the proximal normal cone, there exists $M > 0$ so that

$$\langle (\xi, -1), (y, r) - (x, I_C(x))\rangle \leq M|(y, r) - (x, I_C(x))|^2,$$

for all $(y, r) \in \text{epi}I_C$.

Let $x, y \in C$, then $I_C(x) = I_C(y) = 0$ and $(y, 0) \in \text{epi}I_C$. Replacing $(y, 0)$ in the last inequality we get

$$\langle (\xi, -1), (y - x, 0)\rangle \leq M|(y - x, 0)|^2 \Rightarrow \langle \xi, y - x\rangle \leq M|y - x|^2.$$

As $y \in C$ is arbitrary we have $\partial_P I_C(x) \subseteq N^P_C(x)$.

Now, let $\xi \in N^P_C(x)$. Then, there exists $M > 0$ so that

$$\langle \xi, y - x\rangle \leq M|y - x|^2, \ \ \forall y \in C.$$

There are two cases that we need to consider:

i) For $y \in C$ we have $I_C(y) = 0$. Let $r \geq I_C(y) = 0$, then

$$\langle \xi, y-x \rangle - r = \langle (\xi, -1), (y, r) - (x, I_C(x)) \rangle \leq \langle \xi, y-x \rangle \leq M|y-x|^2 \leq M|(y,r)-(x, I_C(x))|^2,$$

i.e. $\xi \in \partial_P I_C(x)$. Remember that $I_C(x) = 0$ because $x \in C$.

ii) For $y \notin C$ we have $I_C(y) = +\infty$, then $(y, r) \in \text{epi} I_C$ if and only if $r = +\infty$. As above the last inequality holds.
Then, $N_C^P(x) \subseteq \partial_P I_C(x)$.

Our proof is finished. $\square$

Let C be a subset of $\mathbb{R}^n$. The **distance function** $d_C : \mathbb{R}^n \to \mathbb{R}$ is given by

$$d_C(x) := \inf\{\|x - y\| : y \in C\}.$$

When C is closed on $\mathbb{R}^n$ we can change the inf by min.

The **tangent cone** $\mathcal{T}_C(x)$ to the set $C \subset \mathbb{R}^n$ at the point x is given by the Bouligand tangent cone

$$\mathcal{T}_C(x) := \{y \in \mathbb{R}^n : \lim_{h \to 0} \frac{d_C(x + hy)}{h} = 0\}.$$

In the case C is proximally smooth $N_C(x) = N^P(x)$ and $\mathcal{T}_C(x)$ is the negative polar of $N_C(x)$,

$$v \in \mathcal{T}_C(x) \Leftrightarrow \langle \zeta, v \rangle \leq 0 \;\; \forall \zeta \in N_C(x).$$

We can define the **Hausdorff distance** between two compact subsets $A, B \subset \mathbb{R}^m$ as

$$\text{dist}_H(A, B) = \min\{\delta \geq 0 : A \subseteq B + \delta\overline{\mathbb{B}} \;\text{ and }\; B \subseteq A + \delta\overline{\mathbb{B}}\}.$$

1.2 Measure Theory

A family $\mathfrak{X}$ of subsets of X is called a **$\boldsymbol{\sigma}$-algebra** if:
- $\emptyset$, $X \in \mathfrak{X}$;
- If $A \in \mathfrak{X}$, then $A^c \in \mathfrak{X}$;
- If $\{A_N\}_{N \in \mathbb{N}}$ is a sequence of subsets of $\mathfrak{X}$, then $\bigcup_{N \in \mathbb{N}} A_N \in \mathfrak{X}$.

$(X, \mathfrak{X})$ is called measurable space.

If $(X, \mathfrak{X})$ is a measurable space and S is a class of subsets of X then, the intersection of all σ-algebras so that S is contained of them is a σ-algebra, that is called σ-algebra generated by S. In particular, the σ-algebra $\mathfrak{B}$ generated by all the intervals $(a,b) \subset \mathbb{R}$ is called **Borel algebra**.

A function $\mu : X \to \overline{\mathbb{R}}_+$ is a **measure** over a σ-algebra $\mathfrak{X}$ of X if
- $\mu(\emptyset) = 0$;
- $\mu(E) \geq 0$ for all $E \in \mathfrak{X}$;
- Let $\{F_N\}_{N\in\mathbb{N}}$ be a sequence of disjoint subsets of X, that is, $F_j \cap F_i = \emptyset$ for all $j \neq i$, then

$$\mu\left(\bigcup_{N=1}^{\infty} F_N\right) = \sum_{N=1}^{\infty} \mu(F_N);$$

$(X, \mathfrak{X}, \mu)$ is called a measure space. Furthermore, is for all set $F \subset \mathfrak{X}$ we have that $-\infty < \mu(F) < \infty$, μ is called **finite**. If there exists a sequence $\{F_N\}_{N\in\mathbb{N}}$ of $\mathfrak{X}$ so that $X = \bigcup_{N=1}^{\infty} F_N$ and $-\infty < \mu(F_N) < \infty$ for all $N \in \mathbb{N}$, μ is called **$\boldsymbol{\sigma}$-finite**.

In this work we are using some **vectorial measure** that is a natural extension of the real measure, that is, a function $\mu : X \to \overline{\mathbb{R}}_+^m$ is a positive vectorial measure over $\mathfrak{X}$ if

$$\mu(F) = (\mu_1(F), \mu_2(F), ..., \mu_m(F)), \quad F \in \mathfrak{X},$$

where $\mu_j : X \to \overline{\mathbb{R}}_+$ is a measure for all $j = 1, ..., m$. We say that μ has an **atom** if there exists $F \in \mathfrak{X}$ with $\mu(F) \neq 0$ so that if $F_1 \subset F$ and $F_1 \in \mathfrak{X}$ then $\mu(F_1) = 0$ or $\mu(F - F_1) = 0$.

The next example is giving an well known example of impulsive measure called Dirac measure.

Example 1. *Let $X \subset \mathbb{R}^n$ be an nonempty set, $\mathfrak{X}$ be given by the parts of X and $f : X \to [0, \infty]$ be any function. Then f determines a measure μ on $\mathfrak{X}$ by*

$$\mu(E) = \sum_{x\in E} f(x).$$

If, in particular, for some $x_0 \in X$, $f(x_0) = 1$ and $f(x) = 0$ when $x \neq x_0$, μ is called the point mass or **Dirac measure** *at x_0.*

Let $\mu : X \to \overline{\mathbb{R}}_+^m$ be a vectorial measure. We define the set of functions $|\mu_j|$ over $\mathfrak{X}$ by

$$|\mu_j|(F) := \sup \sum_{i=1}^{n} |\mu_j(F_i)|,$$

where the supreme is given over all the disjoint partitions $\{F_i\}$ of F. The **total variation** of μ is given by $\|\mu\| = |\mu|(X) := \sum_{j=1}^{m} |\mu_j|(X)$. Furthermore, we say that a vectorial measure μ is σ-finite if $|\mu|$ is σ-finite. In the same way, μ is finite if $|\mu|$ is finite.

We say that a property holds almost everywhere (a.e.) if there exists a subset $S \subset \mathfrak{X}$ with $\mu(S) = 0$ such that such property holds in the complement of S (S^c).

A sequence of measures $\{\mu_N : X \to \overline{\mathbb{R}}_+^m\}_{N\in\mathbb{N}}$ is **weakly*converging** to a measure $\mu : X \to \overline{\mathbb{R}}_+^m$ if

$$\int_X f(t)d\mu_N \to \int_X f(t)d\mu, \ \ \forall\, f \in C(X;\mathbb{R}^m),$$

where $C(X;\mathbb{R}^m)$ denotes the Banach space of functions $f : X \to \mathbb{R}^m$ continuous on X with the usual norm $|f|_C = \max_{t\in X} |f(t)|$. We denote such convergence by $\mu_N \to^* \mu$.

Take a weak*convergent sequence $\mu_N \to^* \mu$, where μ_N and μ are Borel positive measures, then

$$\int_B h(t)d\mu = \lim_{N\to\infty} \int_B h(t)d\mu_N$$

for any $h \in C(X;\mathbb{R}^m)$ and any $\mu-$continuity set B, that is, $B \subset X$ and $\mu(\mathrm{bdryB}) = 0$, in particular, $B = X$ is a $\mu-$continuity set when $X = [a,b], (a,b], [a,b)$, with $a \leq b$.

1.3 Background in Weak and Strong Invariance and Differential Inclusion

Consider the problem

$$(DI)_\Gamma \begin{cases} \dot{x}(t) \in \Gamma(x(t)) \ \text{ a.e. } \ t \in [0,T], \\ x(0) = x_0, \end{cases}$$

where $\Gamma : \mathcal{M} \rightrightarrows \mathbb{R}^n$ is a multifunction and $x_0 \in \mathcal{M}$ is given. A solution to $(DI)_\Gamma$ is an absolutely continuous function $x : [0,T] \to \mathbb{R}^n$ such that $x(0) = x_0$ and its derivative $\dot{x}(\cdot)$ satisfies the inclusion required almost everywhere. $(DI)_\Gamma$ is called **differential inclusion**.

A function $x : [0,T] \to \mathbb{R}^n$ is called **absolutely continuous** if for each $\epsilon > 0$ given, there exists $\delta > 0$ such that for a countable collection of disjoint subinterval $[a_j, b_j]$ of $[0,T]$ we have

$$\sum_{j=1}^{N}(b_j - a_j) < \delta \Rightarrow \sum_{j=1}^{N} |x(b_j) - x(a_j)| < \epsilon.$$

.

If $T = +\infty$ or $x(t)$ approaches $\overline{\mathcal{M}} \setminus \mathcal{M}$ as $t \nearrow T$, then T is called **the scape time** of $x(\cdot)$ from $\mathcal{M}$ and is denoted by $\mathrm{Esc}(x(\cdot), \mathcal{M}, \Gamma)$.

We say Γ is **lower semicontinuous** at $x \in \mathcal{M}$ if given $\varepsilon > 0$ there exists $\delta > 0$ such that for all $y \in (x + \delta\mathbb{B}) \cap \mathcal{M}$,

$$\Gamma(x) \subseteq \Gamma(y) + \varepsilon\mathbb{B}.$$

The next proposition show us under which conditions there exists a solution to $(DI)_\Gamma$. It is given by [33].

Proposition 2. *Suppose $\mathcal{M} = \mathbb{R}^n$, $\Gamma : \mathcal{M} \rightrightarrows \mathbb{R}^n$ is a multifunction that satisfies*

$$(SH) \begin{cases} (i) \quad \forall\, x \in \mathcal{M},\ \Gamma(x) \textit{ is a nonempty, convex, compact set}; \\ (ii) \quad \textit{The graph } gr\Gamma = \{(x,v); v \in \Gamma(x)\} \textit{ is a closed set relative to } \mathcal{M} \times \mathbb{R}^n; \\ (iii)\ \exists\, r > 0 \textit{ so that} \max\{|v| : v \in \Gamma(x)\} \leq r(1 + |x|). \end{cases}$$

Then there exists $T > 0$ such that $(DI)_\Gamma$ admits at least one solution.

Definition 1.1. *Suppose $\mathcal{M} \subseteq \mathbb{R}^n$ is an embedded manifold, a multifunction $\Gamma : \mathcal{M} \to \mathbb{R}^n$ satisfies (SH), $E \subseteq \mathbb{R}^n$ is closed and $\mathcal{U} \subseteq \mathbb{R}^n$ is open.*

- *Then (Γ, E) is* **weakly invariant** *in $\mathcal{U}$ provided that for all $x \in \mathcal{M} \cap \mathcal{U} \cap E$, there exist a trajectory $x(\cdot)$ to $(DI)_\Gamma$ such that $x(t) \in E$, for all $t \in [0,T)$, where $T = Esc(x(\cdot), \mathcal{M} \cap \mathcal{U}, \Gamma)$;*

- *(Γ, E) is* **strongly invariant** *in $\mathcal{U}$ provided that for all $x \in \mathcal{M} \cap \mathcal{U} \cap E$, every trajectory $x(\cdot)$ to $(DI)_\Gamma$ is such that $x(t) \in E$, for all $t \in [0,T]$, where $T = Esc(x(\cdot), \mathcal{M} \cap \mathcal{U}, \Gamma)$.*

1.4 Theorem and other definitions

In this section we are giving some theorems that are been used in all the theory developed here.

Lemma 1. (Gronwall's Lemma). *Let $x(\cdot) : [0,T] \to \mathbb{R}^n$ be an absolutely continuous function satisfying*

$$|\dot{x}(t)| \leq \gamma|x(t)| + c(t) \ \ q.s. \ \ t \in [0,T]$$

for some $\gamma \geq 0$ and $c(\cdot) \in L^1[0,T]$. Then for all $t \in [0,T]$ we have the following inequality

$$|x(t) - x(0)| \leq (e^{\gamma t} - 1)|x(0)| + \int_0^t e^{\gamma(t-s)} c(s) ds.$$

Proof. See reference [34]. $\square$

Lemma 2. (Discrete Gronwall's Lemma). *Suppose that* $x_0, x_1, ..., x_N$ *are elements in* $\mathbb{R}^n$ *so that*

$$|x_{j+1}| \leq \beta |x_j| + \bar{c},$$

where β *and* $\bar{c}$ *are scales. Then,*

$$|x_N| \leq \bar{c}\frac{1-\beta^N}{1-\beta} + \beta^N |x_0|.$$

Proof. See reference [35]. $\square$

Corollary 1. *If in the Discrete Gronwall's Lemma,* $\beta = 1 + \frac{\alpha}{N}$ *and* $\bar{c} = \frac{\alpha}{N}$*, then*

$$|x_N| \leq e^{\alpha}(1 + |x_0|) - 1.$$

Proof. See reference [35]. $\square$

Theorem 1.1. (The Dominated Convergence Theorem). *Let* $\{f_N : [0,T] \to \mathbb{R}^n\}_{N\in\mathbb{N}}$ *be a sequence in* $L_1^n([0,T])$ *such that*
(i) $f_N \to f$ *a.e.;*
(ii) There exists a nonnegative function $g \in L_1^n([0,T])$ *such that* $|f_N| \leq g$ *a.e. for all* N.
Then, $f \in L_1^n([0,T])$ *and*

$$\int f(t)dt = \lim_{N\to\infty} \int f_N(t)dt$$

Proof. See reference [31]. $\square$

Observation 3. *If* $\{f_N\}_{N\in\mathbb{N}}$ *is dominated by a function* g *in* $L_p^n([0,T])$ *(it is not only almost everywhere), then almost everywhere convergence implies* $L_p^n([0,T])$ *convergence.*

Theorem 1.2. (Bellman-Gronwall Lemma). *Suppose that* $c, K \in [0,\infty)$ *and that the integrable function* $y : \mathbb{R} \to \mathbb{R}$ *satisfies the inequality*

$$y(t) \leq c + K\int_0^t y(s)ds, \quad \forall\, t \in [0,1].$$

Then

$$y(t) \leq ce^K, \quad \forall\, t \in [0,1].$$

Let $x : [a,b] \to \mathbb{R}^n$ and $\Delta = \{a = t_0 < t_1 < ... < t_n = b\}$ be given. The function of bounded variation of x on the interval $[a,b]$ is defined by

$$T(\Delta, x) = \sum_{i=1}^{N} |x(t_i) - x(t_{i-1})|.$$

We denote by

$$V(x) = \sup_{\Delta} T(\Delta, x),$$

where the supreme is given over all the partitions Δ of the compact interval, $[a, b]$, and e call $V(x)$ the total variation of x on $[a, b]$. If $V(x) < \infty$, we say that x has **bounded variation** on $[a, b]$.

PART I

CHAPTER 2 THE IMPULSIVE OPTIMAL CONTROL PROBLEM

In this chapter we study impulsive optimal control problem. In Section 2.1 we introduce the theory of consistent approximations given by [1] where such theory is used to approximate an optimal control problem in the same book. We follow that approach and get approximated problems to an impulsive optimal control problem which is defined in Section 2.3. The approach used here is: we reparametrize the impulsive system, Section 2.2, and then use the Euler discretization method following the consistent approximations techniques provided by [1], chapter 3.

2.1 Theory of Consistent Approximations

Let B be a normed space. Consider the problem

$$(P) \qquad \min_{x \in S_C} f(x), \tag{2.1}$$

where $f : B \to \mathbb{R}$ is continuous and $S_C \subset B$.

Let $\mathcal{N}$ be an infinite subset of $\mathbb{N}$ and $\{S_N\}_{N \in \mathcal{N}}$ be a family of finite dimension subspaces of B such that $S_{N_1} \subset S_{N_2}$ if $N_1 < N_2$ and $\cup S_N$ is dense in B. For all $N \in \mathcal{N}$, let $f_N : S_N \to \mathbb{R}$ be a continuous function that approximates of $f(\cdot)$ over S_N, and let $S_{C,N} \subset S_N$ be an approximation of S_C. Consider the approximated problems family

$$(P_N) \qquad \min_{x \in S_{C,N}} f_N(x), \quad N \in \mathcal{N}. \tag{2.2}$$

Define the epigraph associated to (P) and (P_N), respectively, as

$$E := \{(x,r) : x \in S_C, f(x) \leq r\}$$

and

$$E_N := \{(x,r) : x \in S_{C,N}, f_N(x) \leq r\}.$$

Note that the problems above can be rewritten like

$$(P) \quad \min_{(x,r)\in E} r \qquad\qquad (P_N) \quad \min_{(x,r)\in E_N} r,\ N \in \mathcal{N},$$

and if a sequence or a subsequence of E_N converges to the epigraph E, in the sense of "Kuratowski", we can use the sequence of problems (P_N) because a sequence or a subsequence of it is converging to (P). In Theorem 3.3.2, [1], the epigraph convergence described is equivalent to the item a) of the next theorem that defines the consistent approximations.

Definition 2.1. *Suppose that the functions $f(\cdot)$ and $f_N(\cdot)$ and the sets B, S_C, S_N and $S_{C,N}$ are defined as above.*

- *We say P_N epi-converge to P if:*
  a) For all $x \in S_C$ there exists a sequence $\{x_N\}_{N\in\mathcal{N}}$, with $x_N \in S_{C,N}$, such that $x_N \to^{\mathcal{N}} x$, with $N \to \infty$, and $\overline{\lim} f_N(x_N) \leq f(x)$;
  b) For all infinite sequence $\{x_N\}_{N\in\mathcal{K}}$, $\mathcal{K} \subset \mathcal{N}$, such that $x_N \in S_{C,N}$, for all $N \in \mathcal{K}$, and $x_N \to^{\mathcal{K}} x$, with $N \to \infty$, then $x \in S_C$ and $\underline{\lim}_{N\in\mathcal{K}} f_N(x_N) \geq f(x)$.

- *We say the upper semicontinuous functions $\gamma_N : S_{C,N} \to \mathbb{R}$ are optimality functions for the problems (P_N) if $\gamma_N(\eta) \leq 0$, $\forall\ \eta \in S_{C,N}$ and if $\hat{\eta}_N$ is a local minimizer of (P_N) then $\gamma_N(\hat{\eta}_N) = 0$. We can define the optimality function $\gamma : S_C \to \mathbb{R}$ for (P) in the same way.*

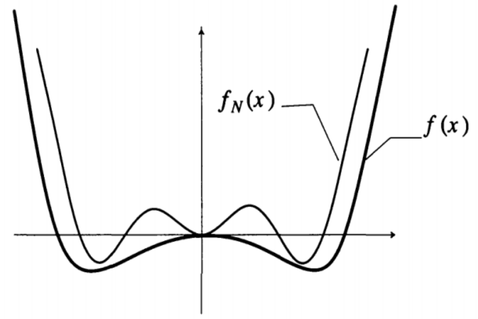
- *The pairs (P_N, γ_N) of the sequence $\{(P_N, \gamma_N)\}_{N\in\mathcal{N}}$ are consistent approximations to the pair (P, γ) if P_N epi-converge to P and for all sequence $\{x_N\}_{N\in\mathcal{N}}$ where $x_N \in S_{C,N}$ and $x_N \to x \in S_C$ we have $\overline{\lim} \gamma_N(x_N) \leq \gamma(x)$.*

The key of that epigraph convergence is given by Theorem 3.3.3, [1], where it is showed that if (P_N) epi-converges to (P) and if $\{x_N\}_{N\in\mathcal{N}}$ is a sequence of local or global solutions to (P_N) so that x_N converges to x, then x is a local or global minimizer of (P) and $f_N(x_N)$ converges $f(x)$, $N \to \infty, N \in \mathcal{N}$. But this property is not conservative as we can see in the next example that is given by [1].

Example 2. *Let $B = \mathbb{R}$ and $S_C = S_{C,N} = \mathbb{R}$ for all $N \in \mathcal{N}$. Define $f_N : \mathbb{R}^n \to \mathbb{R}$ by*

$$f_N(x) = \frac{x^2}{N} - x^4 + x^6.$$

Note that $x = 0$ is a local minimizer of $f_N(\cdot)$, i.e., for each $N \in \mathcal{N}$ there exists $\varepsilon_N > 0$ so that for all $\hat{x} \in B(x, \varepsilon_N)$ we have $f_N(x) \leq f_N(\hat{x})$. Follows the picture



As we can see, 0 is a local maximizer of $f(\cdot)$. This is happening because ε_N is converging to 0, as $N \to \infty$.

It is necessary to define the optimality functions because the epi-convergence alone can't guarantee that stationary points of (P_N) converge to a stationary point of (P), as we can observe in an example given by [1], page 397.

2.2 The Impulsive System

Before we define the impulsive optimal control problem we need to define the impulsive system that is related to it and show some results that are given by [2].

Consider the impulsive system

$$\begin{cases} dx = f(x,u)dt + g(x)d\Omega, t \in [0,T], \\ x(0) = \xi^0 \in \mathcal{C}, \end{cases} \tag{2.3}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is linear in u, $g : \mathbb{R}^n \to \mathcal{M}_{n\times q}$, where $\mathcal{M}_{n\times q}$ is the space of $n \times q$ matrices whose entries are real, $\mathcal{C} \subset \mathbb{R}^n$ is closed and convex, the function $u : [0,T] \to \mathbb{R}^m$ is Borel measurable and essentially bounded, $\Omega := (\mu, |\nu|, \psi_{t_i})$ is the impulsive control, where the first component μ is a vectorial Borel measure with range in a convex, closed cone $K \subset \mathbb{R}^q_+$. The second component is such that there exists $\mu_N : [0,T] \to K$ so that $(\mu_N, |\mu_N|) \to^* (\mu, \nu)$. As $K \subset \mathbb{R}^q_+$ we must have $\nu = |\mu|$. The functions $\psi_{t_i} : [0,1] \to K$ are associated to the measure atoms, that is, $\{\psi_{t_i}\}_{i\in\mathcal{I}}$ where $\mathcal{I}$ is the set of atomic index of the measure μ and we define $\Theta := \{t_i \in [0,T] : \mu(t_i) \neq 0\}$, where $\mu(t)$ is the vectorial value of the measure in K. They are measurable, essentially bounded and satisfy

$$\begin{array}{ll} \text{i)} & \sum_{j=1}^{q} |\psi_{t_i}^{j}(\sigma)| = |\mu|(t_i) \text{ a.e. } \sigma \in [0,1]; \\ \text{ii)} & \int_0^1 \psi_{t_i}^{j}(s)ds = \mu^j(t_i), \quad j = 1, 2, \ldots, q, \end{array}$$

for all $t_i \in \Theta$.

The functions $\psi_{t_i}(\cdot)$ are given us information about the measure μ during the atomic time $t_i \in \Theta$.

2.2.1 The Reparametrized Problem

We obtain a reparametrized problem that is approximated by doing use of the consistent approximations. This can be done, without loss of information, due to a theorem that is enunciated in this subsection and was proved by [36]. It says that the reparametrized problem and the original problem have equivalents solutions, unless than a reparametrization. For more information about it see [36], [35], [6], [37].

