Exercise 1

1. $y(t)=0$ for all t

For linearity, say $u_1 \to y_1$ and $u_2 \to y_2$ then for an input $u_3 = \alpha u_1 + \beta u_2$, the output y_3 must satisfy $y_3 = \alpha y_1 + \beta y_2$ (u_i being inputs to the system, and α & β are constants)

$$y_1 = y_2 = 0$$

$$y(u_3) = y(u_1 + u_2) = y_3 = 0$$

$$\therefore \alpha y_1 + \beta y_2 = \alpha(0) + \beta(0) = 0 = y_3$$

System is linear.

For time variance, given an input $u_1 \to y_1$, if an input $u_2(t) = u_1(t+\tau)$ produces an output y_2 such that $y_2(t) = y_1(t+\tau)$, the system is time invariant.

For u_1, $y_1(t) = 0$.

Now for $u_2(t)$ such that $u_2(t) = u_1(t+\tau)$,

$y_2(t) = 0$.

We know that $y_1(t+\tau) = 0 = y_2(t)$

$\Rightarrow$ for $u_2(t) = u_1(t+\tau)$, $y_2(t) = y_1(t+\tau)$

Thus, the system is time invariant.

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2. $y(t) = u^3(t)$

For linearity, $u_3 = \alpha u_1 + \beta u_2$

where $y_1(t) = u_1^3(t)$, $y_2(t) = u_2^3(t)$

$y_3 = u_3^3(t)$, but $u_3 = \alpha u_1 + \beta u_2$

$$\Rightarrow \quad y_3 = (\alpha u_1 + \beta u_2)^3 = \alpha^3 u_1^3 + \beta_3 u_2^3 + 3\alpha^2 \beta u_1^2 u_2 + 3\alpha\beta^2 u_1 u_2^2$$

For linearity, $y_3 = \alpha y_1 + \beta y_2$ is necessary.

$$\alpha y_1 + \beta y_2 = \alpha u_1^3 + \beta u_2^3$$

Comparing, $\alpha u_1^3 + \beta u_2^3 \overset{?}{=} \alpha_1^3 u_1^3 + \beta_3^3 u_2^3 + 3\alpha^2 \beta u_1^2 u_2 + 3\alpha\beta^2 u_1 u_2^2$

By comparing the coefficients, we can see that the equality doesn't hold.

Thus, the system is not linear.

For time variance, we need: for $u_2 = u_1(t+\tau)$, $y_2 = y_1(t+\tau)$, where $y_2 = u_2^3 t$ & $y_1 = u_1^3 t$

Substituting $u_2 = u_1(t+\tau)$, we get $y_2(t) = u_1^3(t+\tau)$

We know, $y_1(t+\tau) = u_1^3(t+\tau)$

so, $y_1(t+\tau) = u_1^3(t+\tau) = u_2^3(t) = y_2(t)$

$\therefore$ The system is time invariant.

---

3. $y(t) = u(3t)$

For linearity: $y_1 = u_1(3t)$; $y_2 = u_2(3t)$

Let $u_3(t) = \alpha u_1(t) + \beta u_2(t)$

for linearity, we need $y_3(t) = \alpha y_1(t) + \beta y_2(t)$

$$y_3(t) = u_3(3t) = \underbrace{\alpha u_1(3t)}_{\alpha y_1} + \underbrace{\beta u_2(3t)}_{\beta y_2}$$

$$\Rightarrow \quad y_3(t) = \alpha y_1(t) + \beta y_2(t)$$

Hence, the system is linear.

For system to be time invariant, a time shifted input $u(t+\tau)$ should produce an output shifted by the same τ i.e. $u(t+\tau) \rightarrow y(t+\tau)$

If $y(t) = u(3t)$, then for $u(t+\tau)$, we get $y'(t) = u(3t+3\tau)$

Now, the time shifted system would look like:

$$y(t+\tau) = u(3(t+\tau)) = u(3t+3\tau)$$

which is equal to $y'(t)$

Thus, delaying the input and time shifting the system produce the same result.

Thus, the system is time invariant.

---

4. $y(t) = e^{-t}u(t-T)$

Linearity: $y_1 = e^{-t}u_1(t-T)$; $y_2 = e^{-t}u_2(t-T)$

let $u_3 = \alpha u_1 + \beta u_2$

$$y_3 = e^{-t}u_3(t-T) = e^{-t}(\alpha u_1(t-T) + \beta u_2(t-T))$$

$$= \underbrace{\alpha e^{-t}u_1(t-T)}_{\alpha y_1} + \underbrace{\beta e^{-t}u_2(t-T)}_{\beta y_2}$$

$$\Rightarrow y_3(t) = \alpha y_1(t) + \beta y_2(t)$$

$\therefore$ System is linear.

For time variance:

When input is delayed by τ, we get

$$y'(t) = e^{-t}u((t+\tau)-T) = e^{-t}u(t-T+\tau)$$

When system is shifted by the same delay τ, we get:

$$y(t+\tau) = e^{-(t+\tau)} u(t+\tau-T)$$
$$= e^{-\tau} \times \underbrace{e^{-t} u(t-T+\tau)}_{y'(t)}$$
$$= e^{-\tau} y'(t)$$

We can see that the time delay in input does not have the same effect as shifting the input by τ. Thus, system is time variant.

---

5. $y(t) = \begin{cases} 0 & t \le 0 \\ u(t) & t > 0 \end{cases}$

For linearity, $u_1 \to y_1$; $u_2 \to y_2$; $u_3 \to y_3$ where $u_3 = \alpha u_1 + \beta u_2$. If $y_3 = \alpha y_1 + \beta y_2 \Rightarrow$ linear

$$y_3(t) = \begin{cases} 0 & t \le 0 \\ u_3(t) & t > 0 \end{cases} = \begin{cases} 0 & t \le 0 \\ \alpha u_1(t) + \beta u_2(t) & t > 0 \end{cases}$$

$$\alpha y_1 + \beta y_2 = \begin{cases} \alpha(0) + \beta(0) & t \le 0 \\ \alpha u_1(t) + \beta u_2(t) & t > 0 \end{cases} = \begin{cases} 0 & t \le 0 \\ \alpha u_1(t) + \beta u_2(t) & t > 0 \end{cases}$$

$\Rightarrow \alpha y_1 + \beta y_2 = y_3 \Rightarrow$ The system is linear.

For time variance,
for an input delayed by τ, we get

$$y'(t) = \begin{cases} 0 & t+\tau \le 0 \text{ i.e. } t \le -\tau \\ u(t+\tau) & t+\tau > 0 \text{ i.e. } t > -\tau \end{cases}$$

When the system is shifted by the same τ, we get

$$y(t+\tau) = \begin{cases} 0 & t+\tau \leq 0 \text{ i.e. } t \leq -\tau \\ u(t+\tau) & t+\tau > 0 \text{ i.e. } t > -\tau \end{cases}$$

We can see that $y'(t)$ and $y(t+\tau)$ are equivalent. Thus, the system is time invariant.

Exercise 2

1. $p_i(k+1) = p_i(k)(\alpha\gamma / S_i(k))$

Here $S_i(k)$ is the SINR value at receiver i.

