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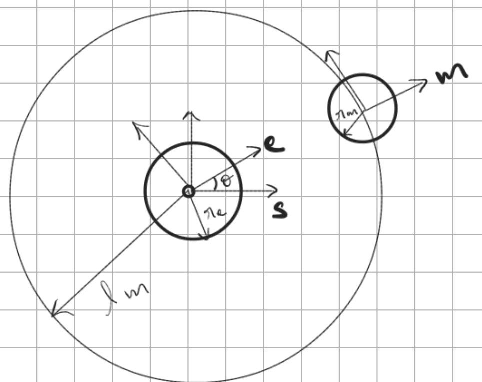
Problems 1) Lunar Motion



Note: 2D homogenous transformations have been used unless 3-D versions are necessary. For ex:

$q = [a \;\; b]^T$, $\bar{q} = [a \;\; b \;\; 1]^T$.

$R = \begin{pmatrix} \cos & -\sin \\ \sin & \cos \end{pmatrix}$; $g = \begin{bmatrix} R & P \\ 0_2 & 1 \end{bmatrix} \rightarrow 3\times3$

1.1) R_{se} is given by $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Now when $+x$ of e is aligned with $-y$ of s, θ would be $270°$.

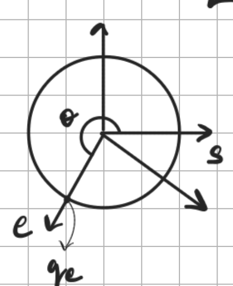
$\Rightarrow R_{se} = \begin{bmatrix} \cos 3\pi/2 & -\sin 3\pi/2 \\ \sin 3\pi/2 & \cos 3\pi/2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (Note: 2D rotation matrix)

g_{se} would be: $\begin{bmatrix} R_{se} & P_{se} \\ 0 \;\; 0 & 1 \end{bmatrix}$

Since, s and e share the same origins, $P_{se} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow g_{se} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\left(\text{In 3D} \quad g_{se} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right)$

1.2)



Given $q_e = [0 \;\; r_e \;\; 0]^T$ and $\theta = 270°$

$q_e = [0 \;\; r_e] \Rightarrow$ Utilizing planarity

$q_s = R_{se} q_e$ and $\bar{q}_s = \bar{g}_{se} \bar{q}_e$ would yield the same result since no translation is involved.

$q_s = R_{se} q_e = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ r_e \end{bmatrix} = \begin{bmatrix} r_e \\ 0 \end{bmatrix}$

With rigid-body translation, $\bar{q}_s = \bar{g}_{se} \bar{q}_e$, where $\bar{q}_e = \begin{bmatrix} q_e \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ r_e \\ 1 \end{bmatrix}$ ← 2D homogenous

$\Rightarrow \bar{q}_s = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ r_e \\ 1 \end{bmatrix} = \begin{bmatrix} r_e \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} q_s \\ 1 \end{bmatrix} \Rightarrow q_s = \begin{bmatrix} r_e \\ 0 \end{bmatrix} \rightarrow$ 2D

1.3) To compute: V^b_{se} and V^s_{se}

$$q_s = g_{se}\, q_e \Rightarrow \dot{q}_s = \dot{g}_{se}\, q_e = \underbrace{\dot{g}_{se}\, g^{-1}_{se}}_{V^s_{se}} \underbrace{g_{se}\, q_e}_{q_s}$$

$$\hat{V}^s_{se} = \dot{g}_{se}\, g^{-1}_{se}$$

$$\dot{g}_{se} = \frac{d}{dt}\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}\dot{\theta}$$

$$g^{-1}_{se} = g_{es} = \begin{bmatrix} R_{es} & p_{es} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_{es} \rightarrow R^{-1}_{se} = R^T_{se}$

$$\therefore \dot{g}_{se}\, g^{-1}_{se} = \begin{bmatrix} -\sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\dot{\theta}$$

$$= \begin{bmatrix} -\sin\theta\cos\theta + \sin\theta\cos\theta & -\sin^2\theta - \cos^2\theta & 0 \\ \cos^2\theta + \sin^2\theta & \sin\theta\cos\theta - \sin\theta\cos\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}\dot{\theta}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\dot{\theta}$$

$\dot{\theta} = 1$ rev/day

$$\Rightarrow \hat{V}^s_{se} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{rev/day}$$

$$\hat{V}^s_{se} = \begin{bmatrix} \omega^s & v^s \\ 0 & 0 \end{bmatrix}$$

$v^s \rightarrow$ translational velocity from translation and rotation

$\omega^s \rightarrow$ rotational velocity

$$V^s_{se} = \begin{bmatrix} v^s \\ \omega^s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\rightarrow$ body velocity of Earth's rotation w.r.t S, measured at S.

