Exercise 1. *Coupling*

(1) Let $0 < p' < p < 1$, let X_p be a Bernoulli (p) random variable. How can you sample $X_{p'} \sim Ber\,(p')$ using X_p and one other Bernoulli random variable Y, so that $X_{p'} \leq X_p$?

(2) Let us consider an infinite random sequence of independent Bernoulli(p). How can you create an infinite random sequence of independent Bernoulli$\left(\frac{1}{2}\right)$?

Remark. This means that if you do not trust the coin of somebody, you can still create a fair "head/tail" process.
Hint: you need to consider a certain number of pairs of independent Bernoulli(p) random variables in order to create one Bernoulli$\left(\frac{1}{2}\right)$ random variable, consider one pair and try to see where $1/2$ could appear.

(3) Let U be a random uniform variable in $[0, 1]$. How can you sample a Bernoulli(p) ?

(4) Let us denote by $\mathbb{P}_p$ be the probability associated with the site percolation on some infinite lattice with probability p (i.e. a site is open independently from the other with probability p). Show that

$$\mathbb{P}_p\left(0 \rightsquigarrow \infty\right)$$

is increasing in p.

Exercise 2. *Connective constant of graphs*

In this exercise we will only work with the graph $\mathbb{Z}^2$ but the result generalizes easily for any regular graph. We want to define a probability measure on the set of self-avoiding random walks (i.e. on the set of paths ω such that $\omega\,(i) \neq \omega\,(j)$ for any $i \neq j$) of the form:

$$P_\beta\left(\omega\right) = \frac{1}{Z_\beta} e^{-\beta|\omega|},$$

where $|\omega|$ is the length of ω and $\beta \in \mathbb{R}$ is a parameter. In order to do so, we need to understand Z_β: if it is infinite, we cannot define this probability measure, if it is finite, we can. We will admit the following lemma (that you can try to prove):

Lemma. *Let $\{a_n\}_{n\geq 1}$ be a sequence of positive real numbers such that:*

(1) *there exists $c \geq 1$, $a_n \geq c^n$ for any n,*

(2) *for any $n, p \geq 1$, $a_{n+p} \leq a_n a_p$.*

Then there exists $\mu \geq c$ such that $a_n^{\frac{1}{n}} \to \mu$ when $n \to \infty$. Besides, $\inf_n \left(a_n\right)^{\frac{1}{n}} = \mu$.

(1) What should be the value of Z_β ? *Hint: we want a probability measure.*

(2) Let us define by λ_N the number of simple walks of size N which start at 0. What is the limit of $\left(\lambda_N\right)^{\frac{1}{N}}$ as N goes to infinity?

(3) Let us define by μ_N the number of self-avoiding walks of size N which start at 0. Prove that $\left(\mu_N\right)^{\frac{1}{N}}$ converges as N goes to infinity to a number $\mu \geq 2$ which is called the connective constant of the lattice.

(4) Deduce that there exists β_c such that

$$\beta > \beta_c \iff Z_\beta < \infty.$$

Give the value of $\beta_c = \beta_c\left(\mu\right)$.

Remark. The connective constant of the honneycomb lattice has been computed in 2010 by H. Duminil-Copin and S. Smirnov with an elegant 6 pages proof (https://arxiv.org/pdf/1007.0575.pdf), using parafermionic observables.

Exercise 3. *From sites to edges and back*

For any graph $G = (V, E)$, the *edge path* is given by a sequence of edges $(e_1, \ldots, e_n)$ such that every consecutive pair shares a vertex. A vertex path $(v_1, \ldots, v_n)$ is a sequence of vertices such that each consecutive pair is connected by an edge.

(1) Show that for each $G = (V, E)$ there exists a graph $G' = (V', E')$ and a bijection $\phi : E \to V'$ which yields a correspondence between edge paths in G and vertex paths in G'.

Remark. This allows us to translate questions about edge percolation on G to questions about site percolation on G'.

(2) What is the modified graph associated with $\mathbb{Z}^2$?

(3) Think of an example of a graph $G' = (V', E')$ such that there exists no graph $G = (V, E)$ whose edge paths would correspond to vertex paths in G'.