**Exercise 1.** *Introduction*

**Binomial coefficients**

1. Let $k$,$n$ be non-negative integers. Give three definitions of $\left(\begin{array}{c} n \\ k \end{array}\right)$: an algebraic one, a combinatorial one, and its value.

   **Solution.** The three definitions are

   (a) The coefficient in front of $x^k$ in $(1+x)^n$ (or the coefficient in front of $a^k b^{n-k}$ in $(a+b)^n$).
   (b) The number of ways to choose $k$ elements in a set of $n$ elements.
   (c) Is equal to
   $$\frac{n!}{k!\,(n-k)!}$$

2. Prove that $\left(\begin{array}{c} n \\ k \end{array}\right) = \left(\begin{array}{c} n \\ n-k \end{array}\right)$.

   **Solution.** We give three solutions

   (a) Since $(a+b)^n = (b+a)^n$ , $\left(\begin{array}{c} n \\ k \end{array}\right)$ is the coefficient in front of $a^k b^{n-k}$ in $(b+a)^n$ so it is the coefficient in front of $b^{n-k} a^k$ in $(b+a)^n$ so it is equal to
   $$\left(\begin{array}{c} n \\ n-k \end{array}\right).$$
   (b) To choose $k$ elements out of $n$ is equivalent to discard $n-k$ out of $n$.
   (c) We have $\left(\begin{array}{c} n \\ n-k \end{array}\right) = \frac{n!}{(n-k)!(n-(n-k))!} = \frac{n!}{k!(n-k)!} = \left(\begin{array}{c} n \\ k \end{array}\right)$.

3. Show that
   $$\left(\begin{array}{c} n \\ k \end{array}\right) + \left(\begin{array}{c} n \\ k+1 \end{array}\right) = \left(\begin{array}{c} n+1 \\ k+1 \end{array}\right).$$

   **Solution.** We give three solutions

   (a) $\left(\begin{array}{c} n+1 \\ k+1 \end{array}\right)$ is the coefficient in front of $x^{k+1}$ in $(1+x)^{n+1}$. Yet, $(1+x)^{n+1} = (1+x)(1+x)^n$ $=(1+x)^n + x(1+x)^n$. Thus $\left(\begin{array}{c} n+1 \\ k+1 \end{array}\right)$ is the sum of the coefficient in front of $x^{k+1}$ in $(1+x)^n$ and the coefficient in front of $x^k$ in $(1+x)^n$.
   (b) Let us consider the integers $\{1, \cdots, n+1\}$. In order to choose $k+1$ elements in $\{1, \cdots, n+1\}$, either one choose $n+1$ and then we need to choose $k$ elements in $\{1, \cdots, n\}$ or one discards $n+1$ and then we need to choose $k+1$ elements in $\{1, \cdots, n\}$.
   (c) We have
   $$\begin{aligned} \left(\begin{array}{c} n \\ k \end{array}\right) + \left(\begin{array}{c} n \\ k+1 \end{array}\right) &= \frac{n!}{k!\,(n-k)!} + \frac{n!}{(k+1)!\,(n-k-1)!} \\ &= \frac{n!}{(k+1)!\,(n-k)!}(k+1+n-k) \\ &= \frac{(n+1)!}{(k+1)!\,(n-k)!} = \left(\begin{array}{c} n+1 \\ k+1 \end{array}\right). \end{aligned}$$

4. What is the value of $\sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right)$ ?

   **Solution.** We give two solutions

(a) We have $\sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix} = \sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix} 1^k 1^{n-k} = (1+1)^n = 2^n$.

(b) $\sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}$ counts the number of subsets in a set with $n$ elements. One has to choose if each element is included or not, thus there are 2 possibilities per element of the set: $\sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix} = 2^n$.

