

Exercise 1

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

To find characteristic polynomial, $\det(A - \lambda I) = 0$

$$\Rightarrow \det \begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix} = 0 \Rightarrow (1-\lambda)(-\lambda)(1-\lambda) = 0$$

$\Rightarrow \lambda = 0, 1, 1$ (can be found directly since A is triangular)

i) A^{10} :

Since $A \in \mathbb{R}^{3 \times 3}$, A^m can be decomposed as:

$$A^m = \beta_2 A^2 + \beta_1 A + \beta_0 I \quad \left. \begin{array}{l} \text{from Cayley-Hamilton} \\ \text{theorem and characteristic} \\ \text{polynomial.} \end{array} \right\}$$

Since λ is also a root of the characteristic polynomial, we can say that $f(A) = \beta_2 A^2 + \beta_1 A + \beta_0 I$ is also satisfied by λ .

$$\Rightarrow \lambda^m = \beta_2 \lambda^2 + \beta_1 \lambda + \beta_0 \text{ at eigenvalues}$$

Here, $m=10$

$$\Rightarrow \lambda^{10} = \beta_2 \lambda^2 + \beta_1 \lambda + \beta_0$$

Since roots of the characteristic polynomial are the eigenvalues λ_1, λ_2 and λ_3 , $\lambda^{10} - \beta_2 \lambda^2 - \beta_1 \lambda - \beta_0 = 0$ at $\lambda_1, \lambda_2, \lambda_3$.

$$\lambda_1 = 1 \Rightarrow 1^{10} = \beta_2 + \beta_1 + \beta_0$$

$$\lambda_2 = 0 \Rightarrow 0^{10} = 0 + 0 + \beta_0$$

$$\lambda_3 = 1 \Rightarrow 1^{10} = \beta_2 + \beta_1 + \beta_0$$

$$\Rightarrow \beta_0 = 0, \beta_1 + \beta_2 = 1 - ①$$

repeated equations.

Since there's a repeated root at λ_1, λ_3 , the derivative of the left and right-hand-sides must also be equal at those points.

$$\Rightarrow \frac{d}{d\lambda}(\lambda^{10}) = \frac{d}{d\lambda}(\beta_2\lambda^2 + \beta_1\lambda) \text{ at } \lambda_1 = \lambda_3 = 1$$

$$\Rightarrow 10 = 2\beta_2 + \beta_1 - ②$$

from ① and ②, we get: $\beta_1 + \beta_2 = 1; 2\beta_2 + \beta_1 = 10$

$$② - ① \Rightarrow \beta_2 = 9;$$

$$\text{Then } \beta_1 = -8$$

$$\therefore \beta_2 = 9, \beta_1 = -8, \beta_0 = 0$$

$$\Rightarrow A^{10} = 9A^2 - 8A$$

$$A^2 \text{ can be computed: } A^2 = A \cdot A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow 9A^2 = \begin{bmatrix} 9 & 9 & 9 \\ 0 & 0 & 9 \\ 0 & 0 & 9 \end{bmatrix}$$

$$9A^2 - 8A = \begin{bmatrix} 9 & 9 & 9 \\ 0 & 0 & 9 \\ 0 & 0 & 9 \end{bmatrix} - \begin{bmatrix} 8 & 8 & 0 \\ 0 & 0 & 8 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 9 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{10} = 9A^2 - 8A = \boxed{\begin{bmatrix} 1 & 1 & 9 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\text{ii) } e^{At}$$

$$e^{At} = \beta_2 A^2 + \beta_1 A' + \beta_0 I$$

$$\text{Also, } e^{At} = \beta_2 \lambda^2 + \beta_1 \lambda + \beta_0 \text{ for } \lambda = 1, 0, 1$$

and for repeated eigenvalue λ , $\frac{\partial(e^{\lambda t})}{\partial \lambda} = \frac{\partial(\beta_2 \lambda^2 + \beta_1 \lambda + \beta_0)}{\partial \lambda}$

$$e^{\lambda t} = \beta_2 \lambda^2 + \beta_1 \lambda + \beta_0 \rightarrow \text{for } \lambda=1,$$

$$e^t = \beta_2 + \beta_1 + \beta_0 \quad \textcircled{1}$$

for $\lambda=0$,

$$e^{0t} = 1 = 0 + 0 + \beta_0 \quad \textcircled{2}$$

And from the gradient equations, we get :

$$te^{\lambda t} = 2\beta_2 \lambda + \beta_1 \quad \text{for } \lambda=1,$$

$$te^t = 2\beta_2 + \beta_1 \quad \textcircled{3}$$

$$\textcircled{2} \Rightarrow \beta_0 = 1$$

$$\textcircled{1} + \textcircled{3} \text{ give : } te^t = 2\beta_2 + \beta_1 \quad \textcircled{4}$$

$$e^t - 1 = \beta_2 + \beta_1 \quad \textcircled{5}$$

$$\textcircled{4} - \textcircled{5} \Rightarrow \beta_2 = te^t - e^t + 1$$

$$\beta_2 = e^t(t-1) + 1$$

Substituting back,

$$\begin{aligned} \beta_1 &= e^t - 1 - (e^t(t-1) + 1) \\ &= e^t - 1 - te^t + e^t - 1 \\ &= e^t(2-t) - 2 \end{aligned}$$

