Exercise 1

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

To find characteristic polynomial, $\det(A - \lambda I) = 0$

$$\Rightarrow \det \begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix} = 0 \Rightarrow (1-\lambda)(-\lambda)(1-\lambda) = 0$$

$\Rightarrow \lambda = 0, 1, 1$ (can be found directly since A is triangular)

i) A^{10}:

Since $A \in \mathbb{R}^{3\times 3}$, A^m can be decomposed as:

$A^m = \beta_2 A^2 + \beta_1 A + \beta_0 I$ } from Cayley-Hamilton theorem and characteristic polynomial.

↳ for any $m \geq 0$, $m \in \mathbb{Z}$

Since λ is also a root of the characteristic polynomial, we can say that $f(A) = \beta_2 A^2 + \beta_1 A + \beta_0 I$ is also satisfied by λ.

$\Rightarrow \lambda^m = \beta_2 \lambda^2 + \beta_1 \lambda + \beta_0$ at eigenvalues

Here, $m = 10$

$$\Rightarrow \lambda^{10} = \beta_2 \lambda^2 + \beta_1 \lambda + \beta$$

Since roots of the characteristic polynomial are the eigenvalues λ_1, λ_2 and λ_3, $\lambda^{10} - \beta_2 \lambda^2 - \beta_1 \lambda - \beta_0 = 0$ at $\lambda_1, \lambda_2, \lambda_3$.

$\lambda_1 = 1 \Rightarrow 1^{10} = \beta_2 + \beta_1 + \beta_0$

$\lambda_2 = 0 \Rightarrow 0^{10} = 0 + 0 + \beta_0$

$\lambda_3 = 1 \Rightarrow 1^{10} = \beta_2 + \beta_1 + \beta_0$

repeated equations.

$\Rightarrow \beta_0 = 0, \quad \beta_1 + \beta_2 = 1$ — ①

Since there's a repeated root at λ_1, λ_3 the derivative of the left and right-hand-sides must also be equal at those points.

$$\Rightarrow \frac{\partial}{\partial \lambda}(\lambda^{10}) = \frac{\partial}{\partial \lambda}(\beta_2\lambda^2 + \beta_1\lambda) \quad \text{at} \quad \lambda_1 = \lambda_3 = 1$$

$$\Rightarrow \quad 10 = 2\beta_2 + \beta_1 \quad -(2)$$

from (1) and (2), we get: $\beta_1 + \beta_2 = 1$; $2\beta_2 + \beta_1 = 10$

(2) $-$ (1) $\Rightarrow \beta_2 = 9$;

Then $\beta_1 = -8$

$\therefore \quad \beta_2 = 9, \beta_1 = -8, \beta_0 = 0$

$$\Rightarrow A^{10} = 9A^2 - 8A$$

A^2 can be computed: $A^2 = A \cdot A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow 9A^2 = \begin{bmatrix} 9 & 9 & 9 \\ 0 & 0 & 9 \\ 0 & 0 & 9 \end{bmatrix}$$

$$9A^2 - 8A = \begin{bmatrix} 9 & 9 & 9 \\ 0 & 0 & 9 \\ 0 & 0 & 9 \end{bmatrix} - \begin{bmatrix} 8 & 8 & 0 \\ 0 & 0 & 8 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 9 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{A^{10} = 9A^2 - 8A = \begin{bmatrix} 1 & 1 & 9 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}}$$

ii) $\underline{\underline{e^{At}}}$

$$e^{At} = \beta_2 A^2 + \beta_1 A^1 + \beta_0 I$$

Also, $e^{\lambda t} = \beta_2\lambda^2 + \beta_1\lambda + \beta_0$ for $\lambda = 1, 0, 1$

and for repeated eigenvalue 1, $\frac{\partial (e^{\lambda t})}{\partial \lambda} = \frac{\partial}{\partial \lambda}(\beta_2\lambda^2 + \beta_1\lambda + \beta_0)$

$e^{\lambda t} = \beta_2\lambda^2 + \beta_1\lambda + \beta_0 \longrightarrow$ for $\lambda = 1$,

$$e^t = \beta_2 + \beta_1 + \beta_0 \quad \text{— ①}$$

for $\lambda = 0$,

$$e^{0t} = 1 = 0 + 0 + \beta_0 \quad \text{— ②}$$

And from the gradient equation, we get:

$t e^{\lambda t} = 2\beta_2\lambda + \beta_1$ for $\lambda = 1$,

$$te^t = 2\beta_2 + \beta_1 \quad \text{— ③}$$

② $\Rightarrow \beta_0 = 1$

① & ③ give: $te^t = 2\beta_2 + \beta_1$ — ④

$e^t - 1 = \beta_2 + \beta_1$ — ⑤

④ - ⑤ $\Rightarrow \beta_2 = te^t - e^t + 1$

$\beta_2 = e^t(t-1) + 1$

Substituting back,

$$\beta_1 = e^t - 1 - (e^t(t-1)+1)$$
$$= e^t - 1 - te^t + e^t - 1$$
$$= e^t(2-t) - 2$$

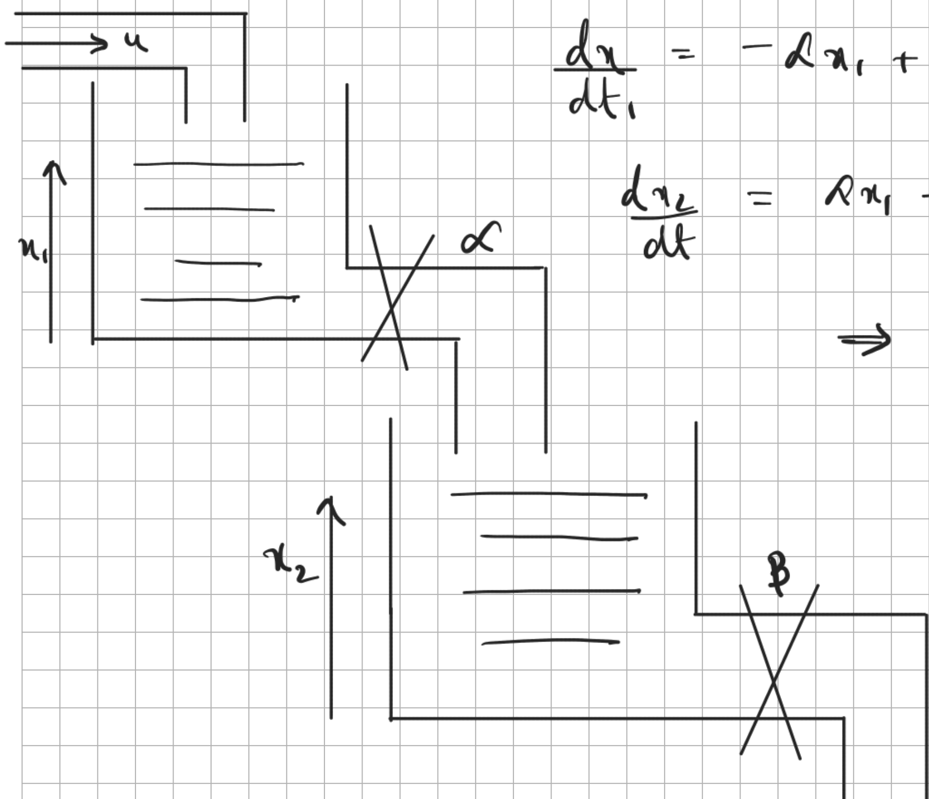
$$\Rightarrow e^{\lambda t} = (e^t(t-1)+1)\lambda^2 + (e^t(2-t)-2)\lambda + 1$$

