System Model

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1 Triangulation

Suppose $(\cos(\varphi_1), \sin(\varphi_1))$ and $(\cos(\varphi_2), \sin(\varphi_2))$ are the direction vector from access points 1 and 2 toward the user. Note that φ is the angle of arrival at the AP. Moreover, (x_1, y_1) and (x_2, y_2) represent the location of these APs. The line equation passing through the AP and user are as follow:

$$(x_{\mathrm{u}}, y_{\mathrm{u}}) = (x_1, y_1) + t_1 \cdot ((\cos(\varphi_1), \sin(\varphi_1)) \tag{1}$$

$$(x_{\mathrm{u}}, y_{\mathrm{u}}) = (x_2, y_2) + t_2 \cdot ((\cos(\varphi_2), \sin(\varphi_2)) \tag{2}$$

To find the intersection between two lines, we equate two parametric form:

$$(x_1, y_1) + t_1 \cdot ((\cos(\varphi_1), \sin(\varphi_1)) = (x_2, y_2) + t_2 \cdot ((\cos(\varphi_2), \sin(\varphi_2)) \tag{3}$$

We can seperate the above equation into two seperate equations:

$$x_1 + t_1 \cdot \cos(\varphi_1) = x_2 + t_2 \cdot \cos(\varphi_2) \tag{4}$$

$$y_1 + t_1 \cdot \sin(\varphi_1) = y_2 + t_2 \cdot \sin(\varphi_2) \tag{5}$$

Writing these equation into matrix form:

$$\underbrace{\begin{bmatrix} \cos(\varphi_1) & -\cos(\varphi_2) \\ \sin(\varphi_1) & -\sin(\varphi_2) \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} t_1 \\ t_2 \end{bmatrix}}_{\mathbf{t}} = \underbrace{\begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}}_{\mathbf{b}} \tag{6}$$

To find user location we should solve the sytem of linear equations for $\mathbf{t}$.

1.1 Generalized Formula

Let us assume $\boldsymbol{p}_u = [x_u, y_u, z_u]^{\mathrm{T}}$ is the location of the unknown user, and $\mathbf{p}_i$ for $i = 1, \dots, P$ denotes the known location of access points. Moreover, $(\hat{\varphi}_i, \hat{\theta}_i)$ represents the estimated angle of arrival at access point i, resulting in direction vector of $\mathbf{k}_i = [\cos(\hat{\varphi}_i)\cos(\hat{\theta}_i), \sin(\hat{\varphi}_i)\cos(\hat{\theta}_i), \sin(\hat{\theta}_i)]^{\mathrm{T}}$. The line equation from the access point considering the estimated direction would be:

$$\boldsymbol{\ell}_i = \mathbf{p}_i + t\mathbf{k}_i \quad \text{for} \quad i = 1, \dots, P \tag{7}$$

We want to estimate $\mathbf{p}_u$ such that it minimize the distance to all the lines extended from all the access points. Therefore

$$\hat{\mathbf{p}}_u = \min_{\mathbf{p}_u} \sum_{i=1}^{P} \|(\mathbf{p}_u - \mathbf{p}_i) - \mathbf{k}_i^{\mathrm{T}}(\mathbf{p}_u - \mathbf{p}_i)\mathbf{k}_i\|^2 \tag{8}$$

which is summation of the distance between point $\hat{\mathbf{p}}_u$ and lines $\boldsymbol{\ell}_i$.

$$\sum_{i=1}^{P} \|(\mathbf{p}_u - \mathbf{p}_i) - \mathbf{k}_i^{\mathrm{T}}(\mathbf{p}_u - \mathbf{p}_i)\mathbf{k}_i\|^2 = \tag{9}$$

$$\sum_{i=1}^{P} \|(\mathbf{p}_u - \mathbf{p}_i) - \mathbf{k}_i\mathbf{k}_i^{\mathrm{T}}(\mathbf{p}_u - \mathbf{p}_i)\|^2 = \tag{10}$$

$$\sum_{i=1}^{P} \|(\mathbf{I} - \mathbf{k}_i\mathbf{k}_i^{\mathrm{T}})(\mathbf{p}_u - \mathbf{p}_i)\|^2 \overset{(a)}{=} \tag{11}$$

$$\sum_{i=1}^{P} \|\mathbf{B}_i(\mathbf{p}_u - \mathbf{p}_i)\|^2 \tag{12}$$

where in (a) we substituted $(\mathbf{I} - \mathbf{k}_i\mathbf{k}_i^{\mathrm{T}}) = \mathbf{B}_i$. Above objective function is quadratic with respect to $\mathbf{p}_u$ and the optimum value can be obtained in closed form.

