

**Exercise 1. General knowledge**

- (1) Let  $h$  be a harmonic function on  $\mathbb{C}$ . Prove that there exists a holomorphic function  $f$  such that  $h = \Re(f)$ .  
*Hint : Prove that if  $f$  exists,  $f'(w) = \partial_x h - i\partial_y h$ . Use the fact that a holomorphic function can be integrated. Conclude.*

**Solution.** We consider  $h$  a harmonic function on  $\mathbb{C}$ . The strategy is to define  $f$  such that  $f(0) = h(0)$  by finding an holomorphic function  $\partial_z f$  and setting

$$f(z) = h(0) + \int_{\gamma} f'(w) dw$$

where  $\gamma$  is any path going from 0 to  $z$ , and where the integral is well defined by holomorphicity of  $f'$ .

If  $f$  is a holomorphic function with real part  $h$  and imaginary part  $g$ , then

$$2 \cdot \partial_z f = \partial_x f - i\partial_y f = (\partial_x h + \partial_y g) + i(\partial_x g - \partial_y h)$$

Moreover if  $f$  is a holomorphic function it satisfies the Cauchy-Riemann equations or equivalently

$$0 = 2 \cdot \partial_{\bar{z}} f = \partial_x f + i\partial_y f = (\partial_x h - \partial_y g) + i(\partial_x g + \partial_y h)$$

Thus

$$\partial_z f = f' = \partial_x h - i\partial_y h$$

Hence we have a candidate for  $f'$  given by  $\partial_x h - i\partial_y h$ .

With this definition  $f'$  is holomorphic since

$$\partial_{\bar{z}} \partial_z f = \frac{1}{2} (\partial_x + i\partial_y) (\partial_x h - i\partial_y h) = \frac{1}{2} (\partial_x^2 h + \partial_y^2 h) = \frac{1}{2} \Delta h = 0.$$

Thus  $f$  defined above is holomorphic as it is the integral of a holomorphic function.

Finally  $f$  has real part  $\Re(f) = h$  since  $f(0) = h(0)$  and  $\partial_x h = \partial_x \Re(f)$  and  $\partial_y h = \partial_y \Re(f)$ . But the holomorphicity of  $f$  implies that  $\partial_z f = \partial_x f - i\partial_y f$ , hence we have  $\partial_x \Re(f) = \Re(\partial_x f) = \Re(\partial_z f) = \partial_x h$  and similarly,  $\partial_y \Re(f) = \Re(\partial_y f) = \Re(i\partial_z f) = -\Im(\partial_z f) = \partial_y h$ .

- (2) Let  $\bar{A} = A \cup \partial A$  be a connected finite graph and let  $\omega = x_0 \xrightarrow{e_1} x_1 \xrightarrow{e_2} \dots \xrightarrow{e_n} x_n$  be a non-self-intersecting path in  $\bar{A}$  such that  $w \cap \partial A = \{x_n\}$ . Describe the set of paths  $\Gamma = \Gamma(\omega)$  in  $\bar{A}$  such that if  $\gamma \in \Gamma$  the loop erased path obtained from  $\gamma$  is  $\omega$ . What is the difference between paths in  $\Gamma$  and trajectories of RW from  $x_0$  stopped at first visit in  $\partial A$  and such that the corresponding LERW is  $\omega$ ?

**Solution.** Let  $\omega = x_0 \rightarrow x_1 \rightarrow \dots \rightarrow x_n$  be a non-self-intersecting path. Any path  $\gamma$  such that the loop erased path obtained from  $\gamma$  is  $\omega$  is of the form :

$$\gamma : x_0 \xrightarrow{\ell_0} x_0 \xrightarrow{e_1} x_1 \xrightarrow{\ell_1} x_1 \xrightarrow{e_2} x_2 \rightarrow \dots \xrightarrow{e_n} x_n \xrightarrow{\ell_n} x_n$$

where  $\ell_i$  is a loop in  $\bar{A} \setminus \{x_0, \dots, x_{i-1}\}$  based at  $x_i$ .

If  $\omega$  is a sample of the LERW from  $x_0$  to  $x_n \in \partial A$  stopped at first visit in  $\partial A$ , then the corresponding random walk trajectory is of the form of  $\gamma$  above with the additional constraints that  $\ell_i$  is actually a loop in  $A \setminus \{x_0, \dots, x_{i-1}\}$  and  $\ell_n$  is the empty loop.