So, $e^{At} = (e^t(t-1)+1)A^2 + (e^t(2-t)-2)A + I$

$$= \begin{bmatrix} e^t(t-1)+1 & e^t(t-1)+1 & e^t(t-1)+1 \\ 0 & 0 & e^t(t-1)+1 \\ 0 & 0 & e^t(t-1)+1 \end{bmatrix} + \begin{bmatrix} e^t(2-t)-2 & e^t(2-t)-2 & 0 \\ 0 & 0 & e^t(2-t)-2 \\ 0 & 0 & e^t(2-t)-2 \end{bmatrix} + I$$

Simplifying $\Rightarrow$ $\boxed{e^{At} = \begin{bmatrix} e^t & e^t - 1 & e^t(t-1)+1 \\ 0 & 1 & e^t - 1 \\ 0 & 0 & e^t \end{bmatrix}}$

Exercise 2



$$\frac{dx}{dt_1} = -\alpha x_1 + u$$

$$\frac{dx_2}{dt} = \alpha x_1 - \beta x_2$$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\alpha x_1 + u \\ \alpha x_1 - \beta x_2 \end{bmatrix}$$

Writing the equations in state-space form, where states are $[x_1 \;\; x_2]^T$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\alpha & 0 \\ \alpha & -\beta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

Given, $\alpha = 0.1$, $\beta = 0.2$, $u = 1$, $x_1(0) = 2$, $x_2(0) = 1$

For any state-space system, we can get the following

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$e^{At}x(t) - e^{At}Ax(t) = e^{At}Bu(t)$$

$$\frac{d\left(e^{At}x(t)\right)}{dt} = e^{At}Bu(t)$$

$$\Rightarrow e^{At}x(t)\Big|_{t_0}^{t} = \int_{t_0}^{t} e^{At}\, Bu(t)\,dt$$

$$\Rightarrow x(t) = e^{-A(t-t_0)}x(t_0) + e^{At}\int_{t_0}^{t} e^{-At}\, Bu(t)\,dt \quad \text{—— (A)}$$

Computing e^{At} using Cayley-Hamilton theorem:

$e^{At} = \beta_0 + \beta_1 A$ for scalar coefficients β_0 and β_1.
This equation is also satisfied by the eigenvalues of A.

To find eigenvalues of A:

$$\det(A - \lambda I) = 0 \Rightarrow (-\alpha - \lambda)(-\beta - \lambda) = 0$$
$$\Rightarrow \lambda = -\alpha, -\beta$$

So, $e^{\lambda t} = \beta_0 + \beta_1 \lambda \Rightarrow$
$$e^{\alpha t} = \beta_0 - \beta_1 \alpha \quad — ①$$
$$e^{-\beta t} = \beta_0 - \beta_1 \beta \quad — ②$$

from ① and ②, $e^{-\alpha t} - e^{-\beta t} = \beta_1(\beta - \alpha)$
$$\beta_1 = \frac{e^{-\alpha t} - e^{-\beta t}}{\beta - \alpha}$$

So
$$\beta_0 = e^{-\alpha t} + \alpha \frac{(e^{-\alpha t} - e^{-\beta t})}{\beta - \alpha}$$
$$\Rightarrow \beta_0 = \frac{\beta e^{-\alpha t} - \alpha e^{-\beta t}}{\beta - \alpha}$$

$\alpha = 0.1, \beta = 0.2$
$$\Rightarrow \beta_1 = 10(e^{-0.1t} - e^{-0.2t})$$
$$\beta_0 = 2e^{-0.1t} - e^{-0.2t}$$

So,
$$e^{At} = (2e^{-0.1t} - e^{-0.2t}) \times I + 10(e^{-0.1t} - e^{-0.2t}) A$$
$$= \begin{bmatrix} 2e^{-0.1t} - e^{-0.2t} & 0 \\ 0 & 2e^{-0.1t} - e^{-0.2t} \end{bmatrix} + \begin{bmatrix} e^{-0.2t} - e^{-0.1t} & 0 \\ e^{-0.1t} - e^{-0.2t} & 2e^{-0.2t} - 2e^{-0.1t} \end{bmatrix}$$
$$= \begin{bmatrix} e^{-0.1t} & 0 \\ e^{-0.1t} - e^{-0.2t} & e^{-0.2t} \end{bmatrix}$$

Now to compute $x(t)$, we need to find $e^{At}\int_{t_0}^{t} e^{-At} Bu(t)dt$. Computing the integral:

$$e^{-At}Bu(t) = \begin{bmatrix} e^{0.1t} & 0 \\ e^{0.1t} - e^{0.2t} & e^{0.2t} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}(1) = \begin{bmatrix} e^{0.1t} \\ e^{0.1t} - e^{0.2t} \end{bmatrix}$$

$$\int_{t_0}^{t} e^{-At}Bu(t)dt = \int_{t_0}^{t} \begin{bmatrix} e^{0.1t} \\ e^{0.1t} - 2e^{0.2t} \end{bmatrix} dt = \begin{bmatrix} 10e^{0.1t} \\ 10e^{0.1t} - 5e^{0.2t} \end{bmatrix}\Bigg|_{t_0}^{t}$$

$$t_0 = 0,\ t = t \Rightarrow \int e^{-At}Bu(t)dt = \begin{bmatrix} 10(e^{0.1t} - 1) \\ 10e^{0.1t} - 5e^{0.2t} - 5 \end{bmatrix}$$

$$\text{Now, } e^{At}\int_0^t e^{-At}Bu(t)dt = \begin{bmatrix} e^{-0.1t} & 0 \\ e^{-0.1t} - e^{-0.2t} & e^{-0.2t} \end{bmatrix}\begin{bmatrix} 10(e^{0.1t} - 1) \\ 10e^{0.1t} - 5e^{0.2t} - 5 \end{bmatrix}$$

$$= \begin{bmatrix} 10(1 - e^{-0.1t}) \\ 1 - e^{-0.1t} - e^{-0.1t} + e^{-0.2t} + 10e^{-0.1t} - 5 - 5e^{-0.2t} \end{bmatrix}$$

$$= \begin{bmatrix} 10(1 - e^{-0.1t}) \\ -4 + 8e^{-0.1t} - 4e^{-0.2t} \end{bmatrix}$$

