

### Exercise 1

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = [1 \ 2 \ 1] x$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix}; B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; C = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$

For controllability:

$$\text{rank}(P) = n, \text{ where } P = [B \ AB \ A^2B]$$

$$A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & -3 & -3 \\ 3 & 8 & 6 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & 0 & 1 \\ -1 & -3 & -3 \\ 3 & 8 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$$

$$\rightarrow P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\text{rank}(P) = 2 < 3 = n$$

So the system is not fully controllable.

For observability:

$$\text{rank}(Q) = n, \text{ where } Q = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$$

$$CA = [1 \ 2 \ 1] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} \\ = [-1 \ -2 \ -1]$$

$$CA^2 = [1 \ 2 \ 1] \begin{bmatrix} 0 & 0 & 1 \\ -1 & -3 & -3 \\ 3 & 8 & 6 \end{bmatrix} = [1 \ 2 \ 1]$$

$$Q = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 1 & 2 & 1 \end{bmatrix} \quad (R_3 = R_1 = -2R_2)$$

$$\text{rank}(Q) = 1$$

so the system is not observable.

## Exercise 2

$$\dot{x} = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 2 & 2 & 1 & 3 & -1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} x$$

To check for controllability:

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & | & 2 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & | & 2 & 1 & 1 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & | & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & | & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & | & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & | & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & | & 1 & 0 & 0 \end{bmatrix}$$

Forming Jordan blocks and checking rank of the matrices formed from  $\hat{B}$ 's for the corresponding lowest rows.

$$\hat{B}^2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & ; \\ 3 & 2 & ; \end{bmatrix}$$

$$\text{rank}(\hat{B}^2) = 3$$

$$\hat{B}' = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\text{rank}(\hat{B}') = 2$$

Since they are both full rank, we have full controllability.

For observability:

$$\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & | & 2 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & | & 2 & 1 & 1 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & | & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & | & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & | & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & | & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & | & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 & 1 & 3 & -1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\hat{C}^2 = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \text{rank}(\hat{C}^2) = 2 \Rightarrow \text{not full rank}$$

$$\hat{C}' = \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \rightarrow \text{rank}(\hat{C}') = 2 \Rightarrow \text{full rank}$$

Since  $\hat{C}^2$  is not full rank. The corresponding states are not fully observable.

### Exercise 3

$$\frac{du_1}{dt} = -\alpha u_1 + u$$

$$\frac{du_2}{dt} = \alpha u_1 - \beta u_2$$

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \begin{bmatrix} -\alpha & 0 \\ \alpha & -\beta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$A = \begin{bmatrix} -\alpha & 0 \\ \alpha & -\beta \end{bmatrix}; B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

For controllability,  $\text{rank}(P) = 2$

$$\text{where } P = [B \ AB]$$

$$AB = \begin{bmatrix} -\alpha & 0 \\ \alpha & -\beta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\alpha \\ \alpha \end{bmatrix}$$

$$\Rightarrow P = \begin{bmatrix} 1 & -\alpha \\ 0 & \alpha \end{bmatrix} \rightarrow \text{rank}(P) = 2 \quad \text{if } \alpha \neq 0$$

So for nonzero  $\alpha$ , the system is controllable.

Modified system:

$$\frac{du_1}{dt} = -\alpha u_1$$

$$\frac{du_2}{dt} = \alpha u_1 - \beta u_2 + u$$

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{bmatrix} = \begin{bmatrix} -\alpha & 0 \\ \alpha & -\beta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

so  $A$  is still the same  $A = \begin{bmatrix} -\alpha & 0 \\ \alpha & -\beta \end{bmatrix}$

$$B \text{ is now: } B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\text{so } AB = \begin{bmatrix} -\alpha & 0 \\ \alpha & -\beta \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -\beta \end{bmatrix}$$

$$\Rightarrow P = [B \ AB] = \begin{bmatrix} 0 & 0 \\ 1 & -\beta \end{bmatrix}$$

$$\text{rank}(P) = 1$$

So system is not controllable anymore.

### Exercise 4

1.

a)  $x + y + z = 3$   
 $x + 2y + 3z = 0$   
 $x + 3y + 2z = 3$

If  $a = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$  and  $b = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$ ,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix} a = b = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$$

To solve for  $a$ :  $\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 0 \\ 1 & 3 & 2 & 3 \end{array} \right]$  making the augmented matrix and performing row operations.

$$R_2 = R_2 - R_1 \Rightarrow \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & -3 \\ 1 & 3 & 2 & 3 \end{array} \right]$$

$$R_3 = R_3 - R_1 - 2R_2 \Rightarrow \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & -3 & 6 \end{array} \right]$$

so we get:  $-3z = 6$ ;  $y + 2z = -3$ ;  $x + y + z = 3$

$$\Rightarrow z = -2, y = 1, x = 4$$

b)  $x + 2y - z = 1$

$$2x + 5y - z = 3$$

$$x + 3y + 2z = 6$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & -1 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$$

Augmented matrix:  $\left[ \begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 5 & -1 & 3 \\ 1 & 3 & 2 & 6 \end{array} \right]$

$$R_2 = R_2 - 2R_1 \Rightarrow \left[ \begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 3 & 2 & 6 \end{array} \right]$$

$$R_3 = R_3 - R_1 - R_2 \Rightarrow \left[ \begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 4 \end{array} \right]$$

$$8, 2z = 4, y + z = 1, x + 2y - z = 1$$

$$\Rightarrow \boxed{z = 2, y = -1, x = 5}$$

$$c) n_1 + n_2 - n_3 + n_4 = 1$$

$$2n_1 + 3n_2 + n_3 - 0n_4 = 4$$

$$3n_1 + 5n_2 + 3n_3 - n_4 = 5$$

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & 3 & 1 & 0 \\ 3 & 5 & 3 & -1 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$$

Augmented matrix:

$$\left[ \begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 2 & 3 & 1 & 0 & 4 \\ 3 & 5 & 3 & -1 & 5 \end{array} \right]$$

$$R_2 = R_2 - 2R_1 \rightarrow \left[ \begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 3 & -2 & 2 \\ 3 & 5 & 3 & -1 & 5 \end{array} \right]$$

$$R_3 = R_3 - 3R_1 - 2R_2 \rightarrow \left[ \begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 3 & -2 & 2 \\ 0 & 0 & 0 & 0 & -2 \end{array} \right]$$

We can see that the last row says  $0 = -2$ , which is a contradiction.

