Decision Theory

Scientific Theory in Informatics

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Basic Blocks

Decision Problem

Cape Town, 1967

After years of experiments on animals, Dr. Barnard offers heart transplant to a 55 year old human patient suffering of severe heart disease.

The operation has never been previously tested on humans.

Decision Matrix

	METHOD WORKS	METHOD FAILS
OPERATION	Live on for some time	Death
NO OPERATION	Death	Death

Figure: Decision matrix - the patient's perspective. No information is available: will the method work or not?

Decision Matrix

		METHOD WORKS	METHOD FAILS
ACTS	OPERATION	Live on for some time	Death
	NO OPERATION	Death	Death

Figure: Decision matrix - the patient's perspective.

Decision Matrix

	STATES of the WORLD	
ACTS	METHOD WORKS	METHOD FAILS
OPERATION	Live on for some time	Death
NO OPERATION	Death	Death

Figure: Decision matrix - the patient's perspective.

Decision Matrix

	STATES of the WORLD	
	METHOD WORKS	**METHOD FAILS**
OPERATION	Live on for some time	Death
ACTS	OUTCOMES	
NO OPERATION	Death	Death

Figure: Decision matrix - the patient's perspective.

Decision Matrix

	Rain (P=.2)	**No rain (P=.8)**
Umbrella	Dry clothes, heavy suitcase	Dry clothes, heavy suitcase
No umbrella	Soaked clothes, light suitcase	Dry clothes, light suitcase

Figure: Decision matrix - an example of decision problem in which the probability of the states of the world is known.

More on Acts

- *Open*, if new alternatives can be discovered by the decision-maker (typical of actual life problems – e.g. how to spend this evening).
- *Closed*, if no new alternatives can be added (e.g. your vote options at the next elections):
  - We can decide to limit our options - *voluntarily closed* alternatives;
  - We might be forced to simply accept the available options - *involuntarily closed* alternatives.
- *Mutually exclusive*: such that no two acts can be simultaneously realized (this makes the problem formulation more clear).

Common hypotheses on Acts: set of alternatives actions is **closed** and its elements are **mutually exclusive**.

Formalization vs Visualization

Three levels of abstraction:

- Decision problem;
- Formalization of the decision problem;
- Visualization of the formalization.

Decisions

"Decision theory is concerned with goal-directed behavior in the presence of options."

- **Goal-directed behavior** implies a set of beliefs and desires (e.g. it is going to rain & I want to stay dry).
- How can an **individual** coordinate decisions over time?
- How can several individuals coordinate their decisions in **social decision** procedures?

Decisions

- Shall I bring my umbrella today?
- Shall I vote for the liberal or conservative party?
- Should I buy the insurance that the electronics store is offering me?
  - How does the cost of the insurance relate to its worth?
  - What information do I need?
  - What information do I have available?

Decisions

Figure: The potential domain of Decision Theory is about all aspect of life!

Certainty vs Uncertainty

Distinction rests in our level of knowledge about the states:

- *Certainty*: an action is known to lead invariably to a specific outcome.
  - i.e. Deterministic knowledge - e.g. p(rain)=1.
- *Uncertainty*: for us, either synonym for or umbrella term indicating both *risk and ignorance* (see next slide).

Certainty vs Uncertainty

- *Risk*: known probability of the possible states of the world - each action leads to one of a set of possible specific outcomes, each outcome occurring with *known* probability.
  - i.e. complete probabilistic knowledge (e.g. weather forecast announces p(rain) $= 0.6$).

- *Ignorance*: probabilities are either unknown or non-existent - each action leads to one of a set of possible specific outcomes, whose probability is *partially known or unknown*.

Descriptive vs Normative Decision Theory

- *Descriptive decision theories* seek to explain and predict how people actually make decisions under uncertainty.
    - Cognitive limitations, time pressure, subjective reference points, etc.;
    - Concerned with internal mechanisms of human decision making.

