Exercise 1. *General knowledge*

(1) Let h be a harmonic function on $\mathbb{C}$. Prove that there exists a holomorphic function f such that $h = \Re(f)$. *Hint : Prove that if f exists, $f'(w) = \partial_x h - i\partial_y h$. Use the fact that a holomorphic function can be integrated. Conclude.*

Solution. We consider h a harmonic function on $\mathbb{C}$. The strategy is to define f such that $f(0) = h(0)$ by finding an holomorphic function $\partial_z f$ and setting

$$f(z) = h(0) + \int_\gamma f'(w)\, dw$$

where γ is any path going from 0 to z, and where the integral is well defined by holomorphicity of f'.

If f is a holomorphic function with real part h and imaginary part g, then

$$2 \cdot \partial_z f = \partial_x f - i\partial_y f = (\partial_x h + \partial_y g) + i(\partial_x g - \partial_y h)$$

Moreover if f is a holomorphic function it satisfies the Cauchy-Riemann equations or equivalently

$$0 = 2 \cdot \partial_{\bar{z}} f = \partial_x f + i\partial_y f = (\partial_x h - \partial_y g) + i(\partial_x g + \partial_y h)$$

Thus

$$\partial_z f = f' = \partial_x h - i\partial_y h$$

Hence we have a candidate for f' given by $\partial_x h - i\partial_y h$.
With this definition f' is holomorphic since

$$\partial_{\bar{z}}\partial_z f = \frac{1}{2}(\partial_x + i\partial_y)(\partial_x h - i\partial_y h) = \frac{1}{2}\left(\partial_x^2 h + \partial_y^2 h\right) = \frac{1}{2}\Delta h = 0.$$

Thus f defined above is holomorphic as it is the integral of a holomorphic function.

Finally f has real part $\Re(f) = h$ since $f(0) = h(0)$ and $\partial_x h = \partial_x \Re(f)$ and $\partial_y h = \partial_y \Re(f)$. But the holomorphicity of f implies that $\partial_z f = \partial_x f = i\partial_y f$, hence we have $\partial_x \Re(f) = \Re(\partial_x f) = \Re(\partial_z f) = \partial_x h$ and similarly, $\partial_y \Re(f) = \Re(\partial_y f) = \Re(i\partial_z f) = -\Im(\partial_z f) = \partial_y h$.

(2) Let $\overline{A} = A \cup \partial A$ be a connected finite graph and let $\omega = x_0 \underset{e_1}{\to} x_1 \underset{e_2}{\to} \ldots \underset{e_n}{\to} x_n$ be a non-self-intersecting path in $\overline{A}$ such that $w \cap \partial A = \{x_n\}$. Describe the set of paths $\Gamma = \Gamma(\omega)$ in $\overline{A}$ such that if $\gamma \in \Gamma$ the loop erased path obtained from γ is ω. What is the difference between paths in Γ and trajectories of RW from x_0 stopped at first visit in ∂A and such that the corresponding LERW is ω?

Solution. Let $\omega = x_0 \to x_1 \to \ldots \to x_n$ be a non-self-intersecting path. Any path γ such that the loop erased path obtained from γ is ω is of the form :

$$\gamma : x_0 \underset{\ell_0}{\to} x_0 \underset{e_1}{\to} x_1 \underset{\ell_1}{\to} x_1 \underset{e_2}{\to} x_2 \to \ldots \underset{e_n}{\to} x_n \underset{\ell_n}{\to} x_n$$

where ℓ_i is a loop in $\overline{A} \setminus \{x_0, \ldots, x_{i-1}\}$ based at x_i.

If ω is a sample of the LERW from x_0 to $x_n \in \partial A$ stopped at first visit in ∂A, then the corresponding random walk trajectory is of the form of γ above with the additional constraints that ℓ_i is actually a loop in $A \setminus \{x_0, \ldots, x_{i-1}\}$ and ℓ_n is the empty loop.

