Game Theory: Lecture #5

Outline:

- Stable Matchings
- The Gale-Shapley Algorithm
- Optimality
- Uniqueness

Stable Matchings

- Example: The Roommate Problem
  - Potential Roommates: $\{A, B, C, D\}$
  - Goal: Divide into two pairs

	A	B	C	D
A	-	1	2	3
B	2	-	1	3
C	1	2	-	3
D	1	2	3	-

Roommates' Preferences

- Questions:
  - Are there any stable matchings?
  - How do you find a stable matching?
  - Multiple stable matchings?
  - Optimal stable matching?
- Definition: A stable matching is one in which there does not exists two potential mates that prefer each other to their proposed mates.
- Question: What are the stable roommate divisions?
- Inspection:
  - (A,B) and (C,D)?
  - (A,C) and (B,D)?
  - (A,D) and (B,C)?
- Conclusion: There are no stable matchings for the roommate problem
- Does this negative result apply to the marriage problem? Differences?

Gale-Shapley Algorithm

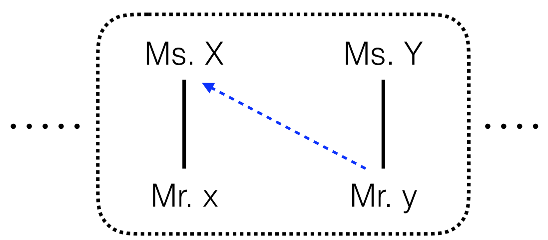
- Setup: The Marriage Problem
  - Set of n men (or applicants)
  - Set of m women (or schools)
  - Preferences for each man over the women
  - Preferences for each woman over the men
- Definition: Gale-Shapley algorithm
  - **First stage:**
    - Each man proposes to woman first on list
    - Each woman with multiple proposals
      - Selects favorite and puts him on waiting list
      - Informs all other that she will never marry them
  - **Second stage:**
    - Each rejected man proposes to woman second on list
    - Each woman with multiple proposals (1st stage WL + 2nd stage proposals)
      - Selects favorite and puts him on waiting list
      - Informs all other that she will never marry them
  - **Third stage:**
    - Each rejected man proposes to next woman on list
      - If rejected in Stage 1 and 2 $\Rightarrow$ 3rd woman on list
      - If on WL Stage 1, rejected Stage 2 $\Rightarrow$ 2nd woman on list
    - Each woman with multiple proposals (2nd stage WL + 3rd stage proposals)
      - Selects favorite and puts him on waiting list
      - Informs all other that she will never marry them
  - **Continuation:** Process continues until no man is rejected in a stage

Gale-Shapley Algorithm (2)

- Questions:
  - Does a stable matching always exist?
  - Does the Gale-Shapley algorithm always terminate?
  - Does the Gale-Shapley algorithm always find a stable matching?
  - How many stages will the Gale-Shapley algorithm take to find a matching?
- **Theorem 1:** The Gale-Shapley algorithm will always terminate in a finite number of steps.
- Assumptions:
  - Same number of men and women, i.e., $n = m$. (simplifying)
  - Analyze case where men propose to women (without loss of generalities)
- Observations:
  1. If at least one woman has no proposals, then there exists at least one woman that has multiple proposals.
  2. Once a woman has a proposal, she will always have a proposal.
  3. Suppose every woman has at least one proposal, then (a) every woman has exactly one proposal and (b) the Gale-Shapley algorithm terminates.
- Proof: Goal is to demonstrate that Observation 3 must happen
  - Suppose that some woman, Ms. X, does not have a proposal
  - Then by Observation 1 there is another woman, Ms. Y, that has at least two proposal, Mr. x and Mr. y
  - Neither Mr. x or Mr. y has ever proposed to Ms. X (Observation 2)
  - Whoever Ms. Y rejects (say Mr. x) will make another proposal.
  - If Mr. x proposes to Ms. X, we are done. Otherwise, we can repeat same arguments.
  - Note that this process can only continue for a finite number of steps.
- Note: Proof extends to the case where $m \neq n$ as well

Gale-Shapley Algorithm (3)

- Questions:
  - Does a stable matching always exist?
  - Does the Gale-Shapley algorithm always terminate? **Yes**
  - Does the Gale-Shapley algorithm always find a stable matching?
  - How many stages will the Gale-Shapley algorithm take to find a matching?
- **Theorem 2:** The Gale-Shapley algorithm will always terminate at a stable matching
- Assumptions:
  - Same number of men and women, i.e., $n = m$. (simplifying)
  - Analyze case where men propose to women (without loss of generalities)
- Observation: A woman's preference of potential match only increases by stages
- Proof:
  - Consider the following matching system



  - Suppose Mr. y prefers Ms. X over Ms. Y
  - Then Mr. y must have proposed to Ms. X before proposing to Ms. Y
  - Since Mr. y proposed to Ms. Y at some point, Ms. X must have rejected Mr. y
  - At the time of rejection, Ms. X preferred Mr. z over Mr. y for some Mr. z
  - By Observation, Ms. X must prefer Mr. x over both Mr. z and Mr. y
  - Hence, Ms. X will not accept Mr. y's proposal