Firstly, we study the impulsive system given by (2.3). For this, let $\Omega = (\mu, \nu, \{\psi_{t_i}\}_{t_i \in \Theta})$ be an impulsive control and an arbitrary vector $\xi^0 \in \mathbb{R}^n$. Denote by $\mathcal{X}_{t_i}(\cdot; \xi^0)$ the solution to the system

$$\begin{cases} \dot{\mathcal{X}}_{t_i}(s) = g(\mathcal{X}_{t_i}(s))\psi_{t_i}(s), \ s \in [0,1], \\ \mathcal{X}_{t_i}(0) = \xi^0. \end{cases}$$

Consider

$$x_\vartheta := (x(\cdot), \{\mathcal{X}_{t_i}(\cdot)\}_{t_i \in \Theta}), \tag{2.4}$$

where $\vartheta := (u, \Omega)$, $x(\cdot) : [0,T] \to \mathbb{R}^n$ is a function of bounded variation with the discontinuity points in the set Θ and $\{\mathcal{X}_{t_i}(\cdot)\}_{t_i \in \Theta}$ is the collection of Lipschitz functions defined above. Follows the definition of solution of the system (2.3).

Definition 2.2. *We say that x_ϑ is a solution of* (2.3) *if*

$$x(t) = \xi^0 + \int_0^t f(x,u)d\sigma + \int_{[0,t]} g(x)d\mu_c + \sum_{t_i \leq t} [\mathcal{X}_{t_i}(1) - x(t_i-)] \ \forall t \in [0,T],$$

where μ_c is a continuous component of μ and $x(t_i-)$ is the left-hand limit of $x(\cdot)$ in t_i.

Now we do a time reparametrization and get a reparametrized system whose solution is equivalent to the solution of the original system (2.3), unless than a reparametrization.

For this, define

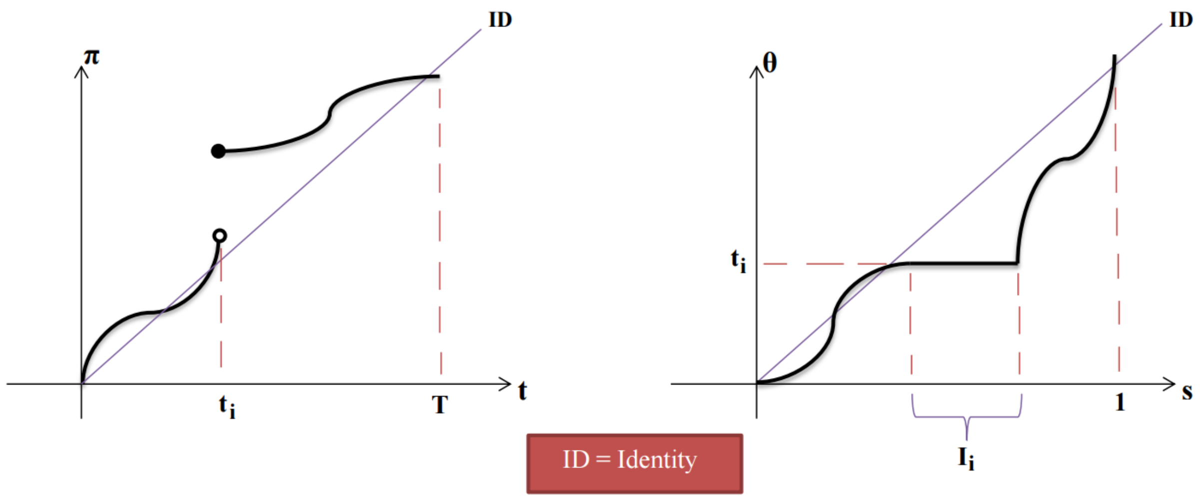
$$\pi(t) := \frac{t + |\mu|([0,t])}{T + \|\mu\|},\ t \in]0,T],\ \pi(0-) = 0.$$

The last equality is a convention because 0 can be an atom of μ.

Then, there exists $\theta : [0,1] \to [0,T]$ such that

- $\theta(s)$ is non-decreasing;
- $\theta(s) = t_i,\ \forall t_i \in \Theta, \forall s \in I_i$, where $I_i = [\pi(t_i-), \pi(t_i)]$.

The next picture shows the functions $\pi(\cdot)$ and $\theta(\cdot)$ if μ has one atom, called t_i. We can also see the interval I_i that is related to t_i.



We define by $F(t;\mu) := \mu([0,t])$ if $t \in]0,T]$, and $F(0;\mu) = 0$ the distribution function of the measure μ.

Let $\phi(\cdot) : [0,1] \to \mathbb{R}^q$ be given by

$$\phi(s) := \begin{cases} F(\theta(s);\mu) \text{ if } s \in [0,1]\backslash(\cup_{i\in\mathcal{I}} I_i), \\ F(\theta(s);\mu) + \int_{[\pi(t_i-),s]} \frac{1}{\pi(t_i)-\pi(t_i-)} \psi_{t_i}(\alpha_{t_i}(\sigma))d\sigma \text{ otherwise,} \end{cases}$$

where $\alpha_{t_i} : [\pi(t_i-), \pi(t_i)] \to [0,1]$ is given by $\alpha_{t_i}(\sigma) = (\sigma - \pi(t_i-))/(\pi(t_i) - \pi(t_i-))$.

According to [37], $\theta(\cdot)$ and $\phi(\cdot)$ are Lipschitz. The Lipschitz constants are given by b and r, respectively, and, furthermore, $\theta(\cdot)$ is absolutely continuous. We say (θ, ϕ) is the **graph completion** of the measure μ.

With all the tools on hands we can define a reparametrized solution of the system(2.3).

Definition 2.3. *Let*

$$y(s) := \begin{cases} x(\theta(s)) & \text{if } s \in [0,1] \setminus (\cup_{i\in\mathcal{I}} I_i)\,, \\ \mathcal{X}_{t_i}(\alpha_{t_i}(s)) & \text{if } s \in I_i, \text{ for some } i \in \mathcal{I}. \end{cases} \tag{2.5}$$

Then $y_\vartheta := y$ *is a reparametrized solution of* (2.3) *since* $y(\cdot)$ *is Lipschitz in* $[0,1]$ *and satisfies*

$$\begin{cases} \dot{y}(s) = f(y(s), u(\theta(s)))\dot{\theta}(s) + g(y(s))\dot{\phi}(s) \text{ a.e. } s \in [0,1], \\ y(0) = \xi^0. \end{cases} \tag{2.6}$$

The next theorem is proved by [36].

Theorem 2.1. *Suppose that the impulsive control* Ω *is given and* x_ϑ *is as defined in* (2.4)*. Then,* y_ϑ *is a reparametrized solution of* (2.3) *if and only if* x_ϑ *is a solution of* (2.3)*.*

2.3 The Impulsive Optimal Control Problem

We need to describe the constrains of the control u. We are following [1]. For this, denote by $L_2^m[0,T]$ the set of all functions defined from $[0,T]$ to $\mathbb{R}^m$ that have integrable square.

Let $\beta_{\max} \in (0,+\infty)$ be such that every control $u : [0,T] \to \mathbb{R}^m$ belongs to the ball $B(0,\beta_{\max}) := \{u \in \mathbb{R}^m; \|u\|_\infty \leq \beta_{\max}\}$.

Define

$$\hat{\mathcal{U}} := \{u \in L_{\infty,2}^m[0,T]; \|u\|_\infty \leq \omega\beta_{\max}\},$$

where $\omega \in (0,1)$ and $L_{\infty,2}^m[0,T]$ is the set of all functions defined from $[0,T]$ to $\mathbb{R}^m$ that are essentially bounded and it is gifted of the L_2 norm.

Now, we define the set of constraints of the control u by

$$\mathcal{U} := \{u \in \hat{\mathcal{U}}; u(t) \in \bar{\mathcal{U}} \subset B(0,\omega\beta_{\max}) \text{ a.e. } t \in [0,T]\},$$

where $\bar{\mathcal{U}} \subset \mathbb{R}^m$ is a convex, compact subset of the ball $B(0,\omega\beta_{\max})$.

Consider the impulsive optimal control problem

$$(P) \qquad \begin{array}{ll} \min & f^0(x(0),x(T)) \\ & dx = f(x,u)dt + g(x)d\Omega \text{ a.e. } t \in [0,T], \\ & x(0) \in \mathcal{C},\ u \in \mathcal{U},\ \mathrm{gc}\sup_{t\in[0,T]} |x(t)| \leq L, \end{array}$$

where $f^0 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is continuous, $L > 0$ is given and the other functions and sets are defined as above. Here,

$$\mathrm{gc}\sup_{t\in[0,T]} |x(t)| = \sup_{s\in[0,1]} |y(s)|.$$

We need the following assumption.

Assumption 1. *a) The functions $f(\cdot,\cdot)$ and $g(\cdot)$ are C^1, and there exist constants $K^{'}, K^{''} \in [1,\infty[$ such that, for all $x, \hat{x} \in \mathbb{R}^n$ and $u, \hat{u} \in B(0,\beta_{\max})$ we have*

$$|f(x,u) - f(\hat{x},\hat{u})| \leq K^{'}[|x-\hat{x}| + |u-\hat{u}|],$$

$$\|g(x) - g(\hat{x})\| \leq K^{''}|x-\hat{x}|,$$

b) The function $f^0(\cdot,\cdot)$ is Lipschitz, has first derivative Lipschitz and is C^1 over limited set.
c) The impulsive system given by

$$\begin{array}{l} dx = f(x,u)dt + g(x)d\Omega \ a.e.\ t \in [0,T], \\ x(0) = \xi^0 \in \mathcal{C},\ u \in \mathcal{U},\ gc\sup_{t\in[0,T]} |x(t)| \leq L, \end{array} \tag{2.7}$$

where all the variables are like above, is controllable.

Let $(\xi_0, \xi_1) \in \mathcal{C} \times \mathbb{R}^n$ be arbitrarily chosen. We say an impulsive system like (2.7) is **controllable** if there exist control $u \in \mathcal{U}$ and an impulsive control Ω so that the trajectory related to such control $x_\vartheta(\cdot)$ satisfies $x(0) = \xi_0$ and $x(T) = \xi_1$ and $\mathrm{gc}\sup_{t\in[0,T]} |x(t)| \leq L$.

Note that if we arbitrarily choose $(\xi_0, \xi_1) \in \mathcal{C} \times \mathbb{R}^n$, there exists a trajectory $x_\vartheta(\cdot)$ of (2.7) satisfying $x(0) = \xi_0$ and $x(T) = \xi_1$. We know there exists a solution of the reparametrized system (2.6) given by $y(\cdot)$ defined by (2.5), then, $y(0) = \xi_0$ and $y(1) = x_\vartheta(T)$, that is, the reparametrized system is controllable.

Because of the item a) we can show that $f(\cdot,\cdot)$ and $g(\cdot)$ have linear growth, that is, there exists a constant $K_1 < \infty$ so that

$$|f(x,u)| \leq K_1(1+|x|) \ \text{ e } \ \|g(x)\| \leq K_1(1+|x|).$$

We want to get the reparametrized impulsive optimal control problem. For this, it is necessary to define the constraints of the control $u \circ \theta$.

We define the set of constraints of the control $u \circ \theta$ by

$$\mathcal{U}_C := \{\hat{u} \in \hat{\mathcal{U}}_1; \hat{u}(s) \in \bar{\mathcal{U}} \subset B(0, \omega\beta_{\max}), \text{ a.e. } s \in [0,1]\},$$

where $\beta_{\max}$, $\bar{\mathcal{U}}$ and ω are the same and $\hat{\mathcal{U}}_1 := \{\hat{u} \in L^m_{\infty,2}[0,1]; \|\hat{u}\|_\infty \leq \omega\beta_{\max}\}$.

Define

$$\tilde{S}_C := \mathcal{C} \times \mathcal{U}_C \times \mathcal{P},$$

whose $\mathcal{U}_C$ is as defined above and $\mathcal{P}$ is the set of all $\Omega := (\mu, |\nu|, \{\psi_{t_i}\})$ that satisfies the assumptions of the system (2.3). We also define

$$S_C := \{\eta \in \tilde{S}_C : \sup_{s\in[0,1]} |y^\eta(s)| \leq L\}.$$

We represent by $y^\eta(\cdot)$ the solution of the system (2.6) for each $\eta \in \tilde{S}_C$.

We obtain the following reparametrized problem

$$(P_{rep}) \ \min_{\eta \in S_C} f^0(y^\eta(0), y^\eta(1)).$$

Note that (P) and (P_{rep}) has the same solution, unless than a reparametrization, because the objective function is the same. So, we will use the consistent approximations in (P_{rep}).

The theorem below guarantee that the system (2.6) has an unique solution.

Theorem 2.2. *Suppose $\eta = (\xi^0, u, \Omega)$ is given, where $\xi^0 \in \mathcal{C}, u \in L^m_{\infty,2}[0,1]$ and $\Omega = (\mu, |\nu|, \psi_{t_i})$ satisfy the assumptions of the system* (2.3)*. Then, the system defined in* (2.6) *has an unique solution.*

Proof. Suppose that η is given and there exist two solutions denoted by y_1^η and y_2^η. We have

$$|y_1^\eta(s) - y_2^\eta(s)| \leq \int_0^s \left(K'|y_1^\eta(\sigma) - y_2^\eta(\sigma)||\dot{\theta}(\sigma)| + K''|y_1^\eta(\sigma) - y_2^\eta(\sigma)||\dot{\phi}(\sigma)|\right) d\sigma$$

$$\leq \int_0^s |y_1^\eta(\sigma) - y_2^\eta(\sigma)| \left(K'b + K''r\right) d\sigma,$$

where in the first inequality we substituted the expressions of $y_1^\eta(\cdot)$ and $y_2^\eta(\cdot)$ given by the system (2.5), and then we used the Assumption 1. By Bellman-Gronwall's Theorem 1.2,

$|y_1^\eta(s) - y_2^\eta(s)| \leq 0$, that is, $y_1^\eta \equiv y_2^\eta$. $\square$

CHAPTER 3 CONSISTENT APPROXIMATIONS AND THE IMPULSIVE PROBLEM

In this chapter we get the consistent approximations to the reparametrized optimal control problem. In Section 3.1 we define some metrics for the spaces that must be approximated and we get some approximations to them and approximated problems to the reparametrized one. After that, in Section 3.2, we show that the approximated problems are consistent approximations to the reparametrized problem when we define appropriated optimality functions. Finally, we show how to get the solution of the original problem from the solution of the reparametrized problem.

3.1 Approximated Problems

We need a metric over the space S_C. Consider $\Omega_1 = (\mu_1, |\mu_1|, \psi^1_{t_i})$, $\Omega_2 = (\mu_2, |\mu_2|, \psi^2_{t_i}) \in \mathcal{P}$. We need to define a metric in the measures space $\mathcal{P}$. Consider the metric given by

$$d_3(p_1, p_2) = d_4(p_1, p_2) + d_5(p_1, p_2),$$

where $d_4(\cdot, \cdot)$ is a metric given by [11],

$$\begin{aligned} d_4(p_1, p_2) = &\ |(\mu_1, |\mu_1|)[0, T] - (\mu_2, |\mu_2|)[0, T]| \\ &+ \int_0^T |F_1(t; (\mu_1, |\mu_1|)) - F_2(t; (\mu_2, |\mu_2|))| dt \\ &+ \max_{s \in [0,1]} |\phi_1(s) - \phi_2(s)|, \end{aligned}$$

and $d_5(\cdot,\cdot)$ is related to the graph-convergence given by [2],

$$d_5(p_1,p_2) = \int_0^1 |\dot{\theta}_1(s) - \dot{\theta}_2(s)|ds + \int_0^1 |\dot{\phi}_1(s) - \dot{\phi}_2(s)|ds,$$

with (θ_1,ϕ_1) and (θ_2,ϕ_2) the graph completion of μ_1 and μ_2, respectively.

According to [11], the set $\mathcal{P}$ with the metric d_4 is a metric space, and, furthermore, is the completion of the absolutely continuous measures given over $[0,T]$ in the metric d_4.

Note that $S_C \subset \mathbb{R}^n \times L^m_{\infty,2}[0,1] \times \mathcal{P} =: B$. Define the metric d over B

$$d = d_1 + d_2 + d_3,$$

where d_3 is given above and

$$d_1(\xi^0,\xi^1) = |\xi^0 - \xi^1|_{\mathbb{R}^n} \text{ and } d_2(u_1,u_2) = \int_0^1 |u_1 - u_2|^2_{\mathbb{R}^m} ds$$

.

We want to get consistent approximations to the problem (P_{rep}). For this, define the sets

$$\mathcal{N} := \{2^k\}_{k=1}^{\infty} \text{ and } S_N := \mathcal{C}_N \times L^m_N \times \mathcal{P}_N \text{ for all } N \in \mathcal{N},$$

where

- $\mathcal{C}_N := \mathbb{R}^n$, $\forall\ N \in \mathcal{N}$. So $\cup_{N\in\mathcal{N}}\mathcal{C}_N$ is dense in $\mathbb{R}^n$;
- $$L^m_N := \{u_N \in L^m_{\infty,2}[0,1]; u_N(s) = \sum_{k=0}^{N-1} u_k \tau_{N,k}(s)\},$$

  with $u_k \in \mathbb{R}^m$ and

  $$\tau_{N,k}(s) := \begin{cases} 1 & \forall\ s \in [k/N,(k+1)/N[\text{ if } k \le N-2, \\ 1 & \forall\ s \in [k/N,(k+1)/N] \text{ if } k = N-1, \\ 0 & \text{otherwise.} \end{cases}$$

  Note that $\cup L^m_N$ is dense in $L^m_{\infty,2}[0,1]$.
- Let $\mathcal{P}_N$ be given by

  $$\mathcal{P}_N := \{(\mu_N, |\mu_N|, 0) : \mu_N([0,t]) := F_N(t)\},$$

where $|\mu_N|$ is the variation of the measure μ_N, $F_N(0) = 0$ and over $]0, T]$

$$F_N(t) := \sum_{k=0}^{N-1} \bar{\tau}_{N,k}(t),$$

and

$$\bar{\tau}_{N,k}(t) := \begin{cases} b_k + \frac{t-\bar{t}_k}{\bar{t}_{k+1}-\bar{t}_k}(b_{k+1} - b_k), \forall\, t \in [\bar{t}_k, \bar{t}_{k+1}], \\ \quad k = 0, ..., N-1, 0 = \bar{t}_0 < ... < \bar{t}_N = T, \\ 0 \quad \text{otherwise}, \end{cases}$$

with $b_k \in K$ for all $k = 0, ..., N-1$. Note that μ_N is an absolutely continuous measure from $[0, T]$ to K (K is convex) for all $N \in \mathcal{N}$. Furthermore, the graph completion of μ_N is defined by $\theta_N : [0, 1] \to [0, T]$ and $\phi_N : [0, 1] \to K$ as

$$\theta_N(s) := \bar{t}_k + \frac{s - s_k}{h}(\bar{t}_{k+1} - \bar{t}_k) \text{ whenever } s \in [s_k, s_{k+1}],$$

$$\phi_N(s) := F_N \circ \theta_N(s),$$

where $h = 1/N$, $s_k = kh$ and $k = 0, ..., N-1$, and it must satisfies:

i) There exists a constant $b > 0$ so that $\theta_N(\cdot)$ is Lipschitz of rank b for all $N \in \mathcal{N}$;

ii) There exists a constant $r > 0$ so that $\limsup_{N\to\infty} \|\dot{\phi}_N(\cdot)\|_\infty \leq r$.

Now, define

$$\tilde{S}_{C,N} := \tilde{S}_C \cap S_N.$$

We can get the following results.

Lemma 3. *$\cup\mathcal{P}_N$ is dense in $\mathcal{P}$.*

Proof. For the first inclusion, let $\Omega \in \overline{\cup\mathcal{P}_N}$, then there exists a sequence $\{\Omega_N\}_{N\in\mathcal{N}} \subset \cup\mathcal{P}_N$ so that $\Omega_N \to^{d_3} \Omega$, that is, $d_4(\Omega_N, \Omega) \to 0$ as $N \to \infty$. By a statement in [11], $\Omega \in \mathcal{P}$.

$$\overline{\cup\mathcal{P}_N} \subseteq \mathcal{P}.$$

Let $\Omega := (\mu, |\mu|, \psi_{t_i}) \in \mathcal{P}$. We need to show there exists a sequence of $\cup\mathcal{P}_N$ converging to Ω in the metric d_3. Let (θ, ϕ) be the gaph completion of μ. By [2], there exists a partition of $[0, T]$, $0 =: \bar{t}_0 < \bar{t}_1 < ... < \bar{t}_N := T$, and functions $\theta_N : [0, 1] \to [0, T]$, $F_N : [0, T] \to \mathbb{R}^n$, $\phi_N : [0, 1] \to K$, given by

$$\theta_N(s) = \bar{t}_k + \frac{s - s_k}{h}(\bar{t}_{k+1} - \bar{t}_k) \text{ whenever } s \in [s_k, s_{k+1}],$$

where $h = 1/N$ and $s_k = kh$,

$$F_N(t;\mu_N) = \phi(s_k) + \frac{t - \bar{t}_k}{\bar{t}_{k+1} - \bar{t}_k}(\phi(s_{k+1}) - \phi(s_k)) \text{ whenever } t \in [\bar{t}_k, \bar{t}_{k+1}],$$

$$\phi_N(s) = (F_N \circ \theta_N)(s),$$

and a measure given by

$$d\mu_N = F_N(t;\mu_N)dt.$$

Note that $\theta_N(\cdot)$ and $\phi_N(\cdot)$ are Lipschitz of rank b and r, respectively, the same rank of the functions $\theta(\cdot)$ and $\phi(\cdot)$, respectively. Such functions satisfy the graph-convergence,

$$\int_0^1 |\dot{\theta}_N(s) - \dot{\theta}(s)|ds \to 0 \quad \text{and} \quad \int_0^1 |\dot{\phi}_N(s) - \dot{\phi}(s)|ds \to 0.$$

We have the inequality

$$0 \le |\phi_N(s) - \phi(s)| \le |\textstyle\int_0^1 (\dot{\phi}_N(\tau) - \dot{\phi}(\tau))d\tau| + |\phi_N(0) - \phi(0)| \le \int_0^1 |\dot{\phi}_N(\tau) - \dot{\phi}(\tau)|d\tau.$$

We can pass the limit in the last inequality and use the graph-convergence and the fact that $\phi_N(0) = \phi(0)$ to get

$$\max_{s \in [0,1]} |\phi_N(s) - \phi(s)| \to 0.$$

By [2], the graph-convergence is stronger than the weak* convergence, so $\mu_N \to^* \mu$. By some assertion cited in [11], as $\mu_N \to^* \mu$, there exists $c > 0$ so that $\|\mu_N\| \le c$ for all N, where $\|\mu_N\|$ is the total variation of the measure μ_N. By Helly's Theorem, [38], it is possible to construct a measure ν on $[0,T]$ and select from $|\mu_N|$ a subsequence such that $|\mu_N| \to^* \nu$. By some assertions listed in [11], as $(\mu_N, |\mu_N|) \to^* (\mu, \nu)$ and the unique measure that this can happen is $|\mu|$, we must have $\nu = |\mu|$. Again, by an assertion in [11], $F_N(t;\mu_N) \to F(t;\mu)$ for all $t \in \text{Cont}|\mu|$, where $\text{Cont}|\mu|$ denotes all the points of the continuity of the scalar-valued measure $|\mu|$. As the set of all points of discontinuity of $|\mu|$ has null Borel measure we can conclude that $F_N(t;\mu_N) \to F(t;\mu)$ a.e. $t \in [0,T]$.