(3) The Laplacian random walk (LARW) started at v is the law of a walk started at v whose first step consists in choosing a neighbour $w \sim v$ with probability

$$\frac{H_{A\setminus\{v\}}(w, \partial A)}{\sum_{w\sim v} H_{A\setminus\{v\}}(w, \partial A)}$$

and if it already did k steps $v_1, \ldots, v_k$, then the next step is to choose a neighbour $w \sim v_k$ with probability

$$\frac{H_{A\setminus\{v_1,\ldots,v_k\}}(w, \partial A)}{\sum_{w\sim v} H_{A\setminus\{v_1,\ldots,v_k\}}(w, \partial A)}$$

where we recall that H is the harmonic measure and $H_{A\setminus\{v_1,\ldots,v_k\}}(w, \partial A)$ denotes the harmonic measure on $A \setminus \{v_1, \ldots, v_k\}$ with boundary $\partial A \cup \{v_1, \ldots, v_k\}$.

(a) Similarly to the previous question, characterize the set of paths Γ ending on ∂A such that if $\gamma \in \Gamma$, the loop erased path obtained from γ begins with $\omega = x_0 \underset{e_1}{\to} x_1 \underset{e_2}{\to} \ldots \underset{e_k}{\to} x_k$.

Solution. A path γ is in Γ if it is of the form

$$\gamma : x_0 \underset{\ell_0}{\rightarrow} x_0 \underset{e_1}{\rightarrow} x_1 \underset{\ell_1}{\rightarrow} x_1 \underset{e_2}{\rightarrow} x_2 \rightarrow \ldots \underset{e_k}{\rightarrow} x_k \underset{\pi}{\rightarrow} y$$

where ℓ_i is a loop in $A \setminus \{x_0, \ldots, x_{i-1}\}$ based at x_i and π is a path from x_k to any $y \in \partial A$ which is in $A \setminus \{x_0, \ldots, x_{k-1}\}$ except for the last point.

(b) Consider the next step that a LERW and a LARW have to take after doing k steps, and prove that LERW and a LARW have the same law. *Hint : in order to understand the first k steps of a LERW γ, we need to consider the whole path π which finishes on ∂A and such that the loop erased path associated to π is γ. Use the previous question to describe the set of such π.*

Solution. We need only to consider the next step that a LERW ω do after doing k steps $x_1, \ldots, x_k$. The probability that it goes to x_{k+1} is

$$\begin{aligned}
\mathbb{P}\left(\omega_{k+1} = x_{k+1} | \omega_1 = x_1, \ldots \omega_k = x_k\right) &= \frac{1}{\mathbb{P}\left(\omega_1 = x_1, \ldots \omega_k = x_k\right)} \sum_{\gamma, LERW(\gamma)_{|[1,k+1]} = (x_1, \ldots, x_{k+1})} p(\gamma) \\
&= \frac{1}{\mathbb{P}\left(\omega_1 = x_1, \ldots \omega_k = x_k\right)} \sum_{\gamma : x_0 \underset{\ell_0}{\rightarrow} x_0 \underset{e_1}{\rightarrow} x_1 \underset{\ell_1}{\rightarrow} x_1 \underset{e_2}{\rightarrow} x_2 \rightarrow \ldots x_k \underset{e_{k+1}}{\rightarrow} x_{k+1} \underset{\pi}{\rightarrow} y} p(\gamma)
\end{aligned}$$

where we have the same conditions as above for the loops and paths (in particular π only hits ∂A at its endpoint y and does not hit $\{x_0, \ldots, x_k\}$) and where $p(\gamma)$ is the probability that the simple random walk follows γ. Considering only the parts which depend on x_{k+1}, and using the notation $\sim$ to say that it is proportional to (with the same constant for any x_{k+1}), we have:

$$\mathbb{P}\left(\omega_{k+1} = x_{k+1} | \omega_1 = x_1, \ldots \omega_k = x_k\right) \sim \sum_{x_k \underset{e_{k+1}}{\rightarrow} x_{k+1} \underset{\pi}{\rightarrow} y} p(\gamma) \sim p(x_k, x_{k+1}) \sum_{x_{k+1} \underset{\pi}{\rightarrow} y} p(\gamma) \sim \sum_{x_{k+1} \underset{\pi}{\rightarrow} y} p(\gamma)$$

