Recall that a simple random walk on a graph is called *recurrent* if it returns to the starting point with probability 1, and *transient* otherwise.

Exercise 1. *General knowledge*

Let G be a general graph (hence locally finite for the scope of this course: every vertex has finite degree), let v be a vertex of G and $(S_n)_{n\geq 0}$ be a simple random walk starting at v. We denote by $\mathbb{P}_v$ the corresponding probability measure.

1. Explain what a simple random walk on G is.

   Solution. A Markov process which jumps at each time, independently from the past, uniformly to one of its neighbours.

2. Prove that $(S_n)_{n\geq 0}$ is recurrent if and only if

$$\sum_{n=0}^{\infty} \mathbb{P}_v (S_n = v) = \infty.$$

   Solution. We first prove that a simple random walk on G is recurrent if and only if the probability of return to the origin is 1, i.e. if and only if the expected number of returns to the origin is ∞. If τ_v^n is the stopping time for the n^{th} visit at v, we have

$$\mathbb{P}(\tau_v^n < \infty) = \mathbb{P}\left(\tau_\nu^n < \infty | \tau_\nu^{n-1} < \infty\right) \mathbb{P}\left(\tau_\nu^{n-1} < \infty\right) = \mathbb{P}\left(\tau_\nu^1 < \infty\right) \mathbb{P}\left(\tau_\nu^{n-1} < \infty\right),$$

   where we used the Markov property in the last equality, hence by recurrence : $\mathbb{P}(\tau_v^n < \infty) = \mathbb{P}\left(\tau_\nu^1 < \infty\right).^n$ If N_v is the number of visits at v, we also have the identity

$$N_v = \sum_{n=1}^{\infty} \mathbf{1}_{\{\tau_v^n < \infty\}}$$

   where $\mathbf{1}_{\{\tau_v^n < \infty\}} = 1$ if $\tau_v^n < \infty$ holds and 0 if not. Therefore on the one hand we have

$$\mathbb{E}_v(N_v) = \sum_{n=1}^{\infty} \mathbb{P}(N_v \geq n) = \sum_{n=1}^{\infty} \mathbb{P}(\tau_v^n < \infty) = \sum_{n=1}^{\infty} \mathbb{P}\left(\tau_\nu^1 < \infty\right)^n = \frac{\mathbb{P}\left(\tau_\nu^1 < \infty\right)}{1 - \mathbb{P}\left(\tau_\nu^1 < \infty\right)}.$$

   (In a shorter way, one can say that N_v is a geometric variable of parameter $\mathbb{P}\left(\tau_v^1 = \infty\right)$.)

   And on the other hand we have

$$\mathbb{E}(N_v) = \mathbb{E}\left[\sum_{k=0}^{\infty} \mathbf{1}_{\{S_n = v\}}\right] = \sum_{n=0}^{\infty} \mathbb{P}_v(S_n = v).$$

   This allows to conclude that $\sum_{n=0}^{\infty} \mathbb{P}_v(S_n = v) = \infty$ if and only if $\mathbb{P}\left(\tau_v^1 < \infty\right) = 1$.

3. Let us suppose that G is connected and v, w are vertices of G.

   (a) Show that the simple random walk on G is recurrent when started from v if and only if it is recurrent when started from w.

   Solution. Since G is connected (and locally finite), the probability that $(S_n)_{n\geq 0}$ goes to w before going back to v is strictly positive. Suppose E_k is the event that our walk started from v passes through w during the k-th excursion from v before coming back the $(k+1)$-th time. By the Markov property, E_k's are independent and has the same non-zero probability. Therefore, an infinite number of E_k happens: with probability 1 the walk from v passes through w infinitely many times.

   Let us consider the random walk after it goes through w for the first time. Because of the previous discussion, it visits w an infinite number of times, but it has also (by the Markov property) the same law as the random walk which starts at w. So we conclude that the simple random walk on G is recurrent when started from w.

