Game Theory: Lecture #2

Outline:

- Information exchange
- Beauty contest
- Social choice

Sociotechnical Systems

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emerging paradigm

"design specification" $\longrightarrow$ "social behavior" $\longrightarrow$ "performance"

- Challenges:
  - Optimal design **must** account for social behavior
  - Predicting social behavior non-trivial (and often not optimal)
  - Emergent social behavior often non-intuitive
  - Information exchange often part of underlying system
- **Example:** Lottery-based system for school assignment
- Goal: Assign students to schools to maximize social benefit (happiness)
  - Schools: $\{1, \ldots, m\}$.
  - Number open spots: $\{n_1, \ldots, n_m\}$, $n_i \geq 0$
  - Students: $\{s_1, \ldots, s_n\}$
  - School ranking for each applicant s: $\{q_1^s, \ldots, q_m^s\}$
    Interpretation: $(q_i^s > q_j^s) \;\Rightarrow\;$ school i is preferred to school j by student s
- Mechanism: Students report their top three school choices
  - *Round #1:* Randomly pick each student. Assign to top choice if available.
  - *Round #2:* Randomly pick each student not assigned in Round #1. Assign to second choice if available.
  - *Round #3:* Randomly pick each student not assigned in Round #1 or #2. Assign to third choice if available.
  - If not assigned, then student assigned to home school.
- Q: Will students report truthfully? **No!**
- Q: Will mechanims ensure optimal or near optimal behavior? **No!**
- **Take away:** Must be extremely careful in system design that interacts with a social system (self-interested, non-controlled decision making entities)

Information-Based System

- Q: Is it really that challenging to acquire accurate information?
- **Example:** Keynesian beauty contest (1936)
- Problem setup:
  - Information: 100 pictures published in local newspaper
  - Goal: Obtain public opinion of most beautiful individual
- Mechanisms:
  - M1: Request readers to send in opinions?
  - M2: Pay readers fixed amount to send in opinions?
  - M3: Pay readers to send in opinions, where the payment amount depends on the quality of their opinions?
- Issues:
  - M1: No incentive to report
  - M2: No incentive to report truthfully
  - M3: Incentive to report according to perceived societal preferences
- Q: Are there any reasonable mechanisms available to attain information?
- Q: M3 seems reasonable... What are the issues?
- **Example:** Mathematical formulation of beauty contest
  - n participants; each selects a number $x_i \in \{1, \dots, 100\}$, $i \in \{1, 2\}$
  - Winner: Participant who is closer to $1/2$ of the average of $\{x_i\}_{i=1}^n$.
- Question: What number will the participants select?
- Answer:

Voting Paradox

- Q: Are there any *reasonable* mechanisms for aggregating the opinions of many?
- **Example:** Voting paradox (Marquis de Condorcet, 1785)
  - Fixed monetary budget can go to one cause: Health, Security, or Education
  - Voters:

Left Party:	3 members
Middle Party:	4 members
Right Party:	5 members

  - Preferences:

Left (3)	Middle (4)	Right (5)
health	education	security
security	health	education
education	security	health

- Hope: individual preferences are ordered, so just need a way to combine these into "group" preferences
- Mechanism #1: Single vote, plurality rules (largest absolute # of votes)
  - Result: Security
  - Issue: More prefer Health (7) to Security (5)
- Mechanism #2: Group voting, strength = population size (dictatorship?)
  - Issue: How do you define the groups? (doesn't solve the problem)
- Mechanism #3: Pairwise voting: look at all the pairs of candidates
  - Result: Health (7) $\succ$ Security (5), Security (8) $\succ$ Education (4)
  - Social preferences: Health $\succ$ Security $\succ$ Education
  - Issue: More prefer Education (9) to Health (3) (preferences can be cyclic!)

