Student Seminar Lecture - Distance Of Graphs From Planarity

Yisrael Haber

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Graph Theory Introduction

A graph is a collection of nodes and edges $G = (V, E)$ such that V is a set of nodes and edges are 2-tuples of distinct elements from the set of nodes. i.e. V is just a set of some elements and E is a set such that

$$E \subseteq V^2 \setminus \{(v, v) \mid v \in V\}$$

The structure of a graph is universal and therefore allows for many applications. These applications are very important, especially nowadays. From electrical networks to the internet and the world wide web, to social media, to electrical circuitry et cetera. The importance of the knowledge of graph theory is therefore quite obvious. The motivations that we use for graph theory also implore us to research the theory of the structure of graphs.

Probability Theory Introduction

In many cases it is useful to use probabilistic models for graph theory, since they tend to be easier to analyze and they can tell us what a "typical" graph looks like, where we have a certain paradigm for typicality we choose. We deal with 2 main models of random graphs (both being finite) and they are as such -

Erdös-Reynii Model

This model is probably the most famous model of random graphs. It takes as input n- the number of vertices, and $0 < p < 1$ a probability constant. First we will need the following definitions

Notation 1. Given $n \in \mathbb{N}$ we define $[n] := \{1, 2, \ldots, n\}$.

Definition 1 (Complete Graph)**.** The complete graph on n vertices is the graph with vertex set $V = [n]$, and edge set being all of the possible edges, formally

$$E := V^2 \setminus \{(v, v) \mid v \in V\}$$

Definition 2 (Graph Percolation)**.** Given a graph, edge/bond percolation on the graph with probability constant p means keeping each edge in our graph with probability p and remove it with probability $1 - p$. This is done independently for each edge.

The model, then, is just applying edge/bond percolation with probability constant p on the complete graph. This model is denoted as $\mathcal{G}(n, p)$. We will discuss only the following type of model - $\mathcal{G}\left(n, \frac{c}{n}\right)$ with $c \geq 1$. Usually this model is considered in certain "regimes", in this case we discuss the sparse non-trivial one.

Random Regular Graph

This model takes as input n- the number of vertices, and $d \in \mathbb{N}$ as a constant. Again we will need initial definitions.

Definition 3 (Graph Regularity)**.** A graph is d-regular if the degree of each node is equal to d, where $d \in \mathbb{N}$. In general we will say a graph is regular if there is a $d \in \mathbb{N}$ such that the graph is d-regular.

Definition 4. We will denote the set of all d-regular graphs with vertex set $V = [n]$ as $F(n, d)$.

The model is then picking a graph uniformly from $F(n, d)$ (each graph has a probability of $\frac{1}{|F(n,d)|}$ of being picked). This model will be denoted as $\mathcal{F}(n, d)$ (it is usually also denoted as $\mathcal{G}(n, d)$ too, we will use different notation to avoid confusion). In this model of course it is much harder to generate the graphs. The way these graphs are usually generated is by using what is called the configuration model. This can be done in the following way:

- Take a vertex set $V = [d] \times [n]$.
- In order to get edges on this set take a random matching. This means we will connect each vertex to only one other vertex in the set. We will connect vertices uniformly and independently.
- We can then get a multi-graph (a graph that isn't restricted to not having multiple edges or self-loops) by merging all vertices in each column to one vertex and connect these meta-vertices if and only if there was a connection between their sub-vertices, with obvious multiplicity of edges.
- Repeat this process until this multi-graph is also a graph (meaning no self-loops or multiple edges). The resulting graph will be a random d-regular graph.

This process is very important for analyzing random regular graphs, and because of it's easy combinatorial structure it can allow us to study the structure of a random regular graph rather simply (at least comparably.)

Planarity

Definition 5 (Planarity)**.** A graph is planar if it can be drawn in the plane such that none of the drawn edges intersect in the plane (of course this can be formalized in the sense of a 2-tuple of functions such that one embeds the vertices in the plane and another generates a curve for each edge that connects the corresponding drawn vertices. The formal definition is not of concern for our research.)

