Algorithms for Telecom

Project 1

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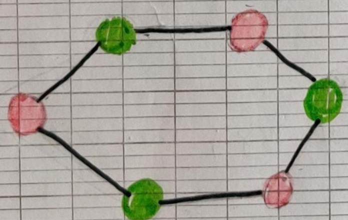
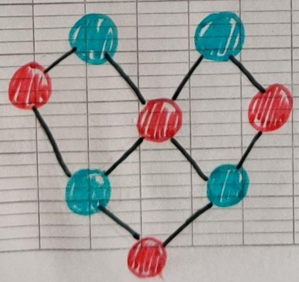
1. Give 2 examples of graphs of order a least 6 for which we have $z(G, \chi(G)) = 0$. That is one can find a coloring π using $k = \chi(G)$ colors and such that $B(G, \pi) = 0$. Recall that $\chi(G)$ is the minimum number of colors needed to color the vertices of G.

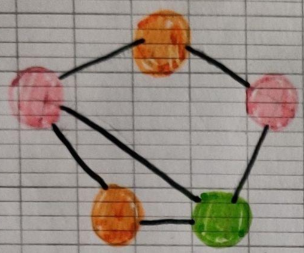


For this example, $K=2$.
The maximum number of vertices a color $= 3$
The minimum number of vertices a color $= 3$.
$z(G, \chi(G)) = 0$



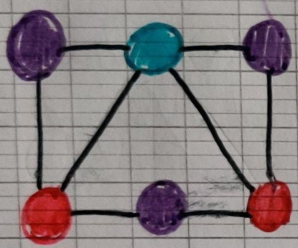
$K=2$. One color is assigned for 4 vertices and the other color is assigned also to 4 vertices
$z(G, \chi(G)) = 0$

2. Give 2 examples of graphs of order a least 5 for which we have z(G, χ(G)) > 0.



We use three colors: one is assigned to 2 vertices, the second is assigned to 2 vertices and the last one is assigned to 1.

So, $z(G, \chi(G)) = 1$



We use 3 colors: one is assigned to 3 vertices, other to 2 vertices and one is assigned to one vertice.

So, the difference between the maximum number of vertices per color and the minimum number of vertices is 2.

3. Write a ILP that, given a graph G = (V, E) and a number k of colors, finds z(G, k). That is, the ILP finds a coloring π of the vertices that uses k colors and such that B(G, π) is minimized. You must ensure that the k colors are used.

→ What we need to model:

- Variables
  - X_{vc} - it's true if and only if vertex v is assigned to color c
  - Y_c - it's true if and only if at least one vertex is assigned to color c
  - m_c - the number of vertex assigned to one color.
  - B - the difference between the max and min m_c

- Objective
  - minimize $B(G, \pi)$. We want a balanced coloring, to color the most equally way possible the graph, so the difference between the larger set of vertices assigned to color and the smallest set of vertices assigned to color is the lowest possible value for the given graph.

$$\min B$$

- Constraints:
  - We need to specify that every node is assigned to one color, exactly one. So, the sum over all colors assigned to one node is equal to one:

$$\forall v \in V, \quad \sum_{c \in C} X_{vc} = 1$$

  - We need to avoid that the same color is assigned to two adjacent nodes. Given a color and a edge, at most one of them can have the color. So, x_{vc} must be true only for one of them, the sum of them must be at most 1.

$$\forall u, v \in E, c \in C \quad x_{uc} + x_{vc} \leq 1$$

• We need to assure that if any node is set for a color, y_c is 1. So, if the vertex v is assigned to c, x_{vc} is true and, if this happens, y_c must be also true. We can set y_c as an upper bound for every x_{vc}, so it's not possible to x_{vc} be true and y_c be false.

$$\forall v \in V, c \in C \quad x_{vc} \leq y_c$$

• We need to take the number of the maximum set of vertices per color and the minimum set of vertices per color. First, we take the sum of vertices per color:

$$\forall c \in C, \quad \sum_{v \in V} x_{vc} = m_c$$

• Then, we take the maximum of the difference between all the m_c, that will be the difference between the larger set of vertices per color and the smallest set:

$$\forall i, j \in C, \quad m_j - m_i \leq B$$

To garantee that K colors are being used, the sum of colors used must be K:

$$\sum_{c \in C} y_c = K$$

$\min B$

subject to: $x_{uc} + x_{vc} \leq 1 \quad , \forall u,v \in E$

$$\sum_{c \in C} x_{vc} = 1 \quad , \forall v \in V$$

$$x_{vc} \leq y_c \quad , \forall v \in V, \forall c \in C$$

$$\sum_{v \in V} x_{vc} = m_c \quad , \forall c \in C$$

$$m_j - m_i \leq B \quad , \forall i,j \in C$$

$$\sum_{c \in C} y_c = K$$

$$x_{vc} \in \{0,1\}$$

$$y_c \in \{0,1\}$$

$$B \geq 0$$

$$m_c \geq 0$$

5. Propose a method to find a coloring π such that B(G, π) ≤ 2 and the number k of colors is minimized.

We modify the constraint used in question 4, setting the number $B = 2$,

$$\forall i,j \in C, \; m_j - m_i \leq 2$$

We inicialize the ILP with the chromatic number, C, using G.chromatic_number().

We need to change the objective. We can sum the value of all variables y_c, counting how many times this variable was true for assigning a color to at least one variable.

$$\min \sum_{c \in C} y_c$$

The ILP is then:

$$\min \sum_{c \in C} y_c$$

$$\text{s.t.} \quad x_{uc} + x_{vc} \leq 1, \; \forall u,v \in E$$

$$\sum_{c \in C} x_{vc} = 1, \; \forall v \in V$$

$$x_{vc} \leq y_c, \; \forall v \in V, \forall c \in C$$

$$\sum_{v \in V} x_{vc} = m_c, \; \forall c \in C$$

$$m_j - m_i \leq 2, \; \forall i,j \in C$$

$$x_{vc} \in \{0,1\}$$

$$y_c \in \{0,1\}$$

$$m_c \geq 0$$