TANA15 Numerical Linear Algebra

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# Computer Exercise 1. The QR factorization and applications

## 1 General information

The programming assignments are intended to take more than two hours to complete. It is therefore important to be well prepared so that the computer sessions are used effectively. The assignment consists of a mixture of theoretical exercises and practical programming. A few of the exercises are based on custom Matlab programs or datasets that are provided in order to save time. These can be downloaded from the course website.

## 2 Computing the QR factorizaton

Given a vector $v$ the Householder reflection matrix $H$ is defined as

$$H = I - 2\frac{vv^T}{v^Tv}.$$

**Exercise 2.1** Let $e_1 = (1, 0, \ldots, 0)^T$. Show that if $v = x + \alpha sign(x_1)e_1$ then $Hx = \alpha e_1$. What is the value of $\alpha$? □

The QR factorization of a matrix $A = (a_1, \ldots, a_n)$ can be computed by a sequence of Householder reflections. In the first step we find a reflection $H_1$ such that $H_1 a_1 = \pm\|a_1\|_2 e_1$. Then

$$H_1 A = (H_1a_1, H_1a_2, \ldots, H_na_n) = (\alpha e_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \begin{pmatrix} x & x & x \\ 0 & x & x \\ 0 & x & x \\ 0 & x & x \end{pmatrix}$$

where the example is for a $4 \times 3$ matrix. Note that the matrix $H_1$ need not be formed explicitly.

**Exercise 2.2** Write a Matlab function `ApplyReflection` that calculates the product $HA$, where $H$ is chosen so that the first column $Ha_1 = \alpha e_1$ as in the example above. □

After the first step described above we have obtained a matrix

$$H_1 A = \begin{pmatrix} r_{11} & c_1^T \\ 0 & A_1 \end{pmatrix},$$

where $A_1$ is $(m-1)\times(n-1)$. We can now proceed and find a Householder matrix $H_2$ such that,

$$\tilde{H}_2 H_1 A = \begin{pmatrix} 1 & 0^T \\ 0 & H_2 \end{pmatrix} \begin{pmatrix} r_{11} & c_1^T \\ 0 & A_1 \end{pmatrix} = \begin{pmatrix} r_{11} & c_1^T \\ 0 & H_2 A_1 \end{pmatrix} = \begin{pmatrix} r_{11} & r_{12} & \tilde{c}_1^T \\ 0 & r_{22} & c_2^T \\ 0 & 0 & A_2 \end{pmatrix},$$

where $A_2$ is a $(m-2)\times(n-2)$ matrix.

**Exercise 2.3** Write a Matlab function `HouseholderQR` that calculates the $R$ matrix of the $QR$ factorization for a rectangular matrix $A$. $\square$

Let,

$$A = \begin{pmatrix} 2 & -5 & 3 \\ -1 & 8 & 5 \\ 4 & -1 & 7 \\ -1 & 3 & 2 \end{pmatrix}, \quad \text{and,} \quad b = \begin{pmatrix} 3 \\ 5 \\ -2 \\ 4 \end{pmatrix}.$$

**Exercise 2.4** Compute the matrix $R$ using both your function and the standard Matlab `qr` routine. Comment on the results. $\square$

**Exercise 2.5** Type

```
>> C = HouseholderQR( [ A , b ] );
>> R = C(1:3,1:3); Qb = C(1:3,4);
>> x = R\Qb;
```

compare `norm(A*x-b)` with `C(4,4)`. Comment on the results. Show that these steps actually solves the least squares problem $\min \|Ax - b\|_2$. $\square$

# 3 Givens Rotations and Row updating

In two dimensions a plane rotation is defined by a angle $\theta$. Given a vector $x$ and an angle such that

$$cos(\theta) = \frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \quad \text{and,} \quad sin(\theta) = \frac{x_2}{\sqrt{x_1^2 + x_2^2}},$$

then

$$G_{12}(\theta)x = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \sqrt{x_1^2 + x_2^2} \\ 0 \end{pmatrix},$$

where $G_{1,2}(\theta)$ is called the *Givens Rotation* matrix. The subscript indicates that the rotation involves the first and the second row.

