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Exercise 1

a)

$$x(k+1) = \begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(k) \longrightarrow \text{discrete time}$$

Computing the eigenvalues of $A = \begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix}$

$\det(A - \lambda I) = 0 \Rightarrow (1-\lambda)(0.5-\lambda) - (-0.5)0 = 0 \Rightarrow \lambda = 1, 0.5$; $m = 0$ for each λ.

For $\lambda = 0.5$, since $\|\lambda\| < 1$, and $r < 1$ ($\lambda = re^{i\theta}$), the associated state is asymptotically stable.

For $\lambda = 1$, $r = 1$, and $m = 0$ (not defective). Thus, this state is state is stable in the sense of Lyapunov.

Thus, the system is stable in the sense of Lyapunov but not asymptotically stable.

b)

$$\dot{x} = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} u \longrightarrow \text{continuous time}$$

Computing the eigenvalues of A, we get $\lambda = -1, -5, -3$; $m = 0$ for each λ.

Writing the eigenvalues as $\lambda = R + i\,I_m$,

for $\lambda = -1$, $\lambda = -1 + i(0)$
$R < 0 \rightarrow$ asymptotically stable.

for $\lambda = -5$, $\lambda = -5 + i(0)$
$R < 0 \Rightarrow$ asymptotically stable.

for $\lambda = -3$, $\lambda = -3 + i(0)$
$R < 0 \Rightarrow$ asymptotically stable.

Since all states are asymptotically stable, the system is asymptotically stable.

Exercise 2

$$\dot{x} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x$$

Computing controllability matrix P: $P = [B \quad AB \quad A^2B]$

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad P = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\rho(P) = 2.$$

Constructing M for controllable decomposition:

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \qquad \rho(M) = 3$$

$$\Rightarrow \quad A' = M^{-1}AM, \quad B' = M^{-1}B, \quad C' = CM$$

$$M^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

So, $A' = M^{-1}AM = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

$$B' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$C' = CM = \begin{bmatrix} 3 & 3 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \left.\dot{x}_c = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} x_c + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u, \quad y = \begin{bmatrix} 3 & 3 & 1 \end{bmatrix} x_c \right\} \text{controllable form}$$

Computing observability matrix Q: $Q = [C' \ C'A' \ C'A'^2]^T$

$$Q = \begin{bmatrix} 3 & 3 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

$\rho(Q) = 2$. Thus, not all states of the system are observable.

To check stabilizability: uncontrollable states must be Lyapunov stable.

The eigenvalues of A' come out to be: $\lambda = 0, -1, -2$

So, all the states are stable in the sense of Lyapunov (two being asymptotically stable).

<u>Stabilizable</u>

Hence, the system is stabilizable in the sense that the uncontrollable state will not diverge. However, it might not be possible to drive this state to a desired equilibrium point.

<u>Detectable</u>

All unobservable modes need to be stable. Here, since all states are stable in the sense of Lyapunov, the system is detectable.

Exercise 3

$$m\ddot{x} = -u_1 \sin\theta + \varepsilon u_2 \cos\theta$$
$$m\ddot{y} = u_1 \cos\theta + \varepsilon u_2 \sin\theta - mg$$
$$J\ddot{\theta} = u_2$$

Writing the nonlinear dynamics in matrix form:

$$\begin{bmatrix} \dot{\theta} \\ \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ (-u_1\sin\theta + \varepsilon u_2\cos\theta)/m \\ (u_1\cos\theta + \varepsilon u_2\sin\theta - mg)/m \\ u_2/J \end{bmatrix} = f(y,u) \quad \left[y = [\theta, x, y, \dot{\theta}]^T \right]$$

Equilibrium point is : $\bar{x}=0, \bar{y}=0, \bar{\theta}=0, \bar{u}_1 = mg, \bar{u}_2 = 0$

