Exercises on the Elvis Problem - Part I

2 The *polar* of a set

Exercise 2.1. *Show the following.*

(a) For any nonempty set $F \subseteq \mathbb{R}^n$, one has F° belonging to $\mathcal{C}$.

(b) $F \in \mathcal{C}$ is bounded if and only if $\mathbf{0} \in \mathrm{int}(F^\circ)$.

(c) $F \in \mathcal{C}_0$ if and only if $F^\circ \in \mathcal{C}_0$.

(d) If $F = r\overline{\mathbb{B}}$ for some $r > 0$, then $F^\circ = \frac{1}{r}\overline{\mathbb{B}}$.

(a) *Proof.* Assume that F is nonempty. Then for every $v \in F$ we have that $\langle \mathbf{0}, v\rangle = 0 < 1$ so that $\mathbf{0} \in F^\circ$ therefore F° is nonempty. Assume that $\zeta_0, \zeta_1 \in F^\circ$. Let $0 \leq \lambda \leq 1$, so that

$$\langle \lambda\zeta_0 v\rangle \leq 1, \langle \zeta_1, v\rangle \leq 1, \text{ and } \langle \lambda\zeta_1, v\rangle \leq 1.$$

Then

$$\langle \lambda\zeta_0 + (1-\lambda)\zeta_1, v\rangle = \langle \lambda\zeta_0, v\rangle + \langle \zeta_1, v\rangle - \langle \lambda\zeta_1, v\rangle \leq 1.$$

so $\lambda\zeta_0 + (1-\lambda)\zeta_1 \in F^\circ$ and F° is convex. **F° closed** □

(b) *Proof.*
($\Rightarrow$) Let $F \in \mathcal{C}$ be bounded. Then there exists some $m \geq 0$ such that for all $v \in F$, $|v| < m$. Let $\zeta \in \mathbb{R}^n$ such that $|\zeta| < 1/m$, then by the Cauchy Schwarz inequality,

$$\langle \zeta, v\rangle \leq |\zeta||v| \leq \frac{1}{m} m \leq 1.$$

Therefore the open ball centered at the origin with radius $\frac{1}{m}$ is contained in F° meaning that $\mathbf{0} \in \mathrm{int}(F^\circ)$.
($\Leftarrow$) Let $\mathbf{0} \in \mathrm{int}(F^\circ)$ and $\epsilon > 0$. Then there is open ball centered at the origin with radius ϵ contained in F°. Let $\zeta \in \mathbb{B}(0, \epsilon)$. If $v \in F$ with $v \neq 0$, then $\zeta = \epsilon\frac{v}{|v|} \in F^\circ$. Furthermore,

$$\langle \zeta, v\rangle = \frac{\epsilon}{|v|}|v|^2 \leq 1 \Rightarrow |v| \leq \frac{1}{\epsilon}.$$

Thus, F is bounded.

□

(c) *Proof.*
($\Rightarrow$) Let $F \in \mathcal{C}_0$, then $F \in \mathcal{C}$ is bounded and $\mathbf{0} \in \text{int}F$. By part (a), since $F \in \mathcal{C}$ it is nonempty thus $F^\circ \in \mathcal{C}$. By part (b), $F \in \mathcal{C}$ is bounded if and only if $\mathbf{0} \in \text{int}(F^\circ)$.

All that is left to show is that F° is bounded. Assume that it is not. Suppose $\zeta_k \in F^\circ$ with $\zeta_k \to +\infty$. Without lost of generality, let $\frac{\zeta_k}{|\zeta_k|} \to \bar{\zeta}$ so that $|\bar{\zeta}| = 1$. Let $v \in F$. Then

$$\langle \frac{\zeta_k}{|\zeta_k|}, v\rangle = \frac{1}{|\zeta_k|}\langle \zeta_k, v\rangle \leq \frac{1}{|\zeta_k|} \to 0.$$

This implies that $\langle \bar{\zeta}, v\rangle \leq 0$ for every $v \in F$.

But for small $\delta > 0$, $\delta\bar{\zeta} \in F$. Which implies that $\langle \bar{\zeta}, \delta\bar{\zeta}\rangle = \delta > 0$. So F° is bounded.

($\Leftarrow$) Let $F^\circ \in \mathcal{C}_0$, then $F^\circ \in \mathcal{C}$ is bounded and $\mathbf{0} \in \text{int}(F^\circ)$. Then by part (b), $F \in \mathcal{C}$ is bounded. Now all that is left to show is that **$\mathbf{0} \in \text{int}(F)$**.

□

(d) *Proof.* Let $F = r\overline{I\!B}$ and $r > 0$. Consider an $x \in \overline{I\!B}$ so that $|x| \leq 1$. Then $rx \in F$ and $\langle rx, \frac{x}{r}\rangle = |x|^2 \leq 1$. This implies that $\frac{x}{r} \in F^\circ$. So for every $x \in \overline{I\!B}$, $rx \in F$ and $\frac{x}{r} \in F^\circ = \frac{1}{r}\overline{I\!B}$. □

Exercise 2.2. *Suppose $\ell(\cdot)$ is a linear functional. Show $\exists$ a unique $\zeta \in \mathbb{R}^n$ with $\ell(\cdot) = \ell_\zeta(\cdot)$.*

Proof. Let $\ell(\cdot)$ be a linear functional. Then let x_i be a basis vector of $\mathbb{R}^n$. Define $\ell(x_i) = \zeta_i \in \mathbb{R}$. Then for any $x = (a_1, ..., a_n) \in \mathbb{R}^n$ and $\zeta = (\zeta_1, ..., \zeta_n)$,

$$\ell(x) = a_1\zeta_1 + ... + a_n\zeta_n = \langle x, \zeta\rangle = \ell_\zeta(x).$$

□