DO DOGS NEED CALCULUS?

Introduction

A few years ago, Tim Pennings [8] wrote in this journal about his dog Elvis. He observed that while fetching a ball, Elvis would optimize his path so that he would reach a thrown ball as quickly as possible. Pennings modeled this behavior mathematically and used differential calculus to predict Elvis's path to the ball.

Recently, two more articles [3] and [9] about Elvis appeared in this journal. These articles illustrate other calculus-based techniques for solving the same problem. The article [10] is a popular account summarizing these three articles.

The point of this note is to show how Elvis's path can be determined without using any methods of calculus. Instead we use elementary methods of algebra. Ultimately, one only needs the algebraic fact that $x^2 \geq 0$ for any real number x. Stated differently, calculus is not needed to minimize a quadratic-based function, since the vertex of a parabola can also be found algebraically. This article emphasizes the elegance of the method, not its practicality.

We don't know if dogs really do know calculus, but we do know that they don't need it for the task of fetching balls—they only need to know a little bit of algebra.

The Problem

Our problem is a slight generalization of the original one, and is illustrated in Figure 1.

The dog starts at point D on the beach when the ball is thrown to point B in the water. The dog runs straight to point P along the shore, then from P, he swims straight to the ball. Given that the dog runs and swims at different speeds r and s respectively, the problem is to find the point P for which the total time to get the ball is smallest. We assume the dog runs faster than he swims, so $r > s$. (The reader might notice the similarity of this problem with Fermat's principle of least time. We return to this later.)

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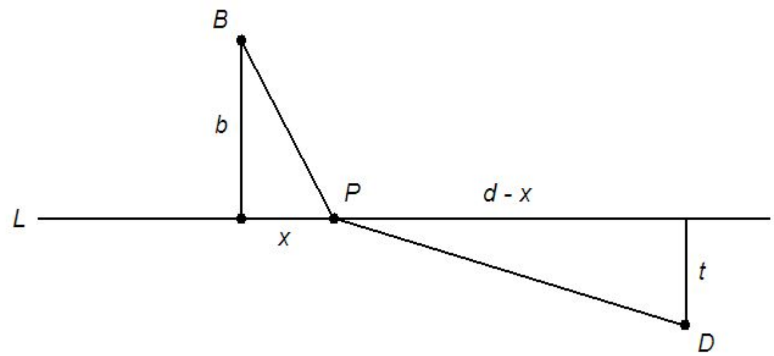

FIGURE 1. The problem as described above. Uppercase letters are points and lines; lowercase letters are lengths of line segments.

We coordinatize the situation as follows. Let L be the x-axis, let $B = (0, b)$, let $D = (d, -t)$, and let $P = (x, 0)$. We assume that b and d are strictly positive (else the problem is easy), and that t is strictly positive, too. After obtaining our solution, we return to the limiting case $t \downarrow 0$; this occurs when D lies on the line L.

As a function of x, the total time it takes for the dog to reach the ball is

$$T(x) = T_r(x) + T_s(x) = \frac{1}{r}\sqrt{(d-x)^2 + t^2} + \frac{1}{s}\sqrt{x^2 + b^2}.$$

Here, $T_r(x)$ and $T_s(x)$ represent the amount of time spent running and swimming, respectively.

Our goal is to identify a unique value x_{min} for which $T(x_{\mathsf{min}}) \leq T(x_{\mathsf{min}} + \epsilon)$ for all ϵ, with equality if and only if $\epsilon = 0$.

LOCATING THE MINIMUM POINT

An easy geometric argument shows that $T(x) > T(d)$ if $x > d$. Similarly, $T(x) > T(0)$ if $x < 0$. It follows that $0 \leq x_{\mathsf{min}} \leq d$. From now on, we assume that $0 \leq x \leq d$.

We proceed to find an *upper* estimate for $T(x + \epsilon)$. In our arguments, it will often be helpful to consider ϵ as the only changing parameter. We will need the following lemma.

