Numerical computing

(1)

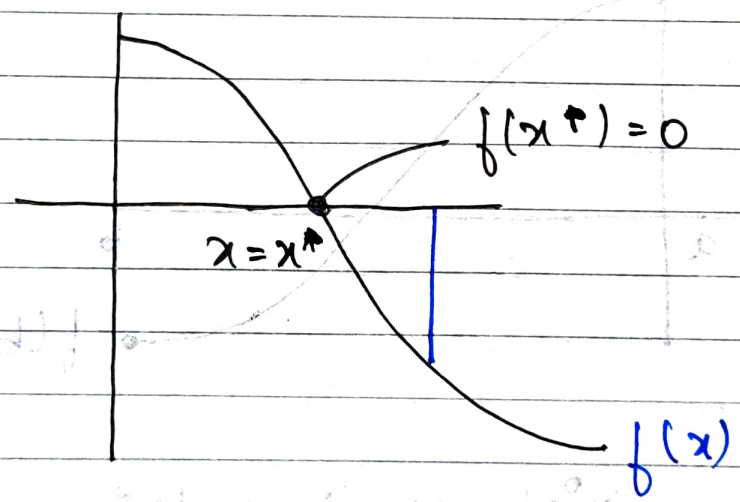
→ Root finding methods using Bisection method

A function f is given

Input is real nos. & output is real nos.

Problem - Find $x = x^*$ so that
$$f(x^*) = 0$$

$$f(x) = x\cos x + x^2 + e^x$$
$$\downarrow$$
Find $x = x^*$ s.t $f(x^*) = 0$



Assumptions - (i) The function $f(x)$ is defined in the interval $[a, b]$ = given.

(ii) The function f is continous in $[a, b]$

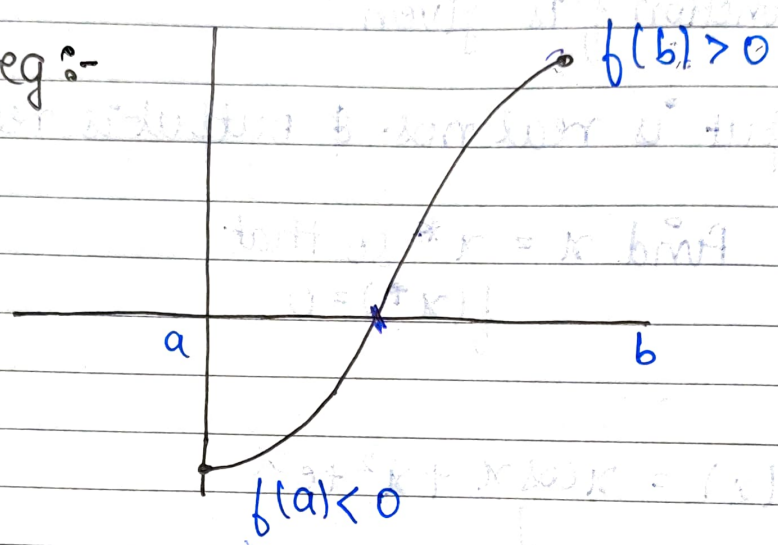
(iii) $f(a)f(b) < 0$ holds.

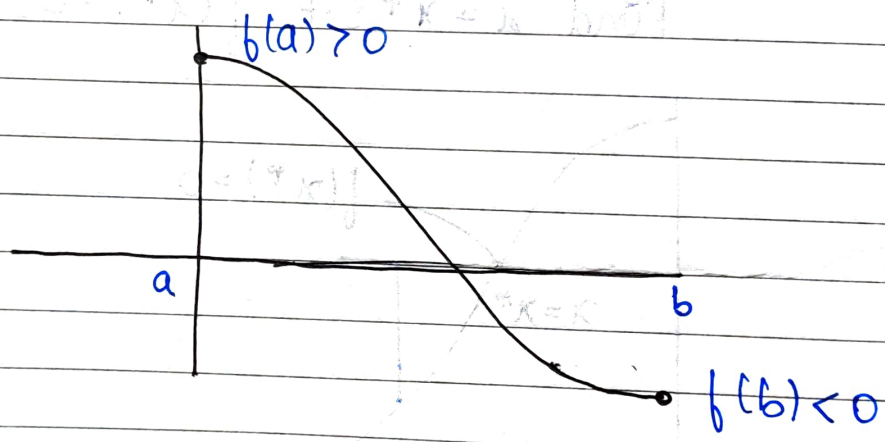
By intermediate value theorem, there exists a root for $f(x)$.

(a) $f(a) > 0,\ f(b) < 0 \Rightarrow f(a)f(b) < 0$

(b) $f(a) < 0,\ f(b) > 0 \Rightarrow f(a)f(b) < 0$

eg :-





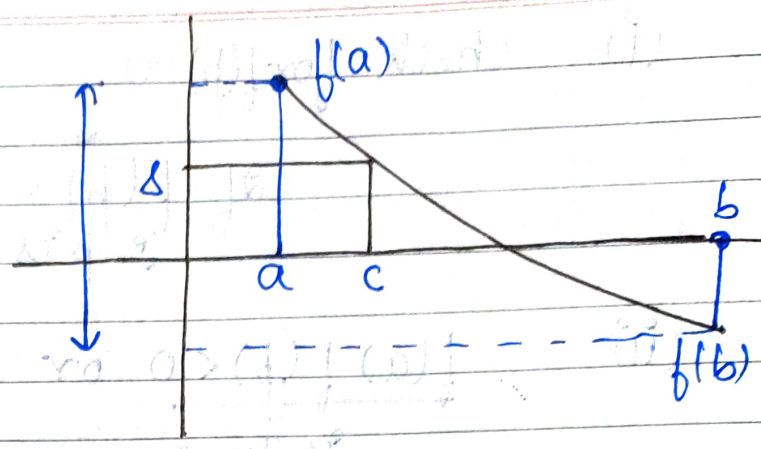
* Checking <u>existence</u> of root :-

$f(c) = s$ will be True.

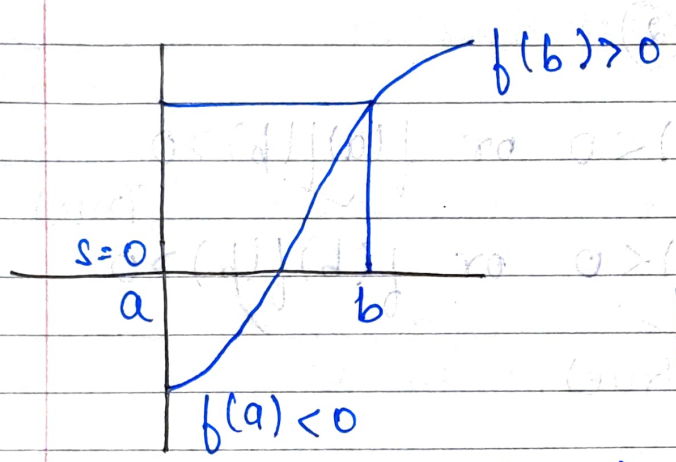
$f(x)$ is continous in $[a, b]$. Then for any s in range b/w $f(a)$ and $f(b)$ there exists c in $[a, b]$

so that

$f(c) = s$ will be True.



This is called intermediate value Theorem.

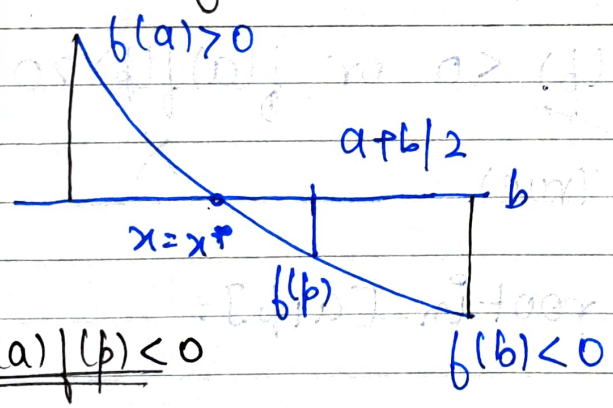


For $s=0$
there exists $c = x^*$
s.t
$f(c) = f(x^*) = s = 0$
will be True

$\Longrightarrow$ x^* will be come a root of $f(x)$.

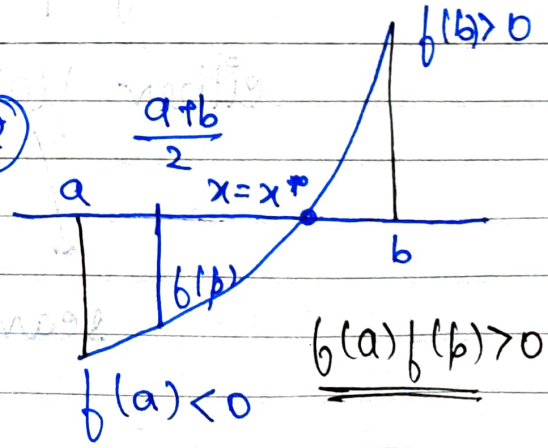
* Finding root of the equation :-

(1)



$f(a)f(p)<0$

(2)



$f(a)f(p)>0$

→ Find the midpt. of $[a,b]$

$$p = \frac{a+b}{2}$$

(i) Check for $f(p)=0$

If $f(p)=0$ we are done
$\therefore$ p is root of eq^n.

(ii) $\underbrace{f(a)f(p)<0}_{\text{is True}}$ or $\underbrace{f(a)f(p)>0}_{\text{is True}}$

→ For graph ②

$f(a)f(p)<0$ or $f(a)f(p)>0$ (True)
✗ ✓

$f(b)f(p)<0$ or $f(b)f(p)>0$
✓ (True) ✗

Search root in $[p, b]$.

→ For graph ①

either $f(a)f(p)<0$ or $f(a)f(p)>0$
✓ (True) ✗

search root in $[a, p]$.

* Function $f(\cdot)$ is given

a, b are endpts. of interval $[a, b]$

$p = (a+b)/2;$

if $f(p) = 0$
print, "p is root"

if $f(a) \cdot f(p) < 0$
root lies in $[a, p]$

else
the root lies in $[p, b]$.

This process keeps on repeating until we reach $f(x^*) = 0$ or until p becomes root

<u>Example</u> → Let

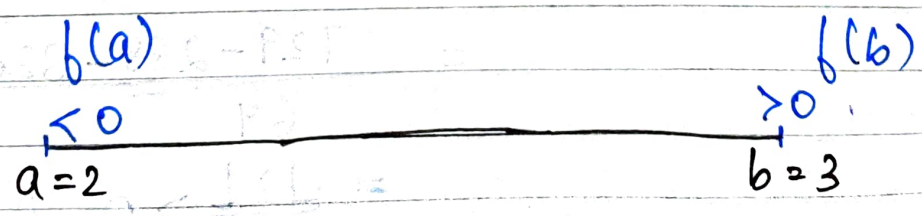
$$f(x) = x^3 - 2x - 5 \text{ in } [2, 3].$$

→ Since it is a polynomial ∴ $f(x)$ is continous in $[2,3]$

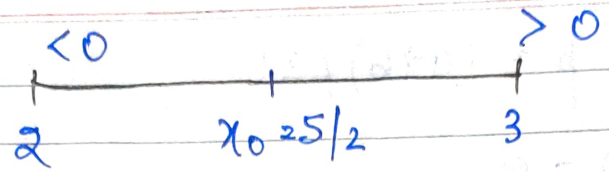
→ $f(a) = f(2) = 8 - 4 - 5$
$= \underline{-1} < 0$

→ $f(b) = f(3) = 27 - 6 - 5 = \underline{16} > 0$

∴ $f(a) f(b) < 0$



$$p = x_0 = \frac{a+b}{2} = \frac{5}{2} = \underline{2.5}$$



Now $f(x_0) = f(5/2)$

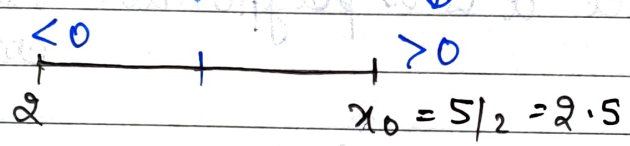
$\therefore f\left(\frac{5}{2}\right) = \left(\frac{5}{2}\right)^3 - \cancel{2}\left(\frac{5}{\cancel{2}}\right) - 5$

$\frac{125}{8} - \frac{10}{1}$

$= \frac{45}{8} > 0$ $\{f(x_0)$ is +ve$\}$

$f(a) = -ve$ $f(b) = +ve$

$\therefore$ root lies in



$x_1 = \frac{2+2.5}{2} = \frac{4.5}{2} = 2.25$

$f(x_1) = f\left(\frac{9}{4}\right) = \left(\frac{9}{4}\right)^3 - 2\left(\frac{9}{4}\right) - 5$

$= \frac{729}{64} - \frac{9}{2} - \frac{5}{1}$

$= \frac{729 - 288 - 320}{64}$

$= \frac{121}{64} > 0$ (1.8970)

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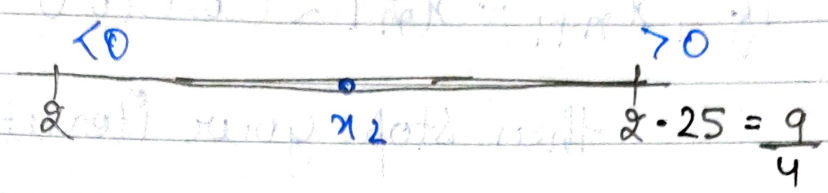
Now root lies in ↓



$$x_2 = \frac{2+\frac{9}{4}}{2} = \frac{8+9}{8}$$

$$= \frac{17}{8} > 0$$

$$\therefore f\left(\frac{17}{8}\right) = \left(\frac{17}{8}\right)^3 - 2\left(\frac{17}{8}\right) - 5$$

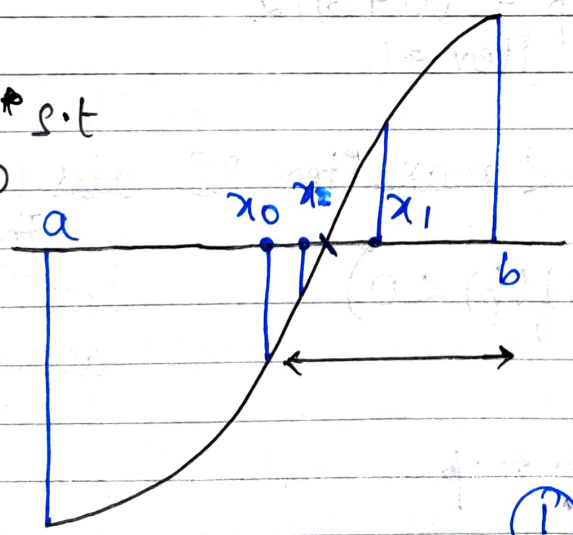
$$= \frac{4913}{512} - \frac{17}{4} - \frac{5}{1}$$

$$= \frac{4913 - 2176 - 2560}{512}$$

$$= \frac{177}{512} > 0$$

Thus, this process continues.

