Exercise 1. *Duality*

(1) Show that the probability in bond percolation at $p = \frac{1}{2}$ on $\mathbb{Z}^2$ restricted to the rectangle $[0, n] \times [0; n+1]$ that there is an open crossing from top to bottom is exactly $\frac{1}{2}$.

Solution. In contrast to the face percolation on honeycomb lattice, notice that for the bond percolation there can be a crossing from top to bottom of black (labelled by 1/present/open) edges and a crossing from left to right of white edges (labelled by 0/absent/closed). Let $\mathcal{E}$ denote the event that there is a black crossing from top to bottom. Our task is to find another event $\mathcal{F}$ which is complementary to $\mathcal{E}$ and which satisfies that $\mathbb{P}(\mathcal{E}) = \mathbb{P}(\mathcal{F})$. This will already imply that $\mathbb{P}(\mathcal{E}) = \frac{1}{2}$.

Suppose that $\mathcal{E}$ does not occur. We define the set

$$B = \{(i,j) \in [0,n] \times [0; n+1] \mid \text{vertex } (i,j) \text{ can be reached from the bottom by a path of black edges}\}$$

and for each x-coordinate $i \in [0, n]$, we define $y_i = max\{j \in [0, n+1] \mid (i,j) \in B\}$. That, is (i, y_i) is the "top-most vertex" with x-coordinate i that is reachable from the bottom using a path of only black edges. Clearly, for each i it holds that $y_i < n+1$ otherwise $\mathcal{E}$ would occur. Let us denote by W the set of vertices that are "above" B; $W = \{(i,j) \in [0,n] \times [0; n+1] \mid \ j > y_i\}$. Now, the important set to consider are the edges with exactly one endpoint in B and one in W. Note that all such edges have to be white. For each such edge e, consider the two square faces it coincides with, and draw a "dual edge" that connects the centers of the faces. This creates a line orthogonal to $e$and this line is exactly an edge in the dual graph of $\mathbb{Z}^2$. The crucial observation (which is slightly technical to prove) is that if $\mathcal{E}$ does not occur, the dual edges form a path in the dual graph of $\mathbb{Z}^2$, connecting the left and right side of the rectangle $[0, n+1] \times [0; n]$. Since the dual graph of $\mathbb{Z}^2$ is $\mathbb{Z}^2$, we can denote by $\mathcal{F}$ the event that there is a white path connecting the left and right side of the rectangle $[0, n+1] \times [0; n]$. The above discussion shows that $\neg\mathcal{E} \implies \mathcal{F}$. Using an analogous argument, we get that $\neg\mathcal{F} \implies \mathcal{E}$. Thus, indeed $\mathcal{E}$ and $\mathcal{F}$ are complementary events. Since we consider percolation at $p = \frac{1}{2}$, we get that $\mathbb{P}(\mathcal{E}) = \mathbb{P}(\mathcal{F})$ which gives us $\mathbb{P}(\mathcal{E}) = \frac{1}{2}$.

(2) Suppose we have a discretisation of a simply connected domain Ω with smooth boundary with three distinct points $a,\ b,\ c$ on $\partial\Omega$ (in counter-clockwise order). Considering the honeycomb face percolation at $p = \frac{1}{2}$, show that the probability that there is an open cluster which connects all three boundary segments $[a,b], [b,c], [c,a]$ is $\frac{1}{2}$.

Solution. Analogously to (1), we need to find two events $\mathcal{E}$ and $\mathcal{F}$ such that:

(a) $\mathcal{E}$ is the event that there is an open cluster connecting all three boundary segments $[a,b], [b,c], [c,a]$,
(b) for any realisation of the percolation, either $\mathcal{E}$ or $\mathcal{F}$ happens,
(c) $\mathcal{E} \cap \mathcal{F} = \emptyset$,
(d) $\mathbb{P}(\mathcal{E}) = \mathbb{P}(\mathcal{F})$ because of some symmetry property.