5. Prove that

$$\sum_{\substack{k_1+k_2=k \\ k_1,k_2\geq 0}} \begin{pmatrix} n_1 \\ k_1 \end{pmatrix}\begin{pmatrix} n_2 \\ k_2 \end{pmatrix} = \begin{pmatrix} n_1+n_2 \\ k \end{pmatrix}$$

**Solution.** We give two solutions

(a) $\sum_{\substack{k_1+k_2=k \\ k_1,k_2\geq 0}} \begin{pmatrix} n_1 \\ k_1 \end{pmatrix}\begin{pmatrix} n_2 \\ k_2 \end{pmatrix}$ is the coefficient of $x^k$ in $(1+x)^{n_1}(1+x)^{n_2} = (1+x)^{n_1+n_2}$ , so it is equal to $\begin{pmatrix} n_1+n_2 \\ k \end{pmatrix}$.

(b) Let us consider the integers $\{1,\cdots,n_1,\cdots,n_1+n_2\}$. In order to choose $k$ elements in $\{1,\cdots,n_1,\cdots,n_1+n_2\}$, one needs to choose $k_1$, the number of elements to take from $\{1,\cdots,n_1\}$ and $k_2$ the number of elements to take from $\{n_1,\cdots,n_1+n_2\}$ (and of course $k_1+k_2=k$) and then chose $k_1$, elements in $\{1,\cdots,n_1\}$ ($\begin{pmatrix} n_1 \\ k_1 \end{pmatrix}$ possibilities)and $k_2$ elements in $\{n_1,\cdots,n_1+n_2\}$ ($\begin{pmatrix} n_2 \\ k_2 \end{pmatrix}$ possibilities). This gives us the equality $\sum_{\substack{k_1+k_2=k \\ k_1,k_2\geq 0}} \begin{pmatrix} n_1 \\ k_1 \end{pmatrix}\begin{pmatrix} n_2 \\ k_2 \end{pmatrix} = \begin{pmatrix} n_1+n_2 \\ k \end{pmatrix}$.

### Stirling approximation

1. Recall the Stirling approximation.

   **Solution.** Stirling's formula is $n! = \sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n}\left(1+O\left(n^{-1}\right)\right)$.

2. Show that

$$\frac{1}{2^{2n}}\begin{pmatrix} 2n \\ n \end{pmatrix} \sim \frac{1}{\sqrt{\pi n}},$$

   as $n\to\infty$.

   **Solution.** This is a simple computation.

### Probabilities

1. Let $A,B\subset(\Omega,\mathcal{A},\mathbb{P})$ , be two events. What does it means that they are independent ?

   **Solution.** It means that $\mathbb{P}(A\cap B)=\mathbb{P}(A)\mathbb{P}(B)$.

2. What is the definition of the conditional probability $\mathbb{P}(A|B)$? What is the value of $\mathbb{P}(A|B)$ if $A$ and $B$ are independent ?

   **Solution.** We have $\mathbb{P}(A|B)=\frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)}$ which is equal to $\mathbb{P}(A)$ if $A$ and $B$ are independent.

3. Let $X$ be a non-negative random variable. State and prove the Markov inequality.

   **Solution.** The Markov inequality is the fact that for any $a\geq 0$,

$$\mathbb{P}(X\geq a)\leq\frac{\mathbb{E}(X)}{a}.$$

   The proof goes as follows: $a\mathbb{1}_{X\geq a}\leq X$ since $X$ is non-negative and computing the expectation, one gets the inequality.

4. Give the definition of a (discrete time) Markov process.

   **Solution.** A random process $(X_n)_{n\in\mathbb{N}}$ such that for any $n\in\mathbb{N}$, any $m\in\mathbb{N}$, any $(x_1,\cdots,x_{n+m})$,

$$\mathbb{P}\left((X_{n+1}=x_{n+1},\cdots,X_{n+m}=x_{n+m})\,|\,(X_1,\cdots,X_n)=(x_1,\cdots,x_n)\right)=\mathbb{P}\left((X_{n+1}=x_{n+1},\cdots,X_{n+m}=x_{n+m})\,|X_n=x$$

5. Let G be a general graph, explain what a simple random walk on $G$ is.

   **Solution.** A Markov process which jumps at each time, independently from the past, uniformly to one of its neighbours.