Find x^* s.t $f(x^*) = 0$



Thus, after a few iterations, we approach very close to the true solution.

very close to root { Stopping cond → (1) $x_0, x_1, \dots x_n$
Stop where $f(x_n) \approx 0$ and $|f(x_n)| < \epsilon$ some given value

(or tolerance) (10^{-8})

(2) if $|x_{n+1} - x_n| <$ close to 0

then stop your iterations.

Code ↓

```
if (iteration < max_iteration or
        abs f(x_n) < tolerance)
                     ↳ (tol)
```

29 July, 19

* Algorithm :- (Bisection method)

```
input a, b    /* interval of definition
                 and is an input */

max_iter = 20
(or accuracy)  tol = 10^-5      /* 10^-5 */

/* the function f(x) is available */

p = (a+b)/2
iter = 1

while ( iter < max_iter && abs(f(p)) > tol)
{
    if (f(a)f(p) < 0)
    {
        b ← p
    }
    else if (f(b) f(p) < 0)
    {
        a ← p
    }
```

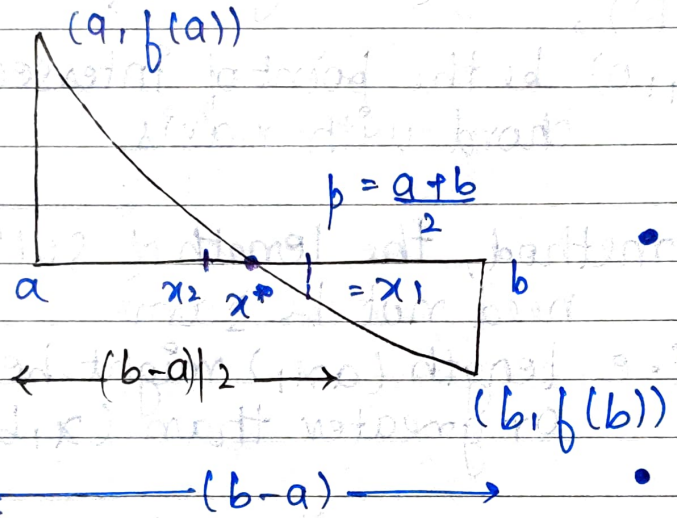
accuracy not met.

```
i = i+1
p = (a+b)/2.
} /* end of while loop.
```

If p is the root of eqn abs(f(p)) will become less than tol. thus while loop terminates and comes out.

```
if ( abs(f(p)) < tol)
    print " p is the root".
```

NOTE:-



We notice that :-

- $|x_1 - x^*| < \frac{(b-a)}{2}$
- $|x_2 - x^*| < \frac{\left(\frac{b-a}{2}\right)}{2}$

We obtain the approximate values of the root x^* being $x_1, x_2 \cdots\cdots x_n \cdots\cdots$

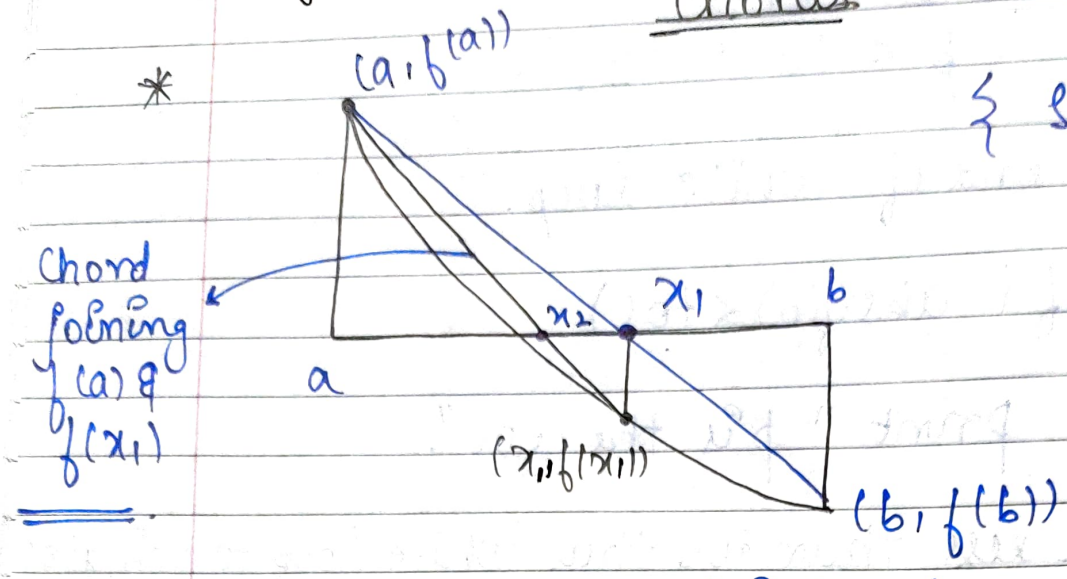
Imp.

$$\boxed{\text{Then } |x_n - x^*| < \left(\frac{b-a}{2^n}\right) \text{ will be True for } n = 1, 2, 3, \cdots}$$

$\Rightarrow$ $x_1, x_2, x_3 \cdots\cdots x_n \cdots$ converge to x^*.

(2) Regula - Falsi or Method of chords

*



{ Same assumptions as in Bisection method }

Consider the chord joining $(a, f(a))$ and $(b, f(b))$.
Let $(x_1, 0)$ be the point of intersection of chord with x axis.

NOTE:- • In this method, the length of subinterval need not be same
i.e length $(a x_1)$ might be smaller or greater than $(x_1 b)$.

(i) Compute $f(x_1)$ and if $abs(f(x_1)) < tol$

$\Downarrow$

x_1 is the root otherwise goto (iii)

(iii) Check $f(a) f(x_1) < 0$

$b \leftarrow x_1$

else check $f(b) f(x_1) < 0$

$a \leftarrow x_1$

proceed to step (i)

- Computation of x_1 :-

Equation of chord joining $(a, f(a))$ and $(b, f(b))$

$$(y - f(a)) = m(x - a)$$

where $m = \dfrac{f(b) - f(a)}{b - a} \longrightarrow (1)$

Now $(x_1, 0)$ lies on (1)

$$0 - f(a) = m(x_1 - a)$$

$$(x_1 - a) = \frac{-f(a)}{m} \quad (m \neq 0)$$

or $x_1 = a - f(a) \times \dfrac{b-a}{f(b) - f(a)}$ {substituting m from (1)}

$$x_1 = \frac{a[f(b) - f(a)] - f(a)(b-a)}{f(b) - f(a)}$$

Imp.

$$\boxed{\text{i.e} \quad x_1 = \frac{a f(b) - b(f(a))}{f(b) - f(a)}}$$

Q Using regula-falsi, find root of the function $f(x) = x^3 - 2x - 5$ in $[2, 3]$.

$a = 2, b = 3$

$f(a) = 2^3 - 4 - 5 = \underline{-1 < 0}$

$f(b) = 27 - 6 - 5 = \underline{16 > 0}$

Now $f(a)f(b) < 0 \Rightarrow$ root exists.

$\longrightarrow$ let x_1 be the x coordinate of the point of intersection of $(a, f(a))$ and $(b, f(b))$

Then, $x_1 = \dfrac{af(b) - bf(a)}{f(b) - f(a)}$

$= \dfrac{2(16) - 3(-1)}{17}$

$= \dfrac{32+3}{17} = \underline{\underline{\dfrac{35}{17}}}$

$f(x_1) = \left(\dfrac{35}{17}\right)^3 - 2\left(\dfrac{35}{17}\right) - 5$

$= 8.7268 - 4.1176 - 5$

$= \underline{-0.3908} < 0$

$\therefore f(x_1) = -0.3908$

$f(a)f(x_1) > 0$

$f(x_1)f(b) < 0$ $\therefore$ root lies in x_1 and b.

i.e $[35/17, 3]$

And proceed with x_2 etc. of the algorithm.

30 July, 2019

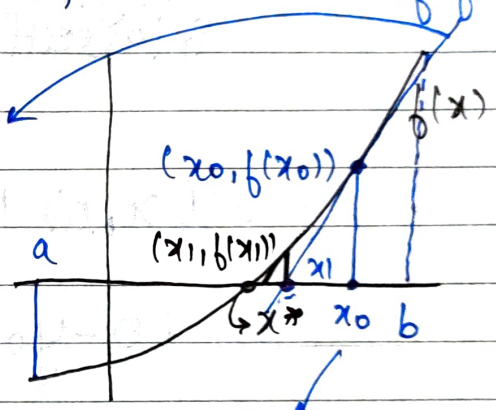
Newton Raphson method
(Iterative method)

In this method, the value of endpoints need not be of opp. signs.

- Assumptions -

→ We assume that the func. is continous as well as $f(x)$ is differentiable.

→ We will explicitly assume here that the root exists, i.e a root of $f(x)$ exists in $[a,b]$.



① {Tangent to $f(x)$ at $(x_0, f(x_0))$ and we find intersection of this tangent to the x axis} i.e x_1

We will determine tangent to curve at $(x_0, f(x_0))$

② Now we again draw tangent to $f(x)$ at $(x_1, f(x_1))$ and find its intersection with x axis ...

- Procedure :-

→ Let x_0 be any initial/ starting pt. in $[a,b]$

→ Consider tangent to $f(x)$ at $(x_0, f(x_0))$.

→ let x_1 be point of intersection of the tangent with x axis

→ x_1 is the next approximation to x

→ The eq^n of tangent at $(x_0, f(x_0))$ is by letting $y_0 = f(x_0)$

$$\boxed{(y - y_0) = m(x - x_0)}$$

where $m = f'(x_0)$.

$$(y - y_0) = f'(x_0)(x - x_0) \quad \text{——— (1)}$$

→ let $(x_1, y_1) = (x_1, 0)$ be the point of intersection of the tangent with x axis.

$(x_1, 0)$ lies in eq^n (1)

$$(0 - y_0) = f'(x_0)(x_1 - x_0)$$

$$(x_1 - x_0) = \frac{-y_0}{f'(x_0)} = -\frac{f(x_0)}{f'(x_0)}$$

⟹ i.e

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$ {Since $y_0 = f(x_0)$}

Proceed with $(x_1, f(x_1))$ and $f'(x_1)$ will be slope

$$(y - y_1) = f'(x_1)(x - x_1)$$

and we are looking for $(x_2, 0)$

i.e $\boxed{x_{(2)} = x_{(1)} - \frac{f(x_{(1)})}{f'(x_{(1)})}}$ {By repeating the calculations}

In general
with x_0 as the initial/
starting point we get

imp.

$$\boxed{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \quad \text{where } k = 0, 1, 2, \ldots}$$

Newton-Raphson formula

Q Using newton Raphson method, find the root of the function.

$$f(x) = x^3 - 2x - 5 \text{ in } [2, 3]$$

Starting point is : $x_0 = 2$ (assume)

$$f(x_0) = 8 - 4 - 5 = \underline{-1}$$

$$f'(x_0) = 3x^2 - 2 = 3(4) - 2 = \underline{10}$$

(Step 1 $k=1$) $\therefore$ By newton Raphson formula

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 2 - \frac{(-1)}{10} = 2 + \frac{1}{10}$$

$$x_1 = \frac{21}{10} = \underline{2.1}$$

Step 2 ($k=2$) Since

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

and $f(x_1) = (2.1)^3 - 2(2.1) - 5$

$= 9.261 - 4.2 - 5$

$= 9.261 - 9.2$

$= \underline{0.061}$

$f'(x_1) = 3(2.1)^2 - 2$

$= 13.23 - 2 = \underline{11.23}$

$\therefore x_2 = 2.1 - \frac{0.061}{11.23}$

$= 2.1 - 0.0054$

$= \underline{2.09456}$

$|x_2 - x_1| = 2.1 - 2.09456$

$= \underline{0.0054} \approx 0$

And proceed for next iterations....

NOTE:- • The rate of convergence (ROC) of Newton Raphson method is faster as compared to other methods.

Q min problem $f(x)$ where x in $[a, b]$

let $x = x^*$ be the point where the minimum is obtained.

Then, necessarily $f'(x^*) = 0$ will be True.

1 Dimensional problem { Solve

$g(x) = f'(x) = 0.$

After this, we perform double derivative.

If $g(x)$ is differentiable, we may go for newton method.

If $g(x)$ is not differentiable, approach with bisection or regula falsi method.

Q $\min f(x_1, \dots x_n)$
$(x_1, x_2 \dots x_n)$ in S and is a subset of $\mathbb{R}^n$.

$\bar{x}^* = (x_1^*, x_2^*, \dots x_n^*)$ be a sol^n of the minimization problem, then

$$\underbrace{\nabla f(x_1^*, \dots x_n^*)}_{\text{gradient of function}} = \begin{pmatrix} \frac{\partial f}{\partial x_1}(x_1^*, \dots x_n^*) \\ \frac{\partial f}{\partial x_2}(x_1^*, \dots x_n^*) \\ \vdots \\ \frac{\partial f}{\partial x_n}(x_1^* \dots x_n^*) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$g_n(\underline{x_1^* \dots x_n}) = 0$

n Dimensional Problem

Q The General root finding problem :-

Find root of $\bar{x}^* = \begin{pmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_n^* \end{pmatrix}$ in R^n s.t

$$\left.\begin{aligned} g_1(x_1^*, \cdots x_n^*) &= 0 \\ &\vdots \\ g_n(x_1^*, \cdots x_n^*) &= 0 \end{aligned}\right\} \quad \text{I}$$

$g_1(x_1 \cdots x_n) = 0$
$g_2(x_1 \cdots x_n) = 0$
$\vdots$
$g_n(x_1 \cdots x_n) = 0$

where $g_1, g_2 \cdots g_n$ are functions in $(x_1, \cdots x_n)$ in R^n and are given/ Known.

→ In this case, we will use Newton Raphson method.

