

Exercise /

1. $y(t) = 0$ for all t

For linearity, say $u_1 \rightarrow y_1$ and $u_2 \rightarrow y_2$ then for an input $u_3 = \alpha u_1 + \beta u_2$, the output y_3 must satisfy $y_3 = \alpha y_1 + \beta y_2$ (u_i being inputs to the system, and α & β are constants)

$$y_1 = y_2 = 0$$

$$y(u_3) = y(u_1 + u_2) = y_3 = 0$$

$$\therefore \alpha y_1 + \beta y_2 = \alpha(0) + \beta(0) = 0 = y_3$$

System is linear.

For time variance, given an input $u_1 \rightarrow y_1$, if an input $u_2(t) = u_1(t + T)$ produces an output y_2 such that $y_2(t) = y_1(t + T)$, the system is time invariant.

For u_1 , $y_1(t) = 0$.

Now for $u_2(t)$ such that $u_2(t) = u_1(t + T)$,

$$y_2(t) = 0$$

We know that $y_1(t + T) = 0 = y_2(t)$

$$\Rightarrow \text{for } u_2(t) = u_1(t + T), y_2(t) = y_1(t + T)$$

Thus, the system is time invariant.

2. $y(t) = u^3(t)$

For linearity, $u_3 = \alpha u_1 + \beta u_2$

where $y_1(t) = u_1^3(t)$, $y_2(t) = u_2^3(t)$

$$y_3 = u_3^3(t), \text{ but } u_3 = \alpha u_1 + \beta u_2$$

$$\Rightarrow y_3 = (\alpha u_1 + \beta u_2)^3 = \alpha^3 u_1^3 + \beta^3 u_2^3 + 3\alpha^2 \beta u_1^2 u_2 + 3\alpha \beta^2 u_1 u_2^2$$

For linearity, $y_3 = \alpha y_1 + \beta y_2$ is necessary.

$$\alpha y_1 + \beta y_2 = \alpha u_1^3 + \beta u_2^3$$

$$\text{Comparing, } \alpha u_1^3 + \beta u_2^3 ?= \alpha^3 u_1^3 + \beta^3 u_2^3 + 3\alpha^2 \beta u_1^2 u_2 + 3\alpha \beta^2 u_1 u_2^2$$

By comparing the coefficients, we can see that the equality doesn't hold.

Thus, the system is **not linear**.

For time variance, we need : for $u_2 = u_1(t+\tau)$,

$$y_2 = y_1(t+\tau), \text{ where } y_2 = u_2^3 t \neq y_1 = u_1^3 t$$

Substituting $u_2 = u_1(t+\tau)$, we get $y_2(t) = u_1^3(t+\tau)$
we know, $y_1(t+\tau) = u_1^3(t+\tau)$

$$\text{so, } y_1(t+\tau) = u_1^3(t+\tau) = u_2^3(t) = y_2(t)$$

\therefore The system is **time invariant**.

$$3. y(t) = u(3t)$$

For linearity : $y_1 = u_1(3t); y_2 = u_2(3t)$

$$\text{Let } u_3(t) = \alpha u_1(t) + \beta u_2(t)$$

for linearity, we need $y_3(t) = \alpha y_1(t) + \beta y_2(t)$

$$y_3(t) = u_3(3t) = \underbrace{\alpha u_1(3t)}_{\alpha y_1} + \underbrace{\beta u_2(3t)}_{\beta y_2}$$

$$\Rightarrow y_3(t) = \alpha y_1(t) + \beta y_2(t)$$

Hence, the system is **linear**.

For system to be time invariant, a time shifted input $u(t+\tau)$ should produce an output shifted by the same τ i.e. $u(t+\tau) \rightarrow y(t+\tau)$

If $y(t) = u(3t)$, then for $u(t+\tau)$, we get $y'(t) = u(3t+3\tau)$

Now, the time shifted system would look like:

$$y(t+\tau) = u(3(t+\tau)) = u(3t+3\tau)$$

which is equal to $y'(t)$

Thus, delaying the input and time shifting the system produce the same result.

