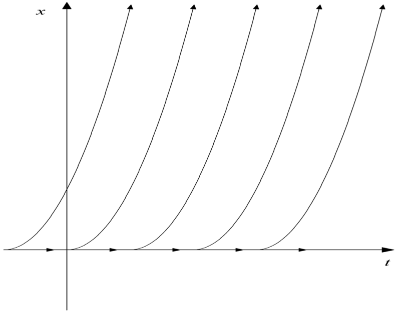
Ordinary differential equations

Marcin Kuta

Existence and Uniqueness

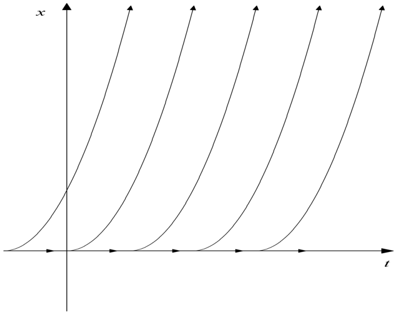
$$\frac{dy}{dt} = y^{1/2}, \; y \geq 0$$
$$y(0) = 0$$



$$y(t) = \begin{cases} \frac{1}{4}(t - C)^2, & \text{for } C \leq t \leq \infty \\ 0, & \text{for } 0 \leq t \leq C \end{cases}$$

Existence and Uniqueness

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y^{\beta},\ 0 < \beta < 1,\ y \geq 0$$
$$y(0) = 0$$



$$y(t) = \begin{cases} (1-\beta)^{\frac{1}{1-\beta}}(t-C)^{\frac{1}{1-\beta}}, & \text{for } C \leq t \leq \infty \\ 0, & \text{for } 0 \leq t \leq C \end{cases}$$

Blow-up

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y^2,\ \ y(0) = 1$$

$$y(t) = \frac{1}{1-t}$$

Existence and Uniqueness

Function $f(t, y) \colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ is *Lipschitz continuous* if there is constant L such, that for any $t \in [a, b]$ and any $y, \hat{y} \in \Omega$

$$||f(t, \hat{y}) - f(t, y)|| \leq L||\hat{y} - y||$$

If f is differentiable:

$$L = \max_{(t,y) \in D} ||J_f(t, y)||$$

Properities of numerical solutions

Consistency	concerns local error
Convergence	concerns global error
Stability	concerns initial conditions

Stability of a problem

$$y' = f(y,t)$$

$$y(t_0) = y_0$$
$$\hat{y}(t_0) = \hat{y}_0$$

Solution to $y' = f(y,t)$ is stable if for every ε there exists δ such that if

$$||\hat{y}(t_0) - y(t_0)|| < \delta$$

then

$$||\hat{y}(t) - y(t)|| < \varepsilon \text{ for } t \geq t_0.$$

Stable solution to $y' = f(y,t)$ is asymptotically stable if

$$||\hat{y}(t) - y(t)|| \to 0 \text{ for } t \to \infty.$$

Explicit Euler method

$$y' = \lambda y$$

$$y_{k+1} = y_k + h_k f(t_k, y_k) \tag{1}$$

Numerical stability:

$$|1 + h\lambda| < 1 \tag{2}$$

Implicit Euler method

$$y' = \lambda y$$

$$y_{k+1} = y_k + h_k f(t_{k+1}, y_{k+1}) \tag{3}$$

Numerical stability:

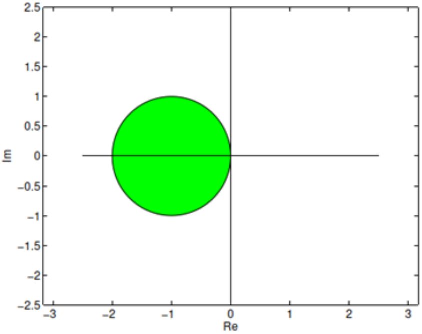
$$\left| \frac{1}{1 - h\lambda} \right| < 1 \tag{4}$$

Amplification factor

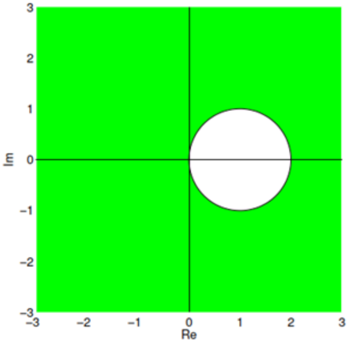
$$\varepsilon_{n+1} = Q(\lambda h)\varepsilon_n \tag{5}$$

Method	Amplification factor
Explicit Euler	$1 + \lambda h$
Implicit Euler	$\frac{1}{1-\lambda h}$
Trapezoidal	$\frac{1+\frac{1}{2}\lambda h}{1-\frac{1}{2}\lambda h}$
Modified Euler	$1 + \lambda h + \frac{1}{2}(\lambda h)^2$
RK4	$1 + \lambda h + \frac{1}{2}(\lambda h)^2 + \frac{1}{6}(\lambda h)^3 + \frac{1}{24}(\lambda h)^4$