Gale-Shapley Algorithm (4)

- Questions:
  - Does a stable matching always exist? **Yes**
  - Does the Gale-Shapley algorithm always terminate? **Yes**
  - Does the Gale-Shapley algorithm always find a stable matching? **Yes**
  - How many stages will the Gale-Shapley algorithm take to find a matching?
- Assumptions:
  - Same number of men and women, i.e., $n = m$. (simplifying)
  - Analyze case where men propose to women (without loss of generalities)
- **Theorem 3:** The Gale-Shapley algorithm will terminate in at most $n^2 - n + 2$ stages.
- Observations:
  - A man can get rejected at most $n - 1$ times
  - Every non-terminal stage there is at least one rejection
  - Every woman will receive a proposal before termination
- Proof:
  - Initial proposal: 1 stage
  - Suppose every man gets rejected exactly $n - 1$ times: $n(n - 1)$ stages
  - Final proposal: 1 stage
  - Total number of stages (worst-case): $1 + n(n - 1) + 1 = n^2 - n + 2$
- Fact: It is impossible to have all men rejected $n-1$ times without having the Gale-Shapley algorithm terminate
- A more careful inspection reveals that the largest number of stages is actually n^2-2n+2
- Specific details of this worst-case scenario not overly important

Optimality

- Q: Is there one stable matching that is everyone's favorite?
- Q: Is there one stable matching that is Men's favorite? Women's favorite?
- Example:

	A	B	C	D
a	3	4	1	1
b	2	2	3	4
c	4	1	2	3
d	1	3	4	2

Women's Preferences

	A	B	C	D
a	2	1	4	3
b	3	2	1	4
c	2	4	3	1
d	4	2	1	3

Men's Preferences

- Stable matchings: (subscript denotes preference of mate in match)
  - Matching #1: Gale-Shapley (M proposing): $\begin{pmatrix} A_3 & B_3 & C_3 & D_3 \\ | & | & | & | \\ a_2 & d_2 & b_1 & c_1 \end{pmatrix}$
  - Matching #2: Gale-Shapley (W proposing): $\begin{pmatrix} A_1 & B_2 & C_2 & D_1 \\ | & | & | & | \\ d_4 & b_2 & c_3 & a_3 \end{pmatrix}$
  - Matching #3: Alternative stable matching: $\begin{pmatrix} A_3 & B_2 & C_2 & D_2 \\ | & | & | & | \\ a_2 & b_2 & c_3 & d_3 \end{pmatrix}$
  - No other stable matches
- Preference over matchings:

	A	B	C	D	a	b	c	d
Matching #1 – GS - Men	3	3	3	3	2	1	1	2
Matching #2 – GS - Women	1	2	2	1	3	2	3	4
Matching #3 – Other	3	2	2	2	2	2	3	3

- Q: Is there an optimal matching? No. Why?
- Q: What is the men's (or women's) favorite matching?

Optimality (2)

- Definition: A stable matching is called optimal for a given (man, woman) if (he, she) is at least as well off under it as any other stable matching
- Recall: Preference over matchings

	A	B	C	D	a	b	c	d
Matching #1 – GS - Men	3	3	3	3	2	1	1	2
Matching #2 – GS - Women	1	2	2	1	3	2	3	4
Matching #3 – Other	3	2	2	2	2	2	3	3

- Questions:
  - Is there an optimal stable matching for the women?
  - Is there an optimal stable matching for the men?
- **Optimality Theorem:** For every preference structure, the matching system obtained by the Gale-Shapley algorithm, when the men propose, is optimal for the men. The matching system obtained by the Gale-Shapley algorithm, when the women propose, is optimal for the women.
- Question: What are the implications of GS (men) and GS (women) producing the same matching system?

Uniqueness

- Questions:
  - Is there a unique stable matching?
  - Are there conditions that give rise to a unique stable matching?
- **Uniqueness Theorem:** Assuming that there is no indifference, if the matching system obtained by the Gale-Shapley algorithm when the men propose is identical to the matching system obtained by the Gale-Shapley algorithm when the women propose, then that matching is the unique stable matching.
- Recall preference structure from previous lecture

	Ann	Beth	Cher	Dot
Al	1	1	3	2
Bob	2	2	1	3
Cal	3	3	2	1
Dan	4	4	4	4

Women's Preferences

	Ann	Beth	Cher	Dot
Al	3	4	1	2
Bob	2	3	4	1
Cal	1	2	3	4
Dan	3	4	2	1

Men's Preferences

- Matching resulting from Gale-Shapley algorithm with men (or women) proposing

Ann	Beth	Cher	Dot
|	|	|	|
Cal	Dan	Al	Bob
(3×1)	(4×4)	(3×1)	(3×1)

- Question: Are there other stable matchings?
- Answer: No, by the uniqueness theorem.
- Utility: No need to do exhaustive check to verify.