Next computing $e^{-A(t-t_0)}x(t_0)$,

$$x(t_0) = x(0) = [2 \ \ 1]^T$$

$$e^{-At}x(0) = \begin{bmatrix} e^{0.1t} & 0 \\ e^{0.1t} - e^{0.2t} & e^{0.2t} \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{0.1t} \\ 2e^{0.1t} - 2e^{0.2t} + e^{0.2t} \end{bmatrix} = \begin{bmatrix} 2e^{0.1t} \\ 2e^{0.1t} - e^{0.2t} \end{bmatrix}$$

$$x(t) = e^{-At}x(0) + e^{At}\int_0^t e^{-At}Bu\,dt$$

$$= \begin{bmatrix} 2e^{0.1t} \\ 2e^{0.1t} - e^{0.2t} \end{bmatrix} + \begin{bmatrix} 10(1-e^{-0.1t}) \\ -4+8e^{-0.1t}-4e^{-0.2t} \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{0.1t}+10-10e^{-0.1t} \\ 2e^{0.1t}+8e^{-0.1t}-4e^{-0.2t}-e^{0.2t}-4 \end{bmatrix}$$

for $x(5)$, substituting $t=5$, we get

$$\boxed{x(5) = \begin{bmatrix} 7.2321 \\ 1.35333 \end{bmatrix}}$$

Exercise 3

$$\begin{matrix} 1 & 12 & 20 \\ 0 & 2 & 0 \\ 0 & 0 & 12 \end{matrix}$$

$$A_1 = \begin{bmatrix} 1 & 4 & 8 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$. Since matrix is triangular, $\lambda = 1, 2, 3$

For $\lambda = 1$, $A_1 - \lambda_1 I = \begin{bmatrix} 0 & 4 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$$(A_1 - \lambda I)v_1 = 0 \Rightarrow \begin{bmatrix} 0 & 4 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = 0 \Rightarrow \begin{matrix} v_{12} + 2v_{13} = 0 \\ v_{13} = 0 \\ \Rightarrow v_{12} = 0 \end{matrix}$$

$$\therefore v_1 = [\alpha \quad 0 \quad 0]^T$$

For $\lambda = 2$, $A_1 - \lambda_2 I = \begin{bmatrix} -1 & 4 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$(A_1 - \lambda_2 I)v_2 = 0 \Rightarrow \begin{bmatrix} -1 & 4 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} = 0 \Rightarrow v_{23} = 0,\ -v_{21} + 4v_{22} + 8v_3 = 0$$

$$v_{21} = 4v_{22}$$

$$\Rightarrow v_2 = [4\beta \quad \beta \quad 0]^T$$

For $\lambda = 3$, $A_1 - \lambda_3 I = \begin{bmatrix} -2 & 4 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$(A_1 - \lambda_3 I)v_3 = 0 \Rightarrow \begin{bmatrix} -2 & 4 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_{31} \\ v_{32} \\ v_{33} \end{bmatrix} = 0$$

$$v_{32} = 0;\ -2v_{31} + 4v_{32} + 8v_{33} = 0$$

$$\Rightarrow v_{31} = 4v_{33}$$

$$\therefore v_3 = [4\gamma \quad 0 \quad \gamma]^T$$

Since all eigenvalues are distinct, the modal matrix is S, where $S = [v_1 \ v_2 \ v_3]$

$$\Rightarrow S = \begin{bmatrix} 1 & 4 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore A_1 = S\Lambda S^{-1} = \begin{bmatrix} 1 & 4 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -4 & -4 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix}$$

To find eigenvalues, $\det(A_2 - \lambda I) = 0$

$$\det(A_2 - \lambda I) = \det \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -2 & -4 & -3-\lambda \end{bmatrix} = -\lambda\left((3+\lambda)\lambda + 4\right) - (2)$$

$$= -\lambda(\lambda^2 + 3\lambda + 4) - 2$$

$$= -(\lambda^3 + 3\lambda^2 + 4\lambda + 2)$$

$$\det(A_2 - \lambda I) = 0 \Rightarrow \lambda^3 + 3\lambda^2 + 4\lambda + 2 = 0$$

$$(\lambda+1)(\lambda^2 + 2\lambda + 2) = 0 \Rightarrow \lambda = -1, -1+i, -1-i$$

All eigenvalues are distinct.

$$\Rightarrow (A - \lambda I)v = 0$$

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -2 & -4 & -3-\lambda \end{bmatrix}$$

for $\lambda_1 = -1$,

$$A - \lambda_1 I = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -2 & -4 & -2 \end{bmatrix}$$

$$(A - \lambda_1 I)v_1 = 0 \Rightarrow v_{11} + v_{12} = 0; \; v_{12} + v_{13} = 0$$

$$-2v_{11} - 4v_{12} - 2v_{13} = 0$$

Substituting, $-2(-v_{12}) - 4v_{12} - 2(-v_{12}) = 0$

$$\Rightarrow 4v_{12} - 4v_{12} = 0$$

This holds for any value of v_{12}

$$\therefore v_1 = [-\alpha \quad \alpha \quad -\alpha]^T$$

For $\lambda_2 = -1+i$, $A - \lambda_2 I = \begin{bmatrix} 1-i & 1 & 0 \\ 0 & 1-i & 1 \\ -2 & -4 & 2-i \end{bmatrix}$

$(A - \lambda_2 I)v_2 = 0$:

$v_{21}(1-i) + v_{22} = 0$ — ①

$v_{22}(1-i) + v_{23} = 0$ — ②

$-2v_{21} - 4v_{22} + (2-i)v_{23} = 0$ — ③

$v_{21}(1-i) - \frac{v_{23}}{(1-i)} = 0$

$-2iv_{21} = v_{23}$

Substituting from ① and ② into ③,

$-2\left(-\frac{v_{23}}{2i}\right) - 4\left(-\frac{v_{23}}{1-i}\right) + (2-i)v_{23} = 0$ } holds for all values of v_{23}

$v_2 = \left[i/2, -\frac{(1+i)}{2}, 1 \right]$

For $\lambda_3 = -1-i$, $A - \lambda_3 I = \begin{bmatrix} 1+i & 1 & 0 \\ 0 & 1+i & 1 \\ -2 & -4 & -2+i \end{bmatrix}$

$(A - \lambda_3 I)v_3 = 0 \Rightarrow$

$(v_{31})(1+i) + v_{32} = 0$

$-\frac{v_{33}(1-i)^2}{4}$

$(1+i)v_{32} + v_{33} = 0$

$-2v_{31} - 4v_{32} + (-2+i)v_{33} = 0$

Substituting, we get

$-2\left(-\frac{v_{33}i}{2}\right) \quad -4\left(-\frac{v_{33}}{2}(1-i)\right) + (-2+i)v_{33} = 0$

$\rightarrow$ holds for all v_{33}.