The noise plus interference power at receiver i is given by: $q_i = \sigma^2 + \sum_{j \neq i} G_{ij} p_j$

And the signal power at receiver i is given by: $s_i = G_{ii} p_i$

$\Rightarrow$ $S_i(k)$ can be written as:

$$S_i(k) = s_i / q_i = G_{ii}\, p_i(k) \Big/ \Big(\sigma^2 + \sum_{i \neq j} G_{ij} p_j\Big)$$

$$\Rightarrow p_i(k+1) = \frac{\cancel{p_i(k)}\,(\alpha\gamma)}{G_{ii}\,\cancel{p_i(k)}} \times \Big(\sigma^2 + \sum_{i \neq j} G_{ij} p_j\Big)$$

$$= \frac{\alpha\gamma}{G_{ii}} \Big(\sigma^2 + \sum G_{ij} p_j\Big)$$

$$p_i(k+1) = \frac{\alpha\gamma\sigma^2}{G_{ii}} + \frac{\sum_{i \neq j} G_{ij} p_j}{G_{ii}}$$

This can be written as:

$$p_1(k+1) = \frac{\alpha\gamma\sigma^2}{G_{11}} + \frac{(G_{12} p_2 + G_{13} p_3)}{G_{11}}$$

$$p_2(k+1) = \frac{\alpha\gamma\sigma^2}{G_{22}} + \frac{(G_{23} p_3 + G_{21} p_1)}{G_{22}}$$

$$p_3(k+1) = \frac{\alpha\gamma\sigma^2}{G_{33}} + \frac{(G_{31} p_1 + G_{32} p_2)}{G_{33}}$$

p_1, p_2, p_3 refer to $p_1(k), p_2(k), p_3(k)$

Writing it in matrix form:

$$\begin{bmatrix} p_1(k+1) \\ p_2(k+1) \\ p_3(k+1) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & \frac{G_{12}}{G_{11}} & \frac{G_{13}}{G_{11}} \\ \frac{G_{21}}{G_{22}} & 0 & \frac{G_{23}}{G_{22}} \\ \frac{G_{31}}{G_{33}} & \frac{G_{32}}{G_{33}} & 0 \end{bmatrix}}_{A} \begin{bmatrix} p_1(k) \\ p_2(k) \\ p_3(k) \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{\alpha\gamma}{G_{11}} \\ \frac{\alpha\gamma}{G_{22}} \\ \frac{\alpha\gamma}{G_{33}} \end{bmatrix}}_{B} \sigma^2$$

---

2. Given $G = \begin{bmatrix} 1 & .2 & .1 \\ .1 & 2 & .1 \\ .3 & .1 & 3 \end{bmatrix}$, $\gamma = 3$, $\alpha = 1.2$, $\sigma = 0.1$

$$\begin{bmatrix} p_1(k+1) \\ p_2(k+1) \\ p_3(k+1) \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & \frac{G_{12}}{G_{11}} & \frac{G_{13}}{G_{11}} \\ \frac{G_{21}}{G_{22}} & 0 & \frac{G_{23}}{G_{22}} \\ \frac{G_{31}}{G_{33}} & \frac{G_{32}}{G_{33}} & 0 \end{bmatrix}}^{G'} \begin{bmatrix} p_1(k) \\ p_2(k) \\ p_3(k) \end{bmatrix} + \overbrace{\begin{bmatrix} \frac{\alpha\gamma}{G_{11}} \\ \frac{\alpha\gamma}{G_{22}} \\ \frac{\alpha\gamma}{G_{33}} \end{bmatrix}}^{B'} \underbrace{\sigma^2}_{u}$$

$$= \begin{bmatrix} 0 & 0.2 & 0.1 \\ 0.05 & 0 & 0.05 \\ 0.1 & 0.1/3 & 0 \end{bmatrix} \begin{bmatrix} p_1(k) \\ p_2(k) \\ p_3(k) \end{bmatrix} + \begin{bmatrix} 3.6 \\ 1.8 \\ 1.2 \end{bmatrix} (0.01)$$

For $\gamma = 5$, the matrix B looks like: $\begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$

For $\gamma = 5$

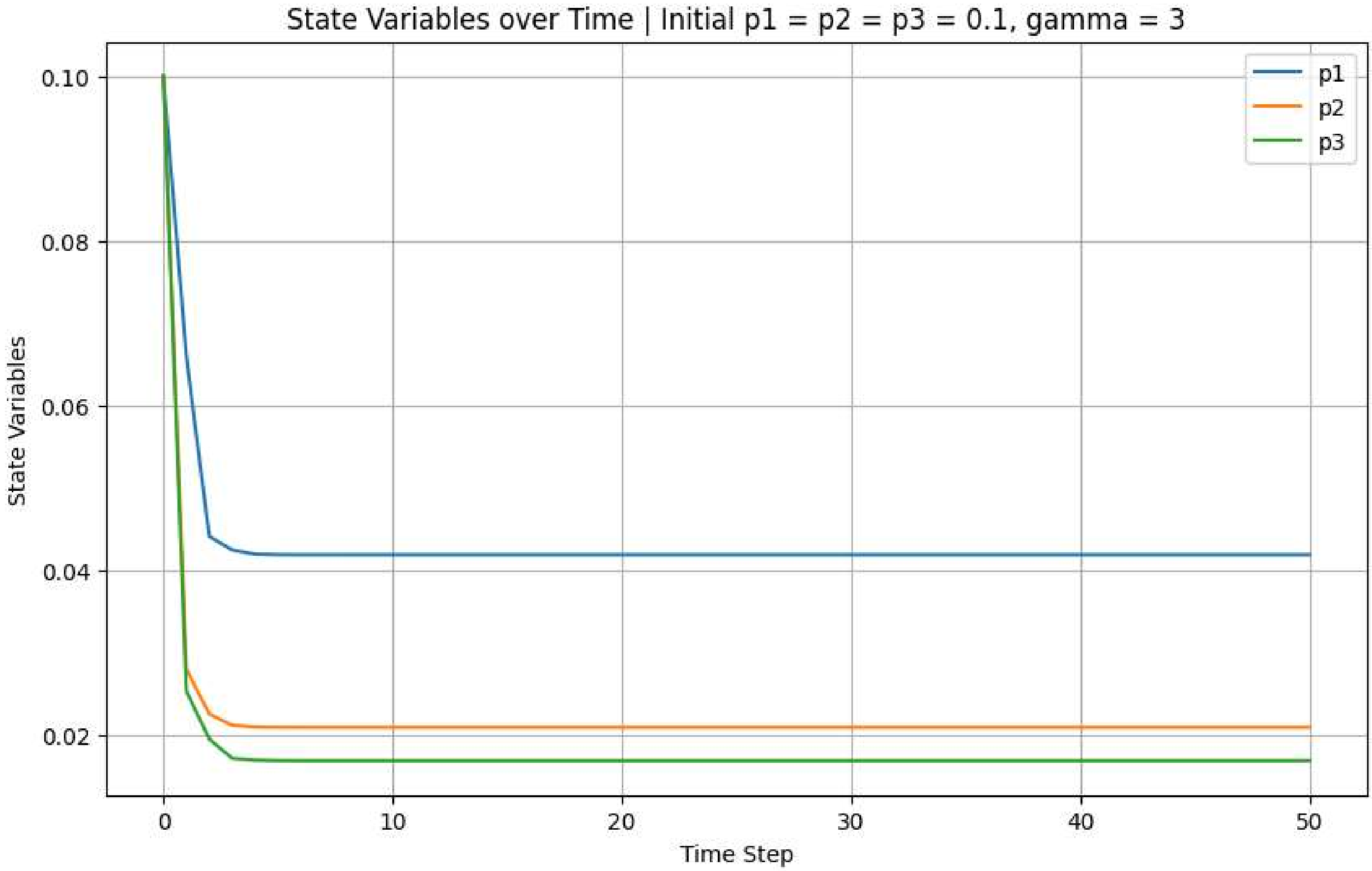
$$\Rightarrow p(k+1) = \begin{bmatrix} 0 & 0.2 & 0.1 \\ 0.05 & 0 & 0.05 \\ 0.1 & 0.1/3 & 0 \end{bmatrix} \begin{bmatrix} p_1(k) \\ p_2(k) \\ p_3(k) \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix} (0.01)$$