Linearizing the system around the equilibrium point:

$$A = \left.\frac{\partial f(y,u)}{\partial y}\right|_{y=\bar{y},u=\bar{u}} = \left.\begin{bmatrix} 0 & 0 & 0 & 1 \\ \frac{-u_1 c\theta - \varepsilon u_2 s\theta}{m} & 0 & 0 & 0 \\ \frac{-u_1 s\theta + \varepsilon u_2 c\theta}{m} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right|_{y=\bar{y},u=\bar{u}} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \left.\frac{\partial f(y,u)}{\partial u}\right|_{y=\bar{y},u=\bar{u}} = \left.\begin{bmatrix} 0 & 0 \\ \frac{-\sin\theta}{m} & \frac{\varepsilon\cos\theta}{m} \\ \frac{\cos\theta}{m} & \frac{\varepsilon\sin\theta}{m} \\ 0 & \frac{1}{J} \end{bmatrix}\right|_{y=\bar{y},u=\bar{u}} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{\varepsilon}{m} \\ \frac{1}{m} & 0 \\ 0 & \frac{1}{J} \end{bmatrix}$$

$y \Rightarrow$ states, $u \Rightarrow$ controls

$$\dot{y} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ -g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} y + \begin{bmatrix} 0 & 0 \\ 0 & \frac{\varepsilon}{m} \\ \frac{1}{m} & 0 \\ 0 & \frac{1}{J} \end{bmatrix} u \longrightarrow \text{linearized system about equilibrium point}$$

To determine the stability, computing the eigenvalues of the Jacobian:

All eigenvalues of A come out to be 0.

So, the system is not asymptotically stable, but it is stable in the sense of Lyapunov.

Exercise 4

$$\dot{x} = \begin{bmatrix} a & 0 \\ 1 & -1 \end{bmatrix} x$$

Lyapunov function: $V = x_1^2 + x_2^2$

$V(x) = 0$ for $x = 0$ only

$V(x) > 0 \;\; \forall \; x \neq 0$

$$\dot{V}(x) = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 = 2x_1(ax_1) + 2x_2(x_1 - x_2)$$

$$= 2ax_1^2 + 2x_1x_2 - 2x_2^2$$

$\dot{V}(x)$ must be < 0 for all $x \neq 0$ for asymptotically stable.

$$\dot{V}(x) = 2ax_1^2 + \frac{x_1^2}{2} - \left(\frac{x_1^2}{2} - 2x_1x_2 + 2x_2^2\right) \quad \text{(adding and subtracting } \tfrac{x_1^2}{2}\text{)}$$

$$= \frac{(4a+1)}{2}x_1^2 - \underbrace{\left(\frac{x_1}{\sqrt{2}} - \sqrt{2}x_2\right)^2}_{<0 \;\forall\; x \neq 0}$$

To ensure $\dot{V}(x) < 0 \;\forall\; x \neq 0$, we need $\frac{4a+1}{2} \leq 0$ always.

$$\frac{4a+1}{2} \leq 0 \Rightarrow a \leq -1/4$$

$\therefore$ For $a \leq -1/4$, $\dot{V}(x) < 0 \;\forall\; x \neq 0 \Rightarrow$ asymptotically stable.

Exercise 5

$\dot{x}_1 = x_2 - x_1 x_2^2$

$\dot{x}_2 = -x_1^3$

Equilibrium point $[\bar{x}_1, \bar{x}_2]^T = [0, 0]^T$

a) $\left.\frac{\partial f}{\partial x}\right|_{\bar{x}} = \left.\begin{bmatrix} -x_2^2 & 1-2x_1x_2 \\ -3x_1^2 & 0 \end{bmatrix}\right|_{\bar{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

linearized state:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x$$

Eigenvalues of state matrix: $\lambda = 0, 0$

So, the linearized system is Lyapunov stable, but not asymptotically stable.

Note: Since the linearization is an approximation of the system, $\lambda = 0$, i.e., stable only in the sense of Lyapunov does not guarantee stability.