- *Normative decision theories* seek to produce prescriptions about what decisions makers are rationally required to do under uncertainty.
    - Concerned with mathematically optimal choice.

Scope of Normative DT

Figure: Aiming for an ambitious formal description (Normative Decision Theory) requires a reduction of the scope!

Scope of Normative DT

- DT is about rational (rather than right) decisions.
- Instrumental rationality: the decision maker has goals to accomplish (e.g. being rich; helping refugees, etc.), and (s)he will do whatever has most reason to expect will fulfil such goals.
- "Normative DT takes care of those normative issues that remain **after the goals have been fixed**."
- The decision maker has **ethical or political norms** that support such goals, which are already fixed too.

Value System

Preference Logic

- Domain: set of entities (A, B, C, ...)

- Preference Logic is a way to express a value system:
  - "Better than" (preference): $>$
  - "At least as good as" (weak preference) : $\geq$
  - "Equal in value to" (indifference): $\equiv$

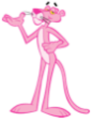
- Value system can be expressed as "better than" order relation over the elements of the domain (e.g. $A > B$, $B > C$, etc.).

Completeness

The preference relation $>$ is *complete* if and only if for any two elements A and B on its domain, either $A > B$ or $B > A$.

- Example:
  - A $>$ C
  - A $>$ B
  - C ? B
- Preference completeness is a common simplifying assumption – problematic!
- In fact, "...most of our preferences have been acquired, and the acquisition of preferences may cost time and effort."

A



B

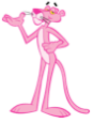
C

Transitivity

Preference relation $>$ is *transitive* if and only if it holds for all elements A, B, and C on its domain that:
if $A > B$ & $B > C$, then $A > C$

- Example: if $A > B$ & $B > C$ $\Rightarrow A > C$

A



B

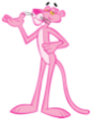
C

Transitivity

"Intransitive preferences are often inadequate to guide actions." - Violation of rationality criteria!

- Example: You offer me a free poster. Assuming completeness but not transitivity, I answer:
  - $B > C$
  - $A > B$
  - $C > A$
- Similar definitions of completeness and transitivity also apply to weak preference and indifference.

A



B

C

Transitivity

Although common, the assumption of transitivity can be problematic, particularly for $\equiv$ and $\geq$.

Example - consider 1000 cups of coffee:

- C_0 (no sugar), C_1 (1 grain of sugar), C_2 (2 grains), ..., C_{999} (999);
- Our taste cannot resolve the difference between C_{i-1} and C_i;
- Therefore: $C_0 \equiv C_1, \ldots, C_{i-1} \equiv C_i, \ldots, C_{998} \equiv C_{999}$;
- From transitivity: $C_0 \equiv C_1$ & $C_1 \equiv C_2 \Rightarrow C_0 \equiv C_2$.
- Similarly, $C_0 \equiv C_3$, ..., $C_0 \equiv C_{999}$
- But we can taste the difference (and express a preference) between C_0 and C_{999}. Hence, it does not seem plausible for the indifference relation to be transitive in this case.

Decision Rule

The rational decision maker tries to obtain as good an outcome as possible, according to some value system, which grades what is good or bad:

- "An alternative is *(uniquely) best* if and only if it is better than all other alternatives. If there is a uniquely best alternative, choose it."
- For multiple best alternatives (e.g. $A > C$, $B > C$, $A \equiv B$): "An alternative is *(among the) best* if and only if it is at least as good as all other alternatives. If there are multiple alternatives that are best, pick one of them."

Values

- To proceed with a decision, outcomes need to be ranked. Ideally, we want to express a *value* for each outcome - that is not an obvious step!
- DT assumes that a value system is at hand, and can be expressed in a precise and pragmatic way.

- We express our value system by a function that associates a numeric value to each possible outcome (see an example in the next slide).
- Therefore, an order relation is implicitly defined and $>$, $\geq$ and $\equiv$ are always complete and transitive.

