# Game Theory: Lecture #4

Outline:

- The Matching Problem
- Stable Matchings
- The Gale-Shapley Algorithm

# The Matching Problem

- Recap: Social choice theory
  - Goal: Derive reasonable mechanism for aggregating opinions of many
  - Conclusion: Any reasonable mechanism for this task has inherent shortcomings
  - Note: Ignored strategic behavior of users (will come shortly)
- Takeaway: Be careful when working with social systems
- New problem setting: The matching problem
- Example: Matching residents to residency programs
  - Residents have preferences over residency programs
  - Residency programs have preferences over potential residents
  - Limited spots available that must be filled
- Example: The marriage problem
  - Men: Al, Bob, Cal, Dan
  - Women: Ann, Beth, Cher, Dot
  - Preferences:

| | Ann | Beth | Cher | Dot |
|---|---|---|---|---|
| Al | 1 | 1 | 3 | 2 |
| Bob | 2 | 2 | 1 | 3 |
| Cal | 3 | 3 | 2 | 1 |
| Dan | 4 | 4 | 4 | 4 |

Women's Preferences

| | Ann | Beth | Cher | Dot |
|---|---|---|---|---|
| Al | 3 | 4 | 1 | 2 |
| Bob | 2 | 3 | 4 | 1 |
| Cal | 1 | 2 | 3 | 4 |
| Dan | 3 | 4 | 2 | 1 |

Men's Preferences

  - Ex: Ann prefers Al to Bob, Bob to Cal, Cal to Dan, etc
- Questions:
  - What is a *reasonable* (or stable) matching for a society?
  - Are there any *reasonable* mechanisms for making matches in society?

# The Marriage Problem

| | Ann | Beth | Cher | Dot |
|---|---|---|---|---|
| Al | 1 | 1 | 3 | 2 |
| Bob | 2 | 2 | 1 | 3 |
| Cal | 3 | 3 | 2 | 1 |
| Dan | 4 | 4 | 4 | 4 |

Women's Preferences

| | Ann | Beth | Cher | Dot |
|---|---|---|---|---|
| Al | 3 | 4 | 1 | 2 |
| Bob | 2 | 3 | 4 | 1 |
| Cal | 1 | 2 | 3 | 4 |
| Dan | 3 | 4 | 2 | 1 |

Men's Preferences

- Proposal #1: Average quality matching

$$\begin{array}{cccc} \text{Al} & \text{Bob} & \text{Cal} & \text{Dan} \\ | & | & | & | \\ \text{Dot} & \text{Ann} & \text{Beth} & \text{Cher} \\ (2\times 2) & (2\times 2) & (2\times 3) & (2\times 4) \end{array}$$

  - Cher displeased (given last choice)
  - Cher can propose to Bob. Will he accept?
  - Cher can propose to Al. Will he accept?
  - Proposal not stable since Cher and Al prefer each other over proposed mates

- Proposal #2: Mens highest choice – Stable matching?

$$\begin{array}{cccc} \text{Al} & \text{Bob} & \text{Cal} & \text{Dan} \\ | & | & | & | \\ \text{Cher} & \text{Dot} & \text{Ann} & \text{Beth} \\ (1\times 3) & (1\times 3) & (1\times 3) & (4\times 4) \end{array}$$

- Proposal #3: Womens highest choice – Stable matching?

$$\begin{array}{cccc} \text{Al} & \text{Bob} & \text{Cal} & \text{Dan} \\ | & | & | & | \\ \text{Ann} & \text{Cher} & \text{Dot} & \text{Beth} \\ (3\times 1) & (4\times 1) & (4\times 1) & (4\times 4) \end{array}$$

# Stable Proposals

- Questions:
  - Are there any stable matchings?
  - How do you find a stable matching?
  - Multiple stable matchings? Optimal stable matching?
- Definition: A stable matching is one in which there do not exist two potential mates that prefer each other to their proposed mates.
- Example: The Roommate Problem
  - Potential Roommates: $\{A, B, C, D\}$
  - Goal: Divide into two pairs

| | A | B | C | D |
|---|---|---|---|---|
| A | - | 1 | 2 | 3 |
| B | 2 | - | 1 | 3 |
| C | 1 | 2 | - | 3 |
| D | 1 | 2 | 3 | - |

Roommates' Preferences

- Question: What are the stable roommate divisions?
- Inspection:
  - (A,B) and (C,D)?
  - (A,C) and (B,D)?
  - (A,D) and (B,C)?
- Conclusion: There are no stable matchings for the roommate problem
- Does this negative result apply to the marriage problem? Differences?

# Gale-Shapley Algorithm

- Setup: The Marriage Problem
  - Set of $n$ men (or applicants)
  - Set of $m$ women (or schools)
  - Preferences for each man over the women
  - Preferences for each woman over the men
- Definition: Gale-Shapley algorithm
  - **First stage:**
    - Each man proposes to woman first on list
    - Each woman with multiple proposals
      - Selects favorite and puts him on waiting list
      - Informs all other that she will never marry them
  - **Second stage:**
    - Each rejected man proposes to woman second on list
    - Each woman with multiple proposals (1st stage WL + 2nd stage proposals)
      - Selects favorite and puts him on waiting list
      - Informs all other that she will never marry them
  - **Third stage:**
    - Each rejected man proposes to next woman on list
      - If rejected in Stage 1 and 2 $\Rightarrow$ 3rd woman on list
      - If on WL Stage 1, rejected Stage 2 $\Rightarrow$ 2nd woman on list
    - Each woman with multiple proposals (2nd stage WL + 3rd stage proposals)
      - Selects favorite and puts him on waiting list
      - Informs all other that she will never marry them
  - **Continuation:** Process continues until no man is rejected in a stage
- Note: Algorithm could proceed with either Men or Women proposing

