

Recall that a simple random walk on a graph is called *recurrent* if it returns to the starting point with probability 1, and *transient* otherwise.

Exercise 1. *General knowledge on the recurrence/transience of random walks*

Let G be a general graph (locally finite), let v be a vertex of G and $(S_n)_{n \geq 0}$ be a simple random walk starting at v . We denote by \mathbb{P}_v the corresponding probability measure.

1. Explain what a simple random walk on G is.
2. Prove that $(S_n)_{n \geq 0}$ is recurrent if and only if

$$\sum_{n=0}^{\infty} \mathbb{P}_v(S_n = v) = \infty.$$

3. Let us suppose that G is connected and v, w are vertices of G .
 - (a) Show that a simple random walk on G is recurrent when started from v if and only if it is recurrent when started from w .
 - (b) Show that if a simple random walk $(S_n)_{n \geq 0}$ on G is recurrent when started from v , then for any vertex w of G , $\mathbb{P}_v(\exists n, S_n = w) = 1$ and $\mathbb{P}_w(\exists n, \tilde{S}_n = v) = 1$ where $(\tilde{S}_n)_{n \geq 0}$ is a simple random walk on G starting at w .
4. Show that a simple random walk on a finite graph is recurrent.
5. Show that a simple random walk $(S_n)_{n \geq 0}$ on \mathbb{Z}^d is recurrent if and only if

$$\sum_{n=0}^{\infty} \mathbb{P}_{\vec{0}}(S_{2n} = \vec{0}) = \infty.$$

Exercise 2. *Universality of the recurrence for random walks on \mathbb{Z}*

Consider a random walk on \mathbb{Z} defined using identically independent jumps : $S_n = Z_1 + \dots + Z_n$ (Z_i are i.i.d. \mathbb{Z} -valued random variables). Let us suppose that Z_1 satisfies $\mathbb{E}(|Z_1|) < \infty$.

1. Prove that if $\mathbb{E}(Z_1) \neq 0$ then S_n is transient.
2. What is the derivative of $\phi(t) = \mathbb{E}(e^{itZ_1})$ at 0 ? Give the Taylor expansion of $\phi(t)$ at 0 at order 1.
3. Using the previous point, prove that if Z_1 is symmetric ($-Z_1$ has the same law as Z_1) then S_n is recurrent.
Hint: use the derivation using the Fourier transform as seen in the lesson.