Exercises on the Elvis Problem - Part I

1 Preliminaries

Exercise 1.1. *Suppose $f : \mathbb{R}^n \to \overline{\mathbb{R}}$.*

(a) The effective domain *of $f(\cdot)$ is defined by* $\mathrm{dom}(f) := \{x : f(x) < \infty\}$. *Show if $f(\cdot) \in \mathcal{F}$, then* $\mathrm{dom}(f)$ *is a convex set.*

(b) If $\mathrm{dom}(f) = \mathbb{R}^n$, *show that $f \in \mathcal{F}$ if and only if*

$$f(x_\lambda) \leq \lambda f(x_0) + (1-\lambda)f(x_1) \tag{1}$$

for all $x_0, x_1 \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1$, and where $x_\lambda := \lambda\, x_0 + (1-\lambda)x_1$.

(c) Define an arithmetic and an order relation on $\overline{\mathbb{R}}$ so that the property that a lsc $f(\cdot)$ is convex is characterized by (1).

(a) *Proof.* Assume that $f \in \mathcal{F}$. Let $0 \leq \lambda \leq 1$ and $x_0, x_1 \in \mathrm{dom}(f)$.

Then since $\mathrm{epi}(f)$ is convex,

$$\begin{aligned} (x_0, f(x_0)), (x_1, f(x_1)) &\in \mathrm{epi}(f) \\ \lambda(x_0, f(x_0)) + (1-\lambda)(x_1, f(x_1) &\in \mathrm{epi}(f) \\ (\lambda x_0 + (1-\lambda)x_1, \lambda f(x_0) + (1-\lambda)f(x_1)) &\in \mathrm{epi}(f) \end{aligned}$$

By definition of $\mathrm{epi}(f)$,

$$f(\lambda x_0 + (1-\lambda)x_1) \leq \lambda f(x_0) + (1-\lambda)f(x_1).$$

Since x_0, $x_1 \in \mathrm{dom}(f)$, $f(x_0) < \infty$ and $f(x_1) < \infty$ so that

$$f(\lambda x_0 + (1-\lambda)x_1) < \infty.$$

Therefore, $\lambda x_0 + (1-\lambda)x_1 \in \mathrm{dom}(f)$ so $\mathrm{dom}(f)$ is a convex set. $\square$

(b) *Proof.*

Let $\mathrm{dom}(f) = \mathbb{R}^n$.

($\Rightarrow$) Assume that $f(\cdot) \in \mathcal{F}$. Let $0 \leq \lambda \leq 1$ and $x_0, x_1 \in \mathbb{R}^n$. Since $\mathrm{dom}(f) = \mathbb{R}^n$, $f(x_0), f(x_1) < \infty$. Then

$$\begin{aligned}(x_0, f(x_0)), (x_1, f(x_1)) &\in \mathrm{epi}(f) \\ (\lambda x_0 + (1-\lambda)x_1, \lambda f(x_0) + (1-\lambda)f(x_1)) &\in \mathrm{epi}(f)\end{aligned}$$

By definition of the $\mathrm{epi}(f)$,

$$f(x_\lambda) = f(\lambda x_0 + (1-\lambda)x_1) \leq \lambda f(x_0) + (1-\lambda)f(x_1).$$

($\Leftarrow$) Assume that $f(x_\lambda) \leq (1-\lambda)f(x_0) + \lambda f(x_1)$. Let $0 \leq \lambda \leq 1$ and $x_0, x_1 \in \mathbb{R}^n$. Since $\mathrm{dom}(f) = \mathbb{R}^n$, $f(x_0), f(x_1) < \infty$. Then $(x_0, f(x_0)), (x_1, f(x_1)) \in \mathrm{epi}(f)$ so that $\mathrm{epi}(f) \neq \emptyset$ meaning $f(\cdot)$ is proper. Using the assumption,

$$f(x_\lambda) \leq \lambda f(x_0) + (1-\lambda)f(x_1)$$

Then $\mathrm{epi}(f)$ is convex since

$$(x_\lambda, \lambda f(x_0) + (1-\lambda)f(x_1)) \in \mathrm{epi}(f).$$

To see that $f(\cdot)$ is lsc consider the sequence $(x_k, f(x_k)) \in \mathrm{epi}(f)$ such that

$$(x_k, f(x_k)) \to (\overline{x}, f(\overline{x})).$$

Since $\overline{x} \in \mathbb{R}^n$ we have that $\overline{x} \in \mathrm{dom}(f)$ so that $f(\overline{x}) < \infty$. Therefore, $(\overline{x}, f(\overline{x})) \in \mathrm{epi}(f)$ so $\mathrm{epi}(f)$ is closed and $f(\cdot)$ is lsc. It follows that $f(\cdot) \in \mathcal{F}$.