Thus, the system is **time invariant**.

$$4. y(t) = e^{-t} u(t-\tau)$$

Linearity: $y_1 = e^{-t} u_1(t-\tau)$; $y_2 = e^{-t} u_2(t-\tau)$

$$\text{let } a_3 = \alpha u_1 + \beta u_2$$

$$y_3 = e^{-t} a_3(t-\tau) = e^{-t} (\alpha u_1(t-\tau) + \beta u_2(t-\tau))$$

$$= \underbrace{\alpha e^{-t} u_1(t-\tau)}_{\alpha y_1} + \underbrace{\beta e^{-t} u_2(t-\tau)}_{\beta y_2}$$

$$\Rightarrow y_3(t) = \alpha y_1(t) + \beta y_2(t)$$

\therefore System is **linear**.

For time variance:

When input is delayed by τ , we get

$$y'(t) = e^{-t} u((t+\tau)-\tau) = e^{-t} u(t-\tau+\tau)$$

When system is shifted by the same delay τ , we get:

$$\begin{aligned} y(t+\tau) &= e^{-(t+\tau)} u(t+\tau-\tau) \\ &= e^{-\tau} \times \underbrace{e^{-t} u(t-\tau+\tau)}_{y'(t)} \\ &= e^{-\tau} y'(t) \end{aligned}$$

We can see that the time delay in input does not have the same effect as shifting the input by τ . Thus, system is **time variant**.

$$5. y(t) = \begin{cases} 0 & t \leq 0 \\ u(t) & t > 0 \end{cases}$$

For linearity, $u_1 \rightarrow y_1; u_2 \rightarrow y_2; u_3 \rightarrow y_3$ where

$$u_3 = \alpha u_1 + \beta u_2. \text{ If } y_3 = \alpha y_1 + \beta y_2 \Rightarrow \text{linear}$$

$$y_3(t) = \begin{cases} 0 & t \leq 0 \\ u_3(t) & t > 0 \end{cases} = \begin{cases} 0 & t \leq 0 \\ \alpha u_1(t) + \beta u_2(t) & t > 0 \end{cases}$$

$$\alpha y_1 + \beta y_2 = \begin{cases} \alpha(0) + \beta(0) & t \leq 0 \\ \alpha u_1(t) + \beta u_2(t) & t > 0 \end{cases} = \begin{cases} 0 & t \leq 0 \\ \alpha u_1(t) + \beta u_2(t) & t > 0 \end{cases}$$

$$\Rightarrow \alpha y_1 + \beta y_2 = y_3 \Rightarrow \text{The system is linear.}$$

For time variance, for an input delayed by τ , we get

$$y'(t) = \begin{cases} 0 & t + \tau \leq 0 \text{ i.e. } t \leq -\tau \\ u(t+\tau) & t + \tau > 0 \text{ i.e. } t > -\tau \end{cases}$$

When the system is shifted by the same τ , we get

$$y(t + \tau) = \begin{cases} 0 & t + \tau \leq 0 \text{ i.e. } t \leq -\tau \\ u(t + \tau) & t + \tau > 0 \text{ i.e. } t > -\tau \end{cases}$$

We can see that $y'(t)$ and $y(t + \tau)$ are equivalent. Thus, the system is time invariant.

Exercise 2

$$1. p_i(k+1) = p_i(k) (\alpha^2 / S_i(k))$$

Here $S_i(k)$ is the SINR value at receiver i .

The noise plus interference power at receiver i is given by:

$$q_i = \sigma^2 + \sum_{j \neq i} G_{ij} p_j$$

And the signal power at receiver i is given by:

$$S_i = G_{ii} p_i$$

$\Rightarrow S_i(k)$ can be written as:

$$S_i(k) = S_i / q_i = G_{ii} p_i(k) / (\sigma^2 + \sum_{j \neq i} G_{ij} p_j)$$

$$\begin{aligned} \Rightarrow p_i(k+1) &= \frac{p_i(k) (\alpha^2)}{G_{ii} p_i(k)} \cdot \left(\sigma^2 + \sum_{j \neq i} G_{ij} p_j \right) \\ &= \underline{\alpha^2} \left(\sigma^2 + \sum_{j \neq i} G_{ij} p_j \right) \end{aligned}$$

