# Bounds for Eigenvalues Using Traces*

Henry Wolkowicz
*Department of Mathematics*
*Dalhousie University*
*Halifax, Nova Scotia, Canada B3H 4H8*

and

George P. H. Styan
*Department of Mathematics*
*McGill University*
*805 Sherbrooke Street West*
*Montréal, Québec, Canada H3A 2K6*

Dedicated to Alston S. Householder
on the occasion of his seventy-fifth birthday.

Submitted by G. W. Stewart

ABSTRACT

Several new inequalities are obtained for the modulus, the real part, and the imaginary part of a linear combination of the ordered eigenvalues of a square complex matrix. Included are bounds for the condition number, the spread, and the spectral radius. These inequalities involve the trace of a matrix and the trace of its square. Necessary and sufficient conditions for equality are given for each inequality.

## 1. INTRODUCTION

The eigenvalues of an $n \times n$ complex matrix $\mathbf{A}$ are the roots of the $n$th-degree polynomial $\det(\mathbf{A}-\lambda\mathbf{I})=0$ and so are difficult to evaluate in general. It is, however, often useful to know the approximate location of the

*This research was supported in part by the Natural Sciences and Engineering Research Council Canada and by the Gouvernement du Québec, Programme de formation de chercheurs et d'action concertée. Some of the results were presented at the Distinguished Lecture Series on Applied Matrix Computations, Baltimore, Maryland, August 1977, and at the International Congress of Mathematicians, Helsinki, Finland, August 1978.



© Elsevier North Holland, Inc., 1980 0024-3795/80/010471+36$01.75

eigenvalues. For example, when solving an equation of the type $\mathbf{Ax}=\mathbf{b}$, one may use the iterative scheme $\mathbf{x}_{h+1}=\mathbf{Tx}_h+\mathbf{c}$, where $\mathbf{T}$ is a certain matrix related to $\mathbf{A}$, and $\mathbf{c}$ is a column vector related to both $\mathbf{A}$ and $\mathbf{b}$; cf. e.g. [17, Chapter 3]. This scheme converges for arbitrary $\mathbf{x}_0$ if and only if all the eigenvalues of $\mathbf{T}$ lie inside the unit circle in the complex plane. This also characterizes the convergence of the geometric series $\Sigma_{h=0}^{\infty}\mathbf{T}^h$ to the matrix $(\mathbf{I}-\mathbf{T})^{-1}$. More generally, $\Sigma_{h=0}^{\infty}\beta_h\mathbf{T}^h$ converges if and only if all the eigenvalues of $\mathbf{T}$ lie inside the circle of convergence of the scalar series $\Sigma_{h=0}^{\infty}\beta_h z^h$; cf. [7]. For a Hermitian positive definite matrix, the ratio of the largest to the smallest eigenvalue is useful in determining whether the equation $\mathbf{Ax}=\mathbf{b}$ is ill-conditioned or not; cf. e.g. [15, p. 185]. In theory of stability of solutions to differential equations, a complex matrix is said to be *stable* if the real parts of all its eigenvalues are negative; cf. e.g. [9, pp. 158–9]. If we know that the matrix has $t$ eigenvalues with negative real part, then it is stable on the eigenspace corresponding to these eigenvalues. Furthermore, if the imaginary part of an eigenvalue is not zero, then it is known that the solution will spiral; cf. e.g. [6].

For certain special types of matrices some information about the eigenvalues is known beforehand, e.g., a stochastic matrix always has at least one eigenvalue equal to 1 and all others lie in or on the unit circle in the complex plane; cf. e.g. [9, p. 133]. In general, though, nothing so specific can be said about where the eigenvalues may lie.

Bounds for eigenvalues have been obtained by many authors over roughly the last hundred years; cf. e.g. [9, Chapter III]. Some of these bounds involve the sums of absolute values of elements in a row and/or column. Following [9, p. 144], let $\mathbf{A}=(a_{kl})$ be an $n\times n$ complex matrix, and write

$$\begin{aligned} R_k &= \sum_{l=1}^{n} |a_{kl}|, \qquad & C_l &= \sum_{k=1}^{n} |a_{kl}|, \\ R &= \max_k R_k, \qquad & C &= \max_l C_l. \end{aligned} \tag{1.1}$$

Let $\lambda(\mathbf{A})$ denote an eigenvalue of $\mathbf{A}$. Then the inequality

$$|\lambda(\mathbf{A})| \leqslant \min(R,C) \tag{1.2}$$

was proved by Alfred Brauer in 1946, though anticipated by Oskar Perron in 1933 (cf. [9, p. 145]).

Possibly the best-known inequality for eigenvalues, however, was found by S. A. Geršgorin in 1931 (cf. e.g. [9, p. 146]): the eigenvalues lie in the closed region of the complex plane consisting of all the discs

$$|z-a_{kk}|\leqslant R_k-|a_{kk}|, \qquad k=1,2,\dots,n. \tag{1.3}$$

In this paper we use traces to obtain various new eigenvalue inequalities. An early result of this kind was derived by Issai Schur in 1909; cf. [10, p. 309]. Let (cf. [1, p. 133])

$$\mathbf{B}=\tfrac{1}{2}(\mathbf{A}+\mathbf{A}^*) \quad \text{and} \quad \mathbf{C}=\tfrac{1}{2}(\mathbf{A}-\mathbf{A}^*)/i, \tag{1.4}$$

where $\mathbf{A}^*=(\bar{a}_{lk})$ is the conjugate transpose of $\mathbf{A}=(a_{kl})$. Then $\mathbf{A}=\mathbf{B}+i\mathbf{C}$, with $\mathbf{B}$ and $\mathbf{C}$ Hermitian. Moreover

$$\Sigma|\lambda(\mathbf{A})|^2\leqslant \operatorname{tr}(\mathbf{A}^*\mathbf{A})=\Sigma|a_{kl}|^2, \tag{1.5}$$

$$\Sigma[\operatorname{Re}\lambda(\mathbf{A})]^2\leqslant \operatorname{tr}\mathbf{B}^2=\Sigma|\tfrac{1}{2}(a_{kl}+\bar{a}_{lk})|^2, \tag{1.6}$$

$$\Sigma[\operatorname{Im}\lambda(\mathbf{A})]^2\leqslant \operatorname{tr}\mathbf{C}^2=\Sigma|\tfrac{1}{2}(a_{kl}-\bar{a}_{lk})|^2. \tag{1.7}$$

Equality in any one of these inequalities implies equality in all three and occurs if and only if $\mathbf{A}$ is normal, i.e., $\mathbf{AA}^*=\mathbf{A}^*\mathbf{A}$.

More recently, L. Mirsky [11] in 1956, and A. Brauer and A. C. Mewborn [2] in 1959, have used traces to derive inequalities for the spread, $\operatorname{sp}(\mathbf{A})=\lambda_{\max}(\mathbf{A})-\lambda_{\min}(\mathbf{A})$; cf. Theorem 2.5. These inequalities follow the work of Popoviciu [12] in 1935 on polynomials with real roots.

In statistics, G. W. Thomson [16] in 1955 obtained related inequalities for the range of a set of random variables. The connection between these inequalities is that the standard deviation of the eigenvalues is a simple function of the trace of the matrix and the trace of its square; cf. Graybill [4, p. 227]. If the matrix $\mathbf{A}$ has real eigenvalues $\lambda_1,\lambda_2,\dots,\lambda_n$, then we may define their variance to be

$$s^2=\frac{1}{n}\left[\sum_{j=1}^{n}\lambda_j^2-\frac{1}{n}\left(\sum_{j=1}^{n}\lambda_j\right)^2\right]=\frac{\operatorname{tr}\mathbf{A}^2-(\operatorname{tr}\mathbf{A})^2/n}{n}. \tag{1.8}$$

Our results are of the following type. Let $\mathbf{A}$ be an $n\times n$ complex matrix, and suppose that its eigenvalues are all real and ordered:

$$\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_n. \tag{1.9}$$

Then (cf. [18] and (2.3) below)

$$m+s/(n-1)^{1/2} \leqslant \lambda_1 \leqslant m+s(n-1)^{1/2}, \tag{1.10}$$

where

$$m=\frac{\operatorname{tr}\mathbf{A}}{n}=\frac{1}{n}\sum_{j=1}^{n}\lambda_j \tag{1.11}$$

is the mean of the eigenvalues, while $s$ is their standard deviation, the positive square root of the variance as defined by (1.8). Equality on the left of (1.10) occurs if and only if $\lambda_1=\lambda_2=\cdots=\lambda_{n-1}$, and on the right if and only if $\lambda_2=\lambda_3=\cdots=\lambda_n$.

In Sec. 2 we present various inequalities which hold when all the eigenvalues are real; these results are extended to the more general complex case in Sec. 3. The paper concludes with a number of examples in Sec. 4.

## 2. REAL EIGENVALUES

Our inequalities are tightest when all the eigenvalues are real; this happens for example when the matrix is Hermitian or is the product of two (semi)definite Hermitian matrices. A diagonable matrix $\mathbf{A}$ has all its eigenvalues real if and only if there exists a positive definite Hermitian matrix $\mathbf{S}$ such that $\mathbf{AS}=\mathbf{SA}^*$; cf. [3].

THEOREM 2.1. *Let* $\mathbf{A}$ *be an* $n\times n$ *complex matrix with real eigenvalues* $\lambda(\mathbf{A})$, *and let*

$$m=\operatorname{tr}\mathbf{A}/n, \qquad s^2=\operatorname{tr}\mathbf{A}^2/n-m^2. \tag{2.1}$$

*Then*

$$m-s(n-1)^{1/2} \leqslant \lambda_{\min}(\mathbf{A}) \leqslant m-s/(n-1)^{1/2}, \tag{2.2}$$

$$m+s/(n-1)^{1/2} \leqslant \lambda_{\max}(\mathbf{A}) \leqslant m+s(n-1)^{1/2}. \tag{2.3}$$

*Equality holds on the left (right) of (2.2) if and only if equality holds on the left (right) of (2.3) if and only if the* $n-1$ *largest (smallest) eigenvalues are equal.*

Notice that when $n=2$ the two inequality strings (2.2) and (2.3) collapse to yield

$$\lambda_{\min}(\mathbf{A}) = m - s \quad \text{and} \quad \lambda_{\max}(\mathbf{A}) = m + s. \tag{2.4}$$

Our inequalities were initially obtained using mathematical programming techniques. These techniques lead to optimal bounds for eigenvalues in terms of the trace of $\mathbf{A}$ and the trace of $\mathbf{A}^2$. Once found, however, the inequalities are more easily proved using a Cauchy-Schwarz type inequality:

LEMMA 2.1. *Let* $\mathbf{w}$ *and* $\boldsymbol{\lambda}$ *be real nonnull* $n\times 1$ *vectors, and let*

$$m = \boldsymbol{\lambda}'\mathbf{e}/n \quad \textit{and} \quad s^2 = \boldsymbol{\lambda}'\mathbf{C}\boldsymbol{\lambda}/n, \tag{2.5}$$

*where* $\mathbf{e}$ *is the* $n\times 1$ *vector of ones, the centering matrix* $\mathbf{C} = \mathbf{I} - \mathbf{e}\mathbf{e}'/n$, *and* $\mathbf{e}'$ *is the transpose of* $\mathbf{e}$. *Then*

$$-s(n\mathbf{w}'\mathbf{C}\mathbf{w})^{1/2} \leqslant \mathbf{w}'\boldsymbol{\lambda} - m\mathbf{w}'\mathbf{e} = \mathbf{w}'\mathbf{C}\boldsymbol{\lambda} \leqslant s(n\mathbf{w}'\mathbf{C}\mathbf{w})^{1/2}. \tag{2.6}$$

*Equality holds on the left (right) of* (2.6) *if and only if*

$$\boldsymbol{\lambda} = a\mathbf{w} + b\mathbf{e} \tag{2.7}$$

*for some scalars* $a$ *and* $b$, *where* $a<0$ $(a>0)$.

