For exercises 1, 2 and 3, we consider the Ising model with + boundary conditions on the square lattice inside the open unit disc $\mathbb{D} \subset \mathbb{R}^2$. We denote by $\mathbb{D}_\delta$ the discretisation $\mathbb{D} \cap \delta\mathbb{Z}^2$.

Exercise 1. *Coupling and stochastic domination*

(1) Recall the Markov Chain for the Ising model that you have seen in class (the Glauber dynamics).

Solution. The Markov Chain you have seen consists of the following steps:
(a) Start from an arbitrary configuration,
(b) Make random flips:
 (i) Compute the energy of the current configuration H_σ.
 (ii) Pick a vertex x at random, consider the configuration ρ obtained by flipping the spin x of σ, and compute its energy H_ρ
 (iii) If $H_\rho \leq H_\sigma$, replace σ by ρ. If $H_\rho > H_\sigma$, replace σ by ρ with probability $e^{-\beta H_\rho}/e^{-\beta H_\sigma}$.

(2) Consider the following Heat Bath Dynamics :
(a) Pick a vertex x at random,
(b) Sample the spin σ_x at random by giving probability

$$\mathbb{P}(\sigma_x = 1) = \frac{e^{-\beta\mathcal{H}(\sigma^+)}}{e^{-\beta\mathcal{H}(\sigma^+)} + e^{-\beta\mathcal{H}(\sigma^-)}}$$

where σ^+ and σ^- denote the configuration σ with the spin σ_x forced to be $+1$ and -1 respectively. Prove that the Ising measure is the invariant probability measure of this dynamics. *Hint: check the detailed balance equation.*

Solution. We will prove the detailed balance equation :

$$\pi_{Ising}(\sigma) P_{HeatBath}(\sigma, \rho) = \pi_{Ising}(\rho) P_{HeatBath}(\rho, \sigma).$$

If ρ is not of the form σ^+ or σ^-, the detailed balance equation is trivially true since $P_{HeatBath}(\sigma, \rho) = P_{HeatBath}(\rho, \sigma) = 0$. Now, let us suppose there exists a vertex x such that $\rho = \sigma^+$, then

$$\pi_{Ising}(\sigma) P_{HeatBath}(\sigma, \rho) = \pi_{Ising}(\sigma) P_{HeatBath}(\sigma, \sigma^+) = \frac{e^{-\beta\mathcal{H}(\sigma)}}{Z} \frac{e^{-\beta\mathcal{H}(\sigma^+)}}{e^{-\beta\mathcal{H}(\sigma^+)} + e^{-\beta\mathcal{H}(\sigma^-)}}$$

and

$$\pi_{Ising}(\rho) P_{HeatBath}(\rho, \sigma) = \pi_{Ising}(\sigma^+) P_{HeatBath}(\sigma^+, \sigma) = \frac{e^{-\beta\mathcal{H}(\sigma^+)}}{Z} \frac{e^{-\beta\mathcal{H}(\sigma)}}{e^{-\beta\mathcal{H}(\sigma^+)} + e^{-\beta\mathcal{H}(\sigma^-)}}.$$

This proves that the detailed balance equation is valid and the Ising measure is the invariant probability measure of this dynamics.

(3) We define a partial ordering between spin configurations $\sigma \in \{\pm 1\}^{\mathbb{D}_\delta}$: $\sigma \leq \sigma'$ if $\sigma_a \leq \sigma'_a$ for all $a \in \mathbb{D}_\delta$. Suppose that we start the chain at a common temperature $\beta > 0$ on two starting configurations $\sigma^0 \leq \sigma'^0$. Show that we can couple the two dynamics such that this ordering is preserved at each step of the Markov Chain, that is

$$\sigma^n \leq \sigma'^n$$

for all the time steps $n \in \mathbb{N}$.

Solution. We will define two Markov Chain σ^n and σ'^n starting from σ^0 and σ'^0 by using the Heat Bath Dynamics and:
(a) picking the same vertex x at random for the two Markov Chain,
(b) sampling the spin σ_x^{n+1} and σ'^{n+1}_x using the same underlying uniform random variable: we consider $U \sim Uni([0, 1])$ and we define

$$\sigma_x^{n+1} = 1 \text{ if } U \leq \frac{e^{-\beta\mathcal{H}(\sigma^{n+})}}{e^{-\beta\mathcal{H}(\sigma^{n+})} + e^{-\beta\mathcal{H}(\sigma^{n-})}}$$

and $\sigma_x^{n+1} = -1$ if not,

$$\sigma'^{n+1}_x = 1 \text{ if } U \leq \frac{e^{-\beta\mathcal{H}(\sigma'^{n+})}}{e^{-\beta\mathcal{H}(\sigma'^{n+})} + e^{-\beta\mathcal{H}(\sigma'^{n-})}}$$

and $\sigma'^{n+1}_x = -1$ if not.

