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Exercise 1

a)

$$x(k+1) = \begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(k) \quad \rightarrow \text{discrete time}$$

Computing the eigenvalues of $A = \begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix}$

$$\det(A - \lambda I) = 0 \Rightarrow (1-\lambda)(0.5-\lambda) - (-0.5)\cdot 0 = 0 \Rightarrow \lambda = 1, 0.5; m=0 \text{ for each } \lambda.$$

For $\lambda = 0.5$, since $\|\lambda\| < 1$, and $m < 1$ ($\lambda = re^{i\theta}$), the associated state is asymptotically stable.

For $\lambda = 1$, $m = 1$, and $n = 0$ (not defective). Thus, this state is stable in the sense of Lyapunov.

Thus, the system is stable in the sense of Lyapunov but not asymptotically stable.

b) $\dot{x} = \begin{bmatrix} -1 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} u \quad \rightarrow \text{continuous time}$

Computing the eigenvalues of A , we get $\lambda = -1, -5, -3; m=0$ for each λ .

Writing the eigenvalues as $\lambda = R + iIm$,

for $\lambda = -1$, $\lambda = -1 + i(0)$
 $R < 0 \rightarrow$ asymptotically stable.

for $\lambda = -5$, $\lambda = -5 + i(0)$
 $R < 0 \rightarrow$ asymptotically stable.

for $\lambda = -3$, $\lambda = -3 + i(0)$
 $R < 0 \rightarrow$ asymptotically stable.

Since all states are asymptotically stable, the system is asymptotically stable.

Exercise 2

$$\dot{x} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix}x + \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}u$$

$$y = [1 \ 1 \ 1]x$$

Computing controllability matrix P : $P = [B \ AB \ A^2B]$

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow P = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & -1 & 0 & 1 \end{bmatrix}$$

$$f(P) = 2.$$

Constructing M for controllable decomposition:

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \quad f(M) = 3$$

$$\Rightarrow A' = M^{-1}AM, \quad B' = M^{-1}B, \quad C' = CM$$

$$M^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\text{So, } A' = M^{-1}AM = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$B' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$C' = CM = [3 \ 3 \ 1]$$

$$\Rightarrow \dot{x}_c = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}x_c + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}u, \quad y = [3 \ 3 \ 1]x_c \quad \left. \right\} \text{ controllable form}$$

Computing observability matrix Q : $Q = [C' \ C'A' \ C'A'^2]'$

$$Q = \begin{bmatrix} 3 & 3 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$\text{r}(Q) = 2$. Thus, not all states of the system are observable.

To check stabilizability: uncontrollable states must be Lyapunov stable.

The eigenvalues of A' come out to be: $\lambda = 0, -1, -2$

So, all the states are stable in the sense of Lyapunov (two being asymptotically stable).

Stabilizable

Hence, the system is stabilizable in the sense that the uncontrollable state will not diverge. However, it might not be possible to drive this state to a desired equilibrium point.

Detectable

All unobservable modes need to be stable. Here, since all states are stable in the sense of Lyapunov, the system is detectable.

Exercise 3

$$\begin{aligned} m\ddot{x} &= -u_1 \sin\theta + \epsilon u_2 \cos\theta \\ m\ddot{y} &= u_1 \cos\theta + \epsilon u_2 \sin\theta - mg \\ J\ddot{\theta} &= u_2 \end{aligned}$$

Writing the nonlinear dynamics in matrix form:

$$\begin{bmatrix} \dot{\theta} \\ \dot{u}_1 \\ \dot{u}_2 \\ \dot{y} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ (-u_1 \sin\theta + \epsilon u_2 \cos\theta)/m \\ (u_1 \cos\theta + \epsilon u_2 \sin\theta - mg)/m \\ u_2/J \\ u_2/J \end{bmatrix} = f(y, u) \quad [y = [x, u, \theta, \dot{x}, \dot{u}]]$$