Thus, $v_3 = \left[-\frac{i}{2}, -\frac{(1-i)}{2}, 1 \right]$

$\Rightarrow A_2 = S \Lambda S^{-1}$, where

$S = \begin{bmatrix} -1 & i/2 & -i/2 \\ 1 & -(1+i)/2 & -(1-i)/2 \\ -1 & 1 & 1 \end{bmatrix}$; $\Lambda = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1+i & 0 \\ 0 & 0 & -1-i \end{bmatrix}$

$$S^{-1} = \begin{bmatrix} -2 & -2 & -1 \\ -1+i & -1+2i & i \\ -1-i & -1-2i & -i \end{bmatrix}$$

$$\Rightarrow A_2 = \begin{bmatrix} -i & i/2 & -i/2 \\ 1 & -(1+i)/2 & -(1-i)/2 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1+i & 0 \\ 0 & 0 & -i-i \end{bmatrix} \begin{bmatrix} -2 & -2 & -1 \\ -1+i & -1+2i & i \\ -1-i & -1-2i & -i \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} ; \quad \lambda = 1, 1, 2$$

Algebraic multiplicity of 1 : 2 and for 2 : 1

For $\lambda_1 = 2$, $A - \lambda_1 I = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$(A - \lambda_1 I) v_1 = 0 \Rightarrow -v_{11} - v_{13} = 0 ; \ v_{12} = 0$

$\therefore v_1 = [1 \quad 0 \quad -1]^T$

For $\lambda_2 = 1$, $A - \lambda_2 I = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$(A - \lambda_2 I) v_2 = 0 \Rightarrow v_3 = 0$

$\therefore$ We can get two vectors that satisfies $(A - \lambda I) v = 0$ for $\lambda = 1$.

$v_2 = [1 \ 1 \ 0]^T$ and $v_3 = [1 \ 2 \ 0]^T$

$$\therefore S = [v_1 \ v_2 \ v_3] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ -1 & 0 & 0 \end{bmatrix}$$

$A_3 = S \Lambda S^{-1}$ where $S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ -1 & 0 & 0 \end{bmatrix}$, $\Lambda = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Writing in order of increasing eigenvalues, we get

$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$; $S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$$S^{-1} = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\Rightarrow A_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \\ -1 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix}$$

$$A_4 - \lambda I = \begin{bmatrix} -\lambda & 4 & 3 \\ 0 & 20-\lambda & 16 \\ 0 & -25 & -20-\lambda \end{bmatrix}$$

$$\det(A_4 - \lambda I) = -\lambda\left((20-\lambda)(-20-\lambda) + (25 \times 16)\right)$$

Setting $\det(A_4 - \lambda I) = 0$, we get

$$\lambda(\lambda^2 - 400 + 400) = 0 \Rightarrow \lambda = 0, 0, 0$$

We have three repeated eigenvalues of value 0.

To find geometric multiplicity, we can do $(A - \lambda I)v = 0$

$$\Rightarrow \begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = 0 \quad \text{where} \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = v_1$$

$\Rightarrow 4v_{12} + 3v_{13} = 0$; $5v_{12} + 4v_{13} = 0$; $5v_{12} + 4v_{13} = 0$

$\rightarrow v_{12} = v_{13} = 0 \Rightarrow v_1 = [1 \quad 0 \quad 0]^T$

So, geometric multiplicty of $\lambda = 0$ is 1.

We need to compute 2 generalized eigenvectors for $\lambda = 0$.

First, $(A - \lambda I) v_2 = v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$4v_{22} + 3v_{23} = 1$; $5v_{22} + 4v_{23} = 0$

Substituting, we get $4v_{22} + 3\left(-\frac{5}{4} v_{22}\right) = 1$

$\Rightarrow v_{22} = 4$ And $v_{23} = -5$

$\therefore v_2 = [0 \quad 4 \quad -5]^T$

Now, $(A - \lambda I) v_3 = v_2$

$\Rightarrow 4v_{32} + 3v_{33} = 0$; $20v_{32} + 16v_{33} = 4$

$-25v_{32} - 20v_{33} = -5$

$\Rightarrow 4v_{32} + 3v_{33} = 0$

$5v_{32} + 4v_{33} = 1$

Substituting, we get

$5v_{32} + 4\left(-\frac{4v_{32}}{3}\right) = 1 \Rightarrow -v_{32} = 3$

And $v_{33} = 4$

$$\Rightarrow v_3 = \begin{bmatrix} 0 & -3 & 4 \end{bmatrix}^T$$

$$\therefore S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & -3 \\ 0 & -5 & 4 \end{bmatrix} \Rightarrow S^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 5 & 4 \end{bmatrix}$$

And $$J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow A_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & -3 \\ 0 & -5 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 5 & 4 \end{bmatrix}$$

Exercise 4

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 2 & 3 \end{bmatrix} x(t)$$

Given $x(0) = 0$
u = unit step input.

i) Continuous Time

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

y at time t is given by:

$$y(t) = Ce^{A(t-t_0)}x(t_0) + C\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

To compute e^{At}, finding diagonal or Jordan form of A:

$$\det(A - \lambda I) = 0 \Rightarrow (-\lambda)(-2-\lambda) - (-2)(1) = 0$$

$$\lambda^2 - 2\lambda + 2 = 0 \Rightarrow \lambda = -1 \pm i$$

Since we have distinct eigenvalues, we can diagonalize A.

$(A - \lambda I)v = 0$ gives eigenvectors.