$$p_i(k+1) = \frac{\alpha \sigma^2 + \sum_{j \neq i} G_{ij} p_j}{G_{ii}}$$

This can be written as:

$$\left. \begin{aligned} p_1(k+1) &= \frac{\alpha \sigma^2 + (G_{12} p_2 + G_{13} p_3)}{G_{11}} \\ p_2(k+1) &= \frac{\alpha \sigma^2 + (G_{23} p_3 + G_{21} p_1)}{G_{22}} \\ p_3(k+1) &= \frac{\alpha \sigma^2 + (G_{31} p_1 + G_{32} p_2)}{G_{33}} \end{aligned} \right\} \begin{array}{l} p_1, p_2, p_3 \\ \text{refer to} \\ p_1(k), p_2(k), \\ p_3(k) \end{array}$$

Writing it in matrix form:

$$\begin{bmatrix} p_1(k+1) \\ p_2(k+1) \\ p_3(k+1) \end{bmatrix} = \begin{bmatrix} 0 & \frac{G_{12}}{G_{11}} & \frac{G_{13}}{G_{11}} \\ \frac{G_{21}}{G_{22}} & 0 & \frac{G_{23}}{G_{22}} \\ \frac{G_{31}}{G_{33}} & \frac{G_{32}}{G_{33}} & 0 \end{bmatrix} \begin{bmatrix} p_1(k) \\ p_2(k) \\ p_3(k) \end{bmatrix} + \begin{bmatrix} \frac{\alpha \sigma^2}{G_{11}} \\ \frac{\alpha \sigma^2}{G_{22}} \\ \frac{\alpha \sigma^2}{G_{33}} \end{bmatrix}$$

$\underbrace{\quad}_{A} \qquad \qquad \qquad \underbrace{\quad}_{B}$

2. Crimes $G_1 = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 3 & 1 & 3 \end{bmatrix}, \gamma = 3, \alpha = 1 \cdot 2, \sigma^2 = 0.1$

$$\begin{bmatrix} p_1(k+1) \\ p_2(k+1) \\ p_3(k+1) \end{bmatrix} =
 \begin{bmatrix} 0 & G_{12} & \frac{G_{13}}{G_{11}} \\ \frac{G_{21}}{G_{22}} & 0 & \frac{G_{23}}{G_{22}} \\ \frac{G_{31}}{G_{33}} & \frac{G_{32}}{G_{33}} & 0 \end{bmatrix}
 \begin{bmatrix} p_1(k) \\ p_2(k) \\ p_3(k) \end{bmatrix} +
 \begin{bmatrix} \frac{\alpha}{G_{11}} \\ \frac{\alpha}{G_{22}} \\ \frac{\alpha}{G_{33}} \end{bmatrix} u$$

$$= \begin{bmatrix} 0 & 0.2 & 0.1 \\ 0.05 & 0 & 0.05 \\ 0.1 & 0.1/3 & 0 \end{bmatrix}
 \begin{bmatrix} p_1(k) \\ p_2(k) \\ p_3(k) \end{bmatrix} +
 \begin{bmatrix} 3.6 \\ 1.8 \\ 1.2 \end{bmatrix} (0.01) u$$

For $\gamma = 5$, the matrix B looks like:

$$\begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$$

For $\gamma = 5$

$$\Rightarrow p(k+1) = \begin{bmatrix} 0 & 0.2 & 0.1 \\ 0.05 & 0 & 0.05 \\ 0.1 & 0.1/3 & 0 \end{bmatrix}
 \begin{bmatrix} p_1(k) \\ p_2(k) \\ p_3(k) \end{bmatrix} +
 \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix} (0.01)$$

Python code and plots are attached below.

From the plots, we can see that controller struggles to take $s_i(t)$ to α_i when the γ is low.

However, when the value of γ is increased, the s_i values are much closer to the target α_i values.