*Proof.* The inequality string (2.6) follows at once from the Cauchy-Schwarz inequality $(\mathbf{w}'\mathbf{C}\boldsymbol{\lambda})^2 \leqslant \mathbf{w}'\mathbf{C}\mathbf{w}\cdot\boldsymbol{\lambda}'\mathbf{C}\boldsymbol{\lambda}$, since $\mathbf{C}$ is symmetric idempotent, while the equality condition (2.7) is equivalent to $\mathbf{C}\boldsymbol{\lambda} = a\mathbf{C}\mathbf{w}$ for some scalar $a$ (the scalar $b = m - a\mathbf{w}'\mathbf{e}/n$). ∎

We will also need

LEMMA 2.2. *Let* $\boldsymbol{\lambda} = (\lambda_j)$, $m$ *and* $s$ *be defined as in Lemma* 2.1, *and*

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n. \tag{2.8}$$

*Then (cf.* (2.2) *and* (2.3))

$$\lambda_n \leqslant m - \frac{s}{(n-1)^{1/2}} \leqslant m + \frac{s}{(n-1)^{1/2}} \leqslant \lambda_1. \tag{2.9}$$

*Equality holds on the left if and only if* $\lambda_2=\lambda_3=\cdots=\lambda_n$, *on the right if and only if* $\lambda_1=\lambda_2=\cdots=\lambda_{n-1}$, *and in the center if and only if* $\lambda_1=\lambda_2=\cdots=\lambda_n \Leftrightarrow s=0$.

*Proof.* We have that

$$\begin{aligned} n^2(m-\lambda_n)^2 &= \left[\sum_{j=1}^{n}(\lambda_j-\lambda_n)\right]^2 = \sum_{j=1}^{n}(\lambda_j-\lambda_n)^2+\sum_{j\neq k}(\lambda_j-\lambda_n)(\lambda_k-\lambda_n)\\ &\geqslant \sum_{j=1}^{n}(\lambda_j-\lambda_n)^2=\sum_{j=1}^{n}(\lambda_j-m+m-\lambda_n)^2\\ &= n\left[s^2+(m-\lambda_n)^2\right], \end{aligned} \tag{2.10}$$

from which the left-hand inequality in (2.9) follows immediately. For the right-hand inequality we expand $n^2(\lambda_1-m)^2$ as we did $n^2(m-\lambda_n)^2$ in (2.10). Equality holds throughout (2.10) if and only if

$$\sum_{j\neq k}(\lambda_j-\lambda_n)(\lambda_k-\lambda_n)=0, \tag{2.11}$$

which holds $\Leftrightarrow \lambda_2=\lambda_3=\cdots=\lambda_n$. The rest of the lemma follows directly. ∎

*Proof of Theorem 2.1.* It is easy to see that $m$ and $s^2$ defined by (2.1) and by (2.5) are equivalent; cf. (1.8). We now use Lemma 2.1 with $\mathbf{w}=\mathbf{e}_j$, the $j$th column of the identity matrix $\mathbf{I}_n$. Then (2.6) becomes

$$-s(n-1)^{1/2}\leqslant\lambda_j-m\leqslant s(n-1)^{1/2}, \tag{2.12}$$

which proves the left-hand side of (2.2) and the right-hand side of (2.3). For equality on the left (right) set $j=n$ and $\mathbf{w}=\mathbf{e}_n$ ($j=1$ and $\mathbf{w}=\mathbf{e}_1$). The right-hand side of (2.2) and the left-hand side of (2.3) follow directly from Lemma 2.2. ∎

As remarked earlier, if $\mathbf{A}$ is positive definite, then $\lambda_{\max}(\mathbf{A})/\lambda_{\min}(\mathbf{A})$ may be used as a "condition number" of $\mathbf{A}$. From the above Theorem 2.1, it is clear that if $m-s(n-1)^{1/2}>0$ (or equivalently if $\operatorname{tr}\mathbf{A}>0$ and $(\operatorname{tr}\mathbf{A})^2/\operatorname{tr}\mathbf{A}^2>n-1$) and if $\mathbf{A}$ is Hermitian, then $\mathbf{A}$ is positive definite. Therefore, (2.2) and (2.3) imply:

COROLLARY 2.1. *Let* $\mathbf{A}$, $m$, *and* $s^2$ *be defined as in Theorem* 2.1.

(i) *If* $\mathbf{A}$ *is positive definite, then*

$$1+\frac{2s/(n-1)^{1/2}}{m-s/(n-1)^{1/2}}\leqslant\frac{\lambda_{\max}(\mathbf{A})}{\lambda_{\min}(\mathbf{A})}. \tag{2.13}$$

*When* $n>2$, *equality holds if and only if all the eigenvalues of* $\mathbf{A}$ *are equal.*

(ii) *If* $\mathbf{A}$ *is Hermitian,* $\operatorname{tr}\mathbf{A}>0$, *and* $(\operatorname{tr}\mathbf{A})^2>(n-1)\operatorname{tr}\mathbf{A}^2$, *then* $\mathbf{A}$ *is positive definite,* (2.13) *holds, and*

$$\frac{\lambda_{\max}(\mathbf{A})}{\lambda_{\min}(\mathbf{A})}\leqslant 1+\frac{2s(n-1)^{1/2}}{m-s(n-1)^{1/2}}. \tag{2.14}$$

*When* $n>2$, *equality holds if and only if* $\mathbf{A}$ *is a scalar matrix.*

The inequalities (2.2) and (2.3) may be extended to linear combinations of the eigenvalues. Let

$$\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_n \tag{2.15}$$

be the ordered eigenvalues of $\mathbf{A}$ [cf. (1.9)], and let

$$\lambda_{(k,l)}=\sum_{j=k}^{l}\frac{\lambda_j}{l-k+1}, \tag{2.16}$$

which we may call a *mid-mean.* Notice that

$$\lambda_{(1,n)}=m=\frac{1}{n}\sum_{j=1}^{n}\lambda_j, \tag{2.17}$$

[cf. (1.11)], while

$$\lambda_{(k,k)}\equiv\lambda_k. \tag{2.18}$$

Then

THEOREM 2.2. *Let* $\mathbf{A}$, $m$, *and* $s^2$ *be defined as in Theorem* 2.1, *and let* $\lambda_{(k,l)}$ *be as in* (2.16). *Then*

$$m-s\left(\frac{k-1}{n-k+1}\right)^{1/2}\leqslant\lambda_{(k,l)}\leqslant m+s\left(\frac{n-l}{l}\right)^{1/2}. \tag{2.19}$$

*When $(k,l)=(1,n)$, the inequality string collapses. When $(k,l)\neq(1,n)$, then equality holds on the left of (2.19) if and only if*

$$\lambda_1=\lambda_2=\cdots=\lambda_{k-1} \quad \text{and} \quad \lambda_k=\lambda_{k+1}=\cdots=\lambda_n, \tag{2.20}$$

*and on the right if and only if*

$$\lambda_1=\lambda_2=\cdots=\lambda_l \quad \text{and} \quad \lambda_{l+1}=\lambda_{l+2}=\cdots=\lambda_n. \tag{2.21}$$

*Furthermore,*

$$m-s\left(\frac{k-1}{n-k+1}\right)^{1/2}\leqslant\lambda_k\leqslant m+s\left(\frac{n-k}{k}\right)^{1/2}. \tag{2.22}$$

*Equality holds on the left if and only if (2.20) holds, and on the right if and only if*

$$\lambda_1=\lambda_2=\cdots=\lambda_k \quad \text{and} \quad \lambda_{k+1}=\lambda_{k+2}=\cdots=\lambda_n. \tag{2.23}$$

*Proof.* Put $\mathbf{w}=\Sigma_{j=k}^{l}\mathbf{e}_j/(l-k+1)$ in Lemma 2.1. Then $\mathbf{w}'\mathbf{e}=1$ and $\mathbf{w}'\mathbf{C}\mathbf{w}=(l-k+1)^{-1}-n^{-1}$. Hence

$$m-s\left(\frac{k-1}{n-k+1}\right)^{1/2}\leqslant\lambda_{(k,n)}\leqslant\lambda_{(k,l)}\leqslant\lambda_{(1,l)}\leqslant m+s\left(\frac{n-l}{l}\right)^{1/2}. \tag{2.24}$$

It follows directly that equality on the left of (2.19) holds if and only if (2.20), and on the right of (2.19) if and only if (2.21). ∎

Mallows and Richter [8] obtained several inequalities for the standard deviation of a set of numbers; in particular, their (6.1) leads directly to our Theorem 2.2. Furthermore, their Corollary 6.1 yields a stronger lower bound than that in (2.19) when $(k,l)=(1,l)$ and a stronger upper bound when $(k,l)=(k,n)$.

THEOREM 2.3. *Let $\mathbf{A}$, $m$, and $s^2$ be defined as in Theorem 2.1, and let $\lambda_{(k,l)}$ be as in (2.16). Then*

$$\lambda_{(1,l)}\geqslant\begin{cases} m+\dfrac{s}{(n-1)^{1/2}} & \text{if } l\leqslant\frac{1}{2}n,\\[2ex] m+\dfrac{s(n-l)}{l(n-1)^{1/2}} & \text{if } l\geqslant\frac{1}{2}n,\end{cases} \tag{2.25}$$

$$\lambda_{(k,n)}\leqslant\begin{cases} m-\dfrac{s(k-1)}{(n-k+1)(n-1)^{1/2}} & \text{if } k\leqslant\frac{1}{2}n+1,\\[2ex] m-\dfrac{s}{(n-1)^{1/2}} & \text{if } k\geqslant\frac{1}{2}n+1.\end{cases} \tag{2.26}$$

*Equality holds in* (2.25) *if and only if*

$$\begin{aligned} &\lambda_1=\lambda_2=\cdots=\lambda_{n-1} && \text{when} \quad l<\tfrac{1}{2}n,\\ &\lambda_1=\lambda_2=\cdots=\lambda_{n-1} \text{ or } \lambda_2=\lambda_3=\cdots=\lambda_n && \text{when} \quad l=\tfrac{1}{2}n,\\ &\lambda_2=\lambda_3=\cdots=\lambda_n && \text{when} \quad l>\tfrac{1}{2}n. \end{aligned} \tag{2.27}$$

*Equality holds in* (2.26) *if and only if*

$$\begin{aligned} &\lambda_1=\lambda_2=\cdots=\lambda_{n-1} && \text{when} \quad k<\tfrac{1}{2}n+1,\\ &\lambda_1=\lambda_2=\cdots=\lambda_{n-1} \text{ or } \lambda_2=\lambda_3=\cdots=\lambda_n && \text{when} \quad k=\tfrac{1}{2}n+1,\\ &\lambda_2=\lambda_3=\cdots=\lambda_n && \text{when} \quad k>\tfrac{1}{2}n+1. \end{aligned} \tag{2.28}$$

*Proof.* The inequalities (2.25) and (2.26) follow directly from Corollary 6.1 in [8, p. 1931]. To obtain the conditions for equality given by (2.27), however, we use the following equality, which is straightforward but tedious to establish:

$$\begin{aligned} n(n-1)(\lambda_{(1,l)}-m)^2-ns^2 &= (n-2l)(n-1)l^{-2}\sum_{i,j=1}^{l} ij\lambda_i\lambda_j+\sum_{i,j=1}^{l}\min(i,j)\left[\max(i,j)-1\right]\lambda_i\lambda_j\\ &\quad +2(n-l-1)l^{-1}\sum_{i=1}^{l}\sum_{j=l+1}^{n-1} i(n-j)\lambda_i\lambda_j\\ &\quad +\sum_{i,j=l+1}^{n-1}\left[n-\max(i,j)\right]\left[n-1-\min(i,j)\right]\lambda_i\lambda_j. \end{aligned} \tag{2.29}$$

The condition (2.28) is obtained similarly. ■

We note that when $k=1$ (respectively $n$), the lower (respectively upper) bound in (2.22) equals $m$. The bounds from Theorem 2.1,

$$\lambda_n \leqslant m-\frac{s}{(n-1)^{1/2}} \leqslant m+\frac{s}{(n-1)^{1/2}} \leqslant \lambda_1, \tag{2.30}$$

are therefore better whenever $s>0$.