2.

$$n_1 + 2n_2 + 4n_3 = 3$$

$$3n_1 + 8n_2 + 14n_3 = 13$$

$$2n_1 + 6n_2 + 13n_3 = 4$$

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$LU = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$$

$$\Rightarrow u_{11} = 1, u_{12} = 2, u_{13} = 4$$

$$l_{21} = 3/1 = 3, u_{22} = 8 - 3 \times 2 = 2, u_{23} = 14 - 3 \times 4 = 2$$

$$l_{31} = 2/1 = 2, l_{32} = \frac{6 - 2 \times 2}{2} = 1, u_{33} = 13 - 2 \times 4 - 1 \times 2 = 3$$

$$\Rightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Now to solve  $Ax = b$ , first we solve  $Ly = b$ , then  $Ux = y$

$$Ly = b \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$$

$$\Rightarrow y_1 = 3, y_2 = 10, y_3 = -12$$

Now,  $Ux = y$ :

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \\ -12 \end{bmatrix}$$

$$\Rightarrow x_3 = -4, \quad x_2 = \frac{10 - 2(-4)}{2} = 9, \quad x_1 = 3 - 2(9) - 4(-4) = 1$$

$$\Rightarrow \boxed{x_1 = 1, x_2 = 9, x_3 = -4}$$

### Exercise 6

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y(t) = [1 \ 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

1. controllable but not observable.

Constructing controllability matrix  $P$ :

$$AB = \begin{bmatrix} -3 & 3 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$

For  $P$  to be full rank, i.e.  $\text{rank}(P) = 2$ ,  $\gamma \neq 0$ .

So if  $\gamma \neq 0$ , system is controllable.

For observability, constructing observability matrix  $Q$ :

$$CA = [1 \ 1] \begin{bmatrix} -3 & 3 \\ 1 & -4 \end{bmatrix} = [(-3+\gamma) \ -1]$$

$$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ (-3+\gamma) & -1 \end{bmatrix}$$

For  $\text{rank}(Q) < 2$ ,  $(-3+\gamma) = -1 \Rightarrow \gamma = 2$

$$\gamma = 2 \Rightarrow Q(\gamma = 2) = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \rightarrow \text{rank}(Q(\gamma = 2)) = 1$$

so at  $\gamma = 2$ , system would not be observable.

And since for  $\gamma = 2$ ,  $\gamma \neq 0$ , system is controllable.

$\boxed{\gamma = 2 \rightarrow \text{controllable, not observable}}$

2. For observable, but not controllable,

$\text{rank}(P) < 2$  and  $\text{rank}(Q) = 2$

$$Q = \begin{bmatrix} 1 & 1 \\ -3+\gamma & -1 \end{bmatrix}, P = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$

For  $\text{rank}(Q) = 2$ ,  $-3+\gamma \neq -1 \Rightarrow \gamma \neq 2$

For  $\text{rank}(P) < 2$ ,  $\gamma = 0$ , as for any other value of  $\gamma$ ,  $(1, -3)$  and  $(0, 1)$  are linearly independant.

So at  $\gamma = 0$ ,  $\text{rank}(P) = 1$  and  $\text{rank}(Q) = 2 \Rightarrow$  observable, not controllable

$\boxed{\gamma = 0 \Rightarrow \text{ob}}$

### Exercise 1

1 | 2 | 3 | 4 | 5

$x_i[k] \rightarrow$  brightness of  $i^{\text{th}}$  LED at  $k^{\text{th}}$  time step  
 $u[k] \rightarrow$  brightness command at  $k^{\text{th}}$  time step

for  $i=2,3,4,5$   $x_i[k+1] = x_{i-1}[k]$

for first LED,  $x_1[k+1] = u[k]$

1) State-space representation:

$$x[k] = \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \\ x_5(k) \end{bmatrix}$$

$$x[k+1] = \begin{bmatrix} x_1[k+1] \\ x_2[k+1] \\ x_3[k+1] \\ x_4[k+1] \\ x_5[k+1] \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1[k] \\ x_2[k] \\ x_3[k] \\ x_4[k] \\ x_5[k] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u[k]$$

$\nwarrow A$        $\nwarrow B$

$$y[k] = [1 \ 1 \ 1 \ 1 \ 1] \begin{bmatrix} x_1[k] \\ x_2[k] \\ x_3[k] \\ x_4[k] \\ x_5[k] \end{bmatrix}$$

$\nwarrow C$

2. For controllability, computing controllability matrix  $P$ :

$$P = [B \ AB \ A^2B \ \dots \ A^4B]$$

using the `ctrb(A,B)` function in MATLAB to compute the controllability matrix:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \text{rank}(P) = 5 \text{ i.e. full rank.}$$

So the system is fully controllable.

In this scenario, full controllability implies that given an initial set of brightnesses for the 5 LED's, we can find a set of input commands  $u(t)$  to take the brightness levels of all 5 LED's to any arbitrary brightness levels within certain limits.