Solution. Other solution : we know that there exists an integer k such that we can go from v to w in k steps. Then for any $n \geq 2k$, if we consider the paths which go from w to v in k steps, then do $n-2k$ steps and come back to v then k steps to come back to w, we get:

$$\mathbb{P}_w\left(S_n = w\right) \geq \mathbb{P}_w\left(S_k = v\right)\mathbb{P}_v\left(S_{n-2k} = v\right)\mathbb{P}_v\left(S_k = w\right)$$

which after summation gives :

$$\sum_n \mathbb{P}_w\left(S_n = w\right) \geq \mathbb{P}_w\left(S_k = v\right)\sum_n \mathbb{P}_v\left(S_{n-2k} = v\right)\mathbb{P}_v\left(S_k = w\right) = \infty$$

which allows to conclude.

(b) Show that if the simple random walk $(S_n)_{n\geq 0}$ on G is recurrent when started from v, then for any vertex w of G, $\mathbb{P}_v\left(\exists n,\, S_n = w\right) = 1$ and $\mathbb{P}_w\left(\exists n,\, \tilde{S_n} = v\right) = 1$ where $(\tilde{S_n})_{n\geq 0}$ is a simple random walk on G starting at w.

Solution. The first assertion was proven in the proof of the previous point 3.(a). For the second, by the result of 3.(a), w is also recurrent, thus we can apply the first assertion and permuting the role of v and w: this gives us the second assertion.

4. Show that a simple random walk on a finite graph is recurrent.

Solution. We can restrict ourself to the case where G is finite and connected. The walk must be somewhere in G at any time n thus $\sum_{v\in G} \mathbb{1}\left(S_n = v\right) = 1$, and therefore

$$\sum_{n\in\mathbb{N}}\sum_{v\in G} \mathbb{1}\left(S_n = v\right) = \infty.$$

Yet, $\sum_{n\in\mathbb{N}}\sum_{v\in G} \mathbb{1}\left(S_n = v\right) = \sum_{v\in G}\sum_{n\in\mathbb{N}} \mathbb{1}_{S_n = v} = \sum_{v\in G} N_v = \infty$, where N_v is the number of visits to v. Since G is finite, there exists w such that $N_w = \infty$ with a positive probability. If we consider the random walk conditioned to visit w (which is possible since we know that $N_w = \infty$ and therefore $\mathbb{P}\left(\exists n, S_n = w\right) > 0$), the law of the walk after the first visit to w is the simple random walk which starts at w. It visits w infinitely many times as the simple random walk is recurrent when started from w and because of the part 3. the simple random walk is recurrent when started from any vertex of G.

5. Show that a simple random walk $(S_n)_{n\geq 0}$ on $\mathbb{Z}^d$ is recurrent if and only if

$$\sum_{n=0}^{\infty} \mathbb{P}_{\vec{0}}\left(S_{2n} = \vec{0}\right) = \infty.$$

Solution. This is due to the fact that for any integer n, $\sum_{i=0}^{d} S_n^i$ has the same parity as n. Thus $S_k = \vec{0}$ can be true only if $k \in 2\mathbb{N}$. One concludes using the point 2.

Exercise 2. *Universality of the recurrence for random walks on* $\mathbb{Z}$

Consider a random walk on $\mathbb{Z}$ defined using identically independent jumps : $S_n = Z_1 + \cdots + Z_n$ (Z_i are i.i.d. $\mathbb{Z}$-valued random variables). Let us suppose that Z_1 satisfies $\mathbb{E}\left(|Z_1|\right) < \infty$.

1. Prove that if $\mathbb{E}\left(Z_1\right) \neq 0$ then S_n is transient.
2. What is the derivative of $\phi(t) = \mathbb{E}\left(e^{itZ_1}\right)$ at 0 ? Give the Taylor expansion of $\phi(t)$ at 0 at order 1.
3. Using the previous point, prove that if Z_1 is symmetric ($-Z_1$ has the same law as Z_1) then S_n is recurrent. *Hint: use the derivation using the Fourier transform as seen in the lesson.*

Solution.

1. Without loss of generality, suppose $\mathbb{E}\left(Z_1\right) = \mu > 0$. By the law of large numbers $\frac{S_n}{n} \overset{a.s.}{\to} \mathbb{E}\left(Z_1\right)$. Therefore, $\forall \epsilon > 0$,

$$\mathbb{P}\left(\exists N > 0, \forall n > N, S_n > n(\mu - \epsilon)\right) \geq \mathbb{P}\left(\exists N > 0, \forall n > N, \left|\frac{S_n}{n} - \mu\right| < \epsilon\right) = 1$$

Therefore, taking $\epsilon = \frac{\mu}{2}$ gives $\mathbb{P}\left(\exists N > 0, \forall n > N, S_n > \frac{n\mu}{2}\right) = 1$. Hence S_n almost surely only visits any integer finitely many times.

