Exercise 1. *General knowledge*

(1) Let h be a harmonic function on $\mathbb{C}$. Prove that there exists a holomorphic function f such that $h = \Re(f)$. *Hint : Prove that if f exists, $f'(w) = \partial_x h - i\partial_y h$. Use the fact that a holomorphic function can be integrated. Conclude.*

(2) Let $\overline{A} = A \cup \partial A$ be a connected finite graph and let $\omega = x_0 \underset{e_1}{\rightarrow} x_1 \underset{e_2}{\rightarrow} \dots \underset{e_n}{\rightarrow} x_n$ be a non-self-intersecting path in $\overline{A}$ such that $w \cap \partial A = \{x_n\}$. Describe the set of paths $\Gamma = \Gamma(\omega)$ in $\overline{A}$ such that if $\gamma \in \Gamma$ the loop erased path obtained from γ is ω. What is the difference between paths in Γ and trajectories of RW from x_0 stopped at first visit in ∂A and such that the corresponding LERW is ω?

(3) The Laplacian random walk (LARW) started at v is the law of a walk started at v whose first step consists in choosing a neighbour $w \sim v$ with probability

$$\frac{H_{A\setminus\{v\}}(w, \partial A)}{\sum_{w\sim v} H_{A\setminus\{v\}}(w, \partial A)}$$

and if it already did k steps $v_1, \dots, v_k$, then the next step is to choose a neighbour $w \sim v_k$ with probability

$$\frac{H_{A\setminus\{v_1,\dots,v_k\}}(w, \partial A)}{\sum_{w\sim v} H_{A\setminus\{v_1,\dots,v_k\}}(w, \partial A)}$$

where we recall that H is the harmonic measure and $H_{A\setminus\{v_1,\dots,v_k\}}(w, \partial A)$ denotes the harmonic measure on $A \setminus \{v_1, \dots, v_k\}$ with boundary $\partial A \cup \{v_1, \dots, v_k\}$.

(a) Similarly to the previous question, characterize the set of paths Γ ending on ∂A such that if $\gamma \in \Gamma$, the loop erased path obtained from γ begins with $\omega = x_0 \underset{e_1}{\rightarrow} x_1 \underset{e_2}{\rightarrow} \dots \underset{e_k}{\rightarrow} x_k$.

(b) Consider the next step that a LERW and a LARW have to take after doing k steps, and prove that LERW and a LARW have the same law. *Hint : in order to understand the first k steps of a LERW γ, we need to consider the whole path π which finishes on ∂A and such that the loop erased path associated to π is γ. Use the previous question to describe the set of such π.*

Exercise 2. *New proof of Wilson's theorem & New proof of Kirchhoff's theorem.*

Let us consider a finite connected graph A with $n+1$ vertices. We will allow ourselves to use generalizations (to any finite connected graph) of the results proven this week.

(1) Show that under Wilson's algorithm, the probability of obtaining a spanning tree T by starting at the root vertex $v_0 = x$ then visiting the other vertices in the order $v_1, \dots v_n$ is

$$\frac{G_{A_0}(v_1, v_1)}{\deg(v_1)} \frac{G_{A_1}(v_2, v_2)}{\deg(v_2)} \cdots \frac{G_{A_{n-1}}(v_n, v_n)}{\deg(v_n)}$$

where $A_i = V(A) \setminus \{v_0, \dots, v_i\}$ and G_A stands for the Green function for A. Note that $G_{A_i} : A \to \mathbb{R}$ considers the vertices $v_{0,} \dots, v_i$ to be now the graph's boundary. *Hint : simply consider the first branch of the tree, and consider the probability that a loop erased random walk gives this branch.*

(2) Prove Wilson's theorem, i.e. Wilson's algorithm samples uniform spanning trees.

(3) Prove Kirchhoff's theorem, i.e

$$\#\{\text{spanning trees of } A\} = \prod_{i=1}^{n} \deg(v_i) \det\left(\Delta_A^{1,1}\right),$$

where Δ_A is the Laplace operator on A.