Equilibrium point is: $\bar{x}=0, \bar{y}=0, \bar{\theta}=0, \bar{u}_1=mg, \bar{u}_2=0$

Linearizing the system around the equilibrium point:

$$A = \frac{\partial f(y, u)}{\partial y} \Big|_{y=\bar{y}, u=\bar{u}} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ \frac{-u_1 \cos\theta - u_2 \sin\theta}{m} & 0 & 0 & 0 \\ \frac{-u_1 \sin\theta + \epsilon u_2 \cos\theta}{m} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Big|_{y=\bar{y}, u=\bar{u}} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \frac{\partial f(y, u)}{\partial u} \Big|_{y=\bar{y}, u=\bar{u}} = \begin{bmatrix} 0 & 0 \\ \frac{-\sin\theta}{m} & \frac{\epsilon \cos\theta}{m} \\ \frac{\cos\theta}{m} & \frac{\epsilon \sin\theta}{m} \\ 0 & \frac{1}{J} \end{bmatrix} \Big|_{y=\bar{y}, u=\bar{u}} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{\epsilon}{m} \\ \frac{1}{m} & 0 \\ 0 & \frac{1}{J} \end{bmatrix}$$

$y \Rightarrow$ states, $u \Rightarrow$ controls

$$y = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} y + \begin{bmatrix} 0 & 0 \\ 0 & \frac{\epsilon}{m} \\ \frac{1}{m} & 0 \\ 0 & \frac{1}{J} \end{bmatrix} u \rightarrow \text{linearized system about equilibrium point}$$

To determine the stability, computing the eigenvalues of the Jacobian:

All eigenvalues of A come out to be 0.

So, the system is not asymptotically stable, but it is stable in the sense of Lyapunov.

Exercise 4

$$\dot{x} = \begin{bmatrix} a & 0 \\ 1 & -1 \end{bmatrix} x$$

Lyapunov function: $V = x_1^2 + x_2^2$

$V(x) = 0$ for $x = 0$ only

$V(x) > 0 \Leftrightarrow x \neq 0$

$$\begin{aligned} \dot{V}(x) &= \frac{\partial V}{\partial x_1} \dot{x}_1 + \frac{\partial V}{\partial x_2} \dot{x}_2 = 2x_1(ax_1) + 2x_2(x_1 - x_2) \\ &= 2ax_1^2 + 2x_1x_2 - 2x_2^2 \end{aligned}$$

$\dot{V}(x)$ must be < 0 for all $x \neq 0$ for asymptotically stable.

$$\begin{aligned} \dot{V}(x) &= 2ax_1^2 + \frac{x_1^2}{2} - \left(\frac{x_1^2}{2} - 2x_1x_2 + 2x_2^2 \right) \quad (\text{adding and subtracting } \frac{x_1^2}{2}) \\ &= \underbrace{\frac{(4a+1)x_1^2}{2}}_{< 0 \Leftrightarrow x \neq 0} - \left(\frac{x_1}{\sqrt{2}} - \sqrt{2}x_2 \right)^2 \end{aligned}$$

To ensure $\dot{V}(x) < 0 \Leftrightarrow x \neq 0$, we need $\frac{4a+1}{2} \leq 0$ always.

$$\frac{4a+1}{2} \leq 0 \Rightarrow a \leq -\frac{1}{4}$$

. For $a \leq -\frac{1}{4}$, $\dot{V}(x) < 0 \Leftrightarrow x \neq 0 \Rightarrow$ asymptotically stable.

Exercise 5

$$\dot{n}_1 = n_2 - n_1 n_2^2$$

$$\dot{n}_2 = -n_1^3$$

Equilibrium point $[\bar{n}_1, \bar{n}_2]^T = [0, 0]^T$

a) $\frac{\partial f}{\partial n} \Big|_{\bar{n}} = \begin{bmatrix} -n_2^2 & 1 - 2n_1 n_2 \\ -3n_1^2 & 0 \end{bmatrix} \Big|_{\bar{n}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

linearized state:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x$$

Eigenvalues of state matrix: $\lambda = 0, 0$

so, the linearized system is Lyapunov stable, but not asymptotically stable.