Recall that a simple random walk on a graph is called *recurrent* if it returns to the starting point with probability 1, and *transient* otherwise. Recall that a simple random walk $(S_n)_{n\geq 0}$ on a connected graph $G$, starting from $v \in G$, is recurrent if and only if

$$\sum_{n=0}^{\infty} \mathbb{P}\left(S_n = v\right) = \infty. \tag{0.1}$$

*Remark.* In the course you saw that a simple random walk $(S_n)_{n\geq\infty}$ is recurrent if and only if $\mathbb{E}\left[N_d\right] = \infty$ where $N_d$ is the number of visits at the starting point $v$. The relation with the statement above is obtained using the relation $N_d = \sum_{n=0}^{\infty} \mathbf{1}_{\{S_n=v\}}$, and using the linearity of the expectation:

$$\mathbb{E}[N_d] = \sum_{n=0}^{\infty} \mathbb{E}[\mathbf{1}_{\{S_n=v\}}] = \sum_{n=0}^{\infty} \mathbb{P}\left(S_n = v\right).$$

**Exercise 2.** *Recurrence/transience theorem for simple random walks on the square lattice* $\mathbb{Z}^d$*,* $d \geq 1$*.*

Let $\left(S_n^{(d)}\right)_{n\geq 0}$ be the simple random walk on $\mathbb{Z}^d$ such that $S_0^{(d)} = 0$.

**1.** $d = 1$ Use Stirling's formula[1] to show that, in one dimension,

$$\mathbb{P}\left(S_{2n}^{(1)} = 0\right) \sim \frac{1}{\sqrt{\pi n}}.$$

Deduce that $(S_n^{(1)})_{n\geq 0}$ is recurrent.

**Solution.** In order for the simple random walk on $\mathbb{Z}$ to come back to 0 in $2n$ steps, it must make $n$ positive steps and $n$ negative steps. Thus among the $2^n$ possible walks (at each step, the walk has 2 choices), the number of walks coming back to the origin in $2n$ steps is equal to the number of ways to choose $n$ positive steps in the $2n$ total steps. So

$$\mathbb{P}\left(S_{2n}^{(1)} = 0\right) = \frac{1}{2^{2n}}\binom{2n}{n} = \frac{(2n)!}{(n!)^2\, 2^{2n}}.$$

Using Stirling's formula on the factorials immediately gives the result. The recurrence property follows from (0.1).

**2.** $d = 2$ The goal is to prove that the simple random walk on $\mathbb{Z}^2$ is recurrent.

1. By enumerating the different cases, show that

$$\mathbb{P}\left(S_{2n}^{(2)} = 0\right) = \left(\frac{1}{2^{2n}}\binom{2n}{n}\right)^2. \tag{0.2}$$

**Solution.** Among the $4^{2n}$ possible walks (4 choices at each step), the number of walks coming back to the origin in $2n$ steps can be obtained by

$$\begin{cases} \text{choosing } 2j \text{ steps in the } x \text{ direction} & \binom{2n}{2j} \\ \text{choosing } j \text{ positive steps among these } 2j \text{ steps} & \binom{2j}{j} \\ \text{choosing } n-j \text{ positive steps among the } 2(n-j) & \binom{2(n-j)}{n-j} \\ \qquad\text{remaining steps (in } y \text{ direction)} & \end{cases}$$

[1]Stirling's formula is $n! = \sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n}\left(1 + O\left(n^{-1}\right)\right)$.

Thus, the number of such walks is equal to

$$\begin{aligned}\sum_{j=0}^{n}\left(\begin{array}{c}2n\\2j\end{array}\right)\left(\begin{array}{c}2j\\j\end{array}\right)\left(\begin{array}{c}2(n-j)\\n-j\end{array}\right) &= \sum_{j=0}^{n}\frac{(2n)!}{(2j)!\,(2n-2j)!}\frac{(2j)!}{j!j!}\frac{(2(n-j))!}{(n-j)!\,(n-j)!}\\ &= \frac{(2n)!}{n!n!}\sum_{j=0}^{n}\frac{n!n!}{j!j!\,(n-j)!\,(n-j)!}\\ &= \frac{(2n)!}{n!n!}\sum_{j=0}^{n}\left(\begin{array}{c}n\\j\end{array}\right)\left(\begin{array}{c}n\\n-j\end{array}\right)\end{aligned}$$

The last sum is equal to $\frac{(2n)!}{n!n!}$ . Indeed either one can see that this is the coefficient of $x^n$ in $(1+x)^n(1+x)^n$ which is equal also to the coefficient of $x^n$ in $(1+x)^{2n}$, or one can use the following combinatorial proof: in order to pick $n$ elements in a bag of $2n$ elements, I split the bag in 2 smaller bags of equal size $n$ (in an arbitrary way) and then I pick $j$ elements in the first bag and $n-j$ in the second bag.