where we recall that π goes from x_{k+1} to y and only hits ∂A at its endpoint y and does not hit $\{x_0, \ldots, x_k\}$. But the last expression $\sum_{x_{k+1} \underset{\pi}{\rightarrow} y} p(\gamma)$ is precisely $H_{A \setminus \{x_0, \ldots, x_k\}}(x_{k+1}, \partial A)$, hence

$$\mathbb{P}\left(\omega_{k+1} = x_{k+1} | \omega_1 = x_1, \ldots \omega_k = x_k\right) \sim H_{A \setminus \{x_0, \ldots, x_k\}}(x_{k+1}, \partial A).$$

This allows us to conclude.

Exercise 2. *New proof of Wilson's theorem & New proof of Kirchhoff's theorem.*

Let us consider a finite connected graph A with $n+1$ vertices. We will allow ourselves to use generalizations (to any finite connected graph) of the results proven this week.

(1) Show that under Wilson's algorithm, the probability of obtaining a spanning tree T by starting at the root vertex $v_0 = x$ then visiting the other vertices in the order $v_1, \ldots v_n$ is

$$\frac{G_{A_0}(v_1, v_1)}{\deg(v_1)} \frac{G_{A_1}(v_2, v_2)}{\deg(v_2)} \cdots \frac{G_{A_{n-1}}(v_n, v_n)}{\deg(v_n)}$$

where $A_i = V(A) \setminus \{v_0, \ldots, v_i\}$ and G_A stands for the Green function for A. Note that $G_{A_i} : A \to \mathbb{R}$ considers the vertices $v_0, \ldots, v_i$ to be now the graph's boundary. *Hint : simply consider the first branch of the tree, and consider the probability that a loop erased random walk gives this branch.*

Solution. We only need to consider the first branch of the tree, the rest is done similarly. Thus we need to understand the probability that a LERW stopped when hitting v_0 is equal to $\omega = v_1 \underset{e_1}{\rightarrow} \ldots \underset{e_{k-1}}{\rightarrow} v_k = v_0$. We have seen that the set Γ of paths of the simple random walk whose loop erasure gives ω is $\Gamma = \mathcal{L}_{v_1}(A \setminus \{v_0\}) . \prod_i e_i . \mathcal{L}_{v_i+1}(A \setminus \{v_0, v_1, \ldots, v_i\})$ where the product denotes the concatenation of paths and $\mathcal{L}_v(A)$ is the set of loops in A based at v. Thus

$$\mathbb{P}(LERW(\gamma) = \omega) = \sum_{\gamma \in \Gamma} p(\gamma)$$

where $p(\gamma)$ is the probability that the simple random walk follows γ, and the γ on the l.h.s. is a simple random walk. Using the description of Γ and the multiplicativity of p:

$$\mathbb{P}(LERW(\gamma)=\omega)=\left(\sum_{\ell_1\in\mathcal{L}_{v_1}(A\setminus\{v_0\})}p(\ell_1)\right)p(e_1)\left(\sum_{\ell_2\in\mathcal{L}_{v_2}(A\setminus\{v_0,v_1\})}p(\ell_2)\right)\dots p(e_{k-1})\left(\sum_{\ell_k\in\mathcal{L}_{v_k}(A\setminus\{v_0,v_1,\dots,v_{k-1}\})}p(\ell_k)\right)$$

Recall that

$$\begin{aligned}\left(\sum_{\ell_1\in\mathcal{L}_{v_1}(A\setminus\{v_0\})}p(\ell_1)\right) &= G_{V(A)\setminus\{v_0\}}(v_1,v_1)\\ &\vdots\\ \left(\sum_{\ell_k\in\mathcal{L}_{v_k}(A\setminus\{v_0,v_1,\dots,v_{k-1}\})}p(\ell_k)\right) &= G_{V(A)\setminus\{v_0,v_1,\dots,v_{k-1}\}}(v_k,v_k).\end{aligned}$$