- (3) The Laplacian random walk (LARW) started at  $v$  is the law of a walk started at  $v$  whose first step consists in choosing a neighbour  $w \sim v$  with probability

$$\frac{H_{A \setminus \{v\}}(w, \partial A)}{\sum_{w' \sim v} H_{A \setminus \{v\}}(w', \partial A)}$$

and if it already did  $k$  steps  $v_1, \dots, v_k$ , then the next step is to choose a neighbour  $w \sim v_k$  with probability

$$\frac{H_{A \setminus \{v_1, \dots, v_k\}}(w, \partial A)}{\sum_{w' \sim v_k} H_{A \setminus \{v_1, \dots, v_k\}}(w', \partial A)}$$

where we recall that  $H$  is the harmonic measure and  $H_{A \setminus \{v_1, \dots, v_k\}}(w, \partial A)$  denotes the harmonic measure on  $A \setminus \{v_1, \dots, v_k\}$  with boundary  $\partial A \cup \{v_1, \dots, v_k\}$ .

- (a) Similarly to the previous question, characterize the set of paths  $\Gamma$  ending on  $\partial A$  such that if  $\gamma \in \Gamma$ , the loop erased path obtained from  $\gamma$  begins with  $\omega = x_0 \xrightarrow{e_1} x_1 \xrightarrow{e_2} \dots \xrightarrow{e_k} x_k$ .

**Solution.** A path  $\gamma$  is in  $\Gamma$  if it is of the form

$$\gamma : x_0 \xrightarrow{\ell_0} x_0 \xrightarrow{e_1} x_1 \xrightarrow{\ell_1} x_1 \xrightarrow{e_2} x_2 \rightarrow \dots \xrightarrow{e_k} x_k \xrightarrow{\pi} y$$

where  $\ell_i$  is a loop in  $A \setminus \{x_0, \dots, x_{i-1}\}$  based at  $x_i$  and  $\pi$  is a path from  $x_k$  to any  $y \in \partial A$  which is in  $A \setminus \{x_0, \dots, x_{k-1}\}$  except for the last point.

- (b) Consider the next step that a LERW and a LARW have to take after doing  $k$  steps, and prove that LERW and a LARW have the same law. *Hint : in order to understand the first  $k$  steps of a LERW  $\gamma$ , we need to consider the whole path  $\pi$  which finishes on  $\partial A$  and such that the loop erased path associated to  $\pi$  is  $\gamma$ . Use the previous question to describe the set of such  $\pi$ .*

**Solution.** We need only to consider the next step that a LERW  $\omega$  do after doing  $k$  steps  $x_1, \dots, x_k$ . The probability that it goes to  $x_{k+1}$  is

$$\begin{aligned} \mathbb{P}(\omega_{k+1} = x_{k+1} | \omega_1 = x_1, \dots, \omega_k = x_k) &= \frac{1}{\mathbb{P}(\omega_1 = x_1, \dots, \omega_k = x_k)} \sum_{\gamma, LERW(\gamma)|_{[1, k+1]} = (x_1, \dots, x_{k+1})} p(\gamma) \\ &= \frac{1}{\mathbb{P}(\omega_1 = x_1, \dots, \omega_k = x_k)} \sum_{\gamma: x_0 \xrightarrow{\ell_0} x_0 \xrightarrow{e_1} x_1 \xrightarrow{\ell_1} x_1 \xrightarrow{e_2} x_2 \rightarrow \dots x_k \xrightarrow{e_{k+1}} x_{k+1} \xrightarrow{\pi} y} p(\gamma) \end{aligned}$$

where we have the same conditions as above for the loops and paths (in particular  $\pi$  only hits  $\partial A$  at its endpoint  $y$  and does not hit  $\{x_0, \dots, x_k\}$ ) and where  $p(\gamma)$  is the probability that the simple random walk follows  $\gamma$ . Considering only the parts which depend on  $x_{k+1}$ , and using the notation  $\sim$  to say that it is proportional to (with the same constant for any  $x_{k+1}$ ), we have:

