For lateral control:

$$\frac{d}{dt}\begin{bmatrix} e_1 \\ \dot{e}_1 \\ e_2 \\ \dot{e}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{4C_\alpha}{m\dot{x}} & \frac{4C_\alpha}{m} & \frac{-2C_\alpha(l_f - l_r)}{m\dot{x}} \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{2C_\alpha(l_f - l_r)}{I_z \dot{x}} & \frac{2C_\alpha(l_f - l_r)}{I_z} & -\frac{2C_\alpha(l_f^2 + l_r^2)}{I_z \dot{x}} \end{bmatrix}}_{A_{lat}} \begin{bmatrix} e_1 \\ \dot{e}_1 \\ e_2 \\ \dot{e}_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ \frac{2C_\alpha}{m} & 0 \\ 0 & 0 \\ \frac{2C_\alpha l_f}{I_z} & 0 \end{bmatrix}}_{B_{lat}} \begin{bmatrix} \delta \\ F \end{bmatrix} + \begin{bmatrix} \cdots \end{bmatrix} \cancelto{0}{\dot{\psi}_{des}}$$

For lateral control:

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A_{long}} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{m} \end{bmatrix}}_{B_{long}} \begin{bmatrix} \delta \\ F \end{bmatrix} + \begin{bmatrix} 0 \\ \cancelto{0}{\dot{\psi}\dot{y} - fg} \end{bmatrix}$$

$e_1 \rightarrow$ distance between vehicle COM and trajectory.
$e_2 \rightarrow$ orientation error with respect to trajectory.

Exercise 1

1) Since the A and B matrices for the longitudinal controller are constant, the controllability matrix does not depend on the longitudinal speed.

$$P_{long} = \begin{bmatrix} B_{long} & A_{long} B_{long} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{m} \\ 0 & \frac{1}{m} & 0 & 0 \end{bmatrix}$$

$$\text{rank}(P_{long}) = 2 \rightarrow \text{full rank}$$

Thus, the longitudinal position and velocity are <u>controllable</u> at all speeds.

For the lateral controller:

$$P_{lat} = \begin{bmatrix} B_{lat} & A_{lat} B_{lat} & A^2_{lat} B_{lat} & A^3_{lat} B_{lat} \end{bmatrix}$$

To check for controllability, the rank of this matrix can be computed at $\dot{x} = 2, 5$ and 8 m/s.

From programming, we get:

$$\text{rank}(Q_{lat, \dot{x}=2}) = \text{rank}(Q_{lat, \dot{x}=5}) = \text{rank}(Q_{lat, \dot{x}=8}) = 4$$

We get: the system describing the lateral error dynamics is <u>controllable</u> at $\dot{x} = 2, 5$ and 8 m/s.

To check observability, assuming $C = [Id]$, i.e., we are able to observe all the states, since we have all the sensing data (GPS, IMU etc.) at this stage.

$\therefore$ For longitudinal dynamics: $C_{long} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow Q_{long} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

$\text{rank}(Q_{long}) = 2 \Rightarrow$ observable

For the lateral dynamics, substituting $\dot{x} = 2, 5$ & 8 m/s and computing $Q_{lat} = [C \;\; CA \;\; CA^2 \;\; CA^3]^T$:

$\text{rank}(Q_{lat, \dot{x}=2}) = \text{rank}(Q_{lat, \dot{x}=5}) = \text{rank}(Q_{lat, \dot{x}=8}) = 4$

$\Rightarrow$ The system is fully observable.

Python code attached below.