Besides, $p(e_k)$ is the probability that a simple random walk starting at v_k goes to v_{k+1} after one step : it is equal to $\frac{1}{\deg(v_k)}$. Thus, we get the desired result:

$$\mathbb{P}(LERW(\gamma)=\omega)=\frac{G_{A_0}(v_1,v_1)}{\deg(v_1)}\frac{G_{A_1}(v_2,v_2)}{\deg(v_2)}\dots\frac{G_{A_{k-1}}(v_k,v_k)}{\deg(v_k)}.$$

By running the rest of Wilson's algorithm, we get the formula for the probability of obtaining a spanning tree T by starting at the root vertex $v_0=x$ then visiting the other vertices in the order $v_1,\dots v_n$:

$$\frac{G_{A_0}(v_1,v_1)}{\deg(v_1)}\frac{G_{A_1}(v_2,v_2)}{\deg(v_2)}\dots\frac{G_{A_{n-1}}(v_n,v_n)}{\deg(v_n)}$$

(2) Prove Wilson's theorem, i.e. Wilson's algorithm samples uniform spanning trees.

Solution. Recall that in the Wilson's algorithm, we have considered an order on the vertices and the algorithm follows this order when it has to pick a new starting point for the LERW. Given that this order is denoted by $v_1,\dots,v_n$ (and v_0 is the root) we just proved that :

$$\mathbb{P}(\text{Wilson's algo samples } T)=\frac{G_{A_0}(v_1,v_1)}{\deg(v_1)}\frac{G_{A_1}(v_2,v_2)}{\deg(v_2)}\dots\frac{G_{A_{n-1}}(v_n,v_n)}{\deg(v_n)}.$$

But for an other tree T', it will also visit all the vertices of G but in an other order : $v_1',\dots,v_n'$. Using the (generalization) of the result of exercise 2 last week we get:

$$\begin{aligned}\mathbb{P}(\text{Wilson's algo samples } T) &= \frac{G_{A_0}(v_1,v_1)}{\deg(v_1)}\frac{G_{A_1}(v_2,v_2)}{\deg(v_2)}\dots\frac{G_{A_{n-1}}(v_n,v_n)}{\deg(v_n)}\\ &= \frac{G_{A_0'}(v_1',v_1')}{\deg(v_1')}\frac{G_{A_1'}(v_2',v_2')}{\deg(v_2')}\dots\frac{G_{A_{n-1}'}(v_n',v_n')}{\deg(v_n')}\\ &= \mathbb{P}(\text{Wilson's algo samples } T')\end{aligned}$$

This proves that Wilson's algorithm samples a uniform spanning tree.

(3) Prove Kirchhoff's theorem, i.e

$$\#\{\text{spanning trees of } A\}=\prod_{i=1}^{n}\deg(v_i)\det\left(\Delta_A^{1,1}\right),$$

where Δ_A is the Laplace operator on A.

Solution. If we have a finite set Ω, and if $\mathbb{P}$ is the uniform probability, for any $\omega\in\Omega$,

$$\mathbb{P}(\omega)=\frac{1}{\#\Omega}.$$

Thus, since Wilson's algorithm samples a uniform spanning tree, and since we know $\mathbb{P}(\text{Wilson algo samples } T)$, we get that

$$\#\{\text{spanning trees of } A\}=\frac{1}{\mathbb{P}(\text{Wilson algo samples } T)}$$

which is equal to

$$\prod_{i=1}^{n} \deg(v_i) \left(\prod_{i=1}^{n} G_{A_{i-1}}(v_i, v_i) \right)^{-1}.$$

By the results of last week, and using the same notations, the latter expression is equal to

$$\prod_{i=1}^{n} \deg(v_i) \det\left(\Delta_A^{1,1}\right)$$

which allows to conclude.