Thus $\mathbb{P}\left(S_{2n}^{(2)}=0\right)=\left(\frac{1}{2^{2n}}\frac{(2n)!}{n!n!}\right)^2$, which is what we needed to prove.

1. Observe that $\mathbb{P}\left(S_{2n}^{(2)}=0\right)$ is equal to $\mathbb{P}\left(S_{2n}^{(1)}=0\right)^2$. Find a probabilistic proof of Equation (0.2).

   **Solution.** The intuition behind the equality

   $$\mathbb{P}\left(S_{2n}^{(2)}=0\right)=\mathbb{P}\left(S_{2n}^{(1)}=0\right)^2$$

   is that one can represent $(S_n^{(2)})_{n\geq 0}$ using two independent uni-dimensional random walks. Beginning at the origin, suppose at every step we do SRW in the $x$ and $y$ directions independently. Then we will move diagonally in $\mathbb{Z}^2$, and the resulting law of the walk in the rotated diagonal lattice is precisely that of a 2 dimensional simple random walk. Then we return to the origin in $2n$ steps if and only if the independent 1 dimensional SRWs both come back to zero, so we get the square of the one dimensional estimate.

   Equivalently, one could have considered the projection of $S_k=(X_k,Y_k)$ on the $x$ and $y$ axis. The projection $X_k$ is clearly not a simple random walk since it stays sometimes at the same place. Yet $X_k+Y_k$ and $X_k-Y_k$ are two processes which always either increase or decrease by 1. Besides $\{S_k=0\}=\{X_k+Y_k=0 \text{ and } X_k-Y_k=0\}$. At last, it is easy to see that $X_k+Y_k$ and $X_k-Y_k$ are two independent random walks (consider the way $(X_k+Y_k, X_k-Y_k)$ moves from time $k$ to time $k+1$).

2. Deduce from Equation (0.2) that $(S_n^{(2)})_{n\geq 0}$ is recurrent.

   **Solution.** From the part 1. we deduce that $\mathbb{P}\left(S_{2n}^{(2)}=0\right)\sim\frac{1}{\pi n}$. The recurrence property follows from (0.2).

**3.** $d=3$ By a simple enumeration argument, show that

$$\mathbb{P}\left(S_{2n}^{(3)}=0\right)=\frac{1}{2^{2n}}\left(\begin{array}{c}2n\\n\end{array}\right)\sum_{\substack{j,k\geq 0\\ j+k\leq n}}\left(\frac{n!}{3^n k!j!\,(n-k-j)!}\right)^2$$

and deduce that a simple random walk on $\mathbb{Z}^3$ is transient.

**Solution.** There exists $\frac{1}{6^{2n}}$ different paths that the random walk can follow during the first $2n$ steps (it has 3 choices at each step). We need to compute the number of paths of length $2n$ in $\mathbb{Z}^3$ which begin and come back to 0. We need to choose :

$$\begin{cases} 2k \text{ times (among the } 2n\text{) at which the path will go in the } x \text{ direction} & \binom{2n}{2k} \\ 2j \text{ times (among the } 2n-2k \text{ left) at which the path will go in the } y \text{ direction} & \binom{2n-2k}{2j} \\ k \text{ times (among the } 2k\text{) at which the path will go "up" in the } x \text{ direction} & \binom{2k}{k} \\ j \text{ times (among the } 2j\text{) at which the path will go "up" in the } y \text{ direction} & \binom{2j}{j} \\ n-k-j \text{ times (among the } 2(n-k-j)\text{) at which the path will go "up"} & \binom{2n-2k-2j}{n-k-j} \\ \qquad \text{in the } z \text{ direction} & \end{cases}$$