$$\hat{V}^b_{se} = g^{-1}_{se}\,\hat{V}^s_{se}\,g_{se}$$

$\hookrightarrow g_{es}$

$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sin\theta & -\cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\sin\theta - \sin\theta\cos\theta & -\sin^2\theta - \cos^2\theta & 0 \\ \cos^2\theta + \sin^2\theta & -\sin\theta\cos\theta + \sin\theta\cos\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\hat{V}^b_{se} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \equiv \begin{bmatrix} \omega^b & v^b \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow V^b_{se} = \begin{bmatrix} v^b \\ \omega^b \end{bmatrix}$$

In 2D, $v^b \in \mathbb{R}^{2\times 1}$

$\omega^b \in \mathbb{R} \rightarrow$ rate of change of angle

$\Rightarrow V^s_{se} \in \mathbb{R}^{3\times 1}$

$$\Rightarrow \boxed{V^s_{se} = V^b_{se} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{array}{l} \rightarrow v_x \\ \rightarrow v_y \\ \rightarrow \omega \end{array}}$$

If written in 3D, V^s_{se} would have 0 at the remaining elements. i.e., v_z, ω_x and $\omega_y = 0$

1.4) $$\bar{v}_{qe} = \dot{\bar{q}}_e = \hat{V}^b_{se}\,\bar{q}_e = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ r_e \\ 1 \end{bmatrix}$$

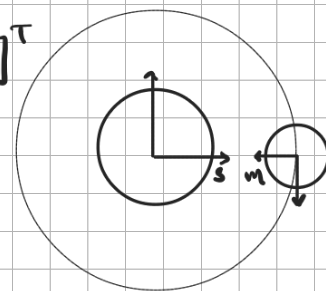
$$\Rightarrow \bar{V}_{qe} = \begin{bmatrix} -r_e \\ 0 \\ 0 \end{bmatrix} \rightsquigarrow \text{2D homogenous coordinates}$$

$$V_{qe} \equiv \begin{bmatrix} -r_e \\ 0 \end{bmatrix}$$

$$\bar{V}_{q/s} = \hat{V}^{s}_{se}\, \bar{q}_{/s} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} r_e \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ r_e \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad V_{q/s} = \begin{bmatrix} 0 \\ r_e \end{bmatrix}$$

1.5) $p_{sm}(t) = [l_m, 0, 0]^T \overset{2D}{\equiv} [l_m, 0]^T$



Angle of rotation between s and m: $\phi = 180^\circ$

$$R_{sm} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$g_{sm} = \begin{bmatrix} -1 & 0 & l_m \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{from } R_{sm} \text{ and } p_{sm}$$

$$g_{se} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{where } \theta = 270^\circ$$

$$g_{se} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$g_{em} = g_{es}\, g_{sm} = (g_{se})^{-1} g_{sm}$$

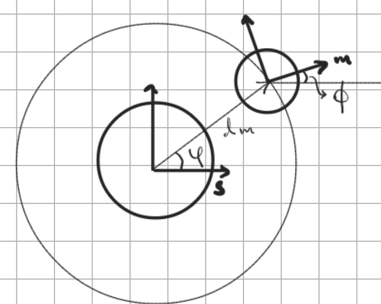
$$g_{es} = \begin{bmatrix} R_{es} & P_{es} \\ 0 \;\; 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{se}^T & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0 \;\; 0 & 0 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

for $\theta = 270^\circ$, $\quad g_{es} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$g_{em} = g_{es}\, g_{sm} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & l_m \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$g_{em} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & l_m \\ 0 & 0 & 1 \end{bmatrix}$$

1.6) To compute V^b_{sm}



$$\hat{V}^b_{sm} = g^{-1}_{sm}\, \dot{g}_{sm} = \begin{bmatrix} R^T_{sm} \dot{R}_{sm} & -R^T_{sm} \dot{p}_{sm} \\ 0 & 0 \end{bmatrix}$$

Note: Switching to 3-D coordinates now,

$$R_{sm} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow R^T_{sm} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\dot{R}_{sm} = \begin{bmatrix} -\sin\phi & -\cos\phi & 0 \\ \cos\phi & -\sin\phi & 0 \\ 0 & 0 & 0 \end{bmatrix} \times \dot{\phi} \quad \text{and} \quad p_{sm} = \begin{bmatrix} l_m \cos\varphi \\ l_m \sin\varphi \\ 0 \end{bmatrix} \Rightarrow \dot{p}_{sm} = \begin{bmatrix} -l_m \dot{\varphi} \sin\varphi \\ l_m \dot{\varphi} \cos\varphi \\ 0 \end{bmatrix}$$

$$\text{Now, } R^T \dot{R} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin\phi & -\cos\phi & 0 \\ \cos\phi & -\sin\phi & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\phi}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\phi}$$