□

(c) Rules of arithmetic on $\overline{\mathbb{R}}$ (to be added to the usual arithmetic rules)

$$\begin{aligned}&\alpha + \infty = \infty + \alpha = \infty \text{ for } -\infty < \alpha \leq \infty, \\ &\alpha - \infty = -\infty + \alpha = -\infty \text{ for } -\infty < \alpha \leq \infty, \\ &\alpha\infty = \infty\alpha = \infty, \alpha(-\infty) = (-\infty)\alpha = -\infty \text{ for } 0 \leq \alpha \leq \infty \\ &\alpha\infty = \infty\alpha = -\infty, \alpha(-\infty) = (-\infty)\alpha = \infty \text{ for } -\infty < \alpha < 0 \\ &\infty - \infty = -\infty + \infty = \infty\end{aligned}$$

The order relation on $\overline{\mathbb{R}}$ is $-\infty \leq \alpha \leq \infty$ for every $\alpha \in \overline{\mathbb{R}}$.

Exercise 1.2. *Suppose $S \subseteq \mathbb{R}^n$. Show $I_S(\cdot)$*

(a) is lsc if and only if S is closed;

(b) is a convex function if and only if S is a convex set; and

(c) belongs to $\mathcal{F}$ if and only if S belongs to $\mathcal{C}$.

(a) *Proof.*
($\Rightarrow$) Let $I_S(\cdot)$ be lsc, then $\operatorname{epi}(I_S)$ is closed. Consider the sequence $(x_k, r_k) \in \operatorname{epi}(I_S)$ with $(x_k, r_k) \to (\overline{x}, \overline{r})$. Then $(\overline{x}, \overline{r}) \in \operatorname{epi}(I_S)$ so that $I_S(\overline{x}) \leq \overline{r} \in \mathbb{R}$. Then $I_S(\overline{x}) = 0$ so that $\overline{x} \in S$. By similar reasoning, the sequence $x_k \in S$ and since $x_k \to \overline{x}$, S is closed.

($\Leftarrow$) Let S be closed. Consider the sequences $x_k \in S$ with $x_k \to \overline{x}$ and $r_k \geq 0$ with $r_k \to \overline{r} \geq 0$. Then $\overline{x} \in S$ and $(x_k, r_k), (\overline{x}, \overline{r}) \in \operatorname{epi}(I_S)$ such that $(x_k, r_k) \to (\overline{x}, \overline{r})$. Therefore $\operatorname{epi}(I_S)$ is closed so that I_S is lsc. □

(b) *Proof.*
Recall that $x_\lambda := (1-\lambda)\, x_0 + \lambda\, x_1$.
($\Rightarrow$) Let $I_S(\cdot)$ be convex function, $0 \leq \lambda \leq 1$, and $x_0, x_1 \in S$. Since $I_S(\cdot)$ is convex, $\operatorname{epi}(I_S)$ is convex and $(x_0, r_0), (x_1, r_1) \in \operatorname{epi}(I_S)$ for $r_0, r_1 \geq 0$. So that $(x_\lambda, \lambda r_0 + (1-\lambda) r_1) \in \operatorname{epi}(I_S)$. Then

$$I_S(x_\lambda) \leq \lambda r_0 + (1-\lambda) r_1 \Rightarrow I_S(x_\lambda) = 0 \Rightarrow x_\lambda \in S.$$

So S is a convex set.