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

$$(x_{k+1} - x_k) = -\frac{f(x_k)}{f'(x_k)}$$

or

Imp.

$$\boxed{f'(x_k)(x_{k+1} - x_k) = -f(x_k)} \quad (2)$$

Iteration no. (superscript)

Let $\bar{x}_0 = \begin{pmatrix} x_1^{(0)} \\ \vdots \\ x_n^{(0)} \end{pmatrix}$ be the initial / starting vector for finding root

column vector

$$\bar{x}^* = \begin{pmatrix} x_1^* \\ \vdots \\ x_n^* \end{pmatrix}$$

Then Newton Raphson method for (I) will be

$$\begin{pmatrix} \frac{\partial g_1}{\partial x_1}(x_1^{(k)} \cdots x_n^{(k)}) & \frac{\partial g_1}{\partial x_2}(x_1^{(k)} \cdots x_n^{(k)}) & \cdots & \frac{\partial g_1}{\partial x_n}(x_1^{(k)} \cdots x_n^{(k)}) \\ \frac{\partial g_2}{\partial x_1}(x_1^{(k)} \cdots x_n^{(k)}) & \frac{\partial g_2}{\partial x_2}(x_1^{(k)} \cdots x_n^{(k)}) & \cdots & \frac{\partial g_2}{\partial x_n}(x_1^{(k)} \cdots x_n^{(k)}) \\ \vdots & \vdots & & \vdots \\ \frac{\partial g_n}{\partial x_1}(x_1^{(k)} \cdots x_n^{(k)}) & \frac{\partial g_n}{\partial x_2}(x_1^{(k)} \cdots x_n^{(k)}) & \cdots & \frac{\partial g_n}{\partial x_n}(x_1^{(k)} \cdots x_n^{(k)}) \end{pmatrix}_{(n \times n)}$$

$$\bar{x}_k = \begin{pmatrix} x_1^k \\ x_2^k \\ \vdots \\ x_n^k \end{pmatrix}_{(n \times 1)}$$

for next iteration

$$f'(x_k)(x_{k+1} - x_k) = -f(x_k)$$

(from ②)

known

$$\begin{pmatrix} \frac{\partial g_1}{\partial x_1}(x_1^{(k)} \cdots x_n^{(k)}) & \frac{\partial g_1}{\partial x_2}(x_1^{(k)} \cdots x_n^{(k)}) & \cdots & \frac{\partial g_1}{\partial x_n}(x_1^{(k)} \cdots x_n^{(k)}) \\ \frac{\partial g_2}{\partial x_1}(x_1^{(k)} \cdots x_n^{(k)}) & \frac{\partial g_2}{\partial x_2}(x_1^{(k)} \cdots x_n^{(k)}) & \cdots & \frac{\partial g_2}{\partial x_n}(x_1^{(k)} \cdots x_n^{(k)}) \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial g_n}{\partial x_1}(x_1^{(k)} \cdots x_n^{(k)}) & \frac{\partial g_n}{\partial x_2}(x_1^{(k)} \cdots x_n^{(k)}) & \cdots & \frac{\partial g_n}{\partial x_n}(x_1^{(k)} \cdots x_n^{(k)}) \end{pmatrix}_{(n \times n)} \begin{pmatrix} x_{k+1} - x_k \end{pmatrix}_{(n \times 1)}$$

known

unknown

$$= \begin{pmatrix} g_1(x_1^k \cdots x_n^k) \\ \vdots \\ g_n(x_1^{(k)} \cdots x_n^{(k)}) \end{pmatrix}_{(n \times 1)}$$

$-$

can be evaluated.

$$\bar{x}_{k+1} = \begin{pmatrix} x_1^{(k+1)} \\ \vdots \\ x_n^{(k+1)} \end{pmatrix}$$

* Termination cond for Newton Raphson :-

Specify the max. no. of iterations = max_iter

$|f(x_k)| < tol$ where

tol = user specified

= (10^{-5}, for example)

5 Aug. 19

Q Solve by Newton Raphson method.

$$\left.\begin{aligned} g_1(x,y) &= 3yx^2 - 10x + 7 = 0 \\ g_2(x,y) &= y^2 - 5y + 7 = 0 \end{aligned}\right\} \text{ in } R^2$$

We will choose an initial guess.

Suppose $\overline{x_0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the initial guess/approximation to the root.

Step 1 :-

$g_1(0,0) = 7$

$g_2(0,0) = 7$

$\frac{\partial g_1}{\partial x} = 6yx - 10 \qquad \frac{\partial g_1}{\partial y} = 3x^2$

$\frac{\partial g_2}{\partial x} = 0 \qquad \frac{\partial g_2}{\partial y} = 2y - 5$

Now evaluating derivatives ↓

$\frac{\partial g_1}{\partial x}(0,0) = -10 \qquad \frac{\partial g_1}{\partial y}(\overline{x_0}) = 0$

$\frac{\partial g_2}{\partial x}(\overline{x_0}) = 0 \qquad \frac{\partial g_2}{\partial y}(\overline{x_0}) = -5$

By Newton Raphson method ↓

$$\begin{pmatrix} \frac{\partial g_1}{\partial x}(\overline{x_0}) & \frac{\partial g_1}{\partial y}(\overline{x_0}) \\ \frac{\partial g_2}{\partial x}(\overline{x_0}) & \frac{\partial g_2}{\partial y}(\overline{x_0}) \end{pmatrix} (\overline{x_1} - \overline{x_0}) = -\begin{pmatrix} g_1(\overline{x_0}) \\ g_2(\overline{x_0}) \end{pmatrix}$$

or $\begin{pmatrix} -10 & 0 \\ 0 & -5 \end{pmatrix} (\bar{x}_1 - \bar{x}_0) = -\begin{pmatrix} 7 \\ 7 \end{pmatrix}$

$$(\bar{x}_1 - \bar{x}_0) = -\begin{pmatrix} -10 & 0 \\ 0 & -5 \end{pmatrix}^{-1} \begin{pmatrix} 7 \\ 7 \end{pmatrix}$$

$$(\bar{x}_1 - \bar{x}_0) = \frac{-1}{50} \begin{pmatrix} -5 & 0 \\ 0 & -10 \end{pmatrix} \begin{pmatrix} 7 \\ 7 \end{pmatrix}$$

$$= \underline{\underline{\frac{-1}{50} \begin{pmatrix} -35 \\ -70 \end{pmatrix}}} = \underline{\underline{\begin{pmatrix} 7/10 \\ 7/5 \end{pmatrix}}}$$

$$\therefore \bar{x}_1 = \bar{x}_0 + \begin{pmatrix} 7/10 \\ 7/5 \end{pmatrix}$$

$$\bar{x}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 7/10 \\ 7/5 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 7/10 \\ 7/5 \end{pmatrix}}}$$

<u>Step 2</u>: $g_1\left(\frac{7}{10}, \frac{7}{5}\right) = 3\left(\frac{7}{5}\right)\left(\frac{7}{10}\right)^2 - 10\left(\frac{7}{10}\right) + 7 = \underline{\underline{\frac{1029}{500}}}$

$$g_2\left(\frac{7}{10}, \frac{7}{5}\right) = \left(\frac{7}{5}\right)^2 - 5\left(\frac{7}{5}\right) + 7$$

$$= \underline{\underline{\frac{49}{25}}}$$

$$\therefore g_1(\bar{x}_1) = \frac{1029}{500}, \quad g_2(\bar{x}) = \frac{49}{25}$$

$$\frac{\partial g_1}{\partial x}(\overline{x_1}) = 6\left(\frac{7}{10}\right)\left(\frac{7}{5}\right) - 10$$

$$\frac{294}{50} - \frac{10}{1} = \underline{\underline{-\frac{206}{50}}}$$

$$\frac{\partial g_1}{\partial y}(\overline{x_1}) = 3\left(\frac{49}{100}\right) = \underline{\underline{\frac{147}{100}}}$$

$$\frac{\partial g_2}{\partial x}(\overline{x_1}) = 0 \qquad \frac{\partial g_2}{\partial y}(\overline{x_1}) = 2\left(\frac{7}{5}\right) - 5$$

$$= \frac{14}{5} - \frac{5}{1} = \underline{\underline{\frac{-11}{5}}}$$

By Newton Raphson method,

$$\begin{pmatrix} \frac{\partial g_1}{\partial x}(\overline{x_1}) & \frac{\partial g_1}{\partial y}(\overline{x_1}) \\ \frac{\partial g_2}{\partial x}(\overline{x_1}) & \frac{\partial g_2}{\partial y}(\overline{x_1}) \end{pmatrix} (\overline{x_2} - \overline{x_1}) = -\begin{pmatrix} g_1(\overline{x}) \\ g_2(\overline{x}) \end{pmatrix}$$

$$\begin{pmatrix} \frac{-206}{50} & \frac{147}{100} \\ 0 & -11/5 \end{pmatrix} (\overline{x_2} - \overline{x_1}) = -\begin{pmatrix} 1029/500 \\ 49/25 \end{pmatrix}$$

$$(\overline{x_2} - \overline{x_1}) = -\frac{250}{206 \times 11} \begin{pmatrix} -11/5 & \frac{-147}{100} \\ 0 & \frac{-206}{50} \end{pmatrix} \begin{pmatrix} \frac{1029}{500} \\ \frac{49}{25} \end{pmatrix}$$

Solve for $\overline{x_2}$

with $\overline{x_1} = \begin{pmatrix} 7/10 \\ 7/5 \end{pmatrix}$

Secant method

- For single dimension problem

By Newton Raphson method, for $f(x)=0$ problem

we have $k = 0, 1, 2 \ldots$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad f'(x_k) \neq 0$$

with x_0 = initial guess for the root.

Suppose $f'(x_k) \approx \dfrac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$

and we replace the approximation in Newton Raphson method :-

$$x_{k+1} = x_k - \frac{f(x_k)}{\left(\dfrac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}\right)}$$

$$= x_k - \frac{(x_k - x_{k-1})\, f(x_k)}{f(x_k) - f(x_{k-1})}$$

$$x_{k+1} = \frac{\cancel{x_k f(x_k)} - x_k f(x_{k-1}) - \cancel{x_k f(x_k)} - f(x_k)\, x_{k-1}}{f(x_k) - f(x_{k-1})}$$

2 initial guess needed.

i.e $$\boxed{x_{k+1} = \frac{x_{k-1} f(x_k) - x_k f(x_{k-1})}{f(x_k) - f(x_{k-1})}} \quad (1)$$

Secant method for finding a root of $f(x)$

In this case, x_k and (x_{k-1}) should be available to compute x_{k+1}

i.e initially x_{-1}, x_0 approximations are to be given and proceed to apply (1)

Q Solve $x^3 - 2x - 5 = 0$ in $[a,b] = [2,3]$ using Bisection & Regula-falsi

(Practical code)

- Fixed point problem :-

→ Suppose we have $f(x)$ is a real valued function.

Find $x = x^*$ in $\mathbb{R}$ so that $f(x^*) = x^*$ is true. } Solve $f(x) = x$ in $[a,b]$

→ Solve $g(x) = f(x) - x = 0$

$x - f(x) = 0$

$y(x) = x - f(x)$ in $[a,b]$.

Solve $f(x) = x$ in $[a, b]$

x_0 = initial guess

for $k = 0, 1, 2, \ldots$

$$\boxed{x_{k+1} = f(x_k)}$$

26 Aug, 19

Solving the matrix equation

$$A\bar{x} = \bar{b}$$

where $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$ $\bar{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

given and

$\bar{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ = unknown vector.

* Gauss elimination method :-

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \longrightarrow \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22}^* & \cdots & a_{2n}^* \\ 0 & a_{32}^* & \cdots & a_{3n}^* \\ \vdots & \vdots & & \\ 0 & a_{n2}^* & \cdots & a_{nn}^* \end{pmatrix} \begin{pmatrix} b_1 \\ b_2^* \\ \vdots \\ b_n^* \end{pmatrix}$$

Try to make it upper Δ matrix.

kth step

$$\begin{array}{c} \\ \\ \\ k^* \rightarrow \\ \\ \\ \end{array} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ 0 & 0 & & & \\ & a_{kk}^{**} & a_{k,k+1}^{*} & \cdots a_{kj}^{**} \cdots & a_{kn}^{**} \quad b_k^{*} \\ & a_{k+1,k}^{**} & a_{k+1,k+1}^{**} & \cdots a_{k+1,j}^{**} \cdots & a_{k+1,n}^{**} \quad b_{k+1}^{*} \\ & a_{ik}^{**} & a_{i,k+1}^{**} & \cdots a_{i,j}^{**} \cdots & a_{i,n}^{**} \quad b_i^{**} \\ 0 & a_{nk}^{**} & a_{n,k+1}^{**} & \cdots a_{nj}^{**} \cdots & a_{nn}^{**} \quad b_n^{**} \end{pmatrix}$$

$$a_{ij}^{(new)} \leftarrow a_{ij}^{**} - \left[\frac{a_{kj}^{**}}{a_{kk}^{**}}\right](a_{ik}^{*})$$

$k = 1, 2, \cdots (n-1)$
$j = k \text{ to } n.$
$i = (k+1) \text{ to } n$

Program ↓

for $k = 1$ to $(n-1)$

for $i = (k+1)$ to n

for $j = k$ to n

$$a_{ij}^{(new)} \leftarrow a_{ij}^{(old)} - \frac{a_{kj}^{(old)} \; a_{ik}^{(old)}}{a_{kk}^{(old)}}$$

provided $a_{kk} \neq 0$.

→ If pivot element is zero ↓

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ 0 & a_{kk}^{*} & & & \\ \vdots & a_{ik}^{*} & & & \\ 0 & a_{nk}^{*} & & & \end{pmatrix}$$

Suppose
$a_{kk}^{*} = 0$
or a_{kk}^{*} is very close to 0.

In such case we compute

$$\max\{|a_{kk}^{*}|, |a_{k+1,k}^{*}| \cdots |a_{ik}^{*}| \cdots |a_{nk}^{*}|\}$$

For suppose max. value occurs at a_{ik}^* then we swap rows a_{kk}^* and a_{ik}^*

This is called method of pivoting

Q For given system of equations.

How to do this method using without pivoting??

$$3y - 2z = 5$$
$$2x + 4y - 7z = 2$$
$$-4x + 5y + 4z = -7$$

(Partial pivoting method)

$$\left(\begin{array}{ccc|c} 0 & 3 & -2 & 5 \\ 2 & 4 & -7 & 2 \\ -4 & 5 & 4 & -7 \end{array}\right)$$

$a_{11} = 0$
$\therefore$ can't divide by this

Among $0, 2, -4$ abs. value of 4 is max.

$\therefore$ $R_1 \longleftrightarrow R_3$ gets swapped.

$$\left(\begin{array}{ccc|c} -4 & 5 & 4 & -7 \\ 2 & 4 & -7 & 2 \\ 0 & 3 & -2 & 5 \end{array}\right)$$

make entry 2 as 0.