Noise or pertrubations might cause instability.

b) $V(x) = x_1^4 + 2x_2^2$

$V(0) = 0$,
$V(x) > 0 \ \forall \ x \neq 0$

$\dot{V}(x) = \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 = 4x_1^3(x_2 - x_1x_2^2) + 4x_2(-x_1^3)$

$= 4x_1^3x_2 - 4x_1^4x_2^2 - 4x_2x_1^3 = -4x_1^4x_2^2$

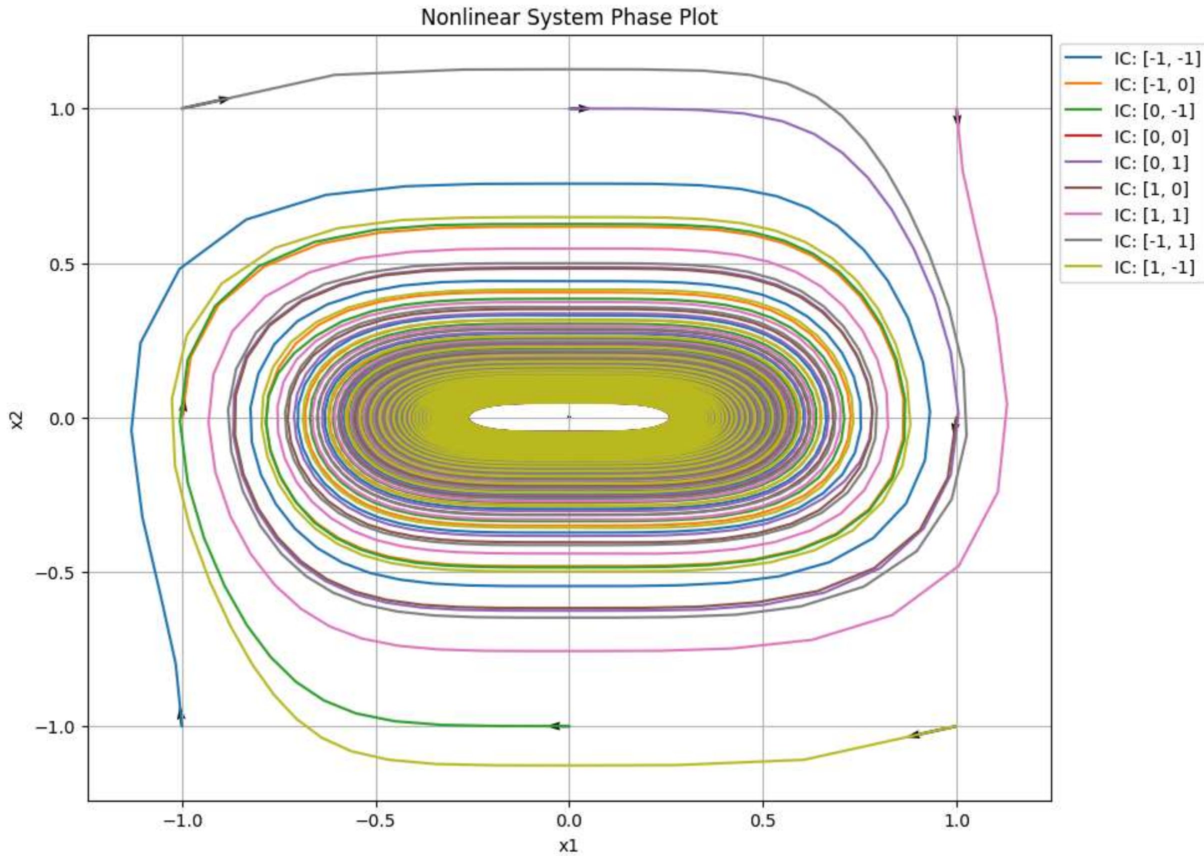
So $\forall \ x_1, x_2 \neq 0$, $\dot{V}(x) = -4x_1^4x_2^2 < 0$

Thus, the system is asymptotically stable.

c) and d) code and plots attached below.