When $n=3$, however, we may combine (2.22) and (2.30) to yield the contiguous bounds

$$m-s\sqrt{2} \leqslant \lambda_3 \leqslant m-s/\sqrt{2} \leqslant \lambda_2 \leqslant m+s/\sqrt{2} \leqslant \lambda_1 \leqslant m+s\sqrt{2}\ . \tag{2.31}$$

When $n \geqslant 4$ our bounds are, unfortunately, no longer contiguous. From (2.22) and (2.30) we obtain, when $n=4$,

$$\begin{gathered} m - s\sqrt{3} \leqslant \lambda_4 \leqslant m - s/\sqrt{3} \leqslant \lambda_2 \leqslant m + s, \\ m - s \leqslant \lambda_3 \leqslant m + s/\sqrt{3} \leqslant \lambda_1 \leqslant m + s\sqrt{3}\,. \end{gathered} \tag{2.32}$$

When $n=5$, we have

$$\begin{aligned} m - 2s \leqslant \lambda_5 \leqslant m - \tfrac{1}{2}s \leqslant \lambda_2 \leqslant m + s\sqrt{\tfrac{3}{2}}\,, \\ m - s\sqrt{\tfrac{3}{2}} \leqslant \lambda_4 \leqslant m + \tfrac{1}{2}s \leqslant \lambda_1 \leqslant m + 2s, \\ m - s\sqrt{\tfrac{2}{3}} \leqslant \lambda_3 \leqslant m + s\sqrt{\tfrac{2}{3}}\,. \end{aligned} \tag{2.33}$$

This suggests that for $n=6$ our bounds for $\lambda_3$ and $\lambda_4$ might be contiguous. However, we obtain

$$\begin{aligned} & m - s\sqrt{5} \leqslant \lambda_6 \leqslant m - s/\sqrt{5} \leqslant \lambda_2 \leqslant m + s\sqrt{2}\,, \\ & m - s\sqrt{2} \leqslant \lambda_5 \leqslant m + s/\sqrt{5} \leqslant \lambda_1 \leqslant m + s\sqrt{5}\,, \\ & m - s \leqslant \lambda_4 \leqslant m + s/\sqrt{2}\,, \\ & m - s/\sqrt{2} \leqslant \lambda_3 \leqslant m + s. \end{aligned} \tag{2.34}$$

Our bounds (2.4), (2.30), (2.32)–(2.34) are plotted in Fig. 1. Each axis is in units of $c$, where $m \pm cs$ is the bound. The center is at $m=0$, and the scale has $s=1$.

Graybill [4, p. 228] considered a matrix $\mathbf{A}$ with real eigenvalues, precisely $t$ of which are nonzero. Then $t>0$ if and only if $\operatorname{tr}\mathbf{A}^2>0$; in this event

$$(\operatorname{tr}\mathbf{A})^2/\operatorname{tr}\mathbf{A}^2 \leqslant t \leqslant \operatorname{rank}\mathbf{A}. \tag{2.35}$$

The following corollary strengthens this result.

COROLLARY 2.2. *Let* $\mathbf{A}$ *have real eigenvalues with precisely* $f$ *being positive and* $g$ *negative. Let* $\operatorname{tr}\mathbf{A}^2>0$.

(i) *When* $\operatorname{tr}\mathbf{A}\geqslant 0$, *then*

$$(\operatorname{tr}\mathbf{A})^2/\operatorname{tr}\mathbf{A}^2 \geqslant f, \tag{2.36}$$

*with equality if and only if all the positive eigenvalues are equal and all the nonpositive eigenvalues are equal.*

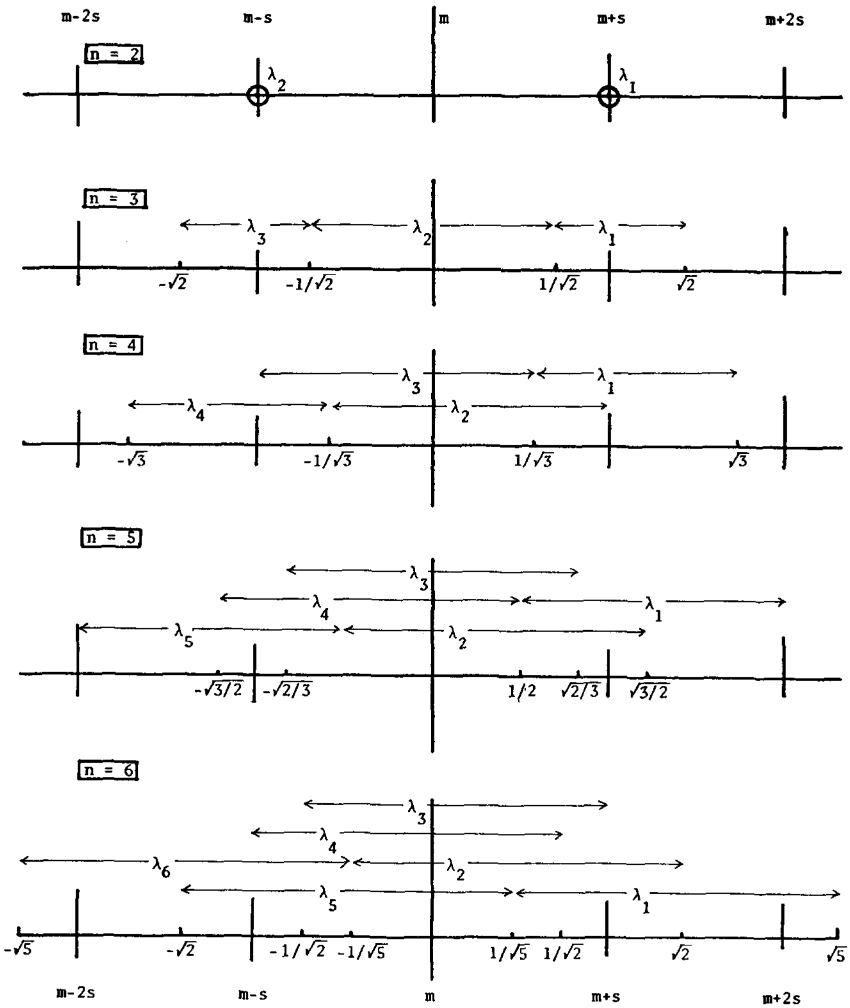


FIG. 1. Bounds for eigenvalues, $n=2(1)6$.

(ii) *When* $\operatorname{tr}\mathbf{A} \leqslant 0$, *then*

$$(\operatorname{tr}\mathbf{A})^2/\operatorname{tr}\mathbf{A}^2 \leqslant g, \tag{2.37}$$

*with equality if and only if all the negative eigenvalues are equal and all the nonnegative eigenvalues are equal.*

*Proof.* It suffices to prove (i), as (ii) then follows by replacing $\mathbf{A}$ with $-\mathbf{A}$. First assume $f=n$. Then (2.36) is just the Cauchy-Schwarz inequality, and equality holds if and only if all the eigenvalues are equal. When $f<n$ it follows that $\lambda_{f+1}\leqslant 0$, and using (2.22) with $k=f+1$, we get $m(n-f)^{1/2}\leqslant sf^{1/2}$. Squaring both sides leads to (2.36) directly; equality holds if and only if $\lambda_{f+1}=0$ and [using (2.20) with $k=f+1$] $\lambda_1=\cdots=\lambda_f$ and $\lambda_{f+1}=\cdots=\lambda_n$. ■

The two inequalities (2.36) and (2.37) may be combined as

$$(\operatorname{tr}\mathbf{A})^2/\operatorname{tr}\mathbf{A}^2\leqslant\max(f,g)\leqslant t\leqslant\operatorname{rank}\mathbf{A}; \tag{2.38}$$

cf. (2.35).

If $\mathbf{A}$ is Hermitian, the inequality (2.36) shows that $\mathbf{A}$ is positive definite when

$$n-1<(\operatorname{tr}\mathbf{A})^2/\operatorname{tr}\mathbf{A}^2 \quad\text{and}\quad \operatorname{tr}\mathbf{A}>0. \tag{2.39}$$

The inequality (2.37) gives a similar criterion for negative definiteness.

Another linear combination of the eigenvalues which is of interest is the difference $\lambda_k-\lambda_l$; when $(k,l)=(1,n)$, then this difference is called the *spread* of the matrix $\mathbf{A}$; we will write this as $\operatorname{sp}(\mathbf{A})$. In statistics the difference $\lambda_1-\lambda_n$ is known as the *range*, and so $\lambda_k-\lambda_l$ may be called a *mid-range*.

THEOREM 2.4. *Let* $\mathbf{A}$ *and* $s^2$ *be defined as in Theorem* 2.1. *Then*

$$\lambda_k-\lambda_l\leqslant sn^{1/2}\left(\frac{1}{k}+\frac{1}{n-l+1}\right)^{1/2},\qquad 1\leqslant k\leqslant l\leqslant n. \tag{2.40}$$

*Equality holds if and only if*

$$\begin{aligned}&\lambda_1=\lambda_2=\cdots=\lambda_k,\\ &\lambda_{k+1}=\lambda_{k+2}=\cdots=\lambda_{l-1}=\operatorname{tr}\mathbf{A}/n,\\ &\lambda_l=\lambda_{l+1}=\cdots=\lambda_n.\end{aligned} \tag{2.41}$$

*Proof.* To prove (2.40) we set $\mathbf{w}=k^{-1}\Sigma_{j=1}^{k}\mathbf{e}_j-(n-l+1)^{-1}\Sigma_{j=l}^{n}\mathbf{e}_j$ in (2.6), with $k<l$. It follows that $\mathbf{w}'\mathbf{e}=0$ and $\mathbf{w}'\mathbf{C}\mathbf{w}=\mathbf{w}'\mathbf{w}=k^{-1}+(n-l+1)^{-1}$. Hence

$$\lambda_k-\lambda_l\leqslant\lambda_{(1,k)}-\lambda_{(l,n)}\leqslant sn^{1/2}\left(\frac{1}{k}+\frac{1}{n-l+1}\right)^{1/2}, \tag{2.42}$$

which yields (2.40). Equality on the right of (2.42) holds if and only if

$$\lambda=\Big(\underbrace{\frac{a}{k}+b,\dots,\frac{a}{k}+b}_{k\text{ terms}},\ \underbrace{b,\dots,b}_{l-k-1\text{ terms}},\ \underbrace{b-\frac{a}{n-l+1},\dots,b-\frac{a}{n-l+1}}_{n-l+1\text{ terms}}\Big)'. \tag{2.43}$$

It follows at once that $m=\mathbf{e}'\lambda/n=b$ and $a=(\lambda_1-\lambda_n)/[k^{-1}+(n-l+1)^{-1}]$, and so equality holds in (2.40) if and only if (2.41). ■