In contrast to the previous exercise, since we consider face percolation now, will find $\mathcal{F}$ directly (without the use of the dual graph). Drawing inspiration from arguments used in the lectures (where it was shown that either black crossing up down of white crossing left right occurs), we should consider

$$\mathcal{F} = \text{ there is a white cluster which connects all three boundary segments } [a,b]\,[b,c]\,[c,a]\,.$$

Clearly $\mathcal{E} \cap \mathcal{F} = \emptyset$. Let us focus on showing that for any realisation of the percolation, either $\mathcal{E}$ or $\mathcal{F}$ happens. If there is no black cluster connecting all three boundary segments, then there is one segment, say $[a, b]$, for which all the clusters of black vertices connected to it do not touch both $[b, c]$ and $[c, a]$. Considering the outermost such black cluster C (relative to $[a, b]$), and assume without loss of generality that it doesn't connect to $[c, a]$. Then the white cluster separating it away from $[a, b]$ must touch $[a, b]$ and $[b, c]$ (since so does the cluster C) as well as the segment $[c, a]$ (since otherwise C wouldn't be outermost relative to $[a, b]$). Besides, since we are considering the $p = \frac{1}{2}$ case, we can flip all the white in black without changing the probabilities: $\mathbb{P}(\mathcal{E}) = \mathbb{P}(\mathcal{F})$. Thus $\mathbb{P}(\mathcal{E}) = \frac{1}{2}$.

(3) Considering again the honeycomb face percolation at $p = \frac{1}{2}$ and Ω a Jordan domain with three distinct points $a,\ b,\ c$ on $\partial\Omega$ (in counter-clockwise order), prove that the two following events have the same probability (black lines represent a black connection, red lines represent a white connection). To be more precise: $\mathcal{E}_1$ is the event that there exists a black path connecting the arc a_1a_2 to the arc a_3a_4. $\mathcal{E}_2$ is the event that there exists a black path connecting the arc a_1a_2 to the arc a_3a_4 and at the same time there

exists a white path connecting the black path to a_4a_1 (note that if the black path directly touches the arc a_4a_1 we still consider the event $\mathcal{E}_2$ to occur as the white path can be empty).

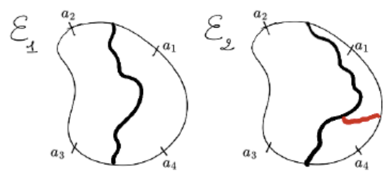
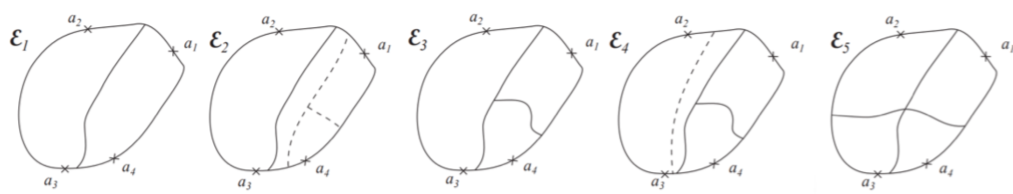


FIGURE 0.1. The events $\mathcal{E}_1$ and $\mathcal{E}_2$

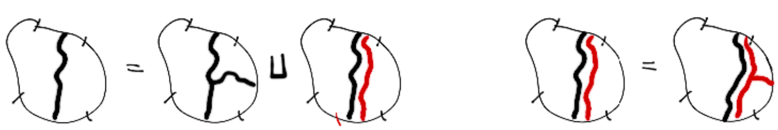
Solution. Actually the two events are equal $\mathcal{E}_1 = \mathcal{E}_2$. Indeed, trivially $\mathcal{E}_2 \subset \mathcal{E}_1$. Now let us suppose that there exists a crossing (i.e. we are in $\mathcal{E}_1$). Let us consider the rightmost black crossing: it defines a right part of Ω, denoted by Ω_r. If Ω_r is not connected, it means that the crossing directly touches a_4a_1 and hence, $\mathcal{E}_2$ occurs with the white path being empty. If Ω_r is connected, either there exists a top-bottom black crossing or a left-right white crossing. But we took the rightmost crossing: there cannot exist a top-bottom black crossing! Thus there exists a left-right white crossing in Ω_r: this proves that $\mathcal{E}_1 \subset \mathcal{E}_2$.