Social Choice

- Q: Are there any *reasonable* mechanisms for aggregating the opinions of many?
- Direction:
  - Shift focus from case-by-base analysis
  - Focus on underlying properties mechanism should possess: what should voting *do?*
  - Temporarily ignore strategic component (no coalitions, tactical voting, etc.)
- **Social Choice Setup:** (Kenneth Arrow, 1951)
  - Set of alternatives: $\mathcal{A} = \{x_1, \ldots, x_m\}$
  - Set of individuals: $N = \{1, \ldots, n\}$
  - Preferences: For each individual $i \in N$ and pair of alternatives $x, x' \in \mathcal{A}$, then exactly one of the following is satisfied (total order):
    - $x \succ_i x'$ (i prefers x to x')
    - $x' \succ_i x$ (i prefers x' to x)
    - $x \sim_i x'$ (i views x and x' as equivalent)
  - Express preferences of individual i as q_i
- **Social Choice Function** ("voting rule"): A function f of the form:

  $$f(\text{Individuals' Preferences}) = \text{Societal Preferences}$$

  or mathematically

  $$f(q_1, \ldots, q_n) = \bar{q}$$

  where $\bar{q}$ encodes a ranking (or order) of the alternatives $\mathcal{A}$.
- Notes:
  - Social choice function returns an order/ranking (not single decision)
  - Ranking is more desirable than purely top choice. Why?
- Goal: Derive social choice function f that satisfies

  $$f(\text{Reasonable Individuals' Preferences}) = \text{Reasonable Social Choice}$$

- Question: What should "reasonable" look like in social choice?

Social Choice (2)

- Goal: Derive social choice function f that satisfies

$$f(\text{Reasonable Individuals' Preferences}) = \text{Reasonable Social Choice}$$

- Question: What are reasonable preferences?
  - Preferences (q_i): For each individual $i \in N$ and pair of alternatives $x, x' \in \mathcal{A}$, then exactly one of the following is satisfied:
    - $x \succ_i x'$ (i prefers x to x')
    - $x' \succ_i x$ (i prefers x' to x)
    - $x \sim_i x'$ (i views x and x' as equivalent)
  - Additional requirements: For all $i \in N$
    - $x \sim_i x$ for all $x \in \mathcal{A}$
    - Completeness: (all pair of alternatives accounted for in q_i)
    - Transitivity:

$$x \succ_i x',\ x' \succ_i x'' \ \Rightarrow\ x \succ_i x'$$

$$x \succ_i x',\ x' \sim_i x'' \ \Rightarrow\ x \succ_i x'$$

$$\ldots$$

  - Summary: Reasonable implies that preferences can be expressed by a list/order

- Reasonable preferences exclude the following phenomena
  - Do you prefer Chicken or Steak? Answer: Chicken
  - Do you prefer Chicken, Steak, or Fish? Answer: Steak
    (which principle is violated?)

Social Choice (3)

- Goal: Derive social choice function f that satisfies

$$f(\text{Reasonable Individuals' Preferences}) = \text{Reasonable Social Choice}$$

- Question: What is a reasonable social choice?
- Alternatively, what properties would we like f to possess?
- What properties should a reasonable solution possess?
- Example #1: What properties should the answer possess?

$$f\left(\begin{bmatrix} x & x & x & x & y \\ y & y & y & y & x \\ z & z & z & z & z \end{bmatrix}\right) = ?$$

- Example #2: Suppose we know that

$$f\left(\begin{bmatrix} x & x & x & x & y \\ y & y & y & y & x \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$$

  What properties should the answer of the following question possess

$$f\left(\begin{bmatrix} x & x & x & x & y \\ y & y & y & y & x \\ z & z & z & z & z \end{bmatrix}\right) = ?$$

- We will refer to these *reasonable properties* as **Axioms**

Social Choice (4)