If we prove that at any time $\frac{e^{-\beta\mathcal{H}\left(\sigma^{n+}\right)}}{e^{-\beta\mathcal{H}(\sigma^{n+})}+e^{-\beta\mathcal{H}(\sigma^{n-})}} \leq \frac{e^{-\beta\mathcal{H}\left(\sigma'^{n+}\right)}}{e^{-\beta\mathcal{H}(\sigma'^{n+})}+e^{-\beta\mathcal{H}(\sigma'^{n-})}}$ then by induction we can conclude that $\sigma^n \leq \sigma'^n$. In order to prove the first inequality, we only need to prove that

$$\frac{e^{-\beta\mathcal{H}\left(\sigma^{n+}\right)}+e^{-\beta\mathcal{H}\left(\sigma^{n-}\right)}}{e^{-\beta\mathcal{H}(\sigma^{n+})}} \geq \frac{e^{-\beta\mathcal{H}\left(\sigma'^{n+}\right)}+e^{-\beta\mathcal{H}\left(\sigma'^{n-}\right)}}{e^{-\beta\mathcal{H}(\sigma'^{n+})}}$$

or

$$\frac{e^{-\beta\mathcal{H}\left(\sigma^{n-}\right)}}{e^{-\beta\mathcal{H}(\sigma^{n+})}} \geq \frac{e^{-\beta\mathcal{H}\left(\sigma'^{n-}\right)}}{e^{-\beta\mathcal{H}(\sigma'^{n+})}}.$$

Let us remark that for a configuration σ and any site x,

$$\frac{e^{-\beta\mathcal{H}\left(\sigma^{-}\right)}}{e^{-\beta\mathcal{H}(\sigma^{+})}} = e^{\beta\left(-\sum_{a\sim b}\sigma^+_a\sigma^+_b+\sum_{a\sim b}\sigma^-_a\sigma^-_b\right)}$$

(Be careful, the energy $\mathcal{H}$ is equal to $-\sum_{x\sim y}\sigma_x\sigma_y$. Do not forget the $-$ sign), yet σ^+ and σ^- only differs at x, thus it is equal to $e^{-2\beta\sum_{a\sim x}\sigma_a}$. This implies that

$$\frac{e^{-\beta\mathcal{H}\left(\sigma^{n-}\right)}}{e^{-\beta\mathcal{H}(\sigma^{n+})}} = e^{-2\beta\sum_{a\sim x}\sigma^n_a} \geq e^{-2\beta\sum_{a\sim x}\sigma'^{n}_a} = \frac{e^{-\beta\mathcal{H}\left(\sigma'^{n-}\right)}}{e^{-\beta\mathcal{H}(\sigma'^{n+})}},$$

which allows us to conclude.

Exercise 2. *Monotonicity property for the boundary conditions*

Show that if $\mathfrak{b}_1, \mathfrak{b}_2 \in \{\pm 1\}^{\partial\mathbb{D}_\delta}$ are boundary conditions such that $\mathfrak{b}_1 \leq \mathfrak{b}_2$ (which means that for any element x of the boundary $\mathfrak{b}_1(x) \leq \mathfrak{b}_2(x)$). Then the corresponding Ising measures satisfy:

$$\mathbb{E}^\beta_{\mathbb{D}_\delta;\mathfrak{b}_1}\left(\sigma_a\right) \leq \mathbb{E}^\beta_{\mathbb{D}_\delta;\mathfrak{b}_2}\left(\sigma_a\right)$$

for any $a \in \mathbb{D}_\delta$. *Hint: Use the Markov chain dynamics seen in the previous exercise; the boundary spins remain unchanged.*

Solution. One just has to use the same coupling of Markov chains (where we never pick x on the boundary) as in the previous exercise, with similar initial condition except for the boundary where one begins with $\mathfrak{b}_1$ and $\mathfrak{b}_2$. The result follows from the general fact about Markov chains that the Glauber or Heat bath dynamics converge to the Ising measure over spin configurations.

Low-temperature expansion

The aim of this exercise is to show that there exists $0 < \beta < \infty$ (large enough) such that

$$\liminf_{\delta\to 0}\mathbb{E}^\beta_{\mathbb{D}_\delta,+}\left(\sigma_{(0,0)}\right) \geq 0.99.$$

Let us fix δ, we will show that $\mathbb{P}^\beta_{\mathbb{D}_\delta,+}\left(\sigma_{(0,0)} = -1\right) \leq \epsilon(\beta)$ where $\epsilon(\beta) \to 0$ as $\beta\to\infty$ is a function independent of δ.

