Game Theory: Lecture #3

Outline:

- Social choice
- Arrow's Impossibility Theorem
- Proof

Recap Social Choice

- Q: Are there any *reasonable* mechanisms for aggregating the opinions of many?
- Social Choice Setup: (Kenneth Arrow, 1951)
  - Set of alternatives: $\mathcal{A} = \{x_1, \ldots, x_m\}$
  - Set of individuals: $N = \{1, \ldots, n\}$
  - Preferences for each individual i: q_i (ordered list)
- Social Choice Function: A function f of the form:

$$f(\text{Individuals' Preferences}) = \text{Societal Preferences}$$

- Q: What constitutes a *reasonable* social choice function?
  - Are there preference profiles q for which $f(q)$ should satisfy certain properties?
  - If $f(q) = \bar{q}$, are there profiles q' for which $f(q')$ should satisfy certain properties?
- Q: What are "reasonable" preferences?
  - Formal definition last time, but "reasonable" means each person *ranks* the alternatives

Social Choice Axioms

- What are the axioms associated with a reasonable social choice function?
- **Axiom 1:** Domain and Range of f
  - Domain: A ranking (or order) of alternatives by each person
  - Range: A single ranking (or order) of discussed alternatives
- **Axiom 2:** Positive Association (or Monotonicity)
  - No voter can *hurt* an alternative by ranking it *higher*
  - Consider two preference profiles q and $\tilde{q}$
  - Suppose for some alternatives x and y, the preference profiles q and $\tilde{q}$ satisfy the following for all $i \in N$:
    - If $x \succ_i y$ in q, then $x \succ_i y$ in $\tilde{q}$
    - If $x \sim_i y$ in q, then either $x \succ_i y$ or $x \sim_i y$ in $\tilde{q}$
    - If $y \succ_i x$ in q, then either $x \succ_i y$, $x \sim_i y$, or $y \succ_i x$ in $\tilde{q}$
  - Further suppose that the preference relation is the same for all alternative $\neq x, y$.
  - Then if $x \succ_i y$ in $f(q)$, then $x \succ_i y$ in $f(\tilde{q})$.
- Ex: Axiom 2 seeks to avoid situations like the following:

$$f\left(\begin{bmatrix} x & x & x & x & y \\ y & y & y & y & x \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$$

and

$$f\left(\begin{bmatrix} x & x & x & x & x \\ y & y & y & y & y \end{bmatrix}\right) = \begin{bmatrix} y \\ x \end{bmatrix}$$

Social Choice Axioms

- **Axiom 3:** Unanimous Decision
  - If all voters prefer x to y, then the social choice should prefer x to y.
- **Axiom 4:** Independence of irrelevant alternative
  - Let q be any preference profile defined over $\{x, y\}$
  - Let $\tilde{q}$ be any preference profile defined over $\{x, y, z\}$
  - Suppose the perference between x and y is the same for each individual in the preference profiles q and $\tilde{q}$, i.e.,
    - $(x \succ_i y \text{ in } q) \;\Leftrightarrow\; (x \succ_i y \text{ in } \tilde{q})$
    - $(x \sim_i y \text{ in } q) \;\Leftrightarrow\; (x \sim_i y \text{ in } \tilde{q})$
    - $(y \succ_i x \text{ in } q) \;\Leftrightarrow\; (y \succ_i x \text{ in } \tilde{q})$
  - If $x \succ_i y$ in $f(q)$, then $x \succ_i y$ in $f(\tilde{q})$
- Ex: Axiom 4 seeks to avoid situations like the following:

$$f\left(\begin{bmatrix} x & x & x & x & y \\ y & y & y & y & x \\ z & z & z & z & z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

  and

$$f\left(\begin{bmatrix} z & x & x & x & z \\ x & z & y & z & y \\ y & y & z & y & x \end{bmatrix}\right) = \begin{bmatrix} y \\ x \\ z \end{bmatrix}$$

- **Axiom 5:** Non-dictatorship
  - In a society of at least three individuals, there is no dictator; that is, there is no individual whose opinion decides all issues even if everyone else opposes his opinion.

Social Choice Axioms

- Definition: A reasonable social choice function f must satisfy Axioms 1-5.
  - Axiom #1: Domain and range of f
  - Axiom #2: Positive association

$$f\left(\begin{bmatrix} x & x & y & y \\ y & y & x & x \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix} \;\Rightarrow\; f\left(\begin{bmatrix} x & x & x & y \\ y & y & y & x \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$$

  - Axiom #3: Unanimous decision
  - Axiom #4: Independence of irrelevant alternative (if someone changes how they rank z, this shouldn't change the social choice of x, y

$$f\left(\begin{bmatrix} x & x & x & x & y \\ y & y & y & y & x \\ z & z & z & z & z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \;\Rightarrow\; f\left(\begin{bmatrix} z & x & x & x & z \\ x & z & y & z & y \\ y & y & z & y & x \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

    then $\bar{q}$ should satisfy $x \succ y$.
  - Axiom #5: Non-dictatorship
- **Theorem (Arrow, 1951):** If any social choice function f satisfies Axioms 1-4, then the social choice function necessarily does not satisfy Axiom 5.

Examples

- Example: Majority rules

$$f\left(\begin{bmatrix} x & x & x & y & y \\ y & y & y & x & x \end{bmatrix}\right) = x$$

$$f\left(\begin{bmatrix} x & x & y & y & y \\ y & y & x & x & x \end{bmatrix}\right) = y$$

  Satisfy Axiom #1? Axiom #2? Axiom #3? Axiom #4? Axiom #5?

