Exercise 1

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} x$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} ; \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} ; \quad C = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$

For controllability:

rank$(P) = n$, where $P = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$

$$A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & -3 & -3 \\ 3 & 8 & 6 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & 0 & 1 \\ -1 & -3 & -3 \\ 3 & 8 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$$

$$\Rightarrow P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\text{rank}(P) = 2 < 3 = n$$

So the system is <u>not</u> fully controllable.

For observability:

rank$(Q) = n$, where $Q = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}$

$$CA = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -1 \end{bmatrix}$$

$$CA^2 = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -1 & -3 & -3 \\ 3 & 8 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 1 & 2 & 1 \end{bmatrix} \qquad \left(R_3 = R_1 = -2R_2 \right)$$

$$\text{rank}(Q) = 1$$

So the system is not observable.

Exercise 2

$$\dot{x} = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 2 & 2 & 1 & 3 & -1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} x$$

To check for controllability:

$$\left[\begin{array}{ccccccc|ccc} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

Forming Jordan blocks and checking rank of the matrices formed from $\hat{B}$'s for the corresponding lowest rows.

$$\hat{B}^2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\text{rank}(\hat{B}^2) = 3$$

$$\hat{B}^1 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\text{rank}(\hat{B}^1) = 2$$

Since they are both full rank, we have full <u>controlability</u>.

For observability:

$$\left[\begin{array}{ccccccc} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 2 & 2 & 1 & 3 & -1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{array}\right]$$

$$\hat{C}^2 = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \longrightarrow \text{rank}(\hat{C}^2) = 2 \Rightarrow \text{not full rank}$$

$$\hat{C}^1 = \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \longrightarrow \text{rank}(\hat{C}^1) = 2 \Rightarrow \text{full rank}$$

Since $\hat{C}^2$ is not full rank. The corresponding states are <u>not</u> fully observable.

Exercise 3

$$\frac{dx_1}{dt} = -\alpha x_1 + u$$

$$\frac{dx_2}{dt} = \alpha x_1 - \beta x_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\alpha & 0 \\ \alpha & -\beta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$A = \begin{bmatrix} -\alpha & 0 \\ \alpha & -\beta \end{bmatrix} ; \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

For controllability, $\text{rank}(P) = 2$

where $P = [B \quad AB]$

$$AB = \begin{bmatrix} -\alpha & 0 \\ \alpha & -\beta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\alpha \\ \alpha \end{bmatrix}$$

$$\Rightarrow P = \begin{bmatrix} 1 & -\alpha \\ 0 & \alpha \end{bmatrix} \longrightarrow \text{rank}(P) = 2 \quad \forall \alpha \neq 0$$

So for nonzero α, the system is controllable.

Modified system:

$$\frac{dx_1}{dt} = -\alpha x_1$$

$$\frac{dx_2}{dt} = \alpha x_1 - \beta x_2 + u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\alpha & 0 \\ \alpha & -\beta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

So A is still the same $A = \begin{bmatrix} -\alpha & 0 \\ \alpha & -\beta \end{bmatrix}$

B is now: $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

So $AB = \begin{bmatrix} -\alpha & 0 \\ \alpha & -\beta \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -\beta \end{bmatrix}$

$$\Rightarrow P = [B \quad AB] = \begin{bmatrix} 0 & 0 \\ 1 & -\beta \end{bmatrix}$$

$\text{rank}(P) = 1$

So system is not controllable anymore.

Exercise 4

1.

a) $x + y + z = 3$
$x + 2y + 3z = 0$
$x + 3y + 2z = 3$

If $a = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $b = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix} a = b = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$$

To solve for a: $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 0 \\ 1 & 3 & 2 & 3 \end{array}\right]$ making the augmented matrix and performing row operations.

