UNIT 1:

FOURIER SERIES

- Introduction
- Periodic Functions
- Fourier series of periodic functions
- Dirichlet's conditions
- Even and odd functions
- Change of intervals

11/July/2019

01) **PERIODIC FUNCTION:** A function $f(x)$ is said to be a periodic function with period T ($T>0\ \forall x$) if $f(x+T)=f(x)$ and where T is the least of all such values of T.

Eg: (1) $f(x)=\sin x$. if $f(x+T)=f(x)$

$T=\pi$, $f(x+T)=f(x+\pi)=\sin(\pi+x)=-\sin x \neq f(x)$

if $T=2\pi$ $f(x+T)=f(x+2\pi)=\sin(2\pi+x)=\sin x=f(x)$

$T=2\pi, 4\pi, 6\pi, 8\pi \ldots (2n)\pi$

Period, $T=2\pi$ for $f(x)=\sin x$.

(2) $f(x)=\cot x$.

$T=\pi$, $f(x+T)=f(x+\pi)=\cot(\pi+x)=\cot x=f(x)$

Period, $T=\pi$ for $f(x)=\cot x$.

02) **SUFFICIENT CONDITION FOR THE FOURIER EXPANSION DIRICHLET'S CONDITIONS**.

A function $f(x)$ has a valid fourier series expansion if it satisfies the following conditions:

(PTO)

(i) $f(x)$ is well defined, periodic, single valued and finite except possibly at a finite number of points in the interval.

(ii) $f(x)$ has finite number of finite discontinuities in any one period.

(iii) $f(x)$ has atmost a finite number of maxima and minima in the given interval

15th July 2019

* Let $f(x)$ be a periodic function with period 2π defined as $(\alpha, \alpha+2\pi)$. Then Fourier series of $f(x)$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

where the constants a_0, a_n, b_n are called fourier coefficients.

17th July 2019

Note:-

$$\int_{\alpha}^{\alpha+2\pi} \sin mx \sin nx \, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$$

$$\int_{\alpha}^{\alpha+2\pi} \cos mx \cos nx \, dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$$

$$\int_{\alpha}^{\alpha+2\pi} \sin mx \cos nx \, dx = 0.$$

* Suppose $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$

$$\int_{\alpha}^{\alpha+2\pi} f(x)\,dx = \int_{\alpha}^{\alpha+2\pi} \left[\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx\right] dx.$$

$$= \int_{\alpha}^{\alpha+2\pi} \frac{a_0}{2} dx + \left[\int_{\alpha}^{\alpha+2\pi} a_1 \cos x + \int_{\alpha}^{\alpha+2\pi} a_2 \cos 2x + \cdots \right] + \left[\int_{\alpha}^{\alpha+2\pi} a_1 \sin nx + a_2 \sin 2x + \cdots \right]$$

$$= \frac{a_0}{2} [x]_{\alpha}^{\alpha+2\pi} + a_1 (\sin x)_{\alpha}^{\alpha+2\pi} + \cdots$$

$$= \frac{a_0}{2} [2\pi] + 0$$

$$\therefore \int_{\alpha}^{\alpha+2\pi} f(x) dx = a_0 \pi$$

$$\Rightarrow \boxed{a_0 = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) dx}$$

(*) Suppose $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$.

Multiplying both sides with $\cos mx$ and integrating from α to $\alpha+2\pi$, we have

$$\int_{\alpha}^{\alpha+2\pi} f(x) \cos mx \, dx = \int_{\alpha}^{\alpha+2\pi} \frac{a_0}{2} \cos mx + \int_{\alpha}^{\alpha+2\pi} \sum_{n=1}^{\infty} a_n \cos nx \cos mx + \int_{\alpha}^{\alpha+2\pi} \sum_{n=1}^{\infty} b_n \sin nx \cos mx$$

$$\int_{\alpha}^{\alpha+2\pi} f(x) \cos mx \, dx = \frac{a_0}{2} [\sin mx]_{\alpha}^{\alpha+2\pi} + \sum_{n=1}^{\infty} a_n \int_{\alpha}^{\alpha+2\pi} \cos nx \cos mx + \sum_{n=1}^{\infty} b_n \int_{\alpha}^{\alpha+2\pi} \sin nx \cos mx$$

(for $m = n$, integral $\to \pi$)

$$\int_{\alpha}^{\alpha+2\pi} f(x) \cos mx \, dx = 0 + a_m [\pi] + 0$$

$$a_m = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cos mx \, dx.$$

$$\boxed{a_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cos nx \, dx}$$

(*) Suppose $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$.

Multiplying both sides with $\sin nx$ and integrating from α to $\alpha+2\pi$, we have.

$$\int_{\alpha}^{\alpha+2\pi} \frac{a_0}{2} \sin mx + \int_{\alpha}^{\alpha+2\pi} a_n \cos nx \sin mx \, dx + \int_{\alpha}^{\alpha+2\pi} \sum b_n \sin nx \sin mx) \, dx$$

$$= 0 + \sum_{n=1}^{\infty} a_n \int_{\alpha}^{\alpha+2\pi} \cos nx \sin mx \, dx] + \sum_{n=}^{\infty} b_n \int_{\alpha}^{\alpha+2\pi} \sin x \sin mx \, dx$$

$$= 0 + 0 + b_m [\pi]$$

$$\Rightarrow b_m = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \sin mx \, dx$$

$$\boxed{b_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \sin nx \, dx}$$

SUFFICIENT CONDITIONS for the existence of FOURIES series expansion.

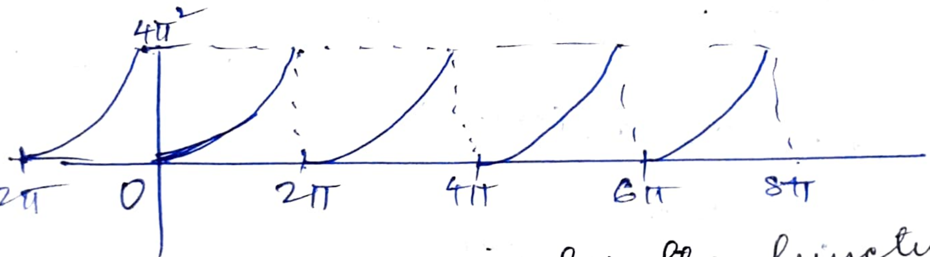
DIRICHLET'S CONDITIONS:

(01) $f(x)$ is periodic function with period 2π

(02) $f(x)$ is piece wise continuous and bounded on $(\alpha, \alpha+2\pi)$

(03) $f(x)$ has finite number of extrema (maxima / minima) in $(\alpha, \alpha+2\pi)$

If $f(x)$



18th July, 2019

Q) Find the fourier series expansion for the function $f(x) = x^2$ in the interval $(0, 2)$.

We know that Fourier Series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

By Euler's Formulae $a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x^2\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi}$$

$$= \frac{1}{3\pi}\left[8\pi^3\right]$$

$$\boxed{a_0 = \frac{8\pi^2}{3}}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx.$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2\cos nx\,dx.$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\underbrace{\frac{x^2\sin nx}{n}}_{=0} + \frac{2x\cos nx}{n^2} - \underbrace{\frac{2\sin nx}{n^3}}_{=0}\right]_0^{2\pi}$$

$[\because \sin n\pi = 0]$

$$= \frac{1}{\pi}\left[\frac{2x\cos nx}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{4\pi(1)}{n^2} - 0\right]$$

$$= \frac{4}{n^2}$$

$$\Rightarrow \boxed{a_n = \frac{4}{n^2}}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2\sin nx\,dx.$$

$$= \frac{1}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right) - \underbrace{2x\left(-\frac{\sin nx}{n^2}\right)}_{=0} + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left[4\pi^2\left(\frac{-1}{n}\right) - 0\right] + 2\left(\frac{1}{n^3} - \frac{1}{n^3}\right)\right]$$

$$= -\frac{4\pi^2}{n\times\pi} = -\frac{4\pi}{n}$$

$$\boxed{b_n = -\frac{4\pi}{n}}$$

$$f(x) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos nx + \sum_{n=1}^{\infty}\left(-\frac{4\pi}{n}\right)\sin nx$$

is the fourier series expansion for the function $f(x) = x^2$

$$f(x) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$$

$$f(x) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4\cos nx - 4\pi n\sin nx}{n^2}\right)$$

$$f(x) = \frac{4\pi^2}{3} + 4\left(\frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x + \cdots\right)$$

$$-4\pi\left(\frac{1}{1}\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots\right)$$

∴ the fourier series expansion for the function $f(x) = x^2$ in the interval $(0, 2)$.