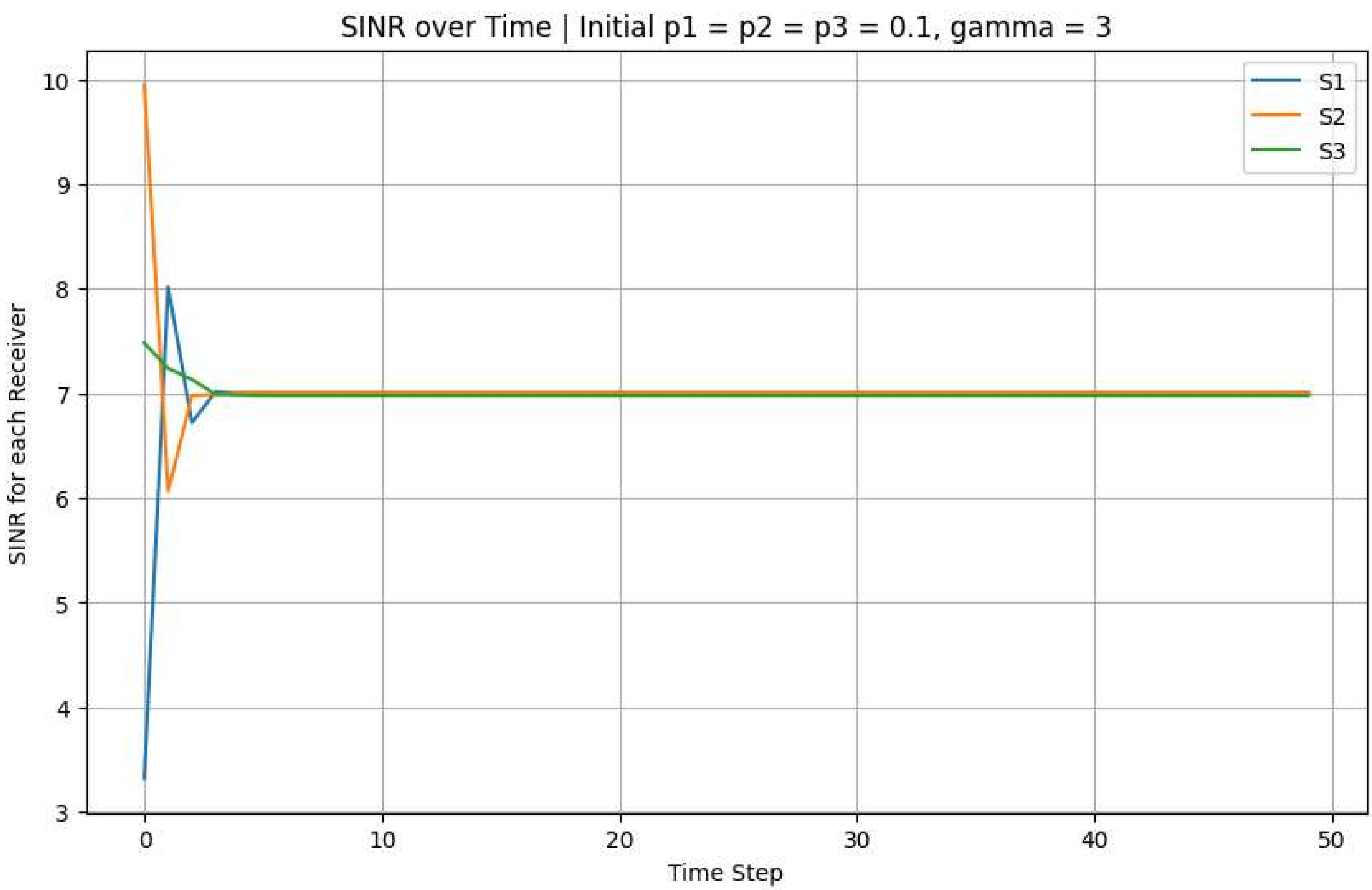
Python code and plots are attached below:

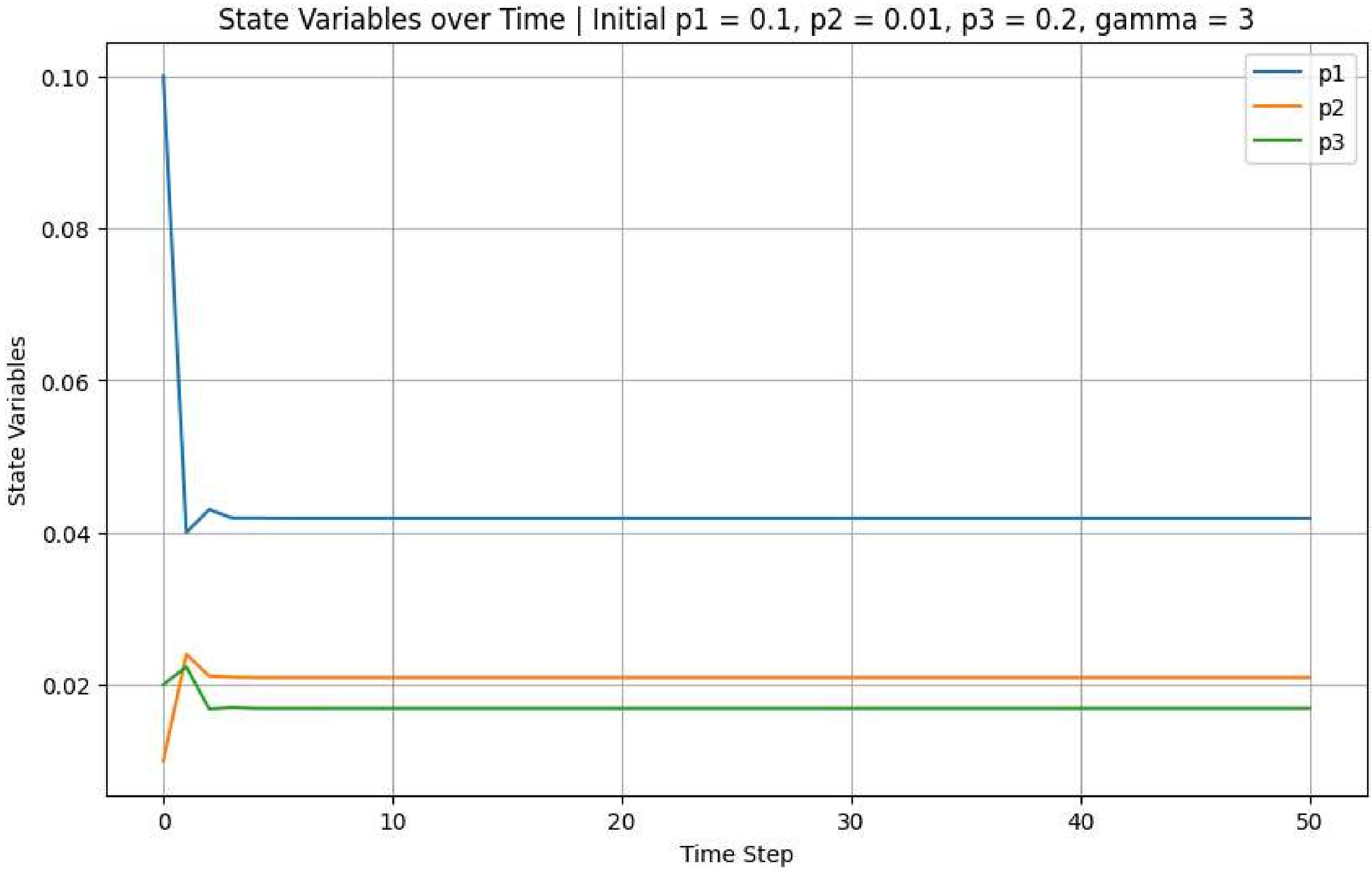
From the plots, we can see that controller struggles to take $S_i(t)$ to $\alpha\gamma$ when the γ is low.