$$\frac{\nabla J}{\mathbf{p}_u} = \sum_{i=1}^{P} \mathbf{B}_i^{\mathrm{H}}\mathbf{B}_i\mathbf{p}_u - \sum_{i=1}^{P} \mathbf{B}_i^{\mathrm{H}}\mathbf{B}_i\mathbf{p}_i = \mathbf{0} \tag{13}$$

Therefore, $\mathbf{p}_u$ can be estimagted as

$$\mathbf{p}_u = (\mathbf{A}^{\mathrm{H}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{H}}\mathbf{b} \tag{14}$$

where $\mathbf{A} = \sum_{i=1}^{P} \mathbf{B}_i^{\mathrm{H}}\mathbf{B}_i$ and $\mathbf{b} = \sum_{i=1}^{P} \mathbf{B}_i^{\mathrm{H}}\mathbf{B}_i\mathbf{p}_i$.

2 Trilateration

Suppose the distance from the user to three access points is denoted by d_i for $i \in \{1, 2, 3\}$. Moreover, (x_i, y_i) denote the location of i-th AP. We have

$$d_1^2 = (x_u - x_1)^2 + (y_u - y_1)^2 \tag{15}$$

$$d_2^2 = (x_u - x_2)^2 + (y_u - y_2)^2 \tag{16}$$

$$d_3^2 = (x_u - x_3)^2 + (y_u - y_3)^2 \tag{17}$$

Expanding the equations:

$$d_1^2 = x_u^2 + x_1^2 - 2x_u x_1 + y_u^2 + y_1^2 - 2y_u y_1 \tag{18}$$

$$d_2^2 = x_u^2 + x_2^2 - 2x_u x_2 + y_u^2 + y_2^2 - 2y_u y_2 \tag{19}$$

$$d_3^2 = x_u^2 + x_3^2 - 2x_u x_3 + y_u^2 + y_3^2 - 2y_u y_3 \tag{20}$$

Subtracting these equation and rearranging:

$$2x(x_1 - x_2) + 2y(y_1 - y_2) = x_1^2 + y_1^2 - x_2^2 - y_2^2 + d_2^2 - d_1^2 \tag{21}$$

$$2x(x_1 - x_3) + 2y(y_1 - y_3) = x_1^2 + y_1^2 - x_3^2 - y_3^2 + d_3^2 - d_1^2 \tag{22}$$

writing it in matrix form

$$\underbrace{\begin{bmatrix} 2(x_1 - x_2) & 2(y_1 - y_2) \\ 2(x_1 - x_3) & 2(y_1 - y_3) \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\mathbf{p}} = \underbrace{\begin{bmatrix} x_1^2 + y_1^2 - x_2^2 - y_2^2 + d_2^2 - d_1^2 \\ x_1^2 + y_1^2 - x_3^2 - y_3^2 + d_3^2 - d_1^2 \end{bmatrix}}_{\mathbf{B}} \tag{23}$$

and solving for $\mathbf{p}$, the user location is estimated.