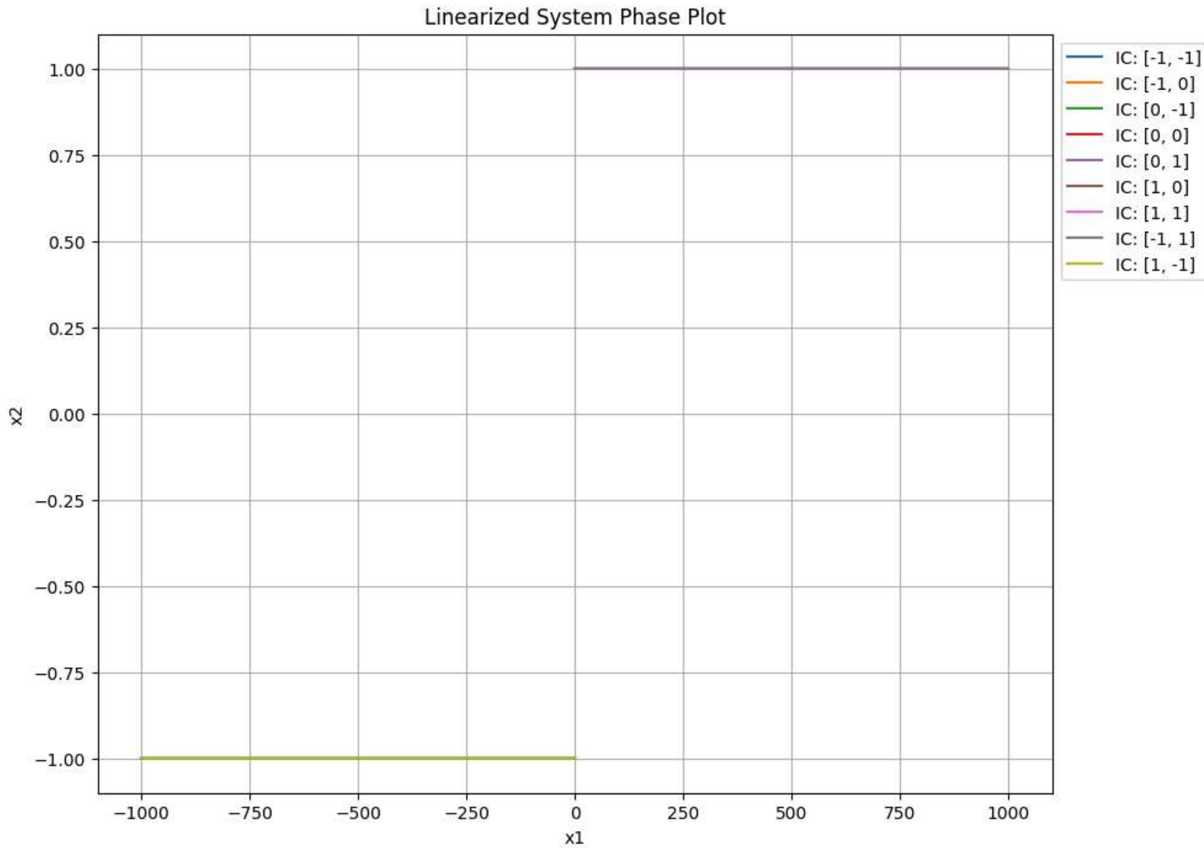
In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp

def nonlinear_dyn(t, x):
    dx1dt = x[1] - x[0] * (x[1] ** 2)
    dx2dt = - x[0] ** 3

    return [dx1dt, dx2dt]

t_span = (0, 1000)
t = np.linspace(0, 1000, 5001)

init_cond = [[-1, -1], [-1, 0], [0, -1], [0, 0], [0, 1], [1, 0], [1, 1], [-1, 1], [
```

In [2]:
```
plt.figure(figsize = (10, 8))
for ic in init_cond:
    sol = solve_ivp(nonlinear_dyn, t_span, ic, t_eval = t)
    plt.plot(sol.y[0], sol.y[1], label = f'IC: {ic}')
    for i in range(0, len(sol.t) - 1, 500):
        plt.quiver(sol.y[0][i], sol.y[1][i], sol.y[0][i+1] - sol.y[0][i], sol.y[1][

