The Elvis Problem

Math 4020

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The Elvis Problem

- Timothy Pennings and Elvis
- *Do Dogs Know Calculus?*
- Two separate regions each with a fixed velocity set.
- Basic scenario

Problem : Find optimal time path such that there is a trajectory from X_1 in first region to X_2 in the second region.

The Minimal Time Problem

Convex Analysis

- Advanced techniques using convex velocity sets as opposed to closed ball velocity sets
- Speed changes depending on angle traveled

Velocity Sets

- In the Elvis case the velocity sets are closed balls with constant radius for speed,

$$F = r\overline{\mathbb{B}}$$

- Ellipse velocity sets are described by the following equation,

$$F = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\}$$

- Conditions for general velocity set
  - F is convex
  - F is bounded
  - $0 \in \operatorname{int}(F)$

Reachable Set

Definition

The reachable set $R^{(T)}(X_1)$ consists of all points X_2 for which there exists a trajectory from X_1 to X_2 in time T.

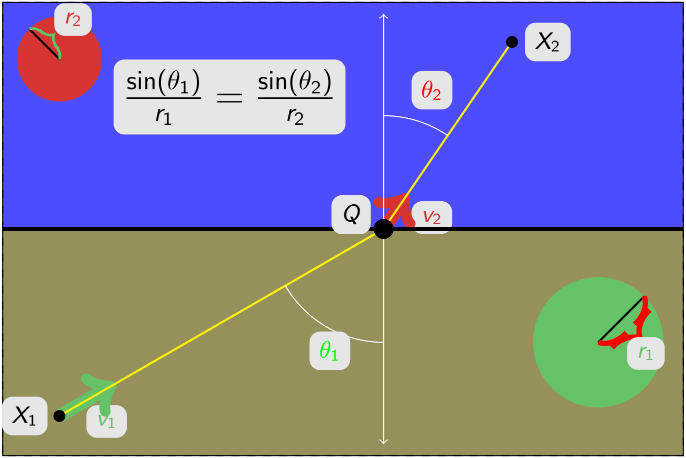
- More general problem - not focused on one single path.

Our Goal

- Find the reachable set given the velocity sets for each region, starting point X_1, and a fixed time.

Snell's Law

- Law that comes from field of optics, describing refraction. Used for regions of constant velocity
- Snell's Law Equation: $\frac{\sin\theta_1}{r_1} = \frac{\sin\theta_2}{r_2}$



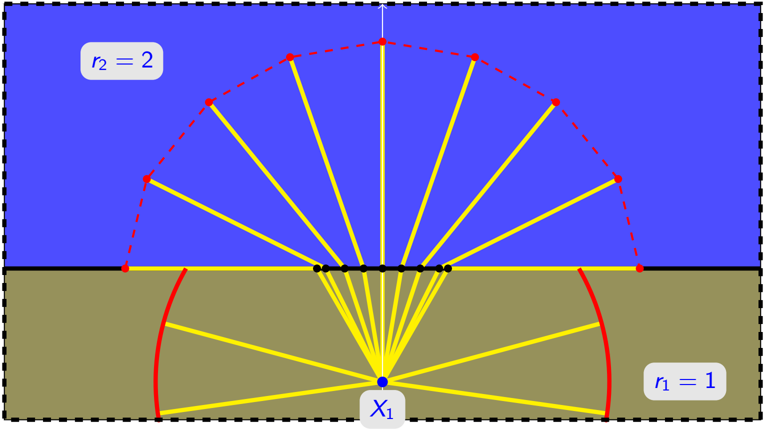


Figure 2: Snell's Law trajectories

Critical Angle

- The critical angle is the angle θ_1 for which θ_2 is 90°.

$$\theta_c = \arcsin\left(\frac{r_1}{r_2}\right)$$

- The critical angle is only defined when $r_1 < r_2$
- Similarly for the ellipse we have:

$$\theta_c = \arcsin\left(\frac{a_1}{a_2}\right) \text{ where } a_1 < a_2$$

Swin Run Swim (SRS) path

- The SRS paths exists when starting in the slower region.
- To find the SRS path
  - Enter the interface using the critical angle.
  - Travel along the interface at the faster velocity.
  - Due to Snell's law when coming back into the 1st region we reflect the critical angle
  - In the GUI we discretize the interval on the interface to give us a representation of the SRS paths.
- Non-optimal points.

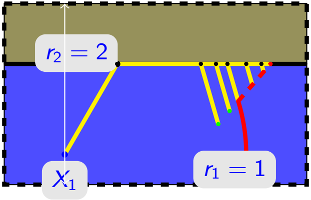


Figure 3: Examples of SRS trajectories

General Problem Formulation

Denote Σ as the interface between the two regions.