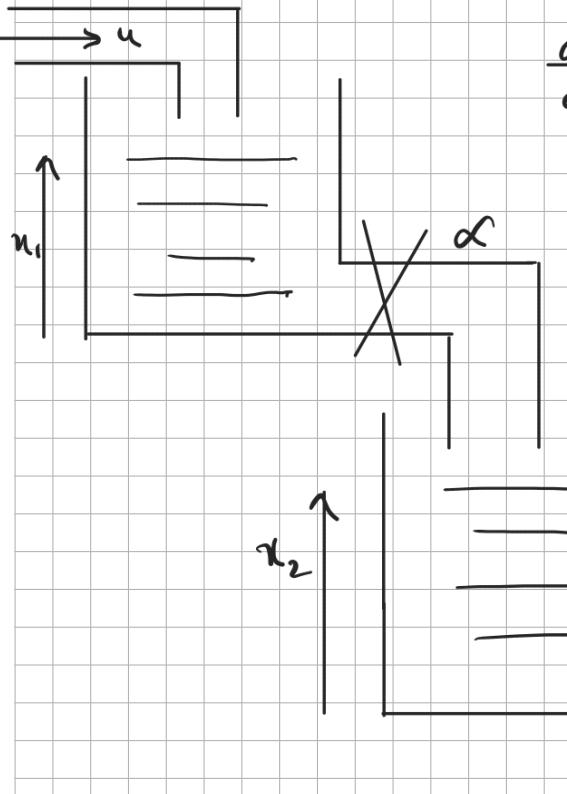
$$\Rightarrow e^{\lambda t} = (e^t(t-1) + 1) \lambda^2 + (e^t(2-t) - 2) \lambda + 1$$

$$\text{So, } e^{At} = (e^t(t-1) + 1) A^2 + (e^t(2-t) - 2) A + I$$

$$= \begin{bmatrix} e^t(t-1)+1 & e^t(t-1)+1 & e^t(t-1)+1 \\ 0 & 0 & e^t(t-1)+1 \\ 0 & 0 & e^t(t-1)+1 \end{bmatrix} + \begin{bmatrix} e^t(2-t)-2 & e^t(2-t)-2 & 0 \\ 0 & 0 & e^t(2-t)-2 \\ 0 & 0 & e^t(2-t)-2 \end{bmatrix} + I$$

Simplifying $\rightarrow e^{At} = \boxed{\begin{bmatrix} e^t & e^t-1 & e^t(t-1)+1 \\ 0 & 1 & e^t-1 \\ 0 & 0 & e^t \end{bmatrix}}$

Exercise 2



$$\frac{dx}{dt}_1 = -\alpha n_1 + u$$

$$\frac{dx_2}{dt} = \beta n_1 - \gamma n_2$$

$$\Rightarrow \begin{bmatrix} \dot{n}_1 \\ \dot{n}_2 \end{bmatrix} = \begin{bmatrix} -\alpha n_1 + u \\ \beta n_1 - \gamma n_2 \end{bmatrix}$$

Writing the equations in state-space form, where states are $[n_1 \ n_2]^T$

$$\begin{bmatrix} \dot{n}_1 \\ \dot{n}_2 \end{bmatrix} = \begin{bmatrix} -\alpha & 0 \\ \beta & -\gamma \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

Given, $\alpha = 0.1$, $\beta = 0.2$, $u = 1$, $n_1(0) = 2$, $n_2(0) = 1$

For any state-space system, we can get the following

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$e^{At}x(t) - e^{At}Ax(t) = e^{At}Bu(t)$$

$$\frac{d(e^{At}x(t))}{dt} = e^{At}Bu(t)$$

$$\Rightarrow e^{At}x(t) \Big|_{t_0}^t = \int_{t_0}^t e^{At}Bu(t) dt$$

$$\Rightarrow x(t) = e^{-A(t-t_0)}x(t_0) + e^{At} \int_{t_0}^t e^{-At}Bu(t) dt \quad \textcircled{A}$$

Computing e^{At} using Cayley - Hamilton theorem:

$$e^{At} = \beta_0 + \beta_1 A \text{ for scalar coefficients } \beta_0 \text{ and } \beta_1.$$

This equation is also satisfied by the eigenvalues of A .