Values

	Fire	**No fire**
Insurance	No house and 100,000 $ (1)	House and 0 $ (4)
No insurance	No house and 100 $ (-100)	House and 100 $ (10)

Where do the numeric values come from? What is the metric for "house" or "having cash"?

Identifying such metrics can be problematic.

Utilitarianism

"According to some moral theorists, all values can be reduced to one single abstract entity: *utility*."

- "Utilitarianism gives rise to a decision theory based on numerical representation of value."
- Utility can be subjective (e.g. in economics) or objective (e.g. risk analysis).
- In economic theory: presumption that everything can be "measured with the measuring rod of money", i.e. utility is assessed as monetary value (willingness to pay).

Diverse Research Traditions

- Philosophy;
- Mathematics and Statistics;
- Economics;
- Political Science
- Social Science
- Psychology
- Social Psychology
- Machine learning
- ...

Decision Under Ignorance

Maximin Principle

Maximin focuses on the worst possible outcome of each alternative.

Maximin P.: Prefer alternatives in which the worst possible outcome is as good as possible - i.e. maximize the minimal value obtainable with each act:
$a_i \succeq a_j \Leftrightarrow min(a_i) \geq min(a_j)$.

NB: An ordinal scale is sufficient to apply the system.

	s_1	s_2	s_3	s_4
a_1	6	9	3	0
a_2	-5	7	4	12
a_3	6	4	5	2
a_4	14	-8	5	7

	Good chef	**Bad chef**
Monkfish (a_1)	Delightful meal (4)	Disgusting meal (1)
Hamburger (a_2)	Decent meal (3)	Decent meal (3)
No main course (a_3)	Hungry (2)	Hungry (2)

Maximax Principle

Example: You send an application to graduate school (the alternative is not to - the outcome will be status quo).

- If you are accepted that will lead to a brilliant career leading to the Nobel Prize.
- If you are rejected you will feel miserable.

Maximax P.: Prefer alternatives in which the best possible outcome is as good as possible (maximize the maximal value).

Decision Under Ignorance: Which Rule?

- There is little consensus on the decision rule to apply to decision under ignorance.
- To the contrary, high convergenc exists about decision making under risk (see next section).

Decision Under Risk

Expected Utility (EU)

- The *utility matrix*, which includes information about acts, states and outcomes, reports a probability for each state of the world (i.e. for each column).
- For each alternative we calculate the *expected utility value*, as the sum of the value of its outcomes weighed by the probability of the different states of the world:
  $EU = \sum p(x_i) \cdot u(x_i)$ for i = 1, ... n.

	Rain (P=.1)	No rain (P=.9)
Umbrella	15	15
No umbrella	0	18

Maximization of Expected Utility (MEU)

According to utilitarian moral theory, individuals should maximize the utility resulting from their actions.

MEU Rule: choose the course of action that maximizes EU.

Example:

- $EU(umbrella) = .1 \cdot 15 + .9 \cdot 15 = 15;$
- $EU(no\ umbrella) = .1 \cdot 0 + .9 \cdot 18 = 16.2.$

	Rain (P=.1)	No rain (P=.9)
Umbrella	15	15
No umbrella	0	18

Maximization of Expected Utility (MEU)

- MEU is an operative method to maximizes the outcome in the long run, over *large numbers* of similar decisions;
- "EU maximization is only meaningful in comparison between options in one and the same decision."
- MEU models are at the core of neoclassical economic theory of rational behavior.

The Neoclassical Decision Maker

Figure: Archetype of neoclassical decision maker: a model of perfect rationality.

- Infinitely rational,
- Infinitely knowledgeable,
- Infinitely selfish,
- Provided with infinite computational capacity (cognitive resources).

- Calculates the consequences of each alternative.
- Ranks all consequences.
- Takes the optimal decision, i.e. maximises EU.
- Anomalies compared to the EU model are **paradoxical**.