$R_2 = R_2 - R_1 \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & -3 \\ 1 & 3 & 2 & 3 \end{array}\right]$

$R_3 = R_3 - R_1 - 2R_2 \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & -3 & 6 \end{array}\right]$

So we get: $-3z = 6$; $y + 2z = -3$; $x + y + z = 3$

$\Rightarrow \boxed{z = -2,\ y = 1,\ x = 4}$

b) $x + 2y - z = 1$
$2x + 5y - z = 3$
$x + 3y + 2z = 6$

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & -1 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$$

Augmented matrix: $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 5 & -1 & 3 \\ 1 & 3 & 2 & 6 \end{array}\right]$

$R_2 = R_2 - 2R_1 \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 3 & 2 & 6 \end{array}\right]$

$R_3 = R_3 - R_1 - R_2 \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 4 \end{array}\right]$

So, $2z = 4$, $y + z = 1$, $x + 2y - z = 1$

$\Rightarrow \boxed{z = 2, \ y = -1, \ x = 5}$

c) $x_1 + x_2 - x_3 + x_4 = 1$

$2x_1 + 3x_2 + x_3 - 0x_4 = 4$

$3x_1 + 5x_2 + 3x_3 - x_4 = 5$

$$\begin{bmatrix} 1 & 1 & -1 & 1 \\ 2 & 3 & 1 & 0 \\ 3 & 5 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$$

Augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 2 & 3 & 1 & 0 & 4 \\ 3 & 5 & 3 & -1 & 5 \end{array}\right]$$

$R_2 = R_2 - 2R_1 \Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 3 & -2 & 2 \\ 3 & 5 & 3 & -1 & 5 \end{array}\right]$

$R_3 = R_3 - 3R_1 - 2R_2 \Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 3 & -2 & 2 \\ 0 & 0 & 0 & 0 & -2 \end{array}\right]$

We can see that the last row says $0 = -2$, which is a contradiction.

2.

$x_1 + 2x_2 + 4x_3 = 3$

$3x_1 + 8x_2 + 14x_3 = 13$

$2x_1 + 6x_2 + 13x_3 = 4$

$A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$; $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; $b = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$

$A = LU = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

$LU = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$

$\Longrightarrow u_{11} = 1$, $u_{12} = 2$, $u_{13} = 4$

$l_{21} = 3/1 = 3$, $u_{22} = 8 - 3 \times 2 = 2$, $u_{23} = 14 - 3 \times 4 = 2$

$l_{31} = 2/1 = 2$, $l_{32} = \frac{6 - 2 \times 2}{2} = 1$, $u_{33} = 13 - 2 \times 4 - 1 \times 2 = 3$

$$\Rightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Now to solve $Ax = b$, first we solve $Ly = b$, then $Ux = y$

$$Ly = b \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$$

$$\Rightarrow y_1 = 3, \; y_2 = 10, \; y_3 = -12$$

Now, $Ux = y$:

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \\ -12 \end{bmatrix}$$

$$\Rightarrow x_3 = -4, \; x_2 = \frac{10 - 2(-4)}{2} = 9, \; x_1 = 3 - 2(9) - 4(-4) = 1$$

$$\Longrightarrow \boxed{x_1 = 1, \; x_2 = 9, \; x_3 = -4}$$

Exercise 5

Compressed to 50% (337 singular values):

Compressed to 10% (67 singular values):

Compressed to 5% (33 singular values):

In [2]:

```python
from PIL import Image
import numpy as np

path = 'CMU_Grayscale.png'
img = Image.open(path).convert('L')
img.show()
img_matrix = np.array(img)

def compress(image_array, factor):
    # Perform SVD on the image
    U, S, V = np.linalg.svd(image_array, full_matrices=False)

    # Number of singular values to keep
    k = int(factor * len(S))
    print("Number of singular values used: ", k)

    # Reconstruct the image using the first k singular values
    S_k = np.diag(S[:k])
    U_k = U[:, :k]
    Vt_k = V[:k, :]

    compressed_image = np.dot(np.dot(U_k, S_k), Vt_k)

    return compressed_image


factors = [0.5, 0.1, 0.05]

for ratio in factors:
    compressed_image = compress(img_matrix, ratio)
    image_array_clipped = np.clip(compressed_image, 0, 255).astype(np.uint8)

    filename = f'compressed_{int(ratio * 100)}.png'
    Image.fromarray(image_array_clipped).save(filename)
```

```
Number of singular values used:  337
Number of singular values used:  67
Number of singular values used:  33
```

