Exercise 1. *Neumann boundary conditions and Dirichlet boundary conditions*

Let us consider the rectangle $A = [|1,n|] \times [|1,m|]$ in $\mathbb{Z}^2$, and its boundary ∂A which is the set of vertices in $\mathbb{Z}^2 \setminus A$ adjacent to a vertex of A. We define the normal derivative at $y \in \partial A$ as :

$$\partial_n f(y) = f(x) - f(y),$$

where x is the unique neighbour of y in A. We denote by $\underline{\partial A}, \partial A|, \overline{\partial A}$ and $|\partial A$ the 4 parts of the boundary, respectively the lower horizontal, the right most vertical, the upper horizontal and the left-most vertical parts of ∂A.

(1) Prove that the problem

$$\begin{cases} \Delta f(x) = 0 & \text{in } A \\ f(y) = 0 & \text{on } \underline{\partial A} \\ f(y) = 1 & \text{on } \overline{\partial A} \\ \partial_n f(y) = 0 & \text{on } \partial A| \cup |\partial A \end{cases}$$

has a unique solution if any.

(2) Let us consider the simple random walk on $A \cup \partial A$ ($x \in A$ and $y \in \partial A$ are linked by an edge if y is adjacent to x in $\mathbb{Z}^2$, two points of ∂A are not linked by an edge). Is $\tau_{\underline{\partial A} \cup \overline{\partial A}}$, the hitting time for $\underline{\partial A} \cup \overline{\partial A}$, finite almost surely ?

(3) Give an explicit formulation of the unique solution of the discrete PDE in terms of random walks. What is the difference with the pure Dirichlet conditions ?

Remark. We used a mix of conditions in order to have a finite stopping time. Yet, one can solve the Neumann problem with pure Neumann boundary conditions. In this case, the boundary conditions must satisfy some additional conditions for a solution to exist, and the solution is unique only up to a constant. This is slightly more technical. (Section 6.7 of https://www.math.uchicago.edu/~lawler/srwbook.pdf)

Remark. Actually we considered a rectangle for simplicity, but one can do the same with any discretisation of any domain Ω with 4 points marked on the boundary in counterclockwise order a, b, c and d and with Dirichlet boundary conditions on $[a,b], [c,d]$ and Neumann conditions on $[b,c], [d,e]$. Then the solution would be the imaginary part of the discretisation of the conformal mapping which sends the domain Ω to a rectangle $[0,L] \times [0,i]$ (for some L) which sends a, b, c, d to the corners of the rectangle.

Exercise 2. *Green's function representation by determinant*

We consider $A \subseteq \mathbb{Z}^d$ finite. The goal of this exercise is to prove that if $x_1, \cdots, x_n \in A$ and $A_k = A \setminus \{x_1, \cdots, x_k\}$ then

$$G_A(x_1,x_1)\, G_{A_1}(x_2,x_2) \cdots G_{A_{n-1}}(x_n,x_n)$$

is independent of the order of $x_1, \ldots, x_n$.

Remark. For this exercise sheet, we will consider :

$$\Delta^+ f(x) = f(x) - \frac{1}{2d} \sum_{y \sim x} f(y)$$

($\Delta^+ = -\Delta$ is the positive definite operator).

(1) We can consider Δ^+ as a linear operator $\Delta^+_A : \mathbb{R}^A \to \mathbb{R}^A$, by considering a vector on A as a function on $A \cup \partial A$ such that $f \restriction_{\partial A} = 0$. Show that :

$$G_A(x,x) = \frac{\det \Delta^+_{A \setminus \{x\}}}{\det \Delta^+_A}.$$

(2) If $x_1, \cdots, x_n \in A$ and $A_k = A \setminus \{x_1, \cdots, x_k\}$, give the value of

$$G_A(x_1,x_1)\, G_{A_1}(x_2,x_2) \cdots G_{A_{n-1}}(x_n,x_n)$$

and prove that it is independent of the order of $x_1, \cdots, x_n$.