However, when the value of γ is increased, the S_i values are much closer to the target $\alpha\gamma$ values.

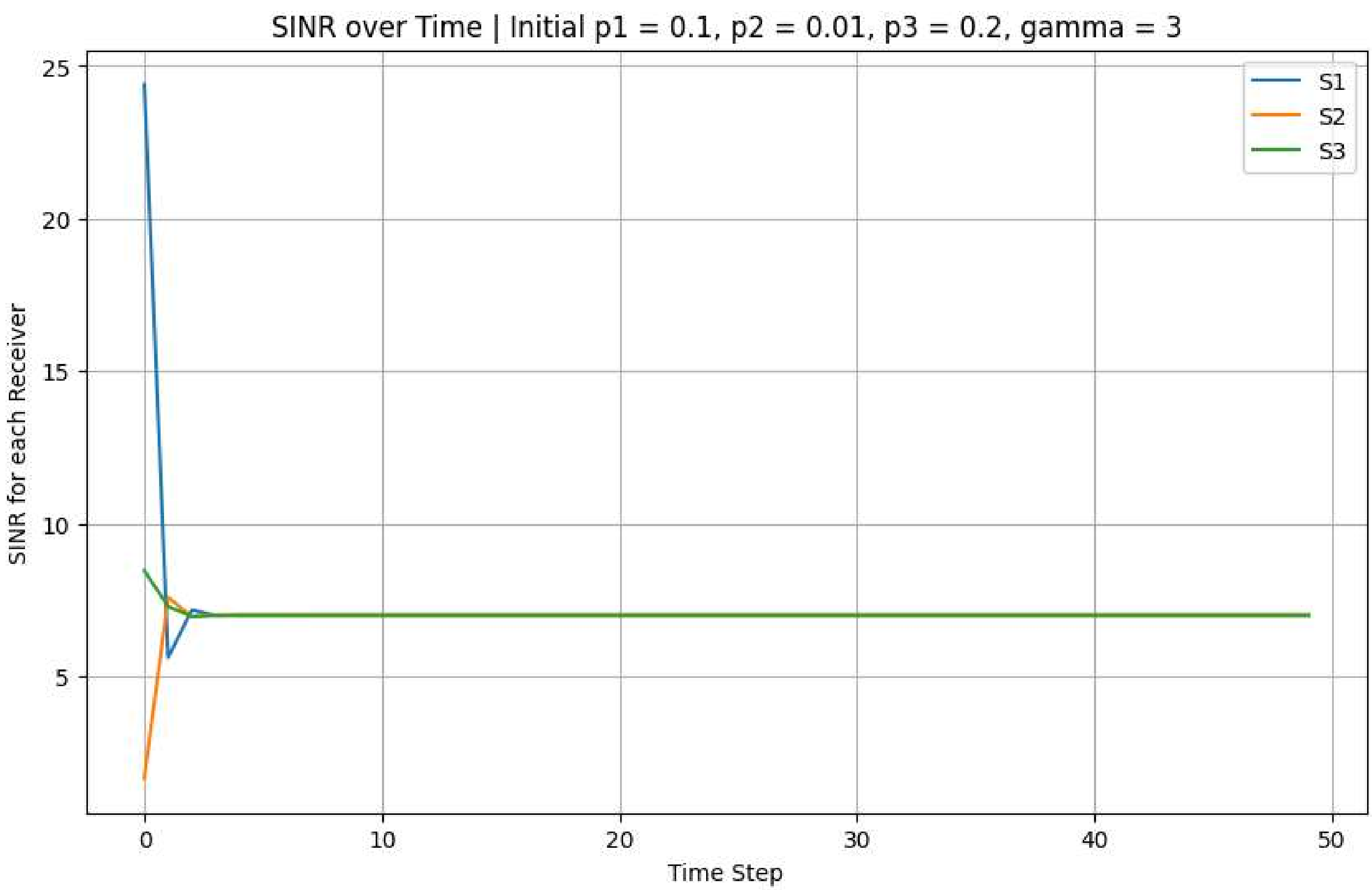
In [39]:
```
import numpy as np
import matplotlib.pyplot as plt

# Defining system parameters
G = np.array([[1, 0.2, 0.1],
              [0.1, 2, 0.1],
              [0.3, 0.1, 3]])
G_prime = np.array([[0, G[0, 1]/G[0, 0], G[0, 2]/G[0, 0]],
                     [G[1, 0]/G[1, 1], 0, G[1, 2]/G[1, 1]],
                     [G[2, 0]/G[2, 2], G[2, 1]/G[2, 2], 0]])
B = np.array([3.6, 1.8, 1.2])
u = 0.01

# Defining simulation parameters (states and other quantities of interest)
num_steps = 50
p = np.zeros((3, num_steps + 1))  # Extra index accounting for initial state
s = np.zeros((3, num_steps))
q = np.zeros((3, num_steps))
S = np.zeros((3, num_steps))

# Initial condition
p[:, 0] = np.array([0.1, 0.1, 0.1])

# Simulating the system
for k in range(num_steps):
    p[:, k + 1] = G_prime @ p[:, k] + B * u  # Updating p value for each step
    s[:, k] = np.array([G[0, 0] * p[0, k],  # s - signal power
                        G[1, 1] * p[1, k],
                        G[2, 2] * p[2, k]])
    q[:, k] = u**2 + np.array([G[0, 1] * p[1, k] + G[0, 2] * p[2, k],  # q - noise plus interfere
                        G[1, 0] * p[0, k] + G[1, 2] * p[2, k],
                        G[2, 0] * p[2, k] + G[2, 1] * p[1, k]])
    S[:, k] = np.divide(s[:, k], q[:, k])  # SINR


# Plotting the values of p
time = np.arange(num_steps + 1)
plt.figure(figsize=(10, 6))
plt.plot(time, p[0, :], label='p1')
plt.plot(time, p[1, :], label='p2')
plt.plot(time, p[2, :], label='p3')
plt.xlabel('Time Step')
plt.ylabel('State Variables')
plt.title('State Variables over Time | Initial p1 = p2 = p3 = 0.1, gamma = 3')
plt.legend()
plt.grid(True)
plt.show()

# Plotting the values of S
plt.figure(figsize=(10, 6))
plt.plot(time[0: -1], S[0, :], label='S1')
plt.plot(time[0: -1], S[1, :], label='S2')
plt.plot(time[0: -1], S[2, :], label='S3')
plt.xlabel('Time Step')
plt.ylabel('SINR for each Receiver')
plt.title('SINR over Time | Initial p1 = p2 = p3 = 0.1, gamma = 3')
plt.legend()
plt.grid(True)
plt.show()
```


State Variables over Time | Initial p1 = p2 = p3 = 0.1, gamma = 3
State Variables
Time Step
p1
p2
p3
0.10
0.08
0.06
0.04
0.02
0
10
20
30
40
50