# Example

| | Ann | Beth | Cher | Dot |
|---|---|---|---|---|
| Al | 1 | 1 | 3 | 2 |
| Bob | 2 | 2 | 1 | 3 |
| Cal | 3 | 3 | 2 | 1 |
| Dan | 4 | 4 | 4 | 4 |

Women's Preferences

| | Ann | Beth | Cher | Dot |
|---|---|---|---|---|
| Al | 3 | 4 | 1 | 2 |
| Bob | 2 | 3 | 4 | 1 |
| Cal | 1 | 2 | 3 | 4 |
| Dan | 3 | 4 | 2 | 1 |

Men's Preferences

- Denote men by $\{a, b, c, d\}$ and women by $\{A, B, C, D\}$
- Algorithm #1: Gale-Shapley algorithm with men proposing
  - Stage 1: $\begin{array}{cccc} A & B & C & D \\ \hline c & & a & b \\ & & & d^* \end{array}$ ($*$ means male is rejected)
  - Stage 2: $\begin{array}{cccc} A & B & C & D \\ \hline c & & a & b \\ & & d^* & \end{array}$
  - Stage 3: $\begin{array}{cccc} A & B & C & D \\ \hline c & & a & b \\ d^* & & & \end{array}$
  - Stage 4: $\begin{array}{cccc} A & B & C & D \\ \hline c & d & a & b \end{array}$
- Resulting proposal

$$\begin{array}{cccc} \text{Ann} & \text{Beth} & \text{Cher} & \text{Dot} \\ | & | & | & | \\ \text{Cal} & \text{Dan} & \text{Al} & \text{Bob} \\ (3 \times 1) & (4 \times 4) & (3 \times 1) & (3 \times 1) \end{array}$$

# Example (2)

| | Ann | Beth | Cher | Dot |
|---|---|---|---|---|
| Al | 1 | 1 | 3 | 2 |
| Bob | 2 | 2 | 1 | 3 |
| Cal | 3 | 3 | 2 | 1 |
| Dan | 4 | 4 | 4 | 4 |

Women's Preferences

| | Ann | Beth | Cher | Dot |
|---|---|---|---|---|
| Al | 3 | 4 | 1 | 2 |
| Bob | 2 | 3 | 4 | 1 |
| Cal | 1 | 2 | 3 | 4 |
| Dan | 3 | 4 | 2 | 1 |

Men's Preferences

- Algorithm #2: Gale-Shapley algorithm with women proposing
  - Stage 1: $\begin{array}{cccc} a & b & c & d \\ \hline A & C & D & \\ B^* & & & \end{array}$
  - Stage 2: $\begin{array}{cccc} a & b & c & d \\ \hline A & B & D & \\ & C^* & & \end{array}$
  - Stage 3: $\begin{array}{cccc} a & b & c & d \\ \hline A & B & C & \\ & & D^* & \end{array}$
  - Stage 4: $\begin{array}{cccc} a & b & c & d \\ \hline D & B & C & \\ A^* & & & \end{array}$
  - Stage 5: $\begin{array}{cccc} a & b & c & d \\ \hline D & A & C & \\ & B^* & & \end{array}$
  - Stage 6: $\begin{array}{cccc} a & b & c & d \\ \hline D & A & B & \\ & & C^* & \end{array}$
  - Stage 7: $\begin{array}{cccc} a & b & c & d \\ \hline C & A & B & \\ D^* & & & \end{array}$
  - Stage 8: $\begin{array}{cccc} a & b & c & d \\ \hline C & D & B & \\ & A^* & & \end{array}$
  - Stage 9: $\begin{array}{cccc} a & b & c & d \\ \hline C & D & A & \\ & & B^* & \end{array}$
  - Stage 10: $\begin{array}{cccc} a & b & c & d \\ \hline C & D & A & B \end{array}$

- Proposal:

| Ann | Beth | Cher | Dot |
|---|---|---|---|
| \| | \| | \| | \| |
| Cal | Dan | Al | Bob |
| $(3 \times 1)$ | $(4 \times 4)$ | $(3 \times 1)$ | $(3 \times 1)$ |

# Example (3)

| | Ann | Beth | Cher | Dot |
|---|---|---|---|---|
| Al | 1 | 1 | 3 | 2 |
| Bob | 2 | 2 | 1 | 3 |
| Cal | 3 | 3 | 2 | 1 |
| Dan | 4 | 4 | 4 | 4 |

Women's Preferences

| | Ann | Beth | Cher | Dot |
|---|---|---|---|---|
| Al | 3 | 4 | 1 | 2 |
| Bob | 2 | 3 | 4 | 1 |
| Cal | 1 | 2 | 3 | 4 |
| Dan | 3 | 4 | 2 | 1 |

Men's Preferences

- Resulting proposal: Same irrespective of proposing party

| Ann | Beth | Cher | Dot |
|---|---|---|---|
| \| | \| | \| | \| |
| Cal | Dan | Al | Bob |
| $(3 \times 1)$ | $(4 \times 4)$ | $(3 \times 1)$ | $(3 \times 1)$ |

- Question: Is resulting proposal stable?
  - Cal will never accept proposal from another potential mate. Why?
  - Al will never accept proposal from another potential mate. Why?
  - Bob will never accept proposal from another potential mate. Why?
- Question: Are there other stable profiles? (Answer = No)
- Questions for next lecture:
  - Does a stable proposal always exist?
  - Is there a unique stable proposal? Conditions for uniqueness?
  - Does the Gale-Shapley algorithm always terminate?
  - Does the Gale-Shapley algorithm always find a stable proposal?
  - How many stages will the Gale-Shapley algorithm take to find a proposal?
  - How does the proposing party impact the quality of the resulting proposals?
  - Is there an optimal proposal?