Note: Since the linearization is an approximation of the system, $\lambda = 0$, i.e., stable only in the sense of Lyapunov does not guarantee stability.

Noise or perturbations might cause instability.

b) $V(n) = n_1^4 + 2n_2^2$

$$V(0) = 0,$$

$$V(n) > 0 \text{ if } n \neq 0$$

$$\begin{aligned} V(n) &= \frac{\partial V}{\partial n_1} n_1 + \frac{\partial V}{\partial n_2} n_2 = 4n_1^3(n_2 - n_1 n_2^2) + 4n_2(-n_1^3) \\ &= 4n_1^3 n_2 - 4n_1^4 n_2^2 - 4n_2 n_1^3 = -4n_1^4 n_2^2 \end{aligned}$$

$$\text{so if } n_1, n_2 \neq 0, V(n) = -4n_1^4 n_2^2 < 0$$

Thus, the system is asymptotically stable.

c) and d) code and plots attached below.

Exercise 6

a) $x(k+1) = \begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(k)$

$$y = [5 \ 5] x(k)$$

Computing the transfer function to check for BIBO stability:

$$G_p(s) = C(zI - A)^{-1}B + D$$

$$= [5 \ 5] \begin{bmatrix} z-1 & 0 \\ 0.5 & z-0.5 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + [0]$$

$$= [5 \ 5] \begin{bmatrix} \frac{1}{(z-1)} & 0 \\ -\frac{1}{(z-1)(z-0.5)} & \frac{2}{(z-1)(z-0.5)} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = [5 \ 5] \begin{bmatrix} \frac{1}{(z-1)} \\ -\frac{1}{(z-1)(z-0.5)} \end{bmatrix} - \frac{2}{(z-1)(z-0.5)}$$

$$= [5 \ 5] \begin{bmatrix} \frac{1}{(z-1)} \\ -\frac{(z-1)}{(z-1)(z-0.5)} \end{bmatrix} = 0$$

The transfer function is 0, indicating that the output doesn't depend on the input.

So looking at the eigenvalues of the system matrix to understand the stability of the system:

$$A = \begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix}$$

$$\lambda = 1, 0.5$$

For $\lambda = 0.5$, eigenvalue is inside the unit circle \Rightarrow stable state.

For $\lambda = 1$, the eigenvalue lies on the unit circle. This indicates that the corresponding state is not asymptotically stable.

Thus, the system is not fully BIBO stable, as perturbations can cause the system to be unstable.

b) $x = \begin{bmatrix} -1 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} u$

$$y = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} x$$

computing the transfer function C_{1c} :

$$C_{1c}(s) = C(sI - A)^{-1}B + D$$

$$= \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} s+7 & 2 & -6 \\ -2 & s+3 & 2 \\ 2 & 2 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 1/s+3 & 0 \end{bmatrix}$$

The pole lies at $-3 \Rightarrow$ BIBO stable.

The system is BIBO stable for the second output. The first output is unaffected by the input.

Checking eigenvalues of the system matrix to ensure that no unstable poles are cancelled out in the transfer function:

Eigenvalues for A : $\lambda = -1, -3, -5$

So the system is stable in the sense of Lyapunov, and it is BIBO stable as well.