For $\lambda = -1+i$, $(A - \lambda I)v_1 = \begin{bmatrix} 1-i & 1 \\ -2 & -1-i \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$

$-2v_{11} = (1+i)v_{12}$

$v_{12} = -(1-i)$

$v_{22} = v_{21}(1+i)$

$$v_{12} = -(1-i)v_{11}$$

$$\Rightarrow v_1 = \begin{bmatrix} 1 & -1+i \end{bmatrix}^T$$

For $\lambda = -1-i$, $(A - \lambda I)v_2 = \begin{bmatrix} 1+i & 1 \\ -2 & -1+i \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = 0$

$$v_{22} = -(1+i)v_{21}$$

$$\Rightarrow v_2 = \begin{bmatrix} 1 & -(1+i) \end{bmatrix}^T$$

So, $A = S\Lambda S^{-1}$, where $\Lambda = \begin{bmatrix} -1+i & 0 \end{bmatrix}$ and $S = \begin{bmatrix} 1 & 1 \\ -1+i & -(1+i) \end{bmatrix}$

Now, $y(t) = Ce^{A(t-t_0)}\cancel{x(t_0)}^{\,0} + C\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau + \cancel{D}^{\,0}u(t)$

$t_0 = 0,\ x(t_0) = \begin{bmatrix}0\\0\end{bmatrix};\ D = 0$

$$\Rightarrow y(t) = C\int_0^t S e^{\Lambda(t-\tau)}S^{-1}Bu(\tau)\,d\tau$$

Also since u is a unit step input, $u(t) = 1\ \forall\, t \geq 0$

Computing $Se^{\Lambda(t-\tau)}S^{-1}$,

$$e^{At} = \begin{bmatrix}1 & 1\\ -1+i & -1-i\end{bmatrix}\begin{bmatrix}e^{(-1+i)t} & 0\\ 0 & e^{-(1+i)t}\end{bmatrix}\begin{bmatrix}\frac{1-i}{2} & -\frac{i}{2}\\ \frac{1+i}{2} & \frac{i}{2}\end{bmatrix}$$

$$= \begin{bmatrix}e^{-(1-i)t} & e^{-(1+i)t}\\ -(1-i)e^{-(1-i)t} & -(1+i)e^{-(1+i)t}\end{bmatrix}\begin{bmatrix}\frac{1-i}{2} & -\frac{i}{2}\\ \frac{1+i}{2} & \frac{i}{2}\end{bmatrix}$$

$(1-i)i$

$-(1+i)\left(\frac{i}{2}\right)$

$-(i-1)$

$$= \begin{bmatrix}e^{-(1-i)t}\left(\frac{1-i}{2}\right) + e^{-(1+i)t}\frac{(1+i)}{2} & -e^{-(1-i)t}\left(\frac{i}{2}\right) + e^{-(1+i)t}\left(\frac{i}{2}\right)\\ e^{-(1-i)t}(i) - e^{-(1+i)t}(i) & \left(\frac{i}{2}+\frac{1}{2}\right)e^{-(1-i)t} + \frac{(1-i)}{2}e^{-(1+i)t}\end{bmatrix}$$

$$= e^{-t}\begin{bmatrix}e^{it}\left(\frac{1-i}{2}\right) + e^{-it}\left(\frac{1+i}{2}\right) & \frac{i}{2}\left(-e^{it} + e^{-it}\right)\\ i\left(e^{it} - e^{-it}\right) & e^{it}\left(\frac{1+i}{2}\right) + e^{-it}\left(\frac{1-i}{2}\right)\end{bmatrix}$$

Writing $e^{it} = \cos t + i\sin t$ and simplifying, we get:

$$= e^{-t}\begin{bmatrix}\frac{(\cos t + \cancel{i\sin t} - \cancel{i\cos t} + \sin t + \cos t - \cancel{i\sin t} + \cancel{i\cos t} + \sin t)}{2}, & \frac{i}{2}(-2i\sin t)\\ i\,(2i\sin t), & \frac{(\cos t + \cancel{i\sin t} + \cancel{i\cos t} - \sin t + \cos t - \cancel{i\sin t} - \cancel{i\cos t} - \sin t)}{2}\end{bmatrix}$$

$$= e^{-t}\begin{bmatrix}\sin t + \cos t & \sin t\\ -2\sin t & \cos t - \sin t\end{bmatrix}$$

So for $e^{A(t-\tau)}$, we get:

$$e^{-(t-\tau)}\begin{bmatrix}\cos(t-\tau)+\sin(t-\tau) & \sin(t-\tau)\\ -2\sin(t-\tau) & \cos(t-\tau)-\sin(t-\tau)\end{bmatrix}$$

$\int S e^{\Lambda(t-\tau)} S^{-1} B u$ becomes: $\displaystyle\int_0^t e^{-A(t-\tau)}\begin{bmatrix}1\\1\end{bmatrix}[1]\,d\tau$

$$= \int_0^t e^{-(t-\tau)}\begin{bmatrix}\cos(t-\tau)+\sin(t-\tau) & \sin(t-\tau)\\ -2\sin(t-\tau) & \cos(t-\tau)-\sin(t-\tau)\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}d\tau$$

$$= \int_0^t e^{-(t-\tau)}\begin{bmatrix}\cos(t-\tau)+2\sin(t-\tau)\\ \cos(t-\tau)-3\sin(t-\tau)\end{bmatrix}d\tau$$

Setting $t-\tau=u$ and computing integral, we get:

$$\int S e^{\Lambda(t-\tau)} S^{-1} B u\, d\tau = \begin{bmatrix}-e^{-u}\dfrac{(3\cos u+\sin u)}{2}\\ e^{-u}(\cos u+2\sin u)\end{bmatrix}\Bigg|_0^t$$

$$= \begin{bmatrix}\dfrac{-e^{-t}(3\cos t+\sin t)+(3)}{2}\\ e^{-t}(\cos t+2\sin t)-(1)\end{bmatrix}$$

Now,

$$y(t) = C\ast\int(\ ) = [2\ \ 3]\begin{bmatrix}\dfrac{3-e^{-t}(3\cos t+\sin t)}{2}\\ e^{-t}(\cos t+2\sin t)-1\end{bmatrix}$$

$$= 3-e^{-t}(3\cos t+\sin t)+3e^{-t}(\cos t+2\sin t)-3$$

$$= 5e^{-t}\sin t$$

$$\Rightarrow y(t) = 5e^{-t}\sin t$$

At $t=5$, $\boxed{y(5) = 5e^{-5}\sin(5)}$

ii) In discrete time, the system is given as:

$$x(k+1) = A_d x(k) + B_d u(k)$$
$$y(k) = C_d x(k) + D_d u(k)$$

where, $C_d = C$, $D_d = D$, $A_d = e^{AT}$, $B_d = A^{-1}(A_d - I)B$

$$A_d = e^{AT} = Se^{\Lambda t}S^{-1} = \begin{bmatrix} e^{-T}(\sin T + \cos T) & e^{-T}\sin T \\ -2e^{-T}\sin T & e^{-T}(\cos t - \sin t) \end{bmatrix}$$

where $T = 1s$ is the sample time.