Lemma 1. *If* $y \geq -1$, *then* $\sqrt{1+y} \leq 1 + y/2$.

Proof. Both sides are nonnegative, so this can be verified by squaring both sides. $\square$

In our application, notice that $x^2 + b^2 > 0$, and

$$T_s(x+\epsilon) = \frac{\sqrt{x^2+b^2}}{s}\sqrt{1+\frac{\epsilon^2+2\epsilon x}{x^2+b^2}}. \tag{1}$$

So applying Lemma 1 with $y = \dfrac{\epsilon^2+2\epsilon x}{x^2+b^2}$ gives

$$T_s(x+\epsilon) \le \frac{\sqrt{x^2+b^2}}{s}\left(1+\frac{1}{2}\frac{\epsilon^2+2\epsilon x}{x^2+b^2}\right). \tag{2}$$

Similar reasoning shows

$$T_r(x+\epsilon) \le \frac{\sqrt{(d-x)^2+t^2}}{r}\left(1+\frac{1}{2}\frac{\epsilon^2-2\epsilon(d-x)}{(d-x)^2+t^2}\right). \tag{3}$$

Combining estimates (2) and (3) with the stipulation $T(x) \le T(x+\epsilon)$ gives, after simplifying,

$$0 \le \epsilon\left(\frac{1}{r}\frac{-(d-x)}{\sqrt{(d-x)^2+t^2}} + \frac{1}{s}\frac{x}{\sqrt{x^2+b^2}}\right) + \epsilon^2\left(\frac{1}{2r}\frac{1}{\sqrt{(d-x)^2+t^2}} + \frac{1}{2s}\frac{1}{\sqrt{x^2+b^2}}\right). \tag{4}$$

We desire a value of x such that the inequality is true for all values of ϵ. We need another lemma.

Lemma 2. *Let α and β be fixed real numbers with $\alpha > 0$. If $\alpha\epsilon^2 + \beta\epsilon \ge 0$ for all ϵ, then $\beta = 0$.*

Proof. The quadratic expression $\alpha\epsilon^2 + \beta\epsilon$ corresponds to a parabola with vertex at $\epsilon = -\beta/2\alpha$. (The vertex can be found algebraically by completing the square.) Substituting this value gives $-\beta^2/4\alpha \ge 0$. It follows that $\beta = 0$. ☐

Now apply Lemma 2 to the combined inequality (4). Since the coefficient of ϵ^2 is always positive, we conclude that

$$0 = \frac{1}{r}\frac{-(d-x)}{\sqrt{(d-x)^2+t^2}} + \frac{1}{s}\frac{x}{\sqrt{x^2+b^2}}. \tag{$\star$}$$

We have so far established the following result.

Proposition 3. *Let x_{min} be a real number such that $T(x_{\mathsf{min}}) \le T(x_{\mathsf{min}}+\epsilon)$ for all ϵ. Then x_{min} satisfies ($\star$).*

Notice that ($\star$) is the same relationship one finds using the methods of calculus—in particular, it is the equation $T'(x) = 0$.

Verifying the solution

In this section we establish a converse to Proposition 3. The converse says that if x_{min} satisfies $(\star)$, then x_{min} is uniquely a minimum point for $T(x)$.

To do this we will find a *lower* estimate for $T(x+\epsilon)$. We will need the following lemma.

Lemma 4. *If $c > 1/4$, then $1 + y/2 \leq \sqrt{1 + y + cy^2}$ for all y. There is equality if and only if $y = 0$.*

Proof. First notice that $1 + y/2 \leq |1 + y/2|$. Then, by squaring both sides, the inequality $|1 + y/2| \leq \sqrt{1 + y + cy^2}$ can be checked. □

