MAI0199 Computational Linear Algebra

Computer Exercise 4. The Singular Value Decomposition

1 General information

The assignment consists of a mixture of theoretical exercises and practical programming. At the end of the exercise a written report with well structured solutions to the theoretical questions and also Matlab programs, and graphs or plots that summarize the computational results should be sent by email to `Fredrik.Berntsson@liu.se`. In order to reduce the number of Matlab programs keep old code as comments when you modify a program in an exercise.

2 Circle fitting

We want to fit a circle to m given points (x_i, y_i). Any circle can be represented as,

$$F(u) = a(x^2 + y^2) + b_1 x + b_2 y + c = 0.$$

Here $u = (a, b_1, b_2, c)^T$ are the unknown parameters. By inserting the data points (x_i, y_i) into the expression for $F(u)$ we get a linear system $Bu = 0$, where $B \in \mathbb{R}^{m\times 4}$. In order to obtain a non–trivial solutuon we put a constraint $\|u\|_2 = 1$ on the solution, and thus solve the over determined system,

$$\min_u \|Bu\|_2, \quad \text{subject to} \quad \|u\|_2 = 1.$$

Hint The Matlab code `CircleData` can be used to create test data.

Exercise 2.1 Do the following

1. Show that the center $z = (z_1, z_2)^T$ and radius r of the circle can be written,

$$z = (z_1, z_2)^T = -(\frac{b_1}{2a}, \frac{b_2}{2a})^T, \qquad r = \sqrt{\frac{b_1^2 + b_2^2}{4a^2} - \frac{c}{a}}.$$

2. Give the expression for the matrix B.
3. Show that the solution of the minization problem is given by the right singular vector v associated with the smallest singular value of B.
4. Use Matlabs `svd` to find the best circle fit given this data set. Display the results. □

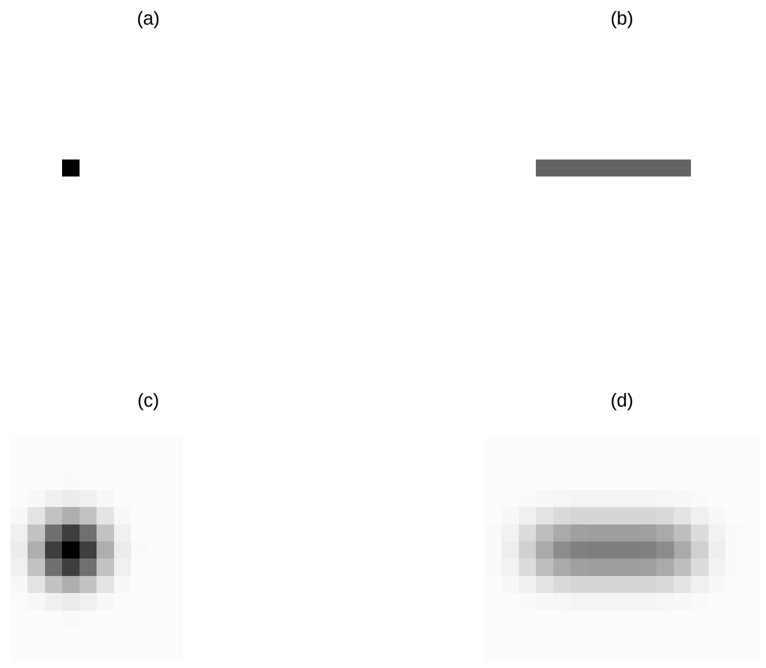

Figure 1: The original image **(a)**, the view of the original object in motion **(b)**, the effect of blurring the original image **(c)**, and the combined effect of blurring and motion **(d)**. The images are 16×16 pixels.

3 Image Reconstruction

Sometimes when looking at an object from a distance we may not see the true shape exactly. The light is diffused and the image may appear "blurred". Also if the object is moving it may appear as if it is spread out in one direction. In *image reconstruction* this is modelled by a point spread function. Using the point spread function we obtain a matrix A such that $b = Ax$, where x is the exact image and b is the image actually recorded by the camera. The image reconstructing problem consists of finding the exact image x by solving the linear system $Ax = b$.