For ease of computing B_d, substituting value of T into A_d

$$A_d = \begin{bmatrix} e^{-1}(\cos 1 + \sin 1) & e^{-1}\sin 1 \\ -2e^{-1}\sin 1 & e^{-1}(\cos 1 - \sin 1) \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -1 & -1/2 \\ 1 & 0 \end{bmatrix}$$

so $$B_d = \begin{bmatrix} -1 & -1/2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^{-1}(\cos 1 + \sin 1) - 1 & e^{-1}\sin 1 \\ -2e^{-1}\sin 1 & e^{-1}(\cos 1 - \sin 1) - 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1/2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^{-1}(\cos 1 + 2\sin 1) - 1 \\ e^{-1}(\cos 1 - 3\sin 1) - 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - e^{-1}(\cos 1 + 2\sin 1) + \frac{1}{2} - \frac{e^{-1}}{2}(\cos 1 - 3\sin 1) \\ e^{-1}(\cos 1 + 2\sin 1) - 1 \end{bmatrix}$$

$$B_d = \begin{bmatrix} \frac{3}{2} - e^{-1}\left(\frac{3}{2}\cos 1 + \frac{1}{2}\sin 1\right) \\ e^{-1}(\cos 1 + 2\sin 1) - 1 \end{bmatrix}$$

And $u(k) = 1 \ \forall\ k \geq 0$

$$\Rightarrow x(k+1) = \begin{bmatrix} e^{-1}(\cos 1 + \sin 1) & e^{-1}\sin 1 \\ -2e^{-1}\sin 1 & e^{-1}(\cos 1 - \sin 1) \end{bmatrix} x(k) + \begin{bmatrix} \frac{3}{2} - \frac{e^{-1}}{2}(3\cos 1 + \sin 1) \\ e^{-1}(\cos 1 + 2\sin 1) - 1 \end{bmatrix}$$

$$y(k) = [2 \quad 3]\, x_k$$

iii) For $y(5)$,

$$y(k) = CA_d^k \cancel{x(0)}^{0} + \sum_{i=0}^{k-1} CA_d^{k-i-1} B_d\, u(i) + \cancel{D_d}^{0} u(k)$$

$x(0) = [0]$, $D_d = [0]$ and $u(i) = 1 \; \forall \; i \geq 0$

$$\Rightarrow y(k) = \sum_{i=0}^{k-1} CA_d^{k-i-1} B_d$$

For $k=5$, $y(5) = \sum_{i=0}^{4} CA_d^{4-i} B_d$

$$= CA_d^4 B_d + CA_d^3 B_d + CA_d^2 B_d + CA_d B_d + CB_d$$

Computing A_d^2, A_d^3 and A_d^4 using computer:

$$A_d^2 = \begin{bmatrix} e^{-2}(\cos 2 + \sin 2) & e^{-2}\sin 2 \\ -2e^{-2}\sin 2 & e^{-2}(\cos 2 - \sin 2) \end{bmatrix}$$

$$A_d^3 = \begin{bmatrix} e^{-3}(\cos 3 + \sin 3) & e^{-3}\sin 3 \\ -2e^{-3}\sin 3 & e^{-3}(\cos 3 - \sin 3) \end{bmatrix}$$

$$A_d^4 = \begin{bmatrix} e^{-4}(\cos 4 + \sin 4) & e^{-4}\sin 4 \\ -2e^{-4}\sin 4 & e^{-4}(\cos 4 - \sin 4) \end{bmatrix}$$

Now, we have to compute $CA_d^m B$ for $m = 0, 1, 2, 3, 4$.
Performing the computations on a computer, we get

$$CA_d^4 B_d = 5e^{-5}\sin 5 - 5e^{-4}\sin 4$$

$$CA_d^3 B_d = 5e^{-4}\sin 4 - 5e^{-3}\sin 3$$

$$CA_d^2 B_d = 5e^{-3}\sin 3 - 5e^{-2}\sin 2$$

$$CA_d B_d = 5e^{-2}\sin 2 - 5e^{-1}\sin 1$$

$$CB_d = 5e^{-1}\sin 1$$

Summing them up, we get:

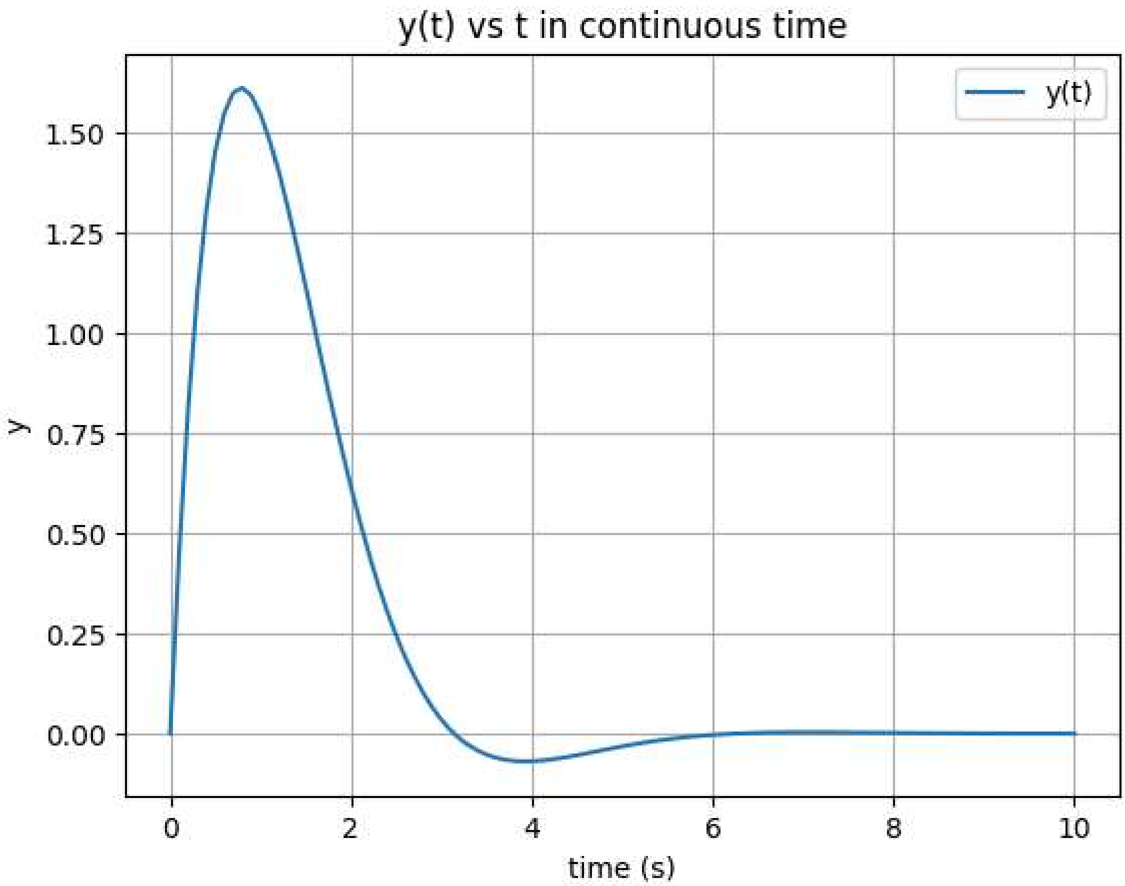
$$\boxed{y(5) = 5e^{-5}\sin 5}$$ matches with continuous time result.