To find eigenvalues of A :

$$\det(A - \lambda I) = 0 \Rightarrow (-\alpha - \lambda)(-\beta - \lambda) = 0$$

$$\Rightarrow \lambda = -\alpha, -\beta$$

$$\text{So, } e^{\lambda t} = \beta_0 + \beta_1 \lambda \Rightarrow e^{-\alpha t} = \beta_0 - \beta_1 \alpha \quad \textcircled{1}$$

$$e^{-\beta t} = \beta_0 - \beta_1 \beta \quad \textcircled{2}$$

$$\text{from } \textcircled{1} \text{ and } \textcircled{2}, e^{-\alpha t} - e^{-\beta t} = \beta_1(\beta - \alpha)$$

$$\beta_1 = \frac{e^{-\alpha t} - e^{-\beta t}}{\beta - \alpha}$$

$$\text{So, } \beta_0 = \frac{e^{-\alpha t} + \alpha(e^{-\alpha t} - e^{-\beta t})}{\beta - \alpha}$$

$$\Rightarrow \beta_0 = \frac{\beta e^{-\alpha t} - \alpha e^{\beta t}}{\beta - \alpha}$$

$$\alpha = 0.1, \beta = 0.2$$

$$\Rightarrow \beta_1 = 10(e^{-0.1t} - e^{-0.2t})$$

$$\beta_0 = 2e^{-0.1t} - e^{-0.2t}$$

$$\text{So, } e^{At} = (2e^{-0.1t} - e^{-0.2t}) \times I + 10(e^{-0.1t} - e^{-0.2t}) A$$

$$= \begin{bmatrix} 2e^{-0.1t} - e^{-0.2t} & 0 \\ 0 & 2e^{-0.1t} - e^{-0.2t} \end{bmatrix} + \begin{bmatrix} e^{-0.2t} - e^{-0.1t} & 0 \\ e^{-0.1t} - e^{-0.2t} & 2e^{-0.2t} - 2e^{-0.1t} \end{bmatrix}$$

$$= \begin{bmatrix} e^{-0.1t} & 0 \\ e^{-0.1t} - e^{-0.2t} & e^{-0.2t} \end{bmatrix}$$

Now to compute $x(t)$, we need to find

$e^{At} \int_{t_0}^t e^{-At} Bu(t) dt$. Computing the integral:

$$e^{-At} Bu(t) = \begin{bmatrix} e^{0.1t} & 0 \\ e^{0.1t} - e^{0.2t} & e^{0.2t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (1) = \begin{bmatrix} e^{0.1t} \\ e^{0.1t} - e^{0.2t} \end{bmatrix}$$

$$\int_{t_0}^t e^{-At} Bu(t) dt = \int_{t_0}^t \begin{bmatrix} e^{0.1t} \\ e^{0.1t} - 2e^{0.2t} \end{bmatrix} dt = \begin{bmatrix} 10e^{0.1t} \\ 10e^{0.1t} - 5e^{0.2t} \end{bmatrix} \Big|_{t_0}^t$$

$$t_0 = 0, t = t \Rightarrow \int e^{-At} Bu(t) dt = \begin{bmatrix} 10(e^{0.1t} - 1) \\ 10e^{0.1t} - 5e^{0.2t} - 5 \end{bmatrix}$$

$$\text{Now, } e^{At} \int_0^t e^{-At} Bu(t) dt = \begin{bmatrix} e^{-0.1t} & 0 \\ e^{-0.1t} - e^{-0.2t} & e^{-0.2t} \end{bmatrix} \begin{bmatrix} 10(e^{0.1t} - 1) \\ 10e^{0.1t} - 5e^{0.2t} - 5 \end{bmatrix}$$

$$= \begin{bmatrix} 10(1 - e^{-0.1t}) \\ 1 - e^{-0.1t} - e^{-0.2t} + e^{-0.2t} + 10e^{-0.1t} - 5 - 5e^{-0.2t} \end{bmatrix}$$

$$= \begin{bmatrix} 10(1 - e^{-0.1t}) \\ -4 + 3e^{-0.1t} - 4e^{-0.2t} \end{bmatrix}$$

Next computing $e^{-A(t-t_0)} x(t_0)$,

$$x(t_0) = x(0) = [2 \ 1]^T$$

$$e^{-At} x(0) = \begin{bmatrix} e^{0.1t} & 0 \\ e^{0.1t} - e^{0.2t} & e^{0.2t} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{0.1t} \\ 2e^{0.1t} - 2e^{0.2t} + e^{0.2t} \end{bmatrix} = \begin{bmatrix} 2e^{0.1t} \\ 2e^{0.1t} - e^{0.2t} \end{bmatrix}$$

$$\begin{aligned}
 x(t) &= e^{-At} x(0) + e^{At} \int_0^t e^{-At} B u dt \\
 &= \left[\begin{array}{c} 2e^{0.1t} \\ 2e^{0.1t} - e^{0.2t} \end{array} \right] + \left[\begin{array}{c} 10(1 - e^{-0.1t}) \\ -4 + 8e^{-0.1t} - 4e^{-0.2t} \end{array} \right] \\
 &= \left[\begin{array}{c} 2e^{0.1t} + 10 - 10e^{-0.1t} \\ 2e^{0.1t} + 8e^{-0.1t} - 4e^{-0.2t} - e^{0.2t} - 4 \end{array} \right]
 \end{aligned}$$

for $x(5)$, substituting $t = 5$, we get

$$x(5) = \boxed{\begin{bmatrix} 7.2321 \\ 1.3533 \end{bmatrix}}$$

Exercise 3

$$\begin{matrix} 1 & 12 & 20 \\ 0 & 2 & 0 \\ 0 & 0 & 12 \end{matrix}$$

$A_1 = \begin{bmatrix} 1 & 4 & 8 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Since matrix is triangular, $\lambda = 1, 2, 3$