Exercise 3

$$\ddot{y} + (1+y)\dot{y} - 2y + 0.5y^3 = 0$$

Firstly, finding equilibrium point : $y = \bar{y} = 0$

$$\Rightarrow 0 + (1+0)\cdot 0 - 2\cdot 0 + 0.5\cdot 0^3 = 0$$

$$\Rightarrow y^3 - 4y = 0 \Rightarrow y(y+2)(y-2) = 0$$

We have equilibrium points at $y = -2, 0, 2$.

For linearizing, taking $y = \bar{y} + \delta y \rightarrow$ small perturbation
 ↴ equilibrium point

$$(\bar{y} + \delta y) + (1 + (\bar{y} + \delta y))(\bar{y} + \delta y) - 2(\bar{y} + \delta y) + 0.5(\bar{y} + \delta y)^3 = 0$$

i) For $\bar{y} = 0$, we get :

$$\ddot{\delta y} + (1 + \delta y)(\delta y) - 2\delta y + 0.5(\delta y)^3 = 0$$

Ignoring higher-order terms since they are small,
 we get

$$\ddot{\delta y} + \delta y - 2\delta y = 0 \quad \left. \begin{array}{l} \text{linearized system around} \\ y = 0 \end{array} \right.$$

ii) For $\bar{y} = -2$, we get $y = -2 + \delta y$

$$(-2 + \delta y) + (1 + (-2 + \delta y))(-2 + \delta y) - 2(-2 + \delta y) + 0.5(-2 + \delta y)^3 = 0$$

$$\ddot{\delta y} + (\delta y - 1)(\delta y) + 4 - 2\delta y + 0.5(-8 + 12\delta y) = 0$$

(\hookrightarrow Ignoring higher-order terms and simplifying)

$$\ddot{\delta y} - \delta \dot{y} + 4 - 2\delta y - 4 + 6\delta y = 0$$

$\Rightarrow \ddot{\delta y} - \delta \dot{y} + 4\delta y = 0$ } linearized system around
 $y = -2$

iii) For $\bar{y} = 2$, we have $y = 2 + \delta y$

$$\begin{aligned} (\ddot{2+\delta y}) + (1 + (2 + \delta y))(2 + \delta y) - 2(2 + \delta y) + \\ 0.5(2 + \delta y) = 0 \\ \ddot{\delta y} + (3 + \delta y)(\delta y) - 4 - 2\delta y + 0.5(8 + 12\delta y) = 0 \end{aligned}$$

$$\ddot{\delta y} + 3\delta y + \cancel{\delta \dot{y}\delta \dot{y}} - 4 - 2\delta y + 4 + 6\delta y = 0$$

$\Rightarrow \ddot{\delta y} + 3\delta y + 4\delta y = 0$ } System linearized around
 $y = 2$.

Alternate solution at the end...

Exercise 4

$$\begin{bmatrix} \dot{n}_1(t) \\ \dot{n}_2(t) \end{bmatrix} = \begin{bmatrix} n_2(t) \\ -g\left(\frac{D}{n_1(t)+D}\right)^2 + \frac{\ln(u)}{m} \end{bmatrix}$$

D, m, g, u are constant for short time periods.

To find equilibrium states, we set $\dot{n}_1 = \dot{n}_2 = 0$

$$\Rightarrow \dot{n}_1(t') = 0 \Rightarrow n_2^* = 0$$

$$n_2^* = 0 \Rightarrow -g\left(\frac{D}{n_1(t')+D}\right)^2 + \frac{\ln(u)}{m} = 0$$

$$\Rightarrow \frac{\ln(u)}{m} = gD^2 / (n_1(t') + D)^2$$

Assuming $\eta_1^* + D \neq 0$, we get

$$(\eta_1^* + D)^2 \ln(u) = g D^2 m$$

$$\Rightarrow (\eta_1^* + D) \sqrt{\ln(u)} = \pm D \sqrt{gm}$$

$$\eta_1^* = \frac{\pm D \sqrt{gm}}{\sqrt{\ln(u)}} - D$$

$$\eta_1^* = D \left(-1 \pm \sqrt{\frac{gm}{\ln(u)}} \right)$$

\Rightarrow The equilibrium points are :

$$\begin{bmatrix} \eta_1^* \\ \eta_2^* \end{bmatrix} = \begin{bmatrix} D(-1 \pm \sqrt{\frac{gm}{\ln(u)}}) \\ 0 \end{bmatrix}$$