First make pivot = 1

$\therefore$ $R_1 \rightarrow R_1/-4$

and then

$$R_2 \leftarrow R_2 - 2\frac{R_1}{-4}$$

$$\left(\begin{array}{ccc|c} -4 & 5 & 4 & -7 \\ 0 & 6\tfrac{1}{2} & -5 & -3/2 \\ 0 & 3 & -2 & 5 \end{array}\right)$$

$$a_{22}^{(new)} = 4 - \frac{5}{(-4)}(2) = \frac{13}{2}$$

$$a_{23}^{(new)} = -7 - \frac{4}{(-4)}(2) = -5$$

$$b_2^{(new)} = b_2^{(old)} - \frac{(-7)}{-4}(2)$$

$$= -3/2$$

Now for choosing next pivot

$$\max\{|6\tfrac{1}{2}|, |3|\} = 6\tfrac{1}{2} = \text{at } 2^{nd} \text{ row}$$

$\therefore$ no need to swap rows

$$\left(\begin{array}{ccc|c} -4 & 5 & 4 & -7 \\ 0 & 6\tfrac{1}{2} & -5 & -3/2 \\ 0 & 3 & -2 & 5 \end{array}\right)$$

$$\left\{ R_3 - \frac{R_2}{6\tfrac{1}{2}}(3) \right\}$$

$$\left(\begin{array}{ccc|c} -4 & 5 & 4 & -7 \\ 0 & 6\tfrac{1}{2} & -5 & -3/2 \\ 0 & 0 & 4/13 & 74/13 \end{array}\right)$$

$$5 - \frac{(-3/2)}{6\tfrac{1}{2}}(3)$$

$$-2 - \frac{(-5)}{6\tfrac{1}{2}}(3) = -2 + \frac{30}{13}$$

$$= \frac{4}{13}$$

$\therefore -4x + 5y + 4z = -7$

$6\tfrac{1}{2}y - 5z = -3/2$

$\frac{4}{13}z = \frac{74}{13}$

$\therefore z = \frac{74}{4} = \frac{37}{2}$

Now $6\tfrac{1}{2}y - 5\left(\frac{37}{2}\right) = \frac{-3}{2}$.

$6\tfrac{1}{2}y = \frac{-3}{2} + \frac{185}{5}$

$\therefore y = \frac{182}{13}$

$-4x + 5\left(\frac{182}{13}\right) + 4\left(\frac{37}{2}\right) \Rightarrow -4x = -7 - \frac{910}{13} - 74$

$x = \frac{7 + \frac{910}{13} + 74}{4} = \frac{151}{4}$.

→ How to check whether obtained values are correct or not??

$$\underbrace{\begin{pmatrix} 0 & 3 & -2 \\ 2 & 4 & -7 \\ -4 & 5 & 4 \end{pmatrix}\begin{pmatrix} 151/4 \\ 182/13 \\ 37/2 \end{pmatrix}} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

multiply & find whether you get same values as b_1, b_2, b_3.

Q (lab) Solve using Gauss elimination

$$\begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

Q Solve eq^n (Gauss Jordan method)

$$\begin{pmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 4 \end{pmatrix}$$

Find inverse of the given matrix.

$$A\bar{x} = \bar{b}$$
$$\bar{x} = A^{-1}\bar{b}$$

and multiply with the right side vector and get the solution.

① Compute inverse

$$\begin{pmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{pmatrix}$$

Procedure is :-

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & -1 & 0 & 1 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right) = \text{augmented matrix.}$$

make entries of this submatrix to be identity matrix.

$R_2 \to R_2 - 4R_1$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -5 & -4 & 1 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right)$$

$R_3 \to R_3 - 3R_1$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -5 & -4 & 1 & 0 \\ 0 & 2 & 0 & -3 & 0 & 1 \end{array}\right)$$

$R_2 \to R_2 / -1$

$R_1 \to R_1 - R_2$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 5 & 4 & -1 & 0 \\ 0 & 2 & 0 & -3 & 0 & 1 \end{array}\right)$$

$R_3 \to R_3 - 2R_2$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -4 & -3 & 1 & 0 \\ 0 & 1 & 5 & 4 & -1 & 0 \\ 0 & 2 & 0 & -3 & 0 & 1 \end{array}\right)$$

$R_3 \to R_3 / -10$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -4 & -3 & 1 & 0 \\ 0 & 1 & 5 & 4 & -1 & 0 \\ 0 & 0 & -10 & -11 & 2 & 1 \end{array}\right)$$

$R_2 \to R_2 - 5R_3$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -4 & -3 & 1 & 0 \\ 0 & 1 & 5 & 4 & -1 & 0 \\ 0 & 0 & 1 & 11/10 & -1/5 & -1/10 \end{array}\right)$$

Date / /

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -4 & -3 & 1 & 0 \\ 0 & 1 & 0 & -3/2 & 0 & 1/2 \\ 0 & 0 & 1 & 11/10 & -1/5 & -1/10 \end{array}\right)$$

$R_1 \to R_1 + 4R_3$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/5 & 1/5 & -2/5 \\ 0 & 1 & 0 & -3/2 & 0 & 1/2 \\ 0 & 0 & 1 & 11/10 & -1/5 & -1/10 \end{array}\right)$$

$\therefore$ Inverse

$$\begin{pmatrix} 7/5 & 1/5 & -2/5 \\ -3/2 & 0 & 1/2 \\ 11/10 & -1/5 & -1/10 \end{pmatrix}$$

Now $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7/5 & 1/5 & -2/5 \\ -3/2 & 0 & 1/2 \\ 11/10 & -1/5 & -1/10 \end{pmatrix} \begin{pmatrix} 1 \\ 6 \\ 4 \end{pmatrix}$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} +1 \\ 1/2 \\ -1/2 \end{pmatrix}$$

NOTE:- Gauss Jordan method - Basically this method is used to find the inverse of a given matrix.

Decomposing a matrix in lower & upper Δ matrix

$$A = LU$$

$$A\bar{x} = \bar{b} \iff (LU)\bar{x} = \bar{b}$$

$$L(U\bar{x}) = \bar{b}$$

$$\iff L\bar{z} = \bar{b} \text{ and } U\bar{x} = \bar{z}.$$

27 Aug. 19 Factorization method / LU factorization method.

$$A\bar{x} = \bar{b}$$

Aim :- $A = LU$ where L is a triangular matrix. U is an upper triangular matrix.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

$$\therefore \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \underbrace{\begin{pmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 \cdots & 0 \\ & & & \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{pmatrix}}_{\frac{n(n+1)}{2}} \underbrace{\begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ & & & \\ - & - & \cdots & u_{nn} \end{pmatrix}}_{\frac{n(n+1)}{2}}$$

$$\therefore \frac{n(n+1)}{2} + \frac{n(n+1)}{2} = n(n+1) = n^2 + n$$

= no. of entries to be found.

no. of equations will be n^2

Having n^2 eq^n is a tedious task to solve ∴ we solve it assuming diagonal elements become 1

↓

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} l_{11} & & & \\ l_{21} & l_{22} & & \\ & & & \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{pmatrix} \begin{pmatrix} 1 & u_{12} & \cdots & u_{1n} \\ & 1 & \cdots & u_{2n} \\ & & & \\ & & & 1 \end{pmatrix}$$

↓

Crout method of factorization.

$$= \underbrace{\begin{pmatrix} 1 & 0 & \cdots & 0 \\ l_{21} & 1 & & 0 \\ & & & \\ l_{n1} & l_{n2} & l_{n,n-1} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ & & & \\ & & & u_n \end{pmatrix}}$$

Factorization of a matrix leads to Doolittle's method of factorization.

$$\rightarrow \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

through multiplication of L and U matrix

$$\left\{ \begin{aligned} l_{11} &= a_{11} \\ l_{21} &= a_{21} \\ l_{31} &= a_{31} \end{aligned} \right\}$$

$l_{11} u_{12} = a_{12}$

$l_{11} u_{13} + 0 + 0 = a_{13}$

$$\left.\begin{aligned} \therefore u_{12} &= \frac{a_{12}}{l_{11}} \\ u_{13} &= \frac{a_{13}}{l_{11}} \end{aligned}\right\} l_{11} \neq 0$$

NOTE: If A is a positive definite matrix, then LU factorization is possible.

Q Consider $A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 2 \end{pmatrix}$

$D_1 = \det(3) = 3 > 0$

$D_2 = \det\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} = 9-4 > 0$

$D_3 = \det\begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 2 \end{pmatrix}$

$= 3(6-4) - 2(4-2) + 1(4-3) > 0$

A is symmetric and $D_1, D_2, D_3 > 0$
i.e A is positive definite.

$\rightarrow \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 2 \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \begin{pmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix}$

Crout's factorization.

Equating the coefficients of LHS matrix with coefficients of product of RHS matrix

$l_{11} = a_{11} = 3$ $\qquad l_{31} = a_{31} = 1$

$l_{21} = a_{21} = 2$

$$l_{11} u_{12} = a_{12}$$

$$\boxed{u_{12} = \frac{2}{3}}$$

$$l_{11} u_{13} = 1$$

$$\boxed{u_{13} = \frac{1}{3}}$$

$$l_{21} u_{12} + l_{22} = 3 \quad (2^{nd} \text{ row}, 2^{nd} \text{ col})$$

$$2\left(\frac{2}{3}\right) + l_{22} = 3$$

$$l_{22} = 3 - \frac{4}{3} = \boxed{\frac{5}{3}}$$

$$l_{21} u_{13} + l_{22} u_{23} = 2 \quad (2^{nd} \text{ row}, 3^{rd} \text{ col})$$

$$2\left(\frac{1}{3}\right) + \left(\frac{5}{3}\right) u_{23} = 2$$

$$\left(\frac{5}{3}\right) u_{23} = 2 - \frac{2}{3}$$

$$u_{23} = \frac{4}{\cancel{3}} \times \frac{\cancel{3}}{5} = \boxed{\frac{4}{5}}$$

$$l_{31} u_{12} + l_{32} = 2 \quad (3^{rd} \text{ row}, 2^{nd} \text{ col})$$

$$1\left(\frac{2}{3}\right) + l_{32} = 2$$

$$l_{32} = 2 - \frac{2}{3} = \boxed{\frac{4}{3}}$$

$l_{31}u_{13} + l_{32}u_{23} + l_{33} = 2$ (3rd row, 3rd col).

$1\left(\frac{1}{3}\right) + \left(\frac{4}{3}\right)\left(\frac{4}{5}\right) + l_{33} = 2$

$2 - \frac{1}{3} - \frac{16}{15} = l_{33}$

$l_{33} = \frac{9}{15} = \boxed{\frac{3}{5}}$

$\therefore$

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 5/3 & 0 \\ 1 & 4/3 & 3/5 \end{pmatrix} \begin{pmatrix} 1 & 2/3 & 1/3 \\ 0 & 1 & 4/5 \\ 0 & 0 & 1 \end{pmatrix}$$

$\Downarrow$

Thus, factorization is possible.

$\rightarrow$ Suppose we need to solve :-

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Since $A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 2 \end{pmatrix} = LU = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 5/3 & 0 \\ 1 & 4/3 & 3/5 \end{pmatrix} \begin{pmatrix} 1 & 2/3 & 1/3 \\ 0 & 1 & 4/5 \\ 0 & 0 & 1 \end{pmatrix}$

$$\left\{ \begin{array}{l} \text{let } U\bar{x} = \bar{z} \\ \text{and } L\bar{z} = \bar{b} \end{array} \right\} \rightarrow (1)$$

Then $L\bar{z} = L(U\bar{x}) = \bar{b}$

i.e $(LU)\bar{x} = \bar{b}$ or $A\bar{x} = \bar{b}$

$\therefore A\bar{x} = \bar{b}$ can be solved by solving (1).

① First we solve

$$L\bar{z} = \bar{b}$$

or
$$\begin{pmatrix} 3 & 0 & 0 \\ 2 & 5/3 & 0 \\ 1 & 4/3 & 3/5 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$3z_1 = 1$$

or $\boxed{z_1 = 1/3}$

$$2z_1 + \frac{5}{3} z_2 = 0$$

$$\cancel{\frac{2}{3} + \frac{5}{3} z_2 = 0}$$

$$\cancel{z_2 =}$$

$$\frac{2}{3} + \frac{5}{3} z_2 = 0$$

$$\boxed{z_2 = -2/5}$$

$$z_1 + \frac{4}{3} z_2 + \frac{3}{5} z_3 = 1$$

$$\frac{1}{3} + \frac{4}{3}\left(\frac{-2}{5}\right) + \frac{3}{5} z_3 = 1$$

$$\boxed{z_3 = 2}$$

$$\left\{ \therefore \; z_1 = 1/3, \; z_2 = -2/5, \; z_3 = 2 \right\}$$

② Consider the upper triangular system

$$U\bar{x} = \bar{z}$$

$$\begin{pmatrix} 1 & 2/3 & 1/3 \\ 0 & 1 & 4/5 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1/3 \\ -2/5 \\ 2 \end{pmatrix}$$

$$x_1 + \frac{2}{3}x_2 + \frac{1}{3}x_3 = \frac{1}{3} \quad \text{——— (1)}$$

$$\boxed{x_3 = 2}$$

$$x_2 + \frac{4}{5}x_3 = -\frac{2}{5}$$

$$x_2 + \frac{8}{5} = -\frac{2}{5}$$

$$\boxed{x_2 = -2}$$

Put x_2 and x_3 in (1)

$$x_1 + \frac{2}{3}(-2) + \frac{1}{3}(2) = \frac{1}{3}$$

$$x_1 = \frac{1}{3} - \frac{2}{3} + \frac{4}{3}$$

$$\boxed{x_1 = 1}$$

$\therefore$ The solution is $(1, -2, 2)$

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Solve for $\bar{x}$

$$A\bar{x} = \bar{b}$$

$A_{n\times n}$ - given

$\bar{b}_{n\times 1}$ = given.

Factorization of A :

Split $A = LU$ → upper triangular.

Lower Triangular

Suppose A is positive definite

(1) A is symmetric

(2) For any $\bar{x}$ in $\mathbb{R}^n$,

$$\bar{x}^T A \bar{x} \geq 0$$

and $\bar{x}^T A \bar{x} = 0 \iff \bar{x} = \bar{0}$.

When A is positive-definite, then we can have

$A = LL^T$ where L is a lower triangular matrix.