For $\lambda=1$, $A_1 - \lambda_1 I = \begin{bmatrix} 0 & 4 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$$(A_1 - \lambda_1 I) v_1 = 0 \Rightarrow \begin{bmatrix} 0 & 4 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = 0 \Rightarrow \begin{array}{l} v_{12} + 2v_{13} = 0 \\ v_{13} = 0 \\ \Rightarrow v_{12} = 0 \end{array}$$

$$\therefore v_1 = [1 \ 0 \ 0]^T$$

For $\lambda=2$, $A_1 - \lambda_2 I = \begin{bmatrix} -1 & 4 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$(A_1 - \lambda_2 I) v_2 = 0 \Rightarrow \begin{bmatrix} -1 & 4 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} = 0 \Rightarrow v_{23} = 0, -v_{21} + 4v_{22} + 8v_3 = 0 \\ v_{21} = 4v_{22}$$

$$\Rightarrow v_2 = [4 \ 1 \ 0]^T$$

For $\lambda=3$, $A_1 - \lambda_3 I = \begin{bmatrix} -2 & 4 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$(A_1 - \lambda_3 I) v_3 = 0 \Rightarrow \begin{bmatrix} -2 & 4 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_{31} \\ v_{32} \\ v_{33} \end{bmatrix} = 0$$

$$v_{32} = 0; -2v_{31} + 4v_{32} + 8v_{33} = 0$$

$$\Rightarrow v_{31} = 4v_{33}$$

$$\therefore v_3 = [4 \ 0 \ 1]^T$$

Since all eigenvalues are distinct, the modal matrix is S , where $S = [v_1 \ v_2 \ v_3]$

$$\Rightarrow S = \begin{bmatrix} 1 & 4 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore A_1 = S A S^{-1} = \begin{bmatrix} 1 & 4 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -4 & -4 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix}$$

To find eigenvalues, $\det(A_2 - \lambda I) = 0$

$$\begin{aligned} \det(A_2 - \lambda I) &= \det \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -2 & -4 & -3-\lambda \end{bmatrix} = -\lambda((3+\lambda)\lambda+4) - (2) \\ &= -\lambda(\lambda^2 + 3\lambda + 4) - 2 \\ &= -(\lambda^3 + 3\lambda^2 + 4\lambda + 2) \end{aligned}$$

$$\det(A_2 - \lambda I) = 0 \Rightarrow \lambda^3 + 3\lambda^2 + 4\lambda + 2 = 0$$

$$(\lambda + 1)(\lambda^2 + 2\lambda + 2) = 0 \Rightarrow \lambda = -1, -1+i, -1-i$$

All eigenvalues are distinct.

$$\Rightarrow (A - \lambda I)v = 0$$

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -2 & -4 & -3-\lambda \end{bmatrix}$$

for $\lambda_1 = -1$,

$$A - \lambda_1 I = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -2 & -4 & -2 \end{bmatrix}$$

$$(A - \lambda_1 I)v_1 = 0 \Rightarrow v_{11} + v_{12} = 0; v_{12} + v_{13} = 0$$

$$-2v_{11} - 4v_{12} - 2v_{13} = 0$$

$$\text{Substituting, } -2(-v_{12}) - 4v_{12} - 2(-v_{12}) = 0$$

$$\Rightarrow 4v_{12} - 4v_{12} = 0$$

This holds for any value of v_{12}

$$\therefore v_1 = [-2 \quad 2 \quad -2]^T$$

$$\text{For } \lambda_2 = -1+i, A - \lambda_2 I = \begin{bmatrix} 1-i & 1 & 0 \\ 0 & 1-i & 1 \\ -2 & -4 & 2-i \end{bmatrix}$$

$$(A - \lambda_2 I) v_2 = 0.$$

$$v_{21}(1-i) + v_{22} = 0 \quad \text{--- (1)}$$

$$v_{22}(1-i) + v_{23} = 0 \quad \text{--- (2)}$$

$$-2v_{21} - 4v_{22} + (2-i)v_{23} = 0 \quad \text{--- (3)}$$

$$v_{21}(1-i) - \frac{v_{23}}{(1-i)} = 0$$

$$-2iv_{21} = v_{23}$$

Substituting from (1) and (2) into (3),

$$\left. \begin{aligned} -2\left(-\frac{v_{23}}{1-i}\right) - 4\left(-\frac{v_{23}}{1-i}\right) + (2-i)v_{23} = 0 \end{aligned} \right\} \text{ holds for all values of } v_{23}$$