(4) For the honeycomb face percolation at $p = \frac{1}{2}$, define events $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4, \mathcal{E}_5$ as in the figure below (solid lines represent a white connection, dashed lines represent a black connection). Show that

$$\mathbb{P}(\mathcal{E}_2) = \mathbb{P}(\mathcal{E}_4) = \frac{\mathbb{P}(\mathcal{E}_1) - \mathbb{P}(\mathcal{E}_5)}{2} \text{ and } \mathbb{P}(\mathcal{E}_3) = \frac{\mathbb{P}(\mathcal{E}_1) + \mathbb{P}(\mathcal{E}_5)}{2}$$



*Hint 1: Prove that $\mathcal{E}_1 = \mathcal{E}_2 \sqcup \mathcal{E}_3$ and then $\mathcal{E}_1 = \mathcal{E}_2 \sqcup \mathcal{E}_4 \sqcup \mathcal{E}_5$. Hint 2: For the first equality, prove the two equalities in the Figure 0.2 (*on the second equality, the black path is the right most black path)*:*



and

FIGURE 0.2. Steps for $\mathcal{E}_1 = \mathcal{E}_2 \sqcup \mathcal{E}_3$.

Solution. Let us follow the steps proposed in the hint:

(1) Let us prove that $\mathcal{E}_1 = \mathcal{E}_2 \sqcup \mathcal{E}_3$: in order to do so, we use the same argument as in the previous question. Indeed, we consider the right-most black top-bottom crossing: it defines Ω_r a right part of Ω. In Ω_r either there exists a black left-right crossing or a white top-bottom crossing (this proves the first equality in the hints). Now in the case that there exists a white top-bottom crossing, this white top-bottom crossing defines a new right part of Ω, denoted by Ω_r'. In Ω_r', either there exists a top-bottom black crossing, or a left-right white crossing. Yet, since we considered at the beginning the right-most black top-bottom crossing, the first case cannot occur: this proves the second equality given in the hint. Putting everything together, this proves that $\mathcal{E}_1 = \mathcal{E}_2 \sqcup \mathcal{E}_3$.
(2) Let us prove that $\mathcal{E}_1 = \mathcal{E}_2 \sqcup \mathcal{E}_4 \sqcup \mathcal{E}_5$. Let us remark that $\mathcal{E}_3 = \mathcal{E}_4 \sqcup \mathcal{E}_5$ since in the left part, either there is a white crossing from up to bottom, of there is a white crossing from left to right : this is exactly what represent $\mathcal{E}_4$ and $\mathcal{E}_5$.

(3) Let us remark that by self-duality argument (i.e., we exchange black and white), $\mathbb{P}(\mathcal{E}_2) = \mathbb{P}(\mathcal{E}_4)$.

(4) Thus $\mathbb{P}(\mathcal{E}_1) = \mathbb{P}(\mathcal{E}_2) + \mathbb{P}(\mathcal{E}_4) + \mathbb{P}(\mathcal{E}_5) = 2\mathbb{P}(\mathcal{E}_2) + \mathbb{P}(\mathcal{E}_5)$. This implies that

$$\mathbb{P}(\mathcal{E}_2) = \frac{\mathbb{P}(\mathcal{E}_1) - \mathbb{P}(\mathcal{E}_5)}{2}$$

and since $\mathbb{P}(\mathcal{E}_1) = \mathbb{P}(\mathcal{E}_2) + \mathbb{P}(\mathcal{E}_3)$, we have that

$$\mathbb{P}(\mathcal{E}_3) = \mathbb{P}(\mathcal{E}_1) - \mathbb{P}(\mathcal{E}_2) = \mathbb{P}(\mathcal{E}_1) - \frac{\mathbb{P}(\mathcal{E}_1) - \mathbb{P}(\mathcal{E}_5)}{2} = \frac{\mathbb{P}(\mathcal{E}_1) + \mathbb{P}(\mathcal{E}_5)}{2}$$

which allows us to conclude.