- Problem: Find the point $Q \in \Sigma$ so that together the time it takes to travel from X_1 in region 1 to Q and Q to X_2 in region 2 is minimized.

Definition

The gauge function, $\gamma_F : X \to [0, \infty]$ is defined by

$$\gamma_F(x) = \inf\left\{ t \geq 0 : \frac{1}{t}\, x \in F \right\}.$$

Using this terminology, we state the minimal time problem as

$$\min \left\{ \gamma_{F_1}(Q - X_1) + \gamma_{F_1}(X_2 - Q) \right\} \quad \text{over } Q \in \Sigma.$$

Definition

The indicator function $I_S : \mathbb{R}^n \to \overline{\mathbb{R}}$ is defined by

$$I_S(x) = \begin{cases} 0 & \text{if } x \in S \\ +\infty & \text{if } x \notin S. \end{cases}$$

- Rephrased problem

$$\min \left\{ \gamma_{F_1}(Q - X_1) + \gamma_{F_1}(X_2 - Q) + \mathcal{I}_{\Sigma}(Q) \right\} \quad \text{over } Q \in \mathbb{R}^n.$$

Now to solve for Q, consider the subgradient $\partial f(x)$ given by

$$\partial f(x) := \{\xi \in X : f(y) \geq f(x) + \langle \xi, y - x \rangle \; \forall y \in X\}. \tag{1}$$

Then we can say that $Q \in \mathbb{R}^n$ solves the problem iff

$$\mathbf{0} \in \partial \left\{ \gamma_{F_1}((\cdot) - X_1) + \gamma_{F_2}(X_2 - (\cdot)) + \mathcal{I}_\Sigma(\cdot) \right\}(Q). \tag{2}$$

Since $\gamma_{F_1}((\cdot) - X_1), \gamma_{F_2}(X_2 - (\cdot)), \mathcal{I}_\Sigma \in \mathcal{F}$ apply Rockafellar's Theorem,

$$\begin{aligned} \partial \Big\{ \gamma_{F_1}((\cdot) - X_1) \; &+ \; \gamma_{F_2}(X_2 - (\cdot)) \; + \; \mathcal{I}_\Sigma(\cdot) \Big\}(Q) \\ &= \; \partial \gamma_{F_1}(Q - X_1) + \partial \gamma_{F_2}(X_2 - Q) + \partial \mathcal{I}_\Sigma(Q) \end{aligned}$$

Note that For $v \in F$, the normal cone $N_F(v)$ is given by

$$N_F(v) := \{\zeta : \langle \zeta, v' - v \rangle \leq 0 \quad \forall v' \in F\}.$$

This together with the subgradient gives

$$\partial \mathcal{I}_\Sigma(Q) = N_\Sigma(Q) := \{\zeta : \langle \zeta, y - Q \rangle \leq 0, \forall y \in \Sigma\}$$

Finally,

$$\mathbf{0} \in \left\{ \partial\gamma_{F_1}((\cdot) - X_1) + \partial\gamma_{F_2}(X_2 - (\cdot)) + N_\Sigma(\cdot) \right\}(Q)$$

and equivalently there exist two vectors $\zeta_1,\, \zeta_2 \in \mathbb{R}^n$ satisfying

$$\begin{aligned} \zeta_1 &\in \partial\gamma_{F_1}(Q - X_1), && (3) \\ -\zeta_2 &\in \partial\gamma_{F_2}(X_2 - Q), \quad \text{and} && (4) \\ \zeta_1 - \zeta_2 &\in N_\Sigma(Q). && (5) \end{aligned}$$

Back to Snell's Law

$F_i = r_i\overline{\boldsymbol{B}}$ and N_Σ is the y-axis.
Now designate all points that are distance 1 from the origin as

