Assignment 2

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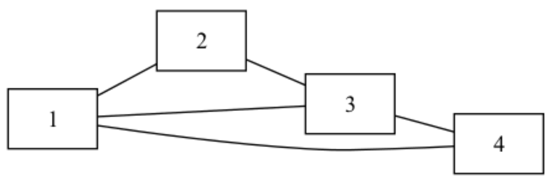
Question 1

Given the following graph



Calculate all walks of size 6 in the given graph

Answer

In order to do so we'll use the following theorem

Theorem 1 (Walks Theorem). *If A is the adjacency matrix of a graph or digraph G with vertices $\{v1, ...vn\}$, then the i, j entry of A^k is the number of walks of length k from v_i to v_j*

By 1 we can just multiple the adjacency matrix by itself 6 times and we'll get all the walks available from node i into node j

Since the problem specify undirected graph we'll have to sum all elements of the matrix and divide it by 2

Algorithm 1 All walks of length 6

$n \leftarrow 6$ ▷ 6 => length of walk
Adj ▷ adjacency matrix
$M \leftarrow I$
while $n \neq 0$ **do**
  $M \leftarrow M \times Adj$ ▷ Matrix Multiples
end while
$sum \leftarrow 0$
while $a \in M$ **do**
  $sum \leftarrow sum + a$
end while

We will apply **algorithm** 1 for getting the number of walk of length 6

```
import numpy
from numpy.linalg import matrix_power
input_array = numpy.array([[0,1,1,1],
                           [1,0,1,0],
                           [1,1,0,1],
                           [1,0,1,0]])
A_to_power6 = matrix_power(input_array, 6)
sum_var = 0
for row in A_to_power6:
    for elem in row:
        sum_var += elem
int(sum_var/2)
# output is 557
```

Question 2

Consider the following quote

> "Undirected Graph can be considered as directed graph"

Prove it

Formally , Given an Undirected graph find a directed graph such that $A_G = A_{G'}$

Answer

Given An undirected graph mark it $G = (V, E)$ and the adjacency matrix of his as A_G

Define the directed graph $G' = (V', E')$ where $V' = V$ and

$$\forall e = (v_1, v_2) \in E : e_1 = (v_1, v_2), e_2 = (v_2, v_1) \in E'$$

The adjancey matrix of the directed graph is equal to the adjancey matrix of the undirected graph

$$A_G = A_{G'}$$

Question 3

Prove that given a directed graph $G = (V, E)$ where $V = (1, 2, .., n)$, let A be the adjacency matrix

$$k \in \mathbb{N} \cup \{0\} : \forall i, j \in V : F(j, i, k) = (A^k)_{i,j}$$

where $F : V \times V \times \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ are all the walks from node j to i of length k

Answer

We'll prove the theorem by induction

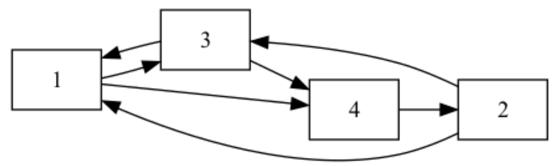
Proof. By induction

Base Case: For k = 1, $A^k = A$, and there is a walk of length 1 between i and j if and only if $a_{ij} = 1$, thus the result holds.

Step Case: Assume the proposition holds for $k = n$ and consider the matrix $A_{n+l} = A_n A$, By the inductive hypothesis, the $(i, j)^{th}$ entry of A_n counts the number of walks of length n between vertices i and j. Now, the number of walks of length n + 1 between i and j equals the number of walks of length n from vertex i to each vertex v that is adjacent to j. But this is the $(i, j)^{th}$ entry of $A^n A = A^{n+1}$ the non-zero entries of the column of A corresponding to v are precisely the first neighbours of v. Thus the result follows by induction on n $\square$

Question 4

Find the number of possible walks of length 8 from 1 to 4 by the following undirected graph



Answer

We'll Write the adjancey matrix and calculate the A^8

To do so we'll use the same code

```
import numpy
from numpy.linalg import matrix_power
input_array = numpy.array([[0,1,1,0],
                           [0,0,0,1],
                           [1,1,0,0],
                           [1,0,1,0]])
A_to_power6 = matrix_power(input_array, 8)
A_to_power6[3][0]
# output is 23
```