$$\mathbb{P}(\omega_{k+1} = x_{k+1} | \omega_1 = x_1, \dots, \omega_k = x_k) \sim \sum_{x_k \xrightarrow{e_{k+1}} x_{k+1} \xrightarrow{\pi} y} p(\gamma) \sim p(x_k, x_{k+1}) \sum_{x_{k+1} \xrightarrow{\pi} y} p(\gamma) \sim \sum_{x_{k+1} \xrightarrow{\pi} y} p(\gamma)$$

where we recall that  $\pi$  goes from  $x_{k+1}$  to  $y$  and only hits  $\partial A$  at its endpoint  $y$  and does not hit  $\{x_0, \dots, x_k\}$ . But the last expression  $\sum_{x_{k+1} \xrightarrow{\pi} y} p(\gamma)$  is precisely  $H_{A \setminus \{x_0, \dots, x_k\}}(x_{k+1}, \partial A)$ , hence

$$\mathbb{P}(\omega_{k+1} = x_{k+1} | \omega_1 = x_1, \dots, \omega_k = x_k) \sim H_{A \setminus \{x_0, \dots, x_k\}}(x_{k+1}, \partial A).$$

This allows us to conclude.

**Exercise 2.** *New proof of Wilson's theorem & New proof of Kirchhoff's theorem.*

Let us consider a finite connected graph  $A$  with  $n+1$  vertices. We will allow ourselves to use generalizations (to any finite connected graph) of the results proven this week.

- (1) Show that under Wilson's algorithm, the probability of obtaining a spanning tree  $T$  by starting at the root vertex  $v_0 = x$  then visiting the other vertices in the order  $v_1, \dots, v_n$  is

$$\frac{G_{A_0}(v_1, v_1)}{\deg(v_1)} \frac{G_{A_1}(v_2, v_2)}{\deg(v_2)} \dots \frac{G_{A_{n-1}}(v_n, v_n)}{\deg(v_n)}$$

where  $A_i = V(A) \setminus \{v_0, \dots, v_i\}$  and  $G_A$  stands for the Green function for  $A$ . Note that  $G_{A_i} : A \rightarrow \mathbb{R}$  considers the vertices  $v_0, \dots, v_i$  to be now the graph's boundary. *Hint : simply consider the first branch of the tree, and consider the probability that a loop erased random walk gives this branch.*

**Solution.** We only need to consider the first branch of the tree, the rest is done similarly. Thus we need to understand the probability that a LERW stopped when hitting  $v_0$  is equal to  $\omega = v_1 \xrightarrow{e_1} \dots \xrightarrow{e_{k-1}} v_k = v_0$ . We have seen that the set  $\Gamma$  of paths of the simple random walk whose loop erasure gives  $\omega$  is  $\Gamma = \mathcal{L}_{v_1}(A \setminus \{v_0\}) \cdot \prod_i e_i \cdot \mathcal{L}_{v_{i+1}}(A \setminus \{v_0, v_1, \dots, v_i\})$  where the product denotes the concatenation of paths and  $\mathcal{L}_v(A)$  is the set of loops in  $A$  based at  $v$ . Thus

$$\mathbb{P}(LERW(\gamma) = \omega) = \sum_{\gamma \in \Gamma} p(\gamma)$$

where  $p(\gamma)$  is the probability that the simple random walk follows  $\gamma$ , and the  $\gamma$  on the l.h.s. is a simple random walk. Using the description of  $\Gamma$  and the multiplicativity of  $p$ :

$$\mathbb{P}(\text{LERW}(\gamma) = \omega) = \left( \sum_{\ell_1 \in \mathcal{L}_{v_1}(A \setminus \{v_0\})} p(\ell_1) \right) p(e_1) \left( \sum_{\ell_2 \in \mathcal{L}_{v_2}(A \setminus \{v_0, v_1\})} p(\ell_2) \right) \dots p(e_{k-1}) \left( \sum_{\ell_k \in \mathcal{L}_{v_k}(A \setminus \{v_0, v_1, \dots, v_{k-1}\})} p(\ell_k) \right)$$