Starting again from (1), we find

$$T_s(x+\epsilon) = \frac{\sqrt{x^2+b^2}}{s}\sqrt{1 + \frac{2\epsilon x}{x^2+b^2} + \frac{\epsilon^2}{x^2+b^2}} \geq \frac{\sqrt{x^2+b^2}}{s}\left(1 + \frac{\epsilon x}{x^2+b^2}\right), \tag{5}$$

where in the second step we have applied Lemma 4 with $y = \dfrac{2\epsilon x}{x^2+b^2}$ and

$$c = \frac{x^2+b^2}{4x^2} > \frac{x^2}{4x^2} = \frac{1}{4}.$$

There is equality in (5) if and only if $y = 0$, and therefore, if and only if $\epsilon = 0$. Similar reasoning shows

$$T_r(x+\epsilon) \geq \frac{\sqrt{(d-x)^2+t^2}}{r}\left(1 - \frac{\epsilon(d-x)}{(d-x)^2+t^2}\right), \tag{6}$$

with equality if and only if $\epsilon = 0$. Combining estimates (5) and (6) with $(\star)$ gives, after simplification,

$$T(x+\epsilon) \;\geq\; T(x) + \epsilon\left(\frac{1}{s}\frac{x}{\sqrt{x^2+b^2}} - \frac{1}{r}\frac{d-x}{\sqrt{(d-x)^2+t^2}}\right) \;=\; T(x),$$

with equality if and only if $\epsilon = 0$.

We have therefore established the following converse to Proposition 3.

Proposition 5. *Let x_{min} be a real number that satisfies $(\star)$. Then $T(x_{\mathsf{min}}) \leq T(x_{\mathsf{min}} + \epsilon)$ for all ϵ, with equality if and only if $\epsilon = 0$.*

At last, we also have our uniqueness result.

Corollary 6. *The function $T(x)$ has a unique minimum point.*

Proof. Suppose that x_0 and x_1 are both minimum points for $T(x)$; in particular, $T(x_0) = T(x_1)$.

Proposition 3 says that x_0 satisfies ($\star$). So we may apply Proposition 5 with $x_{\mathsf{min}} = x_0$. The second part of the proposition, in particular, implies $x_0 = x_1$, since $T(x_0) = T(x_1)$. $\square$

Final Remarks

The complexity of equation ($\star$). In the limiting case $t \downarrow 0$ (i.e., if the dog starts at the water's edge), then ($\star$) simplifies to

$$0 = -\frac{1}{r} + \frac{1}{s}\frac{x}{\sqrt{x^2 + b^2}},$$

which can be easily solved for x to obtain

$$x_{\mathsf{min}} = \frac{bs}{\sqrt{r^2 - s^2}}.$$

Of course this answer is identical to the result obtained in [8].

More generally, ($\star$) admits no easy method of solution because it is equivalent to finding the roots of a polynomial of degree 4. In fact, there are explicit but very unwieldy formulas for finding roots of such polynomials [11]. (We wonder whether dogs know Galois theory? See [1], for example, for the connection.)

Nevertheless, Corollary 6 shows that this polynomial has a unique root, which therefore must be at least a double root.

Snell's Law. Readers familiar with optics may recognize ($\star$) as an algebraic version of Snell's law for the refraction of light as it passes between mediums of differing density. (Usually Snell's law is stated in a simpler trigonometric form.) Recall that light travels at different velocities in two such mediums.

Fermat's principle of least time says that the light will bend in such a way that it follows the path of least time between points. So the problem that is under consideration here is identical to the problem of light refraction.

Ultimately, our argument gives an entirely algebraic proof of Snell's law. This was an important goal for this article. The article [5] provides an additional algebraic verification of Snell's law. Other proofs of Snell's law appear in the literature, too. See, for instance, the books [2], [6], and [12]. Each of those treatments, however, uses some version of calculus or at least relies on limits.

The articles [4] and [7] give elementary geometric proofs of Snell's law. The argument given in [4] was already known to Christiaan Huygens in 1678.

Conclusion. The problem of path optimization illustrates how elementary algebraic considerations can sometimes replace more advanced applications of calculus. We hope that students will recognize the value of pursuing alternate proofs of important results. Such pursuits shed new light on old problems and promote better understanding of the applications and limitations of different mathematical methods.

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