MAI0199 Computational Linear Algebra

Computer Exercise 1. Least Squares

1 General information

The assignment consists of a mixture of theoretical exercises and practical programming. At the end of the exercise a written report with well structured solutions to the theoretical questions and also Matlab programs, and graphs or plots that summarize the computational results should be sent by email to `Fredrik.Berntsson@liu.se`. In order to reduce the number of Matlab programs keep old code as comments when you modify a program in an exercise.

2 Householder Reflections

Given a vector v the Householder reflection matrix H is defined as

$$H = I - 2\frac{vv^T}{v^T v}.$$

The matrix H is symmetric and orthogonal. Also let $e_1 = (1, 0, \ldots, 0)^T$. Then if $v = x + \alpha sign(x_1)e_1$ we have $Hx = \pm\|x\|_2 e_1$.

Exercise 2.1 Write a Matlab function

```
function [HA]=ApplyReflection( A , x );
```

that creates a reflection H such that $HA = \alpha e_1$ and applies the reflection to each column of the A matrix. $\square$

Hint Do not form the matrix H explicitly. Take care to avoid temporary matrices during the computations.

The QR decomposition of a matrix $A = (a_1, \ldots, a_n)$ can be computed by a sequence of Householder reflections. In the first step we find a reflection H_1 such that $H_1 a_1 = \pm\|a_1\|_2 e_1$. This creates zeroes in the first column of A. In the second step we use a reflection H_2 to create zeroes in the second column, etc.

Exercise 2.2 Write a Matlab function

```
function [R,Q]=HouseholderQR( A )
```

that computes the QR decomposition of A using a seqnence of Householder reflections. Use the above `ApplyReflection` function. $\square$

Hint Since $Q^T = H_m \ldots H_2 H_1$ we can obtain Q by starting with the identity matrix I and using `ApplyReflection` to repetedly apply the same reflections as are used to compute the R matrix. Try to make the output Q optional and only compute Q if its required.

Let,

$$A = \begin{pmatrix} 2 & -5 & 3 \\ -1 & 8 & 5 \\ 4 & -1 & 7 \\ -1 & 3 & 2 \end{pmatrix}, \quad \text{and,} \quad b = \begin{pmatrix} 3 \\ 5 \\ -2 \\ 4 \end{pmatrix}.$$

Exercise 2.3 Compute the QR decomposition using both your function and the standard Matlab `qr` routine. Also solve the least squares problem $\|Ax - b\|_2$. $\square$

3 A Comet trajectory

Most celestial objects, e.g. planets or comets, moves along an elliptial orbits. When a new comet is first observed one wants to estimate its path through the solar system to make sure it will not crash into the earth. This is done by collecting a set of positional data and finding an ellipse that fit the data in the least squares sense.

A point $(x, y)^T$ on an ellispe satisfies an equation

$$c_1 x^2 + c_2 xy + c_3 y^2 + c_4 x + c_5 y + 1 = 0.$$

for a specific set of parameters $c = (c_1, c_2, c_3, c_4, c_5)^T$. Not every such equation describes an ellipse. Depending on the values of the coefficients you may get a hyperbola or parabola.

Exercise 3.1 Suppose a set of observed locations (x_k, y_k), $k = 1, .., n$, for the comet is given. Write down an over determined linear system $Ac = b$ that can be solved to find the shape of the elipse. $\square$

Exercise 3.2 Use the Matlab function `CometObservations` to generate a set of observations $\{(x_k, y_k)\}$. Write a function

```
function [c]=CometTrajectory( PositionData )
```

That sets up the above least squares problem and solves it.

In this exercise we assume that we may have many observations. Hence we want to avoid computing the Q matrix. Instead compute only the R matrix from the QR decomposition of the augumented matrix $[A, b]$ and use it to solve the least squares problem. $\square$

Hint You can use the function `DisplayComet` to plot the computed ellipse.

Exercise 3.3 In order to obtain a good ellipse we need enough observations. Experiment a little with different number of observations and also change how well spread out the observations are. Whats most important: Having many observations or having well spread out observations? □