Exercise 1

$$V_c \frac{dT_c}{dt} = f_c(T_{ci} - T_c) + \beta(T_h - T_c)$$

$$V_h \frac{dT_h}{dt} = f_h(T_{hi} - T_h) + \beta(T_c - T_h)$$

$f_c, f_h, V_c, V_h \rightarrow \text{constant}$

$T_c, T_h \rightarrow \text{temperatures in cold and hot compartments respectively}$

$T_{ci}, T_{hi} \rightarrow \text{temperatures of cold and hot inflows respectively}$

$$u_1 = T_{ci}, u_2 = T_{hi}$$

$$y_1 = T_c, y_2 = T_h$$

$$f_c = f_h = 0.1 \text{ m}^3/\text{min}, \beta = 0.2 \text{ m}^3/\text{min}, V_h = V_c = 1 \text{ m}^3$$

i) Rewriting the equations, we get :

$$\dot{T}_c = \frac{1}{V_c} \left(-(f_c + \beta)T_c + \beta T_h + f_c T_{ci} \right)$$

$$\dot{T}_h = \frac{1}{V_h} \left(-(f_h + \beta)T_h + \beta T_c + f_h T_{hi} \right)$$

In state-space form :

$$\begin{bmatrix} \dot{T}_c \\ \dot{T}_h \end{bmatrix} = \begin{bmatrix} -(f_c + \beta)/V_c & \beta/V_c \\ \beta/V_h & -(f_h + \beta)/V_h \end{bmatrix} \begin{bmatrix} T_c \\ T_h \end{bmatrix} + \begin{bmatrix} f_c/V_c & 0 \\ 0 & f_h/V_h \end{bmatrix} \begin{bmatrix} T_{ci} \\ T_{hi} \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} T_c \\ T_h \end{bmatrix}$$

2) When $T_{ci} = T_{hi} = 0$,

$$\underbrace{\begin{bmatrix} \dot{T}_c \\ \dot{T}_h \end{bmatrix}}_x = \underbrace{\begin{bmatrix} -(f_c + \beta)/V_c & \beta/V_c \\ \beta/V_h & -(f_h + \beta)/V_h \end{bmatrix}}_A \underbrace{\begin{bmatrix} T_c \\ T_h \end{bmatrix}}_x$$

$$x = Ax$$

The solution for this equation is $x(t) = e^{At} x(0)$

Substituting the values of f_c, f_h, β , we get : $A = \begin{bmatrix} -0.3 & 0.2 \\ 0.2 & -0.3 \end{bmatrix}$

Eigenvalues of A: $\lambda = -0.5, -0.1$; Eigenvectors: $v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Taking $J = \begin{bmatrix} -0.5 & 0 \\ 0 & -0.1 \end{bmatrix}$, $M = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

$$e^{At} = M^{-1} e^{Jt} M = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} e^{-0.5t} & 0 \\ 0 & e^{-0.1t} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{e^{-t/2}}{2} \begin{bmatrix} e^{0.4t} + 1 & e^{0.4t} - 1 \\ e^{0.4t} - 1 & e^{0.4t} + 1 \end{bmatrix}$$

$$\Rightarrow x(t) = \frac{e^{-t/2}}{2} \begin{bmatrix} e^{0.4t} + 1 & e^{0.4t} - 1 \\ e^{0.4t} - 1 & e^{0.4t} + 1 \end{bmatrix} x(0)$$

$$\rightarrow T_c = \frac{e^{-t/2}}{2} (e^{0.4t} (T_c(0) + T_h(0)) + T_c(0) - T_h(0))$$

$$T_h = \frac{e^{-t/2}}{2} (e^{0.4t} (T_c(0) + T_h(0)) + T_h(0) - T_c(0))$$

3) To investigate the BIBO stability of the system, computing the transfer function:

$$G_1(s) = C(sI - A)^{-1}B$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A = \begin{bmatrix} -0.3 & 0.2 \\ 0.2 & -0.3 \end{bmatrix}, B = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$

$$\Rightarrow G_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s + 0.3 & -0.2 \\ -0.2 & s + 0.3 \end{bmatrix}^{-1} \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$

$$G_1 = \frac{1}{s^2 + 0.6s + 0.1} \begin{bmatrix} 10s + 3 & 2 \\ 2 & 10s + 3 \end{bmatrix}$$

So, the poles of the system lie at $s = -\frac{3 \pm i}{10}$.

Real parts of both poles lie on the negative side. Hence, the system is BIBO stable.