When $(k,l)=(1,n)$ in Theorem 2.4, the inequality (2.40) becomes the upper bound for the spread $\operatorname{sp}(\mathbf{A})=\lambda_1-\lambda_n$ found by Mirsky in [11]. He used the second elementary symmetric function

$$\operatorname{tr}_2\mathbf{A}=\sum_{i<j}\lambda_i\lambda_j=\tfrac{1}{2}\left[(\operatorname{tr}\mathbf{A})^2-\operatorname{tr}\mathbf{A}^2\right], \tag{2.44}$$

so that (cf. [9, p. 167])

$$\operatorname{sp}(\mathbf{A})\leqslant\left\{2\left(1-\frac{1}{n}\right)(\operatorname{tr}\mathbf{A})^2-4\operatorname{tr}_2\mathbf{A}\right\}^{1/2}=(2n)^{1/2}s. \tag{2.45}$$

Corresponding lower bounds for $\operatorname{sp}(\mathbf{A})$ were obtained by Brauer and Mewborn in [2]. We assemble these results in

THEOREM 2.5. *Let* $\mathbf{A}$ *and* $s^2$ *be defined as in Theorem* 2.1. *Then*

$$\lambda_1-\lambda_n\leqslant(2n)^{1/2}s. \tag{2.46}$$

*When* $n>2$, *equality holds if and only if*

$$\lambda_2=\lambda_3=\cdots=\lambda_{n-1}=\tfrac{1}{2}(\lambda_1+\lambda_n). \tag{2.47}$$

*If* $n=2q$ *is even, then*

$$2s\leqslant\lambda_1-\lambda_n, \tag{2.48}$$

*with equality if and only if*

$$\lambda_1=\lambda_2=\cdots=\lambda_q\quad and\quad \lambda_{q+1}=\lambda_{q+2}=\cdots=\lambda_n. \tag{2.49}$$

*If* $n=2q\pm 1$ *is odd, then* (2.48) *holds, but moreover,*

$$2sn/(n^2-1)^{1/2}\leqslant\lambda_1-\lambda_n, \tag{2.50}$$

*with equality if and only if* (2.49) *holds.*

Note that $\lambda_1/\lambda_n=1+(\lambda_1-\lambda_n)/\lambda_n$. Using this fact and the above bounds for the spread, we obtain the following bounds for the condition number.

COROLLARY 2.3. *Let* $\mathbf{A}$ *be Hermitian positive definite, and let* $m$ *and* $s^2$ *be defined as in Theorem* 2.1.

(i) *When* $n$ *is even, then*

$$1+\frac{2s}{m-s/(n-1)^{1/2}}\leqslant\frac{\lambda_{\max}(\mathbf{A})}{\lambda_{\min}(\mathbf{A})}. \tag{2.51}$$

*When* $n>2$ *equality holds if and only if* $\mathbf{A}$ *is a scalar matrix.*

(ii) *When* $n$ *is odd, then* (2.51) *holds, but moreover,*

$$1+\frac{2sn/(n^2-1)^{1/2}}{m-s/(n-1)^{1/2}}\leqslant\frac{\lambda_{\max}(\mathbf{A})}{\lambda_{\min}(\mathbf{A})}. \tag{2.52}$$

*When* $n=3$ *equality holds in* (2.52) *if and only if the two smallest eigenvalues are equal. When* $n>3$ *equality holds if and only if* $\mathbf{A}$ *is a scalar matrix.*

COROLLARY 2.4. *Let* $\mathbf{A}$ *be Hermitian, and let* $m$ *and* $s^2$ *be defined as in Theorem* 2.1. *If* $\operatorname{tr}\mathbf{A}>0$ *and* $(\operatorname{tr}\mathbf{A})^2>(n-1)\operatorname{tr}\mathbf{A}^2$, *then* $\mathbf{A}$ *is positive definite,* (2.51) *holds, and*

$$\frac{\lambda_{\max}(\mathbf{A})}{\lambda_{\min}(\mathbf{A})}\leqslant 1+\frac{(2n)^{1/2}s}{m-s(n-1)^{1/2}}. \tag{2.53}$$

*When* $n>2$, *equality holds if and only if* $\mathbf{A}$ *is a scalar matrix.*

Unfortunately the upper bounds given in (2.14) and (2.53) hold only when $\operatorname{tr}\mathbf{A}>0$ and $(\operatorname{tr}\mathbf{A})^2>(n-1)\operatorname{tr}\mathbf{A}^2$. This is because we need a positive lower bound for $\lambda_n$. If, however, we knew that $\lambda_n\geqslant b>0$ for some $b$, then we could replace $m-s(n-1)^{1/2}$ by $b$ in Corollaries 2.1 and 2.4. For example, when $\mathbf{A}$ is positive definite, then

$$\lambda_n = \frac{\det \mathbf{A}}{\prod_{i=1}^{n-1}\lambda_i} \geqslant \frac{\det \mathbf{A}}{(\lambda_{(1,n-1)})^{n-1}} \geqslant \frac{\det \mathbf{A}}{\left[m+s/(n-1)^{1/2}\right]^{n-1}} > 0 \qquad (2.54)$$

by the arithmetic-geometric mean inequality and (2.19). Setting

$$b = \frac{\det \mathbf{A}}{\left[m+s/(n-1)^{1/2}\right]^{n-1}} \qquad (2.55)$$

yields

COROLLARY 2.5. *Suppose that* $\mathbf{A}$ *is Hermitian positive definite. Then*

$$\frac{\lambda_{\max}(\mathbf{A})}{\lambda_{\min}(\mathbf{A})} \leqslant 1 + \frac{(2n)^{1/2} s\left[m+s/(n-1)^{1/2}\right]^{n-1}}{\det \mathbf{A}}. \qquad (2.56)$$

*When* $n>2$, *equality holds if and only if* $\mathbf{A}$ *is a scalar matrix.*

We note that equality holds in (2.54) if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1}$; but to achieve equality in (2.56) we also need equality in (2.46), and so $\mathbf{A}$ must be a scalar matrix.

It is interesting to compare the inequalities for the condition number provided by Corollaries 2.1, 2.3, 2.4, and 2.5. When $n>2$ and $s>0$, it is clear that the bounds in Corollaries 2.3 and 2.4 are strictly better than those given in Corollary 2.1. These bounds, however, require that

$$(\operatorname{tr}\mathbf{A})^2 > (n-1)\operatorname{tr}\mathbf{A}^2, \qquad (2.57)$$

which is not necessary for the inequality in Corollary 2.5 to hold. We see that that inequality is better than the one in Corollary 2.4 if and only if

$$\det \mathbf{A} > \left[m - s(n-1)^{1/2}\right]\left[m+s/(n-1)^{1/2}\right]^{n-1}. \qquad (2.58)$$

Using Theorem 2.1, we see that this is implied by the product

$$\lambda_2 \cdots \lambda_{n-1} > \left[m+s/(n-1)^{1/2}\right]^{n-2}. \qquad (2.59)$$

Considering the three inequalities (2.14), (2.53), and (2.56), we note that when $n>2$ equality holds if and only if $\mathbf{A}$ is a scalar matrix, and then

$$\operatorname{tr}\mathbf{A}^2=(\operatorname{tr}\mathbf{A})^2/n. \tag{2.60}$$

We now present an inequality which can collapse for any pair of values of $\operatorname{tr}\mathbf{A}^2$ and $\operatorname{tr}\mathbf{A}$ which satisfy (2.57). The inequality (2.62) below is, therefore, always better than (2.14) and (2.53) when $n>2$ and $s>0$.

THEOREM 2.6. *Let* $\mathbf{A}$ *be an* $n\times n$ *Hermitian matrix. If* $\operatorname{tr}\mathbf{A}>0$ *and*

$$p=\frac{(\operatorname{tr}\mathbf{A})^2}{\operatorname{tr}\mathbf{A}^2}-(n-1)>0, \tag{2.61}$$

*then* $\mathbf{A}$ *is positive definite and*

$$\frac{\lambda_{\max}(\mathbf{A})}{\lambda_{\min}(\mathbf{A})}\leqslant\frac{1+(1-p^2)^{1/2}}{p}. \tag{2.62}$$

*When* $n>2$, *equality holds if and only if*

$$\lambda_2=\lambda_3=\cdots=\lambda_{n-1}=\frac{\lambda_1^2+\lambda_n^2}{\lambda_1+\lambda_n}, \tag{2.63}$$

*and then*

$$\frac{\operatorname{tr}\mathbf{A}^2}{\operatorname{tr}\mathbf{A}}=\frac{\lambda_1^2+\lambda_n^2}{\lambda_1+\lambda_n}. \tag{2.64}$$

To prove Theorem 2.6 we use:

LEMMA 2.3. *Let* $A$ *be an* $n\times n$ *nonnull matrix with real eigenvalues* $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_n$. *Then*

$$\frac{(\operatorname{tr}\mathbf{A})^2}{\operatorname{tr}\mathbf{A}^2}\leqslant n-2+\frac{(\lambda_1+\lambda_n)^2}{\lambda_1^2+\lambda_n^2}. \tag{2.65}$$

*When* $n>2$, *equality holds in* (2.65) *if and only if* $\lambda_1+\lambda_n\neq0$ *and*

$$\lambda_2=\cdots=\lambda_{n-1}=\frac{\lambda_1^2+\lambda_n^2}{\lambda_1+\lambda_n}. \tag{2.66}$$

*Proof.* If $\lambda_n = 0$, then (2.65) reduces to

$$(\operatorname{tr} A)^2 \leqslant (n-1)\operatorname{tr} A^2, \tag{2.67}$$

which can be shown to hold by using the Cauchy-Schwarz inequality on the $n-1$ real nonzero eigenvalues of A. Moreover, equality then holds in (2.67) $\Leftrightarrow \lambda_1 = \lambda_2 = \cdots = \lambda_{n-1}$, which is (2.66) when $\lambda_n = 0$. When $\lambda_n \neq 0$ we may write

$$\kappa = \lambda_1/\lambda_n; \qquad \mu_j = \lambda_j/\lambda_n, \quad j = 2,\dots,n-1. \tag{2.68}$$

Then (2.65) $\Leftrightarrow$

$$\gamma = \frac{(\kappa + \mu_2 + \cdots + \mu_{n-1} + 1)^2}{\kappa^2 + \mu_2^2 + \cdots + \mu_{n-1}^2 + 1} \leqslant n - 2 + \frac{(\kappa+1)^2}{\kappa^2+1}. \tag{2.69}$$

Write $\bar{\mu} = \sum_{j=2}^{n-1} \mu_j/(n-2)$. Then, using the Cauchy-Schwarz inequality again, we obtain

$$\mu_2^2 + \cdots + \mu_{n-1}^2 \geqslant (n-2)\bar{\mu}^2. \tag{2.70}$$

Applying (2.70) to the left-hand side of (2.69) yields

$$\gamma \leqslant \frac{[\kappa + 1 + (n-2)\bar{\mu}]^2 + (n-2)[\bar{\mu}(\kappa+1) - (\kappa^2+1)]^2/(\kappa^2+1)}{\kappa^2 + 1 + (n-2)\bar{\mu}^2}, \tag{2.71}$$

since the second term in the numerator of the right-hand side of (2.71) is nonnegative. Simplifying (2.71) yields (2.65). Equality holds in (2.65) if and only if equality holds in (2.70) and $\bar{\mu}(\kappa+1) = \kappa^2 + 1$. Equality holds in (2.70) if and only if $\lambda_2 = \cdots = \lambda_{n-1} = \bar{\lambda}$, say, while $\bar{\mu}(\kappa+1) = \kappa^2+1$ simplifies to $\bar{\lambda}(\lambda_1 + \lambda_n) = \lambda_1^2 + \lambda_n^2$. If $\lambda_1 + \lambda_n \neq 0$, this yields (2.66). If $\lambda_1 + \lambda_n = 0$, then (2.65) reduces to $(\operatorname{tr} A)^2 \leqslant (n-2)\operatorname{tr} A^2$, which collapses if and only if $\lambda_1^2 + \lambda_n^2 = 0$ or $A = 0$. ■