To Calculate the Matrix by power of 8 we will use eigen values $\det(A - \lambda I_n) = 0$ lets apply the calculation

$$\det(A - \lambda I_n) = \begin{vmatrix} -\lambda & 1 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ 1 & 1 & -\lambda & 0 \\ 1 & 0 & 1 & -\lambda \end{vmatrix} = 0$$

In order to find eigen value will python code

```
import numpy as np
from numpy.linalg import eig
input_array = np.array([[0,1,1,0],
                        [0,0,0,1],
                        [1,1,0,0],
                        [1,0,1,0]])
w,v=eig(input_array)

w # The vector of eigen values
# w = | 1.69 | -1 | +0.j | -0.347-1.028j |
```

And finally we do

$$p^{-1} A^8 p = \begin{pmatrix} \lambda_1^8 & 0 & 0 & 0 \\ 0 & \lambda_2^8 & 0 & 0 \\ 0 & 0 & \lambda_3^8 & 0 \\ 0 & 0 & 0 & \lambda_4^8 \end{pmatrix}$$

and change basis to get

$$A^8 = \begin{pmatrix} 19 & 23 & 18 & 13 \\ 13 & 14 & 13 & 10 \\ 18 & 23 & 19 & 13 \\ 23 & 26 & 23 & 14 \end{pmatrix}$$

Question 5

Prove that the probablity to pass from vertex j to vertex i is given by the matrix $\tilde{A_G} = A_G * D_G^{-1}$ where the matrix D_G is define to be

$$D_G = \begin{pmatrix} \deg(1) & 0 & ... & 0 \\ 0 & \deg(2) & ... & 0 \\ 0 & 0 & ... & 0 \\ 0 & 0 & ... & \deg(n) \end{pmatrix}$$

careful $\tilde{A_G}$ might not be symmetric

Answer

Proof. Since G is undirected graph and the probablity to travel from j to i is even distributed , the probability matrix is to divide the matrix A_G column j by $\deg(j)$ for each $j \in \mathbb{N} \cap [1, n]$ or more formally

$$\forall j \in \mathbb{N} \cap [1, n] \, , (\hat{A_G})_{i,j} := \frac{(A_G)_{i,j}}{\deg(j)}$$

But Happily this is exactly equivalent to simply multiply the following matrix

$$\tilde{A_G} = A_G * D_G^{-1}$$

i.e

$$\tilde{A_G} = \hat{A_G}$$

Which is what we want it to be □

Question 6

let $p^{(0)} \in \mathbb{R}$ be the initial probability distribution of the graph

let $p^{(n)} \in \mathbb{R}$ be the distribution of the graph after n walks

prove that

$$p^{(n)} = \tilde{A_G^n} * p^{(0)}$$

Answer

Proof. Given a vertex j and vertex i we first want to find all possible walks from j to i with length n ,

We have already proven that the number of possible walks are $(A_G^n)_{i,j}$

In order to get the required probability we need to find the sum of all walks from j to any vertex i.e

$$\sigma_j = \sum_{i \in V} (A_G)^n_{i,j}$$

$$(Prob)_{i,j} = \frac{(A_G^n)_{i,j}}{\sigma_j}$$

Ultimately , I have proven that the probability of walking n walks from vertex j to i is simply $\tilde{A}^n$

□

Note that we are not yet done we still have to prove that

$$p^{(n)} = \tilde{A}^n * p^{(0)}$$

To skip to that click final

Another proof is by induction

Proof. $\tilde{A}^{m+1} = \tilde{A} * \tilde{A}^m$

Base Case: for n=0 we have $\tilde{A} = \tilde{A}$

Step Case: Given that the claim is true for $n \in \mathbb{N}$ we will prove it for $n + 1$, more specificly given j and i we are searching for the probablity of going from j to i after $n + 1$ walks

Since by assumption we already know the probabilty of walking n long from j to any $v \in V$ which is $(A_G^n)_{v,j}$ and also we know the probablity of getting from any vertex v into vertex i which is $(A_G)_{i,v}$