9) $f(x) = e^x$ in $(0, 2\pi)$

Fourier series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$.

By Euler's formulae.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^x\,dx = \frac{1}{\pi}\left(e^x\right)_0^{2\pi} = \frac{e^{2\pi} - e^0}{\pi} = \frac{e^{2\pi}-1}{\pi}$$

$$\boxed{a_0 = \frac{e^{2\pi}-1}{\pi}}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx. = \frac{1}{\pi}\int_0^{2\pi} \underset{u}{e^x}\,\underset{v}{\cos nx}\,dx.$$

$$\int e^{ax}\cos bx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$$

$$a_n = \frac{1}{\pi}\left[\frac{e^x}{1+n^2}\left(\cos nx + n\sin nx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{e^{2\pi}}{1+n^2}(1+0)\right) - \left(\frac{e^0}{1+n^2}(1+0)\right)\right]$$

$$= \frac{1}{\pi}\left(\frac{e^{2\pi}}{1+n^2} - \frac{1}{1+n^2}\right) = \frac{e^{2\pi}-1}{(1+n^2)\pi}$$

$$\boxed{a_n = \frac{e^{2\pi}-1}{\pi(1+n^2)}}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx. = \frac{1}{\pi}\int_0^{2\pi} e^x \sin nx\,dx.$$

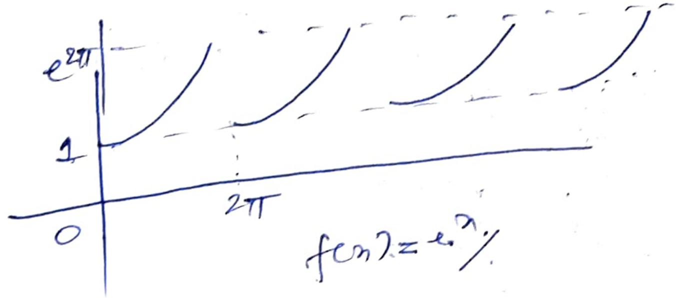
$$\int e^{ax}\sin bx = \frac{e^{ax}}{a^2+b^2}\left(a\sin bx - b\cos bx\right)$$

$$b_n = \frac{1}{\pi}\left[\frac{e^x}{1+n^2}\left(\sin nx - n\cos nx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{2\pi}}{1+n^2}\left((0 - n(1)\right) - \frac{e^0}{1+n^2}\left(0 - n(1)\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{e^{2\pi}}{1+n^2}(-n) - \frac{1}{1+n^2}(-n)\right]$$

$$= \frac{-n}{(1+n^2)\pi}\left[e^{2\pi}-1\right]$$



$$= \frac{1-e^{2\pi}}{\pi(1+n^2)}$$

$$\boxed{b_n = \frac{n(1-e^{2\pi})}{\pi(1+n^2)}}.$$

$$f(x) = \frac{e^{2\pi}-1}{2\pi} + \sum_{n=1}^{\infty} \frac{e^{2\pi}-1}{\pi(1+n^2)} \cos nx + \sum_{n=1}^{\infty} \frac{n(1-e^{2\pi})}{\pi(1+n^2)} \sin x.$$

$$f(x) = \frac{e^{2\pi}-1}{2\pi} + \frac{e^{2\pi}-1}{\pi}\left[\frac{1}{1+1^2}\cos x + \frac{1}{1+2^2}\cos 2x + \frac{1}{1+3^2}\cos 3x + 1 \ldots\ldots\right]$$

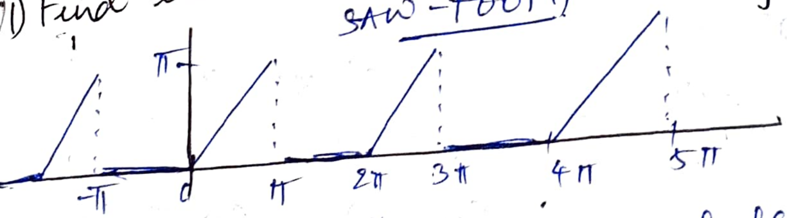
$$+ \frac{1-e^{2\pi}}{\pi}\left[\frac{1}{1+1^2}\sin x + \frac{2}{1+2^2}\sin 2x + \frac{3}{1+3^2}\sin 3x + \ldots\ldots\right]$$

is the fourier series expansion for the function $f(x) = e^x$ in the interval $(0, 2)$.

19th July 2019

1) Find the Fourier Series for SAW-TOOTH

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$$



We know that fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} x\,dx\right]$

$= \frac{1}{\pi}\left(\frac{x^2}{2}\right)_0^{\pi}$

$= \frac{\pi^2}{2\pi} = \frac{\pi}{2}$

$\boxed{a_0 = \frac{\pi}{2}}$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} x\,dx\right]$

$= \frac{1}{\pi}\left[x \sin x + \cos x\right]_{\pi}^{0}$

$= \frac{1}{\pi}\left(1 - (\pi(0) - 1)\right)$

$= \frac{1}{\pi}[1 -$

$= \frac{2}{\pi}$

$= \frac{1}{\pi}\left[\int_0^{\pi} x\cos nx\,dx\right]$

$= \frac{1}{\pi}\left[x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$

$= \frac{1}{\pi}\left[0 + \frac{(-1)^n}{n^2} - \left(0 + \frac{1}{n^2}\right)\right]$

$= \frac{1}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right]$

$= \frac{(-1)^n - 1}{\pi n^2}$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$

$= \frac{1}{\pi}\int_{-\pi}^{0} 0 \sin x\,dx + \int_{0}^{\pi} x \sin x\,dx$

$= \frac{1}{\pi}\left[x(-\cos x) + \sin x\right]_0^{\pi}$

$= \frac{1}{\pi}\left[(\pi(1) + 0) - 0\right]$

$= 1$

$= \frac{1}{\pi}\int_0^{\pi} x \sin nx\,dx$

$= \frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) + \frac{\sin nx}{n^2}\right]_0^{\pi}$

$= \frac{1}{\pi}\left[\left(\frac{\pi}{n}(-1)^n + 0\right) - (0 + 0)\right]$

$= -\frac{(-1)^n}{n}$

$a_0 = \frac{\pi}{2}$, $a_n = \frac{(-1)^n - 1}{\pi n^2}$, $b_n = -\frac{1}{n}(-1)^n = \frac{(-1)^{n+1}}{n}$

Fourier series of $f(x)$ is given

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \, dx + \sum_{n=1}^{\infty} b_n \sin nx \, dx.$

$= \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{\pi n^2} \cos nx + \sum_{n=1}^{\infty} \frac{-1}{n} (-1)^{n+1} \sin nx$

$= \frac{\pi}{4} + \left[\frac{-2}{\pi} \cos x + 0 + \frac{-2}{9\pi} \cos 3x + \frac{-2}{25\pi} \cos 5x + \cdots \right.$

$\left. + \left[+1 \sin x + -\frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots \right] \right]$

$= \frac{\pi}{4} + \frac{-2}{\pi} \left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots \right] + \left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} \right.$

$= \frac{\pi}{4} - \frac{2}{\pi} \sum_{n \text{ odd}} \frac{\cos nx}{n^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin nx}{n}$

$= \frac{\pi}{4} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx.$

Q) Find the F.S. of $f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$ and deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

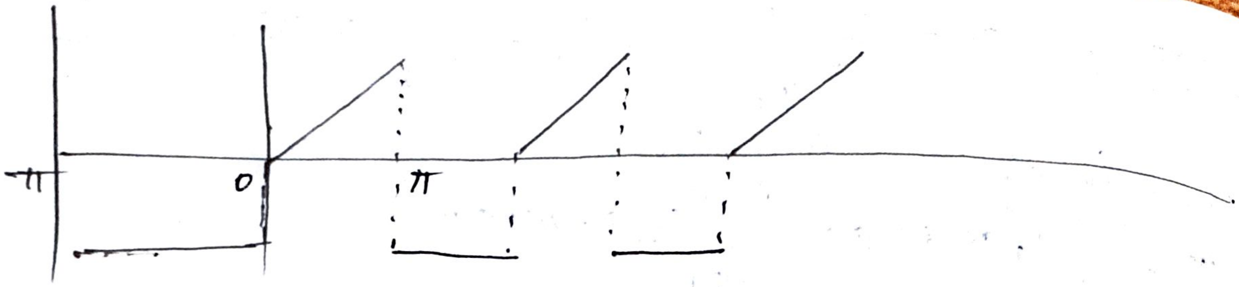
F.S is given by

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \, dx$

$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \left[\int_{-\pi}^{0} -\pi \, dx + \int_{0}^{\pi} x \, dx \right]$