$$R^T p = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -l_m \dot{\varphi} \sin\varphi \\ l_m \dot{\varphi} \cos\varphi \\ 0 \end{bmatrix} = \begin{bmatrix} -l_m \dot{\varphi} \sin\varphi \cos\phi + l_m \dot{\varphi} \cos\varphi \sin\phi \\ l_m \dot{\varphi} \sin\varphi \cos\varphi + l_m \dot{\varphi} \cos\varphi \cos\phi \\ 0 \end{bmatrix}$$

$$R^T p = [\, l_m \dot\varphi \sin(\phi - \varphi) \quad l_m \dot\varphi \cos(\phi - \varphi) \quad 0 \,]^T$$

$$\Rightarrow g_{sm}^{-1} \dot g_{sm} = \begin{bmatrix} 0 & -\dot\phi & 0 & -l_m \dot\varphi \sin(\phi - \varphi) \\ \dot\phi & 0 & 0 & -l_m \dot\varphi \cos(\phi - \varphi) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \hat V^b_{ab}$$

$$V^b_{ab} = \left(\hat V^b_{ab}\right)^\vee = [\, -l_m \dot\varphi \sin(\phi - \varphi) \quad l_m \dot\varphi \cos(\phi - \varphi) \quad 0 \quad 0 \quad 0 \quad \dot\phi \,]^T$$

$$\boxed{V^b_{ab} = \begin{bmatrix} -l_m \dot\varphi \sin(\phi - \varphi) \\ -l_m \dot\varphi \cos(\phi - \varphi) \\ 0 \\ 0 \\ 0 \\ \dot\phi \end{bmatrix}}$$

For the instance mentioned in question 1.5,

$\phi = 180^\circ, \varphi = 0^\circ \Rightarrow \boxed{V^b_{sm} = [\, 0 \quad l_m \dot\varphi \quad 0 \quad 0 \quad 0 \quad \dot\phi \,]^T}$

where, $\dot\varphi = 1/28$ rev/day ; $\dot\phi = 1/28$ rev/day

1.7) $V^s_{sm} = Ad(g_{sm}) \, V^b_{sm}$

Writing in 3D to perform adjoint operation,

$$Ad(g_{sm}) = \begin{bmatrix} R_{sm} & \hat P_{sm} R_{sm} \\ O_{3\times3} & R_{sm} \end{bmatrix}$$

$$R_{sm} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \hat\omega = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

$$P_{sm} = [\, l_m \cos\varphi \quad l_m \sin\varphi \quad 0 \,]^T$$

$$\Rightarrow \hat P_{sm} = \begin{bmatrix} 0 & 0 & l_m \sin\varphi \\ 0 & 0 & -l_m \cos\varphi \end{bmatrix}$$

$$\hat{p}_{sm} R_{sm} = \begin{bmatrix} 0 & 0 & l_m \sin\varphi \\ 0 & 0 & -l_m\cos\varphi \\ -l_m\sin\varphi & l_m\cos\varphi & 0 \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & l_m\sin\varphi \\ 0 & 0 & -l_m\cos\varphi \\ -l_m\sin\varphi\cos\phi + l_m\cos\varphi\sin\phi & l_m(\sin\varphi\sin\phi + \cos\varphi\cos\phi) & 0 \end{bmatrix}$$

$$\hat{p}_{sm} R_{sm} = \begin{bmatrix} 0 & 0 & l_m\sin\varphi \\ 0 & 0 & -l_m\cos\varphi \\ l_m\sin(\phi-\varphi) & l_m\cos(\phi-\varphi) & 0 \end{bmatrix}$$

$$\Rightarrow Ad(g_{sm}) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 & 0 & 0 & l_m\sin\varphi \\ \sin\phi & \cos\phi & 0 & 0 & 0 & -l_m\cos\varphi \\ 0 & 0 & 1 & l_m\sin(\phi-\varphi) & l_m\cos(\phi-\varphi) & 0 \\ 0 & 0 & 0 & \cos\phi & -\sin\phi & 0 \\ 0 & 0 & 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Ad_{g_{sm}} \cdot V^b_{sm} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 & 0 & 0 & l_m\sin\varphi \\ \sin\phi & \cos\phi & 0 & 0 & 0 & -l_m\cos\varphi \\ 0 & 0 & 1 & l_m\sin(\phi-\varphi) & l_m\cos(\phi-\varphi) & 0 \\ 0 & 0 & 0 & \cos\phi & -\sin\phi & 0 \\ 0 & 0 & 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\dot{\varphi}\, l_m\sin(\phi-\varphi) \\ -\dot{\varphi}\, l_m\cos(\phi-\varphi) \\ 0 \\ 0 \\ 0 \\ \dot{\phi} \end{bmatrix}$$