- What are the axioms associated with a reasonable social choice function?
- **Axiom 1:** Domain and Range of f
  - Domain: Reasonable preference profiles
  - Range: Ranking (or order) of discussed alternatives
- Ex: Majority rules does not satisfy Axiom 1. Why?
- **Axiom 2:** Positive Association (or "monotonicity")
  - Consider two preference profiles q and $\tilde{q}$
  - Suppose for some alternatives x and y, the preference profiles q and $\tilde{q}$ satisfy the following for all $i \in N$:
    - If $x \succ_i y$ in q, then $x \succ_i y$ in $\tilde{q}$
    - If $x \sim_i y$ in q, then either $x \succ_i y$ or $x \sim_i y$ in $\tilde{q}$
    - If $y \succ_i x$ in q, then either $x \succ_i y$, $x \sim_i y$, or $y \succ_i x$ in $\tilde{q}$
  - Further suppose that the preference relation is the same for all alternative $\neq x, y$.
  - Then if $x \succ_i y$ in $f(q)$, then $x \succ_i y$ in $f(\tilde{q})$.
- Ex: Axiom 2 seeks to avoid situations like the following:

$$f\left(\begin{bmatrix} x & x & x & x & y \\ y & y & y & y & x \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$$

and

$$f\left(\begin{bmatrix} x & x & x & x & x \\ y & y & y & y & y \end{bmatrix}\right) = \begin{bmatrix} y \\ x \end{bmatrix}$$

- **Axiom 3:** Unanimous Decision
  - If all voters prefer x to y, then the social choice should prefer x to y.

Social Choice (5)

- **Axiom 4:** Independence of irrelevant alternative
  - Let q be any preference profile defined over $\{x, y\}$
  - Let $\tilde{q}$ be any preference profile defined over $\{x, y, z\}$
  - Suppose the perference between x and y is the same for each individual in the preference profiles q and $\tilde{q}$, i.e.,
    - $(x \succ_i y \text{ in } q) \;\Leftrightarrow\; (x \succ_i y \text{ in } \tilde{q})$
    - $(x \sim_i y \text{ in } q) \;\Leftrightarrow\; (x \sim_i y \text{ in } \tilde{q})$
    - $(y \succ_i x \text{ in } q) \;\Leftrightarrow\; (y \succ_i x \text{ in } \tilde{q})$
  - If $x \succ y$ in $f(q)$, then $x \succ y$ in $f(\tilde{q})$
- Ex: Axiom 4 seeks to avoid situations like the following:

$$f\left(\begin{bmatrix} x & x & x & x & y \\ y & y & y & y & x \\ z & z & z & z & z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \text{and} \qquad f\left(\begin{bmatrix} z & x & x & x & z \\ x & z & y & z & y \\ y & y & z & y & x \end{bmatrix}\right) = \begin{bmatrix} y \\ x \\ z \end{bmatrix}$$

- Ex: Plurality voting does not satisfy Axiom 4! (vote-splitting)

Party 1 (2)	Party 2 (3)	Party 3 (4)
C1	C1	C2
C2	C2	C1

C1 (5 votes) beats C2 (4 votes)

Party 1 (2)	Party 2 (3)	Party 3 (4)
C3	C1	C2
C1	C2	C1
C2	C3	C3

C2 (4 votes) beats C1 (3 votes)

- **Axiom 5:** Non-dictatorship
  - In a society of at least three individuals, there is no dictator; that is, there is no individual whose opinion decides all issues even if everyone else opposes his opinion.

Social Choice (6)

- Goal: Derive social choice function f that satisfies

$$f(\text{Reasonable Individuals' Preferences}) = \text{Reasonable Social Choice}$$

- Q: What is a reasonable social choice function?
- Definition: A reasonable social choice function f must satisfy Axioms 1-5.
  - Axiom 1: Domain and Range of f
  - Axiom 2: Positive Association
  - Axiom 3: Unanimous Decision
  - Axiom 4: Independence of irrelevant alternative
  - Axiom 5: Non-dictatorship
- Q: Does there exist a social choice function f that satisfies Axioms 1-5? **No!**
- **Theorem (Arrow, 1951):** If any social choice function f satisfies Axioms 1-4, then the social choice function necessarily does not satisfy Axiom 5.
- Proof - Next Lecture...