Exercise 2. *Arzela-Ascoli theorem*
We will use the same notation as in the lectures:

$$\mathcal{H}^1_\delta(z) = \mathbb{P}_{\frac{1}{2}}\left(\text{a black path disconnects } \{a_1, z\} \text{ from } \{a_2, a_3\}\right)$$

where we consider the face percolation at $p = \frac{1}{2}$ on the honeycomb lattice, Ω_δ the discretization of a Jordan domain Ω, and a_1, a_2, a_3 are anti-clockwise ordered points on the boundary $\partial\Omega$ (z is a vertex of the hexagonal lattice). You have seen in the lesson how to deduce the following Hölder estimate:

$$|\mathcal{H}^1_\delta(x) - \mathcal{H}^1_\delta(y)| \leq C d_{\Omega_\delta}(x, y)^\alpha,$$

where d_{Ω_δ} is calculated by taking the length of the shortest path between x and y in Ω_δ. Our goal is to use this estimate to extract a limit for $\left(\mathcal{H}^1_\delta\right)_\delta$. For this, we extend the discrete function $\mathcal{H}^1_\delta$ defined on the vertices of Ω_δ to a continuous function defined on $\Omega \cup \partial\Omega$ by piecewise linear interpolation.

(1) Recall Arzela-Ascoli theorem.

Solution. We will only give the version we need for the exercise: Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions defined on a compact subset $\overline{\Omega}$ of $\mathbb{R}^2$. If this sequence is uniformly bounded and uniformly equicontinuous, then there exists a subsequence $(f_{k_n})_{n\in\mathbb{N}}$ that converges uniformly.

(2) Show that $\left(\mathcal{H}^1_\delta\right)_\delta$ is uniformly bounded. *Hint: this can be done using the estimate above, but it makes more sense to go back to the definition.*

Solution. Let us remark that $\mathcal{H}^1_\delta$ is a probability (to be more precise, it is a probability on the vertices of the hexagonal lattice): it is thus bounded between 0 and 1.

(3) Show that the above Hölder estimate implies that the family $\left(\mathcal{H}^1_\delta\right)_\delta$ is uniformly equicontinuous.

Solution. It is a well known fact that any family of functions which are Hölder continuous with a constant which does not depend on the function is uniformly equicontinuous. Yet, there are two small points which have to be tackled:

(a) We are considering functions which are piecewise linear interpolation, but the Hölder inequality was proven for x and y on the vertices of hexagonal lattice.

(b) In the Hölder estimate, we used the distance d_Ω which is the length of the shortest path between x and y in Ω and not the usual distance.

Remark. Which linear interpolation do we use? In some sense it does not matter. For example, to the center of each hexagon, we can assign the mean value of its 6 vertices. This gives a triangular tiling. Now, to the center of each triangle, we can assign the mean value of its 3 vertices. Notice this gives us a triangular tiling with a finer mesh size and this process can be iterated.

Solution. Yet, the goal of the lecture is not to be too technical: for the first point, if you can control a function at a lot of points you control its linear interpolation. For the second point let us remark that given a fixed x, $d_\Omega(x, y) \to 0$ as $|y - x| \to 0$. Thus, for any vertex x of the hexagonal graph, you can find a neighbourhood of x such that $d_\Omega(x, y)$ is small on this neighbourhood, and thus a neighbourhood of x such that for any vertices y of the hexagonal graph $|\mathcal{H}^1_\delta(x) - \mathcal{H}^1_\delta(y)|$ is (uniformly in δ) small.

(4) Deduce that we can extract a limit in $\left(\mathcal{H}^1_\delta\right)_\delta$.

Solution. The family is uniformly equicontinuous and thus we can extract a limit using Arzela-Ascoli.