Experimental Support for EU

A classic DT experiment:

- Monetary gambles.
- Likelihood of distributing different payoffs (probability of the states of the world).

The fanning-out pattern of the curves (see figure) is compatible with the EU model.

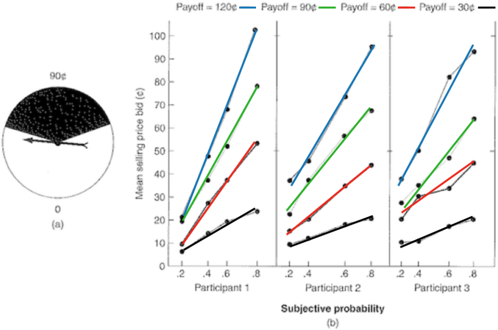


FIGURE 9–3 What's it worth?

(a) The wheel of fortune presented to participants, who were asked to "sell" the opportunity to bet for payoffs of 120¢, 90¢, 60¢, and 30¢ at different probabilities. Only the opportunity for the 90¢ opportunity is shown here; the ratio of dark to light area indicates the probability of winning. (Shanteau, J. (1975). An information-integration analysis or risk decision making. In M. F. Kaplan and S. Schwartz (eds.) Human judgement and decision processes (pp. 109–137). New York: Academic Press. Reprinted with permission from Elsevier.)

(b) The graphs plot the results for three participants. All the graphs show a clear "fan" pattern, indicating that the participants are responding as though they are multiplying the probability and the payoff terms and order the gambles in a manner like that prescribed by the expected utility model.

Towards a Descriptive Theory

EU Anomalies

- With the concept of EU, we have a theory for decision making under risk.
- Over the years, the theory has received massive empirical validation.
- However, many researchers have also produced several examples of "anomalies" and paradoxes.

Violation of Transitivity Axiom

Gamble	Probability of winning	Payoff	Expected value
A	7/24	5.00	1.46
B	8/24	4.75	1.58
C	9/24	4.50	1.69
D	10/24	4.25	1.77
E	11/24	4.00	1.83

(a)

(b)

FIGURE 9–8 Gambles that produce intransitive choices

(a) Participants were given pairwise choices of which of these gambles they wished to play by spinning a wheel of fortune (b). Because the payoff amounts are easy to comprehend, the amounts controlled the pairwise choices and participants consistently preferred the higher payoff gamble in any *adjacent* bets. But the gambles were cleverly designed so that as payoffs went up (from $4.00 to $5.00), probabilities went down (from 11/24 to 7/24), so that the discrepancy between the expected value for the $4.00 bet and the $5.00 bet was dramatic ($1.83 versus $1.46). Participants faced with a choice between the *extreme* bets reversed the preference ordering implied by their pairwise choices, choosing the $4.00 (11/24) bet over the $5.00 (7/24) bet—and showed consistent, and irrational, intransitivities.

(Tversky, A. (1969). The intransitivity of preferences. *Psychological Review, 76,* 31; 48, Table 1 and Figure 1. © 1969 American Psychological Association. Reprinted with permission.)

The Allais Paradox (1953)

Which gamble do you prefer?

	Ticket 1	Tickets 2–11	Tickets 12–100
Gamble A	\$1M	\$1M	\$1M
Gamble B	\$0	\$5M	\$1M

Which gamble do you prefer?

	Ticket 1	Tickets 2–11	Tickets 12–100
Gamble C	\$1M	\$1M	\$0
Gamble D	\$0	\$5M	\$0

The Allais Paradox (1953)

- Most people prefer A over B and D over C.
- However, if we calculate the difference in expected utility for the two gambles:

  $EU(A) - EU(B) =$
  $u(1M) - [0.01 \cdot u(0M) + 0.1 \cdot u(5M) + 0.89 \cdot u(1M)] =$
  $= 0.11 \cdot u(1M) - [0.01 \cdot u(0M) + 0.1 \cdot u(5M)]$

  $EU(C) - EU(D) =$
  $[0.11 \cdot u(1M) + 0.89 \cdot u(0M)] - [0.9 \cdot u(0M) + 0.1 \cdot u(5M)] =$
  $= 0.11 \cdot u(1M) - [0.01 \cdot u(0M) + 0.1 \cdot u(5M)]$

- Therefore, independently on the decision maker's utility for money, it impossible to simultaneously prefer A over B and D over C without a violation of the EU principle.
- Allais thought that his example revealed a deep flaw in the EU principle.