$= \frac{1}{\pi} \left[-\pi (x)_{-\pi}^{0} + \left(\frac{x^2}{2} \right)_0^{\pi} \right] = \frac{1}{\pi} \left[-\pi(+\pi) + \frac{\pi^2}{2} \right] = \frac{-\pi^2 + \frac{\pi^2}{2}}{\pi}$

$a_0 = -\pi/2$



$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \cos nx \, dx + \int_{0}^{\pi} x \cos nx \, dx\right]$$

$$= \frac{1}{\pi}\left[-\pi \left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + \frac{(-1)^n - 1}{n^2}\right]$$

$$= \frac{1}{\pi}\left[-\pi \left[0 - 0\right] + \frac{(-1)^n - 1}{n^2}\right] = \frac{(-1)^n - 1}{\pi n^2}$$

$$a_n = \frac{(-1)^n - 1}{\pi n^2}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{0} -\pi \sin nx \, dx + \int_{0}^{\pi} x \sin nx \, dx.$$

$$= \frac{1}{\pi}\left[+\pi \left(\frac{\cos nx}{n}\right)_{-\pi}^{0} + \frac{-\pi(-1)^n}{n}\right]$$

$$= \left(\frac{\cos nx}{n}\right)_{-\pi}^{0} - \frac{(-1)^n}{n}$$

$$= \frac{1 - (-1)^n}{n} - \frac{(-1)^n}{n}$$

$$= \frac{1 - 2(-1)^n}{n}$$

$$b_n = \frac{1 + 2(-1)^{n+1}}{n}$$

$$a_0 = -\pi/2, \quad a_n = \frac{(-1)^n - 1}{\pi \cdot n^2}, \quad b_n = \frac{1 - 2(-1)^n}{n}$$

F.S of $f(x)$ is given by
$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

$$= -\frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{\pi n^2} \cos nx + \sum_{n=1}^{\infty} \frac{1+2(-1)^{n+1}}{n} \sin nx.$$

$n \to$ odd $\frac{1+2}{1-2}$

$$= -\frac{\pi}{4} + \frac{1}{\pi} \cdot \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos nx + \sum_{n=1}^{\infty} \frac{1+2(-1)^{n+1}}{n} \sin nx$$

$$= -\frac{\pi}{4} + \frac{1}{\pi} \sum_{n \text{ odd}} \frac{-2}{n^2} \cos nx + \left[\frac{3}{1}\sin x - \frac{\sin 2x}{2} + \frac{3}{3}\sin 3x - \frac{\sin 4x}{4} + \frac{3}{5}\sin 5x + \cdots\right]$$

$$= -\frac{\pi}{4} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{-2\cos(2n-1)x}{(2n-1)^2} + \quad ''$$

$$= -\frac{\pi}{4} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2} + \left[\frac{3}{1}\sin x - \frac{\sin 2x}{2} + \frac{3}{3}\sin 3x + \cdots\right]$$

The above series converges to $f(x)\ \forall x \in \mathbb{R} / \{n\pi \mid n \in \mathbb{Z}\}$ and the points of discontinuity $x = n\pi$, the F.S converges to $\frac{1}{2}[f(n\pi^-) + f(n\pi^+)]$

Deduction:

Clearly, $x=0$ is a point of discontinuity. Therefore, the F.S at $x=0$ converges $\frac{1}{2}[f(0^-)+f(0^+)]$ where

$f(0^-) \to$ Left Limit $= \lim_{x\to 0^-} f(x) = -\pi$

$f(0^+) \to$ Right Limit $= \lim_{x\to 0^+} f(x) = 0$. therefore, at $x=0$

F.S of $f(x)$ is given by,

$$-\frac{\pi}{4} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{1}{2}[f(0^-)+f(0^+)] = \frac{1}{2}[-\pi+0]$$

$$-\frac{\pi}{2} \Rightarrow -\frac{\pi}{4} - \frac{2}{\pi} \sum \frac{1}{(2n-1)^2} \Rightarrow -\frac{2}{\pi} \sum \frac{1}{(2n-1)^2} = -\frac{\pi}{4}$$

$$\boxed{\sum \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}}$$

Hence Proved!

22nd July 2019

Q). $f(x) = x^2$ in $(0, 2\pi)$

$$f(x) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}\cos nx - \sum_{n=1}^{\infty} \frac{4\pi}{n}\sin nx. \qquad \forall x \in \mathbb{R}/\{2n\pi\}$$

Deduce that $\frac{\pi^2}{6} = \sum \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

$$f(x) = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{\cos nx}{n^2} - 4\pi\sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

Deduction:

At $x=0$, the Fourier series of $f(x)$ converges to $\frac{1}{2}[f(0^-) + f(0^+)]$ where $f(0^-) = \lim_{x\to 0^-} f(x) = 4\pi^2$

$f(0^+) = \lim_{x\to 0^+} f(x) = 0.$

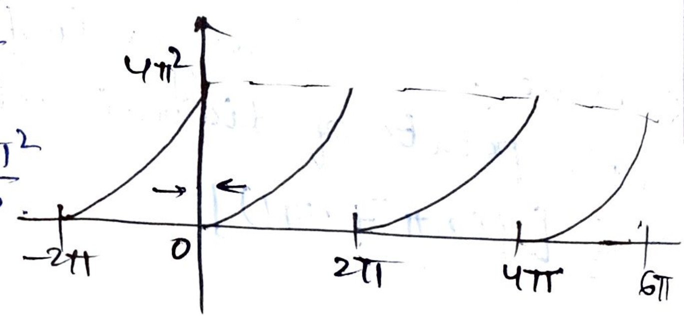
$\therefore$ At $x=0$, RHS $= \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(1) - \left(\sum_{n=1}^{\infty} 0\right) = \frac{1}{2}[4\pi^2 + 0]$

$\frac{4\pi^2}{3} + 4\sum \frac{1}{n^2} = 2\pi^2$

$4\sum \frac{1}{n^2} = 2\pi^2 - \frac{4\pi^2}{3}$

$4\sum \frac{1}{n^2} = \frac{2\pi^2}{3}$

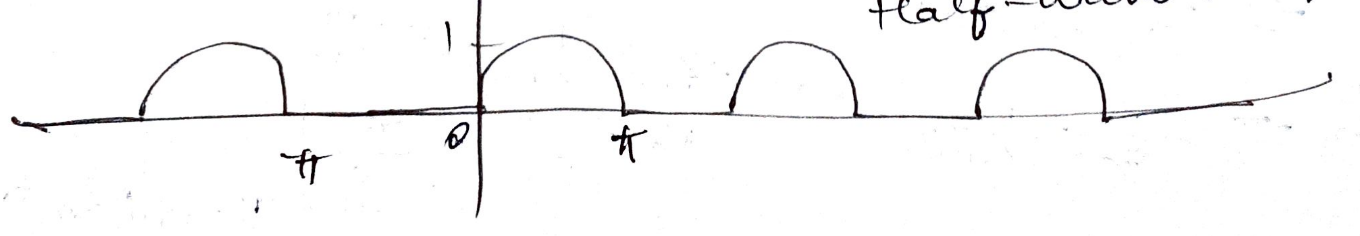
$$\boxed{\sum \frac{1}{n^2} = \frac{\pi^2}{6}}$$



Q) $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \sin x, & 0 < x < \pi \end{cases}$ Find the Fourier series of this function and deduce that

Deduce $\frac{\pi - 2}{4} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots + \cdots$

Half-wave Rectifier



The Fourier series of $f(x)$ is given by.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} (0)\,dx + \int_0^{\pi} \sin x\,dx = \frac{-(\cos x)_0^{\pi}}{\pi}$$

$$= \frac{-(-1-1)}{\pi} = \frac{2}{\pi}$$

$$\boxed{a_0 = \frac{2}{\pi}}.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} \sin x \cos nx\,dx.$$

$$= \frac{1}{2\pi}\int_0^{\pi} 2\cos nx \sin x\,dx = \frac{1}{2\pi}\int_0^{\pi} [\sin(nx+x) - \sin(nx-x)]\,dx.$$

$$= \frac{1}{2\pi}\int_0^{\pi} [\sin(n+1)x - \sin(n-1)x]\,dx = \frac{1}{2\pi}\left[-\left[\frac{\cos(n+1)x}{n+1}\right]_0^{\pi} + \left[\frac{\cos(n-1)x}{(n-1)}\right]_0^{\pi}\right]$$