$$v_2 = \left[\frac{i}{2}, -\frac{(1+i)}{2}, 1 \right]$$

$$\text{For } \lambda_3 = -1-i, A - \lambda_3 I = \begin{bmatrix} 1+i & 1 & 0 \\ 0 & 1+i & 1 \\ -2 & -4 & -2+i \end{bmatrix}$$

$$(A - \lambda_3 I) v_3 = 0 \Rightarrow$$

$$(v_{31})(1+i) + v_{32} = 0$$

$$(1+i)v_{32} + v_{33} = 0$$

$$-\frac{v_{33}(1-i)^2}{4}$$

$$-2v_{31} - 4v_{32} + (-2+i)v_{33} = 0$$

Substituting, we get

$$\left. \begin{aligned} -2\left(-\frac{v_{33}i}{2}\right) - 4\left(-\frac{v_{33}(1-i)}{2}\right) + (-2+i)v_{33} = 0 \end{aligned} \right\}$$

\curvearrowleft holds for all v_{33} .

$$\text{Thus, } v_3 = \left[-\frac{1}{2}, -\frac{(1-i)}{2}, 1 \right]$$

$\Rightarrow A_2 = S \Lambda S^{-1}$, where

$$S = \begin{bmatrix} -1 & \frac{i}{2} & -\frac{i}{2} \\ 1 & -(1+i)/2 & -(1-i)/2 \\ -1 & 1 & 1 \end{bmatrix} \quad \Lambda = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1+i & 0 \\ 0 & 0 & -i-i \end{bmatrix}$$

$$S^{-1} = \begin{bmatrix} -2 & -2 & -1 \\ -1+i & -1+2i & i \\ -1-i & -1-2i & -i \end{bmatrix}$$

$$\Rightarrow A_2 = \begin{bmatrix} -1 & i/2 & -i/2 \\ 1 & -(1+i)/2 & -(1-i)/2 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1+i & 0 \\ 0 & 0 & -i-i \end{bmatrix} \begin{bmatrix} -2 & -2 & -1 \\ -1+i & -1+2i & i \\ -1-i & -1-2i & -i \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \therefore \lambda = 1, 1, 2$$

Algebraic multiplicity of 1 : 2 and for 2 : 1

$$\text{For } \lambda_1 = 2, A - \lambda_1 I = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(A - \lambda_1 I) v_1 = 0 \Rightarrow -v_{11} - v_{13} = 0, v_{12} = 0$$

$$\therefore v_1 = [1 \ 0 \ -1]^T$$

$$\text{For } \lambda_2 = 1, A - \lambda_2 I = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(A - \lambda_2 I) v_2 = 0 \Rightarrow v_3 = 0$$

\therefore we can get two vectors that satisfies

$$(A - \lambda I) v = 0 \text{ for } \lambda = 1.$$

$$v_2 = [1 \ 1 \ 0]^T \text{ and } v_3 = [1 \ 2 \ 0]^T$$

$$\therefore S = [v_1 \ v_2 \ v_3] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ -1 & 0 & 0 \end{bmatrix}$$

$$A_3 = S \Lambda S^{-1} \text{ where } S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ -1 & 0 & 0 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Writing in order of increasing eigenvalues, we get

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}; \quad S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

$$S^{-1} = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\Rightarrow A_3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \\ -1 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix}$$

$$A_4 - \lambda I = \begin{bmatrix} -\lambda & 4 & 3 \\ 0 & 20-\lambda & 16 \\ 0 & -25 & -20-\lambda \end{bmatrix}$$

$$\det(A_4 - \lambda I) = -\lambda((20-\lambda)(-20-\lambda) + (25 \times 16))$$

Setting $\det(A_4 - \lambda I) = 0$, we get

$$\lambda(\lambda^2 - 400 + 400) = 0 \Rightarrow \lambda = 0, 0, 0$$

We have three repeated eigenvalues of value 0.

To find geometric multiplicity, we can do $(A - \lambda I)v = 0$

$$\Rightarrow \begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = 0 \quad \text{where } \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = v_1$$

$$\Rightarrow 4v_{12} + 3v_{13} = 0 ; 5v_{12} + 4v_{13} = 0 ; 5v_{12} + 4v_{13} = 0$$

$$\rightarrow v_{12} = v_{13} = 0 \Rightarrow v_1 = [1 \ 0 \ 0]^T$$

So, geometric multiplicity of $\lambda = 0$ is 1.
We need to compute 2 generalized eigenvectors for $\lambda = 0$.

