Definition

Pagerank is an algorithm used by google search to rank web pages in their search engine

In Simpler words : PageRank is a way of measuring the importance of a website

Notable quote by google

PageRank works by counting the number and quality of links to a page to determine a rough estimate of how important the website is. The underlying assumption is that more important websites are likely to receive more links from other websites.

Nice to know

PageRank and all associated patent are expired !!

So What's the formula ?

The Naive algorithm is the following formula

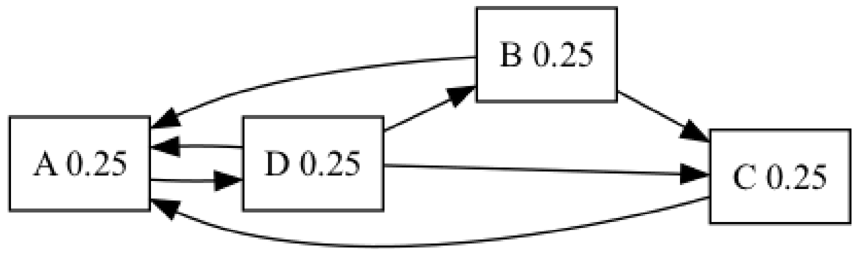
$$PR(u) = \sum_{v \in B_u} \frac{PR(v)}{L(v)}$$

Where $L\colon V \to \mathbb{N}$ is number of outbounds links and $PR\colon V \to [0,1]$ is the PageRank of a vertex and the initial probability is $\forall v \in V, PR(v) = \frac{1}{|V|}$

Simple Example

Note the following graph

```
PR(A) = 0.25
PR(B) = 0.25
PR(C) = 0.25
PR(D) = 0.25
```

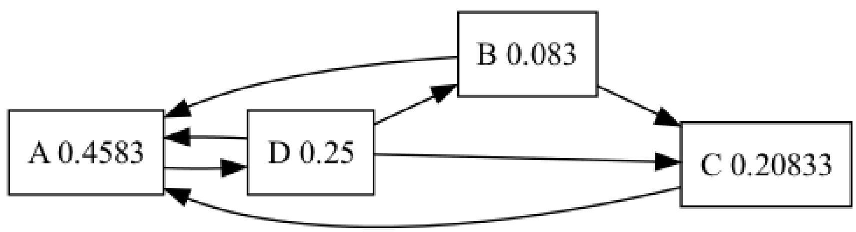


After first iteration

We'll Calculate explicitly for PR(A)

$$PR(A) = \frac{PR(B)}{L(B)} + \frac{PR(C)}{L(C)} + \frac{PR(D)}{L(D)}$$

$$PR(A) = \frac{0.25}{2} + \frac{0.25}{1} + \frac{0.25}{3}$$



Is there a problem ?

- If a page has no links to other pages, it becomes a sink and therefore terminates the random surfing process!

Sound vague .. Why is it a problem ?

intuition

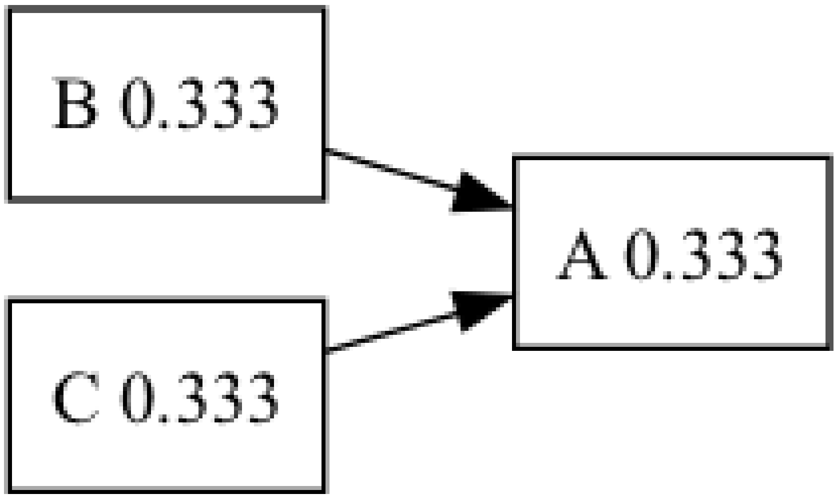
In Each iteration every note "passes" all his probability to it's outbound neighbours

but if one of the notes doesn't have outbound edges he'll receive probability but the note won't export his probablity

So the sum of the probability of all nodes $\sum_{v \in V} PR(v) = 1 - PR(v_i) < 1$ where v_i is the probability of sink in the previous iteration