Note that for $t \in [0,T]$, there exists $k = 0, ..., N-1$ so that $t \in [\bar{t}_k, \bar{t}_{k+1}]$, and

$$|F_N(t;\mu_N)| = \left|\phi(s_k) + \frac{t - \bar{t}_k}{\bar{t}_{k+1} - \bar{t}_k}(\phi(s_{k+1}) - \phi(s_k))\right| \le |\phi(s_k) + \phi(s_{k+1}) - \phi(s_k)| \le |\phi(s_k)|.$$

As $\phi(\cdot)$ is continuous and defined over the compact set $[0,1]$, there exists $M > 0$ so that $|\phi(s)| \le M$ for all $s \in [0,1]$. So, $|F_N(t;\mu_N)| \le M$ for all $N \in \mathcal{N}$, M is integrable and $F_N(\cdot;\mu_N)$ is absolutely continuous, then they are integrable. By the Observation 3,

$$\int_0^T |F_N(t;\mu_N) - F(t;\mu)|dt \to 0, \ N \to \infty.$$

Note that $\nu_N := |\mu_N|$ is a measure from $[0,T]$ to $\mathbb{R}_+$ and $|\nu_N| = \nu_N$, then we have $(\nu_N, |\nu_N|) \to^* (|\mu|, |\mu|)$. By the assertion in [11], $F_N(t;\nu_N) := \nu_N([0,t]) \to F(t;|\mu|) := |\mu|([0,t])$ for all $t \in \text{Cont}(|\mu|)$. As above $F_N(t;\nu_N) \to F(t;|\mu|)$ a.e. $t \in [0,T]$. As ν_N is increasing we must have $\nu_N([0,t]) \leq \|\nu_N\| = \|\mu_N\| \leq c$, i.e., $|F_N(t;\nu_N)| \leq c$ for all N. We can use the same argument above and get

$$\int_0^T |F_N(t;\nu_N) - F(t;|\mu|)|dt \to 0,\ N \to \infty.$$

Then,

$$\begin{aligned} 0 &\leq \textstyle\int_0^T |F_N(t;(\mu_N,\nu_N)) - F(t;(\mu,|\mu|))|dt \\ &\leq \textstyle\int_0^T |F_N(t;\mu_N) - F(t;\mu)|dt + \int_0^T |F_N(t;\nu) - F(t;|\mu|)|dt \to 0,\ N \to \infty. \end{aligned}$$

By [28], $[0,T]$ is a continuity set for any positive measure defined on $[0,T]$, and

$$\int_{[0,T]} d\mu = \lim_{N\to\infty} \int_{[0,T]} d\mu_N \implies |\mu_N([0,T]) - \mu([0,T])| \to 0.$$

The same holds for ν_N. Then, we get

$$|(\mu_N,\nu_N)([0,T]) - (\mu,|\mu|)([0,T])| \to 0.$$

□

By the density of the union of each set, follows that $\cup S_N$ is dense in B.

Lemma 4. *$\tilde{S}_{C,N} \to^{\mathcal{N}} \tilde{S}_C$, $N \to \infty$, where the convergence is in the sense of Kuratowski.*

Proof. Let $\{\eta_N = (\xi_N^0, u_N, \Omega_N)\}_{N\in\mathcal{N}}$ be a sequence in $\tilde{S}_{C,N}$ such that $\eta_N \to^d \eta = (\xi^0, u, \Omega)$. As $\mathcal{C}$ is closed, $\xi^0 \in \mathcal{C}$. The part that $u \in \mathcal{U}_C$ is given by Proposition 4.3.1, [1]. Now, we know that $\Omega_N = (\mu_N, |\mu_N|, 0) \to^{d_3} \Omega = (\mu, |\mu|, \{\psi_{t_i}\})$. As it was said above, $\mathcal{P}$ is the completion of the set of absolutely continuous vector-valued measures given on $[0,T]$ in the metric d_4. As the other part of the metric d_3, d_5, is only completing the other one we can conclude that $\Omega \in \mathcal{P}$.

$$\therefore \overline{\lim} \tilde{S}_{C,N} \subset \tilde{S}_C.$$

Now, take $\eta = (\xi^0, u, \Omega) \in \tilde{S}_C$. We must find a sequence in $\tilde{S}_{C,N}$ that converges to η in the metric d. By Proposition 4.3.1, [1], and by the Lemma (3), follow that there exists such sequence.

$$\therefore \tilde{S}_C \subset \underline{\lim} \tilde{S}_{C,N}.$$

□

Given $\eta = (\xi_N^0, u_N, \Omega_N) \in S_N$, we can use the Euler's descretization to get the discrete dynamic below by the continuous dynamic given by (2.6). In this way, take $N \in \mathcal{N}$, $h = 1/N$ the step size and $s_k = kh$, $k = 0, ..., N$. We have

$$\begin{aligned} y_N^\eta(s_{k+1}) - y_N^\eta(s_k) &= f\left(y_N^\eta(s_k), u_N(s_k)\right)\left(\theta_N(s_{k+1}) - \theta_N(s_k)\right) \\ &\quad + g\left(y_N^\eta(s_k)\right)\left(\phi_N(s_{k+1}) - \phi_N(s_k)\right), \qquad (3.1) \\ k = 0, ..., N-1,\ y_N^\eta(0) = \xi_N^0, & \end{aligned}$$

where $\theta_N : [0,1] \to [0,T]$ and $\phi_N : [0,1] \to K$ are as defined in the definition of $\mathcal{P}_N$.

We associate the function

$$y_N^\eta(s) := \sum_{k=0}^{N-1} \left[y_N^\eta(s_k) + \frac{s - s_k}{h}\left(y_N^\eta(s_{k+1}) - y_N^\eta(s_k)\right)\right] \tau_{N,k}(s), \qquad (3.2)$$

where $\{y_N^\eta(s_k)\}_{k=0}^N$ is the solution of the discrete system (3.1).

Lemma 5. *Let $\eta = (\xi_N^0, u_N, \Omega_N) \in S_{C,N}$ and $\{y_N^\eta(s_k)\}_{k=0}^N$ be the solution of the discretized equation corresponding to this η. Thus, the following inequality holds*

$$|y_N^\eta(s_k)| + 1 \leq e^\beta(1 + |\xi_N^0|),$$

where $\beta := K_1(b + r)$, K_1 is the constant relative to the linear growth of the functions $f(\cdot,\cdot)$ and $g(\cdot)$ and, b and r are the Lipschitz constants of the functions $\theta_N(\cdot)$ and $\phi_N(\cdot)$, respectively.

Proof. This result follows from the Corollary of the Discrete Gronwall's Lemma. □

Define

$$S_{C,N} := \{\eta_N \in \tilde{S}_{C,N} : |y_N^{\eta_N}(s)| \leq L + 1/N,\ \forall s \in [0,1]\},$$

where $y_N^{\eta_N}(\cdot)$ is given by the linear interpolation of the points $y_N^{\eta_N}(s_k)$, $k = 0, ..., N$, (3.2).

We need to show that $S_{C,N}$ is converging to S_C, and, after that, we can define the approximated problems.

Theorem 3.1. *$S_{C,N} \to^{\mathcal{N}} S_C$, $N \to \infty$, where the convergence is in the sense of Kuratowski.*

Proof. Let $\{\eta_N = (\xi_N^0, u_N, \Omega_N)\}_{N \in \mathcal{N}}$ be a sequence in $S_{C,N}$ so that $\eta_N \to^d \eta = (\xi^0, u, \Omega)$. As $S_{C,N} \subset \tilde{S}_{C,N}$, by Lemma 4, $\eta \in \tilde{S}_C$. We know that $|y_N^\eta(s)| \leq L + 1/N$ for all $s \in [0,1]$.

By Theorem 3.2, there exists $\mathcal{K} \subset \mathcal{N}$ so that $y_N^{\eta_N}(\cdot)$ uniformly converges to $y^\eta(\cdot)$ in $\mathcal{K}$, then, given $\varepsilon = 1/N$, there exists $N_0 \in \mathcal{N}$ such that for all $N \geq N_0$, $N \in \mathcal{K}$ we have

$$|y_N^{\eta_N}(s) - y^\eta(s)| \leq 1/N \implies |y^\eta(s)| \leq |y_N^{\eta_N}(s)| + 1/N \to L.$$

$$\therefore \overline{\lim} S_{C,N} \subset S_C.$$

Now, let $\eta = (\xi^0, u, \Omega) \in S_C$. By Lemma 4, there exists a sequence $\{\eta_N = (\xi_N^0, u_N, \Omega_N)\}_{N \in \mathcal{N}} \in \tilde{S}_{C,N}$ so that $\eta_N \to^d \eta$. Again, by Theorem 3.2 there exists $\mathcal{K} \subset \mathcal{N}$ so that $y_N^{\eta_N}(\cdot)$ uniformly converges to $y^\eta(\cdot)$ in $\mathcal{K}$, then, given $\varepsilon = 1/N$ there exists $N_0 \in \mathcal{N}$ such that for all $N \geq N_0$, $N \in \mathcal{K}$ we have

$$|y_N^{\eta_N}(s) - y^\eta(s)| \leq 1/N \implies |y_N^{\eta_N}(s)| \leq |y^\eta(s)| + 1/N \leq L + 1/N.$$

$$\therefore S_C \subset \underline{\lim} S_{C,N}.$$

□

Then, we get the approximated problems

$$(P_{rep}^{C,N}) \quad \min_{\eta \in S_{C,N}} f_N^0(y_N^\eta(0), y_N^\eta(1)),$$

where $f_N^0(y_N^\eta(0), y_N^\eta(1)) := f^0(\xi_N^0, y_N^\eta(1))$.

3.2 Consistent Approximations

In this section, we show that the problems $(P_{rep}^{C,N})$ with some optimality functions $\gamma^{C,N}(\cdot)$ are consistent approximations to the pair (P_{rep}, γ), where $\gamma(\cdot)$ is an optimality function to the problem (P_{rep}).

We begin with a theorem that is giving us a convergence between the polygonal arc given by the Euler's discretization and the solution of the reparametrized problem. Note that $\eta \in \tilde{S}_{C,N}$ is arbitrary and is converging to $\eta \in \tilde{S}_C$ in the metric d. This means that the convergence in the metric d is enough to guarantee the convergence between the solutions.

Theorem 3.2. *Suppose that the Assumption 1 holds, $N \in \mathcal{N}$ and $\eta_N \to^d \eta$, where $\eta_N \in \tilde{S}_{C,N}$ and $\eta \in \tilde{S}_C$. Thus, there exists $\mathcal{K} \subset \mathcal{N}$ such that, $y_N^{\eta_N}(\cdot)$ uniformly converge to $y^\eta(\cdot)$, with $N \in \mathcal{K}$, $N \to \infty$, where $y_N^{\eta_N}(\cdot)$ is defined in* (3.2) *and $y^\eta(\cdot)$ is the solution of* (2.6).

Proof. Note that, over the interval $[s_k, s_{k+1}]$ we have

$$|\dot{y}_N^{\eta_N}(s)| \leq K_1(b+r)|y_N^{\eta_N}(s_k)| + K_1(b+r) =: \beta|y_N^{\eta_N}(s_k)| + \beta, \tag{3.3}$$

and by Lemma 5,

$$|y_N^{\eta_N}(s_k)| \leq e^{\beta}\left(|\xi_N^0| + 1\right) - 1, \ k \in \{0, ..., N-1\}.$$

As ξ_N^0 is convergent, it follows that there exists $\overline{M} > 0$ such that $|\xi_N^0| \leq \overline{M}$, $\forall N \in \mathcal{N}$. So, $|y_N^{\eta_N}(s_{k+1})| \leq e^{\beta}M$. By the equation (3.3), $\dot{y}_N^{\eta_N}(s)$ is uniformly bounded. Using the same argument we have $y_N^{\eta_N}(s)$ is uniformly bounded too. By Arzelà-Ascoli's Theorem, there exist $\mathcal{K} \subset \mathcal{N}$ and $y : [0,1] \to \mathbb{R}^n$ such that $y_N^{\eta_N}(\cdot)$ uniformly converge to $y(\cdot)$, $N \in \mathcal{K}$, $N \to \infty$.

Now, we need to show that $y(\cdot)$ satisfies the system (2.6). For this, define

$$y^{\eta}(s) = f(y(s), u(s))\dot{\theta}(s) + g(y(s))\dot{\phi}(s), \ \ y^{\eta}(0) = \xi^0.$$

For $s \in [s_k, s_{k+1}]$,

$$|y_N^{\eta_N}(s) - y^{\eta}(s)| \leq \left|\int_0^s (\dot{y}_N^{\eta_N}(\sigma) - \dot{y}^{\eta}(\sigma))\, d\sigma\right| + |\xi_N^0 - \xi^0|$$

$$\leq \sum_{j=0}^{k-1} \int_{s_j}^{s_{j+1}} |\dot{y}_N^{\eta_N}(\sigma) - \dot{y}^{\eta}(\sigma)|\, d\sigma + \int_{s_k}^{s} |\dot{y}_N^{\eta_N}(\sigma) - \dot{y}^{\eta}(\sigma)|\, d\sigma + |\xi_N^0 - \xi^0|$$

$$\leq \sum_{j=0}^{k-1} \int_{s_j}^{s_{j+1}} |f(y_N^{\eta_N}(s_j), u_N(s_j)) - f(y(\sigma), u(\sigma))||\dot{\theta}_N(\sigma)|d\sigma$$

$$+ \sum_{j=0}^{k-1}\left[\int_{s_j}^{s_{j+1}} |f(y(\sigma), u(\sigma))|\left|\dot{\theta}_N(\sigma) - \dot{\theta}(\sigma)\right| d\sigma + \int_{s_j}^{s_{j+1}} |g(y(\sigma))|\left|\dot{\phi}_N(\sigma) - \dot{\phi}(\sigma)\right| d\sigma\right]$$

$$+ \sum_{j=0}^{k-1} \int_{s_j}^{s_{j+1}} |g(y_N^{\eta_N}(s_j)) - g(y(\sigma))||\dot{\phi}_N(\sigma)|d\sigma$$

$$+ \int_{s_k}^{s} |\dot{y}_N^{\eta_N}(\sigma) - \dot{y}^{\eta}(\sigma)|d\sigma + |\xi_N^0 - \xi^0|$$

$$= \sum_{j=0}^{k-1}[I + II + III + IV] + \int_{s_k}^{s} |\dot{y}_N^{\eta_N}(\sigma) - \dot{y}^{\eta}(\sigma)|d\sigma + V.$$

Let's check that there exists $\mathcal{K} \subset \mathcal{N}$ such that the equation above converge to zero whenever $N \in \mathcal{K}$.

- For I, as $f(\cdot,\cdot)$ is Lipschitz,

$$I \leq \int_{s_j}^{s_{j+1}} K'b\left(|y_N^{\eta_N}(s_j) - y(\sigma)| + |u_N(s_j) - u(\sigma)|\right)d\sigma \leq bK' \int_{s_j}^{s_{j+1}} \left(|y_N^{\eta_N}(s_j) - y_N^{\eta_N}(\sigma)|\right)d\sigma$$

$$+bK' \int_{s_j}^{s_{j+1}} \left(|y_N^{\eta_N}(\sigma) - y(\sigma)| + |u_N(s_j) - u_N(\sigma)| + |u_N(\sigma) - u(\sigma)|\right)d\sigma,$$

since $\sup|\dot{\theta}_N(s)| \leq b$, for some $b > 0$.
It is easy to verify that $y_N^{\eta_N}(\cdot)$ is Lipschitz. Let's denote its Lipschitz constant by $\kappa > 0$. Then,

$$\int_{s_j}^{s_{j+1}} |y_N^{\eta_N}(s_j) - y_N^{\eta_N}(\sigma)|d\sigma \leq \int_{s_j}^{s_{j+1}} \kappa|s_j - \sigma|d\sigma \leq \kappa h^2 \to 0.$$

As $y_N^{\eta_N}(\cdot)$ uniformly converge to $y(\cdot)$ we have that $|y_N^{\eta_N}(\sigma) - y(\sigma)| \to 0$. As every sequence uniformly converging is bounded, there exists $c > 0$ so that

$$|y_N^{\eta_N}(\sigma) - y(\sigma)| \leq c, \ \ \forall N, \ \forall \tau \in [0,1],$$

then,

$$\int_{s_j}^{s_{j+1}} |y_N^{\eta_N}(\sigma) - y(\sigma)|d\tau \leq ch \to 0.$$

As $u_N(s) = u(s_j)$ for all $s \in [s_j, s_{j+1}[$ we get

$$\int_{s_j}^{s_{j+1}} |u_N(s_j) - u_N(\sigma)|d\sigma \to 0.$$

We know $u_N(\sigma) \to^{d_2} u(\sigma)$. By Hölder's inequality, we get $u_N(\sigma) \to u(\sigma)$ in $L_1^m([0,1])$.

- For II, as $y_N^{\eta_N}(\cdot)$ uniformly converge to $y(\cdot)$, given $\varepsilon = 1$, there exists $N_0 \in \mathbb{N}$ so that for all $N \geq N_0$, $|y(s) - y_N^{\eta_N}(s)| < 1$ for all $s \in [0,1]$, i.e., $|y(s)| < 1 + |y_N^{\eta_N}(s)| < \hat{M}$, for some $\hat{M} > 0$ since $y_N^{\eta_N}(\cdot)$ is uniformly bounded. As f has linear growth,

$$|f(y(\sigma), u(\sigma))| \leq K_1(1 + |y(s)|) \leq K_1(1 + \hat{M}).$$

By the convergence of η_N in the metric d, we have

$$0 \leq \int_{s_j}^{s_{j+1}} |\dot{\theta}_N(s) - \dot{\theta}(s)|ds \leq \int_0^1 |\dot{\theta}_N(s) - \dot{\theta}(s)|ds \to 0.$$

- III and IV are completely analogous to II and I, respectively.

- As $\eta_N \to^d \eta$, then there exists the convergence between the initials conditions, then we have the convergence of V. The last integral is totally analogous to the one that we just showed. □

In the same way that we did in the last Theorem, we can prove the same result but now when the sequence of $\eta's$ belongs to the set $\tilde{S}_C$ and also the point of convergence and when both of them belong to the set $\tilde{S}_{C,N}$.

Proposition 3. *a) Let $\{\eta_N = (\xi_N^0, u_N, \Omega_N)\}_{N\in\mathcal{N}} \in \tilde{S}_C$ be a sequence so that $\eta_N \to^d \eta$, where $\eta \in \tilde{S}_C$. Thus, there exists $\mathcal{K} \subseteq \mathcal{N}$ such that $y_N^\eta(\cdot)$ uniformly converge to $y^\eta(\cdot)$ when $N \to \infty$, $N \in \mathcal{K}$, where $y_N^{\eta_N}(\cdot)$ and $y^\eta(\cdot)$ are the solution of the system* (2.6) *related to η_N and η, respectively.*
b) Let $\{\eta_N = (\xi_N^0, u_N, \Omega_N) \in \tilde{S}_{C,N}$ be so that $\eta_N \to^d \eta$, $\eta \in \tilde{S}_{C,N}$. Let $y_N^\eta(\cdot)$ and $y^\eta(\cdot)$ be the polygonal arc given by the Euler's discretization, equation (3.2)*. Thus, there exists $\mathcal{K} \subseteq \mathcal{N}$ such that $y_N^\eta(\cdot)$ uniformly converge to $y^\eta(\cdot)$ when $N \to \infty$, $N \in \mathcal{K}$.*

Observation 4. *Note that Theorem 3.2 and Proposition 3 hold when we change $\tilde{S}_C$ and $\tilde{S}_{C,N}$ by S_C and $S_{C,N}$, respectively. The proof follows from the Theorem 3.2 and the proof of Theorem 3.1.*

The next result provides the optimality functions to the problems (P_{rep}) and $(P_{rep}^{C,N})$.

Theorem 3.3. *Suppose that Assumption 1 holds. The following statements are satisfied:*
a) Let

$$\gamma(\eta) := \min_{\bar{\eta}\in S_C} \left(\langle \nabla f^0(\xi), \bar{\xi} - \xi \rangle + \frac{1}{2}\bar{d}((\xi, u, \Omega), (\bar{\xi}, \bar{u}, \bar{\Omega})) \right),$$

with $\xi := (\xi^0, y^\eta(1))$, $\bar{\xi} := (\bar{\xi}^0, y^{\bar{\eta}}(1))$, $\bar{d} = d_1 + d_2 + d_4$ and $\gamma : S_C \to \mathbb{R}$.
i) If $\bar{\eta} \in S_C$ is a local minimizer of (P_{rep}), then

$$\langle \nabla f^0(\bar{\xi}), \xi - \bar{\xi} \rangle \geq 0, \ \forall\, \eta \in S_C;$$

ii) $\gamma(\eta) \leq 0, \ \forall\, \eta \in S_C$;
iii) If $\bar{\eta} \in S_C$ is a minimum of (P_{rep}), then $\gamma(\bar{\eta}) = 0$;
iv) $\gamma(\cdot)$ is upper semicontinuous.
Thus, we can conclude that γ is an optimality function to the problem (P_{rep}).

b) Let

$$\gamma^{C,N}(\eta) = \min_{\bar{\eta}\in S_{C,N}} \left(\langle \nabla f_N^0(\xi), \bar{\xi} - \xi \rangle + \frac{1}{2}\bar{d}((\xi, u, \Omega), (\bar{\xi}, \bar{u}, \bar{\Omega})) \right),$$

with $\gamma^{C,N} : S_{C,N} \to \mathbb{R}$.
i) If $\bar{\eta} \in S_{C,N}$ is a local minimizer of $(P_{rep}^{C,N})$, then

$$\langle \nabla f_N^0(\bar{\xi}), \xi - \bar{\xi} \rangle \geq 0, \ \forall\, \eta \in S_{C,N};$$

ii) $\gamma^{C,N}(\eta) \leq 0, \ \forall\, \eta \in S_{C,N}$;

iii) If $\bar{\eta} \in S_{C,N}$ is minimum of $(P_{rep}^{C,N})$, then $\gamma^{C,N}(\bar{\eta}) = 0$;
iv) $\gamma^{C,N}(\cdot)$ is upper semicontinuous;
Thus, we can conclude that $\gamma^{C,N}$ is an optimality function to the problem $(P_{rep}^{C,N})$.

Proof. a). i) Suppose that $\bar{\eta} \in S_C$ is a minimizer of (P_{rep}) and there exists $\eta \in S_C$ so that

$$\langle \nabla f^0(\bar{\xi}), \xi - \bar{\xi} \rangle < 0. \tag{3.4}$$

As $\mathcal{C}$ is convex, $\bar{\xi} + \lambda(\xi - \bar{\xi}) \in \ \mathcal{C} \times \mathbb{R}^n$, $\forall\, \lambda \in [0, 1]$. There exists $\bar{\lambda} \in (0, 1]$ such that

$$f^0(\bar{\xi} + \bar{\lambda}(\xi - \bar{\xi})) - f^0(\bar{\xi}) \leq \bar{\lambda}\langle \nabla f^0(\bar{\xi}), \xi - \bar{\xi} \rangle < 0,$$

where in the last inequality we used the inequality (3.4).