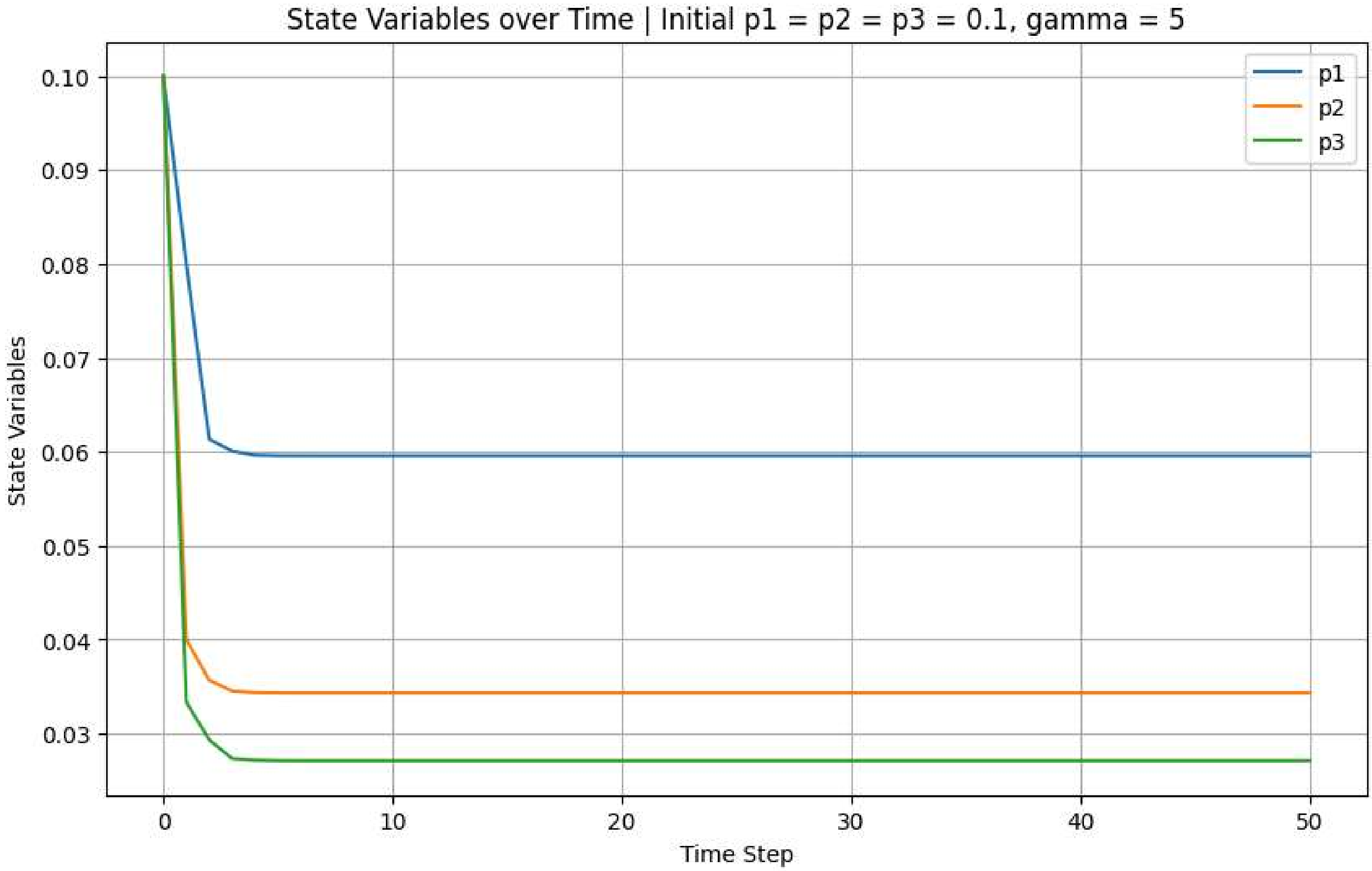

SINR over Time | Initial p1 = p2 = p3 = 0.1, gamma = 3
SINR for each Receiver
Time Step
S1
S2
S3
10
9
8
7
6
5
4
3
0
10
20
30
40
50