Recall that

$$\begin{aligned} \left( \sum_{\ell_1 \in \mathcal{L}_{v_1}(A \setminus \{v_0\})} p(\ell_1) \right) &= G_{V(A) \setminus \{v_0\}}(v_1, v_1) \\ &\vdots \\ \left( \sum_{\ell_k \in \mathcal{L}_{v_k}(A \setminus \{v_0, v_1, \dots, v_{k-1}\})} p(\ell_k) \right) &= G_{V(A) \setminus \{v_0, v_1, \dots, v_{k-1}\}}(v_k, v_k). \end{aligned}$$

Besides,  $p(e_k)$  is the probability that a simple random walk starting at  $v_k$  goes to  $v_{k+1}$  after one step : it is equal to  $\frac{1}{\deg(v_k)}$ . Thus, we get the desired result:

$$\mathbb{P}(\text{LERW}(\gamma) = \omega) = \frac{G_{A_0}(v_1, v_1)}{\deg(v_1)} \frac{G_{A_1}(v_2, v_2)}{\deg(v_2)} \dots \frac{G_{A_{k-1}}(v_k, v_k)}{\deg(v_k)}.$$

By running the rest of Wilson's algorithm, we get the formula for the probability of obtaining a spanning tree  $T$  by starting at the root vertex  $v_0 = x$  then visiting the other vertices in the order  $v_1, \dots, v_n$  :

$$\frac{G_{A_0}(v_1, v_1)}{\deg(v_1)} \frac{G_{A_1}(v_2, v_2)}{\deg(v_2)} \dots \frac{G_{A_{n-1}}(v_n, v_n)}{\deg(v_n)}$$

- (2) Prove Wilson's theorem, i.e. Wilson's algorithm samples uniform spanning trees.

**Solution.** Recall that in the Wilson's algorithm, we have considered an order on the vertices and the algorithm follows this order when it has to pick a new starting point for the LERW. Given that this order is denoted by  $v_1, \dots, v_n$  (and  $v_0$  is the root) we just proved that :

$$\mathbb{P}(\text{Wilson's algo samples } T) = \frac{G_{A_0}(v_1, v_1)}{\deg(v_1)} \frac{G_{A_1}(v_2, v_2)}{\deg(v_2)} \dots \frac{G_{A_{n-1}}(v_n, v_n)}{\deg(v_n)}.$$

But for an other tree  $T'$ , it will also visit all the vertices of  $G$  but in an other order :  $v'_1, \dots, v'_n$ . Using the (generalization) of the result of exercise 2 last week we get:

$$\begin{aligned} \mathbb{P}(\text{Wilson's algo samples } T) &= \frac{G_{A_0}(v_1, v_1)}{\deg(v_1)} \frac{G_{A_1}(v_2, v_2)}{\deg(v_2)} \dots \frac{G_{A_{n-1}}(v_n, v_n)}{\deg(v_n)} \\ &= \frac{G_{A'_0}(v'_1, v'_1)}{\deg(v'_1)} \frac{G_{A'_1}(v'_2, v'_2)}{\deg(v'_2)} \dots \frac{G_{A'_{n-1}}(v'_n, v'_n)}{\deg(v'_n)} \\ &= \mathbb{P}(\text{Wilson's algo samples } T') \end{aligned}$$

This proves that Wilson's algorithm samples a uniform spanning tree.

- (3) Prove Kirchhoff's theorem, i.e

$$\# \{\text{spanning trees of } A\} = \prod_{i=1}^n \deg(v_i) \det(\Delta_A^{1,1}),$$

where  $\Delta_A$  is the Laplace operator on  $A$ .

**Solution.** If we have a finite set  $\Omega$ , and if  $\mathbb{P}$  is the uniform probability, for any  $\omega \in \Omega$ ,

$$\mathbb{P}(\omega) = \frac{1}{\#\Omega}.$$

Thus, since Wilson's algorithm samples a uniform spanning tree, and since we know  $\mathbb{P}(\text{Wilson algo samples } T)$ , we get that

$$\# \{\text{spanning trees of } A\} = \frac{1}{\mathbb{P}(\text{Wilson algo samples } T)}$$

which is equal to

$$\prod_{i=1}^n \deg(v_i) \left( \prod_{i=1}^n G_{A_{i-1}}(v_i, v_i) \right)^{-1}.$$

By the results of last week, and using the same notations, the latter expression is equal to

$$\prod_{i=1}^n \deg(v_i) \det(\Delta_A^{1,1})$$

which allows to conclude.