plt.title('Nonlinear System Phase Plot')
plt.xlabel('x1')
plt.ylabel('x2')
plt.grid()
plt.legend(loc = 'best', bbox_to_anchor=(1, 1))
plt.show()
```



In [3]:
```
A = np.array([[0, 1], [0, 0]])

def linear_dyn(t, x):
    return A @ x
```

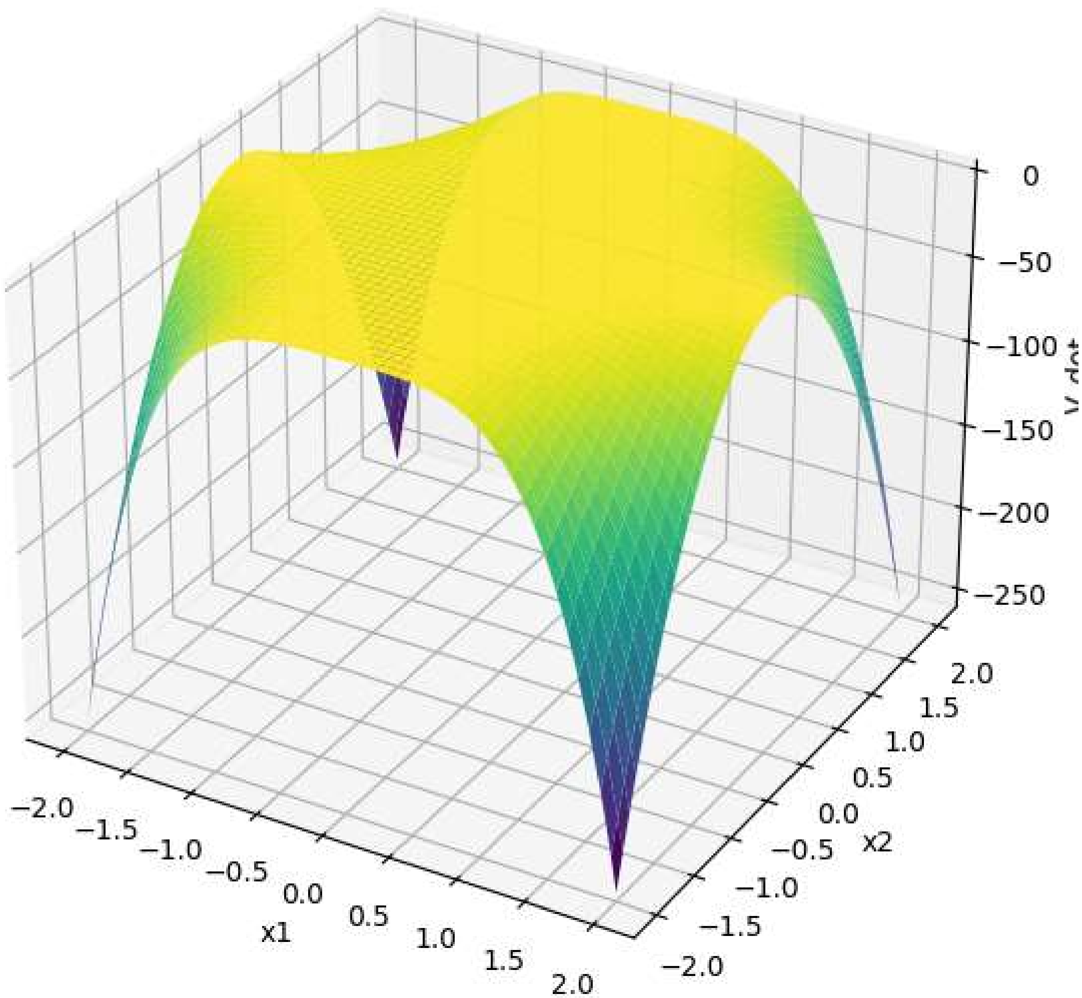
In [4]:
```
plt.figure(figsize = (10, 8))
for ic in init_cond:
    sol = solve_ivp(linear_dyn, t_span, ic, t_eval = t)
    plt.plot(sol.y[0], sol.y[1], label=f'IC: {ic}')