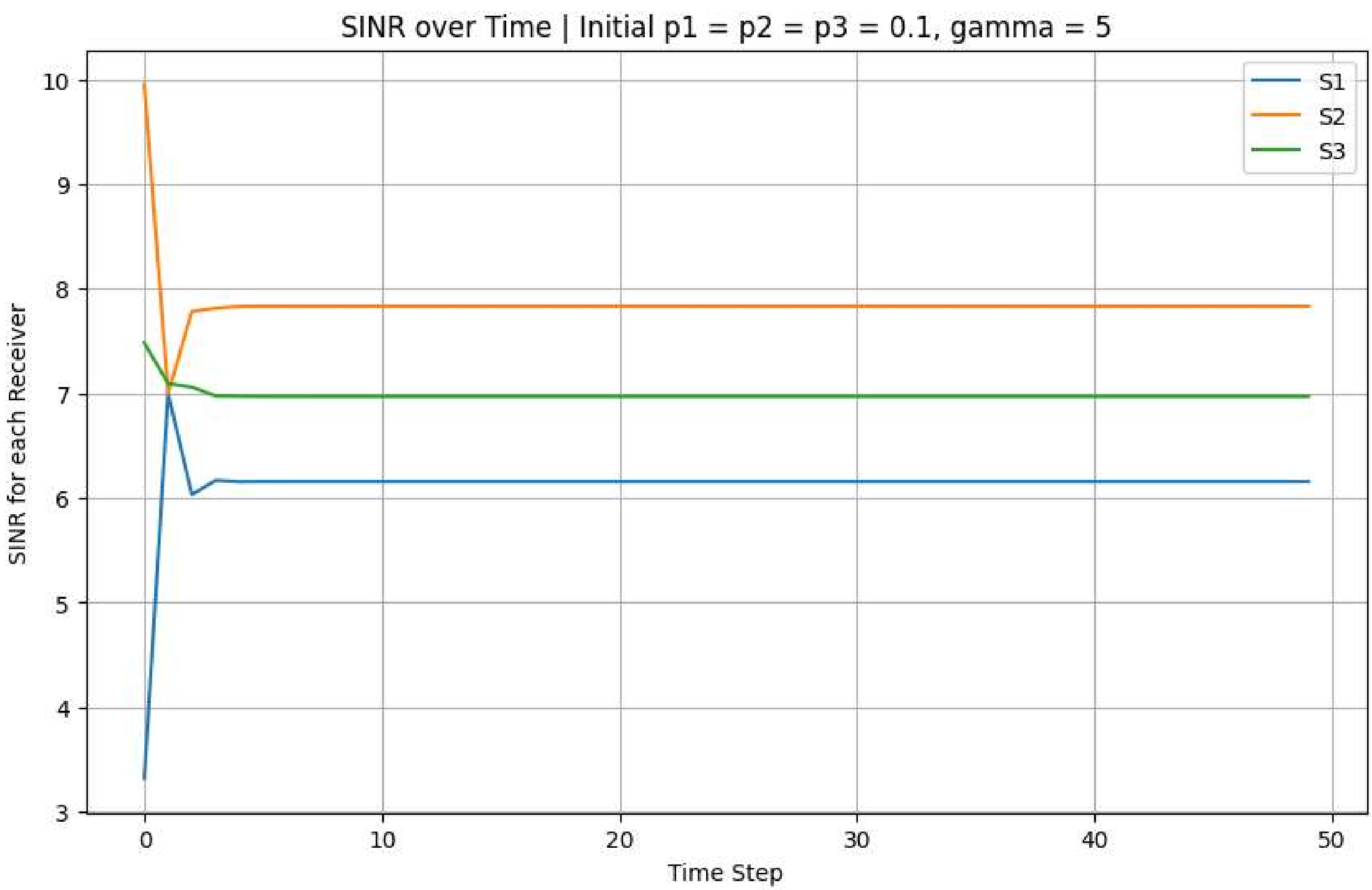


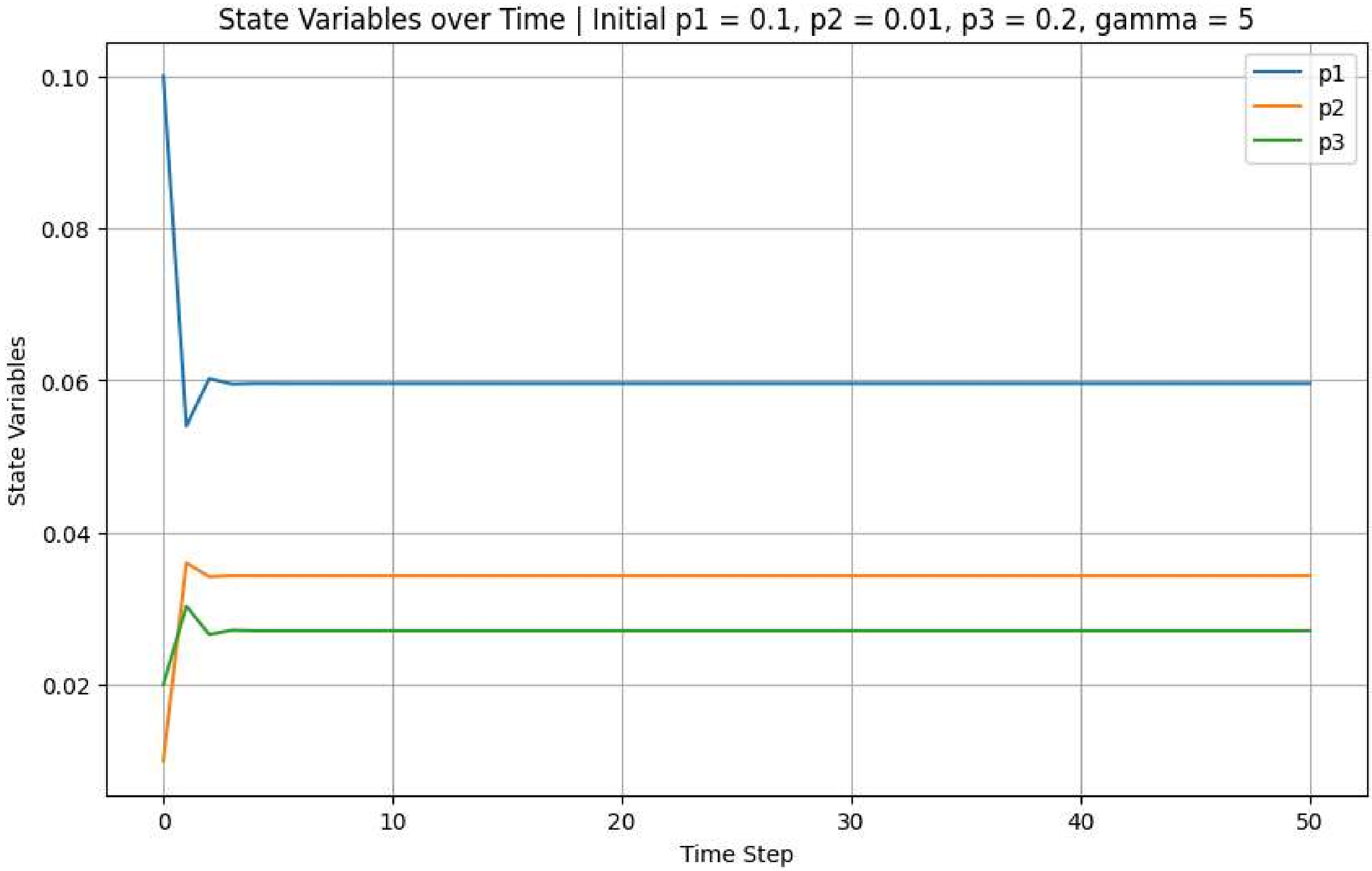
State Variables over Time | Initial p1 = 0.1, p2 = 0.01, p3 = 0.2, gamma = 3
p1
p2
p3
0.10
0.08
0.06
0.04
0.02
State Variables
0
10
20
30
40
50
Time Step

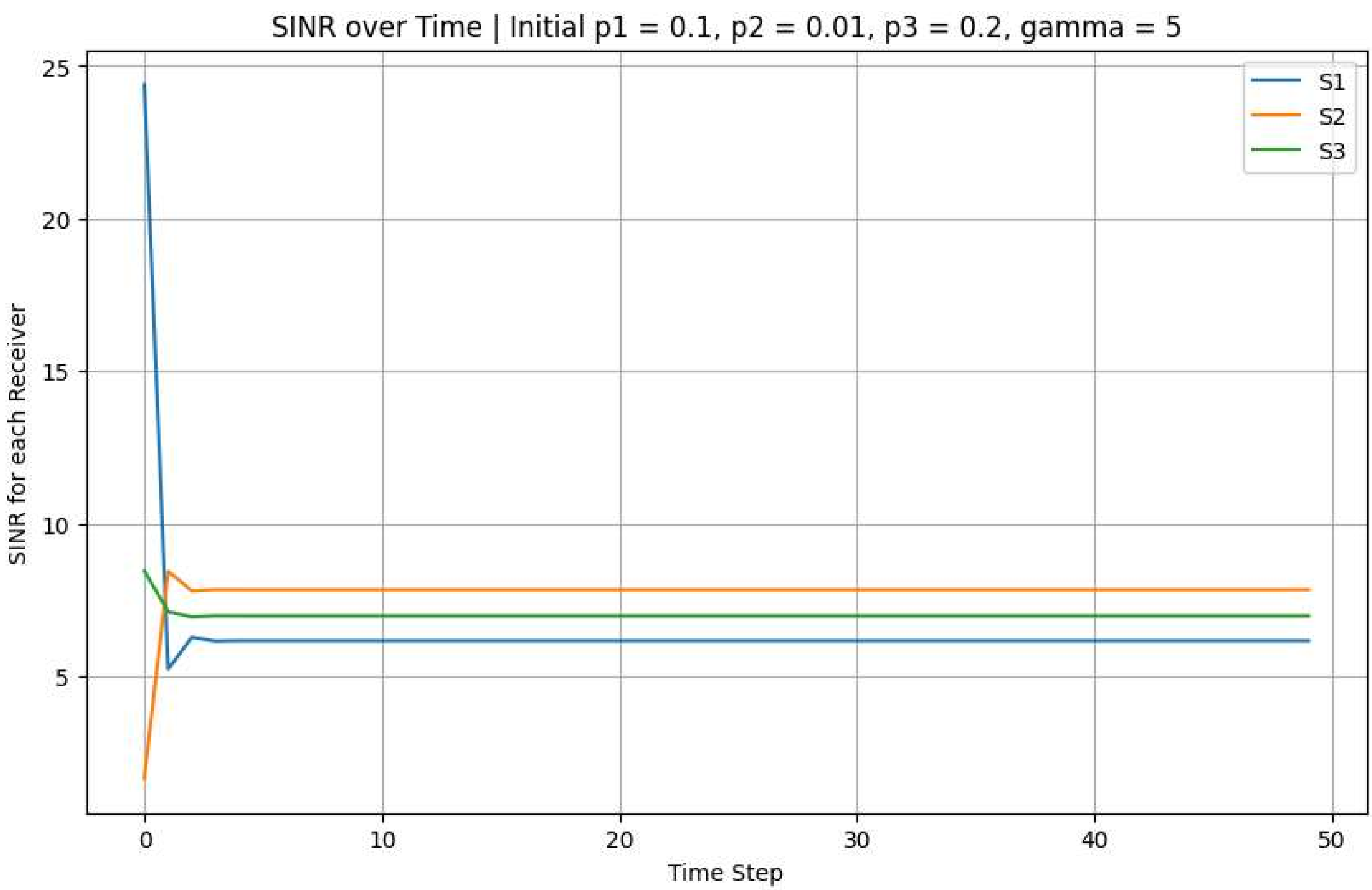


SINR over Time | Initial p1 = 0.1, p2 = 0.01, p3 = 0.2, gamma = 3
S1
S2
S3
25
20
15
10
5
SINR for each Receiver
0
10
20
30
40
50
Time Step