Performing linearization around the equilibrium point :

If the system is given as $\dot{x} = f(x, u)$, the linearized form looks like :

$$\delta \dot{x} = \left. \frac{\partial f}{\partial x} \right|_{x=\bar{x}, u=\bar{u}} \delta x + \left. \frac{\partial f}{\partial u} \right|_{x=\bar{x}, u=\bar{u}} \delta u$$

Here, since u is almost constant, $\delta u \approx 0$

$$\Rightarrow \delta \dot{x} = \left. \frac{\partial f}{\partial x} \right|_{x=\bar{x}} \delta x$$

or $\dot{x} = \left. \frac{\partial f}{\partial x} \right|_{x=\bar{x}} x$

$$f(\eta) = \begin{bmatrix} \eta_1(t) \\ g\left(\frac{D}{\eta_1(t)+D}\right)^2 + \frac{\ln(u)}{m} \end{bmatrix}; \quad \eta(t) = \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \end{bmatrix}$$

$$\frac{\partial f}{\partial \eta} = \begin{bmatrix} \frac{\partial f_1}{\partial \eta_1} & \frac{\partial f_1}{\partial \eta_2} \\ \frac{\partial f_2}{\partial \eta_1} & \frac{\partial f_2}{\partial \eta_2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -2g\left(\frac{D}{\eta_1(t)+D}\right) \times \frac{(-D)}{\left(\eta_1(t)+D\right)^2} & 0 \end{bmatrix}$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ 2gD^2/\left(\eta_1(t)+D\right)^3 & 0 \end{bmatrix}$$

Now substituting the equilibrium point in the Jacobian,

$$\eta_1^* = D \left(-1 \pm \sqrt{\frac{gm}{\ln(u)}} \right)$$

$$\text{Case 1: } \eta_1^* = D \left(-1 + \sqrt{\frac{gm}{\ln(u)}} \right)$$

$$\eta_1^* + D = D \sqrt{\frac{gm}{\ln(u)}}$$

$$\Rightarrow 2gD^2 / (\eta_1^* + D)^3 = \frac{2gD^2}{D^3 \frac{gm}{\ln(u)}} = \frac{2 \sqrt{(\ln(u))^3}}{D \sqrt{gm^3}}$$

$$\Rightarrow \frac{\partial f}{\partial n} = \begin{bmatrix} 0 & 1 \\ \frac{2}{D} \frac{\sqrt{(\ln u)^3}}{\sqrt{gm^3}} & 0 \end{bmatrix}$$

$$\Rightarrow \dot{x} = \begin{bmatrix} \dot{n}_1 \\ \dot{n}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{2}{D} \frac{\sqrt{(\ln u)^3}}{\sqrt{gm^3}} & 0 \end{bmatrix} \begin{bmatrix} \delta n_1 \\ \delta n_2 \end{bmatrix}$$

$$\text{Case 2 : } n_1^* = -D \left(1 + \sqrt{\frac{gm}{\ln(u)}} \right)$$

$$n_1^* + D = -D \sqrt{\frac{gm}{\ln(u)}}$$

$$\Rightarrow \frac{2gD^2}{(n_1^* + D)^3} = \frac{2gD^2 \sqrt{(\ln u)^3}}{-D^3 \sqrt{g^3 m^3}} = -\frac{2}{D} \frac{\sqrt{(\ln u)^3}}{\sqrt{g m^3}}$$

$$\Rightarrow \frac{\partial f}{\partial n} = \begin{bmatrix} 0 & 1 \\ -\frac{2}{D} \frac{\sqrt{(\ln u)^3}}{\sqrt{gm^3}} & 0 \end{bmatrix}$$