This gives a number of paths equal to:

$$\sum_{\substack{j,k\geq 0 \\ j/k\leq n}} \binom{2n}{2k}\binom{2n-2k}{2j}\binom{2k}{k}\binom{2j}{j}\binom{2n-2k-2j}{n-k-j},$$

which after a little massage gives

$$\binom{2n}{n}\sum_{\substack{j,k\geq 0 \\ j/k\leq n}}\left(\frac{n!}{k!j!\,(n-k-j)!}\right)^2,$$

and thus

$$\mathbb{P}\left(S_{2n}^{(3)}=0\right)=\frac{1}{6^{2n}}\binom{2n}{n}\sum_{\substack{j,k\geq 0 \\ j/k\leq n}}\left(\frac{n!}{k!j!\,(n-k-j)!}\right)^2=\frac{1}{2^{2n}}\binom{2n}{n}\sum_{\substack{j,k\geq 0 \\ j/k\leq n}}\left(\frac{n!}{3^n k!j!\,(n-k-j)!}\right)^2.$$

For the assertion about the transience of the random walk, we need to show that $\sum_n \mathbb{P}\left(S_{2n}^{(3)}=0\right)<\infty$: we need to give an upper bound on $\mathbb{P}\left(S_{2n}^{(3)}=0\right)$ which is summable. The first part $\frac{1}{2^{2n}}\binom{2n}{n}$ was already studied it is $O\left(\frac{1}{\sqrt{n}}\right)$. It remains to bound $\sum_{\substack{j,k\geq 0 \\ j/k\leq n}}\left(\frac{n!}{3^n k!j!(n-k-j)!}\right)^2$ and in particular $\frac{n!}{k!j!(n-k-j)!}$. Let us remark that if $a<b$ then $a!b!\geq (a+1)!(b-1)!$ since it is equivalent to $b\geq a+1$. Thus $a!b!c!$ decreases when the distance between any two of $a;b;c$ decreases. We conclude that $\frac{n!}{k!j!(n-k-j)!}$ is maximized among the cases where $j$,$k$, $n-j-k$ are of order $n/3$. Thus $\frac{n!}{3^n k!j!(n-k-j)!}\leq O\left(3^{-n}\frac{n!}{(\lfloor n/3\rfloor !)^3}\right)=O\left(n^{-1}\right)$ using Stirling formula.

Now :

$$\mathbb{P}\left(S_{2n}^{(3)}=0\right)\leq \frac{c}{n^{3/2}}\sum_{\substack{j,k\geq 0 \\ j+k\leq n}}\frac{n!}{3^n k!j!\,(n-k-j)!}=\frac{c}{n^{3/2}}$$

since the sum of the multinomial coefficients is precisely $3^n$. This allows us to conclude about the transience of the random walk in dimension 3.

*Remark.* Let us remark that the brutal majoration which would consist in majoring $\left(\frac{n!}{3^n k!j!(n-k-j)!}\right)^2$ by $O\left(n^{-2}\right)$ and the sum by the number of elements (of order $(\mathrm{O}(n^2))$ ) times $O\left(n^{-2}\right)$ would have given us a majoration $\mathbb{P}\left(S_{2n}^{(3)}=0\right)\leq \frac{c}{n^{1/2}}$ and would have not helped us.

**4.** $d\geq 3$ Prove that it follows from the previous results that $\mathbb{Z}^d$ is transient for $d>3$.

**Solution.** Given an SRW on $\mathbb{Z}^d$ for $d>3$, consider its projection $S_n$ to the first three coordinates. This has a law of a Markov random walk on $\mathbb{Z}^3$ started at the origin which at every step can move to one of its 6 neighbours with probability $\frac{1}{2d}$, or stay at the same point with probability $1-\frac{6}{2d}$. But one obtains a SRW in $\mathbb{Z}^3$ by disregarding the steps where the first three coordinates do not move. So our SRW on $\mathbb{Z}^d$ does not return to zero in its first three coordinates infinitely often, let alone to the origin.