*Proof of Theorem 2.6.* The inequality (2.65) may be written as

$$p \leqslant \frac{2\kappa}{\kappa^2+1}, \tag{2.72}$$

where $p$ is defined in (2.61) and $\kappa$ in (2.68). Simplifying (2.72) yields the quadratic

$$p\kappa^2 - 2\kappa + p \leqslant 0 \tag{2.73}$$

and so (2.62) follows. Equality holds in (2.62) if and only if equality holds in (2.65) and (2.66) = (2.63). ■

To compare (2.62) with (2.53) we may write (2.62) as

$$\frac{\lambda_{\max}(\mathbf{A})}{\lambda_{\min}(\mathbf{A})} \leqslant 1+\frac{(1-p)^{1/2}\left[(1+p)^{1/2}+(1-p)^{1/2}\right]}{p}. \tag{2.74}$$

We may interpret the quantity $p$ defined by (2.61) as follows. Let $\theta$ denote the angle between the Hermitian matrix $\mathbf{A}$ and the identity matrix in the space of $n \times n$ Hermitian matrices with the inner product $\langle \mathbf{A}, \mathbf{B} \rangle = \operatorname{tr} \mathbf{AB}$. Then

$$\cos\theta = \operatorname{tr}\mathbf{A}/(n \operatorname{tr}\mathbf{A}^2)^{1/2}, \tag{2.75}$$

and so $\theta$ is also the angle between the vector of eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ and the equiangular line. Hence

$$p = 1 - n\sin^2\theta, \tag{2.76}$$

and so (2.61) holds, i.e., $p>0$, if and only if

$$\sin^2\theta < 1/n. \tag{2.77}$$

Moreover (2.76) shows immediately that $p \leqslant 1$ and hence (2.62) and (2.74) are well defined. The variance $s^2$ may also be interpreted using this angle $\theta$.

It is easy to see that

$$s^2 = \frac{(\operatorname{tr}\mathbf{A}^2)\sin^2\theta}{n} = \frac{(\operatorname{tr}\mathbf{A})^2\tan^2\theta}{n^2}, \tag{2.78}$$

and

$$\tan\theta = s/m. \tag{2.79}$$

It follows from (2.77) that $\operatorname{tr}\mathbf{A}>0$ and $(\operatorname{tr}\mathbf{A})^2 > (n-1)\operatorname{tr}\mathbf{A}^2$ if and only if

$$0 \leqslant \theta = \arccos\frac{\operatorname{tr}\mathbf{A}}{(n\operatorname{tr}\mathbf{A}^2)^{1/2}} < \arcsin n^{-1/2}. \tag{2.80}$$

TABLE 1

VALUES OF UPPER BOUNDS FOR $\theta$ AND ASSOCIATED PROBABILITIES

| $n$ | Upper bound (deg) (2.80) | Upper bound (deg) (2.81) | Ratio of upper bounds | Monte Carlo probability of (2.80) |
|---|---|---|---|---|
| 2 | 45 | 45 | 1 | 1 |
| 3 | 35.26 | 54.74 | .644 | .835 |
| 4 | 30 | 60 | .5 | .636 |
| 5 | 26.57 | 63.43 | .419 | .482 |
| 6 | 24.09 | 65.91 | .365 | .310 |
| 7 | 22.21 | 67.79 | .328 | .189 |
| 8 | 20.70 | 69.30 | .299 | .124 |
| 9 | 19.47 | 70.53 | .276 | .073 |
| 10 | 18.43 | 71.57 | .258 | .038 |

On the other hand, when $\mathbf{A}$ is positive definite (2.75) has the lower bound $n^{-1/2}$ and so

$$0 \leqslant \theta < \arcsin\left(1 - \frac{1}{n}\right)^{1/2}. \tag{2.81}$$

The range (2.81) exceeds the range (2.80) whenever $n \geqslant 3$; for $n = 2$, however, the two ranges coincide. If $\theta$ is distributed uniformly over the range (2.81), then the probability that (2.80) holds is the ratio of the two upper bounds. Values of these numbers are given in Table 1 for $n = 2(1)10$. Also given is the Monte Carlo probability that (2.80) holds when the eigenvalues of $\mathbf{A}$ are uniformly distributed on $(0,1)$, based on a run length of 2000.

The reason why the two sets of probabilities in Table 1 differ is that if $\theta$ lies between the two upper bounds, then $\mathbf{A}$ is not necessarily positive definite for $n \geqslant 3$. Only when $n = 2$, however, does the set of positive definite matrices form a cone, which is completely determined by the angle $\theta$.

## 3. COMPLEX EIGENVALUES

The inequlities in Sec. 2 may be extended to cover the situation where the eigenvalues are not necessarily all real. We use the matrices [cf. (1.4)]

$$\begin{aligned} \mathbf{B} &= \tfrac{1}{2}(\mathbf{A} + \mathbf{A}^*), \\ \mathbf{C} &= \tfrac{1}{2}(\mathbf{A} - \mathbf{A}^*)/i. \end{aligned} \tag{3.1}$$

We call $\mathbf{B}$ the *Hermitian real part* and $\mathbf{C}$ the *Hermitian imaginary part* of $\mathbf{A}$. Both $\mathbf{B}$ and $\mathbf{C}$ are Hermitian; cf. [1, p. 133]. Notice also that $\mathbf{A}=\mathbf{B}+i\mathbf{C}$.

We again write $\lambda(\mathbf{A})$ for an eigenvalue of $\mathbf{A}$, but order the eigenvalues now according to

$$|\lambda_1| \geqslant |\lambda_2| \geqslant \cdots \geqslant |\lambda_n|. \tag{3.2}$$

This does not necessarily reduce to (2.15) when the eigenvalues are all real. We also consider the real and imaginary parts of the eigenvalues:

$$\mu(\mathbf{A}) = \operatorname{Re}\lambda(\mathbf{A}), \qquad \nu(\mathbf{A}) = \operatorname{Im}\lambda(\mathbf{A}). \tag{3.3}$$

We order these:

$$\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_n, \qquad \nu_1 \geqslant \nu_2 \geqslant \cdots \geqslant \nu_n. \tag{3.4}$$

Notice that

$$\lambda_j = \mu_k + i\nu_l \tag{3.5}$$

for some values of $j$, $k$, and $l$, not necessarily all equal.

When all the eigenvalues are real, however, the $\mu_j$'s correspond directly to the $\lambda_j$'s of (2.15), but not, of course, necessarily to the $\lambda_j$'s in (3.2). Summing (3.5) or taking traces in (3.1) yields (cf. [1, p. 135])

$$\begin{aligned} \operatorname{tr}\mathbf{B} &= \operatorname{Re}\operatorname{tr}\mathbf{A} = \sum_{j=1}^{n} \mu_j, \\ \operatorname{tr}\mathbf{C} &= \operatorname{Im}\operatorname{tr}\mathbf{A} = \sum_{j=1}^{n} \nu_j. \end{aligned} \tag{3.6}$$

The eigenvalues of $\mathbf{B}$ are called the *real singular values* of $\mathbf{A}$ (cf. [1, p. 134]) and need not equal the $\mu$'s; similarly the $\nu$'s need not equal the eigenvalues of $\mathbf{C}$, which are called the *imaginary singular values* of $\mathbf{A}$. The eigenvalues of $\mathbf{B}$ and $\mathbf{C}$ are, however, all real.

Amir-Moéz and Fass [1, p. 135] have shown that

$$\begin{aligned} \lambda_{\min}(\mathbf{B}) &\leqslant \mu(\mathbf{A}) = \operatorname{Re}\lambda(\mathbf{A}) \leqslant \lambda_{\max}(\mathbf{B}), \\ \lambda_{\min}(\mathbf{C}) &\leqslant \nu(\mathbf{A}) = \operatorname{Im}\lambda(\mathbf{A}) \leqslant \lambda_{\max}(\mathbf{C}). \end{aligned} \tag{3.7}$$

We note that if either

$$\lambda_j(\mathbf{B}) = \operatorname{Re}\lambda_j(\mathbf{A}), \qquad j = 1,2,\ldots,n, \tag{3.8}$$

or

$$\lambda_j(\mathbf{C}) = \operatorname{Im}\lambda_j(\mathbf{A}), \qquad j = 1,2,\ldots,n, \tag{3.9}$$

then both sets of equalities hold, and this is possible if and only if $\mathbf{A}$ is normal; cf. (1.6), (1.7).

THEOREM 3.1. *Let* $\mathbf{A}$ *be an* $n \times n$ *complex matrix, and let*

$$m = \operatorname{tr}\mathbf{A}/n \quad \text{and} \quad s_a^2 = \operatorname{tr}\mathbf{A}^*\mathbf{A}/n - |m|^2. \tag{3.10}$$

*Then*

$$|m| - s_a(n-1)^{1/2} \leqslant |\lambda_n| \leqslant (\operatorname{tr}\mathbf{A}^*\mathbf{A}/n)^{1/2}, \tag{3.11}$$

$$|m| \leqslant |\lambda_1| \leqslant |m| + s_a(n-1)^{1/2}. \tag{3.12}$$

*Equality holds on the left of* (3.11) *if and only if* $\mathbf{A}$ *is normal,* $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1}$, *and* $\lambda_n = cm$ *for some real nonnegative scalar* $c \leqslant 1$. *Equality holds on the right of* (3.11) *if and only if* $\mathbf{A}$ *is normal and* $|\lambda_1| = |\lambda_2| = \cdots = |\lambda_n|$. *Equality holds on the left of* (3.12) *if and only if* $\lambda_1 = \lambda_2 = \cdots = \lambda_n$. *Equality holds on the right of* (3.12) *if and only if* $\mathbf{A}$ *is normal,* $\lambda_2 = \lambda_3 = \cdots = \lambda_n$, *and* $\lambda_1 = cm$ *for scalar* $c \geqslant 1$.

We note that when $\mathbf{A}$ is real, then the conditions for equality on the left of (3.11) and throughout (3.12) hold only if the eigenvalues of $\mathbf{A}$ are all real.

To prove Theorem 3.1 we use a complex analogue of Lemma 2.1. We now write

$$s = \frac{\boldsymbol{\lambda}^*\mathbf{C}\boldsymbol{\lambda}}{n} = \frac{1}{n}\sum_{j=1}^{n}|\lambda_j|^2 - \frac{1}{n^2}\left|\sum_{j=1}^{n}\lambda_j\right|^2. \tag{3.13}$$

Then (2.6) becomes, keeping $\mathbf{w}$ real,

$$|\mathbf{w}'\boldsymbol{\lambda} - m\mathbf{w}'\mathbf{e}| = |\mathbf{w}'\mathbf{C}\boldsymbol{\lambda}| \leqslant s(n\mathbf{w}'\mathbf{C}\mathbf{w})^{1/2}, \tag{3.14}$$

with equality if and only if (2.7) holds for some complex scalars $a$ and $b$. We then use Schur's inequality (1.5), so that

$$s^2 \leqslant \operatorname{tr} \mathbf{A}^*\mathbf{A}/n - |\operatorname{tr} \mathbf{A}|^2/n^2 = s_a^2, \tag{3.15}$$

with equality if and only if $\mathbf{A}$ is normal.