{ This factorization is called Cholesky factorization }

Q Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 22 \\ 3 & 22 & 82 \end{bmatrix}$$

Verify A is positive-definite matrix.

$$D_1 = \det(1) = 1 > 0$$

$$D_2 = \det\begin{pmatrix} 1 & 2 \\ 2 & 8 \end{pmatrix} = 8 - 4 > 0$$

$$D_3 = 1[8\times 82 - 22\times 22] - 2[2\times 82 - 3\times 22] + [2\times 22 - 3\times 8](3)$$

$$= (656 - 484) - 2(164 - 66) + (44 - 24)(3)$$

$$= 172 - \underline{196 + 60} > 0$$

A is symmetric and $D_1 > 0$, $D_2 > 0$, $D_3 > 0$

$\Rightarrow$ A is positive definite.

consider,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 22 \\ 3 & 22 & 82 \end{bmatrix} = \underbrace{\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}}_{L^T}$$

$$= \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{21}l_{11} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{11}l_{31} & l_{31}l_{21} + l_{22}l_{32} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

Now comparing coefficients

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$$l_{11}^2 = 1$$
$$\boxed{l_{11} = 1}$$

$$l_{11} l_{21} = 2 \Rightarrow \boxed{l_{21} = 2}$$
$$l_{11} l_{31} = 3 \Rightarrow \boxed{l_{31} = 3}$$

$$l_{21}^2 + l_{22}^2 = 8$$
$$l_{22}^2 = 8 - 4 = 4$$
$$\boxed{l_{22} = 2}$$

$$l_{21} l_{31} + l_{22} l_{32} = 22$$
$$2 \cdot 3 + 2 \cdot l_{32} = 22$$
$$\boxed{l_{32} = 8}$$

Finally,
$$l_{31}^2 + l_{32}^2 + l_{33}^2 = 82$$

$$3^2 + 8^2 + l_{33}^2 = 82$$
$$l_{33}^2 = 82 - 9 - 64$$
$$= 9$$
$$\boxed{l_{33} = 3}$$

$$\therefore A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 8 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 8 \\ 0 & 0 & 3 \end{bmatrix}$$

<u>Remarks</u> :- Assume that
$$A = LL^T$$
and L is lower triangular matrix

Solving
$$A\bar{x} = \bar{b} \text{ becomes}$$

$(LL^T)\bar{x} = \bar{b}$ let $\boxed{L^T\bar{x} = \bar{z}}$

then $\downarrow$ $(LL^T)\bar{x} = \bar{b}$ becomes $\boxed{L\bar{z} = \bar{b}}$

Solve:- $L\bar{z} = \bar{b}$ or we have $\bar{z} = (L^{-1})\bar{b}$

and

$$L^T\bar{x} = \bar{z} \Rightarrow (L^T)^{-1}\bar{z} = (L^{-1})^T\bar{z}$$

$$A = LU$$

If $U = L^T \Rightarrow \underline{A = LL^T}$

Remark :- The methods

{ Gauss elimination
Jordan method
Doolittle, Crout and cholesky }

methods are called "Direct methods" of solving $A\bar{x} = \bar{b}$

→ Iterative methods :-

Start with $\bar{x}^{(0)}$ (initial guess)
Using $\bar{x}^{(0)}$ find $\bar{x}^{(1)}$ as approximation to the solution of $A\bar{x} = \bar{b}$

From $\bar{x}^{(1)} \rightarrow \bar{x}^{(2)} \rightarrow x^{(3)} \rightarrow \dots x^{(k)}$
until $\bar{x}^{(k)}$ is close to solution of $A\bar{x} = \bar{b}$

Q Solve for x :-

$$A\bar{x} = \bar{b}$$

$A_{n\times n}$, $b_{n\times 1}$ is given.

let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}_{n \times n}$

let $B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}_{n \times 1}$

By (1)

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$\cdots$$

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{ii}x_i + \cdots a_{in}x_n = b_i$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

Take,

$$a_{11}x_1 = b_1 - (a_{12}x_2 + \cdots + a_{1n}x_n)$$

$$a_{ii}x_i = b_i - (a_{i1}x_1 + \cdots + a_{i,i-1}x_{i-1} + a_{i,i+1}x_{i+1} + \cdots + a_{in}x_n)$$

$$\vdots$$

$$a_{nn}x_n = b_n - (a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{n-1,n-1}x_{n-1})$$

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$$x_1 = \frac{-1}{a_{11}} [(a_{12}x_2 + \cdots + a_{1n}x_n) - b_1]$$

$$x_i = \frac{-1}{a_{ii}} [(a_{i1}x_1 + \cdots + a_{i,i-1}x_{i-1} + a_{i,i+1}x_{i+1} + \cdots + a_{in}x_n) - b_i]$$

$$x_n = \frac{-1}{a_{nn}} [(a_{n1}x_1 + \cdots + a_{n-1,n-1}x_{n-1}) - b_n]$$

Let $\bar{x}^{(0)} = \begin{pmatrix} x_1^{(0)} \\ \vdots \\ x_n^{(0)} \end{pmatrix}$ be the initial guess (approximation) vector.

we define for $k = 0, 1, 2, \cdots$

$\bar{x}^{(k+1)} = \begin{pmatrix} x_1^{(k+1)} \\ \vdots \\ x_n^{(k+1)} \end{pmatrix}$ as

$$\overline{x_1}^{(k+1)} = -\frac{1}{a_{11}} [a_{12}x_2^{(k)} + \cdots + a_{1n}x_n^{(k)} - b_1]$$

$$\overline{x_i}^{(k+1)} = \frac{-1}{a_{ii}} [a_{i1}x_1^{(k)} + a_{i2}x_2^{(k)} + \cdots a_{i,i-1} x_{i-1}^{(k)} + a_{i,i+1}x_{i+1}^{(k)} + \cdots + a_{in}x_n^{(k)} - b_i]$$

$$x_n^{(k+1)} = \frac{-1}{a_{nn}} [a_{n1}x_1^{(k)} + \cdots + a_{n-1,n-1}x_{n-1}^{(k)} - b_n]$$

Since,

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Let

$$L = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ a_{21} & 0 & \cdots & 0 \\ a_{i1} & a_{i2} & \cdots & 0 \\ a_{n1} & a_{n2} & \cdots a_{n-1,n-1} & 0 \end{pmatrix}$$

ith row

$$D = \begin{pmatrix} a_{11} & & 0 \\ 0 & a_{22} & \\ & & \ddots \; a_{nn} \end{pmatrix} = \text{diagonal matrix}$$

$$U = \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & a_{23} \cdots & a_{2n} \\ & & a_{i,i+1} & a_{in} \\ & & a_{n-1,n} & \\ & & 0 & \end{pmatrix}$$

ith row

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ 7 & 6 & -2 \end{bmatrix}$$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad L = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 7 & 6 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 0 & 2 & 3 \\ 0 & & \end{pmatrix}$$

$$\boxed{A = L + D + U}$$

From the definition of L, D, U

$$\bar{x}^{(k+1)} = -D^{-1}\left((L+U)\bar{x}^{(k)} - \bar{b}\right)$$

$$(a_{i1} \cdots a_{i,i-1}, 0, a_{i,i+1} \cdots a_{in}) \begin{pmatrix} x_1^{(k)} \\ \vdots \\ x_i^{(k)} \\ \vdots \\ x_n^{(k)} \end{pmatrix}$$

i.e we have

$$\bar{x}^{(k+1)} = -D^{-1}\left((L+U)\bar{x}^{(k)} - \bar{b}\right)$$

$$\Rightarrow \bar{x}^{(k+1)} = \bar{x}^{(k)} - \bar{x}^{(k)} - D^{-1}\left((L+U)\bar{x}^{(k)} - \bar{b}\right)$$

$$= \bar{x}^{(k)} - D^{-1}D\bar{x}^{(k)} - D^{-1}\left((L+U)\bar{x}^{(k)} - \bar{b}\right)$$

$$= \bar{x}^{(k)} - D^{-1}\left\{(L+D+U)\bar{x}^{(k)} - \bar{b}\right\}$$

$$= \bar{x}^{(k)} - D^{-1}\left(A\bar{x}^{(k)} - \bar{b}\right)$$

Define $\bar{r}^{(k)} = A\bar{x}^{(k)} - \bar{b}$ then method becomes:

$$\boxed{\bar{x}^{(k+1)} = \bar{x}^{(k)} - D^{-1}\bar{r}^{(k)}} \text{ and } \boxed{\bar{r}^{(k)} = A\bar{x}^{(k)} - \bar{b}}$$

This method is Gauss-Jacobi method

Q Solve using Gauss-Jacobi iterative meth the following systems of equations :-

$$4x_1 + x_2 + x_3 = 2$$
$$x_1 + 5x_2 + 2x_3 = -6$$
$$x_1 + 2x_2 + 3x_3 = -4$$

by taking $\bar{x}^{(0)} = \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \end{pmatrix} = \bar{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

We know :-

$$\bar{x}^{(k+1)} = \bar{x}^{(k)} - D^{-1}\bar{r}^{(k)}$$

and $\bar{r}^{(k)} = A\bar{x}^{(k)} - \bar{b}$, $k = 0, 1, 2, \ldots$

$\bar{x}^{(0)} = \bar{0} \Rightarrow \bar{r}^{(0)} = A\bar{x}^{(0)} - \bar{b} = -\bar{b}$

$$\bar{r}^{(0)} = -\begin{bmatrix} 2 \\ -6 \\ -4 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \\ 4 \end{bmatrix}$$

Also,

$$D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix} \Rightarrow D^{-1} = \begin{pmatrix} 1/4 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}$$

$$\therefore \bar{x}^{(1)} = \bar{x}^{(0)} - \begin{pmatrix} 1/4 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1/3 \end{pmatrix} \bar{r}^{(0)}$$

$$= -\begin{pmatrix} 1/4 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}\begin{pmatrix} -2 \\ 6 \\ 4 \end{pmatrix}$$

i.e $\bar{x}^{(1)} = -\begin{pmatrix} -1/2 \\ 6/5 \\ 4/3 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -6/5 \\ -4/3 \end{pmatrix}$

Now, $\bar{r}^{(1)} = A\bar{x}^{(1)} - \bar{b}$

$$= \begin{pmatrix} 4 & 1 & 1 \\ 1 & 5 & 2 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1/2 \\ -6/5 \\ -4/3 \end{pmatrix} - \begin{pmatrix} 2 \\ -6 \\ -4 \end{pmatrix}$$

$$= \begin{pmatrix} 2-6/4-4/3 \\ 1/2-6-8/3 \\ 1/2-12/5-4 \end{pmatrix} - \begin{pmatrix} 2 \\ -6 \\ -4 \end{pmatrix} = \begin{pmatrix} -8/15 \\ -49/6 \\ -59/10 \end{pmatrix} - \begin{pmatrix} 2 \\ -6 \\ -4 \end{pmatrix}$$

$$= \begin{pmatrix} -38/15 \\ -13/6 \\ -19/10 \end{pmatrix}$$

$$\therefore \bar{r}^{(1)} = \begin{pmatrix} -38/15 \\ -13/6 \\ -19/10 \end{pmatrix}$$

$\therefore \bar{x}^{(2)} = \bar{x}^{(1)} - D^{-1}\bar{r}^{(1)}$

$$\bar{x}^{(2)} = \begin{pmatrix} 1/2 \\ -4/5 \\ -4/3 \end{pmatrix} - \begin{pmatrix} 1/4 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}\begin{pmatrix} -38/15 \\ -13/6 \\ -19/10 \end{pmatrix}$$

$$= \cancel{\begin{pmatrix} 1/2 \\ -6/5 \\ -4/3 \end{pmatrix}} = \begin{pmatrix} 17/15 \\ -23/30 \\ -21/30 \end{pmatrix}$$

3rd Sept, 19 Gauss - Jacobi iterative method

for solving matrix equation : $A\bar{x} = \bar{b}$

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\bar{x}} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}}_{\bar{b}}$$

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{i1}x_1 + a_{i2}x_2 + \cdots a_{ii}x_i + \cdots a_{in}x_n &= b_2 \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}\right\}$$

Define $\bar{x}^{(k+1)} = \begin{pmatrix} x_1^{(k+1)} \\ \vdots \\ x_n^{(k+1)} \end{pmatrix}$ in terms of previous approximation

Solution $\bar{x}^{(k)} = \begin{pmatrix} x_1^{(k)} \\ \vdots \\ x_n^{(k)} \end{pmatrix}$ as $\bar{x}^{(0)} = \begin{pmatrix} x_1^{(0)} \\ \vdots \\ x_n^{(0)} \end{pmatrix}$

(initial approximation, given as 9(P).

$$x_1^{(k+1)} = \frac{-1}{a_{11}} \left[a_{12}x_2^{(k)} + \cdots + a_{1n}x_n^{(k)} - b_1 \right]$$

$$x_i^{(k+1)} = \frac{-1}{a_{ii}} \left[a_{i1}x_1^{(k)} + \cdots a_{i,i-1}x_{i-1}^{(k)} + \cdots a_{i,i+1}x_{i+1}^{(k)} - b_i \right]$$

$$x_n^{(k+1)} = \frac{-1}{a_{nn}} \left[a_{n1}x_1^{(k)} + \cdots + a_{n,n-1}x_{n-1}^{(k)} - b_n \right]$$

or $\bar{x}^{(k+1)} = \bar{x}^{(k)} - D^{-1}\left(A\bar{x}^{(k)} - \bar{b}\right)$

where $D = \text{diag}(a_{11}, a_{22} \cdots a_{nn})$

and $k = 0, 1, 2, \cdots$

→ We consider another new form :-

$$x_1^{(k+1)} = \frac{-1}{a_{11}} \left[a_{12}x_2^{(k)} + \cdots + a_{1n}x_n^{(k)} - b_1 \right]$$

$$x_2^{(k+1)} = \frac{-1}{a_{22}} \left[a_{21}x_1^{(k+1)} + a_{23}x_3^{(k)} + \cdots + a_{2n}x_n^{(k)} - b_2 \right]$$

$\vdots$

$$x_i^{(k+1)} = \frac{-1}{a_{ii}} \left[a_{i1}x_1^{(k+1)} + a_{i2}x_2^{(k+1)} + \cdots + a_{i,i-1}x_{i-1}^{(k+1)} + a_{i,i+1}x_{i+1}^{(k)} + \cdots + a_{in}x_n^{(k)} - b_i \right].$$

$$x_n^{(k+1)} = \frac{-1}{a_{nn}} \left[a_{n1}x_1^{(k+1)} + \cdots + a_{n,n-1}x_{n-1}^{(k+1)} - b_n \right]$$

→ Gauss Seidel Iterative method

Rearranging the terms written earlier,

$$a_{11}x_1^{(k+1)} = -(a_{12}x_2^{(k)} + \cdots + a_{1n}x_n^{(k)} - b_1)$$

$$a_{21}x_1^{(k+1)} + a_{22}x_2^{(k+1)} = -(a_{23}x_3^{(k)} + \cdots + a_{2n}x_n^{(k)} - b_2)$$

$$a_{i1}x_1^{(k+1)} + a_{i2}x_2^{(k+1)} + \cdots + a_{ii}x_i^{(k+1)} = -(a_{i,i+1}x_{i+1}^{(k)} + \cdots + a_{in}x_n^{(k)} - b_i)$$

$$\vdots$$

$$a_{n1}x_1^{(k+1)} + \cdots + a_{nn}x_n^{(k+1)} = b_n$$

$$\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 \cdots & 0 \\ a_{i1} & a_{i2} & \cdots a_{ii} & 0 \cdots 0 \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \\ \vdots \\ x_i^{(k+1)} \\ \vdots \\ x_n^{(k+1)} \end{pmatrix} =$$

$$\underbrace{-\begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & a_{23} \cdots & a_{2n} \\ 0 & & & \\ \vdots & \cdots a_{i,i+1} & \cdots & a_{in} \\ \vdots & & & \\ 0 & \cdots & \cdots & 0 \end{pmatrix} \begin{pmatrix} x_1^{(k)} \\ x_2^{(k)} \\ \vdots \\ \vdots \\ x_n^{(k)} \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_n \end{pmatrix}}_{1}$$