2. Using the derivation under the integral (which holds because $\mathbb{E}(|Z_1|) < \infty$) we have $\phi'(0) = i\mathbb{E}(Z_1)$ and so the Taylor expansion is
$$\phi(x) = 1 + ix\mathbb{E}(Z_1) + o(x).$$

3. For this question, we refer to the lesson for the whole solution. Let P_1 be the law of the jumps of S_n (thus the law of Z_1). Using the lesson, we know that

   (a) in order to show that S_n is recurrent, we prove that $\sum_k r^k \mathbb{P}(X_k = 0) \underset{r\to 1}{\longrightarrow} \infty$,

   (b) $\sum_k r^k \mathbb{P}(X_k = 0) = \sum_k r^k \mathcal{F}^{-1}(\mathcal{F}(P_k))(0)$, where $P_k(\cdot) = \mathbb{P}(S_k = \cdot)$, and $\mathcal{F}$ and $\mathcal{F}^{-1}$ are respectively the Fourier and the inverse Fourier transform,

   (c) $\hat{P}_k = \hat{P_1^{\star k}} = \left(\hat{P}_1\right)^k$ where $\hat{f} = \mathcal{F}(f)$,

   (d) $\hat{P}_1(\xi) = \mathcal{F}(\mathbb{P}(Z_1 = \cdot))(\xi) = \sum_n e^{in\xi}\mathbb{P}(Z_1 = n) = \mathbb{E}\left(e^{iZ_1\xi}\right)$

   (e) thus $\sum_k r^k \mathbb{P}(X_k = 0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_k r^k \mathbb{E}\left(e^{i\xi Z_1}\right)^k = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{1 - r\mathbb{E}\left(e^{i\xi Z_1}\right)} d\xi$.

   (f) As $\sum_k r^k \mathbb{P}(X_k = 0)$ is real we automatically have $\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{1 - r\mathbb{E}\left(e^{i\xi Z_1}\right)} d\xi \in \mathbb{R}$. Furthermore, as Z_1 is symmetric, the integrand $\mathbb{E}\left(e^{i\xi Z_1}\right) \in \mathbb{R}$. Therefore the dampening factor r does imply convergence, justifying the previous swapping of the integral and sum.

   (g) The only possibility for the divergence of this integral (as $r \to 1$) is when $\mathbb{E}\left(e^{i\xi Z_1}\right) = 1$, that is when $\xi = 0$. Actually, we only need to understand the nature of the integral :
   $$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{1 - \mathbb{E}\left(e^{i\xi Z_1}\right)} d\xi.$$
   Let us remark that $\mathbb{E}\left(e^{i\xi Z_1}\right) \leq 1$ thus the integrand is positive : we only need to consider the integral
   $$\frac{1}{2\pi}\int_{0}^{\pi} \frac{1}{1 - \mathbb{E}\left(e^{i\xi Z_1}\right)} d\xi.$$
   Using Point 2 of this exercise, this integral is of the form
   $$\frac{1}{2\pi}\int_{0}^{\pi} \frac{1}{1 - 1 - i\xi E(Z_1) + o(\xi)} d\xi = \frac{1}{2\pi}\int_{0}^{\pi} \frac{1}{O(\xi)} d\xi$$
   where we recall that the integrand is positive. This allows us to conclude : there exist $C > 0$ and $\epsilon > 0$ such that for any $\xi \leq \epsilon$, $0 \leq O(\xi) \leq C\xi$ thus $\frac{1}{O(\xi)} \geq \frac{1}{C\xi}$ on $[0, \epsilon]$ and thus
   $$\frac{1}{2\pi}\int_{0}^{\pi} \frac{1}{O(\xi)} d\xi \geq \frac{1}{2\pi}\int_{0}^{\epsilon} \frac{1}{\xi} = \infty.$$