Exercise 3. *Determinant of Laplacian and uniform spanning trees : Kirchhoff's theorem*

Remark. If M is a matrix, we denote by $M^{i,j}$ the matrix obtained by deleting the i-th row and j-column of M. We also denote by $M^{i,\cdot}$ the matrix obtained by deleting only the i-th row of M. The cofactor $\det^{i,j} M$ is $(-1)^{i+j} \det M^{i,j}$.

Let G be a connected graph with n vertices and m edges (here an edge is a couple of vertices, in particular, we do not consider the case where two vertices are related by two or more edges). Recall that a spanning tree of G is a connected subgraph of G with no loop and which covers all vertices of G. Let us give a unique number between 1 and n to each vertex and a unique number between 1 and m to each edge. For this exercise, we denote by $\tilde{\Delta}_G$ the matrix defined by:

$$\tilde{\Delta}_G(i,j) = \delta_{i,j} \deg(i) + (\delta_{i,j} - 1)\, \delta_{j\sim i},$$

where $1 \leq i, j \leq n$ denote vertices of G and $\deg(i)$ is the degree of i.

We will prove Kirchhoff's theorem :

$$\boxed{\text{Kirchhoff's theorem : } \#\left\{\text{spanning trees of G}\right\} = \det^{i,j}\left(\tilde{\Delta}_G\right)}$$

Actually, we will only show that $\#\left\{\text{spanning trees of G}\right\} = \det^{1,1}\left(\tilde{\Delta}_G\right)$, the general case can be deduced using elementary linear algebra arguments.

(1) Let the $n \times m$ incidence matrix E such that the only non zero elements are given by the following: if the k-th edge goes between i and j and $i < j$ then $E_{ik} = 1$ and $E_{jk} = -1$. Show that $\tilde{\Delta}_G = EE^{\mathrm{T}}$, where E^{T} is the transpose of E.

(2) Show that $\tilde{\Delta}_G^{1,1} = E^{1,\cdot}\left(E^{1,\cdot}\right)^{\mathrm{T}}$.

(3) Prove that $m \geq n-1$ i.e. the matrix $E^{1,\cdot}$ has a horizontal shape more than a vertical shape.

(4) Recall the Cauchy-Binet formula which says that if A and B are two matrices of size $l \times k$ and $k \times l$ then

$$\det(AB) = \sum_{S\subset[k],\#S=l} \det\left(A_{[l],S}\right)\det\left(B_{S,[l]}\right)$$

where $[k] = \{1, \ldots, k\}$, $A_{S,[k]}$ is the matrix obtained by choosing the rows in S and the columns in $[k]$. Use this formula to show that

$$\det \tilde{\Delta}_G^{1,1} = \sum_{S\subset[m],\#S=n-1} \det\left(\left(E^{1,\cdot}\right)_{[n-1],S}\right)^2.$$

(5) What represents a choice of $S \subset [m]$ in the original graph G ? We want to show that S forms a spanning tree of G if and only if $\det\left(\left(E^{1,\cdot}\right)_{[n-1],S}\right) = \pm 1$, and if it does not form a spanning tree then $\det\left(\left(E^{1,\cdot}\right)_{[n-1],S}\right) = 0$.

   (a) Show that if S is not a spanning tree, then there exists a cycle in S. .

   (b) Show that if S is not a spanning tree then $\det\left(\left(E^{1,\cdot}\right)_{[n-1],S}\right) = 0$.

   (c) Let us suppose that S is a spanning tree. Consider the vertex 1 and an edge e in S connected to 1 and to a vertex i. Prove that

$$\det\left(\left(E^{1,\cdot}\right)_{[n-1],S}\right) = \pm\det\left(\left(E^{1,\cdot}\right)_{[n-1]\setminus\{i-1\},S\setminus\{e\}}\right).$$

   (d) Conclude that if S is a spanning tree, then $\det\left(\left(E^{1,\cdot}\right)_{[n-1],S}\right) = \pm 1$.

(6) Prove Kirchhoff's theorem.

(7) How do you write the number $\#\left\{\text{spanning trees of G}\right\}$ using the Laplacian Δ_G ?