($\Leftarrow$) Let $0 \leq \lambda \leq 1$. Assume S is a convex set with $x_0, x_1 \in S$ and $(x_0, r_0), (x_1, r_1) \in \operatorname{epi}(I_S)$. Since S is convex,

$$x_\lambda \in S \Rightarrow I_S(x_\lambda) = 0.$$

Furthermore, $I_S(x_0) = 0$ and $I_S(x_1) = 0$. Then $0 \leq r_0, r_1$ so that $0 \leq \lambda r_0 + (1-\lambda) r_1$. Then we have that

$$I_S(x_\lambda) \leq \lambda r_0 + (1-\lambda) r_1$$

so that $(x_\lambda, \lambda r_0 + (1-\lambda) r_1) \in \operatorname{epi}(I_S)$ meaning $\operatorname{epi}(I_S)$ is convex so that $I_S(\cdot)$ is a convex function. □

(c) *Proof.*
Assume that $S \subseteq \mathbb{R}^n$.
($\Rightarrow$) Let $I_S(\cdot) \in \mathcal{F}$ so that $I_S(\cdot)$ is lsc, convex and proper. In order to show that $S \in \mathcal{C}$, must show that S is closed, convex, and nonempty. By parts (a) and (b), S is closed and convex. Since $I_S(\cdot)$ is proper, $\operatorname{epi}(I_S) \neq \emptyset$ or there exists $(x, r) \in \operatorname{epi}(I_S)$ such that $r \geq I_S(x)$. Then $I_S(x) \neq \infty$ so $x \in S$.

($\Leftarrow$) Let $S \in \mathcal{C}$ so that S is closed, convex, and nonempty. By parts (a) and (b), $I_S(\cdot)$ is lsc and convex. To show that $I_S(\cdot)$ is proper, let $x \in S$. Then for every $r \geq 0$, $I_S(x) = 0 \geq r$ meaning $(x, r) \in \operatorname{epi}(I_S)$. □

Exercise 1.3. *Show that F is closed and convex if and only if*

$$F = \bigcap \Big\{ \mathcal{H}_{\vec{n},r} :\ \vec{n} \in \mathbb{R}^n,\ r \in \mathbb{R} \textit{ are such that } F \subseteq \mathcal{H}_{\vec{n},r} \Big\}$$

Proof.
($\Rightarrow$) Assume that F is closed and convex. Define the following

$$H := \bigcap \Big\{ \mathcal{H}_{\vec{n},r} :\ \vec{n} \in \mathbb{R}^n,\ r \in \mathbb{R} \text{ are such that } F \subseteq \mathcal{H}_{\vec{n},r} \Big\}$$

Let v be arbitrary such that $v \notin F$. Since F is closed, we can find $x = \mathrm{proj}_F(v)$. Then by the Separation Theorem, for each v there exists $\vec{n} \in \mathbb{R}^n$ such that

$$\sup \big\{ \langle v', \vec{n} \rangle :\ v' \in F \big\} < \langle v, \vec{n} \rangle.$$

In particular, $\vec{n} = \frac{v-x}{2}$. Then the half space that separates v from F is

$$\mathcal{H}_{\vec{n},r} := \big\{ w :\ \langle \vec{n}, w \rangle \leq r \big\}.$$

Where $r = \langle \vec{n}, \frac{v+x}{2} \rangle$ so that $F \subseteq \mathcal{H}_{\vec{n},r}$. $\langle \vec{n}, w \rangle = \|\vec{n}\| \|w\| \cos\theta$.) Since $F \subseteq \mathcal{H}_{\vec{n},r}$ for every such $\mathcal{H}_{\vec{n},r}$,

$$F \subseteq H.$$

Furthermore, since above we showed that for every $v \notin F$ we can find a $\mathcal{H}_{\vec{n},r} \not\ni v$ we have $F^c \cap H = \emptyset$. Thus $F = H$.

($\Leftarrow$) Assume that $F = H$. Let $x_0, x_1 \in \mathcal{H}_{\vec{n},r}$ for some $\vec{n} \in \mathbb{R}^n$, $r \in \mathbb{R}$ such that $F \subseteq \mathcal{H}_{\vec{n},r}$. Then $\langle \vec{n}, x_0 \rangle \leq r$ and $\langle \vec{n}, x_1 \rangle \leq r$ so that

$$\begin{aligned}
\langle \vec{n}, \lambda x_0 + (1-\lambda) x_1 \rangle &= \lambda \langle \vec{n}, x_0 \rangle + (1-\lambda) \langle \vec{n}, x_1 \rangle \\
&\leq \lambda r + (1-\lambda) r \\
&= r
\end{aligned}$$

Therefore $\lambda x_0 + (1-\lambda) x_1 \in \mathcal{H}_{\vec{n},r}$ so that each $\mathcal{H}_{\vec{n},r}$ is convex. Since F is the intersection of convex sets, it itself is convex. Similarly since each half space is closed, the intersection of all half spaces is closed. Thus F is both convex and closed.

□