Stability regions

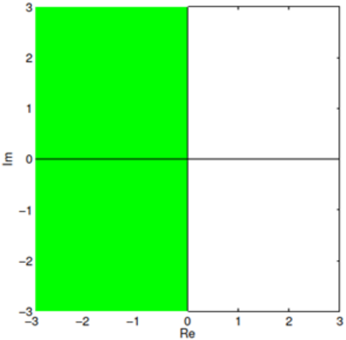


(a) Explicit Euler method

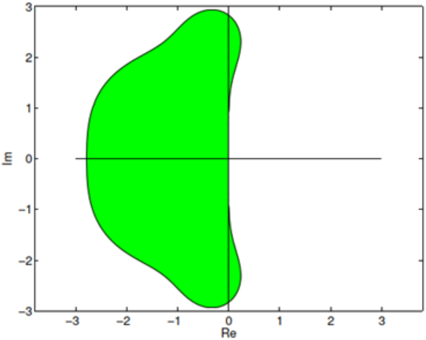


(b) Implicit Euler method

Stability regions



(c) Implicit trapeziodal rule



(d) Explicit Runge-Kutta method (RK4)

Local and global error

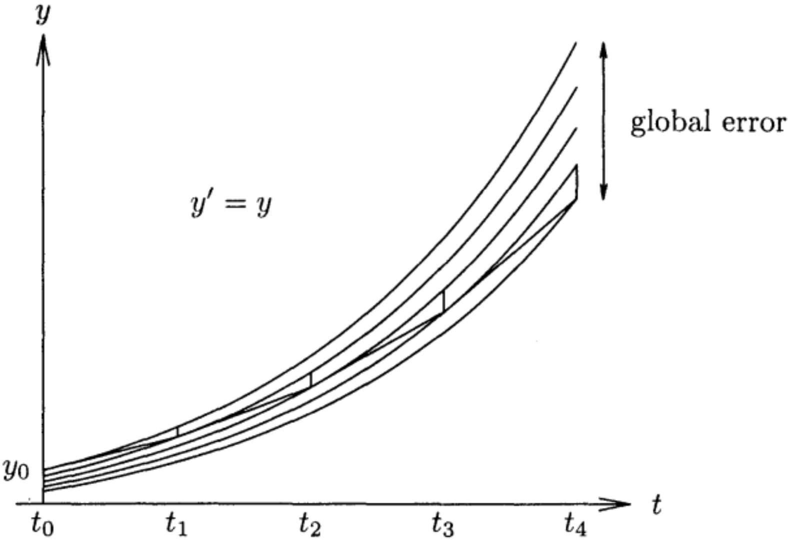
Local error [2]

$$\ell_k = \frac{y_k - u_{k-1}(t_k)}{h_k} \tag{6}$$

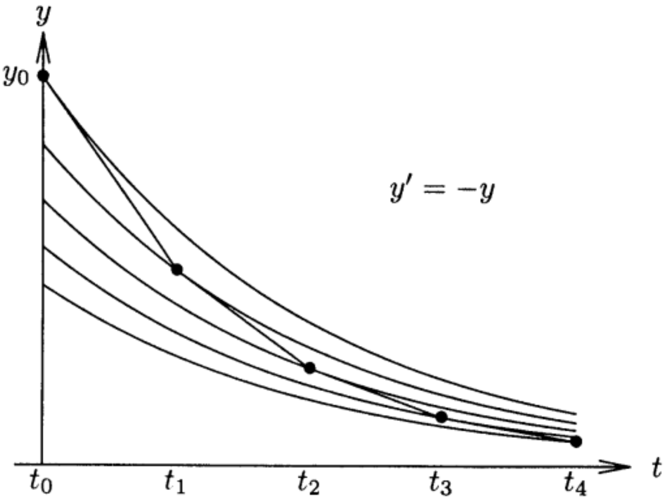
Global error

$$e_k = y_k - y(t_k) \tag{7}$$

Local and global error



Local and global error



Local error

Method	Local error
Explicit Euler	$O(h)$
Implicit Euler	$O(h)$
Trapezoidal	$O(h^2)$
Modified Euler	$O(h^2)$
RK4	$O(h^4)$

Methods

Fixed step	Adaptive step
Euler method	Runge-Kutta 1(2), Adaptive Heun
Midpoint method	Runge-Kutta 2(3), Bogacki-Shampine
Runge-Kutta 4, 3/8 rule	Runge-Kutta 4(5), Dormand-Prince
Explicit Adams-Bashforth	Runge-Kutta 7(8), Dormand-Prince-Shampine
Implicit Adams-Bashforth-Moulton	

Conservation laws

Method	Symplectic	Energy	Angular momentum
Explicit Euler	$\times$	$\nearrow$	$\times$
Implicit Euler	$\times$	$\searrow$	$\times$
Semi explicit Euler	$\checkmark$	$\times$	$\checkmark$
RK4	$\times$		

References

[1] Michael T. Heath,
http://heath.cs.illinois.edu/scicomp/notes/cs450_chapt09.pdf

[2] C. Vuik, F.J. Vermolen, M.B. van Gijzen, M.J. Vuik,
Numerical Methods for Ordinary Differential Equations