Let $L = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ a_{21} & 0 & \cdots & 0 \\ a_{i1} & a_{i,i-1} & \cdots & \\ a_{n1} & \cdots & a_{n,n-1} & 0 \end{pmatrix}$, $D = \begin{pmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{pmatrix}$

and $U = \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & a_{23} \cdots & a_{2n} \\ \vdots & & & \\ 0 & \cdots & 0 & a_{n-1,n} \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$

$\therefore$ (1) becomes $\qquad k = 0, 1, 2, \cdots$

$$\boxed{(L+D)\,\bar{x}^{(k+1)} = -U\bar{x}^{(k)} + \bar{b}}$$

or

$$\bar{x}^{(k+1)} = -(L+D)^{-1}(U\bar{x}^{(k)} - b)$$

$$= -(L+D)^{-1}\underline{U\bar{x}} + (L+D)^{-1}\bar{b}$$

Also,

$$\bar{x}^{(k+1)} = -(L+D)^{-1}(U\bar{x}^{(k)} - \bar{b})$$

$$= \bar{x}^{(k)} - \bar{x}^{(k)} - (L+D)^{-1}(U\bar{x}^{(k)} - \bar{b})$$

$$= \bar{x}^{(k)} - (L+D)^{-1}(L+D)\bar{x}^{(k)} - (L+D)^{-1}(U\bar{x}^{(k)} - \bar{b})$$

$$= \bar{x}^{(k)} - (L+D)^{-1}[\underbrace{(L+D+U)}_{A}\bar{x}^{(k)} - \bar{b}]$$

$$\boxed{\text{i.e } \bar{x}^{(k+1)} = \bar{x}^{(k)} - (L+D)^{-1}(A\bar{x}^{(k)} - \bar{b})}$$

Already, we have

$$\bar{x}^{(k+1)} = -(L+D)^{-1}U\bar{x}^{(k)} + (L+D)^{-1}\bar{b}$$

Q Solve by Gauss Seidel, the system of equations.

$$2x_1 - x_2 + 0x_3 = 7$$
$$-x_1 + 2x_2 - x_3 = 1$$
$$0x_1 - x_2 + 2x_3 = 1$$

$$\bar{x}^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$L+D = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & -1 & 2 \end{pmatrix} \qquad U = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

In Gauss Seidel & Gauss Jacobi, lower Δ and upper Δ both have diagonal as 0 in matrix.

$$L = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$U = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\therefore (L+D)^{-1} = \frac{\begin{pmatrix} 4 & +2 & 1 \\ 0 & +4 & 2 \\ 0 & 0 & 4 \end{pmatrix}^{T}}{8} = \frac{\begin{pmatrix} 4 & 0 & 0 \\ 2 & 4 & 0 \\ 1 & 2 & 4 \end{pmatrix}}{8}$$

$$= \begin{pmatrix} 1/2 & 0 & 0 \\ 1/4 & 1/2 & 0 \\ 1/8 & 1/4 & 1/2 \end{pmatrix}$$

(cont. at end)

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Gauss-Seidel iterative method of solving $A\bar{x} = \bar{b}$

Suppose $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$, $\bar{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

Define :-

$$L = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ a_{21} & 0 & \cdots & 0 \\ \vdots & & & 0 \\ a_{n1} & \cdots & a_{n,n-1} & 0 \end{pmatrix}, \quad D = \begin{pmatrix} a_{11} & & 0 \\ & a_{22} & \\ 0 & & \ddots \; a_{nn} \end{pmatrix}$$

$$U = \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & a_{23} \cdots & a_{2n} \\ \vdots & & & \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$$

Let $\bar{x}^{(0)} = \begin{pmatrix} x_1^{(0)} \\ \vdots \\ x_n^{(0)} \end{pmatrix}$ be the initial approximation vector (input / given)

Then by Gauss Seidel we have

$$\boxed{\bar{x}^{(k+1)} = -(L+D)^{-1}(U\bar{x}^{(k)} - \bar{b}), \quad k = 0, 1, \cdots}$$

or

$$\boxed{\bar{x}^{(k+1)} = \bar{x}^{(k)} - (L+D)^{-1}(A\bar{x}^{(k)} - \bar{b})}$$

Termination cond. is the length of the vector $\bar{r}^{(k)} = A\bar{x}^{(k)} - \bar{b}$ is very small

Q (contd.) $(L+D)^{-1}U = \begin{pmatrix} 1/2 & 0 & 0 \\ 1/4 & 1/2 & 0 \\ 1/8 & 1/4 & 1/2 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 0 & -1/2 & 0 \\ 0 & -1/4 & -1/2 \\ 0 & -1/8 & -1/4 \end{pmatrix}$$

$$(L+D)^{-1}\bar{b} = \begin{pmatrix} 1/2 & 0 & 0 \\ 1/4 & 1/2 & 0 \\ 1/8 & 1/4 & 1/2 \end{pmatrix} \begin{pmatrix} 7 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 7/2 \\ 9/4 \\ 13/8 \end{pmatrix}$$

$$\therefore \bar{x}^{(k+1)} = \begin{pmatrix} 0 & 1/2 & 0 \\ 0 & 1/4 & 1/2 \\ 0 & 1/8 & 1/4 \end{pmatrix} \bar{x}^{(k)} + \begin{pmatrix} 7/2 \\ 9/4 \\ 13/8 \end{pmatrix} \quad k=0,1,\ldots$$

with $\bar{x}^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ we get

$$\bar{x}^{(1)} = (\quad)\,\bar{0} + \begin{pmatrix} 7/2 \\ 9/4 \\ 13/8 \end{pmatrix} = \bar{x}^{(1)}$$

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$$\therefore \bar{x}^{(2)} = \begin{pmatrix} 0 & 1/2 & 0 \\ 0 & 1/4 & 1/2 \\ 0 & 1/8 & 1/4 \end{pmatrix} \bar{x}^{(1)} + \begin{pmatrix} 7/2 \\ 9/4 \\ 13/8 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1/2 & 0 \\ 0 & 1/4 & 1/2 \\ 0 & 1/8 & 1/4 \end{pmatrix} \begin{pmatrix} 7/2 \\ 9/4 \\ 13/8 \end{pmatrix} + \begin{pmatrix} 7/2 \\ 9/4 \\ 13/8 \end{pmatrix}$$

$$= \begin{pmatrix} 9/8 \\ 9/16 + 13/16 \\ 9/32 + 13/32 \end{pmatrix} + \begin{pmatrix} 7/2 \\ 9/4 \\ 13/8 \end{pmatrix}$$

$$= \begin{pmatrix} 9/8 + 7/2 \\ 9/16 + 13/16 + 9/4 \\ 9/32 + 13/32 + 13/8 \end{pmatrix}$$

$$= \underline{\begin{pmatrix} 37/8 \\ 58/16 \\ 37/16 \end{pmatrix}} = \bar{x}^{(2)}$$

Q $\left.\begin{aligned} 2x_1 + 2x_2 + 3x_3 &= 10. \\ 5x_1 + 2x_2 + x_3 &= 12 \\ x_1 + x_2 + x_3 &= 5. \end{aligned}\right\}$ using Gauss <u>Jacobi</u>.

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad D^{-1} = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}$$

Assume $\bar{x}^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ then.

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$$\bar{x}^{(1)} = \bar{x}^{(0)} - D^{-1}(A\bar{x} - \bar{b})$$
$$= \bar{0} - D^{-1}(0 - b) = \underline{D^{-1}b}$$

$$= \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 10 \\ 12 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 5 \end{pmatrix}$$

$$\bar{x}^{(1)} = \begin{pmatrix} 5 \\ 6 \\ 5 \end{pmatrix}$$

Now, $\bar{x}^{(2)} = \bar{x}^{(1)} - D^{-1}(A\bar{x}^{(1)} - \bar{b})$ &

$A\bar{x}^{(1)} - \bar{b} =$

$$\begin{pmatrix} 2 & 2 & 3 \\ 5 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \\ 5 \end{pmatrix} - \begin{pmatrix} 10 \\ 12 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} 27 \\ 30 \\ 11 \end{pmatrix} = \bar{r}^{(1)}$$

$$\begin{pmatrix} 5 \\ 6 \\ 5 \end{pmatrix} - \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 27 \\ 30 \\ 11 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 5 \end{pmatrix} - \begin{pmatrix} 27/2 \\ 30/2 \\ 11 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} 5 - 27/2 \\ 6 - 15 \\ 5 - 11 \end{pmatrix}}}$$

Interpolation methods

$x_0 < x_1 < \cdots\cdots < x_n$ be a set of $(n+1)$ points given.

let us assume that at these points values of an unknown function $f(\,)$ is given.

$$\left.\begin{array}{l} x_0 \longrightarrow f(x_0) \\ x_1 \longrightarrow f(x_1) \\ \vdots \qquad\qquad \vdots \\ x_n \longrightarrow f(x_n) \end{array}\right\}$$

Values of x_0, x_1, x_n and $f(x_0), f(x_1), f(x_n)$ are known.

Problem :- Find $p(x)$ = a polynomial of degree n so that $p(x_0) = f(x_0), \cdots\cdots p(x_n) = f(x_n)$

Find $p(x) = a_0 + a_1 x + \cdots\cdots + a_n x_n$

$(n+1)$ unknowns.

Suppose $x_0 \longrightarrow f(x_0)$
$x_1 \longrightarrow f(x_1)$
2 inputs & their respective outputs.

Find $p(x) = a_0 + a_1 x$ (degree = 1)

S.t ↓

Conditions $\longrightarrow$ $f(x_0) = p(x_0)$
$f(x_1) = p(x_1)$

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$f(x_0) = p(x_0)$

$\Rightarrow a_0 + a_1 x_0 = f(x_0)$ or

$a_0 + a_1 x_0 - f(x_0) = 0$

$f(x_1) = p(x_1) \Rightarrow$

$a_0 + a_1 x_1 = f(x_1)$ or

$a_0 + a_1 x_1 - f(x_1) = 0$

and $a_0 + a_1 x = p(x)$

or $a_0 + a_1 x - p(x) = 0$

for all x.

$$\underset{(3\times 3)}{\begin{pmatrix} 1 & x_0 & -f(x_0) \\ 1 & x_1 & -f(x_1) \\ 1 & x & -p(x) \end{pmatrix}} \underset{(3\times 1)}{\begin{pmatrix} a_0 \\ a_1 \\ 1 \end{pmatrix}} = \underset{(3\times 1)}{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}} \quad \text{--- (1)}$$

Solve eq^n

a_0, a_1 unique values.

$p(x_0) = a_0 + a_1 x_0 = f(x_0)$

$p(x_1) = a_0 + a_1 x_1 = f(x_1)$

2 equations, 2 unknowns a_0, a_1

$$\begin{pmatrix} 1 & x_0 \\ 1 & x_1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} f(x_0) \\ f(x_1) \end{pmatrix}$$

If we assume that x_0 and x_1 have diff. values.

$$\det \begin{pmatrix} 1 & x_0 \\ 1 & x_1 \end{pmatrix} = x_1 - x_0 \neq 0$$

unique solution

Continuing with (I)

$$\det\begin{pmatrix} 1 & x_0 & -f(x_0) \\ 1 & x_1 & -f(x_1) \\ 1 & x & -p(x) \end{pmatrix} \neq 0$$

→ Consider the foll. equation

$$\begin{vmatrix} p(x) & x & 1 \\ f(x_0) & x_0 & 1 \\ f(x_1) & x_1 & 1 \end{vmatrix} = 0$$

we will verify $p(x_0) = f(x_0)$ and $p(x_1) = f(x_1)$ and p has degree $= 1$.

Then at $x = x_0$ we have

$$\begin{vmatrix} \underline{p(x_0)} & x_0 & 1 \\ f(x_0) & x_0 & 1 \\ f(x_1) & x_1 & 1 \end{vmatrix} = \begin{vmatrix} \underline{f(x_0)} & x_0 & 1 \\ f(x_0) & x_0 & 1 \\ f(x_1) & x_1 & 1 \end{vmatrix} = 0$$

When $x = x_1$

$$\underbrace{\begin{vmatrix} \underline{p(x_1)} & x_1 & 1 \\ f(x_0) & x_0 & 1 \\ f(x_1) & x_1 & 1 \end{vmatrix} = \begin{vmatrix} \underline{f(x_1)} & x_1 & 1 \\ f(x_0) & x_0 & 1 \\ f(x_1) & x_1 & 1 \end{vmatrix}}_{\text{2 rows become identical i.e } \Delta = 0} = 0$$

Expanding

$$\begin{vmatrix} p(x) & x & 1 \\ f(x_0) & x_0 & 1 \\ f(x_1) & x_1 & 1 \end{vmatrix} = p(x)(x_0 - x_1) - f(x_0)(x - x_1) + f(x_1)(x - x_0)$$

Suppose $\begin{vmatrix} p(x) & x & 1 \\ f(x_0) & x_0 & 1 \\ f(x_1) & x_1 & 1 \end{vmatrix} = 0$

$$p(x)(x_0 - x_1) = +f(x_0)(x - x_1) - f(x_1)(x - x_0)$$

amp:

$$\boxed{p(x) = \frac{(x - x_1)}{(x_0 - x_1)} f(x_0) + \frac{(x - x_0)}{(x_1 - x_0)} f(x_1)}$$

When $x = x_0$

$$\boxed{p(x_0) = f(x_0) + 0}$$

When $x = x_1$

$$\boxed{p(x_1) = f(x_1) + 0}$$

Lagrange's interpolation formula

$$\boxed{p(x) = l_0(x) f(x_0) + l_1 x f(x_1)}$$

where $l_0(x) = \frac{x - x_1}{x_0 - x_1}$; $l_1(x) = \frac{x - x_0}{x_1 - x_0}$

$$\left\{ \begin{array}{ll} l_0(x_0) = 1 & l_0(x_1) = 0 \\ l_1(x_0) = 0 & l_1(x_1) = 1 \end{array} \right\} \text{ or}$$

$$l_i(x_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{when } i \neq j \end{cases}$$

$i = 0, 1$

Q Suppose

$$\left.\begin{array}{l} x_0 \to f(x_0) \\ x_1 \to f(x_1) \\ x_2 \to f(x_2) \end{array}\right\} \text{given}$$

Then $p(x) = a_0 + a_1 x + a_2 x^2$

To find :- a_0, a_1, a_2 s.t $p(x)$ is interpolated to values $f(x_0), f(x_1), f(x_2)$ at x_0, x_1, x_2.