First, $(A - \lambda I) v_2 = v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$4v_{22} + 3v_{23} = 1 ; 5v_{22} + 4v_{23} = 0$$

Substituting, we get $4v_{22} + 3\left(-\frac{5}{4}v_{22}\right) = 1$

$$\Rightarrow v_{22} = 4 \quad \text{And} \quad v_{23} = -5$$

$$\therefore v_2 = [0 \ 4 \ -5]^T$$

Now, $(A - \lambda I) v_3 = v_2$

$$\Rightarrow 4v_{32} + 3v_{33} = 0 ; 20v_{32} + 16v_{33} = 4$$

$$-25v_{32} - 20v_{33} = -5$$

$$\Rightarrow 4v_{32} + 3v_{33} = 0$$

$$5v_{32} + 4v_{33} = 1$$

Substituting, we get

$$5v_{32} + 4\left(-\frac{4v_{32}}{3}\right) = 1 \Rightarrow -v_{32} = 3$$

$$\text{And} \quad v_{33} = 4$$

$$\Rightarrow V_3 = \begin{bmatrix} 0 & -3 & 4 \end{bmatrix}^T$$

$$\therefore S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & -3 \\ 0 & -5 & 4 \end{bmatrix} \Rightarrow S^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{3}{4} \\ 0 & \frac{5}{4} & \frac{1}{4} \end{bmatrix}$$

And $S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$$\Rightarrow A_F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & -3 \\ 0 & -5 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 5 & 4 \end{bmatrix}$$

Exercise 4

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 2 & 3 \end{bmatrix} x(t)$$

Given $x(0) = 0$
 $u = \text{unit step input}$.

i) Continuous Time

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

y at time t is given by:

$$y(t) = Ce^{\lambda(t-t_0)}x(t_0) + C \int_{t_0}^t e^{\lambda(t-\tau)}Bu(\tau)d\tau + Du(t)$$

To compute $e^{\lambda t}$, finding diagonal or Jordan form of A :

$$\det(A - \lambda I) = 0 \Rightarrow (-\lambda)(-2 - \lambda) - (-2)(1) = 0$$

$$\lambda^2 - 2\lambda + 2 = 0 \Rightarrow \lambda = -1 \pm i$$

Since we have distinct eigenvalues, we can diagonalize A .
 $(A - \lambda I)v = 0$ gives eigenvectors.

$$\text{For } \lambda = -1 + i, (A - \lambda I)v_1 = \begin{bmatrix} 1-i & 1 \\ -2 & -1-i \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$$

$$v_{12} = -(1-i)v_{11}$$

$$\Rightarrow v_1 = [1 \quad -(1+i)]^T$$



$$\text{For } \lambda = -1 - i, (A - \lambda I)v_2 = \begin{bmatrix} 1+i & 1 \\ -2 & -1+i \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = 0$$

$$v_{22} = -(1+i)v_{21}$$

$$\Rightarrow v_2 = [1 \quad -(1+i)]^T$$

So, $A = S \Lambda S^{-1}$, where $\Lambda = \begin{bmatrix} -1+i & 0 \\ 0 & -1-i \end{bmatrix}$ and $S = \begin{bmatrix} 1 & 1 \\ -(1+i) & 1 \end{bmatrix}$

$$\text{Now, } y(t) = Ce^{(t-t_0)}y(t_0) + C \int_{t_0}^t e^{\alpha(t-\tau)}Bu(\tau)d\tau + \overset{\circ}{\beta} u(t)$$

$$t_0 = 0, \quad n(t_0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \quad \overset{\circ}{\beta} = 0$$

$$\Rightarrow y(t) = C \int_0^t S e^{\alpha(t-\tau)} S^{-1} Bu(\tau)d\tau$$

Also since α is a unit step input, $u(t) = 1 \forall t \geq 0$

Computing $S e^{\alpha(t-\tau)} S^{-1}$,

$$\begin{aligned} e^{\alpha(t-\tau)} &= \begin{bmatrix} 1 & 1 \\ -1+i & -1-i \end{bmatrix} \begin{bmatrix} e^{(1+i)t} & 0 \\ 0 & e^{-(1+i)t} \end{bmatrix} \begin{bmatrix} \frac{1-i}{2} \\ \frac{1+i}{2} \end{bmatrix} \begin{bmatrix} \frac{-i}{2} \\ \frac{i}{2} \end{bmatrix} \\ &= \begin{bmatrix} e^{-(1-i)t} & e^{-(1+i)t} \\ -(1-i)e^{-(1-i)t} & -(1+i)e^{-(1+i)t} \end{bmatrix} \begin{bmatrix} \frac{1-i}{2} & \frac{-i}{2} \\ \frac{1+i}{2} & \frac{i}{2} \end{bmatrix} \\ &= \begin{bmatrix} e^{-(1-i)t} \left(\frac{1-i}{2}\right) + e^{-(1+i)t} \left(\frac{1+i}{2}\right) & -e^{-(1-i)t} \left(\frac{i}{2}\right) + e^{-(1+i)t} \left(\frac{i}{2}\right) \\ e^{-(1-i)t} \left(i\right) - e^{-(1+i)t} \left(i\right) & \left(\frac{i+1}{2}\right) e^{-(1-i)t} + \left(\frac{1-i}{2}\right) e^{-(1+i)t} \end{bmatrix} \\ &= e^{-t} \begin{bmatrix} e^{it} \left(\frac{1-i}{2}\right) + e^{-it} \left(\frac{1+i}{2}\right) & \frac{i}{2} \left(-e^{it} + e^{-it}\right) \\ i \left(e^{it} - e^{-it}\right) & e^{it} \left(\frac{1+i}{2}\right) + e^{-it} \left(\frac{1-i}{2}\right) \end{bmatrix} \end{aligned}$$