The Ellesberg Paradox

Ellsberg's ambiguous urn

Daniel Ellsberg proposed to his research participants this urn, which contains balls of three colors. Participants are told the exact probability of drawing a red ball (30 reds out of 90 balls or $p_{\text{red}} = 0.33$) (represented here by blue), but the probabilities of drawing yellow (represented by white) or black balls are ambiguous. The participants know that the total is 60 balls (out of 90, or $p_{\text{yellow or black}} = 0.67$), but not the specific proportions.

- The urn contains a total of 90 balls, 30 of which are red, and the remaining 60 balls are an unknown mixture of black and yellow balls.
- Thus, the probability of drawing a red ball is 0.333, but the probability of drawing a black ball or the probability of drawing a yellow ball remains, ambiguous.

The Ellesberg Paradox

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Choose:

- Gamble 1: you win 100$ if red, nothing if yellow or black.
- Gamble 2: you win nothing if red, 100$ if black, nothing if yellow.

Choose:

- Gamble 3: you win 100$ if red or yellow, nothing if black.
- Gamble 4: you win 100$ if black or yellow, nothing if red.

The Ellesberg Paradox

- Typical choice is gamble G1 and G4 (people seem to seek certainty over ambiguity).
- However, independently on utility for money and beliefs on proportion of black/yellow, EU predicts consistency in the choice (either G1 and G3, or G2 and G4).
- In fact, the difference in expected utility for the two gambles, when labeling $u(100\$) = M$ and setting $u(0\$) = 0$ is:

$$EU(G1) - EU(G2) = (30/90) \cdot M - (B/90) \cdot M =$$
$$= (30 - B) \cdot M/90$$

$$EU(G3) - EU(G4) = ((30 + 60 - B)/90) \cdot M - (60/90) \cdot M =$$
$$= (30 - B) \cdot M/90$$

Individual Risk Attitude

Typically, humans show *risk aversion* for gains and *risk seeking* for losses. A sure gain is typically chosen over a probabilitic gain, and a probabilistic loss over a sure loss.

Example - toss a fair coin and consider the following two couples of gambles:

- Gamble 1: you win 10$ if head is tossed vs. 50$ if tail.
- Gamble 2: you receive 30$ for certain.

- Gamble 3: you lose10$ if head is tossed vs. 50$ if tail.
- Gamble 4: you lose 30$ for certain.

People tend to choose 2 over 1 (risk aversion for gains) and 3 over 4 (risk seeking for losses), although EU are the same over all the gambles.

Framing Effect

Description invariance: The description a the problem should be irrelevant to the final choice (i.e. a decision problem described in logically equivalent terms should lead to the same choice).

Framing Effect

PROBLEM 1 (Tversky & Kahneman, 1981)
The government is preparing for the outbreak of an unusual disease, which is expected to kill 600 people. Two alternative programs to combat the disease have been proposed.

Choose between programs A and B:

- If Program A is adopted, 200 people will be saved.
- If Program B is adopted, there is a 1/3 probability that 600 people will be saved, and a 2/3 probability that no people will be saved.

PROBLEM 2 (Tversky & Kahneman, 1981)
The government is preparing for the outbreak of an unusual disease, which is expected to kill 600 people. Two alternative programs to combat the disease have been proposed.