$(n \neq 1)$

$$= \frac{1}{2\pi}\left[-\left(\frac{\cos(n+1)\pi}{(n+1)} - \frac{\cos 1}{n+1}\right) + \left(\frac{\cos(n-1)\pi - 1}{n-1}\right)\right]$$

$$= \frac{1}{2\pi}\left[\left[-\frac{(-1)^{n+1}}{n+1} - \frac{1}{n+1}\right] + \left(\frac{(-1)^{n-1}}{n-1} - \frac{1}{n-1}\right)\right]$$

$$= \frac{1}{2\pi}\left[(-1)^n\left[\frac{1}{n+1} - \frac{1}{n-1}\right] + \left[\frac{1}{n+1} - \frac{1}{n-1}\right)\right]$$

$$= \frac{1}{2\pi}\left[\left(\frac{1}{n+1} - \frac{1}{n-1}\right) \times \left((-1)^n + 1\right)\right]$$

$n-1-n-1$

-2

$$= \frac{-2}{2\pi(n^2-1)}\left((-1)^n + 1\right)$$

$$= \frac{-((-1)^n+1)}{\pi(n^2-1)} = \frac{((-1)^n+1)}{\pi(1-n^2)} \quad (n \neq 1)$$

$(n = 2, 3, 4 \dots)$

$b_n = \frac{1}{\pi}$ $\quad a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos 1x\,dx. = \frac{1}{\pi}\int_0^{\pi} \sin x \cos nx\,dx.$

$$= \frac{1}{2\pi}\int_0^{\pi} \sin 2x\,dx = \frac{-1}{2\pi}\left(\frac{\cos 2x}{2}\right)_0^{\pi} = \frac{-1}{4\pi}(\cos 2x)_0^{\pi} = \frac{-1}{4\pi}(1-1)$$

$$\boxed{a_1 = 0}.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{0}^{\pi}\sin x\sin nx\,dx = \frac{1}{2\pi}\int_0^{\pi} 2\sin nx\sin x$$

$$= \frac{+1}{2\pi}\int_0^{\pi}[\cos(n-1)x\,dx - \cos(n+1)x]\,dx. = \frac{1}{2\pi}\left[\frac{\sin(n-1)x}{(n-1)} - \frac{\sin(n+1)x}{(n+1)}\right]$$

$$= \frac{1}{2\pi}\left[\left(\frac{\sin(n-1)\pi}{n-1} - \frac{\sin(n+1)\pi}{(n+1)}\right) - (0-0)\right] = 0.$$

$n = 2, 3, 4$

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi}\sin x\sin x\,dx = \frac{1}{\pi}\int_0^{\pi}\sin^2 x\,dx = \frac{1}{\pi}\int_0^{\pi}\frac{1-\cos 2x}{2}\,dx$$

$$b_1 = \frac{1}{2\pi}\int_0^{\pi} 1\,dx - \int_0^{\pi}\cos 2x\,dx = \frac{1}{2\pi}\left[(x)_0^{\pi} - \left(\frac{\sin 2x}{2}\right)_0^{\pi}\right]$$

$$= \frac{1}{2\pi}[(\pi-0) - (0-0)] = \frac{1}{2}$$

$a_0 = \frac{2}{\pi}$

$a_1 = 0$

$a_n = \frac{(-1)^n+1}{\pi(1-n^2)}$ for $n = 2, 3, \ldots$

$b_1 = \frac{1}{2}$

$b_n = 0$

Therefore Fourier Series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

$$= \frac{1}{\pi} + \sum_{n=2}^{\infty}\frac{(-1)^n+1}{\pi(1-n^2)} + \frac{1}{2}\sin x.$$

$$f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx + b_1 \sin x + \sum_{n=2}^{\infty} b_n \sin nx$$

$$= \frac{1}{2}\left(\frac{2}{\pi}\right) + (0)\cos x + \sum_{n=2}^{\infty} \frac{(-1)^n + 1}{\pi(1-n^2)} \cos nx + \frac{1}{2}\sin x.$$

$$= \frac{1}{\pi} + \sum_{n=2}^{\infty} \frac{(-1)^n + 1}{\pi(1-n^2)} \cos nx + \frac{1}{2}\sin x.$$

$$= \frac{1}{\pi} + \frac{2}{\pi} \sum_{n=\text{even}} \frac{\cos nx}{(1-n^2)} + \frac{1}{2}\sin x.$$

From the graph it is clear that, ~~Fourier~~ $f(x)$ is continuous $\forall x \in \mathbb{R}$. Therefore, the Fourier series converges to $f(x) \; \forall \mathbb{R}$

$$f(x) = \frac{1}{\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} \left(\frac{\cos 2nx}{1-(2n)^2}\right) + \frac{1}{2}\sin x.$$

$$= \frac{1}{\pi} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2nx}{(2n+1)(2n-1)} + \frac{\sin x}{2}$$

Put $x = 0$

$$f(0) = 0 = \frac{1}{\pi} - \frac{2}{\pi} \sum \frac{1}{(2n+1)(2n-1)}$$

$$-\frac{1}{\pi} = -\frac{2}{\pi} \sum \frac{1}{(2n+1)(2n-1)}$$

$$\sum \frac{1}{(2n+1)(2n-1)} = \frac{1}{2} \quad (\times)$$

Put $x = \pi/2$

$$f(\pi/2) = 1 = \frac{1}{\pi} - \frac{2}{\pi} \sum \frac{1}{(2n+1)(2n-1)} \cos nx + \frac{1}{2}$$

$$\frac{1}{2} - \frac{1}{\pi} = -\frac{2}{\pi} \sum \frac{(-1)^n}{(2n+1)(2n-1)}$$

$$\frac{\pi - 2}{2\pi} = \frac{-2}{\pi} \sum \frac{(-1)^n}{(2n+1)(2n-1)}$$

$$\frac{\pi - 2}{2} = 2 \sum \frac{(-1)^{n+1}}{(2n+1)(2n-1)}$$

$$\frac{\pi-2}{4} = \frac{1}{1\cdot3} - \frac{1}{3\cdot5} + \frac{1}{5\cdot7} - \frac{1}{7\cdot9} + \cdots$$

★★ Q) $f(x) = x\sin x$ in $(0, 2\pi)$

Fourier series for $f(x)$ is given by $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n \sin nx$.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} x\sin x\,dx = x(-\cos x)_0 - \int(-\cos x)$$

$$= \left[-x\cos x + \sin x\right]_0^{2\pi} = \left[-2\pi\cos 2\pi + \sin 2\pi\right] - (0-0)]$$

$$a_0 = \frac{-2\pi}{\pi} = -2$$

$$\boxed{a_0 = -2}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi}(x\sin x)\cos nx\,dx.$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[2\cos nx\sin x\right]dx = \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin(n+1)x - \sin(n-1)x\right]$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right] - 1\left[-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right]$$

$$= \frac{1}{2\pi}\left[\left(2\pi\left(\frac{-1}{n+1} + \frac{1}{n-1}\right) - 1(0+0)\right) - \left(0 - 1(0+0)\right)\right]$$

$$= \frac{-n+1+n+1}{n^2-1} = \frac{2}{n^2-1} \quad (n \neq 1)$$

$n = 2, 3, 4 \ldots$

$$\boxed{a_n = \frac{2}{n^2-1} \quad n = 2,3,4}$$

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos x\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\sin 2x = \frac{1}{2\pi}\left[x\left(-\frac{\cos 2x}{2}\right) + \left(\frac{\sin 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left(\frac{2\pi(-1)}{2} + 0\right) - (0+0)\right]$$

$$\boxed{a_1 = \frac{-1}{2}}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin x \sin nx\,dx.$$