(1) Verify that showing $\mathbb{P}^\beta_{\mathbb{D}_\delta,+}\left(\sigma_{(0,0)} = -1\right) \leq \epsilon(\beta)$ is already enough to prove $\liminf_{\delta\to 0}\mathbb{E}^\beta_{\mathbb{D}_\delta,+}\left(\sigma_{(0,0)}\right) \geq 0.99$.

Solution. We have $\mathbb{E}^\beta_{\mathbb{D}_\delta,+}\left(\sigma_{(0,0)}\right) = \mathbb{P}^\beta_{\mathbb{D}_\delta,+}\left(\sigma_{(0,0)} = 1\right) - \mathbb{P}^\beta_{\mathbb{D}_\delta,+}\left(\sigma_{(0,0)} = -1\right) = 1 - 2\mathbb{P}^\beta_{\mathbb{D}_\delta,+}\left(\sigma_{(0,0)} = -1\right) \geq 1 - 2\epsilon(\beta)$. To get the desired estimate, it suffices to choose β large enough such that $2\epsilon(\beta) < 0.01$.

(2) Recall the partition function of the Ising model on $\mathbb{D}_\delta$ with $+$ boundary conditions:

$$Z_{\mathbb{D}_\delta,+} = \sum_{\sigma\in\{\pm 1\}^{\mathbb{D}_\delta,+}} e^{\beta\sum_{xy\in\mathcal{E}}\sigma_x\sigma_y}.$$

Using the relation $\sum_{xy\in\mathcal{E}}\sigma_x\sigma_y = |\mathcal{E}| - \sum_{xy\in\mathcal{E}}(1-\sigma_x\sigma_y)$, express the Hamiltonian and the partition function in terms of the loops of σ.

Solution. We have:

$$\mathcal{H}_{\mathbb{D}_\delta,+}(\sigma) = -\sum_{\{x,y\}\in\mathcal{E}}\sigma_x\sigma_y = -|\mathcal{E}| + \sum_{\{x,y\}\in\mathcal{E}} 1 - \sigma_x\sigma_y = -|\mathcal{E}| + \sum_{\{x,y\}\in\mathcal{E},\sigma_x\neq\sigma_y} 2 = -|\mathcal{E}| + 2|\{\{x,y\}\in\mathcal{E},\sigma_x\neq\sigma_y\}|.$$

As we have seen in the lecture, each configuration σ defines a set of loops in the dual graph of $\mathbb{D}_\delta$, each loop edge corresponds to an original edge between vertices whose spins differ; we denote this set of loops as $\mathcal{C}(\sigma)$. We can thus rewrite the last sum as:

$$H_{\mathbb{D}_\delta,+}(\sigma) = -|\mathcal{E}| + 2 \sum_{\gamma \in \mathcal{C}(\sigma)} |\gamma|.$$

Thus, the partition function can be expressed as:

$$Z \quad = \sum_{\sigma \in \{\pm 1\}^{\mathbb{D}_\delta,+}} e^{\beta|\mathcal{E}| - 2\beta \sum_{\gamma \in \mathcal{C}(\sigma)} |\gamma|} = e^{\beta|\mathcal{E}|} \sum_{\sigma \in \{\pm 1\}^{\mathbb{D}_\delta,+}} \prod_{\gamma \in \mathcal{C}(\sigma)} e^{-2\beta|\gamma|}.$$

Notice that when we consider the probability of a particular configuration σ using this expression, the term $e^{\beta|\mathcal{E}|}$ cancels out in the numerator and denominator, so we can disregard it.

(3) What is en equivalent way to describe the event $\sigma_{(0,0)} = -1$ in terms of the contours surrounding $(0,0)$? Conclude that $\mathbb{P}^{\beta}_{\mathbb{D}_\delta,+}\left(\sigma_{(0,0)} = -1\right) \leq \mathbb{P}^{\beta}_{\mathbb{D}_\delta,+}(\exists \gamma^* \in \mathcal{C}(\sigma) \text{ a loop surrounding}(0,0))$. *Hint: What has to be the parity of the number of loops?*

Solution. In order for $\sigma_{(0,0)} = -1$, the vertex $(0,0)$ has to be surrounded by an odd number of loops. This in particular implies, that there needs to exist at least one loop surrounding (0,0).