In this exercise the blurring is assumed to be described by a Gaussian function. A single pixel $P(i_0, j_0)$ in the original image will be diffused over a few nearby pixles in a manner described by a function.

$$P_b(i,j) = \exp\left(\frac{-(i-i_0)^2 - (j-j_0)^2}{2\sigma^2}\right).$$

The Matlab function `BlurMatrix` calculates a matrix representing the "blurring part" of the point spread function for a given image size. Type

```
>>[A1]=BlurMatrix(N,band,sigma)
```

and an $N^2 \times N^2$ matrix A is computed. The parameters `band` and `sigma` control the shape of the

Gaussian. If we store the original image as an $N^2 \times 1$ vector x then the "blurred image" is the vector $b = A_1 x$.

Furthermore, we will assume that the object we look at moves parallel to the x axis. As a result of the movement a pixel $P(i_0, j_0)$ in the original image will be seen as a stripe $P(i_0, j_0 + w)$ where the speed of the object determine the length of the stripe, i.e. the parameter `w`. The function

```
>>[A2]=MoveMatrix(N,w);
```

creates the matrix that represents the movement effects in the image.

Note The effects of camera movement is significant in astronomy where, sometimes, it takes several hours to record a single image. The rotation of the earth means that the stars move in relation to the telescope. The movement is easily modelled.

The effect of blurring and movement, as well as the combined effect, on an 16×16 image consisting of a single colored pixel is seen in Figure 1.

Exercise 3.1 The combined point spread function is represented by the product $A = A_1 A_2$. Let $N = 16$, $\sigma = 1.3$ and `band`$= 7$ for the blurring, and $w = 4$ for the movement. Investigate the properties of the two matrices by computing their condition numbers $\kappa_2(A_1)$ and $\kappa_2(A_2)$. Note that the matrices need to be stored as full matrices or `cond()` will not work.

What are the condition numbers $\kappa(A_1)$ and $\kappa(A_2)$? Do you expect both of these effects to be difficult to compensate for? $\square$

The function `TestImage` creates a simple test image; supposedly showing a photo of a sattelite. This will be our original image in this exercise. Display the image using

```
>> X=TestImage(N);
>> imshow(X,'InitialMagnification','fit')
```

The vector x representing the image is created using

```
>> x = reshape( X , N*N , 1 );
```

The image we "record" with our camera is created as follows.

```
>> b=A1*A2*x+randn(size(x))*10^-4;
```

where random noise representing the errors in the recording device, i.e. the camera, and modelling errors in the point spread function has been added.

Exercise 3.2 Let $N = 32$, $\sigma = 1.3$ and `band`$= 7$ for the blurring, and $w = 4$ for the movement. Compute the image we will see in the camera with noise as detailed above. Look at the image. Is it possible to see any details of the original image? $\square$

Hint In order to display the image use `reshape` to transform the vector x back into a quadratic matrix X.

Let b be the vector representing the image recorded by the camera. Since the combined matrix $A = A_1A_2$ is very ill-conditioned we can't solve the linear system $Ax = b$ using Guaussian elimination and expect a good solution. Try this and see what happens. Instead we will use Tikhonov regularization. For a given value of λ minimize,

$$\|Ax^\lambda - b\|_2^2 + \lambda^2\|x^\lambda\|_2^2.$$

This minimization problem can be solved efficiently using the singular value decomposition. This is the goal in the next few exercises.

Exercise 3.3 The normal equations for the above minimization problem are

$$(A^T A + \lambda I)x = A^T b.$$

Suppose the decomposition $A = U\Sigma V^T$ is known and derive a solution formula for the normal equations. $\square$

Exercise 3.4 Use a image of size $N = 32$ with blurring, movement and noise as detailed above. Experiment, in the range $10^{-1} < \lambda < 10^{-7}$, and compute the Tikhonov solutions x^λ. Look at the resulting images and find a good value for the reqularization parameter λ? Is it possible to find a good reconstruction of the image?

Remark The SVD requires dense matrices. Hence the problem sizes have to be relatively small. Hence the small size of the image. In the next computer exercise we will demonstrate how to do the same thing for more realistic problem sizes using iterative methods.