- Example: Pairwise Majority rules
  - Compare each pair of alternative (x, y) independently
  - If x is preferred to y by the majority, then social preference satisfies $x \succ y$.
  - Example #1:

$$f\left(\begin{bmatrix} z & x & x & y & y \\ x & z & y & z & x \\ y & y & z & x & z \end{bmatrix}\right) =?$$

  - Example #2:

$$f\left(\begin{bmatrix} z & x & x & y & y \\ x & z & y & z & z \\ y & y & z & x & x \end{bmatrix}\right) =?$$

  - Satisfy Axiom #1? Axiom #2? Axiom #3? Axiom #4? Axiom #5?

Proof

- Roadmap:
  - Starting point: Social choice rule f that satisfies Axioms #1-4
  - Analysis: Investigate properties of f for specific preference profiles
  - Conclusion: f can only satisfy Axioms #1-4 if Axiom #5 is not satisfied
  - Central argument hinges on idea of "*Minimal Decisive Set*"
- Definition: A set of individuals V is decisive for the pair (x, y) if for any preference profile $q = (q_1, \ldots, q_n)$ where $x \succ_i y$ for all $i \in V$, then the social choice $f(q)$ must satisfy $x \succ y$.
- Interpretation: If all individuals in V prefer x to y, then the social choice must favor x to y.
- Questions:
  - If f satisfies Axioms #1-4 is there a decisive set?
  - Is N a decisive set? If so, for what pairs?
  - Are there "smaller" decisive sets?
- Definition: Minimal decisive set V
  - V is decisive for some pair (x, y)
  - Any set Q, $|Q| < |V|$, is not decisive for *any* pair (x, y).
- Fact: Since a decisive set exists (due to Unanimity), then there must exist a minimal decisive set.
- Question: Can $V = \emptyset$ be the minimal decisive set?

Proof (2)

- Knowledge of social choice rule f
  - Satisfies Axioms #1-4
  - V is the minimal decisive set for some pair (x, y), $V \neq \emptyset$
- Let z be any alternative. Consider the following preference profile where $V = \{j\} \cup W$ and U is all individuals not in V

$\{j\}$	W	U
x	z	y
y	x	z
z	y	x

- Question: What is the resulting social choice?
  - $x \succ y$ (because $V = \{j\} \cup W$ is decisive for set (x, y))
  - What about the pair (z, y)? Could $z \succ y$?
  - Answer: No! Why? If so, W would be a decisive set.
  - Conclusion: $x \succ y$ and $y \succ z$ or $y \sim z$.
- Question: How does the pair (x, z) relate?
- Answer: $x \succ z$ by transitivity.
- Implications:
  - Only player $\{j\}$ chose alternative x over z
  - Social choice chose x over z
  - $\{j\}$ is a decisive set for the pair (x, z)
  - $W = \emptyset$. Why?
- Take away: If Axioms #1-4 are satisfied, there is an individual j that is decisive for every pair of alternatives of the form (x, z), $z \neq x$

Proof (3)

- Knowledge of social choice rule f
  - Satisfies Axioms #1-4
  - There is an individual j that is decisive for every pair of alternatives of the form (x, z)
- Let z be any alternative. Consider the following preference profile where U is all individuals not including j

$\{j\}$	U
w	z
x	w
z	x

- Question: What is the resulting social choice?
  - $x \succ z$ (because $\{j\}$ is decisive for set (x, z))
  - $w \succ x$ (because of Axiom #3 – Unanimous)
  - $w \succ z$ (by transitivity)
- Conclusion: j is also decisive for every pair of alternative of the form (w, z), $w, z \neq x$

Proof (4)

- Knowledge of social choice rule f
  - Satisfies Axioms #1-4
  - There is an individual j that is decisive for:
    - Every pair of alternatives of the form (x, z)
    - Every pair of alternatives of the form (w, z), $w, z \neq x$
- Question: Is j a dictator?
- Let $w, z \neq x$ be any alternatives. Consider the following preference profile where U is all individuals not including j

$\{j\}$	U
w	z
z	x
x	w

- Question: What is the resulting social choice?
  - $w \succ z$ (because $\{j\}$ is decisive for the set (w, z))
  - $z \succ x$ (because of Axiom #3 – Unanimous)
  - $w \succ x$ (by transitivity)
- Conclusion: j is also decisive for every pair of alternative of the form (w, x), $w \neq x$
- Accordingly, there is an individual j that is decisive for:
  - Every pair of alternatives of the form (x, z)
  - Every pair of alternatives of the form (w, z), $w, z \neq x$
  - Every pair of alternatives of the form (z, x)
- Conclusion: $\{j\}$ is a dictator, and hence Axiom #5 is not satisfied!

Recap Social Choice

- Q: Are there any *reasonable* mechanisms for aggregating the opinions of many?
- Social Choice Function: A function f of the form:

$$f(\text{Individuals' Preferences}) = \text{Societal Preferences}$$

- "Reasonable" Axioms:
  - Axiom #1: Domain and range of f
  - Axiom #2: Positive association
  - Axiom #3: Unanimous decision
  - Axiom #4: Independence of irrelevant alternative
  - Axiom #5: Non-dictatorship
- **Theorem (Arrow, 1951):** If any social choice function f satisfies Axioms 1-4, then the social choice function necessarily does not satisfy Axiom 5.
- Take aways:
  - Arrow identifies fundamental limitation in the design of social choice functions
  - Impossible to design social choice function that satisfies Axioms #1-5
  - Aggregating societal opinions hard $\Rightarrow$ Controlling societal response very hard
- Questions: What do we do now?
  - Limit domain of f?
  - Introduce lotteries?
- Fact: Research had demonstrated by imposing appropriate limitations, social choice rules could be established that satisfy Axioms #1-5
- Arrow's foundational research has prompted all this work!