State Variables over Time | Initial p1 = p2 = p3 = 0.1, gamma = 5
State Variables
Time Step
p1
p2
p3
0.10
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0
10
20
30
40
50


SINR over Time | Initial p1 = p2 = p3 = 0.1, gamma = 5
SINR for each Receiver
Time Step
S1
S2
S3
10
9
8
7
6
5
4
3
0
10
20
30
40
50


State Variables over Time | Initial p1 = 0.1, p2 = 0.01, p3 = 0.2, gamma = 5
p1
p2
p3
State Variables
0.10
0.08
0.06
0.04
0.02
0
10
20
30
40
50
Time Step


SINR over Time | Initial p1 = 0.1, p2 = 0.01, p3 = 0.2, gamma = 5
S1
S2
S3
SINR for each Receiver
25
20
15
10
5
0
10
20
30
40
50
Time Step

Exercise 3

$$\ddot{y} + (1+y)\dot{y} - 2y + 0.5y^3 = 0$$

Firstly, finding equilibrium point : $\dot{y} = \ddot{y} = 0$

$$\Rightarrow 0 + (1+y)0 - 2y + 0.5y^3 = 0$$

$$\Rightarrow y^3 - 4y = 0 \Rightarrow y(y+2)(y-2) = 0$$

$\therefore$ We have equilibrium points at $y = -2, 0, 2$.

For linearizing, taking $y = \underbrace{\bar{y}}_{\text{equilibrium point}} + \delta y \rightarrow$ small pertrubration

$$(\ddot{\bar{y}} + \delta\ddot{y}) + (1 + (\bar{y} + \delta y))(\dot{\bar{y}} + \delta\dot{y}) - 2(\bar{y} + \delta y) + 0.5(\bar{y} + \delta y)^3 = 0$$

i) For $\bar{y} = 0$, we get :

$$\delta\ddot{y} + (1 + \delta y)(\delta\dot{y}) - 2\delta y + 0.5(\delta y)^3 = 0$$

Ignoring higher-order terms since they are small, we get

$$\left.\delta\ddot{y} + \delta\dot{y} - 2\delta y = 0\right\}$$ linearized system around $y = 0$.

ii) For $\bar{y} = -2$, we get $y = -2 + \delta y$

$$(\ddot{-2} + \delta\ddot{y}) + (1 + (-2 + \delta y))(\dot{-2} + \delta\dot{y}) - 2(-2 + \delta y) + 0.5(-2 + \delta y)^3 = 0$$

$$\delta\ddot{y} + (\delta y - 1)(\delta\dot{y}) + 4 - 2\delta y + 0.5(-8 + 12\delta y) = 0$$

$\hookrightarrow$ Ignoring higher-order terms and simplifying

$\delta\ddot{y} - \delta\dot{y} + 4 - 2\delta y - 4 + 6\delta y = 0$

$\Rightarrow \delta\ddot{y} - \delta\dot{y} + 4\delta y = 0$ } linearized system around $y = -2$

iii) For $\bar{y} = 2$, we have $y = 2 + \delta y$

$(2+\ddot{\delta y}) + (1+(2+\delta y))(2+\dot{\delta y}) - 2(2+\delta y) + 0.5(2+\delta y)^3 = 0$

$\delta\ddot{y} + (3+\delta y)(\delta\dot{y}) - 4 - 2\delta y + 0.5(8 + 12\delta y) = 0$

$\delta\ddot{y} + 3\delta\dot{y} + \cancel{\delta y\,\delta\dot{y}}^{\,0} - \cancel{4} - 2\delta y + \cancel{4} + 6\delta y = 0$

$\Rightarrow \delta\ddot{y} + 3\delta\dot{y} + 4\delta y = 0$ } System linearized around $y = 2$.

Alternate solution at the end...

Exercise 4

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -g\left(\frac{D}{x_1(t)+D}\right)^2 + \frac{\ln(u)}{m} \end{bmatrix}$$

D, m, g, u are constant for short time periods.

To find equilibrium states, we set $\dot{x}_1 = \dot{x}_2 = 0$

$\Rightarrow \dot{x}_1(t') = 0 \Rightarrow x_2^* = 0$

$x_2^* = 0 \Rightarrow -g\left(\frac{D}{x_1(t')+D}\right)^2 + \frac{\ln(u)}{m} = 0$

$\Rightarrow \frac{\ln(u)}{m} = gD^2 / (x_1(t') + D)^2$

Assuming $x_1^* + D \neq 0$, we get

$$(x_1^* + D)^2 \ln(u) = g D^2 m$$

$$\Rightarrow \quad (x_1^* + D)\sqrt{\ln(u)} = \pm D\sqrt{gm}$$

$$x_1^* = \frac{\pm D\sqrt{gm}}{\sqrt{\ln(u)}} - D$$

$$x_1^* = D\left(-1 \pm \sqrt{\frac{gm}{\ln u}}\right)$$

$\Rightarrow$ The equilibrium points are:

$$\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} D\left(-1 \pm \sqrt{\frac{gm}{\ln u}}\right) \\ 0 \end{bmatrix}$$

Performing linearization around the equilibrium point:

If the system is given as $\dot{x} = f(x,u)$, the linearized form looks like:

$$\delta\dot{x} = \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x},\, u=\bar{u}} \delta x + \left.\frac{\partial f}{\partial u}\right|_{x=\bar{x},\, u=\bar{u}} \delta u$$

Here, since u is almost constant, $\delta u \simeq 0$

$$\Rightarrow \quad \delta\dot{x} = \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}} \delta x$$

$$f(x) = \begin{bmatrix} x_2(t) \\ -g\left(\frac{D}{x_1(t)+D}\right)^2 + \frac{\ln(u)}{m} \end{bmatrix} \quad ; \quad x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -2g\left(\frac{D}{x_1(t)+D}\right) \times \frac{(-D)}{(x_1(t)+D)^2} \times 1 & 0 \end{bmatrix}$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ 2gD^2/(x_1(t)+D)^3 & 0 \end{bmatrix}$$

Now substituting the equilibrium point in the Jacobian,

$$x_1^* = D\left(-1 \pm \sqrt{\frac{gm}{\ln(u)}}\right)$$

Case 1 : $x_1^* = D\left(-1 + \sqrt{\frac{gm}{\ln(u)}}\right)$

$$x_1^* + D = D\sqrt{\frac{gm}{\ln(u)}}$$

$$\Rightarrow \quad 2gD^2/(x_1^*+D)^3 = \frac{2gD^2}{D^3 \left(\frac{gm}{\ln(u)}\right)^{3/2}} = \frac{2\sqrt{(\ln u)^3}}{D\sqrt{gm^3}}$$

$$\Rightarrow \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ \frac{2}{D}\frac{\sqrt{(\ln u)^3}}{\sqrt{gm^3}} & 0 \end{bmatrix}$$

$$\Rightarrow \quad \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{2}{D}\frac{\sqrt{(\ln u)^3}}{\sqrt{gm^3}} & 0 \end{bmatrix} \begin{bmatrix} \partial x_1 \\ \partial x_2 \end{bmatrix}$$