$$\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$$

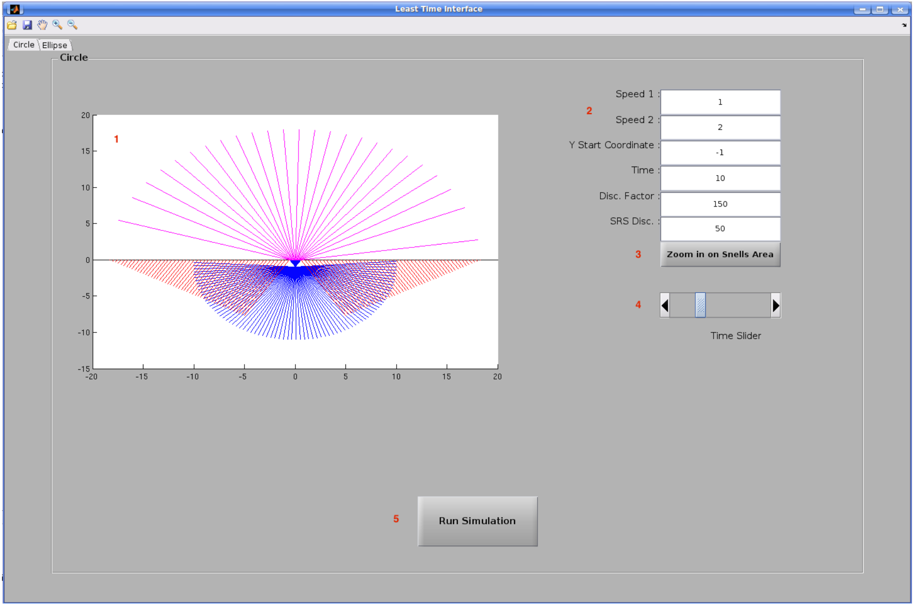
Then

$$\begin{aligned} (3) \quad &\Rightarrow \quad \zeta_1 = \frac{1}{r_1}\begin{pmatrix} \sin\theta_1 \\ \cos\theta_1 \end{pmatrix} \\ (4) \quad &\Rightarrow \quad \zeta_2 = \frac{1}{r_2}\begin{pmatrix} \sin\theta_2 \\ \cos\theta_2 \end{pmatrix}. \end{aligned}$$

Since (5) $\Rightarrow \zeta_1 - \zeta_2 \in$ y-axis we Snell's Law, which is stated in the form

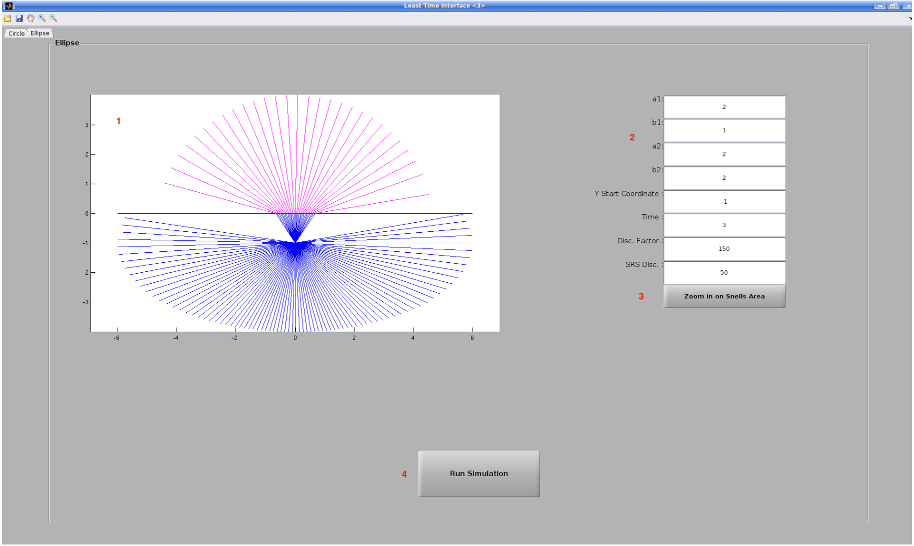
$$\frac{\sin\theta_1}{r_1} = \frac{\sin\theta_2}{r_2}.$$

GUI



Least Time Interface
Circle
Ellipse
Circle
1
20
15
10
5
0
-5
-10
-15
-20
-15
-10
-5
0
5
10
15
20
Speed 1 :
1
2
Speed 2 :
2
Y Start Coordinate :
-1
Time :
10
Disc. Factor :
150
SRS Disc. :
50
3
Zoom in on Snells Area
4
Time Slider
5
Run Simulation

GUI



Least Time Interface <3>
Circle
Ellipse
Ellipse
1
2
a1: 2
b1: 1
a2: 2
b2: 2
Y Start Coordinate : -1
Time : 3
Disc. Factor : 150
SRS Disc. : 50
3
Zoom in on Snells Area
4
Run Simulation

Math 4020 Project Manual

"Do Dogs Know Calculus?"

(no)

Overview

The inspiration for this research project comes from Timothy Pennings and his dog Elvis. In particular, Pennings had a seemingly ridiculous question: Do dogs know calculus? Make no mistake, *it is* a ridiculous question, however this thought experiment proved interesting.

Essentially, our project's genesis comes down to an overeager dog on the beach. When Pennings would throw a tennis ball into the water, he noticed that Elvis did not travel in a straight line from the beach to the ball, rather Elvis entered the water at an angle with respect to his path on the beach. Understanding the canine's propensity for enthusiasm, Pennings conjectured that this was an instinctive response on Elvis's part to minimize the time to reach the ball, as opposed to minimizing the distance. It was fairly obvious that Elvis could run in the sand much faster than

User Manual

- Ultimately chose a more general format
- Introduces the reader to the project and its motivation
- In this respect it serves more as a general documentation of the class
- Goes through the software and how to use it as any manual should

GUI Demo