*Proof of Theorem 3.1.* Put $\mathbf{w} = \mathbf{e}_n$ in (3.14) to obtain

$$|\lambda_n - m| \leqslant s(n-1)^{1/2}, \tag{3.16}$$

with equality if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1}$. The left of (3.11) then follows, since

$$\begin{aligned} |\lambda_n| &\geqslant |m| - |\lambda_n - m| \\ &\geqslant |m| - s(n-1)^{1/2} \\ &\geqslant |m| - s_a(n-1)^{1/2}. \end{aligned} \tag{3.17}$$

Equality holds in the first inequality in (3.17) if and only if $m - \lambda_n = k\lambda_n$ for some real scalar $k \geqslant 0$. Hence set $c = 1/(k+1)$. The right-hand side of (3.12) follows similarly. To prove the right-hand inequality in (3.11) we note that

$$|\lambda_n| \leqslant \Sigma|\lambda_j|/n \leqslant \left(\Sigma|\lambda_j|^2/n\right)^{1/2} \leqslant (\operatorname{tr}\mathbf{A}^*\mathbf{A}/n)^{1/2}, \tag{3.18}$$

using the Cauchy-Schwarz and Schur inequalities. The equality condition follows at once. The left-hand inequality in (3.12) follows from

$$|m| = |\Sigma\lambda_j/n| \leqslant \Sigma|\lambda_j|/n \leqslant |\lambda_1|, \tag{3.19}$$

with equality if and only if

$$\lambda_j = c_j\lambda, \qquad c_j \geqslant 0, \tag{3.20}$$

$$|\lambda_1| = |\lambda_2| = \cdots = |\lambda_n|. \tag{3.21}$$

These two conditions reduce to $\lambda_1 = \lambda_2 = \cdots = \lambda_n$. ■

We now extend Theorem 3.1 to linear combinations of the absolute values of the eigenvalues.

THEOREM 3.2. *Let* $\mathbf{A}$, $m$, *and* $s_a^2$ *be as in Theorem* 3.1. *Let*

$$|\lambda|_{(k,l)} = \frac{1}{l-k+1} \sum_{j=k}^{l} |\lambda_j|. \tag{3.22}$$

*Then*

$$|m| - s_a\left(\frac{k-1}{n-k+1}\right)^{1/2} \leqslant |\lambda|_{(k,l)} \leqslant \left(\frac{\operatorname{tr}\mathbf{A}^*\mathbf{A}}{n}\right)^{1/2} + s_a\left(\frac{n-l}{l}\right)^{1/2}. \tag{3.23}$$

*Equality holds on the left if and only if* $\mathbf{A}$ *is normal,*

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{k-1}, \quad \text{and} \quad \lambda_k = \lambda_{k+1} = \cdots = \lambda_n = c\lambda_1, \tag{3.24}$$

*with $c$ real and nonnegative. Equality holds on the right if and only if* $\mathbf{A}$ *is a scalar matrix.*

*Furthermore*

$$|m| - s_a\left(\frac{k-1}{n-k+1}\right)^{1/2} \leqslant |\lambda_k| = |\lambda|_{(k,k)} \leqslant \left(\frac{\operatorname{tr}\mathbf{A}^*\mathbf{A}}{n}\right)^{1/2} + s_a\left(\frac{n-k}{k}\right)^{1/2}. \tag{3.25}$$

*Equality holds on the left if and only if* $\mathbf{A}$ *is normal and* (3.24) *holds. Equality holds on the right if and only if* $\mathbf{A}$ *is a scalar matrix.*

*Proof.* Let $\boldsymbol{\lambda}^{(a)} = \{|\lambda_j|\}$, $m_a = \Sigma|\lambda_j|/n$, and $s_d^2 = \Sigma|\lambda_j|^2/n - (\Sigma|\lambda_j|)^2/n^2$. Then applying (2.19) yields

$$m_a - s_d\left(\frac{k-1}{n-k+1}\right)^{1/2} \leqslant |\lambda|_{(k,l)} \leqslant m_a + s_d\left(\frac{n-l}{l}\right)^{1/2}. \tag{3.26}$$

Now

$$m_a = \Sigma|\lambda_j|/n \geqslant |\Sigma\lambda_j/n| = |\operatorname{tr}\mathbf{A}/n| = |m|, \tag{3.27}$$

while

$$\begin{aligned} s_d^2 &\leqslant \operatorname{tr}\mathbf{A}^*\mathbf{A}/n - |\Sigma\lambda_j|^2/n^2 \\ &= \operatorname{tr}\mathbf{A}^*\mathbf{A}/n - |m|^2 = s_a^2. \end{aligned} \tag{3.28}$$

This proves the left-hand side of (3.23). To prove the right-hand side we use (3.28) and

$$m_a = \Sigma|\lambda_j|/n \leqslant \left(\Sigma|\lambda_j|^2/n\right)^{1/2} \leqslant (\operatorname{tr}\mathbf{A}^*\mathbf{A}/n)^{1/2}, \tag{3.29}$$

where the first inequality is Cauchy-Schwarz. Equality holds on the left of (3.23) if and only if equality holds in (3.27), in (3.28), and on the left of (3.26). This means that, respectively, (3.20) holds,

$$\mathbf{A} \text{ is normal,} \tag{3.30}$$

and

$$|\lambda_1| = |\lambda_2| = \cdots = |\lambda_{k-1}| \quad \text{and} \quad |\lambda_k| = |\lambda_{k+1}| = \cdots = |\lambda_n|, \tag{3.31}$$

using (2.20). Substituting (3.20) into (3.31) yields (3.24).

Equality holds on the right of (3.23) if and only if equality holds in (3.28), in (3.29), and on the right of (3.26). This means that (3.20) and (3.30) hold,

$$|\lambda_j| = c, \tag{3.32}$$

and

$$|\lambda_1| = |\lambda_2| = \cdots = |\lambda_l| \quad \text{and} \quad |\lambda_{l+1}| = |\lambda_{l+2}| = \cdots = |\lambda_n| \tag{3.33}$$

[cf. (2.21)] must hold. This can be only if $\mathbf{A}$ is a scalar matrix. ■

We notice that putting $k=1$ on the left of (3.23) and $l=n$ on the right yields

$$|m| \leqslant |\lambda|_{(1,l)}, \qquad l = 1,2,\ldots,n, \tag{3.34}$$

$$|\lambda|_{(k,n)} \leqslant (\operatorname{tr}\mathbf{A}^*\mathbf{A}/n)^{1/2}, \qquad k = 1,2,\ldots,n. \tag{3.35}$$

Equality holds in (3.34) if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_n$, and in (3.35) if and only if $\mathbf{A}$ is normal and $|\lambda_1| = |\lambda_2| = \cdots = |\lambda_n|$.

We presented "better" bounds for the real case in Theorem 2.3. In [19] we strengthen (3.34) and (3.35) using Theorem 2.3 and the trace of $(\mathbf{AA}^* - \mathbf{A}^*\mathbf{A})^2$.

Unfortunately the lower bound in (3.25) for $|\lambda_k|$ may be negative. This bound will be positive if and only if

$$|m|^2 > \left(\frac{k-1}{n-k+1}\right)s_a^2 = \left(\frac{k-1}{n-k+1}\right)\left[\frac{\operatorname{tr}\mathbf{A}^*\mathbf{A}}{n} - |m|^2\right], \tag{3.36}$$

which reduces to

$$|\operatorname{tr}\mathbf{A}|^2 > (k-1)\operatorname{tr}\mathbf{A}^*\mathbf{A}. \tag{3.37}$$

We therefore get the following

COROLLARY 3.1. *Let* $\mathbf{A}$ *be a nonnull* $n\times n$ *complex matrix with exactly* $k$ *nonzero eigenvalues. Then*

$$|\operatorname{tr}\mathbf{A}|^2/\operatorname{tr}\mathbf{A}^*\mathbf{A} \leqslant k \leqslant \operatorname{rank}\mathbf{A}. \tag{3.38}$$

*Equality holds on the left if and only if* $\mathbf{A}$ *is normal and* $|\lambda_1| = |\lambda_2| = \cdots = |\lambda_k|$. *Equality holds on the right if and only if* rank $\mathbf{A} = \operatorname{rank}\mathbf{A}^2$, *i.e.*, $\mathbf{A}$ *has "index"* 1.

We will now find analogous results for the real and imaginary parts of the eigenvalues. To ease the notation, let

$$\lambda_j^{(a)} = |\lambda_j|, \quad \lambda_j^{(b)} = \mu_j, \text{ and } \lambda_j^{(c)} = \nu_j \qquad \text{for} \quad j = 1, 2, \ldots, n, \tag{3.39}$$

where $\mu_j$ and $\nu_j$ are as in (3.3) and (3.4).

THEOREM 3.3. *Let* $\mathbf{A}$ *be an* $n\times n$ *complex matrix. Let* $\mathbf{B}$ *and* $\mathbf{C}$ *be as in* (3.1) *and*

$$m_b = \operatorname{Re}\operatorname{tr}\mathbf{A}/n = \operatorname{tr}\mathbf{B}/n, \qquad m_c = \operatorname{Im}\operatorname{tr}\mathbf{A}/n = \operatorname{tr}\mathbf{C}/n,$$

$$s_b^2 = \operatorname{tr}\mathbf{B}^2/n - m_b^2, \qquad s_c^2 = \operatorname{tr}\mathbf{C}^2/n - m_c^2. \tag{3.40}$$

*Then, for* $t = b$ *or* $c$,

$$m_t - s_t(n-1)^{1/2} \leqslant \lambda^{(t)}(\mathbf{A}) \leqslant m_t + s_t(n-1)^{1/2}. \tag{3.41}$$

*Moreover, for* $t=b,c$

$$m_t - s_t(n-1)^{1/2} = \lambda_n^{(t)}(\mathbf{A}) \tag{3.42}$$

*if and only if* $\mathbf{A}$ *is normal and*

$$\lambda_1^{(t)}(\mathbf{A}) = \lambda_2^{(t)}(\mathbf{A}) = \cdots = \lambda_{n-1}^{(t)}(\mathbf{A}), \tag{3.43}$$

*while*

$$\lambda_1^{(t)}(\mathbf{A}) = m_t + s_t(n-1)^{1/2} \tag{3.44}$$

*if and only if* $\mathbf{A}$ *is normal and*

$$\lambda_2^{(t)}(\mathbf{A}) = \lambda_3^{(t)}(\mathbf{A}) = \cdots = \lambda_n^{(t)}(\mathbf{A}). \tag{3.45}$$

*Furthermore, if* $\mathbf{A}$ *is normal, then*

$$\lambda_{\min}^{(t)}(\mathbf{A}) \leqslant m_t - s_t/(n-1)^{1/2}, \tag{3.46}$$

$$m_t + s_t/(n-1)^{1/2} \leqslant \lambda_{\max}^{(t)}(\mathbf{A}). \tag{3.47}$$

*Equality holds in* (3.46) (*in* (3.47)) *if and only if the* $n-1$ *smallest* (*largest*) $\lambda_j^{(t)}$*'s are equal.*

The proofs of Theorem 3.3 and the subsequent results in this section follow our proofs for real eigenvalues, with $\boldsymbol{\lambda}^{(t)}$, $t=a,b,c$, replacing the vector $\boldsymbol{\lambda}$ of real eigenvalues. The variance of the $\lambda_j^{(t)}$ cannot, however, in general be computed in terms of traces; to obtain our inequalities we use

$$\frac{1}{n}\Sigma\left(\lambda_j^{(t)}\right)^2 - \left(\frac{1}{n}\Sigma\lambda_j^{(t)}\right)^2 \leqslant s_t^2, \qquad t=a,b,c. \tag{3.48}$$

For $t=b,c$ equality in (3.48) holds if and only if $\mathbf{A}$ is normal [cf. (1.6), (1.7)], while for $t=a$ equality holds if and only if $\mathbf{A}$ is normal and $\lambda_j = c_j\lambda_1$ for some nonnegative scalars $c_j$, $j=2,\ldots,n$ [cf. (1.5), (3.20)].