It is much easier to visualize a planar graph and it is probably the way most people imagine graphs. There are many other advantages of a graph being planar - for example it is easier to solve routing problems and so on. Another advantage was discovered by Peter Ungar in the 1950's - the planar separator theorem. The most significant improvement was notably that of Lipton and Tarjan in 1979, it appears in the following form -

Theorem 1 (Improved Planar Separator)**.** Let G be any n-vertex planar graph. The vertices of G can be partitioned into three sets A, B, C such that no edge joins a vertex in A with a vertex in B, neither A nor B contains more than $\frac{2}{3}n$ vertices, and C contains no more than $\sqrt{8}n$ vertices.

Notice that the constants $\frac{1}{3}, \frac{2}{3}$ are strict since we can take a line graph with 4 nodes in which case it has to be these constants. The next natural step is to try and compose these separators over and over until we separate the graph into smaller and smaller sizes. This can be important for many problems, for example graph compression and divide and control algorithms. Using this idea they prove another theorem that inspires the definition of what we will call "shattering" a graph -

Theorem 2 (Shattering Of Planar Graphs)**.** Let G be an n-vertex planar graph and let $k(n) \leq n$ be a natural number dependent on n. Then there is some set C of size at most

$$\frac{\sqrt{24}}{\sqrt{3}-\sqrt{2}} \cdot \frac{n}{\sqrt{k(n)}} \approx 15.4134 \cdot \frac{n}{\sqrt{k(n)}} \left(= O\left(\frac{n}{\sqrt{k(n)}} \right) \right)$$

such that its removal leaves components of size at most $k(n)$.

Notice that if we then take $k(n) \approx n^{\alpha}$ for some $0 < \alpha < 1$, then this theorem says that we can remove a set C of size $O(n^{1-\frac{\alpha}{2}})$ from the graph such that the size of every connected component is of size at most n^{α}. This means that for every planar graph we can remove a (asymptotically) negligible amount of vertices such that all of the connected components are also of (asymptotically) negligible size.

Shattering

What we just saw above will fuel our definition of shattering. Given a graph or model of graphs, our purpose will be to see how many vertices we need to remove asymptotically in order to render the connected components to be of negligible size. In order to do this correctly we need to create a formal set of definitions that will correspond to the intuition.

Definition 6 (Simple Shattering)**.** Given a graph $G = (V, E)$ we say that a subset of nodes $F \subseteq V$ simply m-shatters G for $m \in \mathbb{N}$ if the size of the largest connected component of $G \setminus F$ is at most m. We then say that the graph can be simply (m, l)-shattered for $m, l \in \mathbb{N}$ if there is a subset of nodes of size at most l the simply m-shatters G.

Definition 7 (Series Shattering)**.** Given a countably infinite series of graphs $\{G_n\}_{n\in\mathbb{N}}$, and functions $f, g : \mathbb{N} \longrightarrow \mathbb{N}$ we will say that the series can be (f, g)-shattered if and only if for every $n \in \mathbb{N}$, G_n can be simply $(f(n), g(n))$-shattered. We will use shorthand and say that the series can be g-shattered if and only if the above occurs and additionally

$$\lim_{n\to\infty} \frac{f(n)}{|V(G_n)|} = 0$$

Meaning the size of the largest component will be negligible. Notice that when we are referring to a specific graph we add the phrase "simple", otherwise we are referring to a series of graphs.

Definition 8 (ratio-shattering)**.** We will say that a series of graphs can be c-ratio-shattered for $c \in [0, 1]$ if and only if there is g such that the series can be g-shattered and additionally

$$\lim_{n\to\infty} \frac{g(n)}{|V(G_n)|} = c$$

Meaning that asymptotically there is a subset of nodes of size about cn, and not any significant amount less, whose removal leaves the largest component of negligible size. This in a sense tells us if the order of magnitude of nodes that need to be removed to really disconnect the graph is comparable to that of the size of the original graph (of course we are dealing with series of graphs, but this is true asymptotically). Under this definition of shattering, the previous theorem gives us the following result

Corollary 1. Any series of planar graphs $\{G_i\}_{i\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} |V(G_n)| = \infty$$

can be 0-ratio-shattered.