Writing $e^{it} = \cos t + i \sin t$ and simplifying, we get:

$$= e^{-t} \begin{bmatrix} (\cos t + i \sin t) - i(\cos t + \sin t) + \cos t - i \sin t + i \cos t + \sin t \\ i(2i \sin t) \end{bmatrix}, \quad \frac{i}{2} \left(-2i \sin t\right)$$

$$= e^{-t} \begin{bmatrix} \sin t + \cos t & \sin t \\ -2 \sin t & \cos t - \sin t \end{bmatrix}$$

So for $e^{A(t-\tau)}$, we get:

$$e^{-(t-\tau)} \begin{bmatrix} \cos(t-\tau) + \sin(t-\tau) & \sin(t-\tau) \\ -2\sin(t-\tau) & \cos(t-\tau) - \sin(t-\tau) \end{bmatrix}$$

$$\int s e^{n(t-\tau)} s^{-1} B u \text{ becomes : } \int_{-\infty}^t e^{-A(t-\tau)} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} d\tau$$

$$= \int_{-\infty}^t e^{-(t-\tau)} \begin{bmatrix} \cos(t-\tau) + \sin(t-\tau) & \sin(t-\tau) \\ -2\sin(t-\tau) & \cos(t-\tau) - \sin(t-\tau) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} d\tau$$

$$= \int_0^t e^{-(t-\tau)} \begin{bmatrix} \cos(t-\tau) + 2\sin(t-\tau) & \sin(t-\tau) \\ \cos(t-\tau) - 3\sin(t-\tau) & \end{bmatrix} d\tau$$

Setting $t-\tau = u$ and computing integral, we get:

$$\int s e^{n(t-\tau)} s^{-1} B u d\tau = \left[-e^{-u} \left(\frac{3\cos u + \sin u}{2} \right) \right]_0^t + \left[e^{-u} (\cos u + 2\sin u) \right]_0^t$$

$$= \left[-e^{-t} \left(\frac{3\cos t + \sin t}{2} \right) + (3) \right] + \left[e^{-t} (\cos t + 2\sin t) - (1) \right]$$

Now,

$$y(t) = c \times \int () = [2 \ 3] \left[\frac{3 - e^{-t}(3\cos t + \sin t)}{2} \right] + \left[e^{-t}(\cos t + 2\sin t) - 1 \right]$$

$$= 3 - e^{-t}(3\cos t + \sin t) + 3e^{-t}(\cos t + 2\sin t) - 3$$

$$= 5e^{-t} \sin t$$

$$\Rightarrow y(t) = 5e^{-t} \sin t$$

$$\text{At } t=5, \boxed{y(5) = 5e^{-5} \sin(5)}$$

ii) In discrete time, the system is given as:

$$x(k+1) = A_d x(k) + B_d u(k)$$

$$y(k) = C_d x(k) + D_d u(k)$$

where, $C_d = C$, $D_d = D$, $A_d = e^{AT}$, $B_d = A^{-1}(A_d - I)B$

$$A_d = e^{AT} = Se^{1t}s^{-1} = \begin{bmatrix} e^{-T}(\cos T + i \sin T) & e^{-T}i \sin T \\ -2e^{-T}i \sin T & e^{-T}(cos T - i \sin T) \end{bmatrix}$$

where $T = 1s$ is the sample time.

For ease of computing B_d , substituting value of T into A_d

$$A_d = \begin{bmatrix} e^{-1}(\cos 1 + i \sin 1) & e^{-1}i \sin 1 \\ -2e^{-1}i \sin 1 & e^{-1}(\cos 1 - i \sin 1) \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -1 & -\frac{1}{2} \\ 1 & 0 \end{bmatrix}$$

$$\text{so } B_d = \begin{bmatrix} -1 & -\frac{1}{2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^{-1}(\cos 1 + i \sin 1) - 1 & e^{-1}i \sin 1 \\ -2e^{-1}i \sin 1 & e^{-1}(\cos 1 - i \sin 1) - 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -\frac{1}{2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^{-1}(\cos 1 + 2i \sin 1) - 1 \\ e^{-1}(\cos 1 - 3i \sin 1) - 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 - e^{-1}(\cos 1 + 2i \sin 1) + \frac{1}{2} - \frac{e^{-1}(\cos 1 - 3i \sin 1)}{2} \\ e^{-1}(\cos 1 + 2i \sin 1) - 1 \end{bmatrix}$$

$$B_d = \begin{bmatrix} \frac{3}{2} - e^{-1}\left(\frac{3}{2}\cos 1 + \frac{1}{2}i \sin 1\right) \\ e^{-1}(\cos 1 + 2i \sin 1) - 1 \end{bmatrix}$$