As remarked earlier, a matrix is stable if and only if the real parts of its eigenvalues are all less than zero. By the above Theorem 3.3, this will occur if $m_b<0$ and $s_b^2(n-1)<m_b^2$. This reduces to

$$\operatorname{Re}\operatorname{tr}\mathbf{A}/n = \operatorname{tr}\mathbf{B}/n < 0 \quad \text{and} \quad (n-1)\operatorname{tr}\mathbf{B}^2 < (\operatorname{tr}\mathbf{B})^2.$$

We summarize this in

COROLLARY 3.2. *Let* $\mathbf{A}=(a_{kl})$ *be an* $n\times n$ *complex matrix. Then* $\mathbf{A}$ *is stable if* $\operatorname{Re}\operatorname{tr}\mathbf{A}<0$ *and*

$$(n-1)\Sigma|\tfrac{1}{2}(a_{kl}+\bar{a}_{lk})|^2<(\operatorname{Re}\operatorname{tr}\mathbf{A})^2. \tag{3.49}$$

Let us now define the real and imaginary mid-means [cf. (3.39)]

$$\lambda_{(k,l)}^{(b)}=\frac{1}{l-k+1}\sum_{j=k}^{l}\lambda_j^{(b)}=\frac{1}{l-k+1}\sum_{j=k}^{l}\mu_j,$$

$$\lambda_{(k,l)}^{(c)}=\frac{1}{l-k+1}\sum_{j=k}^{l}\lambda_j^{(c)}=\frac{1}{l-k+1}\sum_{j=k}^{l}\nu_j. \tag{3.50}$$

We then get

THEOREM 3.4. *Let* $\mathbf{A}$, $m_t$, *and* $s_t$ *for* $t=b,c$ *be as in Theorem* 3.3. *Then for* $t=b,c$,

$$m_t-s_t\left(\frac{k-1}{n-k+1}\right)^{1/2}\leqslant\lambda_{(k,l)}^{(t)}\leqslant m_t+s_t\left(\frac{n-l}{l}\right)^{1/2}. \tag{3.51}$$

*When* $(k,l)=(1,n)$ *the inequality string collapses. When* $(k,l)\neq(1,n)$, *then equality holds on the left if and only if* $\mathbf{A}$ *is normal and*

$$\lambda_1^{(t)}=\lambda_2^{(t)}=\cdots=\lambda_{k-1}^{(t)}\quad\text{and}\quad\lambda_k^{(t)}=\lambda_{k+1}^{(t)}=\cdots=\lambda_n^{(t)}. \tag{3.52}$$

*Equality holds on the right if and only if* $\mathbf{A}$ *is normal and*

$$\lambda_1^{(t)}=\lambda_2^{(t)}=\cdots=\lambda_l^{(t)}\quad\text{and}\quad\lambda_{l+1}^{(t)}=\lambda_{l+2}^{(t)}=\cdots=\lambda_n^{(t)}. \tag{3.53}$$

*Furthermore, for* $t=b,c$,

$$m_t-s_t\left(\frac{k-1}{n-k+1}\right)^{1/2}\leqslant\lambda_k^{(t)}\leqslant m_t+s_t\left(\frac{n}{k}-1\right)^{1/2}. \tag{3.54}$$

*Equality holds on the left if and only if* $\mathbf{A}$ *is normal and* (3.52) *holds. Equality holds on the right if and only if* $\mathbf{A}$ *is normal,*

$$\lambda_1^{(t)}=\lambda_2^{(t)}=\cdots=\lambda_k^{(t)}\quad\text{and}\quad\lambda_{k+1}^{(t)}=\lambda_{k+2}^{(t)}=\cdots=\lambda_n^{(t)}. \tag{3.55}$$

THEOREM 3.5. *Let* $\mathbf{A}$, $m_t$, *and* $s_t$ *for* $t=b,c$ *be as in Theorem* 3.3. *If* $\mathbf{A}$ *is normal, then for* $t=b,c$,

$$\lambda_{(1,l)}^{(t)} \geqslant \begin{cases} m_t + \dfrac{s_t}{(n-1)^{1/2}} & \text{if} \quad l \leqslant \frac{1}{2}n, \\ m_t + \dfrac{s_t(n-l)}{l(n-1)^{1/2}} & \text{if} \quad l \geqslant \frac{1}{2}n, \end{cases} \tag{3.56}$$

$$\lambda_{(k,n)}^{(t)} \leqslant \begin{cases} m_t - \dfrac{s_t(k-1)}{(n-k+1)(n-1)^{1/2}} & \text{if} \quad k \leqslant \frac{1}{2}n+1, \\ m_t - \dfrac{s_t}{(n-1)^{1/2}} & \text{if} \quad k \geqslant \frac{1}{2}n+1. \end{cases} \tag{3.57}$$

*Equality holds in* (3.56) *if and only if*

$$\begin{array}{ll} \lambda_1^{(t)} = \lambda_2^{(t)} = \cdots = \lambda_{n-1}^{(t)} & \textit{when} \quad l < \frac{1}{2}n, \\ \lambda_1^{(t)} = \lambda_2^{(t)} = \cdots = \lambda_{n-1}^{(t)} \ \textit{or} \ \lambda_2^{(t)} = \lambda_3^{(t)} = \cdots = \lambda_n^{(t)} & \textit{when} \quad l = \frac{1}{2}n, \\ \lambda_2^{(t)} = \lambda_3^{(t)} = \cdots = \lambda_n^{(t)} & \textit{when} \quad l > \frac{1}{2}n. \end{array} \tag{3.58}$$

*Equality holds in* (3.57) *if and only if*

$$\begin{array}{ll} \lambda_1^{(t)} = \lambda_2^{(t)} = \cdots = \lambda_{n-1}^{(t)} & \textit{when} \quad k < \frac{1}{2}n+1, \\ \lambda_1^{(t)} = \lambda_2^{(t)} = \cdots = \lambda_{n-1}^{(t)} \ \textit{or} \ \lambda_2^{(t)} = \lambda_3^{(t)} = \cdots = \lambda_n^{(t)} & \textit{when} \quad k = \frac{1}{2}n+1, \\ \lambda_2^{(t)} = \lambda_3^{(t)} = \cdots = \lambda_n^{(t)} & \textit{when} \quad k > \frac{1}{2}n+1. \end{array} \tag{3.59}$$

We note the similarity between Theorems 3.5 and 2.3. The difference is the need for normality. This is due to the fact that equality occurs in (3.48) for $t=b,c$ if and only if $\mathbf{A}$ is normal.

As mentioned in the introduction, a matrix is stable on the eigenspace corresponding to those eigenvalues with negative real parts. Furthermore, the solutions to the ordinary differential equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ will be spirals along the eigenvectors in the phase plane corresponding to those eigenvalues with nonzero imaginary parts.

We conclude this section with the following inequalities on differences (cf. Theorem 2.4).

THEOREM 3.6. *Let* $\mathbf{A}$ *and* $s_t$ *be as in Theorem* 3.3. *If* $1 \leqslant k < l \leqslant n$, *then for* $t = a$, $b$, *or* $c$,

$$\lambda_k^{(t)} - \lambda_l^{(t)} \leqslant s_t n^{1/2} \left( \frac{1}{k} + \frac{1}{n-l+1} \right)^{1/2}. \tag{3.60}$$

*Equality holds if and only if* $\mathbf{A}$ *is normal and*

$$\begin{aligned} \lambda_1^{(t)} &= \lambda_2^{(t)} = \cdots = \lambda_k^{(t)} \\ \lambda_{k+1}^{(t)} &= \lambda_{k+2}^{(t)} = \cdots = \lambda_{l-1}^{(t)} = m_t \\ \lambda_l^{(t)} &= \lambda_{l+1}^{(t)} = \cdots = \lambda_n^{(t)}, \end{aligned} \tag{3.61}$$

*where* $m_a = \Sigma|\lambda_j|/n$, *and for* $t = a$,

$$\lambda_j = c_j \lambda_1, \qquad j = 2, \ldots, n, \tag{3.62}$$

*for some nonnegative scalars* $c_j$, $j = 2, \ldots, n$.

*Furthermore, if* $\mathbf{A}$ *is normal and* $n = 2q$ *is even, then for* $t = b$ *or* $c$,

$$2s_t \leqslant \lambda_1^{(t)} - \lambda_n^{(t)} \tag{3.63}$$

*with equality if and only if*

$$\lambda_2^{(t)} = \lambda_3^{(t)} = \cdots = \lambda_q^{(t)} \quad \text{and} \quad \lambda_{q+1}^{(t)} = \lambda_{q+2}^{(t)} = \cdots = \lambda_n^{(t)}. \tag{3.64}$$

*If, however,* $n = 2q \pm 1$ *is odd, then* (3.63) *holds, but moreover,*

$$2s_t n/(n^2-1)^{1/2} \leqslant \lambda_1^{(t)} - \lambda_n^{(t)} \tag{3.65}$$

*with equality if and only if* (3.64) *holds.*

We note that if $\mathbf{A}$ is nonsingular and normal, then

$$\frac{\max|\lambda(\mathbf{A})|}{\min|\lambda(\mathbf{A})|} \tag{3.66}$$

may be used as a condition number of **A**. We may now follow the approach in Sec. 2 and obtain upper and lower bounds for (3.66). We use Theorems 3.1 and 3.6 to see that

$$1+\frac{2s_a}{(\operatorname{tr}\mathbf{A}^*\mathbf{A}/n)^{1/2}} \leqslant \frac{\max|\lambda(\mathbf{A})|}{\min|\lambda(\mathbf{A})|}. \tag{3.67}$$

Furthermore, if

$$|\operatorname{tr}\mathbf{A}|^2 > (n-1)\operatorname{tr}\mathbf{A}^*\mathbf{A}, \tag{3.68}$$

then

$$\frac{\max|\lambda(\mathbf{A})|}{\min|\lambda(\mathbf{A})|} \leqslant 1+\frac{(2n)^{1/2}s_a}{|m|-s_a(n-1)^{1/2}}. \tag{3.69}$$

## 4. EXAMPLES

To illustrate our bounds for eigenvalues we present five numerical examples.

Example 1. In his recent paper Scheffold [14] obtained bounds for the subdominant eigenvalues of a matrix with nonnegative elements. To illustrate his findings, he considered the matrix

$$\begin{bmatrix} 6 & 0 & 0 \\ 1 & 3 & 1 \\ 2 & 4 & 0 \end{bmatrix}, \tag{4.1}$$

and found

$$|\lambda_2|, |\lambda_3| \leqslant 5. \tag{4.2}$$

Our bounds (3.25), however, yield

$$\begin{aligned} 3 \leqslant |\lambda_1| &\leqslant 9.89, \\ 0.89 \leqslant |\lambda_2| &\leqslant 7.31, \\ 0 \leqslant |\lambda_3| &\leqslant 4.73. \end{aligned} \tag{4.3}$$

We observe, moreover, that the eigenvalues of (4.1) are $\lambda_1 = 6$ and the two eigenvalues of $\begin{pmatrix} 3 & 1 \\ 4 & 0 \end{pmatrix}$, and so all the eigenvalues must be real. Applying (2.31) to (4.1) then yields

$$-1.16 \leqslant \lambda_3 \leqslant 0.92 \leqslant \lambda_2 \leqslant 5.08 \leqslant \lambda_1 \leqslant 7.16. \tag{4.4}$$

It is easily seen that the subdominant eigenvalues of (4.1) are $\lambda_2 = 4$ and $\lambda_3 = -1$.