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} x\,[2\sin nx\,dx\,\sin x]\,dx = \frac{1}{2\pi}\int_0^{2\pi} x\,[\cos(n-1)dx - \cos(n+1)dx]$$

$$b_n = \frac{1}{2\pi}\left[x\left[\frac{\sin(n-1)x}{(n-1)} - \frac{\sin(n+1)x}{(n+1)}\right] - 1\left[-\frac{\cos(n-1)x}{(n-1)^2} + \frac{\cos(n+1)x}{(n+1)^2}\right]\right]_0^{2\pi}$$

$$b_n = \frac{1}{2\pi}\left[\left(2\pi(0-0) - 1\left(\frac{-1}{(n-1)^2} + \frac{1}{(n+1)^2}\right)\right) - \left(0 - 1\left(\frac{-1}{(n-1)^2} + \frac{1}{(n+1)^2}\right)\right)\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} + \frac{1}{(n+1)^2} - \frac{1}{(n-1)^2}\right] = 0$$

$$\boxed{b_n = 0}$$

$n = 2, 3, 4 \ldots$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin^2 x\,dx.$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\left(\frac{1 - \cos 2x}{2}\right)dx.$$

$$= \frac{1}{2\pi}\int_0^{2\pi}(x - x\cos 2x)\,dx$$

$$= \frac{1}{2\pi}\left[\left(\frac{x^2}{2}\right)_0^{2\pi} - \int_0^{2\pi} x\cos 2x\,dx\right]$$

$$= \frac{1}{2\pi}\left[\left(\frac{x^2}{2}\right)_0^{2\pi} - \left[x\,\frac{\sin 2x}{2} + \left(\frac{\cos 2x}{4}\right)\right]_0^{2\pi}\right]$$

$$= \frac{1}{2\pi}\left[\frac{4\pi^2}{2} - \left[2\pi(0) + \cancel{\frac{1}{4}} - \left(0 + \cancel{\frac{1}{4}}\right)\right]\right]$$

$$= \frac{1}{2\pi}\left[2\pi^2 - \left[\cancel{2\pi}\right]\right]$$

$$\boxed{b_1 = \pi}$$

Fourier Series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx + b_1 \sin x + \sum_{n=1}^{\infty} b_n \sin nx$$

$$= -1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} \frac{2}{n^2-1} \cos nx + \pi \sin x + 0$$

$$= -1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} \frac{2}{n^2-1} \cos nx + \pi \sin x //$$

H(W)

Q1) $f(x) = x\cos x$ in $(0, 2\pi)$

Q2). $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in $(0, 2\pi)$

Deduce $\frac{\pi^2}{6} = \sum \frac{1}{n^2}$

Q3) $f(x) = \begin{cases} -x, & -\pi < x \leq 0 \\ 0, & 0 < x \leq \pi \end{cases}$

Fourier series for $f(x)$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x\cos x\,dx.$$

$$= \frac{(x \sin x + \cos x)_0^{2\pi}}{\pi} = \frac{1}{\pi}\left[2\pi \sin 2\pi + \cos 2\pi - (0+1)\right]$$

$$= \frac{1}{\pi}[0 + 1 - 1] = 0 //$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{2\pi}\int_0^{2\pi} 2x \cos x \cos nx\,dx.$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x[\cos(n+1)x + \cos(n-1)x].\,dx.$$

$$= \frac{1}{2\pi} \int_0^{2\pi} x \left[\frac{\sin(n+1)x}{(n+1)} + \frac{\sin(n-1)x}{(n-1)} \right] - 1 \left[-\frac{\cos(n+1)x}{(n+1)^2} - \frac{\cos(n-1)x}{(n-1)^2} \right]$$

$$\frac{1}{2\pi} \left[2\pi (0-0) - 1 \left(\frac{\cos(n+1)2\pi}{(n+1)^2} - \frac{\cos(n-1)2\pi}{(n-1)^2} \right) - \left[0 - 1 \left(\frac{-1}{(n+1)^2} - \frac{1}{(n-1)^2} \right) \right] \right.$$

$$= \frac{1}{2\pi} \left[\frac{1}{(n+1)^2} + \frac{1}{(n-1)^2} + \frac{-1}{(n+1)^2} - \frac{1}{(n-1)^2} \right]$$

$$\boxed{a_n = 0}$$

$$= 0.$$

$$a_1 = \frac{1}{\pi} \int_0^{2\pi} x \cos^2 x \, dx.$$

$$= \frac{1}{\pi} \int_0^{2\pi} x \left(\frac{1+\cos 2x}{2} \right) dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} x + x\cos 2x \, dx$$

$$= \frac{1}{2\pi} \left[\left(\frac{x^2}{2} \right)_0^{2\pi} + \left(\frac{x \sin 2x}{2} + \frac{\cos 2x}{4} \right)_0^{2\pi} \right]$$

$$\boxed{a_1 = \pi}$$

$$= \frac{1}{2\pi} \left[\frac{4\pi^2}{2} + \left(\frac{1}{4} - \frac{1}{4} \right) \right] = \frac{4\pi^2}{4\pi} = \pi$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \int_0^{2\pi} x \cos x \sin nx \, dx.$$

$$= \frac{1}{2\pi} \int_0^{2\pi} x [2 \sin nx \cos nx] dx = \frac{1}{2\pi} \int_0^{2\pi} x [\sin(n+1)x - \sin(n-1)x] dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} (x \sin(n+1)x \, dx - x \sin(n-1)x \, dx)$$

$$= \frac{1}{2\pi} \left[x \left(-\frac{\cos(n+1)x}{(n+1)} + \frac{\cos(n-1)x}{n-1} \right) - 1 \left(-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2} \right) \right]_0^{2\pi}$$

$$= \frac{1}{2\pi} \left[2\pi \left(\frac{(-1)(1)}{n+1} + \frac{1}{n-1} \right) + 0 \right)$$

$$= \frac{1}{n-1} - \frac{1}{n+1} = \frac{n+1-n+1}{n^2-1} = \frac{2}{n^2-1}, \quad n = 2, 3, 4 \ldots$$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x\cos x \sin x \, dx = \frac{1}{2\pi}\int_0^{2\pi} x \sin 2x \, dx$$

$$= \frac{1}{2\pi}\left(x\left(\frac{-\cos 2x}{2}\right) + 1 \times \frac{\sin 2x}{4}\right)_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left(2\pi\left(\frac{-1}{2}\right) + 0\right) - \left(0 + 0\right)\right]$$

$$= -1/2$$

Fourier series of $f(x)$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

$$= \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx + b_1 \sin x + \sum_{n=2}^{\infty} b_n \sin nx.$$

$$= 0 + \pi\cos x + 0 - \frac{1}{2}\sin x + \sum_{n=2}^{\infty} \frac{2}{n^2-1} \sin x.$$

$$= \pi\cos x - \frac{\sin x}{2} + 2\sum_{n=2}^{\infty} \frac{\sin x}{n^2-1}$$

12) $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in $(0, 2\pi)$

Fourier series for $f(x)$ is given by $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} \left(\frac{\pi - x}{2}\right)^2 dx = \frac{1}{4\pi}\int_0^{2\pi} (\pi^2 - 2\pi x + x^2)\, dx = \frac{1}{4\pi}\left[\left(\pi^2 x - \pi x^2 + \frac{x^3}{3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left(2\pi^3 - 4\pi^3 + \frac{8\pi^3}{3}\right) = \frac{2\pi^3}{3 \times 4\pi} = \frac{\pi^2}{6}$$

$8\pi^3 - 6$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)^2 \cos nx\, dx = \frac{1}{4\pi}\int_0^{2\pi} (\pi - x)^2 \cos nx\, dx$$

$$= \frac{1}{4\pi}\left[(\pi - x)^2 \frac{\sin nx}{n} - \left[2(\pi - x)(-1) - \frac{\cos nx}{n^2}\right]\right]$$

$$= \frac{1}{4\pi}\left[(\pi - x)^2 \frac{\sin nx}{n} - \frac{2(\pi - x)\cos nx}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left((\pi - 2\pi)^2 (0) - \frac{2(\pi - 2\pi)}{n^2}\cos 2\pi x\right) - \left(\pi^2 (0) - \frac{2\pi}{n^2}(1)\right)\right]$$

$$= \frac{1}{4\pi}\left[\frac{2\pi}{n^2} + \frac{2\pi}{n^2}\right] = \frac{4\pi}{n^2} \times \frac{1}{4\pi} = \frac{1}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)^2 \sin nx\, dx = \frac{1}{4\pi}\int_0^{2\pi} (\pi - x)^2 \sin nx\, dx$$

$$= \frac{1}{4\pi}\left[(\pi - x)^2\left(\frac{-\cos nx}{n}\right) - \left[2(\pi - x)(-1)\left(-\frac{\sin nx}{n^2}\right)\right]\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left((\pi - 2\pi)^2 \frac{-\cos 2\pi x}{n} - 0\right) - \left(\frac{-\pi^2}{n} - (2\pi \times 0)\right)\right]$$

