GATE 2023 ST 17

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Question

Let X be a random variable having Poisson distribution with mean $\lambda > 0$. Then $\mathrm{E}\left(\frac{1}{1+X}\middle| X > 0\right)$ equals

1. $\frac{1-e^{-\lambda}-\lambda e^{-\lambda}}{\lambda\left(1-e^{-\lambda}\right)}$
2. $\frac{1-e^{-\lambda}}{\lambda}$
3. $\frac{1-e^{-\lambda}-\lambda e^{-\lambda}}{\lambda}$
4. $\frac{1-e^{-\lambda}}{\lambda+1}$

Defining Poisson random variable

Let X be a Poisson random variable, then pmf of X is defined as

$$X \sim Pois\left(\lambda\right) \tag{1}$$

$$\Pr\left(X = k\right) = e^{-\lambda}\frac{\lambda^k}{k!}; k \geq 0 \tag{2}$$

where λ is the Poisson parameter and also,

$$E\left(X\right) = \lambda \tag{3}$$

Conditional Expectance

Conditional Expectance $E\left(X|Y\right)$ is defined as

$$E\left(X \mid A\right) = \sum_{x} xP\left(X = x \mid A\right) \tag{4}$$

$$= \sum_{x} x\frac{P\left(X = x, A\right)}{P\left(A\right)} \tag{5}$$

$$\implies E\left(\frac{1}{1+X}\middle| X > 0\right) = \frac{\sum_{k=1}^{\infty} \frac{1}{k+1} \Pr\left(X = k\right)}{\Pr\left(X > 0\right)} \tag{6}$$

$$= \frac{\sum_{k=1}^{\infty} \frac{1}{k+1} e^{-\lambda} \frac{\lambda^k}{k!}}{1 - \Pr\left(X \le 0\right)} \tag{7}$$

$$= \frac{e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k+1)!}}{1 - \Pr\left(X = 0\right)} \tag{8}$$

from equation (2)

$$= \frac{e^{-\lambda}\sum_{k=1}^{\infty}\frac{\lambda^k}{(k+1)!}}{1-e^{\lambda}} \tag{9}$$

Now simplifying Just the Summation

$$\implies \sum_{k=1}^{\infty}\frac{\lambda^k}{(k+1)!} \tag{10}$$

$$= \frac{1}{\lambda}\sum_{k=1}^{\infty}\frac{\lambda^{k+1}}{k+1} \tag{11}$$

Letting $k+1=m$,

$$\implies \frac{1}{\lambda}\sum_{m=2}^{\infty}\frac{\lambda^m}{m!} \tag{12}$$

We Know from Taylor series

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \tag{13}$$

$$\implies \frac{1}{\lambda} \sum_{m=2}^{\infty} \frac{\lambda^m}{m!} = \frac{1}{\lambda} \left(e^{\lambda} - 1 - \lambda \right) \tag{14}$$

Substituting back we get,

$$= \frac{e^{-\lambda}}{(1 - e^{\lambda})} \left(\frac{1}{\lambda} \left(e^{\lambda} - 1 - \lambda \right) \right) \tag{15}$$

$$= \frac{e^{-\lambda}}{\lambda (1 - e^{\lambda})} \left(e^{\lambda} - 1 - \lambda \right) \tag{16}$$

$$= \frac{1 - e^{-\lambda} - \lambda e^{-\lambda}}{\lambda (1 - e^{\lambda})} \tag{17}$$