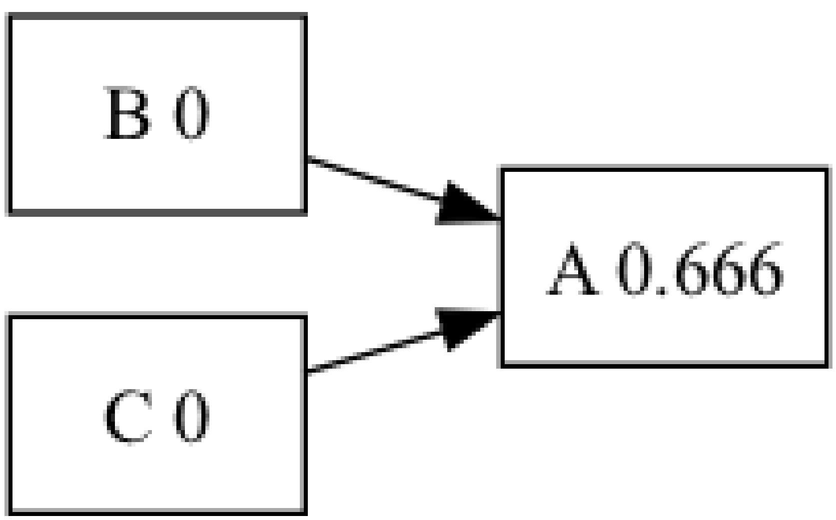
Example

Example with sink



in the first iteration

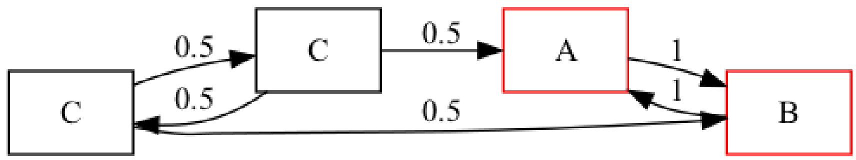
Note that node B and node C passes all it's probability to node A And node A passes 0.333 to nobody !



Another Problem

Another Problem might occur if we have inside our graph infinite cycle like in the following graph

All the PageRank would go into node a and node b



Damping Factor

solution : if we'll reach a sink we'll jump to a random note (Uniformly) by the formula

$$PR(p_i) = \frac{1-d}{N} + d \sum_{p_j \in B_{p_i}} \frac{PR(p_j)}{L(p_j)}$$

Where d is usually 0.85 i.e $d = 0.85$ and B_{p_i} is the set containing all pages linking to page u and $L(p_j)$ is the number of links from p_j

Why will it work ?? how ??

A more general notation

note that we can write the formula more compactly like so

$$\mathbf{R} = \begin{bmatrix} PR(p_1) \\ PR(p_2) \\ \vdots \\ PR(p_N) \end{bmatrix} \tag{1}$$

where "'R"' is the solution of the equation

The full equation

$$\mathbf{R} = \begin{bmatrix} (1-d)/N \\ (1-d)/N \\ \vdots \\ (1-d)/N \end{bmatrix} + d \begin{bmatrix} \ell(p_1,p_1) & \ell(p_1,p_2) & \cdots & \ell(p_1,p_N) \\ \ell(p_2,p_1) & \ddots & & \vdots \\ \vdots & & \ell(p_i,p_j) & \\ \ell(p_N,p_1) & \cdots & & \ell(p_N,p_N) \end{bmatrix} \mathbf{R} \tag{2}$$

Continue

where the adjacency function $\ell(p_i, p_j)$ is the ratio between number of links outbound from page j to page i to the total number of outbound links of page j.

$$\sum_{i=1}^{N} \ell(p_i, p_j) = 1 \tag{3}$$

The Algorithm

If the matrix $\mathcal{M}$ is a transition probability, i.e., column-stochastic and $\mathbf{R}$ is a probability distribution

- $\|\mathbf{R}\| = 1, \mathbf{ER} = \mathbf{1}$ where $\mathbf{E}$ is matrix of all ones

$$\mathbf{R} = \left(d\mathcal{M} + \frac{1-d}{N}\mathbf{E} \right) \mathbf{R} =: \widehat{\mathcal{M}}\mathbf{R}$$

Continue

Hence PageRank $\mathbf{R}$ is the principal eigenvector of $\widehat{\mathcal{M}}$. A fast and easy way to compute this is using the power method: starting with an arbitrary vector x(0), the operator $\widehat{\mathcal{M}}$ is applied in succession, i.e.,

$$x(t+1) = \widehat{\mathcal{M}}x(t),$$

until

$$|x(t+1) - x(t)| < \epsilon$$

Simple Implementation

```
import numpy as np
def pagerank(M, num_iterations: int = 100, d: float = 0.85):
    N = M.shape[1]
    v = np.ones(N) / N
    M_hat = (d * M + (1 - d) / N)
    for i in range(num_iterations):
        v = M_hat @ v
    return v
M = np.array([[0  , 0, 0 , 0, 1],
```

```
              [0.5, 0, 0 , 0, 0],
              [0.5, 0, 0 , 0, 0],
              [0  , 1,0.5, 0, 0],
              [0  , 0,0.5, 1, 0]])
v = pagerank(M, 100, 0.85)
print(v)
```