plt.title('Linearized System Phase Plot')
plt.xlabel('x1')
plt.ylabel('x2')
plt.grid()
plt.legend(loc = 'best', bbox_to_anchor=(1, 1))
plt.show()
```



In [5]:
```
from mpl_toolkits.mplot3d import Axes3D

def dV(x1, x2):
    dv = 4 * x1**3 * (x2 - x1 * x2**2) + 4 * x2 * (-x1**3)
    return dv

x1 = np.linspace(-2, 2, 100)
x2 = np.linspace(-2, 2, 100)
X1, X2 = np.meshgrid(x1, x2)

V_dot = dV(X1, X2)
```

In [6]:
```
fig = plt.figure(figsize=(10, 7))
ax = fig.add_subplot(111, projection='3d')

ax.plot_surface(X1, X2, V_dot, cmap='viridis', edgecolor='none')

ax.set_xlabel('x1')
ax.set_ylabel('x2')
ax.set_zlabel('V_dot')
ax.set_title('Variation of V_dot with respect to x1 and x2')

plt.show()
```

Variation of V_dot with respect to x1 and x2



Exercise 6

a) $x(k+1) = \begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(k)$

$y = [5 \;\; 5]\, x(k)$

Computing the transfer function to check for BIBO stability:

$$G_D(s) = C(zI-A)^{-1}B + D$$

$$= [5 \;\; 5] \begin{bmatrix} z-1 & 0 \\ 0.5 & z-0.5 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + [0]$$

$$= [5 \;\; 5] \begin{bmatrix} 1/(z-1) & 0 \\ -1/(2z-1)(z-1) & 2/(2z-1) \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = [5 \;\; 5] \begin{bmatrix} 1/(z-1) \\ -1/(2z-1)(z-1) \quad - 2/(2z-1) \end{bmatrix}$$

$$= [5 \;\; 5] \begin{bmatrix} 1/(z-1) \\ -(2z-1)/(2z-1)(z-1) \end{bmatrix} = 0$$

The transfer function is 0, indicating that the output doesn't depend on the input.

So looking at the eigenvalues of the system matrix to understand the stability of the system:

$A = \begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix}$

$\lambda = 1, 0.5$

For $\lambda = 0.5$, eigenvalue is inside the unit circle $\Rightarrow$ stable state.

For $\lambda = 1$, the eigenvalue lies on the unit circle. This indicates that the corresponding state is not asymptotically stable.

Thus, the system is not fully BIBO stable, as pertrubations can cause the system to be unstable.

b) $\dot{x} = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} u$

$y = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} x$

Computing the transfer function G_c:

$$G_c(s) = C(sI - A)^{-1}B + D$$

$$= \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} s+7 & 2 & -6 \\ -2 & s+3 & 2 \\ 2 & 2 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 1/s+3 & 0 \end{bmatrix}$$

The pole lies at $-3 \Rightarrow$ BIBO stable.

The system is BIBO stable for the second output. The first output is unaffected by the input.

Checking eigenvalues of the system matrix to ensure that no unstable poles are cancelled out in the transfer function:

Eigenvalues for A: $\lambda = -1, -3, -5$

So the system is stable in the sense of Lyapunov, and it is BIBO stable as well.

Exercise 7

$$V_c \frac{dT_c}{dt} = f_c(T_{ci} - T_c) + \beta(T_H - T_c)$$

$$V_H \frac{dT_H}{dt} = f_H(T_{Hi} - T_H) + \beta(T_c - T_H)$$

$f_c, f_H, V_c, V_H \rightarrow$ constant