Choose between programs C and D

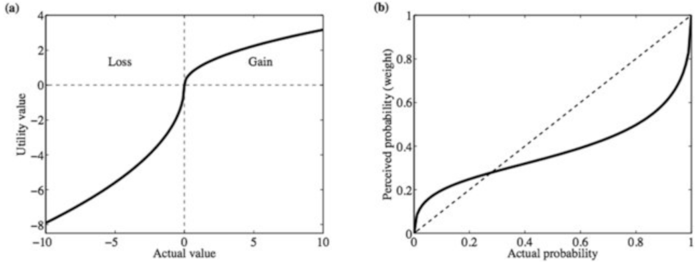
- If Program C is adopted, 400 people will die.
- If Program D is adopted, there is a 1/3 probability that nobody will die, and a 2/3 probability that 600 people will die.

Framing Effect

- A and C have objectively identical outcome (200 survivors, 400 deaths) and so B and D.
- Problem 1: A and B are designed to have same EU, but we are *risk adverse for gains* and we tend to choose A.
- Problem 2: C and D are designed to have same EU, but we are *risk seeker for losses* and we tend to choose D.

Prospect Theory

Tversky & Kahneman (1977) introduced a descriptive theory including two corrective functions: a weighting function for utility and one for probability.



Prospect Theory

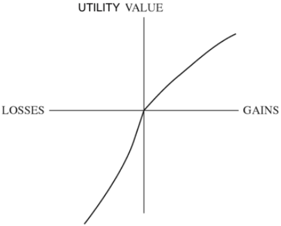


Figure 14.2 A hypothetical weighting function for values.

- This non-linear weighting function rescales the "net worth" of the outcome (value) in terms of subjective utility.
- *Diminishing marginal utility* (saturation).
- *Loss aversion* - loss is perceived as more significant than the equivalent gain (different 1^{st} derivative for gain/loss).

Prospect Theory

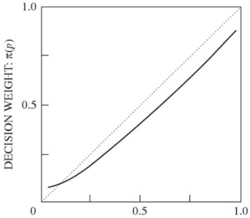


Figure 14.1 A hypothetical weighting function for probabilities.

- We tend to overweight small probabilities (prob. fatalities for airplane vs. car accidents) and underweight higher probabilities.
- Violation of probability axioms (e.g. $p(head) + p(tail) < 1$).

Prospect Theory

- First stage in DM is framing the terms of the decision, i.e. estimating prospective gains and losses in relation to a reference point - *anchor* – typically the current situation.
- Anchors are frequently updated.
- Subjective utilities and probabilities are mentally represented for the prospect under consideration.
- Expected value calculation is performed on the prospect – analogous to EU model, but now framing effects and decision weights violate mathematical and economic rationality.

- Prospect theory yields fairly accurate predictions on many behavioral patterns in human (and other animals') DM, as long as we consider rather simple decision problems.
- However, the theory does not provide a specific account for violations of transitivity and completeness, nor for preference reversal.

Bounded Rationality

The theory of *Bounded Rationality* builds on a reflection on people's limited cognitive resources (e.g. limited access to information, limited capacity to process information, limited working memory, etc.)

- Humans are far from perfect computers (e.g limited attention and working memory).
- Trade-off between searching for and processing information, and choice of the absolute best alternative.
- The decision process can be halted on anything that is good enough (*satisficing*) to meet the standards of the decision maker.
- Based on empirical evidence, BR theory provides a fairly good description of everyday decision making behavior.

The Contemporary Decision Maker

Even non-pathological people (Beinhoker, Origin of wealth, 2006):

- Framing biases
- Representativeness
- Availability biases
- Difficulties judging risks
- Dynamic inconsistency
- Superstitious reasoning
- ...

Figure: Archetype of contemporary decision maker.

Suggested Readings

Suggested Readings

Available on Canvas:

- Hansson S.O. (2005) Decision Theory, a brief introduction.
- Smith E.E. and Kosslyn S.M.: Cognitive psychology (Chapt. 9). Pearson Education, Inc., Upper Saddle River, New Jersey, 2007.