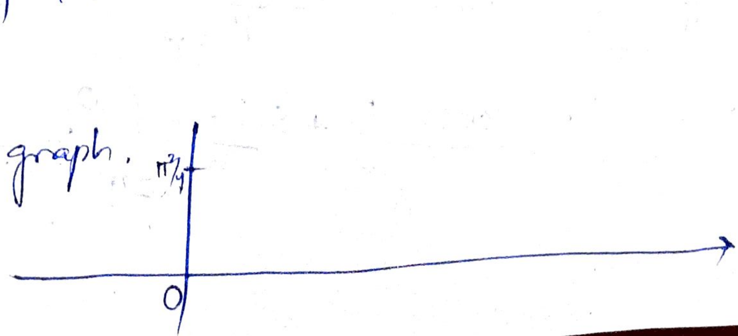
$$= \cancel{\frac{-\pi^2}{4\pi n}} \quad \frac{1}{4\pi}\left[-\frac{\pi^2}{n} + 0\right] = \frac{-\pi}{4n}.$$

Fourier series for $f(x)$ is given by.

$$\frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2}\cos nx + \sum_{n=1}^{\infty} \frac{-\pi}{4n}\sin nx.$$

Deduction:

At $x = 0$, From the graph.



$$f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2} - \frac{\pi}{4}\sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

Put $x=0$

$$f(0) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} + 0.$$

$$\lim_{x\to 0} f(0) = \frac{\pi^2}{4}$$

$$\frac{\pi^2}{4} - \frac{\pi^2}{12} = \sum \frac{1}{n^2}$$

$$\sum \frac{1}{n^2} = \frac{\pi^2}{6}.$$

$$\frac{\pi^2}{4} - \frac{\pi^2}{12}$$

$$\frac{2\pi^2}{12}$$

03). $f(x) = \begin{cases} -x, & -\pi < x \leq 0 \\ 0, & 0 < x \leq \pi \end{cases}$

Fourier series for $f(x)$ is given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0}(-x)\,dx + \int_{0}^{\pi} 0\,dx = \frac{1}{\pi}\left[\frac{-x^2}{2}\right]_{-\pi}^{0} = \frac{-1}{\pi}\left[0 - \frac{\pi^2}{2}\right]$$

$$\boxed{a_0 = \frac{\pi}{2}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0}(-x)\cos nx\,dx = \frac{-1}{\pi}\int_{-\pi}^{0} x\cos nx\,dx$$

$$a_n = \frac{-1}{\pi}\left[x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_{-\pi}^{0} = \frac{-1}{\pi}\left[0 + \frac{1}{n^2} - \left(-\pi(0) + \frac{(-1)^n}{n^2}\right)\right]$$

$$= \cancel{\frac{-1}{\pi}\left[\frac{2}{n^2}\right]} = \cancel{\frac{-2}{\pi n^2}} \quad a_n = \frac{-1}{\pi}\left[\frac{1}{n^2} - \frac{(-1)^n}{n^2}\right] = \frac{(-1)^n - 1}{\pi n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0}(-x)\sin nx\,dx = \frac{-1}{\pi}\int_{-\pi}^{0} x\sin nx\,dx$$

$$= \frac{-1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) + \frac{\sin nx}{n^2}\right]_{-\pi}^{0} = \frac{-1}{\pi}\left[(0+0) - \left(-\pi\left(\frac{-(-1)^n}{n}\right) + 0\right)\right]$$

$$= \frac{\pi}{n} \times \frac{(-1)^n}{\pi} = \frac{(-1)^n}{n}$$

Fourier series of $f(x)$ is given by

$$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

$$\frac{\pi}{4}+\sum_{n=1}^{\infty}\frac{(-1)^n-1}{\pi n^2}\cos nx+\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

25th July '2019

FOURIER SERIES OF EVEN / ODD PERIODIC FUNCTIONS DEFINED ON $(-\pi, \pi)$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

Case-1) $f(x)$ is even periodic function on $(-\pi,\pi)$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx.$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx=0.$$

$\therefore$ Fourier series of $f(x)$ becomes

$$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos nx \rightarrow \text{Fourier Cosine Series (FCS)}$$

where $a_0=\frac{2}{\pi}\int_0^{\pi} f(x)\,dx$, $a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$, $n=1,2,3\ldots$

Case-2) $f(x)$ is odd periodic function on $(-\pi,\pi)$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx=0, \quad a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx=0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx.$$

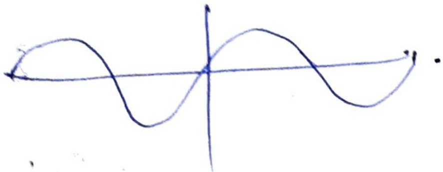
$\therefore$ Fourier series of $f(x)$ becomes Fourier Sine Series (FSS).

$$\sum_{n=1}^{\infty} b_n \sin nx \rightarrow \text{Fourier Sine Series}$$

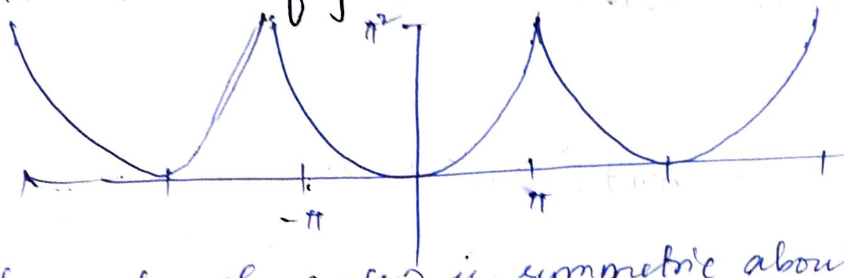
where $b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx.$

Odd.

01) $f(-x) = -f(x)$

02) 

01) Find Fourier series of $f(x) = x^2$ in $(-\pi, \pi)$



Since the graph of $y=f(x)$ is symmetric about y-axis, the given function is an even periodic function on $(-\pi,\pi)$. Therefore, Fourier series of $f(x)$ becomes Fourier Cosine Series given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi} = \frac{2}{\pi} \times \frac{\pi^3}{3} = \frac{2\pi^2}{3}.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx. = \frac{2}{\pi}\left(x^2 \frac{\sin nx}{n} - 2x\left(-\frac{\cos nx}{n^2}\right) + 2 - \frac{\sin nx}{n^3}\right)_0^{\pi}$$

$$= \frac{2}{\pi}\left(0 + \frac{2\pi(-1)^n}{n^2}\right) = \frac{4\pi}{\pi n^2} = \frac{+4(-1)^n}{n^2}$$

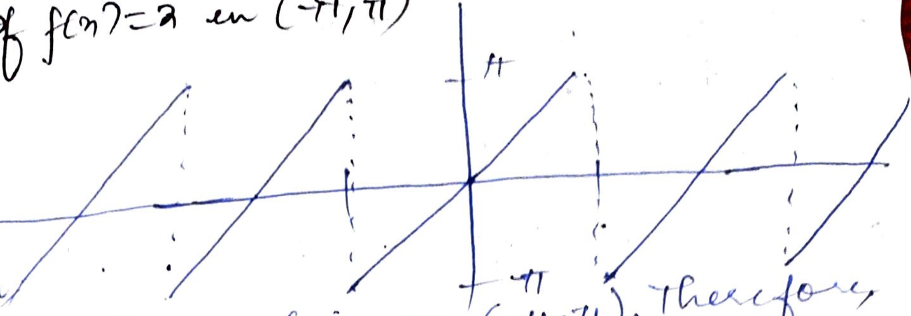
$\therefore$ Fourier Cosine Series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n 4}{n^2} \cos nx.$$

$$= \frac{\pi^2}{3}$$

12) Find Fourier series of $f(x) = x$ in $(-\pi, \pi)$



Since the graph of $y = f(x)$ is symmetric about the opposite quadrants, the given function is an odd periodic function on $(-\pi, \pi)$. Therefore, Fourier series of $f(x)$ becomes Fourier sine series given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \cancel{x \sin nx\, dx} \quad \frac{2}{\pi}\cancel{\left(\frac{x^2}{2}\right)_0^{\pi}} = \cancel{\frac{2}{\pi}\frac{\pi^2}{2}} = \cancel{\pi}$$

$\therefore$ ~~Fourier Sine series~~

$$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx\, dx = \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - 1\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi(-1)^{n-1}}{n} + \cancel{\frac{\sin n\pi}{n^2}}\right] \left\{ \cancel{0} - 0\ 0 \right]$$

$$= \frac{2(-1)^{n+1}}{n}$$

Fourier sine series for $f(x) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin nx$, in the ~~interval~~ $(-\pi, \pi)$.