Let $\xi^0 = \bar{\xi}^0 + \bar{\lambda}(\xi^0 - \bar{\xi}^0)$ and $\xi_1 = y^{\bar{\eta}}(1) + \bar{\lambda}(y^{\eta}(1) - y^{\bar{\eta}}(1))$ be the fixed initial and final points of the reparametized system (2.6). As such system is controllable it must exists a solution of the reparametrized system $y(\cdot)$ satisfying $y(0) = \xi_0$ and $y(1) = \xi_1$, i.e.,

$$f^0(\bar{\xi} + \bar{\lambda}(\xi - \bar{\xi})) < f^0(\bar{\xi}).$$

This is a contradiction with the assumption.

ii) Note that

$$\gamma(\eta) \leq \left(\langle \nabla f^0(\xi), \xi - \xi \rangle + \frac{1}{2}\bar{d}((\xi, u, v, \Omega), (\xi, u, v, \Omega)) \right) = 0,\ \forall\, \eta \in S_C,$$

because as $\gamma(\eta)$ is the minimum, it is smaller than when calculated in η.

iii) Suppose that $\bar{\eta} \in S_C$ is a minimizer of (P_{rep}). We have

$$0 \geq \gamma(\bar{\eta}) \geq \min_{\eta \in S_C} \langle \nabla f^0(\bar{\xi}), \xi - \bar{\xi} \rangle \geq 0,$$

where in the first and third inequality we used the items ii) and i), respectively.

iv) Let $\{\eta_N\}_{N \in \mathcal{N}}$ be a sequence of S_C so that $\eta_N \to^d \eta$, $N \in \mathcal{N}$, and $\mathcal{K} \subset \mathcal{N}$ such that

$$\overline{\lim} \gamma(\eta_N) = \lim_{N \in \mathcal{K}} \gamma(\eta_N).$$

By Proposition 3 part a), there exists $\bar{\mathcal{K}} \subset \mathcal{K}$ such that $y_N^{\eta_N}(1) \to y^{\eta}(1)$ whenever $N \in \bar{\mathcal{K}}$. Then, by the definition of $\gamma(\cdot)$ and Assumption 1, follows that $\lim_{N \in \bar{\mathcal{K}}} \gamma(\eta_N) = \gamma(\eta)$. But, as the limit $\lim_{N \in \mathcal{K}} \gamma(\eta_N)$ exists, we must have that all the subsequence are converging to the same point, that is,

$$\overline{\lim} \gamma(\eta_N) = \gamma(\eta).$$

Part b) is totally analogous to the part a). □

The following Lemma is very important to the next result.

Lemma 6. *Let $\varphi : S_C \to \mathbb{R}$ be given by*

$$\varphi(\bar{\eta}) = \langle \nabla f^0(\xi), \bar{\xi} - \xi \rangle + \frac{1}{2}\bar{d}((\xi^0, u, \Omega), (\bar{\xi}^0, \bar{u}, \bar{\Omega})),$$

for each $\eta \in S_C$ fixed and $\xi = (\xi^0, y^\eta(1)) \in \mathcal{C} \times \mathbb{R}^n$. Then there exists $\hat{\eta} \in S_C$ such that

$$\varphi(\hat{\eta}) = \min_{\bar{\eta} \in S_C} \varphi(\bar{\eta}).$$

Proof. Let $\alpha = \inf_{\bar{\eta} \in S_C} \varphi(\bar{\eta})$, then by the definition of the infimum, there exists $\alpha_N = \varphi(\eta_N)$ so that $\alpha_N \to \alpha$ (in $\mathbb{R}$), and $\eta_N \in S_C$ for all $N \in \mathcal{N}$. As α_N is a convergent sequence of $\mathbb{R}$, it must exists $M > 0$ so that $|\alpha_N| \leq M$ for all $N \in \mathcal{N}$, that is

$$\langle \nabla f^0(\xi), \xi_N - \xi \rangle + \frac{1}{2}\bar{d}((\xi^0, u, \Omega), (\xi^0_N, u_N, \Omega_N)) \leq M.$$

As $\eta_N \in S_C$ for all $N \in \mathcal{N}$, we must have $\sup_{s \in [0,1]} |y^{\eta_N}(s)| \leq L$ for all $N \in \mathcal{N}$, then ξ_N is uniformly bounded, that is, $\langle \nabla f^0(\xi), \xi_N - \xi \rangle$ is uniformly bounded. We have

$$d_1(\xi^0, \xi^0_N) \leq M_1, \quad d_2(u_N, u) \leq M_1 \text{ and } d_4(\Omega_N, \Omega) \leq M_1,$$

for some $M_1 > 0$.

We have some points:

- $d_1(\xi^0, \xi^0_N) \leq M_1 \implies |\xi^0_N| \leq M_2$, M_2 is some positive constant. Then there exist $\mathcal{K}_1 \subset \mathcal{N}$ and $\bar{\xi}^0 \in \mathcal{C}$ so that $\xi^0_N \to \bar{\xi}^0$;

- $d_2(u_N, u) \leq M_1 \implies \int_0^1 |u_N(s)|^2 \leq M_3$ if we use Minkoviski's inequality, M_3 is some positive constant. By the sequential compactness in $L_2^m([0,1])$, there exist $\mathcal{K}_2 \subset \mathcal{K}_1$ and $\bar{u} \in L_2^m([0,1])$ so that

$$\int_0^1 \langle u_N(s) - \bar{u}(s), h(s) \rangle ds \to 0, \quad \forall h \in L_2^m([0,1]), N \in \mathcal{K}_2.$$

We need to show that $\bar{u} \in \mathcal{U}_C$. We know $u_N(s) \in \overline{\mathcal{U}}$ a.e., for all $N \in \mathcal{K}_2$. Define

$$W := \{\omega \in L_2^m([0,1]) : \omega(t) \in \overline{\mathcal{U}} \text{ a.e.}\}$$

Then W is strongly closed in $L_2^m([0,1])$, because a strongly convergent sequence admits a subsequence converging almost everywhere, and since $\overline{\mathcal{U}}$ is closed by assumption. W is convex because $\overline{\mathcal{U}}$ is convex. By Theorem III.7, [39], W is weakly closed. As $\bar{u}$ is the weak limit of the sequence u_N, $\bar{u}$ belongs to W, and then $\bar{u} \in \mathcal{U}_C$.

- $d_4(\Omega_N, \Omega) \leq M_1$.
  By a statement given by [11], there exist $\mathcal{K}_3 \subset \mathcal{K}_2$ and $\bar{\Omega} \in \mathcal{P}$ so that $d_4(\Omega_N, \bar{\Omega}) \to 0, N \in \mathcal{K}_3$.

Then, when $N \in \mathcal{K}_3$, we have
1) $\xi_N^0 \to^{d_1} \bar{\xi}^0$;
2) $u_N \to \bar{u}$ weakly in $L_2^m([0,1])$;
3) $\Omega_N \to^{d_4} \bar{\Omega}$.

By an observation given in [11], if f is linear in u, 1), 2) and 3) hold and $\sup_{s\in[0,1]} |y^{\eta_N}(s)| \leq L$, we can still apply Lemma 3.2, [11] and get that $y^{\eta_N}(1) \to y^{\bar{\eta}}(1)$, where $y^{\bar{\eta}}(\cdot)$ is the trajectory of the reparametrized system related to $\bar{\eta} = (\bar{\xi}^0, \bar{u}, \bar{\Omega})$. Moreover, $\sup_{s\in[0,1]} |y^{\eta_N}(s)| \to \sup_{s\in[0,1]} |y^{\bar{\eta}}(s)|$, and as $\sup_{s\in[0,1]} |y^{\eta_N}(s)| \leq L$, we must have that $\sup_{s\in[0,1]} |y^{\bar{\eta}}(s)| \leq L$. Then, $\bar{\eta} \in S_C$.

We have strong convergence in $\mathcal{C}$ and $\mathcal{P}$ but we have weak convergence in $L_2^m([0,1])$. Let's work on that.

We know $d_2 : \mathcal{U}_C \to \mathbb{R}$, $\mathcal{U}_C \subset L_{\infty,2}^m([0,1]) \subset L_2^m([0,1])$ that is a Banach space. Let λ be so that there exists $\hat{u} \in \mathcal{U}_C$ satisfying $d_2(u, \hat{u}) = \lambda$. Define

$$A := \{\tilde{u} \in \mathcal{U}_C : d_2(u, \tilde{u}) \leq \lambda\}.$$

As $\mathcal{U}_C$ and d_2 are convex, A is convex.

If we take a sequence $\{\tilde{u}_N\}_{N\in\mathcal{N}}$ in A so that $\tilde{u}_N \to^{d_2} \tilde{\tilde{u}}$, we must have that $\tilde{u}_N \in \mathcal{U}_C$ for all $N \in \mathcal{N}$ and $d_2(\tilde{u}_N, u) \leq \lambda$. As d_2 is continuous we have that $d_2(\tilde{\tilde{u}}, u) \leq \lambda$. We need to show that $\tilde{\tilde{u}} \in \mathcal{U}_C$. In the same way we showed $\bar{u} \in W$, we can get that $\tilde{\tilde{u}} \in \mathcal{U}_C$. Then, A is strongly closed. By Theorem III.7, [39], A is weakly closed. In particular we have that if $u_N \to \bar{u}$ weakly in $L_2^m([0,1])$ then

$$d_2(u, \bar{u}) \leq \lim_{N\to\infty} \inf d_2(u_N, u).$$

We can write

$$\begin{aligned}
\varphi(\bar{\eta}) &= \langle \nabla f^0(\xi), \bar{\xi} - \xi \rangle + d_1(\xi^0, \bar{\xi}^0) + d_2(u, \bar{u}) + d_4(\Omega, \bar{\Omega}) \\
&\leq \lim_{N\to\infty} [\langle \nabla f^0(\xi), \xi_N - \xi \rangle + d_1(\xi_N^0, \xi^0) + d_4(\Omega_N, \Omega)] + \lim_{N\to\infty} \inf d_2(u_N, u) \\
&= \lim_{N\to\infty} \inf [\langle \nabla f^0(\xi), \xi_N - \xi \rangle + d_1(\xi_N^0, \xi^0) + d_2(u_N, u) + d_4(\Omega_N, \Omega)] \\
&= \lim_{N\to\infty} \inf \varphi(\eta_N) = \alpha.
\end{aligned}$$

Therefore, φ achieves its minimum over S_C. □

Look that, analogously, the same result can be proved when we change the domain of φ for $S_{C,N}$.

Theorem 3.4. *Suppose that the Assumption 1 holds. Then, $\{(P_{rep}^{C,N}, \gamma^{C,N})\}_{N\in\mathcal{N}}$ is a sequence of consistent approximations to the pair (P_{rep}, γ).*

Proof. To get this result, we need to show that the problems $(P_{rep}^{C,N})$ epi-converge to (P_{rep}) and $\overline{\lim}\gamma^{C,N}(\eta_N) \leq \gamma(\eta)$, when $\eta_N \to^d \eta$, $N \to \infty$, $N \in \mathcal{N}$.

Epi-convergence;

i) Let $\eta \in S_C$ be arbitrary. By the own construction of $S_{C,N}$ and by the proof of Lemma 4, there exists $\{\eta_N\}_{N\in\mathcal{N}}$ such that $\eta_N \in S_{C,N}$, for all $N \in \mathcal{N}$, and $\eta_N \to^d \eta$, $N \to \infty$. Let $\mathcal{K} \subset \mathcal{N}$ be such that

$$\overline{\lim} f_N^0(\xi_N^0, y_N^{\eta_N}(1)) = \lim_{N\in\mathcal{K}} f_N^0(\xi_N^0, y_N^{\eta_N}(1)).$$

By Theorem 3.2, there exist $\mathcal{K}' \subset \mathcal{K}$ such that $y_N^{\eta_N}(1) \to y^\eta(1)$, $N \in \mathcal{K}'$. Then, we have

$$\lim_{N\in\mathcal{K}'} f_N^0(\xi_N^0, y_N^{\eta_N}(1)) = f^0(\xi^0, y^\eta(1)),$$

because of the Assumption 1. It follows that

$$\lim_{N\in\mathcal{K}} f_N^0(\xi_N^0, y_N^{\eta_N}(1)) = f^0(\xi^0, y^\eta(1)),$$

because if the limit exist, all the subsequences must converge to the same point. This gives us

$$\overline{\lim} f_N^0(\xi_N^0, y_N^{\eta_N}(1)) = f^0(\xi^0, y^\eta(1)).$$

ii) Let $\{\eta_N\}_{N\in\mathcal{K}}$ be a sequence such that $\eta_N \in S_{C,N}$ for all $N \in \mathcal{K}$ and $\eta_N \to^d \eta$, $N \to \infty$. By Lemma 4, we must have $\eta \in S_C$. Take $\bar{\mathcal{K}} \subset \mathcal{K}$ such that

$$\underline{\lim}_{N\in\mathcal{K}} f^0(\xi_N^0, y_N^{\eta_N}(1)) = \lim_{N\in\bar{\mathcal{K}}} f^0(\xi_N^0, y_N^{\eta_N}(1)).$$

As $\eta_N \to^d \eta$, $N \in \bar{\mathcal{K}}$, it follows by Theorem 3.2 that there exist $\bar{\bar{\mathcal{K}}} \subset \bar{\mathcal{K}}$ such that $y_N^{\eta_N}(1) \to y^{\eta}(1)$, $N \in \bar{\bar{\mathcal{K}}}$. By the same arguments used in the proof of the item i), it follows that $\underline{\lim}_{N\in\mathcal{K}} f^0(\xi_N^0, y_N^{\eta_N}(1)) = f^0(\xi^0, y^{\eta}(1))$.

$$\therefore P_{rep}^{C,N} \to^{Epi} P_{rep}.$$

Now, let $\{\eta_N\}_{N\in\mathcal{N}}$ be a sequence such that $\eta_N \in S_{C,N}$ for all $N \in \mathcal{N}$ and it converges to $\eta \in S_C$. We must show that $\overline{\lim}\gamma^{C,N}(\eta_N) \leq \gamma(\eta)$, $N \to \infty$. By Lemma 6, there exists $\bar{\eta} \in S_C$ such that

$$\gamma(\eta) = \langle \nabla f^0(\xi^0, y^{\eta}(1)), (\bar{\xi}^0, y^{\bar{\eta}}(1)) - (\xi^0, y^{\eta}(1))\rangle + \frac{1}{2}\bar{d}((\xi^0, u, \Omega), (\bar{\xi}^0, \bar{u}, \bar{\Omega})).$$

Let $\overline{\mathcal{K}} \subset \mathcal{N}$ be such that $\overline{\lim}\gamma^{C,N}(\eta_N) = \lim_{N\in\overline{\mathcal{K}}} \gamma^{C,N}(\eta_N)$. By Lemma 4, there exists $\{\bar{\eta}_N\}_{N\in\overline{\mathcal{K}}}$ so that $\bar{\eta}_N \to \bar{\eta}$, and $\bar{\eta}_N \in S_{C,N}$ for all $N \in \overline{\mathcal{K}}$. By the definition of $\gamma^{C,N}(\cdot)$,

$$\gamma^{C,N}(\eta_N) \leq \langle \nabla f^0(\xi_N^0, y_N^{\eta_N}(1)), (\xi_N^0, y_N^{\eta_N}(1)) - (\bar{\xi}_N^0, y_N^{\bar{\eta}_N}(1))\rangle$$
$$+\tfrac{1}{2}\bar{d}((\xi_N^0, u_N, \Omega_N), (\bar{\xi}_N^0, \bar{u}_N, \bar{\Omega}_N)).$$

By Theorem 3.2, there exists $\mathcal{K} \subset \overline{\mathcal{K}}$ such that $y_N^{\eta_N}(1) \to y^{\eta}(1)$, and $y_N^{\bar{\eta}_N}(1) \to y^{\bar{\eta}}(1)$, $N \in \mathcal{K}$, then passing the limit with $N \to \infty$, $N \in \overline{\mathcal{K}}$, in the last inequality, we get

$$\lim_{N\in\mathcal{K}} \gamma^{C,N}(\eta_N) \leq \langle \nabla f^0(\xi^0, y^{\eta}(1)), (\xi^0, y^{\eta}(1)) - (\bar{\xi}^0, y^{\bar{\eta}}(1))\rangle$$
$$+\tfrac{1}{2}\bar{d}((\xi^0, u, \Omega), (\bar{\xi}^0, \bar{u}, \bar{\Omega})) = \gamma(\eta),$$

which give us

$$\overline{\lim}\gamma^{C,N}(\eta_N) \leq \gamma(\eta).$$

□

The next Theorem shows that a local or global minimizers sequence of $(P_{rep}^{C,N})$ that has a convergent subsequence is converging to a local or global minimizer of (P_{rep}).

Theorem 3.5. *Let $(P_{rep}^{C,N})$ and (P_{rep}) be defined as before. Let $\{\eta_N\}_{N\in\mathcal{N}}$ be a sequence of local or global minimizers of $(P_{rep}^{C,N})$ such that $\eta_N \to^d \eta$, with $N \to \infty$ and $\eta \in S_C$. Then η is a local or global minimizer of (P_{rep}) and there exists $\mathcal{K} \subset \mathcal{N}$ such that $f_N^0(\xi_N^0, y_N^{\eta_N}(1)) \to f^0(\xi^0, y^{\eta}(1))$, with $N \to \infty$, $N \in \mathcal{K}$.*

Proof. Let $\{\eta_N\}_{N\in\mathcal{N}}$ be a sequence of local minimizers of the problems $(P_{rep}^{C,N})$, that is, there exists $\varepsilon > 0$ so that for all $\hat{\eta} \in S_{C,N}$ satisfying $d(\eta_N, \hat{\eta}) \leq \varepsilon$ we have $f_N^0(\xi_N^0, y_N^{\eta_N}(1)) \leq f_N^0(\hat{\xi}^0, y^{\hat{\eta}}(1))$, such that $\eta_N \to^d \eta$. By Theorem 3.2 and Assumption 1, there exists $\mathcal{K} \subset \mathcal{N}$

such that $y_N^{\eta_N} \to y^\eta$ uniformly and $f_N^0(\xi_N^0, y_N^{\eta_N}(1)) \to f^0(\xi^0, y^\eta(1))$, with $N \in \mathcal{K}$, $N \to \infty$. We need to show that η is a local minimizer to the problem (P_{rep}). Let's suppose that η is not a local minimizer, then given $\varepsilon > 0$ there exists $\bar{\eta} \in S_C$ with $d(\eta, \bar{\eta}) < \varepsilon/3$ such that

$$f^0(\bar{\xi}^0, y^{\bar{\eta}}(1)) = f^0(\xi^0, y^\eta(1)) - 3\varepsilon,$$

where $y^{\bar{\eta}}(\cdot)$ is the associated trajectory to $\bar{\eta}$.
As $(P_{rep}^{C,N})$ epi-converge to (P_{rep}) and $\eta_N \to^d \eta$ with $\eta_N \in S_{C,N}$, we have

$$\underline{\lim}_{N\in\mathcal{K}} f_N^0(\xi_N^0, y^{\eta_N}(1)) \geq f^0(\xi^0, y^\eta(1)). \tag{3.5}$$

By the epi-convergence again, there exists a sequence $\{\bar{\eta}_N\}_{N\in\mathcal{N}}$ in $S_{C,N}$ such that $\bar{\eta}_N \to^d \bar{\eta}$ and

$$\overline{\lim}_{N\in\mathcal{K}} f^0(\bar{\xi}_N^0, y^{\bar{\eta}_N}(1)) \leq \overline{\lim} f^0(\bar{\xi}_N^0, y^{\bar{\eta}_N}(1)) \leq f^0(\bar{\xi}^0, y^{\bar{\eta}}(1)). \tag{3.6}$$

Let $\mathcal{K}' \subset \mathcal{K}$ be such that

$$\underline{\lim}_{N\in\mathcal{K}} f_N^0(\xi_N^0, y^{\eta_N}(1)) = \lim_{N\in\mathcal{K}'} f_N^0(\xi_N^0, y^{\eta_N}(1)).$$

As $\eta_N \to^d \eta$, given $\varepsilon/3$ there exists $N_1 \in \mathcal{K}'$ so that $d(\eta_N, \eta) \leq \varepsilon/3$ for all $N \geq N_1$, $N \in \mathcal{K}'$. As $\bar{\eta}_N \to^d \bar{\eta}$, given $\varepsilon/3$ there exists $N_2 \in \mathcal{K}'$ so that $d(\bar{\eta}_N, \bar{\eta}) < \varepsilon/3$ for all $N \geq N_2$, $N \in \mathcal{K}'$. Let $N_3 = \max\{N_1, N_2\}$ and $N \leq N_3$, $N \in \mathcal{K}^1$, then

$$d(\bar{\eta}_N, \eta_N) \leq d(\eta_N, \eta) + d(\eta, \bar{\eta}) + d(\bar{\eta}, \bar{\eta}_N) \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon,$$

and there exists $N_4 \in \mathcal{K}'$ so that for all $N \leq N_4$, $N \in \mathcal{K}'$,

$$(4.8) \Rightarrow f_N^0(\bar{\xi}_N^0, y^{\bar{\eta}_N}(1)) \leq f^0(\bar{\xi}^0, y^{\bar{\eta}}(1)) + \varepsilon = f^0(\xi^0, y^\eta(1)) - 2\varepsilon.$$

$$(3.5) \Rightarrow f_N^0(\xi_N^0, y_N^\eta(1)) \geq f^0(\xi^0, y^\eta(1)) - \varepsilon.$$

It follows that

$$f_N^0(\bar{\xi}_N^0, y_N^{\bar{\eta}_N}(1)) \leq f_N^0(\xi_N^0, y_N^\eta(1)) - \varepsilon$$

for all $N \geq N_0$, $N \in \mathcal{K}'$, where $N_0 = \max\{N_3, N_4\}$. Contradiction! □

Suppose $\{\eta_N\}_{N\in\mathbb{N}} \subset S_{C,N}$ is a sequence of global minimizers of $(P_{rep}^{C,N})$ that is converging to $\eta \in S_C$ in the metric d. By Theorem 3.5, η is a global minimizer of (P_{rep}). Then $y^\eta(\cdot)$ given by (2.6) is the function that minimizes (P_{rep}).

Define $\pi : [0,T] \to [0,1]$ by

$$\pi(t) = \frac{t + |\mu|([0,t])}{T + \|\mu\|},$$

and $x : [0,T] \to \mathbb{R}^n$ by

$$x(t) = y^\eta(\pi(t)).$$

Because of Theorem 2.1, as $y^\eta(\cdot)$ is a solution of the system (2.6), then $x(\cdot)$ is a solution of the original system (2.3) related to $p = (\mu, |\mu|, \psi_{t_i})$ and $\bar{u} : [0,T] \to \mathbb{R}^m$ given by

$$\bar{u}(t) = u(\pi(t)).$$

As $y^\eta(\cdot)$ minimizes the reparametrized problem (P_{rep}) and $f^0(\xi^0, x(T)) = f^0(\xi^0, y^\eta(1))$, we have that $x(\cdot)$ minimizes (P).

Now, we can show that a subsequence of discrete-time approximated functions graph-converges to a solution.

Theorem 3.6. *Suppose* $\eta_N \to^d \eta$ *and define*

$$\Lambda^N := \{(t_j, y_j) : j = 0, ..., N\},$$

and

$$\mathbb{X}_\mu := (x(\cdot), \phi(\cdot), \{\mathcal{X}_{t_i}\}_{t_i \in \Theta}),$$

where $y_j = y_N^{\eta_N}(s_j)$, $j = 0, ..., N$, *and* $x(\cdot)$ *is as defined above. Then, there exists* $\mathcal{K} \subset \mathcal{N}$ *so that*

$$dist_H(\Lambda^N, gr\mathbb{X}_\mu) \to 0 \quad as \;\; N \to \infty, N \in \mathcal{K}.$$

Proof. For each $N \in \mathcal{N}$, define

$$\tilde{\Lambda}^N := \{(s_j, y_j) : j = 0, ..., N\}.$$

Note that as $\eta_N \to^d \eta$, by Theorem 3.2, there exists $\mathcal{K} \subset \mathcal{N}$ so that $y_N^{\eta_N} \to y^\eta$ uniformly when $N \in \mathcal{K}$, then we have

$$\text{dist}_H(\text{gr}y_N^{\eta_N}, \text{gr}y^\eta) \to 0 \;\text{ as } N \to \infty, N \in \mathcal{K}.$$

Observe the second coordinates of Λ^N and $\tilde{\Lambda}^N$ are equal for each $j = 0, ..., N$. In the same way, the second coordinates of $\text{gr}\mathbb{X}_\mu$ and $\text{gr}y$ are the same for each $t_i \notin \Theta$, and when $t \in \Theta$, the set of projections onto the second coordinate are the same. Then, we have the following

$$\mathrm{dist}_H(\Lambda^N, \mathrm{gr}\mathbb{X}_\mu) \leq \mathrm{dist}_H(\tilde{\Lambda}^N, \mathrm{gr}y).$$

We can get

$$\mathrm{dist}_H(\tilde{\Lambda}^N, \mathrm{gr}y) \leq \mathrm{dist}_H(\tilde{\Lambda}^N, \mathrm{gr}y_N^{\eta_N}) + \mathrm{dist}_H(\mathrm{gr}y_N^{\eta_N}, \mathrm{gr}y).$$

Passing the limit when $N \in \mathcal{K}$ in the last inequality we get the desired result. $\square$

PART II

CHAPTER 4

THE MINIMAL TIME PROBLEM ON STRATIFIED DOMAINS

In this chapter we are beginning a new subject. In Section 4.1 we introduce the stratified domains and the differential inclusion that is included in the minimal time problem. In Section 4.2 the minimal time problem is defined, some results given by [26] are cited and other condition is assumed. The addition of such condition is explained in the same section. We follow to show that the H-J inequalities hold for such minimal time problem. This is explained in Section 4.3.