Exercise 6

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ \gamma & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

1. Controllable but not observable.

Constructing controllability matrix P:

$$AB = \begin{bmatrix} -3 & 3 \\ \gamma & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ \gamma \end{bmatrix}$$

$$P = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & \gamma \end{bmatrix}$$

For P to be full rank, i.e. $\text{rank}(P) = 2$, $\gamma \neq 0$.

So if $\gamma \neq 0$, system is controllable.

For observability, constructing observability matrix Q:

$$CA = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} -3 & 3 \\ \gamma & -4 \end{bmatrix} = \begin{bmatrix} (-3+\gamma) & -1 \end{bmatrix}$$

$$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ (-3+\gamma) & -1 \end{bmatrix}$$

For $\text{rank}(Q) < 2$, $(-3+\gamma) = -1 \Rightarrow \gamma = 2$

$$\gamma = 2 \Rightarrow Q(\gamma = 2) = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \rightarrow \text{rank}(Q(\gamma = 2)) = 1$$

So at $\gamma = 2$, system would not be observable.

And since for $\gamma = 2$, $\gamma \neq 0$, system is controllable.

$\boxed{\gamma = 2 \Rightarrow \text{controllable, not observable}}$

2. For observable, but not controllable,

$\text{rank}(P) < 2$ and $\text{rank}(Q) = 2$

$$Q = \begin{bmatrix} 1 & 1 \\ -3+\gamma & -1 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & -3 \\ 0 & \gamma \end{bmatrix}$$

For $\text{rank}(Q) = 2$, $-3+\gamma \neq -1 \Rightarrow \gamma \neq 2$

For $\text{rank}(P) < 2$, $\gamma = 0$, as for any other value of γ, $(1,-3)$ and $(0,\gamma)$ are linearly independant.

So at $\gamma = 0$, $\text{rank}(P) = 1$ and $\text{rank}(Q) = 2 \Rightarrow$ observable, not controllable

$\gamma = 0 \Rightarrow$ ob

Exercise 1

1	2	3	4	5

$x_i[k] \rightsquigarrow$ brightness of i^{th} LED at k^{th} time step

$u[k] \rightsquigarrow$ brightness command at k^{th} time step

for $i = 2, 3, 4, 5$ $\quad x_i[k+1] = x_{i-1}[k]$

for first LED, $\quad x_i[k+1] = u[k]$

1) State-space representation:

$$x[k] = \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \\ x_5(k) \end{bmatrix}$$

$$x[k+1] = \begin{bmatrix} x_1[k+1] \\ x_2[k+1] \\ x_3[k+1] \\ x_4[k+1] \\ x_5[k+1] \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1[k] \\ x_2[k] \\ x_3[k] \\ x_4[k] \\ x_5[k] \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{B} u[k]$$

$$y[k] = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}}_{C} \begin{bmatrix} x_1[k] \\ x_2[k] \\ x_3[k] \\ x_4[k] \\ x_5[k] \end{bmatrix}$$

2. For controllability, computing controllability matrix P:

$$P = [B \;\; AB \;\; A^2B \; \cdots \; A^4B]$$

using the ctrb(A,B) function in MATLAB to compute the controllability matrix:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \text{rank}(P) = 5 \text{ i.e. full rank.}$$

So the system is fully controllable.

In this scenario, full controllability implies that given an initial set of brightnesses for the 5 LED's, we can find a set of input commands $u(t)$ to take the brightness levels of all 5 LED's to any arbitrary brightness levels within certain limits.