The solution to:

$$x \cdot \left(\frac{a}{b}\right)^x = \frac{c}{d}$$

Can be computed via:

$$x = \frac{W\left(\log\left(\frac{a}{b}\right) \cdot \frac{c}{d}\right)}{log\left(\frac{a}{b}\right)}$$

Where W is the Lambert W Function.

In order to approximate this solution, we split the input domain $z = \log\left(\frac{a}{b}\right) \cdot \frac{c}{d}$ into:

$$\underbrace{-\infty \ldots -1/e}\Big|\underbrace{-1/e \ldots 0}\Big|\underbrace{0 \ldots +1/e}\Big|\underbrace{+1/e \ldots 24+1/e}\Big|\underbrace{24+1/e \ldots +\infty}$$

For $z < -1/e$, the value of $W(z)$ is not real.

Respectively, the equation $x \cdot \left(\frac{a}{b}\right)^x = \frac{c}{d}$ has no real solution.

This is because $x \cdot \left(\frac{a}{b}\right)^x \leq \frac{1}{e \cdot \log\left(\frac{b}{a}\right)} < \frac{c}{d}$ for every real value of x.

For $-1/e \leq z \leq +1/e$, you may observe that $x = \frac{W\left(\log\left(\frac{a}{b}\right) \cdot \frac{c}{d}\right)}{log\left(\frac{a}{b}\right)} = \frac{c}{d} \cdot \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} \cdot \left(\log\left(\frac{a}{b}\right) \cdot \frac{c}{d}\right)^{n-1}$:

- For $-1/e \leq z \leq 0$, which implies that $a \leq b$, we compute $x = \frac{c}{d} \cdot \sum_{n=1}^{\infty} \frac{(+n)^{n-1}}{n!} \cdot \left(\log\left(\frac{b}{a}\right) \cdot \frac{c}{d}\right)^{n-1}$
- For $0 \leq z \leq +1/e$, which implies that $a \geq b$, we compute $x = \frac{c}{d} \cdot \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} \cdot \left(\log\left(\frac{a}{b}\right) \cdot \frac{c}{d}\right)^{n-1}$

As you can see, when $a = b$, both formulas can be reduced to $x = \frac{c}{d}$.

Note that as the value of z approaches $-1/e$, the value of $W(z)$ approaches -1 but it never actually reaches there.

The function becomes increasingly steep, making its approximation via the Taylor series above increasingly inaccurate.

Hence for the smallest %1 of the range $-1/e \leq z \leq 0$, we approximate $W(z)$ via Newton-Raphson convergence instead:

$$\begin{cases} y_0 = z \\ y_{n+1} = \frac{z \cdot e^{y_n} - y_n^2}{1 - y_n} \end{cases} \Big| n = 1, \ldots, 4 \Big| \text{ return } \frac{y_4}{z}$$

This method is a lot more accurate but also a lot more expensive, and it is therefore used only for small portion of the range.

For $+1/e < z < 24 + 1/e$, we use a lookup table which maps 128 uniformly distributed values of z.

Then, we calculate $W(z')$ as the weighted-average of $W(z_0)$ and $W(z_1)$, where $z_0 \leq z' < z_1$.

For $z \geq 24 + 1/e$, we rely on the fact that $W(z) \approx p - q + \frac{q^2 + 2pq - 2q}{2p^2}$, where $p = \log(z)$ and $q = \log(p)$.

Since this method requires the calculation of $\log(log(z))$, it is actually applicable for as low as $z = e$.

However, a higher starting value ($z = 24 + 1/e$) is used here in order to achieve higher accuracy.