EE1205

EE22BTECH11014 - Barath Surya M

Q2) The sum of the third and the seventh terms of AP is 6 and their product is 8. Find the sum of first sixteen terms of the AP

Solution: The general Term of an ap is

$$x(n) = x(0) + nd \tag{1}$$

from the given values

$$x(2) = x(0) + 2d \tag{2}$$

$$x(6) = x(0) + 6d \tag{3}$$

$$x(2) + x(6) = x(2) + x(6) \tag{4}$$

on solving (4) using (2) and (3) we get

$$x(6) = 2 \text{ or } 4 \tag{5}$$

$$x(2) = 4 \text{ or } 2 \tag{6}$$

then when $x(6) = 2$ and $x(2) = 4$

$$x(0) = 5 \tag{7}$$

$$d = \frac{-1}{2} \tag{8}$$

then when $x(6) = 4$ and $x(2) = 2$

$$x(0) = 1 \tag{9}$$

$$d = \frac{1}{2} \tag{10}$$

Sum of ap till n terms is

$$s(n) = \sum_{0}^{n} x(k) \tag{11}$$

$$= \sum_{k=-\infty}^{\infty} x(k)\, u(n-k) \tag{12}$$

$$= x(n) * u(n) \tag{13}$$

Taking Z transform,

$$S(z) = X(z)\, U(z) \tag{14}$$

from the z-transformation of $x(n)$

$$S(z) = \left(\frac{x(0)}{1 - z^{-1}} + \frac{dz^{-1}}{(1 - z^{-1})^2}\right)\left(\frac{1}{1 - z^{-1}}\right) \quad |z| > |1| \tag{15}$$

$$= \frac{x(0)}{(1 - z^{-1})^2} + d\frac{z^{-1}}{(1 - z^{-1})^3} \quad |z| > |1| \tag{16}$$

taking inverse z-transform using contour integration

$$s(n) = \frac{1}{2\pi j} \oint_C S(z) z^{n-1} dz \tag{17}$$

where C is clockwise closed contour in region of convergence of $S(z)$.
Contour integrals of this form can be solved using Cauchy's residue theorem, which states

$$s(n) = \frac{1}{2\pi j} \oint_C S(z) z^{n-1} dz \tag{18}$$

$$= \sum \left[\text{residues of } S(z) z^{n-1} \text{ at the poles inside } C\right] \tag{19}$$

Residue of a mth order pole is

$$X(z) z^{n-1} = \frac{X'(z)}{(z - z_0)^m} \tag{20}$$

$$Res\left[X(z) z^{n-1} \text{ at } z = z_0\right] = \frac{1}{(m-1)!} \left(\frac{d^{m-1}(X'(z))}{dz^{m-1}}\right) \tag{21}$$

for simple first order pole

$$Res\left[X(z) z^{n-1} \text{ at } z = z_0\right] = X'(z_0) \tag{22}$$

So for (16)

$$S(z) z^{n-1} = \frac{x(0) z^{n+1} (z-1) + dz^{n+1}}{(z-1)^3} \tag{23}$$

$$s(n) = \frac{1}{2} \left(\frac{d^2 \left(S(z)(z)^{n-1}\right)}{dz^2}\right) \quad \text{at } z = 1 \tag{24}$$

$$= \frac{1}{2} \frac{d^2 \left(x(0) z^{n+2} - x(0) z^{n+1} + dz^{n+1}\right)}{dz^2} \quad \text{at } z = 1 \tag{25}$$

$$= \frac{1}{2} \left((n+1) z^{n-1} ((n+2) x(0) z - nx(0) + dn)\right) \quad \text{at } z = 1 \tag{26}$$

$$= \frac{1}{2} ((n+1)((n+2) x(0) - nx(0) + dn)) \tag{27}$$

$$= \frac{n+1}{2} (2x(0) + dn) \tag{28}$$

Sum of first 16 terms if $x(6) = 2$ and $x(2) = 4$

$$S(15) = \frac{16}{2} \left(2(5) + 15 \left(\frac{-1}{2}\right)\right) \tag{29}$$

$$= 20 \tag{30}$$

Sum of first 16 terms if $x(6) = 4$ and $x(2) = 2$

$$s(15) = \frac{16}{2} \left(2(1) + 15 \left(\frac{1}{2}\right)\right) \tag{31}$$

$$= 76 \tag{32}$$