Suppose that we have an over determined linear system $Ax = b$ and know the the $QR$ factorization of $A$. Then

$$Q^T Ax = \begin{pmatrix} R \\ 0 \end{pmatrix} x = Q^T b = \begin{pmatrix} \widetilde{b}_1 \\ \widetilde{b}_2 \end{pmatrix}.$$

and the least squares problem $\min \|Ax - b\|_2$ has the solution $x = R^{-1}\tilde{b}_1$. As explained previously such a $QR$ factorization can be computed using, e.g., Householder transformations.

In applications each *row* in a least squares problem correspond to a specific measurement. Suppose we perform an additional measurement and obtain an equation $a_{m+1}^T x = \beta_{m+1}$ then we have a new over determined linear system,

$$\begin{pmatrix} A \\ a_{m+1}^T \end{pmatrix} x = \begin{pmatrix} b \\ \beta_{m+1} \end{pmatrix} \quad \text{or,} \quad \begin{pmatrix} R \\ 0 \\ a_{m+1}^T \end{pmatrix} x = \begin{pmatrix} \widetilde{b}_1 \\ \widetilde{b}_2 \\ \beta_{m+1} \end{pmatrix}.$$

We wish to solve this problem, in the sense of least squares, while taking advantage of the known factorization of $A$.

**Exercise 3.1** Let

$$\widetilde{A} = \begin{pmatrix} R & \widetilde{b}_1 \\ a_{m+1}^T & \beta_{m+1} \end{pmatrix},$$

where $R$ is upper triangular. Show how a sequence of Givens rotations can be used to restore $\widetilde{A}$ to upper triangular form. $\square$

**Exercise 3.2** Write a Matlab function `QRUpdate` that uses Givens rotations to transform into upper triangular form as described above. $\square$.

**Exercise 3.3** Consider the matrix $A$ and righthand side $b$ given earlier. Solve the least squares problem as follows:

```
>> R = HouseholderQR( [ A , b ] );
>> R=R(1:3,:);    % Remmove the "zero" row.
>> x = R(:,1:3)\R(:,4);
```

We want to add a new equation $x_1 + 2x_2 + 6x_3 = 5$. Extend the matrix $R$ and use your function to restore the upper triangular shape to $R$ by

```
>> R = [R ; 1 2 6 5];
>> R = QRUpdate(R); R=R(1:3,:);    % Remmove the "zero" row.
>> x = R(:,1:3)\R(:,4);
```

Verify that you get the same solution if you add the new row to the matrix $A$ and righthand side $b$ directly by

```
>> x2 = [A; 1 2 6]\[b;5];
```

Comment on the results. $\square$

# 4 Real time tracking of a comet trajectory

Most celestial objects, e.g. planets or comets, moves along an elliptial orbits. When a new comet is first observed one wants to estimate its path through the solar system to make sure it will not crash into the earth. This is done by collecting a set of positional data and finding an ellipse that fit the data in the least squares sense.

A point $(x, y)^T$ on an ellispe satisfies an equation

$$c_1x^2 + c_2xy + c_3y^2 + c_4x + c_5y + 1 = 0.$$

for a specific set of parameters $c = (c_1, c_2, c_3, c_4, c_5)^T$. Not every such equation describes an ellipse. Depending on the values of the coefficients you may get a hyperbola or parabola.

**Exercise 4.1** Suppose a set of observed locations $(x_k, y_k)$, $k = 1, .., n$, for the comet is given. Write down an over determined linear system $Ac = b$ that can be solved to find the shape of the ellipse. □

**Exercise 4.2** On the course library there is a file `CometTracking.m`. Open that file in the editor and add code that creates the matrix $A$ and right hand side $b$. Run the program by tying

```
>> CometTracking();
```

The program attempts to use $n = 10$ observations of the comet to find the elliptic orbit. Note that by only using observations that are close together and that only covers a small part of the ellipse we get an ill–conditioned problem. Was the attempt to find an ellipse successful? □

**Exercise 4.3** We will improve the ellipse fitting by adding more observations of the comet position. Each new observation of the comet will add another row to the linear system $Ac = b$.

First update the program `CometTracking` so that the new row is computed as additional position measurements are made.

Second use your function `QRUpdate` to restore the system to upper triangular form after adding the new row.

When you are done you should be able to run `CometTracking` and see how the ellipse fit is improved as more observations are used. How does the condition number of the least squares problem depend on the number of observations used? □