4.1 Differential Inclusion with Stratified Domains

Let $\{\mathcal{M}_1, \mathcal{M}_2, ..., \mathcal{M}_M\}$ be a collection of linear and convex manifolds embedded on $\mathbb{R}^n$ given by the intersection of half spaces

$$\mathcal{M}_i = \bigcap_{j=1}^{K_i} \mathcal{H}_{c_j^i, \alpha_j^i}, \tag{4.1}$$

where $c_j^i \in \mathbb{R}^n$, $\alpha_j^i \in \mathbb{R}$, $K_i \in \mathbb{N}$, $i \in \{1, ..., M\}$ and $\mathcal{H}_{c,\alpha}$ is a half space given by

$$\mathcal{H}_{c,\alpha} = \{x : \langle c, x\rangle \leq \alpha\} \ \text{ or } \ \mathcal{H}_{c,\alpha} = \{x : \langle c, x\rangle < \alpha\},$$

such that

- $\cup_{i=1}^M \mathcal{M}_i = \mathbb{R}^n, \mathcal{M}_i \cap \mathcal{M}_j = \emptyset, i \neq j;$

- If $\overline{\mathcal{M}}_i \cap \mathcal{M}_j \neq \emptyset$, then $\mathcal{M}_j \subset \overline{\mathcal{M}}_i$.

Define $d_i = \dim \mathcal{M}_i$ for all $i \in \{1, ..., M\}$.

Let $F_i : \mathcal{M}_i \rightrightarrows \mathbb{R}^n$ be a constant multifunction such that (SH) is satisfied for all $i \in \{1, ..., M\}$. With such hypotheses F_i is Lipschitz of rank 0.

Consider the differential inclusion

$$(DI)_F \begin{cases} \dot{x}(t) \in F(x(t)) \text{ a.e. } t \in [0, T], \\ x(0) = x_0, \end{cases}$$

where $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a multifunction given by $F(x(t)) = F_i(x(t))$ whenever $x(t) \in \mathcal{M}_i$, and $x_0 \in \mathbb{R}^n$ is given.

In general, the multifunction F is not Lipschitz and does not have closed graph, because of these we need to define another multifunction related to F that will be used in the next analysis.

Let $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be the Filippov Regularization given by

$$G(x) := \bigcap_{\varepsilon > 0} \overline{\text{co}} \bigcup \{F(y) : |y - x| < \varepsilon\}.$$

Because of the nature of the stratification structure, G can be written like

$$G(x) = \text{co}\{\overline{F}_i(x) : x \in \overline{\mathcal{M}}_i\} = \text{co}\{F_i(x) : x \in \overline{\mathcal{M}}_i\},$$

where $\overline{F}_i : \overline{\mathcal{M}}_i \rightrightarrows \mathbb{R}^n$ is an extension of F_i to the boundary of $\mathcal{M}_i$. In our case, as F_i is constant we have that $F_i = \overline{F}_i$ for all $i \in \{1, ..., M\}$.

Observation 5. *The multifunction G defined above satisfies (SH).*

Now, consider the differential inclusion

$$(DI)_G \begin{cases} \dot{x}(t) \in G(x(t)) \text{ a.e. } t \in [0, T], \\ x(0) = x_0. \end{cases}$$

We can follow [26] and to assume that the stratified dynamics satisfy the Structural Condition (SC) given by

$$G(x) \cap \mathcal{T}_{\mathcal{M}_k}(x) = F_k(x) \text{ whenever } x \in \mathcal{M}_k.$$

In addition, we are supposing that the stratified dynamics satisfy the Projection Condition (PC) given by

$$\forall x \in \mathcal{M}_j \cap \overline{\mathcal{M}}_i \text{ and } v \in F_i(x),\ \mathrm{proj}_{F_j(x)} v \text{ is not empty.}$$

The next example shows that the Structural Condition does not imply the Projection Condition.

Example 3. *Consider $\mathbb{R}^2$. Define the manifolds as following*

$$\mathcal{M}_1 = \{(x,y) \in \mathbb{R}^2 : y > 0, x \in \mathbb{R}\},\ \mathcal{M}_2 = \{(x,y) \in \mathbb{R}^2 : y < 0, x \in \mathbb{R}\},$$

$$\mathcal{M}_3 = \{(x,y) \in \mathbb{R}^2 : y = 0, x \in \mathbb{R}\}.$$

Let the dynamics be given by

$$F_1(x,y) = \{(1,v) \in \mathbb{R}^2 : -1 \leq v \leq 1\},\quad F_2(x,y) = \{(2,-1)\},$$

$$F_3(x,0) = \{(u,0) \in \mathbb{R}^2 : 0 \leq u \leq 3/2\}.$$

Note that the dynamic satisfy the Structural Condition but for $(0,0) \in \mathcal{M}_3 \cap \overline{\mathcal{M}}_2$ we have

$$proj_{F_3(x,0)}(u,v) \ \textit{ is empty}, \quad (u,v) = (2,-1) \in F_2.$$

The following Lemma shows that in our specific case the Structural Condition implies another inclusion.

Lemma 7. *Suppose that (SC) holds. Then the following inclusion holds*

$$\overline{F}_i(x) \cap \mathcal{T}_{\overline{\mathcal{M}}_j}(x) \subseteq \overline{F}_j(x) \qquad \forall\, x \in \overline{\mathcal{M}}_j \cap \overline{\mathcal{M}}_i,\ \mathcal{M}_j \subseteq \overline{\mathcal{M}}_i. \tag{4.2}$$

Proof. Let $x \in \overline{\mathcal{M}}_j \cap \overline{\mathcal{M}}_i$ be so that $\mathcal{M}_j \subseteq \overline{\mathcal{M}}_i$. As $\mathcal{M}_j \cap \overline{\mathcal{M}}_i \neq \emptyset$ there exists y on it. As our dynamic is constant, we have

$$F_j(y) = F_j(x) = \overline{F}_j(x) \quad \text{and} \quad \overline{F}_i(y) = \overline{F}_i(x).$$

By the Structural Condition follows the equality

$$G(y) \cap \mathcal{T}_{\mathcal{M}_j}(y) = F_j(y) = F_j(x). \tag{4.3}$$

As $\mathcal{M}_j$ is linear, $y \in \mathcal{M}_j$ and $x \in \mathrm{bdry}(\mathcal{M}_j)$ we have that $\mathcal{T}_{\overline{\mathcal{M}}_j}(x) \subseteq \mathcal{T}_{\mathcal{M}_j}(y)$. Note that, by the definition of G,

$$\overline{F}_i(y) \subseteq G(y).$$

With all these facts in our hands we can get

$$\overline{F}_i(x) \cap \mathcal{T}_{\overline{\mathcal{M}}_j}(x) = \overline{F}_i(y) \cap \mathcal{T}_{\overline{\mathcal{M}}_j}(x) \subseteq G(y) \cap \mathcal{T}_{\mathcal{M}_j}(y) = \overline{F}_j(x),$$

where in the second equality we used the equality (4.3). □

In the general case, with general manifolds, the Structural Condition does not imply the inclusion (4.2). See example below.

Example 4. *Consider $\mathbb{R}^2$. Define the manifolds as following*

$$\mathcal{M}_1 = \{(x,y) \in \mathbb{R}^2 : y = \sqrt{x},\ \ x > 0\},\ \mathcal{M}_2 = \{(x,y) \in \mathbb{R}^2 : 0 < y < \sqrt{x}, x > 0\},$$

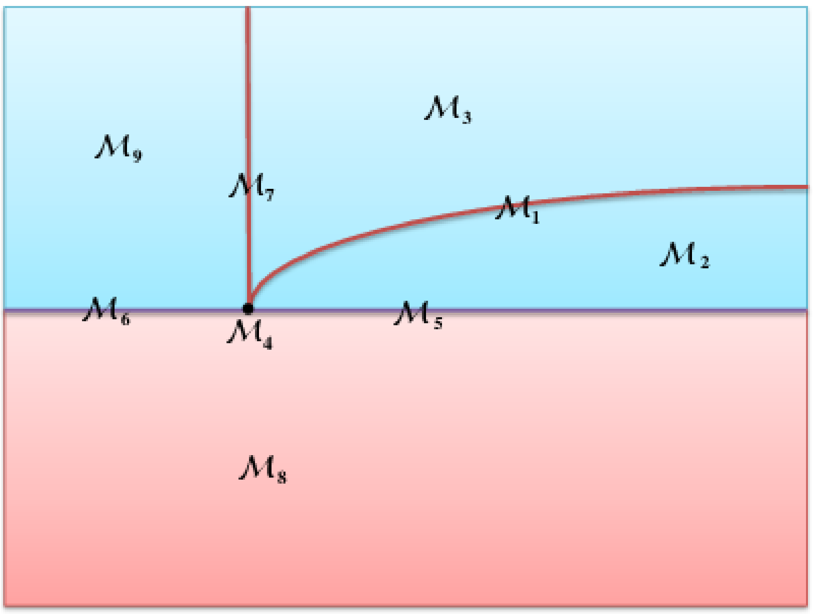
$$\mathcal{M}_3 = \{(x,y) \in \mathbb{R}^2 : y > \sqrt{x}, x > 0\},\ \mathcal{M}_4 = \{(0,0)\},$$

$$\mathcal{M}_5 = \{(x,y) \in \mathbb{R}^2 : y = 0, x > 0\},\ \mathcal{M}_6 = \{(x,y) \in \mathbb{R}^2 : y = 0, x < 0\}$$

$$\mathcal{M}_7 = \{(x,y) \in \mathbb{R}^2 : y > 0, x = 0\},\ \ \mathcal{M}_8 = \{(x,y) \in \mathbb{R}^2 : y < 0, x \in \mathbb{R}\},$$

$$\mathcal{M}_9 = \{(x,y) \in \mathbb{R}^2 : y > 0, x < 0\}.$$

The following picture helps us to understand better.



The related dynamics are given by

$$F_1 = F_3 = F_4 = F_5 = F_6 = F_7 = F_8 = F_9 = \{(0,0)\},\ \ F_2 = \{(0,1)\},$$

It is easy to show that the Structural Condition (SC) *holds for such dynamic.*

Note that $(0,0) \in \overline{\mathcal{M}}_1 \cap \overline{\mathcal{M}}_2$, $\mathcal{M}_1 \subseteq \overline{\mathcal{M}}_2$, *and* $(0,1) \in \mathcal{T}_{\overline{\mathcal{M}}_1}(0,0)$ *but*

$$\overline{F}_2(0,0) \cap \mathcal{T}_{\overline{\mathcal{M}}_1}(0,0) \nsubseteq \overline{F}_1.$$

We are going to explain latter why that inclusion (4.2) is so important. Actually, in the general case we need the projection assumption or the inclusion (4.2) to get an important result.

As it was said before, the multifunction F doesn't have the desired properties, but G satisfies (SH). By Proposition 2, there exists $T > 0$ so that $(DI)_G$ admits at least one solution. The next result was proved by [26] and because of it we can study the inclusion $(DI)_G$.

Proposition 4. *Suppose* $x(\cdot) : [0,T] \to \mathbb{R}^n$ *is a Lipschitz arc. Then the following are equivalent*

- $x(\cdot)$ *satisfies* $(DI)_G$*;*
- $x(\cdot)$ *satisfies* $(DI)_F$*;*
- *For each* i, $x(\cdot)$ *satisfies* $x(0) = x_0$ *and*

$$\dot{x}(t) \in F_i(x(t)) \;\; \textit{whenever}\; x(t) \in \mathcal{M}_i.$$

The essential velocity over $\overline{\mathcal{M}}_i$ for all $i = 1, ..., M$ is given by

$$F_i^\sharp(x) = F_i(x) \cap \mathcal{T}_{\overline{\mathcal{M}}_i}(x).$$

Because of the definition of the manifolds, $\overline{\mathcal{M}}_i$ is convex for all $i \in \{1, ..., M\}$ then the proximal normal and limiting normal cones are equal. By Proposition 4.2.7, [28], the limiting normal cone to $\overline{\mathcal{M}}_i$ at x contains nonzero elements for all $x \in \text{bdry}\overline{\mathcal{M}}_i$. By [26], $\overline{\mathcal{M}}_i$ is proximally smooth then the Bouligand and Clarke tangent cones coincide, and $\mathcal{T}_{\overline{\mathcal{M}}_i}(x)$ is closed and convex, and thus has a relative interior denoted by r-int$\mathcal{T}_{\overline{\mathcal{M}}_i}(x)$. The relative boundary is given by r-bdry$\mathcal{T}_{\overline{\mathcal{M}}_i}(x) := \mathcal{T}_{\overline{\mathcal{M}}_i}(x)/$r-int$\mathcal{T}_{\overline{\mathcal{M}}_i}(x)$.

The values of $F_i^\sharp$ are exactly those velocities that can be realized by trajectories of the system over $\overline{\mathcal{M}}_i$. Because of the type of our stratification for each $i \in \{1, ..., M\}$ all the velocities in $F_i^\sharp$ is pointing to $\overline{\mathcal{M}}_i$. The essential velocity multifunction $G^\sharp : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$

is defined by

$$G^\sharp(x) = \bigcup_i F_i^\sharp(x).$$

In the proof of Proposition 5.1, [26], was used a projection that can be empty if we don't suppose the projection condition (PC). Another way to prove it is using the inclusion (4.3). Such proof is given below.

Proposition 5. *Let $v \in G^\sharp(x_0)$, then there exists $T > 0$ and a C^1 solution to $(DI)_G$ with $\dot{x}(0) = v$.*

Proof. Let $v \in G^\sharp(x_0)$, then there exists $i \in \{1, ..., M\}$ so that $x_0 \in \overline{\mathcal{M}_i}$ and $v \in F_i^\sharp(x_0) = \overline{F_i}(x_0) \cap \mathcal{T}_{\overline{\mathcal{M}_i}}(x_0)$. Suppose $v \neq 0$ (if $v = 0$ the result follows). If $v \in$ r-int$\mathcal{T}_{\overline{\mathcal{M}_i}}(x_0)$ then the result follows by Lemma 5.1, [26]. If $v \notin$ r-int$\mathcal{T}_{\overline{\mathcal{M}_i}}(x_0)$, by Lemma 3.1, [26], there exists $j \in \{1, ..., M\}$ which $\mathcal{M}_j \subseteq \overline{\mathcal{M}_i}$, $x_0 \in \overline{\mathcal{M}_j}$, $v \in \mathcal{T}_{\overline{\mathcal{M}_j}}(x_0)$ and $d_j < d_i$. By the inequality (4.2), $v \in \overline{F}_j(x_0)$, i.e., $v \in F_j^\sharp(x_0)$. Again, if $v \in$ r-int$\mathcal{T}_{\overline{\mathcal{M}_j}}(x_0)$, then the result follows by Lemma 5.1, [26]. If $v \notin$ r-int$\mathcal{T}_{\overline{\mathcal{M}_j}}(x_0)$, then $v \in$ r-bdry$\mathcal{T}_{\overline{\mathcal{M}_j}}(x_0)$, and, again, we can apply Lemma 3.1, [26]. This can be repeated as necessary but must terminate because the dimension is decreasing each step. $\square$

The last Proposition shows why $G^\sharp$ is called essential velocity, such multifunction is given us every velocity at every point that matters.

4.2 The Minimal Time Problem

We can define the minimal time function $\hat{T}_S : \mathbb{R}^n \to [0, \infty]$ related to a closed set $S \subset \mathbb{R}^n$ as follows. If $x_0 \notin S$, then

$$\hat{T}_S(x_0) := \inf\{T : \text{ there exists } x(\cdot) \text{ satisfying } (DI)_G \text{ with } x(T) \in S\}.$$

The next Theorem gives us the result that for every optimal trajectory to the minimal time problem we can find another optimal trajectory that is regular, the sense of regular is given below.

Theorem 4.1. *Let (x^*, u^*) be an optimal pair to the minimal time problem, where $x^*, u^* : [0, T] \to \mathbb{R}^n$. Then, there exists a trajectory $y : [0, T] \to \mathbb{R}^n$ of G related to a control function $v : [0, T] \to \mathbb{R}^n$ satisfying $y(0) = x_0$, $y(T) = x^*(T)$, that is regular, i.e., there exists a partition of $[0, T]$, $\{0 =: t_0 < t_1 < ... < t_N < t_{N+1} := T\}$ so that for any $k = 0, ..., N$ we can find $i_k \in \{1, ..., M\}$ such that $y(t) \in \mathcal{M}_{i_k}$ on (t_k, t_{k+1}).*

Proof. Let us assume there exists $i \in \{1, ..., M\}$ so that $J_i := \{t \in [0,T] : x^*(t) \in \mathcal{M}_i\}$ contains infinitely many disjoints open intervals. If it is not the case $(x^*(\cdot), u^*(\cdot))$ satisfies the conclusion. We can assume $\mathcal{M}_i$ is unique because the manifolds are disjoints and the stratification is locally finite.

Let $\mathcal{M}$ be the union of all manifolds $\mathcal{M}_j$ so that $\mathcal{M}_i \subseteq \overline{\mathcal{M}_j}$ and define $J := \{t \in [0,T] : x^*(t) \in \mathcal{M}\} = [0,T] \backslash J_i$. Because of the minimality of the dimension of $\mathcal{M}_i$, J is an open set and we have that $a = \min J_i$ and $b = \max J_i$ are well-defined.

We can construct a partition of $[a, b]$

$$b_0 := a \leq a_1 < b_1 \leq a_2 < b_2 \leq ... \leq a_{l-2} < b_{l-2} \leq b =: a_{l-1}$$

with $x^*(a_k), x^*(b_k) \in \mathcal{M}$ and $(a_k, b_k) \subseteq J$ for any $k = 1, ..., l-2$.

For the intervals (b_k, a_{k+1}) we have $x^*(b_k), x^*(a_{k+1}) \in \mathcal{M}_i$ and $[b_k, a_{k+1}] \in (J \cup J_i)$, $k = 0, ..., l-2$. By (PC), as $x^*(b_k) \in \overline{(\cup \mathcal{M}_j)} \cap \mathcal{M}_i$ and $\dot{x}^*(t) \in \cup F_j$ a.e. $t \in [b_k, a_{k+1}]$, then $\text{proj}_{F_i(x^*(b_k))}\dot{x}^*(t)$ is nonempty a.e. $t \in [b_k, a_{k+1}]$. Define

$$z_k(t) = \text{proj}_{F_i}(\dot{x}^*(t)),$$

that exists almost everywhere on $[b_k, a_{k+1}]$. Let $t \in [b_k, a_{k+1}]$ be so that $z_k(t)$ doesn't exist. Let $\{t_j\}$ be a sequence in $[b_k, a_{k+1}]$ converging to t. As the set of all points in $[b_k, a_{k+1}]$ such that $z_k(\cdot)$ exists has positive measure, there exists $\mathcal{K} \subseteq \mathbb{N}$ so that for all $j \in \mathcal{K}$, $z_k(t_j)$ exists and $t_j \to t$. As $z_k(t_j) \in F_i$ for all $j \in \mathcal{K}$ and F_i is compact, there exists $\mathcal{K}' \subseteq \mathcal{K}$ and $z_k^t \in F_i$ so that $z_k(t_j) \to z_k^t$, $j \in \mathcal{K}'$. Define $v_k : [b_k, a_{k+1}] \to \mathbb{R}^n$ by

$$v_k(t) = \begin{cases} z_k(t) & \text{if it exists,} \\ z_k^t & \text{otherwise.} \end{cases}$$

Now define $y_k : [b_k, a_{k+1}] \to \mathbb{R}^n$ by

$$y_k(t) = x^*(b_k) + \int_{b_k}^{t} v_k(s)ds,$$

then $y_k(b_k) = x^*(b_k)$, $y_k(a_{k+1}) = x^*(a_{k+1})$ and $\dot{y}_k(t) = v_k(t) \in F_i$ a.e. $t \in [b_k, a_{k+1}]$, i.e., $y_k(\cdot)$ is a trajectory of F_i for all $k = 0, ..., l-2$.

Define $y : [0, T] \to \mathbb{R}^n$ by

$$y(t) = \begin{cases} x^*(t) & \text{if} \quad t \in [0, b_0] \\ y_k(t) & \text{if} \quad t \in [b_k, a_{k+1}], \quad k = 0, ..., l-2 \\ x^*(t) & \text{if} \quad t \in [a_k, b_k], \quad k = 1, ..., l-2 \\ x^*(t) & \text{if} \quad t \in [a_{l-1}, T], \end{cases}$$

and $v : [0, T] \to \mathbb{R}^n$ by

$$v(t) = \begin{cases} u^*(t) & \text{if} \quad t \in [0, b_0] \\ v_k(t) & \text{if} \quad t \in [b_k, a_{k+1}], \quad k = 0, ..., l-2 \\ u^*(t) & \text{if} \quad t \in [a_k, b_k], \quad k = 1, ..., l-2 \\ u^*(t) & \text{if} \quad t \in [a_{l-1}, T]. \end{cases}$$

Note that $y(0) = x_0$, $y(T) = x^*(T)$ and $\dot{y}(t) = v(t) \in G(y(t))$ a.e. $t \in [0, T]$. We can reindex the intervals,

$$t_{2k} = a_k \quad t_{2k-1} = b_{k-1}, \quad k = 1, ..., l-1.$$

Let $N := 2l - 1$, then

$$0 = t_0 < t_1 < ... < t_N = T,$$

and, for each $k = 0, ..., N$, there exists $i_k \in \{1, ..., M\}$ so that $y(t) \in \mathcal{M}_{i_k}$ for all $t \in (t_k, t_{k+1})$. Therefore, $(y(\cdot), v(\cdot))$ is a regular optimal pair to the minimal time problem. $\square$

Because of the Theorem 4.1 we are going to consider only regular trajectories. From here we are considering the following differential inclusion

$$(DI)_{G_r} \begin{cases} \dot{x}(t) \in G(x(t)) \ \text{ a.e. } \ t \in [0, T] \\ x(0) = x_0, \ x(\cdot) \text{ is regular}, \end{cases}$$

where the restriction that $x(\cdot)$ is regular is in the sense of the Theorem 4.1.

As before, $(DI)_{G_r}$ and $(DI)_{F_r}$ are equivalent. Then we can define the minimal time function $T_S : \mathbb{R}^n \to [0, \infty]$ related to a closed set $S \subset \mathbb{R}^n$ as follows. If $x_0 \notin S$, then

$$T_S(x_0) := \inf\{T : \text{ there exists } x(\cdot) \text{ satisfying } (DI)_{G_r} \text{ with } x(T) \in S\}.$$

The minimal time problem is given when we fix x_0. According to the reference [22], Proposition 2.6, as G satisfies (SH) the minimum in $T_S(x_0)$ is attained.

The next result is used to prove the weak invariance statement.