EXAMPLE 2. In their book [9, p. 158] Marcus and Minc compared various bounds for the dominant eigenvalue of a matrix with positive elements:

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 5 \end{bmatrix}. \tag{4.5}$$

Their best bounds are

$$5.162 \leqslant \lambda_1 \leqslant 9.359. \tag{4.6}$$

Our bounds (3.25) yield

$$\begin{aligned} 2.33 \leqslant |\lambda_1| &\leqslant 9.67, \\ 0 \leqslant |\lambda_2| &\leqslant 7.04, \\ 0 \leqslant |\lambda_3| &\leqslant 4.40. \end{aligned} \tag{4.7}$$

The matrix in (4.5) is, however, singular; hence all its eigenvalues are real. Applying (2.31) yields

$$-2.87 \leqslant \lambda_3 \leqslant -0.27 \leqslant \lambda_2 \leqslant 4.93 \leqslant \lambda_1 \leqslant 7.54. \tag{4.8}$$

The nonzero eigenvalues are $3\frac{1}{2} \pm \frac{1}{2}\sqrt{65}$, or approximately 7.531 and $-0.531$.

EXAMPLE 3. Marcus and Minc [9, p. 148] also consider the complex matrix

$$\mathbf{A} = \begin{bmatrix} 7+3i & -4-6i & -4 \\ -1-6i & 7 & -2-6i \\ 2 & 4-6i & 13-3i \end{bmatrix}. \tag{4.9}$$

Using results due to Hirsch [5], they obtain

$$\begin{aligned} |\lambda(\mathbf{A})| &\leqslant 40.03, \\ |\operatorname{Re}\lambda(\mathbf{A})| &\leqslant 39, \\ |\operatorname{Im}\lambda(\mathbf{A})| &\leqslant 20.12, \end{aligned} \tag{4.10}$$

while Geršgorin's discs are

$$\begin{aligned} |z-7-3i| &\leqslant 11.21, \\ |z-7| &\leqslant 12.40, \\ |z-13+3i| &\leqslant 9.21. \end{aligned} \tag{4.11}$$

Applying (3.25) yields

$$\begin{aligned} 9 \leqslant \lambda_1^{(a)} &\leqslant 25.46, \\ 2.64 \leqslant \lambda_2^{(a)} &\leqslant 19.09, \\ 0 \leqslant \lambda_3^{(a)} &\leqslant 12.73, \end{aligned} \tag{4.12}$$

while (3.54) yields

$$\begin{aligned} 9 \leqslant \lambda_1^{(b)} &\leqslant 14.20, \\ 6.40 \leqslant \lambda_2^{(b)} &\leqslant 11.60, \\ 3.81 \leqslant \lambda_3^{(b)} &\leqslant 9 \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} 0 \leqslant \lambda_1^{(c)} &\leqslant 11.62, \\ -5.81 \leqslant \lambda_2^{(c)} &\leqslant 5.81, \\ -11.62 \leqslant \lambda_3^{(c)} &\leqslant 0. \end{aligned} \tag{4.14}$$

Recall that $\lambda_j^{(t)}$, $t=a,b,c$, and $j=1,2,3$ are the ordered modulus, real part, and imaginary part, respectively.

The bounds (4.13) and (4.14) define the rectangle [cf. (3.41)]

$$\begin{aligned} 3.81 \leqslant \operatorname{Re}\lambda(\mathbf{A}) &\leqslant 14.20, \\ -11.62 \leqslant \operatorname{Im}\lambda(\mathbf{A}) &\leqslant 11.62, \end{aligned} \tag{4.15}$$

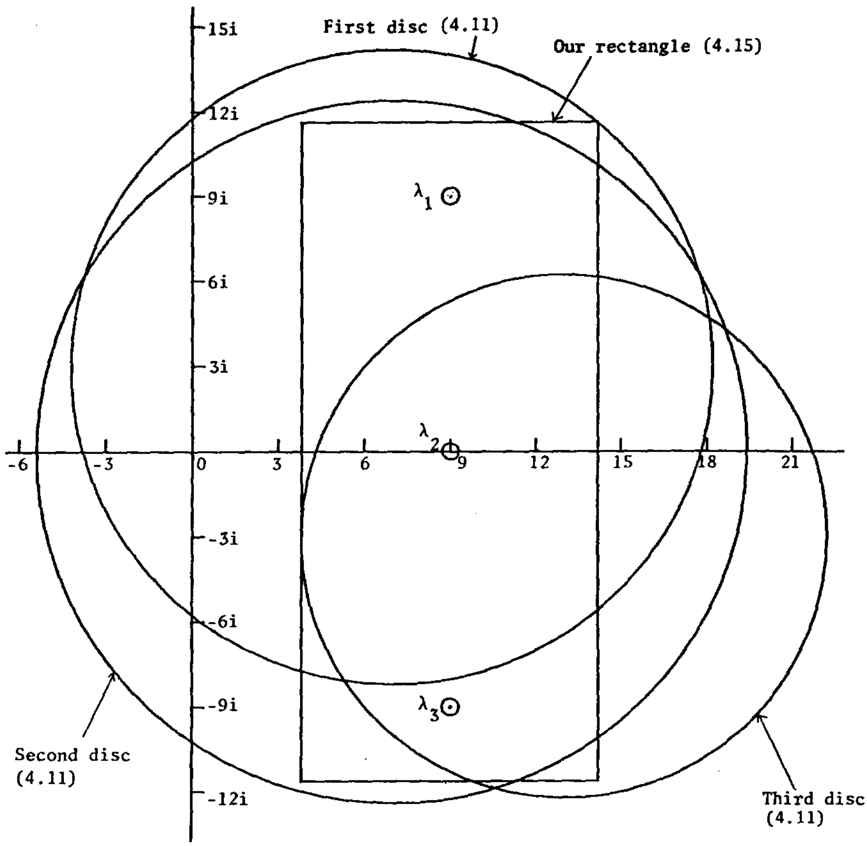


FIG. 2. Gerŝgorin discs (4.11) and our bounds (4.15) for Example 3.

which sits almost entirely within the union of the three Gerŝgorin discs as given by (4.11); cf. Fig. 2. the eigenvalues are 9, $9+9i$, $9-9i$, with $\lambda_1^{(a)}=12.73$.

EXAMPLE 4. To illustrate our bound (2.62) for the condition number we consider the symmetric matrix

$$\begin{bmatrix} 4 & 0 & 2 & 3 \\ 0 & 5 & 0 & 1 \\ 2 & 0 & 6 & 0 \\ 3 & 1 & 0 & 7 \end{bmatrix}. \tag{4.16}$$

Our bounds (2.32) yield

$$\begin{aligned} 7.158 &\leqslant \lambda_1 \leqslant 10.475, \\ 3.842 &\leqslant \lambda_2 \leqslant 8.372, \\ 2.628 &\leqslant \lambda_3 \leqslant 7.158, \\ 0.525 &\leqslant \lambda_4 \leqslant 3.842. \end{aligned} \tag{4.17}$$

We find that [cf. (2.61)]

$$p = 0.1429 \tag{4.18}$$

and so $\theta = 27.57° < 30°$; cf. Table 1. Thus (4.16) is positive definite. Moreover [cf. (2.62)]

$$\kappa \leqslant 13.928. \tag{4.19}$$

Frobenius' theorem [9, p. 152] indicates that $\lambda_1$ must lie between the smallest and largest row sums, i.e.,

$$6 \leqslant \lambda_1 \leqslant 11, \tag{4.20}$$

while from the separation theorem [13, p. 64], using the top left $2\times 2$ and bottom right $2\times 2$ submatrices of (4.16), we obtain

$$\begin{aligned} 7 &\leqslant \lambda_1, \\ 6 &\leqslant \lambda_2, \\ &\lambda_3 \leqslant 5, \\ &\lambda_4 \leqslant 4. \end{aligned} \tag{4.21}$$

The eigenvalues are 9.376, 6.423, 4.775, and 1.426, and so $\kappa = 6.575$.

EXAMPLE 5. Our last example is the symmetric matrix

$$\begin{bmatrix} 4 & 1 & 1 & 2 & 2 \\ 1 & 5 & 1 & 1 & 1 \\ 1 & 1 & 6 & 1 & 1 \\ 2 & 1 & 1 & 7 & 1 \\ 2 & 1 & 1 & 1 & 8 \end{bmatrix}. \tag{4.22}$$

Our bounds (2.33) yield

$$\begin{aligned} 7.449 &\leqslant \lambda_1 \leqslant 11.797, \\ 4.551 &\leqslant \lambda_2 \leqslant 9.550, \\ 3.634 &\leqslant \lambda_3 \leqslant 8.366, \\ 2.450 &\leqslant \lambda_4 \leqslant 7.449, \\ 0.203 &\leqslant \lambda_5 \leqslant 4.551. \end{aligned} \tag{4.23}$$

Moreover [cf. (2.61)]

$$p = 0.0541. \tag{4.24}$$

Furthermore $\theta = 25.78° < 26.57°$; cf. Table 1. Thus (4.22) is positive definite. And so [cf. (2.62)]

$$\kappa \leqslant 36.973. \tag{4.25}$$

Frobenius' theorem [cf. (4.20)] here gives

$$9 \leqslant \lambda_1 \leqslant 13. \tag{4.26}$$

The eigenvalues of (4.22) are 11.171, 6.527, 5.434, 4.296, and 2.571, and so $\kappa = 4.345$.

*The authors wish to thank Patricia J. Babecki for suggesting Examples 4 and 5, and Paul J. Brockman for drawing Fig. 2.*

## REFERENCES

1 A. R. Amir-Moéz and A. L. Fass, *Elements of Linear Spaces*, Pergamon, Oxford, 1962.

2 A. Brauer and A. C. Mewborn, The greatest distance between two characteristic roots of a matrix, *Duke Math. J.* 26:653–661 (1959).

3 M. P. Drazin and E. V. Haynsworth, Criteria for the reality of matrix eigenvalues, *Math. Z.* 78:449–452 (1962).

4 F. A. Graybill, *Introduction to Matrices with Applications in Statistics*, Wadsworth, Belmont, Calif., 1969.

5 A. Hirsch, Sur les racines d'une équation fondamentale, *Acta Math.* 25:367–370 (1902).

6 M. W. Hirsch and S. Smale, *Differential Equations, Dynamical Systems, and Linear Algebra*, Academic, New York, 1974.

7 A. S. Householder, Localization of the characteristic roots of matrices, in *Recent Advances in Matrix Theory* (H. Schneider, Ed.), Univ. of Wis. Press, Madison, 1964, pp. 39–60.

8 C. L. Mallows and Donald Richter, Inequalities of Chebyshev type involving conditional expectations, *Ann. Math. Statist.* 40:1922–1932 (1969).

9 M. Marcus and H. Minc, *A Survey of Matrix Theory and Matrix Inequalities*, Prindle, Weber & Schmidt, Boston, 1964.

10 L. Mirsky, *An Introduction to Linear Algebra*, Oxford U.P., 1955.

11 L. Mirsky, The spread of a matrix, *Mathematika* 3:127–130 (1956).

12 T. Popoviciu, Sur les équations algébriques ayant toutes leurs racines réelles, *Mathematica* 9:129–145 (1935).

13 C. R. Rao, *Linear Statistical Inference and Its Applications*, 2nd ed., Wiley, New York, 1973.

14 E. Scheffold, Eine Abschätzung für die subdominanten Eigenwerte nichtnegativer Matrizen, *Linear Algebra and Appl.* 19:91–93 (1978).

15 G. W. Stewart, *Introduction to Matrix Computations*, Academic, New York, 1973.

16 G. W. Thomson, Bounds for the ratio of range to standard deviation, *Biometrika* 42:268–269 (1955).

17 R. S. Varga, *Matrix Iterative Analysis*, Prentice-Hall, Englewood Cliffs, N.J., 1962.

18 H. Wolkowicz and G. P. H. Styan, Extensions of Samuelson's inequality, *Amer. Stat.* 33:143–144 (1979).

19 H. Wolkowicz and G. P. H. Styan, More bounds for eigenvalues using traces, *Linear Algebra and Appl.* to appear. Vol. pp. of LA817 H.


*Received 1 August 1978; revised 3 August 1979*