Plots and python code attached below

In [7]:
```python
## ChatGPT was used to understand the working of SymPy, its syntax and helper funct

import numpy as np
from scipy import linalg
import sympy as sp
import matplotlib.pyplot as plt

## for continuous time

# using the sympy library to define a continuous time state space system
# variables
t = sp.symbols('t')
x1 = sp.Function('x1')(t)
x2 = sp.Function('x2')(t)
y = sp.Function('y')(t)

# matrices
A = sp.Matrix([[0, 1], [-2, -2]])
B = sp.Matrix([[1], [1]])
C = sp.Matrix([2, 3]).reshape(1, 2)
D = sp.Matrix([0])
x = sp.Matrix([[x1], [x2]])
u = sp.Matrix([1])

# dynamics and observation
dxdt = A * x + B * u
y = C * x + D * u

# solving analytically
solns = sp.dsolve([sp.Eq(x1.diff(t), dxdt[0]), sp.Eq(x2.diff(t), dxdt[1])], ics = {

# substituting the solution of x into y's equation
y_t = y.subs({x1: solns[0].rhs, x2: solns[1].rhs})
y_t5 = y_t.subs(t, 5)
```

In [8]:
```python
print('y(t) = \n')
sp.pprint(y_t)
print('\ny(5) = \n')
sp.pprint(y_t5)

# converting to numerical solution to plot
y_numeric = sp.lambdify(t, y_t[0], modules='numpy')

# generating time array for plotting
time = np.linspace(0, 10, 101)
y_values = y_numeric(time)
```

```
y(t) =

⎡    -t         ⎤
⎣5⋅ℯ   ⋅sin(t)⎦

y(5) =

⎡    -5         ⎤
⎣5⋅ℯ   ⋅sin(5)⎦
```



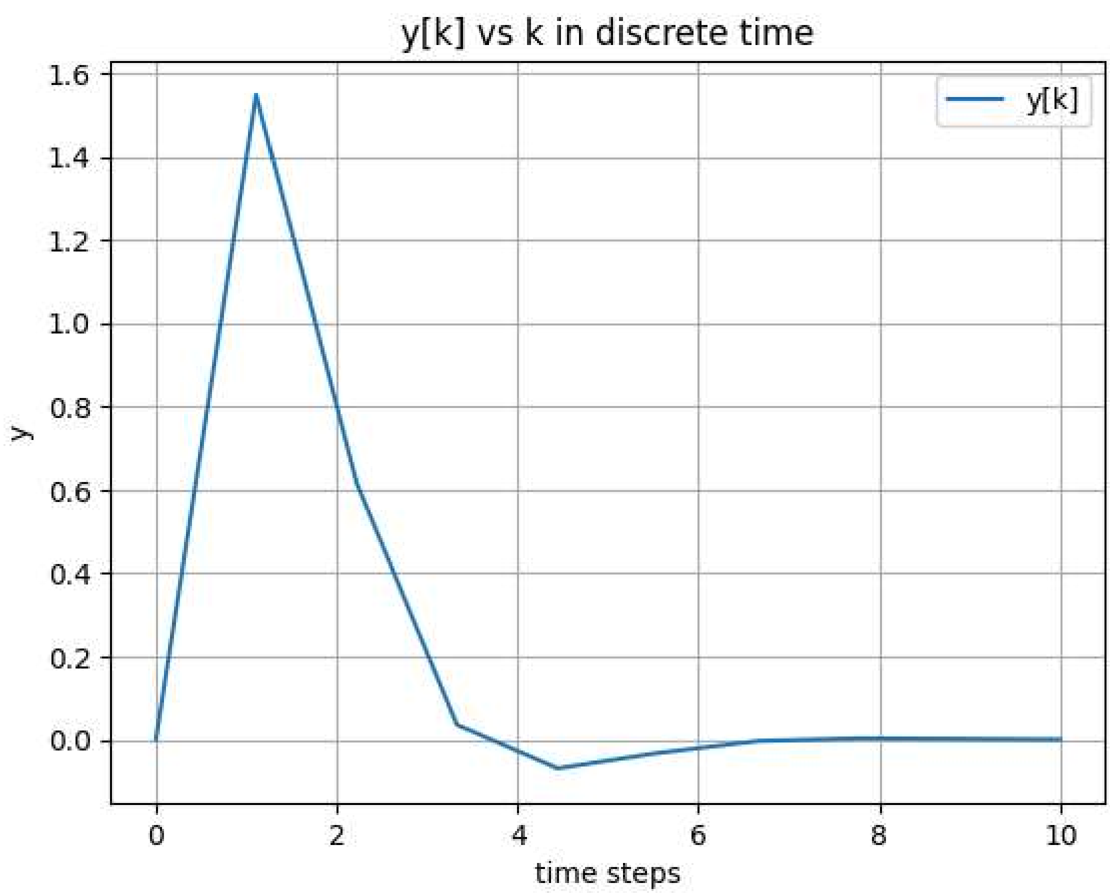
In [34]:

```
## discrete time

# deriving the matrices for discrete time from the continuous time matrices
A_d = np.array([[0, 1],
               [-2, -2]])
B_d = np.array([[1],
                [1]])
C_d = np.array([[2, 3]])
D_d = np.array([[0]])

xd = np.zeros([2, 1])
xd_new = np.zeros_like(x)
yd = np.zeros([10, 1])

T = 1  # sample time
ud = np.ones([1, 1])  # constant input for all k >= 0

# defining matrices for discrete time system
Ad = linalg.expm(A_d * T)
```