Complex value derivative identities

$$\frac{\partial \mathbf{a}^{\mathrm{H}}\mathbf{x}}{\partial \mathbf{x}} = \mathbf{a}^* \tag{1}$$

$$\frac{\partial \mathbf{x}^{\mathrm{H}}\mathbf{a}}{\partial \mathbf{x}} = \mathbf{0} \tag{2}$$

$$\frac{\partial \mathbf{x}^{\mathrm{H}}\mathbf{A}\mathbf{x}}{\partial \mathbf{x}} = \mathbf{A}^{\mathrm{T}}\mathbf{x}^* \tag{3}$$

$$\frac{\partial \mathbf{y}^{\mathrm{H}}\mathbf{X}\mathbf{a}}{\partial \mathbf{X}} = \mathbf{y}^*\mathbf{a}^{\mathrm{T}} \tag{4}$$

Minimum Mean Square Error (MMSE)

The system model:

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n} \tag{5}$$

The MMSE estimator of $\mathbf{x}$ is:

$$\arg\min_{\mathbf{W}} \mathbb{E}\left\{\|\mathbf{W}\mathbf{y} - \mathbf{x}\|^2\right\} \tag{6}$$

where the expectation is with respect to $\mathbf{x}$. To find it:

$$\begin{aligned} &\frac{\partial \mathbb{E}\left\{\|\mathbf{W}^{\mathrm{H}}\mathbf{y} - \mathbf{x}\|^2\right\}}{\partial \mathbf{W}} = \\ &\mathbb{E}\left\{\frac{\partial\left(\mathbf{W}^{\mathrm{H}}\mathbf{y}\mathbf{y}^{\mathrm{H}}\mathbf{W} - \mathbf{y}^{\mathrm{H}}\mathbf{W}\mathbf{x} - \mathbf{x}^{\mathrm{H}}\mathbf{W}^{\mathrm{H}}\mathbf{y} + \|\mathbf{x}\|^2\right)}{\partial \mathbf{W}}\right\} = 0 \end{aligned} \tag{7}$$

$$\left(\mathbf{y}\mathbf{y}^{\mathrm{H}}\right)^{\mathrm{T}}\mathbf{W}^* - \mathbf{y}^*\mathbf{x}^{\mathrm{T}} = 0 \tag{8}$$

which results in:

$$\mathbf{W} = \mathbb{E}\left\{(\mathbf{y}\mathbf{y}^{\mathrm{H}})^{-1}\mathbf{y}\mathbf{x}^{\mathrm{H}}\right\} \tag{9}$$

Substitute $\mathbf{y}$ with $\mathbf{H}\mathbf{x} + \mathbf{n}$ and we get:

$$\mathbf{W} = \left(\mathbf{H}\mathbf{R}_{\mathbf{xx}}\mathbf{H}^{\mathrm{H}} + \mathbf{R}_{\mathbf{nn}}\right)^{-1}\mathbf{H}\mathbf{R}_{\mathbf{xx}} \tag{10}$$

Therefore, to estimate $\mathbf{x}$:

$$\hat{\mathbf{x}} = \mathbf{R}_{\mathbf{xx}}\mathbf{H}^{\mathrm{H}}\left(\mathbf{H}\mathbf{R}_{\mathbf{xx}}\mathbf{H}^{\mathrm{H}} + \mathbf{R}_{\mathbf{nn}}\right)^{-1}\mathbf{y} \tag{11}$$

3 GPS for beginners

There are two codes being used in GPS. They are called C/A (Coarse/Acquisition code) and P codes (Precision code).

	C/A-code	P-code
Cyclic period	1 ms	7 days
Length	1023 chips	$\approx 2.35e^{14}$
Transmission frequency	1.023 MHz	10.23 MHz
Carrier frequency	1575.42 MHz	1227.60 MHz

Table 1: Comparison of two codes

4 Carrier Frequency Offset

$$y(t) = x(t)\cos(2\pi f_c t)\cos(2\pi(f_c + \Delta f)t) \overset{(a)}{=} x(t)\cos(2\pi \Delta f t) \tag{12}$$

where in (a) we applied a based band filter with passing band more than Δf. If we use BPSK, $x(t) \in \{-1, 1\}$ then the output detection would be:

$$y(t) = \begin{cases} x(t) & 0 < t < \frac{1}{4\Delta f} \quad \text{or} \quad \frac{3}{4\Delta f} < t < \frac{1}{\Delta f} \\ -x(t) & \frac{1}{4\Delta f} < t < \frac{3}{4\Delta f} \end{cases} \tag{13}$$

When designing a communication system, it is essential to consider CFO to prevent faulty symbol detection.