The probabilty of getting from j to i after n+1 walks is by conditional probabilty

$$Prob^{(n+1)} = \sum_{v \in V} (A_G)_{i,v} * (A_G^n)_{v,j}$$

since

$$P(B) = \sum_i P(B|A_i) \, , \sum_i P(A_i) = 1$$

This equation is nothing but $\tilde{A}^{m+1} = \tilde{A} * \tilde{A}^m$ □

Now Given a vertices $i \in V$ by the complete probability theorem the probability of getting into vertex i is

$$p_i^{(n)} = \sum_{v \in V} \tilde{A}_{i,v} * p_v^{(0)}$$

But this equation is nothing but what we needed to prove which is

$$p^{(n)} = \tilde{A}^n * p^{(0)} \tag{1}$$

Question 7

Prove that $\tilde{A}$ is diagolizable over $\mathbb{R}$

Answer

We will use the spectral theory for the prove

Spectral Theory

Reminder: Please Note that $\tilde{A} = A_G * D^{-1}$

By assumption , we know A is a symmetric matrix ,

lets mark the matrix

$$Q \in \mathbb{M}_n(\mathbb{R}) : Q := \operatorname{diag}(\sqrt{\deg(v_1)}, \ldots, \sqrt{\deg(v_n)})$$

Observe the following logic:

$$Q^{-1} * \tilde{A} * Q = Q^{-1} * A * D^{-1} * Q = Q^{-1} * A * Q^{-1}$$

where $A = A^t$

$$\begin{aligned}(Q^{-1} * A * Q^{-1})^t &= (A * Q^{-1})^t * (Q^{-1})^t = \\ &= (Q^{-1})^t * A^t * (Q^{-1})^t \\ &= (Q^{-1})^t * A * (Q^{-1})^t \\ &= Q^{-1} * A * Q^{-1}\end{aligned}$$

By this equation we can infer that the matrix

$Q^{-1} * \tilde{A} * Q$ is a symmetric matrix which means that we can use on it the spectral theory

$$\begin{aligned}\exists P \in \mathbb{M}_n(\mathbb{R}) : P^{-1}*(Q^{-1}*\tilde{A}*Q)*P &= \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \\ &= (QP)^{-1}\tilde{A}QP\end{aligned}$$

So I have found a matrix QP which diagolize the matrix $\tilde{A}$

Question 8

let λ be an eigen value of $\tilde{A}$ proof that $\lambda \in [-1, 1]$

Answer

I'll prove it by contradiction , on the matrix $\tilde{A}^t$, let λ be eigen value of $\tilde{A}$

$$\exists v \in V : \tilde{A}v = \lambda v$$

for some $\lambda > 1$

Since the row of the matrix $\tilde{A}^t$ sums to 1 , each element of $\tilde{A}^t x$ is an convex combination of the components of x , which can't be greater than a x_{max} where x_{max} is the maxium element of the vector x .

Let's see it formally

$$x_{max} := \max_{j \in \mathbb{N} \cap [1,n]} |x_j|$$

$$\alpha := \tilde{A}^t$$

$$\forall \mathbb{N} \cap [1, n] \sum_{j=1}^{n} \alpha_{ij} = 1$$

$$\begin{aligned}\forall x \in \mathbb{R}^n , \forall i \in \mathbb{N} \cap [1, n] , \sum_{j=1}^{n} \alpha_{ij} x_j \leq \sum_{j=1}^{n} \alpha_{ij} x_{max} &= \\ = x_{max} \sum_{j=1}^{n} \alpha_{ij} &= x_{max}\end{aligned}$$

In conclusion

$$(\tilde{A}^t x)_{max} \leq x_{max} \tag{2}$$

On the other hand , At least one element of λx is greater than x_{max} , which proves that $\lambda > 1$ is impossible

Or , more precisely , let $\lambda > 1$ be an eigen value of the matrix $\tilde{A}^t$ and let v_λ be the normalized eigen vector of eigen value λ then

$$(v_\lambda)_{max} < \lambda (v_\lambda)_{max} = (\lambda v_\lambda)_{max} = (\tilde{A}^t v_\lambda)_{max} \tag{3}$$

By equation 2 we can select $x := v_\lambda$ and we'll have $(\tilde{A}^t v_\lambda)_{max} \leq (v_\lambda)_{max}$ which contradict equation 3

Introduction

Simple pseudo code