Markov Chain

Let's be more formal ! Given a Graph and initial probability vector $\pi_0 \in \mathbb{R}^n$ where n is the number of vertaces , Define the Matrix P to be $[P]_{ij}$ the probability of going from node i to node j Define $\forall j \in [0, n] \bigcap \mathbb{N}, \vec{\pi}_{n+1}(j) = \vec{\pi}_n \cdot \vec{P}_{i,j}$

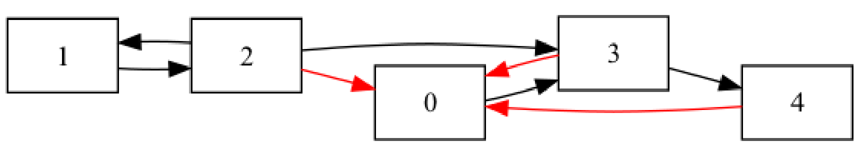
More definition

we want to find a unique stationary distribution $\lim_{n\to\infty} \pi_n = \pi$ and Rank the Web Pages via that unique stationary distribution !

Example

Observe the following graph

$\pi_{n+1}(0) = \pi_n \cdot P(0,0) + \pi_n \cdot P(1,0) + \pi_n \cdot P(2,0) + \pi_n \cdot P(3,0) + \pi_n \cdot P(4,0)$



Serious questions

- Is there unique stationary distribution ? Certainly it would be hard to evaluate web pages if there are couple of stationary distribution
- Does every initial distribution converges to the stationary one ? if so how to pick initial distribution ?

serious question . . . maybe mathematics could help us ??

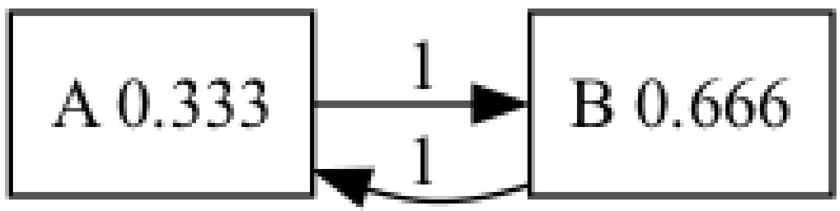
First question

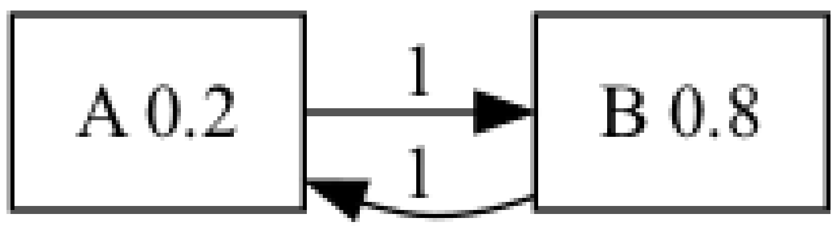
Can you define a Markov chain with multiple stationary distributions ?

Think of markov chain with only 2 states . . .

The Answer

This graph has multiple stationary solutions





Another Example

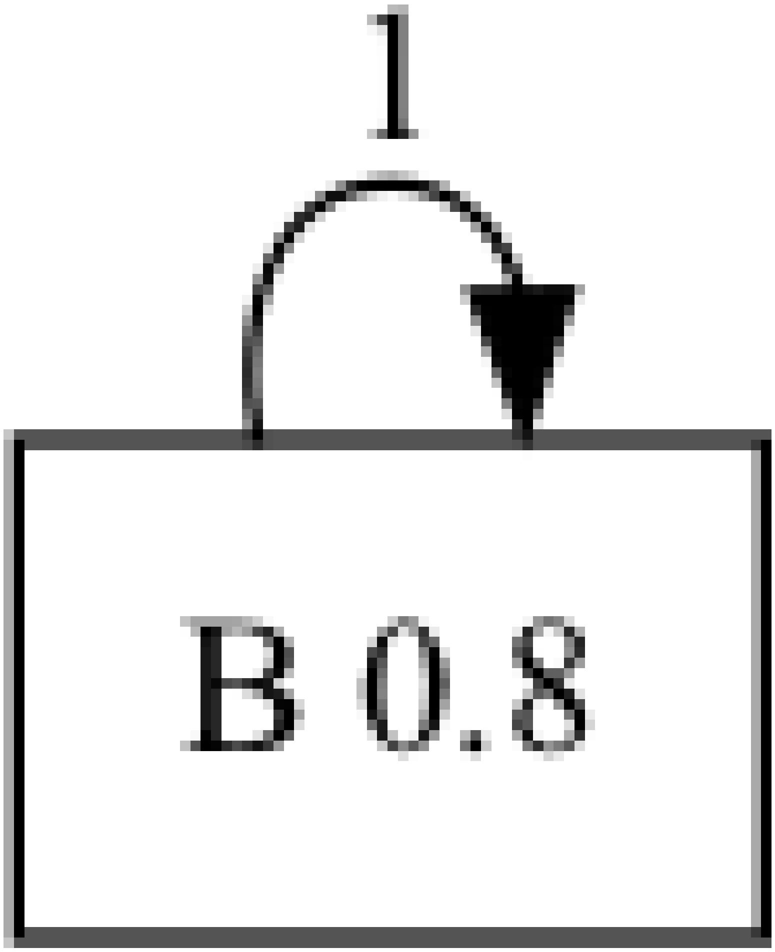
Can we Think of another example ?? Graph that isn't strongly connect .i.e exists 2 nodes such that we cannot create a path from the former to the latter