$$\text{So, } \dot{x} = \begin{bmatrix} \dot{n}_1 \\ \dot{n}_2 \end{bmatrix} = \frac{\partial f}{\partial n} \delta n = \begin{bmatrix} 0 & 1 \\ -\frac{2}{D} \frac{\sqrt{(\ln u)^3}}{\sqrt{gm^3}} & 0 \end{bmatrix} \begin{bmatrix} \delta n_1 \\ \delta n_2 \end{bmatrix}$$

$$\dot{x} = \begin{bmatrix} \dot{n}_1 \\ \dot{n}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{2}{D} \frac{\sqrt{(\ln u)^3}}{\sqrt{gm^3}} & 0 \end{bmatrix} \begin{bmatrix} \delta n_1 \\ \delta n_2 \end{bmatrix}$$

Exercise 5

$$\ddot{r} = \dot{\theta}^2 - k/r^2 + u_1$$

$$\ddot{\theta} = -\frac{2\dot{\theta}\dot{r}}{r} + \frac{u_2}{r}$$

i) For the reference orbit, $u_1 = u_2 = 0$ and $r(t) = p$ and $\theta(t) = \omega t$

$$\ddot{r}(t) = \ddot{r}(t) = 0, \quad \ddot{\theta}(t) = \omega; \quad \dot{\theta}(t) = 0$$

$$\ddot{r} = \dot{\theta}^2 - k/r^2 + u_1 \text{ becomes } 0 = p\omega^2 - k/p^2 + 0$$

$$\Rightarrow k = p^3\omega^2$$

ii) Linearizing about a trajectory, we get

$$\dot{r} = \ddot{r} = 0, \quad \ddot{\theta} = 0, \quad \dot{\theta} = \omega$$

Also, equilibrium conditions $\Rightarrow u_1 = u_2 = 0, r = p, k = p^3\omega^2$

Choosing the states to be $x = \begin{bmatrix} \dot{r} \\ r \\ \dot{\theta} \\ \theta \end{bmatrix}$, we get

$$\dot{x} = \begin{bmatrix} \ddot{r} \\ \dot{r} \\ \ddot{\theta} \\ \dot{\theta} \end{bmatrix} = f(r, u)$$

$$f(r, u) = \begin{bmatrix} \dot{\theta}^2 - k/r^2 + u_1 \\ \dot{r} \\ -\frac{2\dot{\theta}\dot{r}}{r} + \frac{u_2}{r} \\ \dot{\theta} \end{bmatrix}$$

The system linearized about the equilibrium point can be represented as:

$$\dot{x}_i = \left. \frac{\partial f}{\partial x} \right|_{x=\bar{x}, u=\bar{u}} x_i + \left. \frac{\partial f}{\partial u} \right|_{x=\bar{x}, u=\bar{u}} u_i$$

$$\left. \frac{\partial f}{\partial x} \right|_{x=\bar{x}, u=\bar{u}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_4} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_4}{\partial x_1} & \dots & \frac{\partial f_4}{\partial x_4} \end{bmatrix} = \begin{bmatrix} 0 & \frac{\dot{\theta}^2 + \omega^2}{r^3} & 2\dot{\theta}\eta & 0 \\ 1 & 0 & 0 & 0 \\ -\frac{2\dot{\theta}}{r} & \frac{2\dot{\theta}\eta - u_2}{r^2} & -\frac{2\dot{\theta}}{r} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\left. \frac{\partial f}{\partial u} \right|_{x=\bar{x}, u=\bar{u}} = \begin{bmatrix} 0 & 3\omega^2 & 2\rho\omega & 0 \\ 1 & 0 & 0 & 0 \\ -\frac{d\omega}{P} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\left. \frac{\partial f}{\partial u} \right|_{x=\bar{x}, u=\bar{u}} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \vdots & \vdots \\ \frac{\partial f_4}{\partial u_1} & \frac{\partial f_4}{\partial u_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1/r \\ 0 & 0 \end{bmatrix}$$