13) Find Fourier Series of $f(x) = x + x^2$ in the

H.W

$$f(+x) = x + x^2$$

$$f(-x) = -x + x^2$$

$$f(-x) \neq f(x)\ \forall x \quad ; \quad f(-x) = -f(x)\ \forall x.$$

The given function is neither even nor odd.

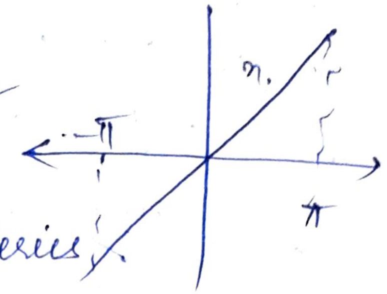
$$\boxed{f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n 4}{n^2}\cos nx + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\sin nx}$$

Calculate / Define Fourier series for both x & x^2 separately as they are odd and even functions respectively. Superimpose them to get the result

Define $g(x) = x$ in $(-\pi, \pi)$
$h(x) = x^2$ in $(-\pi, \pi)$

then $f(x) = g(x) + h(x)$

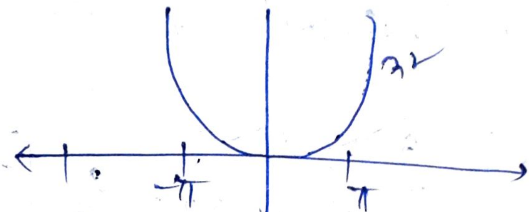
Fourier series of $f(x)$ = FSS of $g(x)$ + FCS of $h(x)$



$g(x) = x$.

Since the graph of $y = g(x)$ is symmetric about origin the function $g(x)$ is odd periodic function on $(-\pi, \pi)$. Therefore Fourier series of $g(x)$ becomes Fourier Sine Series, given by $g(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$$g(x) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nx.$$



$h(x) = x^2$

Since the graph of $y = \cancel{h}(x)$ is symmetric about y-axis, the function $h(x)$ is even periodic on $(-\pi, \pi)$. Therefore, Fourier series of $h(x)$ becomes Fourier Cosine series given by

$$h(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$h(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{(-1)4}{n^2} \cos nx.$$

$$f(x) = g(x) + h(x)$$

Fourier series of $f(x)$ = FSS of $g(x)$ + FCS of $h(x)$

Fourier series of $f(x) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nx + \frac{\pi^2}{3} \cancel{\frac{a_0}{2}} \sum_{n=1}^{\infty} \cancel{a_n \cos nx} \frac{(-1)^n 4 \cos nx}{n^2}$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n 4 \cos nx}{n^2} + \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nx$$

$\therefore$ Fourier series of $f(x) = x + x^2$ in the interval $(-\pi, \pi)$ is given as $\frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n 4}{n^2} \cos nx + \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin nx$

14) Find Fourier Series of $f(x) = x\sin x$. in $(-\pi, \pi)$

$f(x) = x\sin x$.

$f(-x) = (-x)\sin(-x) = (-x)(-\sin x) = x\sin x = f(x)\ \forall x$

$\therefore$ $f(x)$ is even periodic function on $(-\pi, \pi)$

Therefore, Fourier Series of $f(x)$ becomes Fourier Cosine Series given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx. \qquad a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx = \frac{2}{\pi}\left[x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi} = \frac{2}{\pi}\left[\left(0 + \frac{(-1)^n}{n^2}\right) - \left(\frac{1}{n^2}\right)\right.$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right] = \frac{2((-1)^n - 1)}{\pi n^2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos x\cos$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\sin x\,dx = \frac{2}{\pi}\left[x(-\cos x) - 1(-\sin x)\right]_0^{\pi} =$$

$$= \frac{2}{\pi}\left[(-\pi(-1) + 0) - (0 + 0)\right] = 2 \qquad \boxed{a_0 = 2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx\,dx. = \frac{1}{\pi}\int_0^{\pi} x[2\cos nx\sin x]\,dx.$$

$$= \frac{1}{\pi}\int_0^{\pi} x[\sin(n+1)x - \sin(n-1)x]\,dx.$$

$$= \frac{1}{\pi}\left\{x\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right] - 1\left[-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right]\right\}_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi\left(\frac{-1\times(-1)^{n+1}}{(n+1)} + \frac{(-1)^{n-1}}{(n-1)}\right) - 1(0 - 0)\right]$$

$n+2$
$-n+1$
3

$$= \frac{(-1)^{n+2}}{n+1} + \frac{(-1)^{n-1}}{(n-1)} = (-1)^{n-1}\left[\frac{-1}{n+1} + \frac{1}{n-1}\right]$$

$$= (-1)^{n-1}\left[\frac{-n+1+n+1}{n^2-1}\right]$$

$$= (-1)^{n-1}\times\frac{2}{n^2-1}$$

$$\boxed{a_n = \frac{2(-1)^{n-1}}{n^2-1} = \frac{2(-1)^n}{1-n^2}}$$

$a_1 = \frac{2}{\pi}\int_0^{\pi} f(x)\cos x\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos x\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx$

Fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx.$$

$$= \frac{a_0}{2} + a_1\cos x + \sum_{n=2}^{\infty} a_n\cos nx.$$

$$= 1 - \frac{1}{2}\cos 2x. + \sum_{n=2}^{\infty} \frac{2(-1)^n}{1-n^2}\cos nx.$$

Q) $f(x) = x\cos x$ in $(-\pi, \pi)$

H.W

$\underbrace{x}_{\text{odd}}\ \underbrace{\cos x}_{\text{even}}$ — odd

$f(x) = x\cos x$

$f(-x) = (-x)\cos(-x) = (-x)\cos x = -f(x)$

$\therefore f(x)$ is odd periodic function in $(-\pi, \pi)$

Therefore, Fourier series of $f(x)$ becomes Fourier sine series given by $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\cos x\sin nx.$$

$$= \frac{1}{\pi}\int_0^{\pi} x\,[2\sin nx\cos x] = \frac{1}{\pi}\int_0^{\pi}\left[x\left[\sin(n+1)x - \sin(n-1)x\right]dx\right]$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)}{(n-1)}\right) - 1\left(-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)}{(n-1)^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\pi\frac{(-1)(-1)^{n+1}}{(n+1)} + \frac{\pi(-1)^{n-1}}{(n-1)} - 1(0-0)\right) - 0(\ ) - 1(0)\right]$$

$$= (-1)^n\left[\frac{(-1)(-1)}{n+1} - \frac{1}{n-1}\right] = (-1)^n\left[\frac{1}{n+1} - \frac{1}{n-1}\right] = \frac{(-1)^n - 2}{n^2-1}$$

$$= (-1)^n\,\frac{n-1-n-1}{n^2-1}$$

$b_n = \frac{(-1)^{n+1} 2}{n^2-1}$ $n=2,3,4\ldots$

$b_1 = \frac{2}{\pi}\int_0^\pi x\cos x \cancel{\cos x} \sin x\,dx$

$= \frac{1}{\pi}\int_0^\pi x \sin 2x. = \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - 1\left(\frac{-\sin 2x}{4}\right)\right]_0^\pi$

$= \frac{1}{\pi}\left[\frac{\pi(-1)(+1)}{2} - 1(0)\right]$

$b_1 = \frac{-1}{2}$

Fourier series of $f(x)$ is given by.

$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

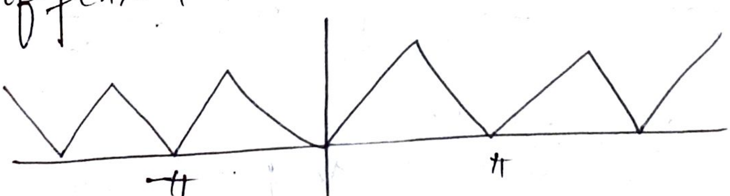
$= b_1 \sin x + \sum_{n=2}^{\infty} b_n \sin nx$

$= \frac{-1}{2}\sin x + 2\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^2-1}$

$= \frac{-1}{2}\sin x + 2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}$ //

27th July 2019

Find the Fourier series of $f(x) = |x|$ is $(-\pi < x < \pi)$



Clearly given function is even periodic function Fourier series is Fourier Cosine series given by

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$a_0 = \frac{2}{\pi}\int_0^\pi f(x)dx = \frac{2}{\pi}\int_0^\pi |x| dx = \frac{2}{\pi}\int_0^\pi x\,dx = \frac{2}{\pi}\left[\frac{\pi^2}{2}\right] = \pi$

$a_n = \frac{2}{\pi}\int_0^\pi |x|\cos nx\,dx = \frac{2}{\pi}\int_0^\pi x\cos nx\,dx =$

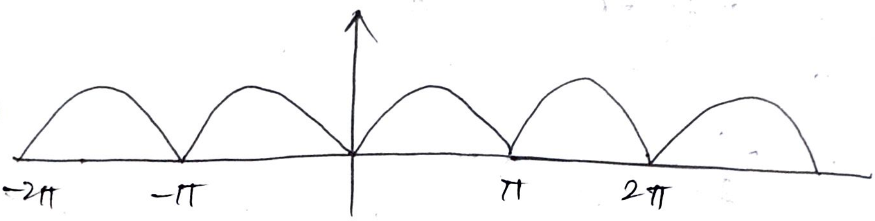
$$= \frac{2}{\pi}\left[x\left[\frac{\sin nx}{n}\right] - 1\left[-\frac{\cos nx}{n^2}\right]\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi\,(0) + \left[\frac{(-1)^n}{n^2}\right]\right] - \left[0 + \frac{1}{n^2}\right] = \frac{2}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right]$$

Fourier series of given function is $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$.