Case 2 : $x_1^* = -D\left(1 + \sqrt{\frac{gm}{\ln(u)}}\right)$

$$x_1^* + D = -D\sqrt{\frac{gm}{\ln(u)}}$$

$$\Rightarrow \frac{2gD^2}{(x_1^*+D)^3} = \frac{2gD^2\sqrt{(\ln u)^3}}{-D^3\sqrt{g^3m^3}} = \frac{-2}{D}\frac{\sqrt{(\ln u)^3}}{\sqrt{gm^3}}$$

$$\Rightarrow \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ \frac{-2}{D}\frac{\sqrt{(\ln u)^3}}{\sqrt{gm^3}} & 0 \end{bmatrix}$$

So, $\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \frac{\partial f}{\partial x}\,\delta x = \begin{bmatrix} 0 & 1 \\ \frac{-2}{D}\frac{\sqrt{(\ln u)^3}}{\sqrt{gm^3}} & 0 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix}$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-2}{D}\frac{\sqrt{(\ln u)^3}}{\sqrt{gm^3}} & 0 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix}$$

Exercise 5

$$\ddot{r} = r\dot{\theta}^2 - k/r^2 + u_1$$
$$\ddot{\theta} = -\frac{2\dot{\theta}\dot{r}}{r} + \frac{u_2}{r}$$

i) For the reference orbit, $u_1 = u_2 = 0$ and $r(t) \equiv p$ and $\theta(t) = \omega t$

$$\dot{r}(t) = \ddot{r}(t) = 0, \quad \dot{\theta}(t) = \omega; \ \ddot{\theta}(t) = 0$$

$\ddot{r} = r\dot{\theta}^2 - k/r^2 + u_1$ becomes $0 = p\omega^2 - k/p^2 + 0$

$$\Rightarrow k = p^3\omega^2$$

ii) Linearizing about a trajectory, we get

$\dot{r} = \ddot{r} = 0, \ \ddot{\theta} = 0, \ \dot{\theta} = \omega$

Also, equilibrium conditions $\Rightarrow$ $u_1 = u_2 = 0, \ r = p, \ k = p^3\omega^2$

Choosing the states to be $x = \begin{bmatrix} \dot{r} \\ r \\ \dot{\theta} \\ \theta \end{bmatrix}$, we get

$$\dot{x} = \begin{bmatrix} \ddot{r} \\ \dot{r} \\ \ddot{\theta} \\ \dot{\theta} \end{bmatrix} = f(x,u)$$

$$f(x,u) = \begin{bmatrix} r\dot{\theta}^2 - k/r^2 + u_1 \\ \dot{r} \\ -\frac{2\dot{\theta}\dot{r}}{r} + \frac{u_2}{r} \\ \dot{\theta} \end{bmatrix}$$

The system linearized about the equilibrium point can be represented as:

$$\delta \dot{x} = \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x},\, u=\bar{u}} \delta x + \left.\frac{\partial f}{\partial u}\right|_{x=\bar{x},\, u=\bar{u}} \delta u$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_4} \\ \vdots & \ddots & \\ \frac{\partial f_4}{\partial x_1} & \cdots & \frac{\partial f_4}{\partial x_4} \end{bmatrix} = \begin{bmatrix} 0 & \dot{\theta}^2 + \frac{2k}{r^3} & 2\dot{\theta} r & 0 \\ 1 & 0 & 0 & 0 \\ -\frac{2\dot{\theta}}{r} & \frac{2\dot{\theta}\dot{r} - u_2}{r^2} & -\frac{2\dot{r}}{r} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{x=\bar{x},\, u=\bar{u}} = \begin{bmatrix} 0 & 3\omega^2 & 2p\omega & 0 \\ 1 & 0 & 0 & 0 \\ \frac{-2\omega}{p} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\frac{\partial f}{\partial u} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \vdots & \vdots \\ \frac{\partial f_4}{\partial u_1} & \frac{\partial f_4}{\partial u_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1/r \\ 0 & 0 \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial u}\right|_{x=\bar{x},\, u=\bar{u}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1/p \\ 0 & 0 \end{bmatrix}$$

Now, $\delta \dot{x} = \left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}, u=\bar{u}} \delta x + \left.\frac{\partial f}{\partial u}\right|_{x=\bar{x}, u=\bar{u}} \delta u$

$$\delta \dot{x} = \begin{bmatrix} 0 & 3\omega^2 & 2\omega p & 0 \\ 1 & 0 & 0 & 0 \\ -2\omega/p & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \delta x + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1/r \\ 0 & 0 \end{bmatrix} \delta u$$

Thus, the linearized system looks like:

$$\boxed{\begin{bmatrix} \ddot{r} \\ \dot{r} \\ \ddot{\theta} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 3\omega^2 & 2p\omega & 0 \\ 1 & 0 & 0 & 0 \\ -2\omega/p & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{r} \\ r \\ \dot{\theta} \\ \theta \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1/r \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}$$

<u>Exercise 3</u> (alternate solution)

$\ddot{y} + (1+y)\dot{y} - 2y + 0.5y^3 = 0$

$\ddot{y} = -0.5y^3 + 2y - (1+y)\dot{y}$

Defining states as $\begin{bmatrix} y \\ \dot{y} \end{bmatrix} \equiv \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

We can write the system equations as: $\dot{y} = f(y)$, given by

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ -0.5y_1^3 + 2y_1 - (1+y_1)y_2 \end{bmatrix}$$

To find the equilibrium point, setting $\dot{y} = \ddot{y} = 0$

So, $\ddot{y} + (1+y)\dot{y} - 2y + 0.5y^3 \equiv 0 + (1+y)0 - 2y + 0.5y^3 = 0$

$\Rightarrow \quad y = 0, -2, 2 \}$ equilibrium points.

Since there is no external input, $u = 0$. Thus linearization looks like:

$$\frac{\partial f}{\partial y} = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1.5y_1^2 + 2 - y_2 & -(1+y_1) \end{bmatrix}$$

Linearized system given by:

$$\delta\dot{y} = \left.\frac{\partial f}{\partial y}\right|_{y=\bar{y}} \delta y$$

i) $\bar{y} = 0 \Rightarrow y_1 = y = 0, \; y_2 = \dot{y} = 0$

$$\Rightarrow \delta\dot{y} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} \delta y_1 \\ \delta y_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} \delta y_2 \\ 2\delta y_1 - \delta y_2 \end{bmatrix} \rightarrow \delta\dot{y}_2 = 2\delta y_1 - \delta y_2$$

$\Rightarrow \ddot{y} = 2y - \dot{y} \;\}$ linearized system around $y=0$.

ii) $\bar{y} = 2 \Rightarrow y_1 = y = 2, \; y_2 = \dot{y} = 0$

$$\Rightarrow \delta\dot{y} = \begin{bmatrix} 0 & 1 \\ -4 & -3 \end{bmatrix} \begin{bmatrix} \delta y_1 \\ \delta y_2 \end{bmatrix} \rightarrow \delta\dot{y}_2 = -4\delta y_1 - 3\delta y_2$$

$\Rightarrow \ddot{y} = -4y - 3\dot{y} \;\}$ linearized system around $y=2$

iii) $\bar{y} = -2 \Rightarrow y_1 = -2, \; y_2 = 0$

$$\Rightarrow \delta\dot{y} = \begin{bmatrix} 0 & 1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} \delta y_1 \\ \delta y_2 \end{bmatrix} \rightarrow \delta\dot{y}_2 = -4\delta y_1 + \delta y_2$$

$\Rightarrow \ddot{y} = -4y + \dot{y} \;\}$ linearized system around $y = -2$