Exercise 1. *Getting familiar with dual graphs: Euler's theorem*

A graph $\mathbb{G}= (\mathbb{V},\mathbb{E})$ is planar, if it can be embedded in the plane; i.e., if you can draw it on a plane in such a way that its edges intersect only at the endpoints. A face is any region delimited by a set of edges (usually, there is one "outer unbounded region" which is considered a face also).

For a planar graph we define its dual $\mathbb{G}'= (\mathbb{V}',\mathbb{E}')$ that has a vertex for each face of $\mathbb{G}$ and has an edge for each pair of faces in $\mathbb{G}$ that are separated from each other by an edge, and a self-loop when the same face appears on both sides of an edge. Thus, each edge e of $\mathbb{G}$ has a corresponding dual edge, whose endpoints are the dual vertices corresponding to the faces on either side of e.

(1) Show that there is a natural bijection between $\mathbb{E}$ and $\mathbb{E}'$.
(2) Let T be a spanning tree of $\mathbb{G}$ and let us consider its edge complement $T^c = \mathbb{E}\setminus T$. Show that edges in the dual graph corresponding to T^c also form a spanning tree.
(3) Remember that each tree with k vertices has $k-1$ edges.
(4) Using the dual graph, prove Euler's formula; i.e. let V, E, F denote the number of vertices, edges, and faces of $\mathbb{G}$, respectively; show that $V-E+F=2$.

Recall that if $\mathbb{G}$ is a graph, $\mathbb{V}$ is the set of vertices of $\mathbb{G}$, and $\mathbb{E}$ is the set of edges of $\mathbb{G}$, if $p\in[0,1]$, we can consider:

(1) *the vertex (or site) percolation*, which is a percolation on the vertices of $\mathbb{G}$. It means we consider $(X_v)_{v\in V}$ a sequence of i.i.d. Bernoulli random variables of parameter p.
(2) *the bond (or edge) percolation*, which is a percolation on the edges of $\mathbb{G}$. It means we consider $(X_e)_{e\in E}$ a sequence of i.i.d. Bernoulli random variables of parameter p.

In a site percolation, a path is composed of nearest neighbours whose labels are equal to 1, whereas in the bond percolation, a path is a concatenation of edges whose labels are equal to 1.

Exercise 2. *Probability $\theta(p)$*

Let us consider the bond percolation on $\mathbb{Z}^d$ of parameter $p\in[0,1]$. We therefore consider i.i.d Bernoulli random variables $(X_e)_{e\in\mathbb{E}^d}$ of parameter p where $\mathbb{E}^d$ is the edge set of $\mathbb{Z}^d$. If $X_e=1$ we say that the edge is open.

The critical probability p_c is such that if $p>p_c$, then there almost surely exists an infinite open cluster in the percolation of parameter p, and if $p<p_c$ there almost surely exists no infinite open cluster in the percolation of parameter p.

We define

$$\theta(p)=\mathbb{P}_p(0\to\infty)$$

where by $0\to\infty$ we mean that 0 belongs to an infinite open cluster.

(1) Show that $\mathbb{P}_p(\exists \text{ infinite open cluster})=0$ or 1. *Hint : Think about a general theorem which involves* 0 *and* 1.
(2) Recall why $p\to\mathbb{P}_p(\exists \text{ infinite open cluster})$ is non-decreasing. What is the value of this function at $p=0$ and 1? Deduce that p_c exists.
(3) Show that $\theta(p)>0 \iff \mathbb{P}_p(\exists \text{ infinite open cluster})=1$.
(4) Show that $p\to\theta(p)$ is right continuous. *Hint : Use some exchange of limits.*
(5) Draw the shape of $\theta(p)$: can we say anything for now at $p=p_c$?

Remark. Similar arguments can be applied to site percolation on "gentle" infinite graphs, typically the triangle percolation.

Exercise 3. *Existence of a phase transition for $\mathbb{Z}^d$ for $d\geq 2$: $p_c\in(0,1)$.*

(1) We want to prove that $\theta(p)=0$ when p is small enough:
    (a) Show that for all $N\in\mathbb{N}^*, \mathbb{P}_p(0\to\infty)\leq\mathbb{P}_p(\exists\gamma,\ \text{self avoiding walk starting from 0 of length } N \text{ which is open})$.
    (b) Prove that if $p<\frac{1}{\mu_d}$ where $\mu_d\leq 2d$ is the connectivity constant of $\mathbb{Z}^d$ then $\mathbb{P}_p(0\to\infty)=0$. Deduce that $p_c\geq\frac{1}{\mu_d}$.

We still consider the bond percolation of parameter p. We have shown that p_c exists yet we have only shown that $p_c\in[0,1]$. The goal is to prove that $p_c\in]0,1[$. Recall that the connective constant (μ) of a graph was defined in the previous exercise sheet (Exercise 2).

(1) We want to prove that $\theta(p)>0$ when p is big enough:
    (a) Why do we only need to prove the case where $d=2$?

(b) Consider the dual graph of $\mathbb{Z}^2$: which graph is it ? Explain why a percolation on $\mathbb{Z}^2$ induces a natural percolation on the dual graph (there should be two possibilities, but one is not really interesting if you look at the next question and if you think about the crossing arguments in the square which were used in the lesson). What is the parameter of the dual percolation ?

(c) Show that 0 is not in an infinite cluster if and only if there exists a self-avoiding cycle which surrounds 0 in the dual percolation.

(d) Prove that

$$1-\theta(p) \leq C \sum_{\ell \geq 0} \ell 4^{\ell} (1-p)^{\ell}$$

where the constant C does not depend on p. *Hint : I did not choose the letter ℓ for nothing, what can it be ?*

(e) Show that if p is big enough, $\theta(p) > 0$.

(2) Summarize the result we got.

Remark. Again similar arguments can be used for site percolation on “gentle” infinite graphs, typically the triangular site percolation. One can show that for the triangle percolation, $p_c = \frac{1}{2}$. In the following exercise sheets, we will only consider the case $p_c = \frac{1}{2}$.