Proposition 6. *Let $x_0 \in \mathbb{R}^n$. Suppose $x(\cdot)$ is a regular trajectory of G with $x(0) = x_0$ and let $\{t_k\}_{N\in\mathbb{N}}$ be a sequence converging to 0 when $k \to \infty$. Then, there exists $i \in \{1, ..., M\}$ so that $J_i^k := \{t \in [0, t_k] : x(t) \in \mathcal{M}_i\}$ has positive measure for all $k \in \mathbb{N}$ and a subsequence $\{t_k^i\}_{k\in\mathcal{K}}$ of t_k, $\mathcal{K} \subseteq \mathbb{N}$, $t_k^i \to 0$ when $k \to \infty$ satisfying $x(t) \in \mathcal{M}_i$ for all $t \in (0, t_k^i]$, for all $k \in \mathcal{K}$.*

Proof. Suppose $x(\cdot)$ is a regular trajectory of G with $x(0) = x_0$ and let $\{t_k\}_{n\in\mathbb{N}}$ be a sequence converging to 0. By the regularity assumption, there exists a partition of $[0, T]$, $\{0 =: s_0 < s_1 < ... < s_N < s_{N+1} := T\}$ so that $x(t) \in \mathcal{M}_{j_l}$ for all $t \in (t_l, t_{l+1})$, $l = 0, ..., N$, for some $j_l \in \{1, ..., M\}$. Then, there exists $i \in \{1, ..., M\}$ such that $x(t) \in \mathcal{M}_i$ for all $t \in (s_0, s_1)$, i.e., J_i^k has positive measure for all $k \in \mathbb{N}$. Consider such i and let N be the first index so that $0 < t_N \leq s_1$, and define

$$\mathcal{K} = \{k \in \mathbb{N} : k \geq N\}.$$

Then, $\{t_k^i = t_k\}_{k\in\mathcal{K}}$ is a subsequence of $\{t_k\}_{k\in\mathbb{N}}$ so that $t_k^i \to 0$, $k \to \infty$ and $x(t) \in \mathcal{M}_i$ for all $t \in (0, t_k^i]$. □

4.3 Hamilton-Jacobi Inequalities

In this section we are proving the Hamilton-Jacobi inequalities. Remember, we are supposing our trajectories are regular.

Theorem 4.2. *Suppose a closed set $E \subseteq \mathbb{R}^n$, an open set $\mathcal{U} \subseteq \mathbb{R}^n$, and a stratified system are given. Then (G, E) is weakly invariant in $\mathcal{U}$ if and only if for all $x_0 \in E \cap \mathcal{U}$ there exists $i \in \{1, ..., M\}$ so that $x_0 \in \overline{\mathcal{M}}_i$ and*

$$h_{F_i^\sharp}(x_0, \xi) \leq 0, \quad \forall \xi \in N^P_{E\cap\overline{\mathcal{M}}_i}(x_0). \tag{4.4}$$

Proof. ($\Rightarrow$) Suppose (G, E) is weakly invariant in $\mathcal{U}$. Then there exists a trajectory $x(\cdot)$ of G so that $x(0) = x_0$, $x(t) \in E$ for all $t \in [0, T]$ and $x(\cdot)$ is regular.

Let $\{t_k\}_{k\in\mathbb{N}}$ be a sequence so that $t_k \to 0$ and

$$\frac{x(t_k) - x_0}{t_k} \to v \text{ as } k \to \infty,$$

for some $v \in \mathbb{R}^n$.

Define $J_i^k := \{t \in [0, t_k] : x(t) \in \mathcal{M}_i\}$. By Proposition 6, there exists $j \in \{1, ..., M\}$ so that measure$(J_j^k) > 0$ for all $k \in \mathbb{N}$ and a subsequence $\{t_k^j\}_{k\in\mathcal{K}}$ of $\{t_k\}_{k\in\mathbb{N}}$ with $t_k^j \in J_k^j$ for all $k \in \mathcal{K}$, $\mathcal{K} \subseteq \mathbb{N}$, so that $x(t) \in \mathcal{M}_j$ for all $t \in (0, t_k^j]$, for all $k \in \mathcal{K}$.

Let $\xi \in N^P_{E\cap\overline{\mathcal{M}}_j}(x_0)$, then

$$\begin{aligned}\langle v, \xi\rangle &= \langle \lim_{k\to\infty} \tfrac{x(t_k^j)-x_0}{t_k^j}, \xi\rangle = \lim_{k\to\infty} \tfrac{1}{t_k^j}\sigma|x(t_k^j) - x_0|^2 \\ &\leq \lim_{k\to\infty} \tfrac{1}{t_k^j}\sigma\left(\int_0^{t_k^j} |\dot{x}(t)|dt\right)^2 \leq \lim_{k\to\infty} \tfrac{1}{t_k^j}\sigma\|G\|^2(t_k^j)^2 = 0,\end{aligned}$$

where in the first inequality we used the proximal inequality for ξ since $x(t_k^j) \in \mathcal{M}_j$ and as $x(t_k^j) \to x_0$ we have that $x_0 \in \overline{\mathcal{M}}_j$ because $x(\cdot)$ is continuous and $\|G\| := \sup\{|\gamma| : \gamma \in G(x), x \in rB(0, 1)\}$, with r sufficiently large.

We need to show that $v \in F_j^\sharp(x_0)$.

We can use the equivalent definition of the Bouligand tangent cone given by [28] and get that $v \in \mathcal{T}_{\overline{\mathcal{M}}_j}(x_0)$. Now we are going to show that $v \in F_j$,

$$\begin{aligned}v &= \lim_{k\to\infty} \tfrac{x(t_k^j)-x_0}{t_k^j} = \lim_{k\to\infty} \tfrac{1}{t_k^j}\int_0^{t_k^j} \dot{x}(t)dt \\ &\in \lim_{k\to\infty} \tfrac{1}{t_k^j}\int_0^{t_k^j} F_j dt = \lim_{k\to\infty} \tfrac{t_k^j}{t_k^j} F_j \in F_j.\end{aligned}$$

($\Leftarrow$) We know that (G, E) is weakly invariant in $\mathcal{U}$ if and only if

$$\min_{v\in G(x_0)} \langle v, \xi\rangle \leq 0, \quad \forall\, x_0 \in E \cap \mathcal{U},\ \xi \in N_E^P(x_0).$$

Let $x_0 \in E \cap \mathcal{U}$. By assumption, there exists $i \in \{1, 2, ..., M\}$ so that $x_0 \in \overline{\mathcal{M}}_i$ and the inequality (4.4) holds. Let $\xi \in N_E^P(x_0)$, then for all $y \in E$,

$$\langle \xi, y - x_0\rangle \leq \sigma|y - x_0|^2. \tag{4.5}$$

As $E \cap \overline{\mathcal{M}}_i \subset E$ the inequality (4.5) holds when $y \in E \cap \overline{\mathcal{M}}_i$, that is, $\xi \in N^P_{E\cap\overline{\mathcal{M}}_i}(x_0)$. Since $F_i^\sharp(x_0) \subseteq G(x_0)$, we have

$$\forall\, \xi \in N_E^P(x_0), \quad \min_{v\in G(x_0)} \langle v, \xi\rangle \leq \min_{v\in F_i^\sharp(x_0)} \langle v, \xi\rangle \leq 0,$$

by assumption (inequality (4.4)). □

In the last Theorem we have that there exists $i \in \{1, ..., M\}$ so that the inequality (4.4) holds. Now, we are giving an example that is weakly invariant but the inequality

(4.4) doesn't hold for all i.

Example 5. *Consider $\mathbb{R}^2$. Let the manifolds be*

$$\mathcal{M}_1 = \{(x,y) \in \mathbb{R}^2 : y > 0, x \in \mathbb{R}\},\ \mathcal{M}_2 = \{(x,y) \in \mathbb{R}^2 : y < 0, x \in \mathbb{R}\}$$

$$\mathcal{M}_3 = \{(x,y) \in \mathbb{R}^2 : y = 0, x \in \mathbb{R}\}.$$

Define the multifunction related to those manifolds by

$$F_1(x,y) = \{(u,v) \in \mathbb{R}^2 : u^2 + v^2 \leq 1\},$$

$$F_2(x,y) = \{(u,v) \in \mathbb{R}^2 : 0 \leq u \leq 1/2, -3/4 \leq v \leq 3/4\},$$

$$F_3(x,0) = \{(u,0) \in \mathbb{R}^2 : -1 \leq u \leq 1\}.$$

Let E be given by $E = \{(x,y) \in \mathbb{R}^2 : (x-1)^2 + (y-1)^2 \leq 1\}$, and consider the point $(1,0) \in E$. Note that for $i = 2$, $N^P_{E\cap\overline{\mathcal{M}}_2}(1,0) = \mathbb{R}^2$ and for $\xi = (1/2, 0)$ we have

$$h_{F_2^\sharp}((1,0),\xi) > 0.$$

But for $i = 1$, $N^P_{E\cap\overline{\mathcal{M}}_1}(1,0) = \{c(0,-1)\}$, where $c \geq 0$, and

$$h_{F_1^\sharp}((1,0), c(0,-1)) \leq 0.$$

In this case, (G,E) is weakly invariant but the inequality (4.4) doesn't hold for all i.

Lemma 8. *Consider a stratified system. Suppose $E \subseteq \mathbb{R}^n$ is closed and assume that for all $x_0 \in E \cap \overline{\mathcal{M}}_i$, $i \in \{1, ..., M\}$,*

$$\max_{v\in F_i^\sharp(x_0)} \langle v, \xi\rangle = -h_{F_i^\sharp}(x_0, -\xi) \leq 0, \ \ \forall\ \xi \in N^P_{E\cap\overline{\mathcal{M}}_i}(x_0). \tag{4.6}$$

For a fixed i, if $x(\cdot) : [0,T] \to \overline{\mathcal{M}}_i$ is a trajectory of G with $x(0) = x_0$ and $x(t) \in \mathcal{M}_i$ for all $t \in (0,T)$, then there exists $r > 0$ such that

$$(d_E(x(t)))^2 \leq (d_{E\cap\overline{\mathcal{M}}_i}(x(t)))^2 \leq e^{rt} d_{E\cap\overline{\mathcal{M}}_i}(x_0), \quad \forall\, t \in [0,T]. \tag{4.7}$$

Proof. Fix i and let $y \in \mathcal{M}_i$, $v \in F_i$, $s \in \operatorname{proj}_{E\cap\overline{\mathcal{M}}_i}(y)$ and $u \in F_i^\sharp(s)$. Note that

$$\langle y - s, v\rangle = \langle y - s, u\rangle + \langle y - s, v - u\rangle \leq 0 + \langle y - s, v - u\rangle,$$

because $s \in E \cap \overline{\mathcal{M}}_i$, $y - s \in N^P_{E\cap\overline{\mathcal{M}}_i}(s)$ and then we used the inequality (4.6). As F_i is compact we have

$$\bar{r} := \sup\{|v| : v \in F_i\} < \infty,$$

and

$$|v-u| \leq 2\bar{r}, \ \text{ since } \ F_i^\sharp(s) \subseteq F_i, \ \forall\, s \in E \cap \overline{\mathcal{M}}_i.$$

Then,

$$\langle y-s, v\rangle \leq |y-s||v-u| \leq 2\bar{r} d_{E\cap\overline{\mathcal{M}}_i}(y) =: r d_{E\cap\overline{\mathcal{M}}_i}(y), \tag{4.8}$$

for all $y \in \mathcal{M}_i, v \in F_i$ and $s \in \ \text{proj}_{E\cap\overline{\mathcal{M}}_i}(y)$. If $\text{proj}_{E\cap\overline{\mathcal{M}}_i}(y) = \emptyset$ then (4.8) holds by vacuity.

Let $x : [0,T] \to \overline{\mathcal{M}}_i$ be a trajectory of G with $x(0) = x_0$ and $x(t) \in \mathcal{M}_i$ for all $t \in (0,T)$. For $\varepsilon > 0$ small, $x(\cdot)$ restricted to $[\varepsilon, T-\varepsilon]$ is contained in $\mathcal{M}_i$ and therefore is a trajectory of F_i. We can use the Theorem 4.1 in the reference [26] where $\Gamma = G = F_i$ satisfies the assumption $(SH)_+$ then satisfies the assumption (v) of the Theorem 4.1, $C = E \cap \overline{\mathcal{M}}_i$ is closed and the inequality (4.8) holds when $y = x(\varepsilon) \in \mathcal{M}_i$. Thus,

$$(d_E(x(t)))^2 \leq (d_{E\cap\overline{\mathcal{M}}_i}(x(t)))^2 \leq e^{rt} d_{E\cap\overline{\mathcal{M}}_i}(x(\varepsilon)), \ \ t \in [\varepsilon, T-\varepsilon].$$

Let $\varepsilon \downarrow 0$ finishes the proof. □

Observation 6. *In Theorem 4.1, [26], the result holds without square in the middle term of an inequality like inequality* (4.7). *Here, we have the square because $F_i^\sharp$ is not necessarily Lipschitz. The proof is exactly the same but in the end we don't need to take the square root, because of this it is not presented here.*

Now, we can characterize the strong invariance by a Hamiltonian inequality. To prove that we need to use the last Lemma.

Theorem 4.3. (G,E) *is strongly invariant in* $\mathcal{U} \subseteq \mathbb{R}^n$ *open set if and only if for all* $x_0 \in E \cap \mathcal{U} \cap \overline{\mathcal{M}}_i$, $i \in \{1, ..., M\}$,

$$h_{F_i^\sharp}(x_0, -\xi) \geq 0, \quad \forall \xi \in N^P_{E\cap\overline{\mathcal{M}}_i}(x_0). \tag{4.9}$$

Proof. $(\Rightarrow)$ Suppose (G,E) is strongly invariant in $\mathcal{U}$, $x_0 \in E \cap \mathcal{U} \cap \overline{\mathcal{M}}_i$, $v \in F_i^\sharp(x_0)$ and $\xi \in N^P_{E\cap\overline{\mathcal{M}}_i}(x_0)$. By Proposition 5, there exist $T > 0$ and a C^1 solution to $(DI)_G$ on the interval $[0,T]$ such that $x(0) = x_0$, $\dot{x}(0) = v \in F_i^\sharp(x_0)$ and $x(t) \in \mathcal{U}$ for all $t \in [0,T)$. By the strong invariance, $x(t) \in E$ for all $t \in [0,T)$. As $\mathcal{M}_i$ is linear there exists $\bar{t} \in (0,T]$ such that $x(t) \in \overline{\mathcal{M}}_i$ for all $t \in [0,\bar{t}]$. Then

$$\langle v, \xi\rangle = \left\langle \lim_{t\downarrow 0} \frac{x(t)-x_0}{t}, \xi \right\rangle = \lim_{t\downarrow 0} \left\langle \frac{x(t)-x_0}{t}, \xi \right\rangle = \lim_{t\downarrow 0} \frac{1}{t}\langle x(t)-x_0, \xi\rangle$$

$$\leq \lim_{t\downarrow 0} \frac{\sigma}{t}|x(t)-x_0|^2 \leq \lim_{t\downarrow 0} \frac{\sigma}{t}\left(\int_0^t |\dot{x}(s)|ds\right)^2 \leq \lim_{t\downarrow 0} \frac{\sigma}{t}\|F_i^\sharp(x_0)\|^2 t^2 = 0,$$

where in first inequality we used the fact that $\xi \in N^P_{E\cap\overline{\mathcal{M}}_i}(x_0)$ and in the last equality we used that $F_i(x_0)$ is bounded and $F_i^\sharp(x_0) \subseteq F_i(x_0)$.
As $v \in F_i^\sharp(x_0)$ is arbitrary, the proof is complete.

($\Leftarrow$) Let $x(\cdot)$ be a trajectory of G with $x(0) = x_0$ so that there exists a partition $0 =: t_0 < t_1 < ... < t_N := T$ such that $x(t) \in \mathcal{M}_{i_k}$ for all $t \in (t_k, t_{k+1})$, $i_k \in \{1, ...M\}$ and $k = 1, ..., N-1$, and $x(t) \in \mathcal{U}$ for all $t \in [0, T]$. Suppose for all $i \in \{1, ...M\}$ so that $x_0 \in E \cap \mathcal{U} \cap \overline{\mathcal{M}}_i$ the inequality (4.9) holds. Consider the interval $[t_0, t_1]$. Then there exists $j \in \{1, ..., M\}$ so that $x(\cdot) \in \overline{\mathcal{M}}_j$ for all $t \in [t_0, t_1]$. Define $\bar{x}_1 : [0, t_1] \to \mathbb{R}^n$ by $\bar{x}_1(t) = x(t)$ for all $t \in [t_0, t_1]$. As $\bar{x}_1(\cdot)$ is a trajectory of G we can apply the Lemma 8, that is, there exists $r > 0$ so that

$$\left(d_{E\cap\overline{\mathcal{M}}_j}(\bar{x}_1(t))\right)^2 \leq e^{rt} d_{E\cap\overline{\mathcal{M}}_j}(x_0), \quad \forall t \in [t_0, t_1].$$

As $x_0 \in E \cap \overline{\mathcal{M}}_j$ and $\bar{x}_1(t) = x(t)$ for all $t \in [t_0, t_1]$, we have that $x(t) \in E$ for all $t \in [t_0, t_1]$. We know there exists $l \in \{1, ..., M\}$ so that $x(t) \in \mathcal{M}_l$ for all $t \in (t_1, t_2)$. As $x(t_1) \in E \cap \mathcal{U} \cap \overline{\mathcal{M}}_l$ the inequality (4.9) holds. Again, we can define $\bar{x}_2 : [t_1, t_2] \to \mathbb{R}^n$ by $\bar{x}_2(t) = x(t)$ for all $t \in [t_1, t_2]$, apply Lemma 8 and get that $x(t) \in E$ for all $t \in [t_1, t_2]$. We can proceed in the same way for all $N's$ intervals and get the result. □

We want to characterize the weak invariance for the multifunction $(G \times \{-1\}, E)$ and the strong invariance for $(-G \times \{1\}, E)$, where E is the epigraph of T_S. We are considering such multifunction because of the next proposition that gives us the weak and strong invariance and relate these concepts to the minimal time problem. The proof was given by [22] and it was used some results of the Dynamical Programming. With the next result we can get the H-J inequality to the the minimal time function that are related to the proximal subdifferential of it.

Proposition 7. *Suppose G satisfies (SH) and let $E := epiT_S$.*
a) $(G \times \{-1\}, E)$ is weakly invariant in $\mathcal{U} := S^c \times \mathbb{R}$;
b) $(-G \times \{1\}, E)$ is strongly invariant in $\mathbb{R}^{n+1}$.

Define $T_i : \mathbb{R}^n \to [0, \infty]$, $i = 1, ..., M$, by

$$T_i(x) = \begin{cases} T_S(x) & \text{if } x \in \overline{\mathcal{M}}_i \\ +\infty & \text{if } x \notin \overline{\mathcal{M}}_i. \end{cases}$$

Theorem 4.4. *Let $E := epiT_S$ and $\mathcal{U} := S^c \times \mathbb{R}$. Then $(G \times \{-1\}, E)$ is weakly invariant* *in $\mathcal{U}$ if and only if for all $x_0 \notin S$, there exists $i \in \{1, ..., M\}$ so that $x_0 \in \overline{\mathcal{M}}_i$ and*

$$1 + h_{F_i^\sharp}(x_0, \xi) \leq 0, \quad \forall \xi \in \partial_P T_i(x_0). \tag{4.10}$$

Proof. ($\Rightarrow$) Suppose $(G \times \{-1\}, E)$ is weakly invariant in $\mathcal{U}$. Let $x_0 \notin S$, by Theorem 4.2 there exists $i \in \{1, ..., M\}$ so that $x_0 \in \overline{\mathcal{M}}_i$ and

$$h_{(F_i \times \{-1\})^\sharp}((x_0, r), (\xi_1, \xi_2)) \leq 0 \quad \forall\ (x_0, r) \in E \cap (\overline{\mathcal{M}}_i \times \mathbb{R}) =: \overline{E}_i \text{ and } (\xi_1, \xi_2) \in N^P_{\overline{E}_i}(x_0, r).$$

Note that if $\xi_2 < 0$ we have

$$\begin{aligned} h_{(F_i \times \{-1\})^\sharp}((x_0, r), (\xi_1, \xi_2)) &= \min_{(u,v) \in (F_i^\sharp \times \{-1\})(x_0, r)} \langle (u, v), (\xi_1, \xi_2) \rangle = \min_{u \in F_i^\sharp(x_0)} \langle u, \xi_1 \rangle - \xi_2 \\ &= -\xi_2 [\min_{u \in F_i^\sharp(x_0)} \langle u, -\xi_1/\xi_2 \rangle + 1] \leq 0. \end{aligned} \tag{4.11}$$

Take the values $(x_0, r) = (x_0, T_i(x_0))$ and $(\xi_1, -1) \in N^P_{\overline{E}_i}(x_0, T_i(x_0))$, this is $\xi_1 \in \partial_P T_i(x_0)$, then

$$h_{F_i^\sharp}(x_0, \xi_1) + 1 \leq 0, \quad \forall\ \xi_1 \in \partial_P T_i(x_0),$$

because of the inequality (4.11).

($\Leftarrow$) Let $(x_0, r) \in E \cap \mathcal{U}$, $x_0 \notin S$, and $\xi = (\xi_1, \xi_2) \in N^P_E(x_0, r)$. We know that $\xi_2 \leq 0$. First we can suppose that $\xi_2 < 0$. Then $r = T_S(x_0)$ (because $\xi_2 < 0$ implies that $(x_0, r) \in$ bdry$E$). By assumption, there exists $i \in \{1, ..., M\}$ so that the inequality (4.10) hods and $x_0 \in \overline{\mathcal{M}}_i$. Note that $\xi \in N^P_{\overline{E}_i}(x_0, T_i(x_0))$. Then, as $N^P_{\overline{E}_i}(x_0, T_i(x_0))$ is a cone we have $(\xi_1/(-\xi_2), -1) \in \partial_P T_i(x_0)$. Then

$$\begin{aligned} h_{(G \times \{-1\})}((x_0, T_i(x_0)), (\xi_1, \xi_2)) &\leq h_{(F_i \times \{-1\})^\sharp}((x_0, T_i(x_0)), (\xi_1, \xi_2)) \\ &= -\xi_2 [\min\nolimits_{u \in F_i^\sharp(x_0)} \langle u, -\xi_1/\xi_2 \rangle + 1] \leq 0, \end{aligned} \tag{4.12}$$

where we used the inequality (4.11) and the assumption.

Now, suppose $\xi_2 = 0$. Since $(\xi_1, 0) \in N^P_E(x_0, r)$, there exists $\sigma > 0$ such that

$$\langle (\xi_1, 0), (y, q) - (x_0, r) \rangle \leq \sigma |(y, q) - (x_0, r)|^2 \ \ \forall\ \ (y, q) \in E.$$

But

$$\langle (\xi_1, 0), (y, q) - (x_0, r) \rangle = \langle (\xi_1, 0), (y, q) - (x_0, T_S(x_0)) \rangle$$

and

$$\sigma |(y, q) - (x_0, r)|^2 \leq \sigma [|y - x_0| + |q - T_S(x_0)|]^2 = \sigma |(y, q) - (x_0, T_S(x_0))|^2,$$

that is, $(\xi_1, 0) \in N_E^P(x_0, T_S(x_0))$.