$T_c, T_H \rightarrow$ temperatures in cold and hot compartments respectively

$T_{ci}, T_{Hi} \rightarrow$ temperatures of cold and hot inflows respectively

$u_1 = T_{ci}$, $u_2 = T_{Hi}$

$y_1 = T_c$, $y_2 = T_H$

$f_c = f_H = 0.1\, m^3/min$, $\beta = 0.2\, m^3/min$, $V_H = V_c = 1\, m^3$

1) Rewriting the equations, we get:

$$\dot{T}_c = \frac{1}{V_c}\left(-(f_c + \beta)T_c + \beta T_H + f_c T_{ci}\right)$$

$$\dot{T}_H = \frac{1}{V_H}\left(-(f_H + \beta)T_H + \beta T_c + f_H T_{Hi}\right)$$

In state-space form:

$$\begin{bmatrix} \dot{T}_c \\ \dot{T}_H \end{bmatrix} = \begin{bmatrix} -(f_c+\beta)/V_c & \beta/V_c \\ \beta/V_H & -(f_H+\beta)/V_H \end{bmatrix} \begin{bmatrix} T_c \\ T_H \end{bmatrix} + \begin{bmatrix} f_c/V_c & 0 \\ 0 & f_H/V_H \end{bmatrix} \begin{bmatrix} T_{ci} \\ T_{Hi} \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} T_c \\ T_H \end{bmatrix}$$

2) When $T_{ci} = T_{Hi} = 0$,

$$\underbrace{\begin{bmatrix} \dot{T}_c \\ \dot{T}_H \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} -(f_c+\beta)/V_c & \beta/V_c \\ \beta/V_H & -(f_H+\beta)/V_H \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} T_c \\ T_H \end{bmatrix}}_{x}$$

$$\dot{x} = Ax$$

The solution for this equation is $x(t) = e^{At}x(0)$

Substituting the values of f_c, f_H, β, we get: $A = \begin{bmatrix} -0.3 & 0.2 \\ 0.2 & -0.3 \end{bmatrix}$

Eigenvalues of A : $\lambda = -0.5, -0.1$; Eigenvectors : $v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Taking $J = \begin{bmatrix} -0.5 & 0 \\ 0 & -0.1 \end{bmatrix}$, $M = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

$$e^{At} = M^{-1} e^{Jt} M = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} e^{-0.5t} & 0 \\ 0 & e^{-0.1t} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{e^{-t/2}}{2} \begin{bmatrix} e^{0.4t}+1 & e^{0.4t}-1 \\ e^{0.4t}-1 & e^{0.4t}+1 \end{bmatrix}$$

$$\Rightarrow x(t) = \frac{e^{-t/2}}{2} \begin{bmatrix} e^{0.4t}+1 & e^{0.4t}-1 \\ e^{0.4t}-1 & e^{0.4t}+1 \end{bmatrix} x(0)$$

$$\Rightarrow \boxed{\begin{aligned} T_c &= \frac{e^{-t/2}}{2}\left(e^{0.4t}(T_c(0)+T_H(0)) + T_c(0) - T_H(0)\right) \\ T_H &= \frac{e^{-t/2}}{2}\left(e^{0.4t}(T_c(0)+T_H(0)) + T_H(0) - T_c(0)\right) \end{aligned}}$$

3) To investigate the BIBO stability of the system, computing the transfer function:

$$G(s) = C(sI-A)^{-1}B$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} -0.3 & 0.2 \\ 0.2 & -0.3 \end{bmatrix}, \quad B = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$

$$\Rightarrow G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s+0.3 & -0.2 \\ -0.2 & s+0.3 \end{bmatrix}^{-1} \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$$

$$G = \frac{1}{5(20s^2+12s+1)} \begin{bmatrix} 10s+3 & 2 \\ 2 & 10s+3 \end{bmatrix}$$

So, the poles of the system lie at $s = \frac{-3 \pm i}{10}$.

Real parts of both poles lie on the negative side. Hence, the system is BIBO stable.