(4) Let us fix a particular loop γ^* which surrounds $(0,0)$. Show that

$$\frac{\sum_{\sigma:\gamma^* \in \mathcal{C}(\sigma)} \prod_{\gamma \in \mathcal{C}(\sigma)\setminus\{\gamma^*\}} e^{-2\beta|\gamma|}}{\sum_{\sigma'} \prod_{\gamma' \in \mathcal{C}(\sigma')} e^{-2\beta|\gamma'|}} \leq 1.$$

Solution. We will show that the numerator is less than or equal to the denominator as follows: for every configuration σ that contains the loop γ^* we construct a new configuration σ' such that $\prod_{\gamma \in \mathcal{C}(\sigma)\setminus\{\gamma^*\}} e^{-2\beta|\gamma|} = \prod_{\gamma' \in \mathcal{C}(\sigma')} e^{-2\beta|\gamma'|}$. In other words, we want σ' to have the exact same loops as σ, just without γ^*. To construct σ', simply take σ and flip all the spins in the interior of the γ^* loop; it is not hard to see that the resulting configuration σ' will have all the loops as σ, except for γ^*. Moreover, from the construction, it is apparent that the mapping $\sigma \mapsto \sigma'$ is injective. Thus, each product $\prod_{\gamma \in \mathcal{C}(\sigma)\setminus\{\gamma^*\}} e^{-2\beta|\gamma|}$ in the numerator is also present in the denominator as $\prod_{\gamma' \in \mathcal{C}(\sigma')} e^{-2\beta|\gamma'|}$which concludes the proof.

(5) Show that $\mathbb{P}^{\beta}_{\mathbb{D}_\delta,+}\left(\sigma_{(0,0)} = -1\right)$ is bounded above by $\sum_{\ell \geq 1} \ell 4^\ell e^{-2\beta\ell}$.

Solution. We have that:

$$\mathbb{P}^{\beta}_{\mathbb{D}_\delta,+}\left(\sigma_{(0,0)} = -1\right) \leq \mathbb{P}^{\beta}_{\mathbb{D}_\delta,+}(\exists \gamma^* \in \mathcal{C}(\sigma) \text{ a loop surrounding}(0,0)) \;\leq \sum_{\gamma^*} \mathbb{P}^{\beta}_{\mathbb{D}_\delta,+}(\gamma^*) = \sum_{\gamma^*} \sum_{\sigma:\gamma^* \in \mathcal{C}(\sigma)} \mathbb{P}^{\beta}_{\mathbb{D}_\delta,+}(\sigma)$$

Using (2), we can rewrite the probability of σ in terms of its contours:

$$\sum_{\gamma^*} \sum_{\sigma:\gamma^* \in \mathcal{C}(\sigma)} \mathbb{P}^{\beta}_{\mathbb{D}_\delta,+}(\sigma) = \sum_{\gamma^*} \frac{\sum_{\sigma:\gamma^* \in \mathcal{C}(\sigma)} \prod_{\gamma \in \mathcal{C}(\sigma)} e^{-2\beta|\gamma|}}{\sum_{\sigma'} \prod_{\gamma' \in \mathcal{C}(\sigma')} e^{-2\beta|\gamma'|}} = \sum_{\gamma^*} e^{-2\beta|\gamma^*|} \frac{\sum_{\sigma:\gamma^* \in \mathcal{C}(\sigma)} \prod_{\gamma \in \mathcal{C}(\sigma)\setminus\{\gamma^*\}} e^{-2\beta|\gamma|}}{\sum_{\sigma'} \prod_{\gamma' \in \mathcal{C}(\sigma')} e^{-2\beta|\gamma'|}} \leq \sum_{\gamma^*} e^{-2\beta|\gamma^*|}.$$

Where the last inequality uses point (4). Now, it remains to estimate the number of loops around $(0,0)$ with a fixed length l (you have seen this result in the lecture as well as in Exercise sheet 8, exercise 3.). This gives the final inequality

$$\sum_{\gamma^*} e^{-2\beta|\gamma^*|} \leq \sum_{\ell \geq 1} \ell 4^\ell e^{-2\beta\ell}.$$

(6) Conclude that there exists $0 < \beta < \infty$ (large enough) such that

$$\liminf_{\delta \to 0} \mathbb{E}^{\beta}_{\mathbb{D}_\delta,+}\left(\sigma_{(0,0)}\right) \geq 0.99.$$

Solution. Since $\sum_{\ell \geq 1} \ell 4^\ell e^{-2\beta\ell}$ converges to 0 as $\beta \to \infty$ we have shown that $\mathbb{P}^{\beta}_{\mathbb{D}_\delta,+}\left(\sigma_{(0,0)} = -1\right) \leq \epsilon(\beta)$ where $\epsilon(\beta) \to 0$ as $\beta \to \infty$ is a function independent of δ. Thus, using (1), the proof is finished.