By Rockafellar's horizontality Theorem, [40], there exist sequences $\{x_j\}_{j\in\mathbb{N}}$, $\{\xi_j\}_{j\in\mathbb{N}}$ and $\{\gamma_j\}_{j\in\mathbb{N}}$ such that $x_j \to x_0$, $T_S(x_j) \to T_S(x_0)$, $\xi_j \to \xi$, $\gamma_j < 0$ with $\gamma_j \uparrow 0$, and $-\xi_j/\gamma_j \in \partial_P T_S(x_j)$. By assumption for each $j \in \mathbb{N}$ there exists $k_j \in \{1, ..., M\}$ so that $x_j \in \overline{\mathcal{M}}_{k_j}$ and the inequality (4.10) holds. We have by the inequality (4.10)

$$-\gamma_j[h_{F^\sharp_{k_j}}(x_j, -\xi_j/\gamma_j) + 1] \leq 0,$$

that is, for each j there exists $v_j \in F^\sharp_{k_j}(x_j)$ so that the last inequality holds. As $F^\sharp_{k_j}(x_j) \subseteq G(x_j)$ for all j and $G(\cdot)$ is upper semicontinuous, there exists a subsequence of $\{v_j\}_{j\in\mathbb{N}}$ that is converging to $v \in G(x_0)$. Then, in the last inequality,

$$\min_{u\in G(x_0)} \langle u, \xi\rangle \leq \langle v, \xi\rangle = \lim_{j\to\infty} \langle v_j, \xi_j\rangle = \lim_{j\to\infty} -\gamma_j[h_{F^\sharp_{k_j}}(x_j, -\xi_j/\gamma_j) + 1] \leq 0,$$

and then

$$h_{(G\times\{-1\})}((x_0, r), (\xi, 0)) = h_G(x_0, \xi) \leq 0. \tag{4.13}$$

By inequality (4.12), (4.13) and Theorem 3.1, [22], $(G \times \{-1\}, E)$ is weakly invariant in $S^c \times \mathbb{R}$. □

Before the next theorem we need to prove an auxiliary result. First, note that $(-F_i)^\sharp : \overline{\mathcal{M}}_i \rightrightarrows \mathbb{R}^n$ is defined from $F_i^\sharp$ for all $i = 1, ..., M$ and is given by

$$(-F_i)^\sharp(x) := \overline{(-F_i)} \cap \mathcal{T}_{\overline{\mathcal{M}}_i}(x).$$

Lemma 9. *The multifunction $(-F_i)^\sharp$ defined above is lower semicontinuous.*

Proof. Let $x_0 \in \overline{\mathcal{M}}_i$. We first show that $\overline{(-F_i)}$ and $\mathcal{T}_{\overline{\mathcal{M}}_i}(x_0)$ are l.s.c.

As $-F_i$ is closed and constant since F_i is, we get $\overline{(-F_i)}$ is constant, that is, $\overline{(-F_i)}$ is lower semicontinuous at x_0.

By Theorem 4.10.4, [28],

$$N_{\overline{\mathcal{M}}_i}(x_0) = [\mathcal{T}_{\overline{\mathcal{M}}_i}(x_0)]^*.$$

By Proposition 4.2.6, [28], the set-valued mapping $x \mapsto N_{\overline{\mathcal{M}}_i} : \overline{\mathcal{M}}_i \rightrightarrows \mathbb{R}^n$ has a closed graph. By Proposition 1.2.2, [33], the set-valued map $x \mapsto \mathcal{T}_{\overline{\mathcal{M}}_i}(x) : \overline{\mathcal{M}}_i \rightrightarrows \mathbb{R}^n$ is lower semicontinuous at x_0.

Let $v \in (-F_i)^\sharp$ and $\varepsilon > 0$. As $\overline{(-F_i)}(x_0)$ is l.s.c. there exists $\delta_1 > 0$ so that

$$\bar{x} \in \mathrm{dom}\overline{(-F_i)},\ \bar{x} \in x_0 + \delta_1 B(0,1) \implies v \in \overline{(-F_i)} + \varepsilon B(0,1). \tag{4.14}$$

As $\mathcal{T}_{\overline{\mathcal{M}}_i}(x_0)$ is l.s.c there exists $\delta_2 > 0$ so that

$$\bar{x} \in \mathrm{dom}\mathcal{T}_{\overline{\mathcal{M}}_i},\ \bar{x} \in x_0 + \delta_2 B(0,1) \implies v \in \mathcal{T}_{\overline{\mathcal{M}}_i}(x_0) + \varepsilon B(0,1). \tag{4.15}$$

Define $\delta := \min\{\delta_1, \delta_2\}$, then (4.14) and (4.15) hold and we have

$$\bar{x} \in \mathrm{dom}[\overline{(-F_i)} \cap \mathcal{T}_{\overline{\mathcal{M}}_i}],\ \bar{x} \in x_0 + \delta B(0,1) \implies v \in \overline{(-F_i)} \cap \mathcal{T}_{\overline{\mathcal{M}}_i}(x_0) + \varepsilon B(0,1),$$

that is, $(-F_i)^\sharp$ is l.s.c at x_0. As x_0 is arbitrary, $(-F_i)^\sharp$ is l.s.c. □

Theorem 4.5. *Let $E = epiT_S$. $(-G \times \{1\}, E)$ is strongly invariant in $\mathbb{R}^{n+1}$ if and only if for all $i \in \{1, ..., M\}$ so that $x_0 \in \overline{\mathcal{M}}_i$ we have*

$$h_{(-F_i)^\sharp}(x_0, -\xi) + 1 \geq 0, \quad \forall \xi \in \partial_P T_i(x_0). \tag{4.16}$$

Proof. $(\Rightarrow)$ Suppose $(-G \times \{1\}, E)$ is strongly invariant in $\mathbb{R}^{n+1}$. By Theorem 4.3, we have

$$h_{(-F_i \times \{1\})^\sharp}((x_0, r), (-\xi_1, -\xi_2)) \geq 0 \quad \forall\ (x_0, r) \in \overline{E}_i \text{ and } (\xi_1, \xi_2) \in N^P_{\overline{E}_i}(x_0, r).$$

Note that

$$0 \leq h_{(-F_i \times \{1\})^\sharp}((x_0, r), (-\xi_1, -\xi_2)) = \min_{(u,v) \in ((-F_i)^\sharp \times \{1\})(x_0, r)} \langle (u, v), (-\xi_1, -\xi_2) \rangle \tag{4.17}$$

$$= \min_{u \in (-F_i)^\sharp(x_0)} \langle u, -\xi_1 \rangle - \xi_2 = -\xi_2 [\min_{v \in (-F_i)^\sharp(x_0)} \langle v, \xi_1/\xi_2 \rangle + 1].$$

Let $(\xi_1, -1) \in N^P_{\overline{E}_i}(x_0, T_i(x_0))$, this is, $\xi_1 \in \partial_P T_i(x_0)$ and by the inequality (4.17)

$$h_{(-F_i)^\sharp}(x_0, -\xi_1) + 1 \geq 0, \quad \forall\ x_0 \in \overline{\mathcal{M}}_i,\ i = 1, ..., M,\ \ \forall\ \xi_1 \in \partial_P T_i(x_0).$$

$(\Leftarrow)$ Fix $i \in \{1, ..., M\}$. Let $(x_0, r) \in \overline{E}_i$ and $\xi = (\xi_1, \xi_2) \in N^P_{\overline{E}_i}(x_0, r)$. We know $\xi_2 \leq 0$. First we suppose that $\xi_2 < 0$. Then $r = T_i(x_0)$ (because $\xi_2 < 0$ implies that $(x_0, r) \in$ bdry$\overline{E}_i$). As $N^P_{\overline{E}_i}(x_0, T_i(x_0))$ is a cone, we have $(\xi_1/(-\xi_2), -1) \in N^P_{\overline{E}_i}(x_0, T_i(x_0))$, this is $\xi_1/(-\xi_2) \in \partial_P T_i(x_0)$. Then

$$h_{(-F_i \times \{1\})^\sharp}((x_0, T_i(x_0)), (-\xi_1, -\xi_2)) = -\xi_2 \left(h_{(-F_i)^\sharp}(x_0, \xi_1/\xi_2) + 1 \right) \geq 0,$$

where we used the inequality (4.17) and the assumption.

Now, suppose $\xi_2 = 0$. Using the same arguments above, $(\xi_1, 0) \in N^P_{E_i}(x_0, T_i(x_0))$.

By Rockafellar's horizontality Theorem, [40], there exist sequences $\{x_j\}_{j\in\mathbb{N}}$, $\{\xi_j\}_{j\in\mathbb{N}}$ and $\{\gamma_j\}_{j\in\mathbb{N}}$ such that $x_j \to x_0$, $T_i(x_j) \to T_i(x_0)$, $\xi_j \to \xi_1$, $\gamma_j < 0$ with $\gamma_j \uparrow 0$, and $-\xi_j/\gamma_j \in \partial_P T_i(x_j)$. By assumption the inequality (4.16) holds.

We want to use Theorem 1.2.4., [33]. For this define $V : \overline{\mathcal{M}}_i \times \mathbb{R}^n \to \mathbb{R}$ by

$$V(x,\xi) := H_{(-F_i)^\sharp}(x,\xi) = \sup_{v\in\hat{G}(x,\xi)} W(v,x,\xi),$$

where $W : \mathbb{R}^n \times \overline{\mathcal{M}}_i \times \mathbb{R}^n \to \mathbb{R}$ and $\hat{G} : \overline{\mathcal{M}}_i \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ are given by

$$W(v,x,\xi) := \langle v, \xi\rangle, \quad \hat{G}(x,\xi) := (-F_i)^\sharp(x).$$

As W is given by the inner product it is continuous. By Lemma 9 the multifunction $\hat{G}$ is l.s.c. Now, we can use Theorem 1.2.4., [33], and get the result that V is lower semicontinuous at (x_0, ξ_1).

Let $v = H_{(-F_i)^\sharp}(x_0,\xi_1)$, that is a real number since the inner product is continuous and $(-F_i)^\sharp(x_0)$ is compact. As the Hamiltonian maximized is l.s.c. at (x_0,ξ_1), for those sequences above there exists $v_j = H_{(-F_i)^\sharp}(x_j,\xi_j)$ so that $v_j \to v$, $j \to \infty$.

By assumption, for all $i \in \{1, ..., M\}$ so that $x \in \overline{\mathcal{M}}_i$ and for all $\xi \in \partial_P T_i(x)$ we have

$$h_{(-F_i)^\sharp}(x,-\xi) + 1 \geq 0 \Leftrightarrow -h_{(-F_i)^\sharp}(x,-\xi) - 1 \leq 0 \Leftrightarrow H_{(-F_i)^\sharp}(x,\xi) - 1 \leq 0. \tag{4.18}$$

Then, we have

$$H_{(-F_i\times\{1\})^\sharp}((x_j,T_i(x_j)),(\xi_j,\gamma_j)) = -\gamma_j\left(-1 + \sup_{u_j\in(-F_i)^\sharp(x_j)}\left\langle u_j, \frac{-\xi_j}{\gamma_j}\right\rangle\right) \leq 0,$$

because $\gamma_j < 0$ for all j and the inequality (4.18) holds.

Passing the limit with $j \to \infty$ we get

$$0 \geq \lim_{j\to\infty}\left(\gamma_j + \sup_{u_j\in(-F_i)^\sharp(x_j)}\langle u_j,\xi_j\rangle\right) = \lim_{j\to\infty}\left(\gamma_j + H_{(-F_i)^\sharp}(x_j,\xi_j)\right) = \lim_{j\to\infty}(\gamma_j + v_j) = v.$$

Therefore,

$$h_{(-F_i\times\{1\})^\sharp}((x_0,T_i(x_0)),(-\xi_1,0)) = \min_{v\in(-F_i)^\sharp(x_0)}\langle v,-\xi_1\rangle = -H_{(F_i)^\sharp}(x_0,\xi_1) = -v \geq 0.$$

By Theorem 4.3 follows the desired result. $\square$

By Theorems 4.4 and 4.5 if $x_0 \notin S$ there exists $i \in \{1, ..., M\}$ so that $x_0 \in \overline{\mathcal{M}}_i$ and

$$\partial_P T_i(x_0) \subseteq \{\xi \in \mathbb{R}^n : 1 + h_{F_i^\sharp}(x_0, \xi) \leq 0\} \cap \{\xi \in \mathbb{R}^n : 1 + h_{(-F_i)^\sharp}(x_0, -\xi) \geq 0\}. \quad (4.19)$$

If there exists an unique i so that $x_0 \in \overline{\mathcal{M}}_i$, then $x_0 \in \mathcal{M}_i$ and

$$\partial_P T_i(x_0) \subseteq \{\xi \in \mathbb{R}^n : 1 + h_{F_i}(x_0, \xi) = 0\}.$$

In this case, as F_i is Lipschitz, by Theorem 5.1, [22], whenever $r > 0$ and $T_S(x_0) = r$ we have

$$\partial_P T_i(x_0) = \left(N^P_{S(r)}(x_0) \cap \overline{\mathcal{M}}_i\right) \cap \{\xi \in \mathbb{R}^n : \min_{v \in F_i(x_0)} \langle v, \xi \rangle = -1\},$$

where $S(r) := \{x \in \mathbb{R}^n : T_S(x) \leq r\}$ is the r-level set of $T_S(\cdot)$.

When $F \subseteq X$ is closed, convex, bounded, and with $0 \in \mathrm{int}F$ and X is a real Hilbert space, by [23],

$$\partial_P T_S(x_0) = N^P_{S(r)}(x_0) \cap \{\xi : \rho_{F^\circ}(-\xi) = 1\},$$

where $\rho_F : X \to [0, \infty]$ is the (Minkowski) *gauge* function given by

$$\rho_F(\zeta) = \min\{t \geq 0 : \frac{1}{t}\zeta \in F\},$$

and the polar F° of F is the set

$$F^\circ := \{\zeta : \langle \zeta, v \rangle \leq 1 \ \forall v \in F\}.$$

That is, in the constant case the H-J equality can be replaced by the gauge equality, and then when there exists an unique i so that $x_0 \in \overline{\mathcal{M}}_i$ we can rewrite the proximal subdefferential of T_i by

$$\partial_P T_i(x_0) = (N^P_{S(r)} \cap \overline{\mathcal{M}}_i) \cap \{\xi \in \mathbb{R}^n : \rho_{F_i^\circ}(-\xi) = 1\}.$$

Denote by $\Pi^F_S(\cdot)$ the F-projection set, $\Pi^F_S(x) = S \cap \{x + T_S(x)F\}$. Suppose $S \subseteq X$ is closed, $x_0 \notin S$, and $\Pi^F_S(x_0) \neq \emptyset$. By [23], the following inclusion holds:

$$\partial_P T_S(x_0) \subseteq -\partial \rho_F(s - x_0) \ \ \forall s \in \Pi^F_S(x_0).$$

In the classical theory, see for example [41], the adjoint arc given by the Maximum Principle is related to the minimal time function

$$-p(t) \in \partial_P T_S(x(t)) \ \ t \in [0, T_S(x_0)),$$

here the Hamiltonian is like our Hamiltonian but in [41] it is defined with a signal of minus.

If $\Pi_S^F(x(t)) \neq \emptyset$ and F is constant we have

$$-p(t) \in -\partial\rho_F(s - x(t)) \;\; \forall s \in \Pi_S^F(x(t)).$$

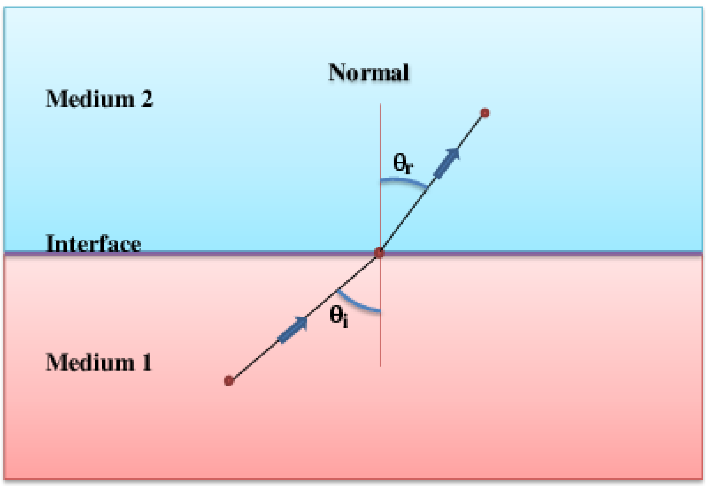
By the inclusion (4.19) we can realize that the adjoint arc for our particular case is related to some function in $F_i^\sharp$, and by the last inclusion we can see that it can be related to the gauge function of F_i, and now we are able to get the necessary conditions for our minimal time problem.

CHAPTER 5
MINIMUM TIME PROBLEM: NECESSARY AND SUFFICIENT CONDITIONS

In this chapter there are the main theorems of our research about the minimal time problem. In Section 5.1 we show some real examples and an introduction about the Snell's law that can motivate our study. Section 5.2 has the main results, the necessary and sufficient conditions for the minimal time problem. In this section we also show that the Snell's law is a particular case of our case.

5.1 Motivation

5.1.1 Snell's Law



Suppose we have two different medium and a beam of light is travelling between them, for example from air into glass or water. The Snell's law determines the relationship between the incidence and the refraction angles, that is related with the velocities of each medium.

Let $v_1 > 0$ be the velocity in Medium 1 and $v_2 > 0$ be the velocity in Medium 2. Suppose the beam originates from $P_1 \in \mathcal{M}_1$ and goes into the second medium through a point $P_2 \in \mathcal{M}_2$. Denote by I the point on the interface. Then, the minimal time function is given by $T = T_1 + T_2$, where

$$T_1 = \frac{|I - P_1|_{\mathcal{M}_1}}{v_1} \text{ and } T_2 = \frac{|P_2 - I|_{\mathcal{M}_2}}{v_2}.$$

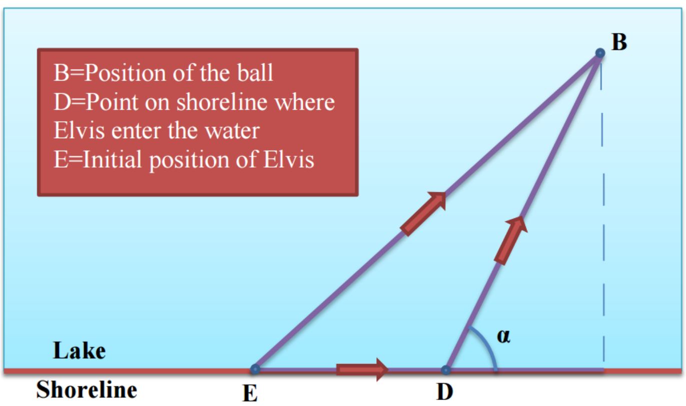
If we minimize the minimal function over I, we get the Snell's Law,

$$\frac{\sin\theta_i}{v_1} = \frac{\sin\theta_r}{v_2} = \text{constant},$$

that is, the angles are independent of the initial and final points. Fermat's principle in optics asserts that light rays propagating in inhomogeneous media follow paths that minimize time. Then, if you have the incidence and refraction angles you have the optimal path for a minimal time problem.

5.1.2 Elvis Example

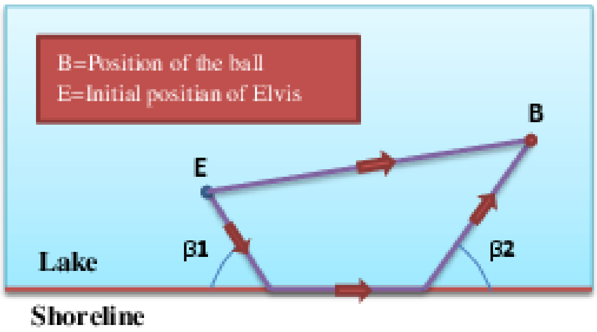
Timothy J. Pennings from Hope College in Holland, Michigan, has a Welsh Corgi, a dog called Elvis, and they used to go to the Lake Michigan to play fetch with a tennis ball. Elvis was on the shoreline of the lake and he needed to fetch the ball that was thrown into the lake. Professor Timothy realized that Elvis wasn't using the shortest distance to get the ball. The picture below illustrates this case.



He took Elvis to the lake on a day when the waves were small. He threw the ball into the lake for sometimes. He measured the distance between the points E and D and raced him to the ball. The distance between E and the projection of B on the shoreline was constant equal 15 meters. He put all the results in his article called “Do dogs know calculus?", [25], but he confess that although Elvis made good choices, he doesn't know calculus. Good choices mean that Elvis was choosing a path very close to the minimal time path most of the time.

The angle α at the picture above is the optimal angle, that is, if Elvis takes that angle from the point D he is taking the minimal time path to fetch the ball. Such angle is the θ_r found in Snell's law of optics, and $\theta_i = 90°$. Snell's law holds in this situation, that is, there exists a connection between the path someone must take to arrive in some place as soon as possible and the path light has always travelled.

Another interesting case is when Elvis and the ball are both already in the water. Then, which option will take the smallest amount of time: swim directly to the ball or swim to the shoreline, run and swim to the ball? The picture below illustrates this case.



Roland Minton and Timothy J. Pennings studied this case in the article called "The Dogs Know Bifurcations"?, [42]. They discovered that β_1 and β_2 needs to be equal when Elvis uses the velocity from the shoreline. They didn't mention Snell's law but it hods in this particular case too.

These examples are interesting because Elvis can achieve different velocities in different places and then, these examples can be seen as some particular case of our case.

5.2 Necessary and Sufficient Conditions for the Minimal Time Problem

We are going to consider the minimal time problem defined before with regular trajectories. To get the result that we desire we need to add an assumption in our dynamics. From here we are considering that $0 \in \text{int} F_i$ for all $i \in \{1, ..., M\}$.

The (Minkowski) *gauge* function $\rho_{F_i} : \overline{\mathcal{M}}_i \to [0, \infty]$ associated to F_i, $i \in \{1, ..., M\}$, is as defined in the previous chapter.

The necessary conditions for our particular problem are stated in the next theorem.

Theorem 5.1. (Necessary Conditions - Maximum Principle). *Consider the minimal time problem. Suppose $S \subset \mathbb{R}^n$ is closed and convex and (x^*, u^*) is an optimal pair to it, where $x^*, u^* : [0, T] \to \mathbb{R}^n$ are such that there exists a partition of $[0, T]$, $0 =: t_0 < t_1 < ... < t_N < t_{N+1} := T$ so that for each $k = 1, ..., N+1$, $x(t) \in \mathcal{M}_{i_k}$ for all $t \in (t_{k-1}, t_k)$, for some $i_k \in \{1, ..., M\}$, and for $k = 2, ..., N$, $x^*(t_{k-1}) =: x^*_{k-1} \in \overline{\mathcal{M}}_{i_{k-1}} \cap \overline{\mathcal{M}}_{i_k}, x^*(t_k) =: x^*_k \in \overline{\mathcal{M}}_{i_k} \cap \overline{\mathcal{M}}_{i_{k+1}}$, $x_0 \in \overline{\mathcal{M}}_1$ and $x(T) =: x_{N+1} \in S \cap \overline{\mathcal{M}}_{i_{N+1}}$. Then there exists $p_k \in \mathbb{R}^n$, $k = 1, ..., N+1$, such that*

i) $\rho_{F_i^\circ}(p_k) = 1, \ \ \forall k = 1, ..., N+1;$

ii) $-p_k + p_{k+1} \in N_{\Sigma_k}(x^*_k), \ \ \Sigma_k := \overline{\mathcal{M}}_{i_k} \cap \overline{\mathcal{M}}_{i_{k+1}}, \forall k = 1, ..., N,$
$-p_{k+1} \in N_{\Sigma_{N+1}}(x^*_{N+1}), \Sigma_{N+1} := S \cap \overline{\mathcal{M}}_{i_{N+1}};$

iii) $u^*(t) = u_k$ for all $t \in [t_{k-1}, t_k)$, where the interval is closed in t_{N+1}, $k = 1, ..., N+1$, is so that $u_k \in arg\max \sigma_{F_{i_k}}(p_k)$, $k = 1, ..., N+1$, where $\sigma_{F_{i_k}}(\cdot)$ is the support function of the unmaximized Hamiltonian $H_k : (t_{k-1}, t_k) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by

$$H_k(t, x, p, u) := \langle p, u \rangle,$$

and

$$u_k = \frac{x^*_k - x^*_{k-1}}{\rho_{F_{i_k}}(x^*_k - x^*_{k-1})}, \ \ \forall k = 1, ..., N+1.$$

Proof. By reference [23] we can rewrite the minimal time problem like below

$$(P) \qquad \min f(X) + I_C(X),$$

where $X := (x_1, x_2, ..., x_N, x_{N+1}) \in \mathbb{R}^{n(N+1)}$, $C := \prod_{k=1}^{N+1} \Sigma_k \subset \mathbb{R}^{n(N+1)}$ with $\Sigma_k := (\overline{\mathcal{M}}_{i_k} \cap \overline{\mathcal{M}}_{i_{k+1}})$, $k = 1, ..., N$, $\Sigma_{N+1} = S \cap \overline{\mathcal{M}}_{i_{N+1}}$, $x_0 \in \overline{\mathcal{M}}_{i_1}$ is given, and $f : \mathbb{R}^{n(N+1)} \to \mathbb{R}$ is given by

$$f(X) = \sum_{k=1}^{N+1} \rho_{F_{i_k}}(x_k - x_{k-1}).$$

By reference [23] the gauge function is lower semicontinuous and convex. As C is closed, convex and nonempty we have that $I_C(\cdot)$ is lower semicontinuous and convex, [29], then $f(\cdot) + I_C(\cdot)$ is lower semicontinuous and convex.