$$\left. \frac{\partial f}{\partial u} \right|_{x=\bar{x}, u=\bar{u}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1/P \\ 0 & 0 \end{bmatrix}$$

$$\text{Now, } \delta \dot{x} = \frac{\partial f}{\partial n} \Big|_{n=\bar{n}, u=\bar{u}} \delta x + \frac{\partial f}{\partial u} \Big|_{n=\bar{n}, u=\bar{u}} \delta u$$

$$\delta \dot{x} = \begin{bmatrix} 0 & 3\omega^2 & 2\omega p & 0 \\ 1 & 0 & 0 & 0 \\ -2\omega/p & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \delta x + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1/g \\ 0 & 0 \end{bmatrix} \delta u$$

Thus, the linearized system looks like:

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \\ \ddot{x}_5 \end{bmatrix} = \begin{bmatrix} 0 & 3\omega^2 & 2\omega p & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -2\omega/p & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1/g \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Exercise 3 (alternate solution)

$$\ddot{y} + (1+y)\dot{y} - 2y + 0.5y^3 = 0$$

$$\ddot{y} = -0.5y^3 + 2y - (1+y)\dot{y}$$

Defining states as $\begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

We can write the system equation as: $\dot{y} = f(y)$, given by

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_2 \\ -0.5y_1^3 + 2y_1 - (1+y_1)y_2 \end{bmatrix}$$

To find the equilibrium point, setting $\dot{y} = \ddot{y} = 0$

$$\begin{aligned} & \text{So, } \dot{y} + (1+ty)\ddot{y} - 2y + 0.5y^3 \equiv 0 + (1+ty)0 - 2y + 0.5y^3 = 0 \\ & \Rightarrow y = 0, -2, 2 \} \text{ equilibrium points.} \end{aligned}$$

Since there is no external input, $a=0$. Thus linearization looks like:

$$\frac{\partial f}{\partial y} = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1.5y_1^2 + 2 - y_2 & -(1+ty_1) \end{bmatrix}$$

Linearized system given by:

$$\delta \dot{y} = \left. \frac{\partial f}{\partial y} \right|_{y=\bar{y}} \delta y$$

$$i) \bar{y} = 0 \Rightarrow y_1 = y = 0, y_2 = \dot{y} = 0$$

$$\Rightarrow \delta \dot{y} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} \delta y_1 \\ \delta y_2 \end{bmatrix}$$

$$\begin{bmatrix} \dot{\delta y}_1 \\ \dot{\delta y}_2 \end{bmatrix} = \begin{bmatrix} \delta \dot{y}_2 \\ 2\delta y_1 - \delta y_2 \end{bmatrix} \rightarrow \dot{\delta y}_2 = 2\delta y_1 - \delta y_2$$

$$\Rightarrow \ddot{\delta y} = 2\dot{\delta y}_1 - \dot{\delta y}_2 \} \text{ linearized system around } y=0.$$

$$ii) \bar{y} = 2 \Rightarrow y_1 = y = 2, y_2 = \dot{y} = 0$$

$$\Rightarrow \delta \dot{y} = \begin{bmatrix} 0 & 1 \\ -4 & -3 \end{bmatrix} \begin{bmatrix} \delta y_1 \\ \delta y_2 \end{bmatrix} \rightarrow \dot{\delta y}_2 = -4\delta y_1 - 3\delta y_2$$

$$\Rightarrow \ddot{\delta y} = -4\dot{\delta y}_1 - 3\dot{\delta y}_2 \} \text{ linearized system around } y=2$$

$$iii) \bar{y} = -2 \Rightarrow y_1 = -2, y_2 = 0$$

$$\Rightarrow \delta \dot{y} = \begin{bmatrix} 0 & 1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} \delta y_1 \\ \delta y_2 \end{bmatrix} \rightarrow \dot{\delta y}_2 = -4\delta y_1 + \delta y_2$$

$$\Rightarrow \ddot{\delta y} = -4\dot{\delta y}_1 + \dot{\delta y}_2 \} \text{ linearized system around } y=-2$$