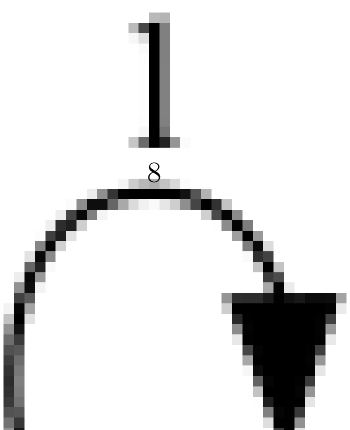
interestingly , That is one of the definition of "Reducible Markov Chain"

Let's see the example !

Visulaize

There isn't a path between node 0 into node 1 i.e this graph isn't reducible So again the stationary solution isn't unique !!


1
B 0.8


1

Conclusion

The first graph was periodic markov chain and the second graph was reducible markov chain

apparently The answer is yes to the 2 previous question if the graph is aperiodic and irreducible markov chain

Theorem

if graph is Irreducible markov chain i.e all state are reachable $\rightarrow$ there is unique stationary distribution

Note , The first example has a unique stationary distribution (0.5 0.5) All other solutions aren't stationary for example (0.3 0.7) would oscilate between them

The second question

Does every initial distribution converge to the stationary one ?

Clarification ! for all initial distribution we want to converge to **the** stationary one

Give me an irreducible markov chain where the stationary distribution does not converge we have already counter-example , so what else we need to assume on the graph ?

Periodic Markov Chain

- Must be an irreducible markov chain
- User visits states in regular interval (period) > 1 (There is a better definition)

So Given an Periodic Markov Chain There is no guarantee of convergence to stationary distribution

If no such period exists > 1 , then we said that the graph is Aperiodic Markov Chain

Ergodic Theorem

For Irreducible and aperiodic markov chains:

1. A unique stationary distribution π exists

2. All initial distribution π_0 converges to that unique stationary distribution π

How fast does the solution converges ?

Because of the large eigengap of the modified adjacency matrix above, the values of the PageRank eigenvector can be approximated to within a high degree of accuracy within only a “few” iterations.

Davis–Kahan theorem

But What is “few” ?

log(n)

The End

End !

To delete

Column 1 Column 2

To delete

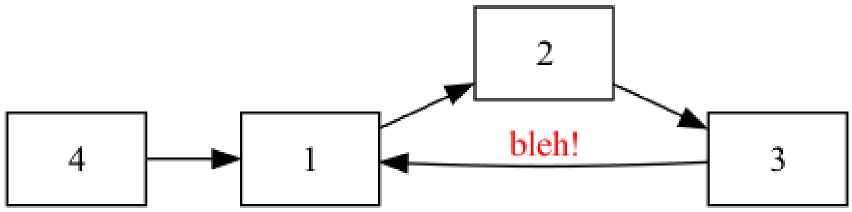
Box 1 Box 2 Box 3

Some Notable code

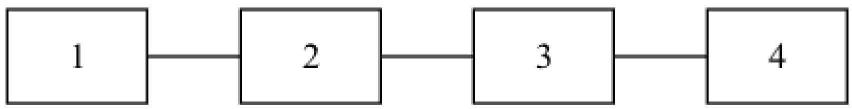
```
int main()
{
  cout << "Hello" << endl;
}
```

Slide 1

- List item 1



- List item 2



Nice Code Animation

```
let index = 1
```

Added value

```
let index = 1
let value = 2
```

some equation

$$\sin(x) = \frac{1}{n}$$