Proof. For each graph take the subset of nodes F_i implied by the improved planar separator theorem where $k(n) = \lceil V(|G_n|)^{3/4} \rceil$. Then we know that the series can be $(f(n), g(n))$-shattered for $f(n) \approx |V(G_n)|^{3/4}$ and where $g(n) \leq 16 \cdot |V(G_n)|^{5/8}$, these of course satisfy the conditions for 0-ratio-shattering a series of graphs. ■

Comment 1. This also implies that, asymptotically, the ratio of nodes needed to be removed from a graph to make it planar is at least the number of nodes needed to be removed so that the graph is shattered. This is not true in reverse - take for example a collection of separate $\sqrt{n}$ complete graphs. This graph is already shattered yet in order to make it planar we will need to remove almost all nodes.

As it turns out for some basic models though these ratio will be the same. This is a consequence of the following theorem by Janson -

Theorem 3. For both models $\mathcal{G}\left(n, \frac{c}{n}\right)$ with $c > 1$, and $\mathcal{G}(n, d)$ with $d > 3$ constant the following occurs - For every $\epsilon > 0$ there is small enough $\delta > 0$ such that each set T that spans at most δn vertices, spans at most $\left(1 + \frac{\epsilon}{3}\right)|T|$ edges w.h.p.

Using this it can be proved that every shattering of these models leaves every connected component of sublinear size is at most uni-cyclic. Every uni-cyclic graph is planar. And so this means that all together that shattering and making the graph planar are equivalent asymptotically. Additionally it is known that for these models are "hard-to-shatter" which means that they are already hard to make planar. This gives us the following result, which also points to the distance from planarity -

Theorem 4. For every sparse model $\mathcal{G}_n$ that is hard to shatter the number of pairs of edges that cross each other in an embedding is $O(n^\alpha)$ for every $\alpha < 1.5$.

Proof. Add all the crossing points to our graph we now get a new graph. This is therefore equivalent to proving that the number of nodes is $O(n^\alpha)$ for every $\alpha < 1.5$. Assume otherwise, meaning there is $\alpha < 1.5$ such that the number of nodes is $o(n^\alpha)$. This new graph is planar. Using the shattering of planar graphs theorem we can shatter this graph to a size of $o(n^\alpha)^{2/3}$ with $o(n^\alpha)^{2/3}$ nodes. Notice that both these sizes are sublinear (in the original graph size). Now using this set we can construct a corresponding shattering of the original graph in the following way. If the node from the shattering set is in the original graph add it to the new shattering set, otherwise remove the 4 original nodes whose edges intersect at that point. This will shatter the original graph with a sublinear set which is of course in contradiction to the model being hard to shatter. ■

It turns out that the crossing number of the 2 earlier mentioned models is $O(n^2)$, but this is a decent result in and of itself. It is quite possible that this argument can be augmented so that for many models we can shatter with more then linear or allow the sizes of the components to be more than linear yet still produce a non-trivial shattering of such models. But this is quite a straight-forward and strong result as is. (We believe that with other methods you can achieve this crossing number and in a way that is extendable to many other models but that gets into more complicated territory.)

If There Is Time Left

The proof of why for our models the crossing numbers is $\Omega(n^2)$ is due to the following inequality -

Definition 9 (bisection-width)**.** Given a graph G, the bisection width $b(G)$ is defined to be the minimal number of edges that intersect distinctly between any division of the graph to 2 sets of vertices of sizes between $\frac{1}{3}$ and $\frac{2}{3}$.

A known inequality (that uses the improved separator theorem) is that

$$b(G) \leq 10\sqrt{\mathrm{Cr}(g)} + 2\sqrt{\sum_{v \in V} \deg^2(v)}$$

Notice that the second expression is $o(n)$ with high probability and so this means that if we can show that the bisection width is $O(n)$ then the crossing number is at least $O(n^2)$. And so classically it was proven for this models exactly that this bisection width has to be linear ("by hand") and that is how they proved it. We believe that a similar, but more general, approach can show using planarization and shattering methods that this crossing number is as such. This is yet to be fully proven but the extension "seems" quite natural and with some more work will hopefully turn out to be true. Notice that this result of $n^{1.5}$ still has significant place since it doesn't assume anything else about the structure of the graph other than hard shattering.