Consider
$$\begin{vmatrix} p(x) & x^2 & x & 1 \\ f(x_0) & x_0^2 & x_0 & 1 \\ f(x_1) & x_1^2 & x_1 & 1 \\ f(x_2) & x_2^2 & x_2 & 1 \end{vmatrix} = 0.$$

Then, we have

$$(1)\quad \left\{\begin{array}{l} p(x)\underbrace{\begin{vmatrix} x_0^2 & x_0 & 1 \\ x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \end{vmatrix}}_{D_0} - f(x_0)\underbrace{\begin{vmatrix} x^2 & x & 1 \\ x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \end{vmatrix}}_{D_1} + \\ f(x_1)\underbrace{\begin{vmatrix} x^2 & x & 1 \\ x_0^2 & x_0 & 1 \\ x_2^2 & x_2 & 1 \end{vmatrix}}_{D_2} - f(x_2)\underbrace{\begin{vmatrix} x^2 & x & 1 \\ x_0^2 & x_0 & 1 \\ x_1^2 & x_1 & 1 \end{vmatrix}}_{D_3} = 0 \end{array}\right.$$

Consider $\begin{vmatrix} x^2 & x & 1 \\ x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \end{vmatrix}$ is a polynomial of degree 2.

roots of this polynomial = x_1 and x_2 as $\Delta = 0$.

$$p_1(x) = \begin{vmatrix} x^2 & x & 1 \\ x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \end{vmatrix} = (x-x_1)(x-x_2)$$

where c is a constant.

$$\underbrace{a_0 + a_1 x + a_2 x^2 = c_0 + c_1 x + c_2 x^2}$$

while comparison

$a_0 = c_0,\ a_1 = c_1,\ a_2 = c_2$

Similarly LHS = RHS

$$\underbrace{\begin{vmatrix} & & \\ & & \\ & & \end{vmatrix}} = (x-x_1)(x-x_2)c$$

Solve this eqn & equate coeff. to find value of c.

Equating coeff. of x^2 both sides

$c = (x_1 - x_2)$ $\left\{ \because \underline{x^2}(x_1-x_2) \equiv \underline{x^2} c_1 \right\}$

$(x-x_1)(x-x_2)c$
$= x^2 c + \ldots\ldots$

Imp.

$$\begin{vmatrix} x^2 & x & 1 \\ x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \end{vmatrix} = (x-x_1)(x-x_2)(x_1-x_2)$$

let us write (1) as :-

$$P(x)\,D_0 - D_1 f(x_0) + D_2 f(x_1) - D_3 f(x_2) = 0 \quad (3)$$

from (2),

$$D_0 = (x_0 - x_1)(x_0 - x_2)(x_1 - x_2)$$

$\{\because$ Put $x = x_0\}$

solve similarly as of (2)

$$D_1 = (x - x_1)(x - x_2)(x_1 - x_2)$$

$$D_2 = (x - x_0)(x - x_2)(x_0 - x_2)$$

$$D_3 = (x - x_0)(x - x_1)(x_0 - x_1)$$

For D_1 roots $\rightarrow$ x_1, x_2 and $(x_1 - x_2)$

Rewritting (3)

$$P(x) = \frac{D_1}{D_0} f(x_0) + \frac{(-D_2)}{D_0} f(x_1) + \frac{D_3}{D_0} f(x_2)$$

But $\underline{\underline{\frac{D_1}{D_0}}} = \dfrac{(x - x_1)(x - x_2)\cancel{(x_1 - x_2)}}{(x_0 - x_1)(x_0 - x_2)\cancel{(x_1 - x_2)}}$

$$= \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = l_0(x) \text{ (say)}$$

Similarly $\underline{\underline{\frac{D_2}{D_0}}} = \dfrac{(x - x_0)(x - x_2)}{(x_0 - x_1)(x_1 - x_2)} = l_1(x)$ say

$$\frac{-D_2}{D_0} = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

$$\frac{D_2}{D_0} = \frac{(x-x_0)(x-x_1)}{(x_0-x_2)(x_1-x_2)} = l_2(x) \text{ say.}$$

$$\frac{D_2}{D_0} = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

(where $n=2$) $\left\{ \therefore P(x) = l_0(x) f(x_0) + l_1 x f(x_1) + l_2(x) f(x_2) \right\}$

$$\therefore l_0(x_0) = \frac{(x_0-x_1)(x_0-x_2)}{(x_0-x_1)(x_0-x_2)} = 1$$

Similarly

$$l_0(x_1) = 0$$
$$l_0(x_2) = 0$$

$$l_1(x_0) = 0, \quad l_1(x_1) = 1, \quad l_1(x_2) = 0$$

$$l_2(x_0) = 0, \quad l_2(x_1) = 0, \quad l_2(x_2) = 1$$

Verify :-

$P(x_0) = f(x_0)$
$P(x_1) = f(x_1)$
$P(x_2) = f(x_2)$

or $l_i(x_j) = \delta_{ij}$ $\left\{ \begin{array}{l} 1 \text{ if } i=j \\ 0 \text{ if } i \neq j \end{array} \right\}$

$(i, j = 0, 1, 2)$

NOTE :- How to remember l_0, l_1, l_2 values

$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$$

For numerator, replace x_0 by x keeping denominator exp. same.

Q It is given that :-

$x = 2$, $f(2) = 4$

$x = 2.5$, $f(2.5) = 5.5$

i.e $\left.\begin{array}{l} x_0 = 2, \quad f(x_0) = 4 \\ x_1 = 2.5, \quad f(x_1) = 5.5 \end{array}\right\}$ given.

Find $p(x)$ = a linear polynomial $= a_0 + a_1 x$ using lagrange's interpolation formula.

We know that

$$P(x) = l_0(x) f(x_0) + l_1(x) f(x_1)$$

where $l_0(x) = \dfrac{x - x_1}{x_0 - x_1}$

and $l_1(x) = \dfrac{x - x_0}{x_1 - x_0}$

Now $l_0(x) = \dfrac{(x - 2.5)}{-0.5}$

$l_1(x) = \dfrac{(x-2)}{0.5}$

$$\therefore P(x) = \frac{(x-2.5)}{-0.5}(4) + \frac{(x-2)}{0.5}(5.5)$$

$$= \frac{1}{0.5}\left[4(2.5-x) + 5.5(x-2)\right]$$

$$= \frac{1}{0.5}(10-4x+5.5x-11)$$

$$= \frac{1}{0.5}(1.5x-1)$$

$$= 2(1.5x-1) = \underline{\underline{3x-2}}$$

{Put given values of ques in polynomial & check your polynomial}.

NOTE:
- The methods - lagrange's interpolation and Newton divided difference are for unequal interval width b/w data points.

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Interpolation

$a < x_0 < x_1 \cdots\cdots < x_n < b$.

Input		Output
x_0	$\longrightarrow$	$f(x_0)$
x_1	$\longrightarrow$	$f(x_1)$
$\vdots$		
x_n	$\longrightarrow$	$f(x_n)$

$(n+1)$ points (inputs) $x_i \longrightarrow f(x_i)$
$= (n+1)$ outputs.

Let $p(x)$ be an nth degree polynomial.
s.t

$$p(x_i) = f(x_i) \quad i = 0, \cdots n$$

$$p(x) = a_0 + a_1 x + \cdots a_n x^n$$

$p(x_i)$ is known at $x_i = 0, \cdots n$
then $p(x)$ is uniquely determined.

→ Lagrange's interpolating polynomial :-

$$p(x) = l_0(x) f(x_0) + l_1(x) f(x_1) + \cdots$$
$$\cdots l_n(x) f(x_n)$$

where

$$l_i(x) = \frac{(x-x_0)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}$$

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or $$l_i(x) = \frac{\prod_{\substack{j=0 \\ j\neq i}}^{n} (x - x_j)}{\prod_{\substack{j=0 \\ j\neq i}}^{n} (x_i - x_j)}$$

$x_0 \rightarrow f(x_0)$
$x_1 \rightarrow f(x_1)$
$x_2 \rightarrow f(x_2)$

$p(x)$ of degree 2

Consider $$\begin{vmatrix} p(x) & x^2 & x & 1 \\ f(x_0) & x_0^2 & x_0 & 1 \\ f(x_1) & x_1^2 & x_1 & 1 \\ f(x_2) & x_2^2 & x_2 & 1 \end{vmatrix} = 0$$

$$p(x)\begin{vmatrix} x_0^2 & x_0 & 1 \\ x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \end{vmatrix} - f(x_0)\begin{vmatrix} x^2 & x & 1 \\ x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \end{vmatrix}$$

$$+ f(x_1)\begin{vmatrix} x^2 & x & 1 \\ x_0^2 & x_0 & 1 \\ x_2^2 & x_2 & 1 \end{vmatrix} - f(x_2)\begin{vmatrix} x^2 & x & 1 \\ x_0^2 & x_0 & 1 \\ x_1^2 & x_1 & 1 \end{vmatrix}$$

$= 0$

$$p(x) = l_0(x) f(x_0) + l_1(x) f(x_1) + l_2(x) f(x_2)$$

$$\left.\begin{array}{l} x_0 \rightarrow f(x_0) \\ x_1 \rightarrow f(x_1) \end{array}\right\} \text{ 2 conditions.}$$

$p(x)$ is of degree 1

consider

$$\begin{vmatrix} p(x) & x & 1 \\ f(x_0) & x_0 & 1 \\ f(x_1) & x_1 & 1 \end{vmatrix} = 0$$

$$p(x)\begin{vmatrix} x_0 & 1 \\ x_1 & 1 \end{vmatrix} - x\begin{vmatrix} f(x_0) & 1 \\ f(x_1) & 1 \end{vmatrix} + 1\begin{vmatrix} f(x_0) & x_0 \\ f(x_1) & x_1 \end{vmatrix} = 0$$

$$\therefore p(x)(x_0 - x_1) = x\{f(x_0) - f(x_1)\} - \{f(x_0)x_1 - x_0 f(x_1)\}$$

$$= (x - x_1) f(x_0) + (x_0 - x) f(x_1)$$

$$\boxed{p(x) = \frac{(x - x_1) f(x_0)}{(x_0 - x_1)} + \frac{(x - x_0) f(x_1)}{(x_1 - x_0)}}$$

Now $p(x) = \dfrac{(x - x_0 + x_0 - x_1) f(x_0)}{(x_0 - x_1)} + \dfrac{(x - x_0) f(x_1)}{(x_1 - x_0)}$

$$= \frac{(x - x_0) f(x_0)}{(x_0 - x_1)} + f(x_0) + \frac{(x - x_0) f(x_1)}{(x_1 - x_0)}$$

$$= f(x_0) + \frac{(x_0 - x) f(x_0)}{(x_1 - x_0)} + \frac{(x - x_0) f(x_1)}{(x_1 - x_0)}$$

$$= f(x_0) + \frac{(x - x_0)[f(x_1) - f(x_0)]}{(x_1 - x_0)}$$

$$= f(x_0) + (x - x_0) \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$f[x_0, x_1]$
(1st order divided difference)

$\therefore$ we have

$$\boxed{p(x) = f(x_0) + (x - x_0) f[x_0, x_1]}$$

Newton's divided diff. formula

<u>Define</u> :-

let $x_0 < x_1 < x_2 \cdots < x_n$ be a set of points.

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{(x_2 - x_0)}$$

Consider

$$f[x_1, x_2] - f[x_0, x_1]$$

$$= \frac{f(x_2) - f(x_1)}{(x_2 - x_1)} - \frac{f(x_1) - f(x_0)}{(x_1 - x_0)}$$

$$= \frac{f(x_2)}{(x_2-x_1)} - f(x_1)\left\{\frac{1}{x_2-x_1} + \frac{1}{x_1-x_0}\right\} + \frac{f(x_0)}{x_1-x_0}$$

$$= \frac{f(x_2)}{(x_2-x_1)} - \frac{f(x_1)(x_2-x_0)}{(x_2-x_1)(x_1-x_0)} + \frac{f(x_0)}{(x_1-x_0)}$$

$$\therefore f[x_0, x_1, x_2]$$

$$= \frac{f(x_2)}{(x_2-x_1)(x_2-x_0)} + \frac{f(x_1)}{(x_1-x_2)(x_1-x_0)} + \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)}$$

$$\boxed{\text{i.e } f[x_0, x_1, x_2] = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)} + \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)} + \frac{f(x_2)}{(x_2-x_1)(x_2-x_0)}}$$

<u>Remark</u> :- $f[x_0, x_1 \cdots x_{n-1}, x_n] = \dfrac{f[x_1 \cdots x_n] - f[x_0, x_1 \cdots x_{n-1}]}{(x_n - x_0)}$

n^{th} <u>divided difference</u>

For 1 degree interpolating polynomial $p(x)$, we have :-

$$p(x) = f(x_0) + (x-x_0)\, f[x_0, x_1].$$

<u>Q</u> What about n^{th} degree interpolating polynomial?

$$p(x) = f(x_0) + (x - x_0) f[x_0, x_1]$$

let us write,

$$p(x) = a_0 + (x - x_0) a_1 + (x - x_0)(x - x_1) a_2 + \cdots + (x - x_0) \cdots (x - x_{n-1}) a_n$$

Thus at $x = x_0$ we have

$$P(x_0) = f(x_0)$$

$$p(x_0) = f(x_0) = a_0 + a_1(0) = a_0$$

$$\boxed{a_0 = f(x_0)}$$

$$p(x_1) = f(x_1) = a_0 + (x_1 - x_0) a_1 + 0$$

$$\boxed{= f(x_0) + (x_1 - x_0) a_1}$$

$$(x_1 - x_0) a_1 = f(x_1) - f(x_0)$$

$$a_1 = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)} = f[x_0, x_1]$$

i.e $\boxed{a_1 = f[x_0, x_1]}$

Similarly,

$$p(x_2) = f(x_2) = a_0 + (x_2 - x_0)\, a_1 + (x_2 - x_0)(x_2 - x_1)\, a_2 + 0$$

$$f(x_2) = f(x_0) + (x_2 - x_0)\, f[x_0, x_1] + (x_2 - x_0)(x_2 - x_1)\, a_2$$

$$(x_2 - x_0)(x_2 - x_1)\, a_2 = f(x_2) - f(x_0) - (x_2 - x_0)\, f[x_0, x_1].$$

Now simplification, we get

$$a_2 = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)} = f[x_0, x_1, x_2]$$

For $x_0 < x_1 < \cdots\cdots < x_n$ given,
the nth degree interpolating polynomial $p(x)$ becomes,

$$p(x) = f(x_0) + (x - x_0)\, f[x_0, x_1] + \cdots\cdots + (x - x_0)(x - x_1)\cdots\cdots(x - x_{n-1})\, f[x_0, x_1 \cdots x_n]$$

is Newton's formula with divided difference

$$f[x_0, \cdots x_n] = \frac{f(x_0)}{(x_0 - x_1)\cdots(x_0 - x_n)} + \frac{f(x_1)}{(x_1 - x_0)\cdots(x_1 - x_n)} + \cdots\cdots + \frac{f(x_n)}{(x_n - x_0)(x_n - x_{n-1})}$$

$$= \sum_{i=0}^{n} \left(\frac{f(x_i)}{\prod_{j=0}^{n} (x_i - x_j)} \right)$$

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Newton divided difference interpolation

Let $x_0 < x_1 \cdots\cdots < x_n$ be $(n+1)$ of distinct points.