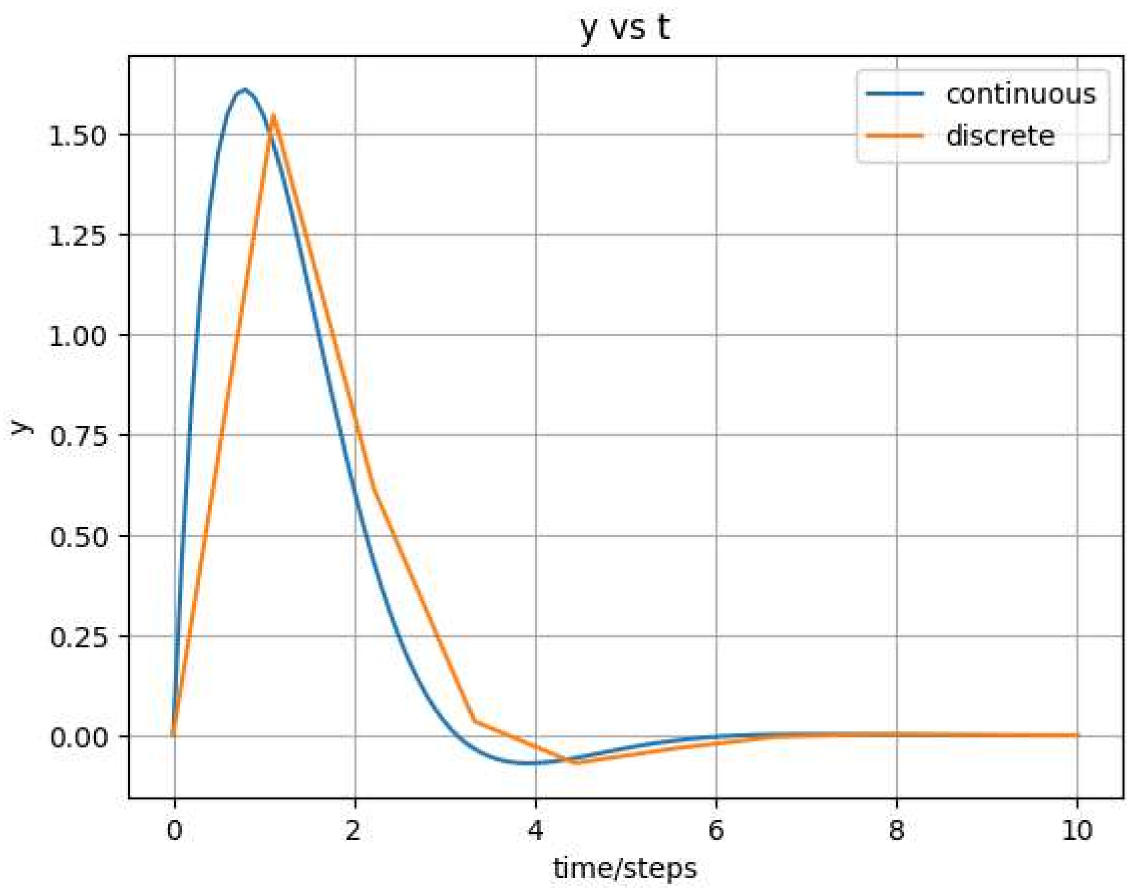
```python
Bd = np.linalg.inv(A_d) @ (Ad - np.identity(2)) @ B_d
Cd = C_d
Dd = D_d

# x[k + 1] = Ad * x[k] + Bd * u[k]; y[k] = Cd * x[k] + Dd * u[k]
# To get y[5], let's iterate the system 4 times

for i in range(10):
    xd_new = Ad @ xd + Bd * ud
    yd[i] = Cd @ xd + Dd @ u
    xd = xd_new
```

In [39]:
```python
print('y(5) =', yd[5])

# generating time array for plotting
time_d = np.linspace(0, 10, 10)

# plotting
plt.plot(time_d, yd, label='y[k]')
plt.xlabel('time steps')
plt.ylabel('y')
plt.title('y[k] vs k in discrete time')
plt.grid()
plt.legend()
plt.show()
```

```
y(5) = [-0.0323059]
```



In [40]:
```
# plotting in same figure
plt.plot(time, y_values, label = 'continuous')
plt.plot(time_d, yd, label = 'discrete')
plt.xlabel('time/steps')
plt.ylabel('y')
plt.title('y vs t')
plt.grid()
plt.legend()
plt.show()
```



Exercise 5

$$F_{k+2} = F_k + F_{k+1}$$

To compute : F_{20}

Denoting F_m as $x(m)$, we get :

$$x(m+2) = x(m) + x(m+1)$$

If we choose a state matrix $X(k) = \begin{bmatrix} x(k+1) \\ x(k) \end{bmatrix}$

we get : $X(k+1) = \begin{bmatrix} x(k+2) \\ x(k+1) \end{bmatrix}$ given by

$$X(k+1) = \begin{bmatrix} x(k+2) \\ x(k+1) \end{bmatrix} = \begin{bmatrix} x(k+1) + x(k) \\ x(k+1) \end{bmatrix}$$

Writing in state-space form :

$$\underbrace{\begin{bmatrix} x(k+2) \\ x(k+1) \end{bmatrix}}_{X(k+1)} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x(k+1) \\ x(k) \end{bmatrix}}_{X(k)} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We need to compute $x(20)$, which will be present in $X(19)$ as $X(19) = \begin{bmatrix} x(20) \\ x(19) \end{bmatrix}$

For a discrete system, $X(k)$ is given by :

$$X(k) = A_d^k X(0) + \sum_{m=0}^{k-1} A_d^{k-m-1} B_d u(m)$$

Since B_d is zero, this reduces to :

$$X(k) = A_d^k X(0)$$

So, $x(19) = A_d^{19}\, x(0) = \left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right)^{19} \begin{bmatrix} x(1) \\ x(0) \end{bmatrix}$

Using a calculator to compute A_d^{19}, we get

$$A_d^{19} = \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right)^{19} = \begin{bmatrix} 6765 & 4181 \\ 4181 & 2584 \end{bmatrix}$$

Thus, $A_d^{19}\, x(0) = A_d^{19} \begin{bmatrix} x(1) \\ x(0) \end{bmatrix}$

$x(1) = F_1 = 1, \quad x(0) = F_0 = 0$

$$\Rightarrow \quad A_d^{19}\, X(0) = \begin{bmatrix} 6765 & 4181 \\ 4181 & 2584 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 6765 \\ 4181 \end{bmatrix}$$

So, $\boxed{x(20) = F_{20} = 6765}$