This library implements two different methods for solving the equation $x \cdot \left(\frac{a}{b}\right)^x = \frac{c}{d}$:

- Function `solveExact(a, b, c, d)`, which opts for accuracy over performance
- Function `solveQuick(a, b, c, d)`, which opts for performance over accuracy

The solution to:

$$x \cdot \left(\frac{a}{b}\right)^x = \frac{c}{d}$$

Can be computed via:

$$x = \frac{W\left(\log\left(\frac{a}{b}\right) \cdot \frac{c}{d}\right)}{log\left(\frac{a}{b}\right)}$$

Where $W(z)$ computes a value of x which solves the equation $x \cdot e^x = z$.

For $z < -1/e$, the value of $W(z)$ is not real.

Respectively, the equation $x \cdot \left(\frac{a}{b}\right)^x = \frac{c}{d}$ has no real solution.

This is because $x \cdot \left(\frac{a}{b}\right)^x \leq \frac{1}{e \cdot \log\left(\frac{b}{a}\right)} < \frac{c}{d}$ for every real value of x.

In order to handle the rest of the input domain, we split it into several sections.

When opting for accuracy, we split the input domain $z = \log\left(\frac{a}{b}\right) \cdot \frac{c}{d}$ into:

$$\underbrace{-1/e \ldots 0} \Big| \underbrace{0 \ldots \infty}$$

For $-1/e \leq z < 0$, we approximate $W(z)$ via the Newton-Raphson converging sequence:

$$\begin{cases} y_0 = -z \\ y_{n+1} = \frac{y_n^2 + e^{y_n} \cdot z}{y_n + 1} \end{cases}$$

For $0 \leq z < \infty$, we approximate $W(z)$ via the Newton-Raphson converging sequence:

$$\begin{cases} y_0 = \begin{cases} z & z < 1 \\ \log(z) & z \geq 1 \end{cases} \\ y_{n+1} = \frac{y_n^2 \cdot e^{y_n} + z}{(y_n + 1) \cdot e^{y_n}} \end{cases}$$

When opting for performance, we split the input domain $z = \log\left(\frac{a}{b}\right) \cdot \frac{c}{d}$ into:

$$\underbrace{-1/e \ldots 0}\Big|\underbrace{0 \ldots 1/e}\Big|\underbrace{1/e \ldots 24+1/e}\Big|\underbrace{24+1/e \ldots \infty}$$

For $-1/e \le z \le +1/e$, you may observe that $x = \frac{W\left(\log\left(\frac{a}{b}\right)\cdot\frac{c}{d}\right)}{log\left(\frac{a}{b}\right)} = \frac{c}{d} \cdot \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} \cdot \left(\log\left(\frac{a}{b}\right) \cdot \frac{c}{d}\right)^{n-1}$:

- For $-1/e \le z \le 0$, which implies that $a \le b$, we compute $x = \frac{c}{d} \cdot \sum_{n=1}^{\infty} \frac{(+n)^{n-1}}{n!} \cdot \left(\log\left(\frac{b}{a}\right) \cdot \frac{c}{d}\right)^{n-1}$
- For $0 \le z \le +1/e$, which implies that $a \ge b$, we compute $x = \frac{c}{d} \cdot \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} \cdot \left(\log\left(\frac{a}{b}\right) \cdot \frac{c}{d}\right)^{n-1}$

As you can see, when $a = b$, both formulas can be reduced to $x = \frac{c}{d}$.

Note that as the value of z approaches $-1/e$, the value of $W(z)$ approaches -1 but it never actually reaches there.

The function becomes increasingly steep, making its approximation via the Taylor series above increasingly inaccurate.

Hence for the smallest $\%1$ of the range $-1/e \le z \le 0$:

- We precalculate a lookup table which maps 16 uniformly distributed values of z to values of $W(z)$
- During runtime, we calculate $W(z')$ as the weighted-average of $W(z_0)$ and $W(z_1)$, where $z_0 \le z' < z_1$

For $1/e < z < 24 + 1/e$:

- We precalculate a lookup table which maps 128 uniformly distributed values of z to values of $W(z)$
- During runtime, we calculate $W(z')$ as the weighted-average of $W(z_0)$ and $W(z_1)$, where $z_0 \le z' < z_1$

For $z \ge 24 + 1/e$, we rely on the fact that $W(z) \approx p - q + \frac{q^2+2pq-2q}{2p^2}$, where $p = \log(z)$ and $q = \log(p)$.

Since this method requires the calculation of $\log(log(z))$, it is actually applicable for as low as $z = e$.

However, a higher starting value ($z = 24 + 1/e$) is used here in order to achieve higher accuracy.