$$= \begin{bmatrix} -\dot{\varphi}\, l_m\cos\phi\sin(\phi-\varphi) + \dot{\varphi}\, l_m\sin\phi\cos(\phi-\varphi) + \dot{\phi}\, l_m\sin\varphi \\ -\dot{\varphi}\, l_m\sin\phi\cos(\phi-\varphi) - \dot{\varphi}\, l_m\cos\phi\cos(\phi-\varphi) - \dot{\phi}\, l_m\cos(\varphi) \\ 0 \\ 0 \\ 0 \\ \dot{\phi} \end{bmatrix}$$

$$\boxed{V_{sm}^{s} = \begin{bmatrix} l_m \sin\varphi\,(\dot\phi + \dot\varphi) \\ -l_m \cos\varphi\,(\dot\phi + \dot\varphi) \\ 0 \\ 0 \\ 0 \\ \dot\phi \end{bmatrix}}$$

At $\varphi = 0^\circ$ and $\phi = 180^\circ$

$$V_{sm}^{s}\Big|_{\varphi=0^\circ,\ \phi=180^\circ} = \begin{bmatrix} 0 \\ -l_m(\dot\phi + \dot\varphi) \\ 0 \\ 0 \\ 0 \\ \dot\phi \end{bmatrix} = \begin{bmatrix} 0 \\ -2\dot\phi l_m \\ 0 \\ 0 \\ 0 \\ \dot\phi \end{bmatrix}$$

where $\dot\phi = 1/28$ rev/day

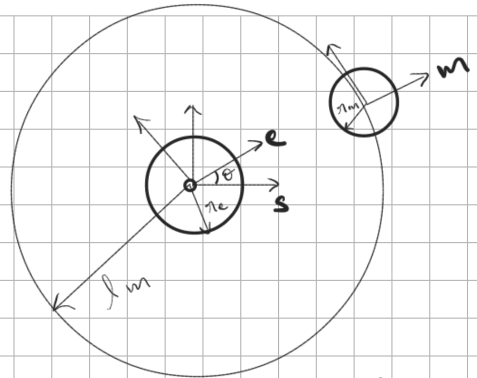
$$v_{sm}^{s} = [l_m \sin\varphi(\dot\phi + \dot\varphi) \quad -l_m \cos\varphi(\dot\phi + \dot\varphi) \quad 0]^T$$

i) $v_{sm}^{s} \cdot p_{sm} = 0 \Rightarrow$ translational velocity is always tangential to the orbit.

ii) Since $\dot\phi$ and $\dot\varphi$ are in the same direction, the same side of the moon would face the Earth always

iii) In the given instance where $\phi = 180$ and $\varphi = 0$;

$v_{sm}^{s} = [0 \quad -2 l_m \dot\phi \quad 0]^T$

1.3) 

To compute: for a point q on earth's surface, how long does it take between two instances where q_s lies on p_{sm}?

In polar coordinates, $q_s = (r_e, \theta, 0)$
$$p_{sm} = (l_m, \varphi, 0)$$

We need to find the time period between 2 instances, where $\theta = \varphi$ (assuming $\theta = \varphi$ at $t = t_0$)

Since we know the angular velocity of both the earth's rotation and the moon's revolution, and since they are constant, we can say that:

$$\theta\big|_t = \theta_0 + \dot{\theta} t$$
$\to$ known constants
and $\varphi\big|_t = \varphi_0 + \dot{\varphi} t$

For simplicity, we can take $\theta_0 = \varphi_0 = 0$ (first instance when they are equal)

For moon to reach above the same point on earth's surface, we need $\theta\big|_t = \varphi\big|_t + 2n\pi$, where $n \in \mathbb{Z}$

$$\longrightarrow \theta\big|_t = \dot{\theta} t = \varphi\big|_t + 2n\pi = \dot{\varphi} t + 2n\pi$$

$$\Rightarrow t = \frac{2n\pi}{(\dot{\theta} - \dot{\varphi})} \quad \text{or,}$$

For the shortest time, we set $n = 1 \Rightarrow t = \dfrac{2\pi}{(\dot{\theta} - \dot{\varphi})}$

Now, $\dot{\theta} = 1 \text{ rev/day} = 2\pi \text{ rads/day}$

$\dot{\varphi} = 1/28 \text{ rev/day} = (2\pi/28) \text{ rev/day}$

So, $$\frac{2\pi}{(\dot{\theta} - \dot{\varphi})} = \frac{2\pi}{\left(2\pi - \frac{2\pi}{28}\right)} = (28/27) \text{ days}$$

$$\Rightarrow \boxed{1 \text{ lunar day} = 28/27 \text{ Earth days}}$$