$$\text{And } u(k) = 1 \quad \forall k \geq 0$$

$$\Rightarrow \mathbf{x}(k+1) = \begin{bmatrix} e^{-1}(\cos 1 + \sin 1) & e^{-1}\sin 1 \\ -2e^{-1}\sin 1 & e^{-1}(\cos 1 - \sin 1) \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} \frac{3}{2} & -\frac{e^{-1}(3\cos 1 + \sin 1)}{2} \\ e^{-1}(\cos 1 + 2\sin 1) - 1 & \end{bmatrix}$$

$$y(k) = [2 \ 3] \mathbf{x}_k$$

iii) For $y(5)$,

$$y(k) = C A_d^k \mathbf{x}(0) + \sum_{i=0}^{k-1} C A_d^{k-i-1} B_d u(i) + \overset{\circ}{D_d} u(k)$$

$$\mathbf{x}(0) = [0] ; D_d = [0] \text{ and } a(i) = 1 + i \geq 0$$

$$\Rightarrow y(k) = \sum_{i=0}^{k-1} C A_d^{k-i-1} B_d$$

$$\text{For } k=5, y(5) = \sum_{i=0}^4 C A_d^{4-i} B_d$$

$$= C A_d^4 B_d + C A_d^3 B_d + C A_d^2 B_d + C A_d B_d + C B_d$$

Computing A_d^2 , A_d^3 and A_d^4 using computer:

$$A_d^2 = \begin{bmatrix} e^{-2}(\cos 2 + \sin 2) & e^{-2}\sin 2 \\ -2e^{-2}\sin 2 & e^{-2}(\cos 2 - \sin 2) \end{bmatrix}$$

$$A_d^3 = \begin{bmatrix} e^{-3}(\cos 3 + \sin 3) & e^{-3}\sin 3 \\ -2e^{-3}\sin 3 & e^{-3}(\cos 3 - \sin 3) \end{bmatrix}$$

$$A_d^4 = \begin{bmatrix} e^{-4}(\cos 4 + \sin 4) & e^{-4}\sin 4 \\ -2e^{-4}\sin 4 & e^{-4}(\cos 4 - \sin 4) \end{bmatrix}$$

Now, we have to compute $C A_d^m B$ for $m=0, 1, 2, 3, 4$.
Performing the computations on a computer, we get

$$CA_d^4 B_d = 5e^{-5} \sin 5 - 5e^{-4} \sin 4$$

$$CA_d^3 B_d = 5e^{-4} \sin 4 - 5e^{-3} \sin 3$$

$$CA_d^2 B_d = 5e^{-3} \sin 3 - 5e^{-2} \sin 2$$

$$CA_d B_d = 5e^{-2} \sin 2 - 5e^{-1} \sin 1$$

$$CB_d = 5e^{-1} \sin 1$$

Jamming them up, we get:

$$y(5) = 5e^{-5} \sin 5$$

↗ matches with continuous time result.

Plots and python code attached below

Exercise 5

$$F_{k+2} = F_k + F_{k+1}$$

To compute: F_{20}

Denoting F_m as $x(m)$, we get:

$$x(m+2) = x(m) + x(m+1)$$

If we choose a state matrix $x(k) = \begin{bmatrix} x(k+1) \\ x(k) \end{bmatrix}$

we get: $x(k+1) = \begin{bmatrix} x(k+2) \\ x(k+1) \end{bmatrix}$ given by

$$x(k+1) = \begin{bmatrix} x(k+2) \\ x(k+1) \end{bmatrix} = \begin{bmatrix} x(k+1) + x(k) \\ x(k+1) \end{bmatrix}$$

Writing in state-space form:

$$\underbrace{\begin{bmatrix} x(k+2) \\ x(k+1) \end{bmatrix}}_{x(k+1)} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x(k+1) \\ x(k) \end{bmatrix}}_{x(k)} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We need to compute $x(20)$, which will be present in $x(19)$ as $x(19) = \begin{bmatrix} x(20) \\ x(19) \end{bmatrix}$

For a discrete system, $x(k)$ is given by:

$$x(k) = A_d^k x(0) + \sum_{m=0}^{k-1} A_d^{k-m-1} B_d u(m)$$

Since B_d is zero, this reduces to:

$$x(k) = \dots$$

$$\text{So, } x(19) = A_d^{19} x(0) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{19} \begin{bmatrix} x(1) \\ x(0) \end{bmatrix}$$

Using a calculator to compute A_d^{19} , we get

$$A_d^{19} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{19} = \begin{bmatrix} 6765 & 4181 \\ 4181 & 2584 \end{bmatrix}$$

$$\text{Thus, } A_d^{19} x(0) = A_d^{19} \begin{bmatrix} x(1) \\ x(0) \end{bmatrix}$$

$$x(1) = F_1 = 1, \quad x(0) = F_0 = 0$$

$$\begin{aligned} \Rightarrow A_d^{19} x(0) &= \begin{bmatrix} 6765 & 4181 \\ 4181 & 2584 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 6765 \\ 4181 \end{bmatrix} \end{aligned}$$

$$\text{So, } x(20) = F_{20} = 6765$$