As (x^*, u^*) is an optimal pair to the minimal time problem, $X^* := (x_1^*, x_2^*, ..., x_N^*, x_{N+1}^*)$ is optimal for (P). By Proposition 5.1.1, [28],

$$0 \in \partial(f(X^*) + I_C(X^*)).$$

By Theorem 4.10, [29], we have

$$0 \in \partial f(X^*) + \partial I_C(X^*),$$

that is,

$$-\partial f(X^*) \subseteq \partial I_C(X^*) = N_C(X^*), \tag{5.1}$$

where the last equality is given by Proposition 1.

We can calculate the subgradient above and get

$$-\partial f(X^*) = -\left(\prod_{k=1}^{N}\left(\partial\rho_{F_{i_k}}(x_k^* - x_{k-1}^*) - \partial\rho_{F_{i_{k+1}}}(x_{k+1}^* - x_k^*)\right), \partial\rho_{F_{i_{N+1}}}(x_{N+1}^* - x_N^*)\right).$$

By Proposition 4.2.8, [28],

$$N_C(X^*) = \prod_{k=1}^{N+1} N_{\Sigma_k}(x_k^*).$$

By the last two equalities and the inclusion (5.1)

$$-\partial\rho_{F_{i_k}}(x_k^* - x_{k-1}^*) + \partial\rho_{F_{i_{k+1}}}(x_{k+1}^* - x_k^*) \in N_{\Sigma_k}(x_k^*), \quad \forall k = 1, ..., N,$$

and

$$-\partial\rho_{F_{i_{N+1}}}(x_{N+1}^* - x_N^*) \in N_{\Sigma_{N+1}}(x_{N+1}^*).$$

Let $p_k \in \partial\rho_{F_{i_k}}(x_k^* - x_{k-1}^*)$ for all $k = 1, ..., N+1$, then

$$-p_k + p_{k+1} \in N_{\Sigma_k}(x_k^*), \quad \forall k = 1, ..., N,$$

i.e., ii) holds. By the Proposition 2.2, [23],

$$p_k \in \partial\rho_{F_{i_k}}(x_k^* - x_{k-1}^*) = \{\xi : \rho_{F_{i_k}^\circ}(\xi) = 1\} \cap N_{F_{i_k}}\left(\frac{x_k^* - x_{k-1}^*}{\rho_{F_{i_k}}(x_k^* - x_{k-1}^*)}\right),$$

then i) holds and we only need to show iii).

As p_k belongs to that normal cone to F_{i_k} at $u_k(x_k^* - x_{k-1}^*)/[\rho_{F_{i_k}}(x_k^* - x_{k-1}^*)]$, for all $z \in F_{i_k}$ the following inequality holds

$$\langle p_k, z - u_k \rangle \leq 0.$$

In other words,

$$\langle p_k, z \rangle \leq \langle p_k, u_k \rangle\,,\ \forall z \in F_{i_k}.$$

This means that u_k maximizes the support function of the unmaximized Hamiltonian on F_{i_k}.

We have

$$s_k := \rho_{F_{i_k}}(x_k^* - x_{k-1}^*),\ k = 1, ..., N+1, \qquad s_0 := t_0$$

is the minimal time between the points x_{k-1}^* and x_k^* crossing the manifold $\mathcal{M}_{i_k}$. Then, $x : [0, T] \to \mathbb{R}^n$ given by

$$x(t) = x_{k-1}^* + (t - (s_0 + s_1 + ... + s_{k-1}))u_k,\ \ t \in [q_{k-1}, q_k],\ k = 1, ..., N+1,$$

where $q_k = s_0 + s_1 + ... + s_{k-1} + s_k$, $k = 0, ..., N+1$, is an optimal trajectory of (P).

Note that all the optimal trajectories of (P) are linear and as (P) and the minimal time problem are equivalent the optimal trajectories of the minimal time problem are linear too. As $x(\cdot)$ and $x^*(\cdot)$ have the same extremal we can conclude that $x(t) = x^*(t)$ for all $t \in [0, T]$, then $u_k = \dot{x}(t) = \dot{x}^*(t) = u^*(t)$ a.e. $t \in [t_{k-1}, t_k)$, $k = 1, ..., N+1$. □

Observation 7. *a) By the item iii) of the last theorem, u_k maximizes the Hamiltonian, that is,*

$$\max_{u \in F_{i_k}} H_k(t, x, p_k, u) = H_k(t, x, p_k, u_k).$$

Note that $u_k \in F_{i_k}^\sharp(x_k^*)$ for all $k = 1, ..., N+1$. This agree with our H-J inequalities and the conclusion that p_k should be related to the gauge function of $F_{i_k}^\sharp$.

b) Suppose p_k and u_k satisfy the items i) and iii). By Proposition 2.2, [43], and the item iii),

$$p_k \in N_{F_{i_k}}\left(\frac{x_k^* - x_{k-1}^*}{\rho_{F_{i_k}}(x_k^* - x_{k-1}^*)}\right).$$

By the item i),

$$p_k \in \{\xi : \rho_{F_{i_k}^\circ}(\xi) = 1\} \cap N_{F_{i_k}}\left(\frac{x_k^* - x_{k-1}^*}{\rho_{F_{i_k}}(x_k^* - x_{k-1}^*)}\right) = \partial\rho_{F_{i_k}}(x_k^* - x_{k-1}^*).$$

Example 6. (Snell's law is a particular case of our case) *Consider* $\mathbb{R}^2$. *Define the manifolds as following*

$$\mathcal{M}_1 = \{(x,y) \in \mathbb{R}^2 : y < 0, x \in \mathbb{R}\}, \qquad \mathcal{M}_2 = \{(x,y) \in \mathbb{R}^2 : y > 0, x \in \mathbb{R}\},$$

$$\mathcal{M}_3 = \{(x,y) \in \mathbb{R}^2 : y = 0, x \in \mathbb{R}\}.$$

Let the dynamics be given by

$$F_1 = \{(u,v) \in \mathbb{R}^2 : u^2+v^2 \leq v_1^2\},\ F_2 = \{(u,v) \in \mathbb{R}^2 : u^2+v^2 \leq v_2^2\},\ F_3 = co(F_1 \cup F_2) \cap \mathcal{M}_3,$$

where $v_1, v_2 > 0$.

Let's suppose that the initial point $(x_0, y_0) \in \mathcal{M}_1$ *and the final point* $(x_2, y_2) \in \mathcal{M}_2$ *are given. We need to find* $p_1 = (p_{11}, p_{12})$ *and* $p_2 = (p_{21}, p_{22})$ *so that i) and ii) are satisfied. Let's called* $(x_1, 0)$ *the point on the interface* $\Sigma_1 = \overline{\mathcal{M}}_1 \cap \overline{\mathcal{M}}_2$.

We know that $F_1^\circ = \{(\xi_1,\xi_2) \in \mathbb{R}^2 : \langle(\xi_1,\xi_2),(u,v)\rangle \leq 1, \forall (u,v) \in F_1\} = \{(\xi_1,\xi_2) \in \mathbb{R}^2 : v_1^2\xi_1^2 + v_1^2\xi_2^2 \leq 1\}$. *By Proposition 2.1, [23],*

$$1 = \rho_{F_1^\circ}(p_1) = \max_{v \in F_1}\langle p_1, v\rangle = \sigma_{F_1}(p_1).$$

Then, we have the system

$$\begin{cases} v_1^2 p_{11}^2 + v_1^2 p_{12}^2 = 1 \\ p_{11}u_{11} + p_{12}u_{12} = 1, \end{cases}$$

where by i) $p_k \in F_{i_k}^\circ$ *and then it must satisfies the first equality and* $u_1 = (u_{11}, u_{12})$ *is given by iii). We can solve such system and get*

$$p_{11} = \frac{(x_1 - x_0)}{v_1^2\rho_{F_1}((x_1 - x_0, -y_0))} \qquad \text{and} \qquad p_{12} = \frac{y_0}{v_1^2\rho_{F_1}((x_1 - x_0, -y_0))},$$

with $\rho_{F_1}((x_1 - x_0, -y_0)) = \sqrt{(x_1 - x_0)^2 + (-y_0)^2}$.

We have the same system for p_2. *We can solve it and get*

$$p_{21} = \frac{(x_2 - x_1)}{v_2^2\rho_{F_2}((x_2 - x_1, y_2))} \qquad \text{and} \qquad p_{22} = \frac{y_2}{v_2^2\rho_{F_2}((x_2 - x_1, y_2))},$$

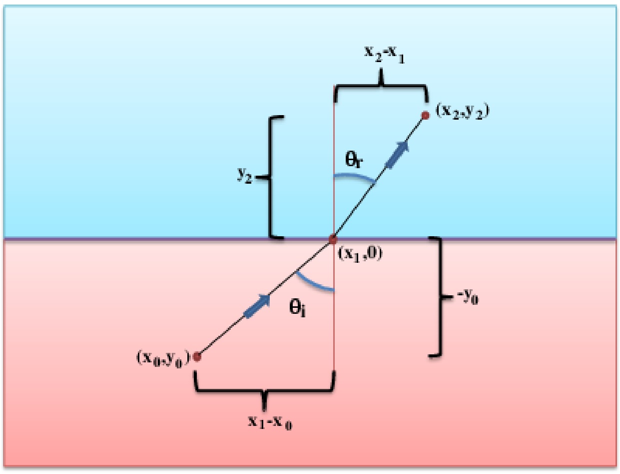
with $\rho_{F_1}((x_2 - x_1, y_2)) = \sqrt{(x_2 - x_1)^2 + (y_2)^2}$.

By ii), as $N_{\Sigma_1}(x_1, 0) = \{(0, \xi) : \xi \in \mathbb{R}\}$,

$$-p_{11} + p_{21} = 0.$$

Then we can find the following equality

$$x_1 = \frac{x_2 v_1^2 \rho_{F_1}((x_1 - x_0, -y_0)) + x_0 v_2^2 \rho_{F_2}((x_2 - x_1, y_2))}{v_1^2 \rho_{F_1}((x_1 - x_0, -y_0)) + v_2^2 \rho_{F_2}((x_2 - x_1, y_2))}.$$



We have all the necessary information to get the relation between the incidence and the refraction angles. With simple calculation and substituting the value of x_1 *in the last equality we get*

$$\frac{\sin \theta_i}{\sin \theta_r} = \frac{\frac{x_1 - x_0}{\sqrt{(x_1 - x_0)^2 + (0 - y_0)^2}}}{\frac{x_2 - x_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - 0)^2}}} = \frac{v_1}{v_2}.$$

Now we can prove that our necessary conditions are sufficient if we have the sequence of manifolds that an optimal solution should follow.

Theorem 5.2. (Sufficient Conditions). *Consider the minimal time problem where* $S \subset \mathbb{R}^n$ *is closed and convex. Suppose a sequence of manifolds is given* $\{i_1, i_2, ..., i_{N+1}\}$. *Let* $(x(\cdot), u(\cdot))$ *be a feasible pair for the minimal time problem so that* $x_k := x(t_k) \in \overline{\mathcal{M}}_{i_k} \cap \overline{\mathcal{M}}_{i_{k+1}}$, $k = 1, ..., N$, $x(T) \in S \cap \overline{\mathcal{M}}_{i_{N+1}}$, *and* $x(t) \in \mathcal{M}_{i_k}$ *when* $t \in (t_{k-1}, t_k)$, $k = 1, ..., N+1$, $t_0 = 0$, $t_{N+1} = T$ *and suppose there exists* $p_k \in \mathbb{R}^n$, $k = 1, ..., N+1$, *so that*

i) $\rho_{F_i^\circ}(p_k) = 1$, $\forall k = 1, ..., N+1$;

ii) $-p_k + p_{k+1} \in N_{\Sigma_k}(x_k)$, $\Sigma_k := \overline{\mathcal{M}}_{i_k} \cap \overline{\mathcal{M}}_{i_{k+1}}$, $\forall k = 1, ..., N$,
$-p_{N+1} \in N_{\Sigma_{N+1}}(x_{N+1})$, $\Sigma_{N+1} := S \cap \overline{\mathcal{M}}_{i_{N+1}}$;

iii) $u(t) \in arg \max \sigma_{F_{i_k}}(p_k)$, $t \in (t_{k-1}, t_k)$, *and*

$$u(t) = \frac{x_k - x_{k-1}}{\rho_{F_{i_k}}(x_k - x_{k-1})} \in F_{i_k}, t \in [t_{k-1}, t_k) \; \forall k = 1, ..., N+1.$$

Then $(x(\cdot), u(\cdot))$ *is an optimal pair to the minimal time problem.*

Proof. Let $(x(\cdot), u(\cdot))$ satisfying i), ii) and iii). By Observation 7, b), we have

$$p_k \in \partial \rho_{F_{i_k}}(x_k - x_{k-1}), k = 1, ..., N+1.$$

By Proposition 4.3.6, [28], for all $y \in \overline{\mathcal{M}}_{i_k} \cap \overline{\mathcal{M}}_{i_{k+1}}$, $k = 1, ..., N$,

$$\langle p_k, (y - x_{k-1}) - (x_k - x_{k-1}) \rangle \leq \rho_{F_{i_k}}(y - x_{k-1}) - \rho_{F_{i_k}}(x_k - x_{k-1}). \tag{5.2}$$

The same holds for p_{k+1},

$$\langle p_{k+1}, (x_{k+1} - y) - (x_{k+1} - x_k) \rangle \leq \rho_{F_{i_{k+1}}}(x_{k+1} - y) - \rho_{F_{i_{k+1}}}(x_{k+1} - x_k). \tag{5.3}$$

We can sum the inequalities (5.2) and (5.3) and get

$$\langle -p_k + p_{k+1}, (x_k - y) \rangle \leq \rho_{F_{i_k}}(y - x_{k-1}) + \rho_{F_{i_{k+1}}}(x_{k+1} - y) - (\rho_{F_{i_k}}(x_k - x_{k-1}) + \rho_{F_{i_{k+1}}}(x_{k+1} - x_k)).$$

As $x_k - y \in \mathcal{T}_{\Sigma_k}(x_k)$, by item ii), $\langle -p_k + p_{k+1}, x_k - y \rangle = 0$, then

$$\rho_{F_{i_k}}(x_k - x_{k-1}) + \rho_{F_{i_{k+1}}}(x_{k+1} - x_k) \leq \rho_{F_{i_k}}(y - x_{k-1}) + \rho_{F_{i_{k+1}}}(x_{k+1} - y), k = 1, ..., N.$$

For all $y \in S \cap \overline{\mathcal{M}}_{i_{N+1}}$,

$$\langle p_{N+1}, (y - x_N) - (x_{N+1} - x_N) \rangle \leq \rho_{F_{i_{N+1}}}(y - x_N) - \rho_{F_{i_{N+1}}}(x_{N+1} - x_N). \tag{5.4}$$

As Σ_{N+1} is convex and $-p_{N+1} \in N_{\Sigma_{N+1}}(x_{N+1})$ we have

$$\langle -p_{N+1}, y - x_{N+1} \rangle \leq 0 \implies \langle p_{N+1}, y - x_{N+1} \rangle \geq 0$$

if we substitute the last inequality in the inequality (5.4) we get

$$\rho_{F_{i_{N+1}}}(x_{N+1} - x_N) \leq \rho_{F_{i_{N+1}}}(y - x_N).$$

Therefore, $(x(\cdot), u(\cdot))$ minimizes the minimal time problem. $\square$

Now we are giving an example showing that when we don't give a sequence of manifolds we can't say that a pair satisfying the items i), ii) and iii) of the last theorem is an optimal pair to the minimal time problem.

Example 7. *Consider* $\mathbb{R}^2$ *and define the manifolds like*

$$\mathcal{M}_1 = \{(x,y) \in \mathbb{R}^2 : 0 < x < 20, y = 10\},\ \mathcal{M}_2 = \{(x,y) \in \mathbb{R}^2 : x = 20, -5 < y < 10\},$$

$$\mathcal{M}_3 = \{(x,y) \in \mathbb{R}^2 : 0 < x < 20, y = -5\},\ \mathcal{M}_4 = \{(x,y) \in \mathbb{R}^2 : x = 0, -5 < y < 10\},$$

$$\mathcal{M}_5 = \{(x,y) \in \mathbb{R}^2 : 0 < x < 20, -5 < y < 10\},\ \mathcal{M}_6 = \{(0,10)\},$$

$$\mathcal{M}_7 = \{(20,10)\},\ \mathcal{M}_8 = \{(20,-5)\},\ \mathcal{M}_9 = \{(0,-5)\},$$

we need to define more manifolds to cover $\mathbb{R}^2$ *but they are not necessary here because we can define all the velocities in such manifolds as being* 0.

Let the multifunction be given by

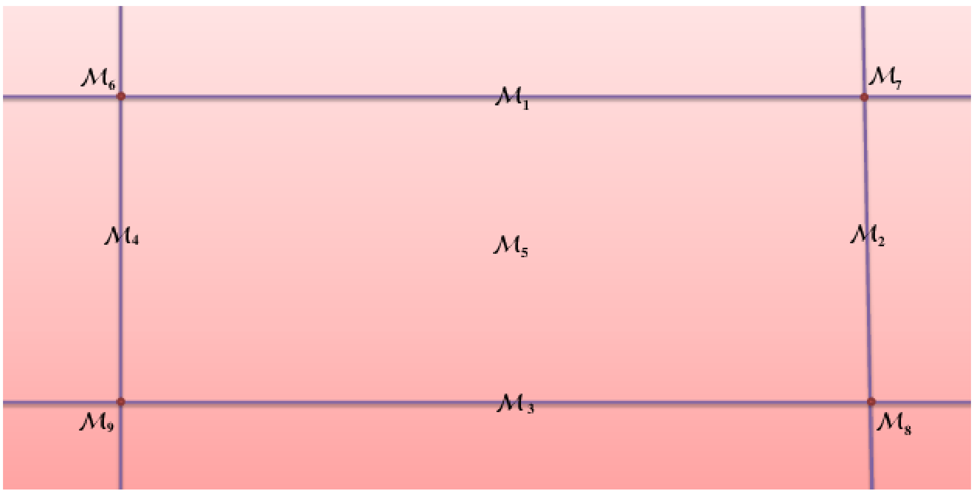
$$F_1 = \{(u,v) : -2 \le u \le 2, v = 0\},\ F_2 = \{(u,v) : u = 0, -5 \le v \le 5\},$$

$$F_3 = \{(u,v) : -10 \le u \le 10, v = 0\},\ F_4 = \{(u,v) : u = 0, -8 \le v \le 8\},$$

$$F_5 = \{(u,v) : u^2 + v^2/4 \le 1\},\ F_6 = \{(0,0)\},$$

$$F_7 = \{(0,0)\},\ F_8 = \{(0,0)\},\ F_9 = \{(0,0)\}.$$

We have the following picture:



Let $(x_0, y_0) = (18, -3)$ *be the initial point and* $(x_f, y_f) = (2, 8)$ *be the target. Consider the minimal time problem with the dynamics described above.*

Define $x : [0, 16.9189] \to \mathbb{R}^2$ *by*

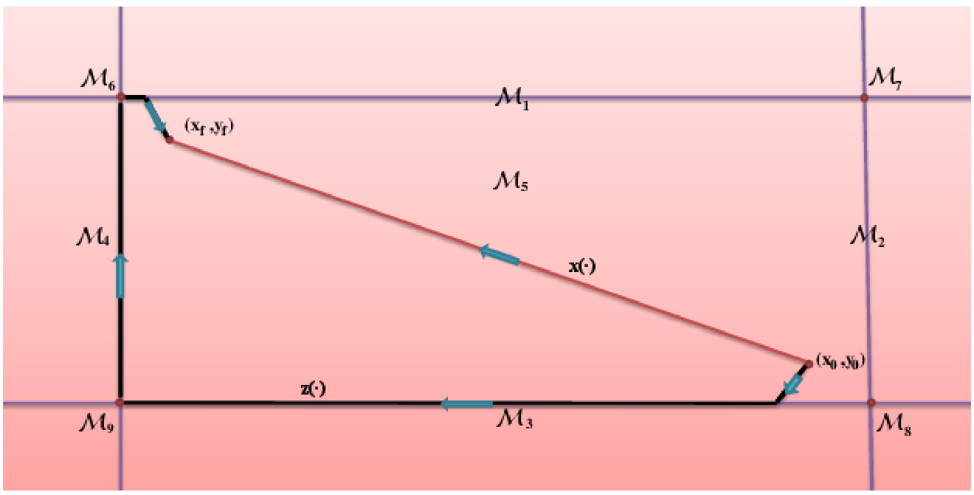
$$x(t) = (x_0, y_0) + tu,$$

where $u = (-0.945687, 0.65016)$. *Note that* $x(\cdot)$ *is a linear trajectory between the initial and final points.*

We can make some calculations and get $p = (-0.945687, 0.16254)$ *so that the items i) and ii) of the last theorem hold. It is easy to show that* u *satisfies iii). But* $(x(\cdot), u(\cdot))$ *is not an optimal pair for the minimal time problem.*
Define $z : [0, 6.53601] \to \mathbb{R}^2$ *by*

$$z(t) = \begin{cases} (x_0, y_0) + tu_1, & t \in [0, 1.00504] \\ (17.8995, -1) + tu_2, & t \in [1.00504, 2.79499] \\ (0, -5) + tu_3, & t \in [2.79535, 4.66999] \\ (0, 10) + tu_4, & t \in [4.66999, 5.38131] \\ (1.42265, 10) + tu_5, & t \in [5.38131, 6.53601], \end{cases}$$

where $u_1 = (-0.1, -1.98997)$, $u_2 = (-10, 0)$, $u_3 = (0, 8)$, $u_4 = (2, 0)$ *and* $u_5 = (0.5, -1.73205)$. *It is easy to show they satisfy iii). We can make some calculations and get* $p_1 = (-0.1, -0.497494)$, $p_2 = (-0.1, 0)$, $p_3 = (0, 0.125)$, $p_4 = (0.5, 0)$ *and* $p_5 = (0.5, -0.433013)$ *satisfying i) and ii). As we can see* $z(\cdot)$ *attains the target in a smaller time than* $x(\cdot)$. *This is happening because* $x(\cdot)$ *and* $z(\cdot)$ *are following different manifolds. Then, if we don't give a sequence of manifolds we can't guarantee that our necessary conditions are also sufficient.*



CHAPTER 6

CONCLUSIONS

The contribution of this work is twofold. First we study an impulsive optimal control problem in which we show we can get approximated problems to the reparametrized problem by Euler's discretization and if the sequence of approximated problems converge, they will converge to a solution of the reparametrized problem. We also show that a subsequence of discrete-time approximated functions graph-converge to a solution. The results obtained in this part of the Thesis come from a mix of ideas from consistent approximations given by [1] and Euler approximation and graph-convergence for impulsive differential inclusion given by [2].

On the other hand we consider the minimal time problem with stratified domains where we get necessary and sufficient condition as well as H-J inequality. We mention that there is another work providing such inequalities. They give H-J inequalities for the Mayer and minimum time problem but the stratification is different from ours. Such work is private communication, and is called "The Mayer and minimum time problems with stratified state constraints" and is given by Hermosilla, C., Wolenski, P. R. and Zidani, H. It has been submitted. It is interesting to note that there are a lot of work that can be done if we consider our type of stratification or nonconstant case. For example, necessary conditions, H-J theory, feedback synthesis, regularity theory, numerics, etc, even in the constant case.

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