1 Triangulation

Suppose $(\cos(\varphi_1), \sin(\varphi_1))$ and $(\cos(\varphi_2), \sin(\varphi_2))$ are the direction vector from access points 1 and 2 toward the user. Note that φ is the angle of arrival at the AP. Moreover, (x_1, y_1) and (x_2, y_2) represent the location of these APs. The line equation passing through the AP and user are as follow:

$$(x_{\mathrm{u}}, y_{\mathrm{u}}) = (x_1, y_1) + t_1 \cdot ((\cos(\varphi_1), \sin(\varphi_1)) \tag{1}$$

$$(x_{\mathrm{u}}, y_{\mathrm{u}}) = (x_2, y_2) + t_2 \cdot ((\cos(\varphi_2), \sin(\varphi_2)) \tag{2}$$

To find the intersection between two lines, we equate two parametric form:

$$(x_1, y_1) + t_1 \cdot ((\cos(\varphi_1), \sin(\varphi_1)) = (x_2, y_2) + t_2 \cdot ((\cos(\varphi_2), \sin(\varphi_2)) \tag{3}$$

We can seperate the above equation into two seperate equations:

$$x_1 + t_1 \cdot \cos(\varphi_1) = x_2 + t_2 \cdot \cos(\varphi_2) \tag{4}$$

$$y_1 + t_1 \cdot \sin(\varphi_1) = y_2 + t_2 \cdot \sin(\varphi_2) \tag{5}$$

Writing these equation into matrix form:

$$\underbrace{\begin{bmatrix} \cos(\varphi_1) & -\cos(\varphi_2) \\ \sin(\varphi_1) & -\sin(\varphi_2) \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} t_1 \\ t_2 \end{bmatrix}}_{\mathbf{t}} = \underbrace{\begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}}_{\mathbf{b}} \tag{6}$$

To find user location we should solve the sytem of linear equations for $\mathbf{t}$.

1.1 Generalized Formula

Let us assume $\boldsymbol{p}_u = [x_u, y_u, z_u]^{\mathrm{T}}$ is the location of the unknown user, and $\mathbf{p}_i$ for $i = 1, \ldots, P$ denotes the known location of access points. Moreover, $(\hat{\varphi}_i, \hat{\theta}_i)$ represents the estimated angle of arrival at access point i, resulting in direction vector of $\mathbf{k}_i = [\cos(\hat{\varphi}_i)\cos(\hat{\theta}_i), \sin(\hat{\varphi}_i)\cos(\hat{\theta}_i), \sin(\hat{\theta}_i)]^{\mathrm{T}}$. The line equation from the access point considering the estimated direction would be:

$$\boldsymbol{\ell}_i = \mathbf{p}_i + t\mathbf{k}_i \quad \text{for} \quad i = 1, \ldots, P \tag{7}$$

We want to estimate $\mathbf{p}_u$ such that it minimize the distance to all the lines extended from all the access points. Therefore

$$\hat{\mathbf{p}}_u = \min_{\mathbf{p}_u} \sum_{i=1}^{P} \|(\mathbf{p}_u - \mathbf{p}_i) - \mathbf{k}_i^{\mathrm{T}}(\mathbf{p}_u - \mathbf{p}_i)\mathbf{k}_i\|^2 \tag{8}$$

which is summation of the distance between point $\hat{\mathbf{p}}_u$ and lines $\boldsymbol{\ell}_i$.

$$\sum_{i=1}^{P} \|(\mathbf{p}_u - \mathbf{p}_i) - \mathbf{k}_i^{\mathrm{T}}(\mathbf{p}_u - \mathbf{p}_i)\mathbf{k}_i\|^2 = \tag{9}$$

$$\sum_{i=1}^{P} \|(\mathbf{p}_u - \mathbf{p}_i) - \mathbf{k}_i\mathbf{k}_i^{\mathrm{T}}(\mathbf{p}_u - \mathbf{p}_i)\|^2 = \tag{10}$$

$$\sum_{i=1}^{P} \|(\mathbf{I} - \mathbf{k}_i\mathbf{k}_i^{\mathrm{T}})(\mathbf{p}_u - \mathbf{p}_i)\|^2 \overset{(a)}{=} \tag{11}$$

$$\sum_{i=1}^{P} \|\mathbf{B}_i(\mathbf{p}_u - \mathbf{p}_i)\|^2 \tag{12}$$

where in (a) we substituted $(\mathbf{I} - \mathbf{k}_i\mathbf{k}_i^{\mathrm{T}}) = \mathbf{B}_i$. Above objective function is quadratic with respect to $\mathbf{p}_u$ and the optimum value can be obtained in closed form.

$$\frac{\nabla J}{\mathbf{p}_u} = \sum_{i=1}^{P} \mathbf{B}_i^{\mathrm{H}}\mathbf{B}_i\mathbf{p}_u - \sum_{i=1}^{P} \mathbf{B}_i^{\mathrm{H}}\mathbf{B}_i\mathbf{p}_i = \mathbf{0} \tag{13}$$

Therefore, $\mathbf{p}_u$ can be estimagted as

$$\mathbf{p}_u = (\mathbf{A}^{\mathrm{H}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{H}}\mathbf{b} \tag{14}$$

where $\mathbf{A} = \sum_{i=1}^{P} \mathbf{B}_i^{\mathrm{H}}\mathbf{B}_i$ and $\mathbf{b} = \sum_{i=1}^{P} \mathbf{B}_i^{\mathrm{H}}\mathbf{B}_i\mathbf{p}_i$.

2 Trilateration

Suppose the distance from the user to three access points is denoted by d_i for $i \in \{1, 2, 3\}$. Moreover, (x_i, y_i) denote the location of i-th AP. We have

$$d_1^2 = (x_u - x_1)^2 + (y_u - y_1)^2 \tag{15}$$

$$d_2^2 = (x_u - x_2)^2 + (y_u - y_2)^2 \tag{16}$$

$$d_3^2 = (x_u - x_3)^2 + (y_u - y_3)^2 \tag{17}$$

Expanding the equations:

$$d_1^2 = x_u^2 + x_1^2 - 2x_u x_1 + y_u^2 + y_1^2 - 2y_u y_1 \tag{18}$$

$$d_2^2 = x_u^2 + x_2^2 - 2x_u x_2 + y_u^2 + y_2^2 - 2y_u y_2 \tag{19}$$

$$d_3^2 = x_u^2 + x_3^2 - 2x_u x_3 + y_u^2 + y_3^2 - 2y_u y_3 \tag{20}$$

Subtracting these equation and rearranging:

$$2x(x_1 - x_2) + 2y(y_1 - y_2) = x_1^2 + y_1^2 - x_2^2 - y_2^2 + d_2^2 - d_1^2 \tag{21}$$

$$2x(x_1 - x_3) + 2y(y_1 - y_3) = x_1^2 + y_1^2 - x_3^2 - y_3^2 + d_3^2 - d_1^2 \tag{22}$$

writing it in matrix form

$$\underbrace{\begin{bmatrix} 2(x_1 - x_2) & 2(y_1 - y_2) \\ 2(x_1 - x_3) & 2(y_1 - y_3) \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\mathbf{p}} = \underbrace{\begin{bmatrix} x_1^2 + y_1^2 - x_2^2 - y_2^2 + d_2^2 - d_1^2 \\ x_1^2 + y_1^2 - x_3^2 - y_3^2 + d_3^2 - d_1^2 \end{bmatrix}}_{\mathbf{B}} \tag{23}$$

and solving for $\mathbf{p}$, the user location is estimated.