The Newton divided difference interpolation

$$\boxed{f(x) = f(x_0) + (x-x_0) f[x_0, x_1] + (x-x_0)(x-x_1) f[x_0, x_1, x_2] + \cdots + (x-x_0)(x-x_1)\cdots\cdots(x-x_{n-1}) f[x_0, x_1, \cdots\cdots x_n].}$$

Recall,

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{(x_2 - x_0)}$$

$$f[x_0, x_1, \cdots\cdots x_n] = \frac{f[x_1, x_2 \cdots x_n] - f[x_0, x_1 \cdots\cdots x_{n-1}]}{(x_n - x_0)}$$

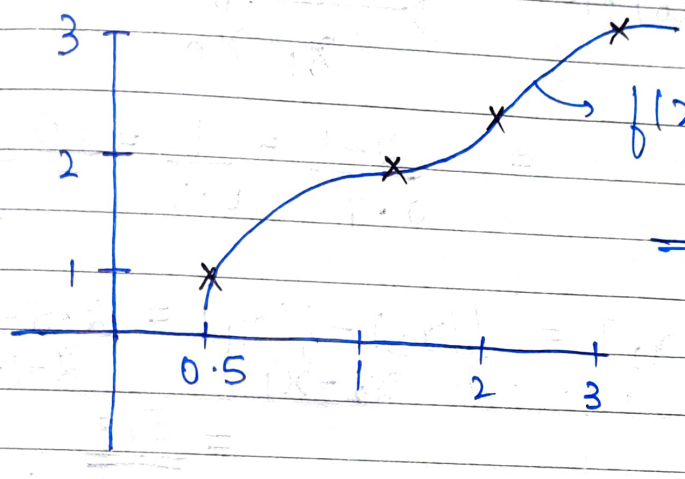
Also,

$$\boxed{f[x_0, x_1, x_2] = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)}}$$

$$f[x_0, x_1, \cdots x_n] = \sum_{i=0}^{n} \left(\frac{f(x_i)}{(x_i - x_0)(x_i - x_1) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_n)} \right)$$

Example -

x	x_0	x_1	x_2	x_3
	0.5	1.2	2	3
$f(x)$	1	2	2.5	3.2



$f(x)$ is the reqd. polynomial

The required interpolation polynomial will be of degree 3. By Newton Divided difference interpolation formula :-

$$① \left\{ \begin{aligned} f(x) = f(x_0) &+ (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] \\ &+ (x - x_0)(x - x_1)(x - x_2) f[x_0, x_1, x_2, x_3]. \end{aligned} \right\}$$

$$f(x_0) = a_0 + a_1 x_0 + \cdots + a_n x_0^n$$

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n$$

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(n+1) coefficients.

$$x_0 \longrightarrow f(x_0) \quad (1)$$
$$x_1 \longrightarrow f(x_1) \quad (2)$$
$$\vdots$$
$$x_n \longrightarrow f(x_n) \quad (n+1)$$

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$= \frac{2-1}{0.7} = \frac{1}{0.7} = \frac{10}{7} = 1\frac{3}{7}$$

$$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{2.5-2}{2-1.2}$$

$$= 5/8$$

$$\therefore f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{(x_2 - x_0)}$$

$$= \frac{5/8 - 10/7}{2.5-1} = \frac{-15}{28}$$

Now,

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{(x_3 - x_0)}$$

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But $f[x_1, x_2, x_3] = \dfrac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$

$f[x_2, x_3] = \dfrac{f(x_3) - f(x_2)}{x_3 - x_2} = \dfrac{3.2 - 2.5}{1}$

$= \underline{\underline{0.7}}$

$f[x_1, x_2, x_3] = \dfrac{0.7 - 5/8}{3 - 1.2} = \underline{\underline{\dfrac{1}{24}}}$

$\therefore f[x_0, x_1, x_2, x_3]$

$= \dfrac{\frac{1}{24} + \frac{15}{28}}{3 - 0.5}$

$= \dfrac{\frac{97}{168}}{2.5} = \dfrac{97}{168 \times 2.5}$

$= \underline{\underline{0.2309}}$

$\therefore$ Substituting values in (1),

$f(x) = 1 + (x - 0.5)\left(\dfrac{10}{7}\right) + (x - 0.5)(x - 1.2)\left(\dfrac{-15}{28}\right)$

$+ (x - 0.5)(x - 1.2)(x - 2)(0.2309)$

Solve & find out the polynomial.

Let $x_0 < x_1 < x_2 < \cdots < x_n$ be $(n+1)$ set of pts.

Gregory newton

Assume that for $i = 0, \ldots n-1$.

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$x_{i+1} = x_i + h$ and $h > 0$.

$x_0 \xrightarrow{h} x_1 \xrightarrow{h} x_2$

Define

$$\Delta f_0 = f(x_1) - f(x_0) = f_1 - f_0$$

$x_1 - x_0$

$$\Delta f_i = f(x_{i+1}) - f(x_i) = f_{i+1} - f_i$$

$$\Delta f_{n-1} = f(x_n) - f(x_{n-1}) = f_n - f_{n-1}$$

We know that,

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{\Delta f_0}{h};$$

$$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{(x_2 - x_1)} = \frac{\Delta f_1}{h}$$

Now,

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{(x_2 - x_0)}$$

$$= \frac{\left(\frac{\Delta f_1}{h} - \frac{\Delta f_0}{h}\right)}{(x_2 - x_0)} = \frac{\Delta f_1 - \Delta f_0}{2h^2}$$

i.e $f[x_0, x_1, x_2] = \frac{\Delta f_1 - \Delta f_0}{2h^2}$

Define $\Delta^2 f_0 = \Delta f_1 - \Delta f_0$

$$\Downarrow$$

$$\boxed{f[x_0, x_1, x_2] = \frac{\Delta^2 f_0}{2h^2}}$$

$\therefore f(x) = f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] + \cdots + (x - x_0) \cdots (x - x_{n-1}) f[x_0, x_1, \cdots x_n]$

Writing this in form of h, we have.

Consider $f[x_0, x_1, x_2, x_3]$

$$= \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{(x_3 - x_0)}$$

$$= \frac{\frac{\Delta^2 f_1}{2h^2} - \frac{\Delta^2 f_0}{2h^2}}{3h}$$

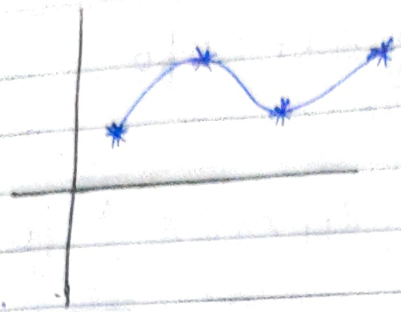
$$= \frac{\Delta^2 f_1 - \Delta^2 f_0}{(3.2) h^3}$$

$$= \frac{\Delta^2 f_1 - \Delta^2 f_0}{3! h^3} = \frac{\Delta^3 f_0}{3! h^3}$$

Gregory-Newton forward diff. interpolation formula

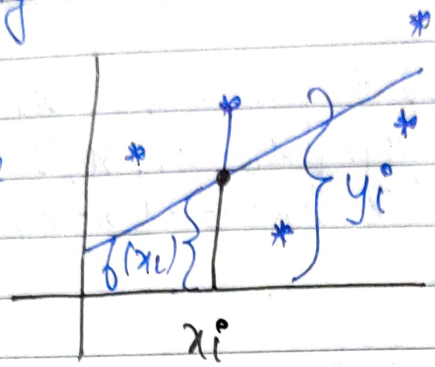
where $\Delta^3 f_0 = \Delta^2 f_1 - \Delta^2 f_0$

$$\therefore f(x) = f(x_0) + (x - x_0) \frac{\Delta f_0}{h} + (x - x_0)(x - x_1) \frac{\Delta^2 f_0}{2! h^2} + (x - x_0)(x - x_1)(x - x_2) \frac{\Delta^3 f_0}{3! h^3} + \cdots + (x - x_0)(x - x_1) \cdots (x - x_{n-1}) \Delta^n f_0 / n! h^n$$



curve fitting.

Straight line fitting



$f(x) = wx + b$

$$\left.\begin{array}{l} x_1 \to y_1 \\ x_2 \to y_2 \\ \vdots \\ x_m \to y_m \end{array}\right\} \text{given}$$

$\{(x_i, y_i)\}_{i=1}^{m}$ be given.

At x_i, the error is

$$e_i = |f(x_i) - y_i|$$

Find total error as :-

$$\sum_{i=1}^{m} e_i^2 = \sum_{i=1}^{m} (f(x_i) - y_i)^2$$

Sum of squares of error terms.

Q Find w and b so that the total error is minimum.

i.e. $\min_{w,b} \sum_{i=1}^{m} (wx_i + b - y_i)^2$

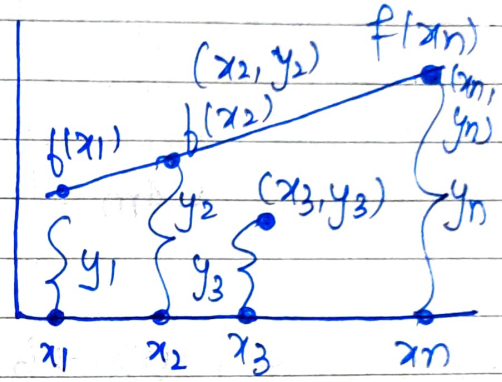
1 Oct, 19

CURVE FITTING

We have,

$$\left.\begin{array}{l} x_1 \rightarrow y_1 \\ x_2 \rightarrow y_2 \\ \vdots \\ x_m \rightarrow y_m \end{array}\right\} \text{Given.}$$

Assume that a straight line $f(x) = wx + b$ is fitted against the data.



Here w, b are unknowns and are to be determined.

Let $e_1 = |y(x_i) - y_1|$ ..... $e_n = |f(x_n) - y_n|$

Consider sum of squared error :-

$$\sum_{i=1}^{m} |f(x_i) - y_i|^2 = \sum_{i=1}^{m} (f(x_i) - y_i)^2$$

$$= \sum_{i=1}^{m} (wx_i + b - y_i)^2 \quad \text{—} (1)$$

Let $A = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_m & 1 \end{bmatrix}_{m \times 2}$ matrix.

$$\bar{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}_{m\times 1} \quad \text{matrix.}$$

consider I

$$A \begin{bmatrix} w \\ b \end{bmatrix} - y$$

$$= \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_m & 1 \end{bmatrix}_{m\times 2} \begin{bmatrix} w \\ b \end{bmatrix}_{2\times 1} - \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}_{m\times 1}$$

$$= \begin{bmatrix} wx_1 + b \\ \vdots \\ wx_i + b \\ wx_n + b \end{bmatrix}_{(m\times 1)} - \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_i \\ y_m \end{bmatrix}_{(m\times 1)} = \begin{bmatrix} wx_1 + b - y_1 \\ \vdots \\ wx_i + b - y_i \\ wx_m + b - y_m \end{bmatrix}_{(m\times 1)}$$

We know if :- $\bar{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{bmatrix}$

then $\bar{z}^T z = [z_1 \; z_2 \cdots z_n] \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \sum_{i=1}^{m} z_i^2$

So, if we choose $\bar{z} = \begin{bmatrix} wx_1 + b - y_1 \\ wx_2 + b - y_2 \\ \vdots \\ wx_i + b - y_i \\ \vdots \\ wx_m + b - y_m \end{bmatrix}$

$$= \bar{z}^T z = [wx_1 + b - y_1 \cdots wx_m + b - y_m] * \begin{bmatrix} wx_1 + b - y_1 \\ \vdots \\ wx_m + b - y_m \end{bmatrix}$$

$$= \sum_{i=1}^{m} (wx_i + b - y_i)^2 = \bar{z}^T \bar{z} = \left(A\begin{bmatrix} w \\ b \end{bmatrix} - \bar{y}\right)^T \left(A\begin{bmatrix} w \\ b \end{bmatrix} - \bar{y}\right)$$

Let $\bar{U} = \begin{bmatrix} w \\ b \end{bmatrix}_{2\times 1}$

$$\Rightarrow \sum_{i=1}^{m} (wx_i + b - y_i)^2 = (A\bar{U} - \bar{y})^T (A\bar{U} - \bar{y})$$

$$= (\bar{U}^T A^T - \bar{y}^T)(A\bar{U} - \bar{y})$$

$$= \bar{U}^T A^T A \bar{U} - \bar{U}^T A^T \bar{y} - \bar{y}^T A \bar{U} + \bar{y}^T \bar{y}$$

$$= \bar{U}^T A^T A \bar{U} - 2\bar{U}^T A^T \bar{y} + \bar{y}^T \bar{y}$$

$$\Rightarrow \boxed{\sum_{i=1}^{m} (wx_i + b - y_i)^2 = \bar{U}^T A^T A \bar{U} - 2\bar{U}^T A^T \bar{y} + \bar{y}^T y.}$$

Q Find w, b so that $\sum_{i=1}^{m} e_i^2$ is minimum.

Find $\bar{U} = \begin{bmatrix} w \\ b \end{bmatrix}$ so that $\sum_{i=1}^{m} e_i^2 = \bar{U}^T A^T A \bar{U} - 2\bar{U}^T A^T \bar{y} + \bar{y}^T y$ is minimum

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Then, $\frac{d}{dw}\sum_{i=1}^{m} e_i^2 = 0$, $\frac{d}{dw}\sum e_i^2 = 0$.

or $\frac{d}{d\bar{U}}\sum e_i^2 = \bar{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$2A^TA\bar{U} - 2A^T\bar{y} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or

$$\boxed{A^TA\bar{U} = A\bar{y} \text{ should be True.}}$$

For a given set of values,
$x_1 \to y_1$, $x_2 \to y_2$, ... $x_m \to y_m$.

and fitting the data with a function $f(x) = wx + b$ and $\bar{U} = \begin{bmatrix} w \\ b \end{bmatrix}$ is determined by solving.

eq^n $A^TA\bar{U} = A^T\bar{y}$ where

$$A = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_m & 1 \end{bmatrix} \quad \bar{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

is called "least square method".

<u>Remark</u> → Suppose $\bar{x}_1 = (x_{11}, x_{12})^T \to y_1$
$\bar{x}_2 = (x_{21}, x_{22})^T \to y_2$
$\vdots$
$\bar{x}_m = (x_{m1}, x_{m2})^T \to y_m$