When switch and sampling between antennas is performed to collect I/Q samples, the time instances of collected symbols are not the same. Therefore, there is an extra phase addition to the symbol which needs to be reduced/corrected.

$$\boldsymbol{\varphi} = [\varphi_1, \varphi_2, \ldots, \varphi_M] \in [0, 2\pi] \tag{14}$$

$$\boldsymbol{\varphi}' = [\varphi'_1, \varphi'_2, \ldots, \varphi'_M] \in [0, 2M\pi] \tag{15}$$

$$\boldsymbol{\varphi}'' = [\varphi''_1, \varphi''_2, \ldots, \varphi''_M] \in [0, 2\pi] \tag{16}$$

where $\varphi''_i = \varphi'_i - 2\pi f_{\text{dev}} \cdot t_i$ where t_i is the time elapsed from the first sampling time of reference antenna.

$$\boldsymbol{y}_{\text{V}} = \begin{bmatrix} b_{11}e^{j\varphi''_{11}} & b_{21}e^{j\varphi''_{21}} & b_{31}e^{j\varphi''_{31}} & b_{41}e^{j\varphi''_{41}} \\ b_{12}e^{j\varphi''_{12}} & b_{22}e^{j\varphi''_{22}} & b_{32}e^{j\varphi''_{32}} & b_{42}e^{j\varphi''_{42}} \\ b_{13}e^{j\varphi''_{13}} & b_{23}e^{j\varphi''_{23}} & b_{33}e^{j\varphi''_{33}} & b_{43}e^{j\varphi''_{43}} \\ \vdots & \vdots & \vdots & \vdots \\ b_{19}e^{j\varphi''_{19}} & b_{29}e^{j\varphi''_{29}} & b_{39}e^{j\varphi''_{39}} & b_{49}e^{j\varphi''_{49}} \end{bmatrix} \tag{17}$$

$$\boldsymbol{y}'_{\text{V}} = \begin{bmatrix} b_{11}e^{j\varphi''_{11}} & b_{21}e^{j\varphi''_{21}} & b_{31}e^{j\varphi''_{31}} & b_{41}e^{j\varphi''_{41}} \end{bmatrix} \tag{18}$$

$$\boldsymbol{y}''_{\text{V}} = \begin{bmatrix} b_{12}e^{j\varphi''_{12}} & b_{22}e^{j\varphi''_{22}} & b_{32}e^{j\varphi''_{32}} & b_{42}e^{j\varphi''_{42}} \\ b_{13}e^{j\varphi''_{13}} & b_{23}e^{j\varphi''_{23}} & b_{33}e^{j\varphi''_{33}} & b_{43}e^{j\varphi''_{43}} \\ \vdots & \vdots & \vdots & \vdots \\ b_{19}e^{j\varphi''_{19}} & b_{29}e^{j\varphi''_{29}} & b_{39}e^{j\varphi''_{39}} & b_{49}e^{j\varphi''_{49}} \end{bmatrix} \tag{19}$$

$$\boldsymbol{p} = \frac{\boldsymbol{y}'_{\text{V}}(\boldsymbol{y}''_{\text{V}})^{\text{H}}}{\boldsymbol{y}''_{\text{V}}(\boldsymbol{y}''_{\text{V}})^{\text{H}}} \tag{20}$$

$$\boldsymbol{e} = [1 \quad \boldsymbol{p}]^{\text{T}} \tag{21}$$

$$P_{\text{PDDA}} = \left|\boldsymbol{a}(\varphi, \theta)^{\text{H}} \boldsymbol{e}\right| \tag{22}$$

5 Time of Flight Estimation

In many application where time of flight is estimated, a correlation between the delayed signal and the original copy of the signal is used. The obtained peak corresponds to time of flight.

$$y(t) = \alpha_0 x(t - \tau_0) + n(t) \quad \text{for} \quad t \in [0, T] \tag{23}$$

Since noise $n(t)$ is Gaussian, $y(t) - \alpha_0 x(t - \tau_0)$ has a Gaussian distribution. Therefore the log-likelihood follows:

$$\mathcal{L}(y(t), \tau, \alpha) = \int_0^T \left(y(t) - \alpha x(t - \tau)\right)^2 d_t \tag{24}$$