(12) $f(x) = |\sin x|$ in $-\pi < x < \pi$

Fourier series of the function $f(x)$ is F.C.S.



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} |\sin x|\,dx = \frac{2}{\pi}\left[-\cos x\right]_0^{\pi} = -\frac{2}{\pi}\left[(-1) - 1\right] = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\,dx = \frac{1}{\pi}\int_0^{\pi}\left[\sin(n+1)x - \sin(n-1)x\right]dx.$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi} = \frac{1}{\pi}\left[\left(-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}\right) - \left(\frac{-1}{n+1} + \frac{1}{n-1}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n + 1}{n+1} + \frac{(-1)^{n-1} - 1}{n-1}\right] = \frac{1}{\pi}\left[\frac{(-1)^n n + n - (-1)^n - 1) + (-1)^{n-1} n - n + (-1)^n - 1}{(n+1)(n-1)}\right]$$

$$a_n = \frac{1}{\pi}\left[\frac{-2(-1)^n - 1}{n^2 - 1}\right] \quad \forall\, n = 2, 3, 4$$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x\,dx = \frac{1}{\pi}\int_0^{\pi} \sin 2x\,dx = \frac{1}{\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi} = \frac{-1}{2\pi}\left[1 - 1\right]$$

$$\boxed{a_1 = 0}$$

∴ Fourier Cosine Series is

$$\frac{2}{\pi} + \sum_{n=2}^{\infty} \frac{1}{\pi}\left[\frac{-2(-1)^n - 1}{n^2 - 1}\right].$$

$f(x) = x^3$ in $-\pi < x < \pi$

$f(-x) = (-x)^3 = -x^3 = -f(x)$ odd periodic function

F.S is F.S.S $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x^3 \sin nx \, dx = \frac{2}{\pi}\left[x^3\left(\frac{-\cos nx}{n}\right) - 3x^2\left[\frac{-\sin nx}{n^2}\right] + 6x\left[\frac{\cos nx}{n^3}\right] - 6\left[\frac{\sin nx}{n^4}\right]\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left[\pi^3\left(\frac{-1(-1)^n}{n}\right) - 0 + 6\pi\left(\frac{(-1)^n}{n^3}\right) - 6(0)\right] - (0)\right]$$

$$= \frac{2}{\pi}\left[\pi^3\left(\frac{-(-1)^n}{n}\right] + \frac{6\pi(-1)^n}{n^3}\right] = 2(-1)^n\left[\frac{6}{n^3} - \frac{\pi^2}{n}\right]$$

Fourier series is $\sum_{n=1}^{\infty} 2(-1)^n\left[\frac{6}{n^3} - \frac{\pi^2}{n}\right] \sin nx$.

FOURIER SERIES OF PERIODIC FUNCTION ON $(\alpha, \alpha+2L)$

Let $f(x)$ be a periodic function with a period $T = 2L$ defined on $(\alpha, \alpha+2L)$

Let $z = \frac{\pi x}{L} \Rightarrow dz = \frac{\pi\, dx}{L}$

Put $x = \alpha$ then $z = \frac{\pi \alpha}{\pi} = d$ (say)

Put $x = \alpha + 2L$ the $z = \frac{\pi}{L}(\alpha + 2L) = \frac{\pi\alpha}{L} + 2\pi = d + 2\pi$

$\alpha < x < \alpha + 2\pi \Leftrightarrow d < z < d + 2\pi$

$$f(x) = f\left(\frac{zL}{\pi}\right) = F(z)$$

$f(x+2L) = f(x) \Rightarrow F(z+2\pi) = F(z) \; \forall z$

$\therefore F(z)$ is periodic function on $(d, d+2\pi)$

$\therefore$ F.S of $F(z)$ is $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nz + \sum_{n=1}^{\infty} b_n \sin nz$.

$$a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} F(z)\, dz = \frac{1}{\pi}\int_{\alpha}^{\alpha+2L} f(x)\frac{\pi}{L}\, dx = \frac{1}{L}\int_{\alpha}^{\alpha+2L} f(x)\, dx$$

$$\boxed{a_0 = \frac{1}{L}\int^{\alpha+2L} f(x)\, dx}$$

$$a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} F(z)\cos nz\,dz = \frac{1}{\pi}\int_{\alpha}^{\alpha+2L} f(x)\cos\frac{n\pi}{L}x \times \frac{\pi}{L}\,dx$$

$$= \frac{1}{L}\int_{\alpha}^{\alpha+2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$n = 1, 2, 3 \dots$

$$b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} F(z)\sin nz\,dz = \frac{1}{\pi}\int_{\alpha}^{\alpha+2L} f(x)\sin\left(\frac{n\pi x}{L}\right)\frac{\pi}{L}\,dx$$

$$= \frac{1}{L}\int_{\alpha}^{\alpha+2L} f(x)\sin\left(\frac{n\pi}{L}\right)x\,dx$$

$n = 1, 2, 3 \dots$

$\therefore$ F.S is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{L}\right)$$

<u>Special Cases:</u>

01) Fourier Series of even periodic function defined on $(-L, L)$, in this cases F.S of $f(x)$ becomes F.C.S given by $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi}{L}\right)x$

where $a_0 = \frac{2}{L}\int_0^L f(x)\,dx$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx$$

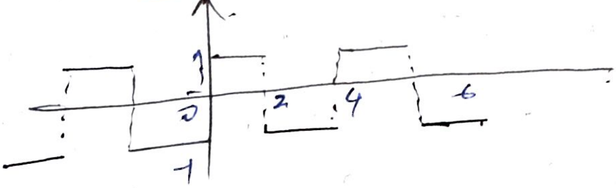
02) Fourier series of odd periodic function defined on $(-L, L)$
In this case F.S of $f(x)$ becomes F.S.S is given by

$$\sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{L}\right)$$

where $b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$. $n = 1, 2, 3 \dots$

Q) Find the Fourier series of $f(x) = \begin{cases} -1, & -2 < x < 0 \\ 1, & 0 < x < 2 \end{cases}$

Odd periodic function defined on $(-2,2)$



$f(x+4)=f(x)$

$T=2L=4 \Rightarrow \boxed{L=2}$

$F.S.S = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$

$b_n = \frac{2}{L}\int_0^L \sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\left[\frac{-\cos\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right]_0^2 = \frac{2}{L}\left[-\left[\frac{\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}} - 1\right]\right]$

$= \frac{-2}{n\pi}\left[\cos(n\pi) - \cos(0)\right] = \frac{-2}{n\pi}\left[(-1)^n - 1\right]$

$\therefore$ Fourier Sine Series is $\sum_{n=1}^{\infty} \frac{-2\left[(-1)^n - 1\right]}{n\pi}\cos nx.$

01st August 2019

$f(x)$ be periodic on $(\alpha, \alpha+2L)$

FS is given by $\frac{a_0}{2} + \sum a_n \cos\left(\frac{n\pi}{L}x\right) + \sum b_n \sin\left(\frac{n\pi}{L}x\right)$.

$a_0 = \frac{1}{L}\int_\alpha^{\alpha+2L} f(x)\,dx$, $a_n = \frac{1}{L}\int_\alpha^{\alpha+2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$, $b_n = \int_\alpha^{\alpha+2L} f(x).\sin\left(\frac{n\pi x}{L}\right)dx$

Special case-1 $f(x)$ is even periodic on $(-L, L)$

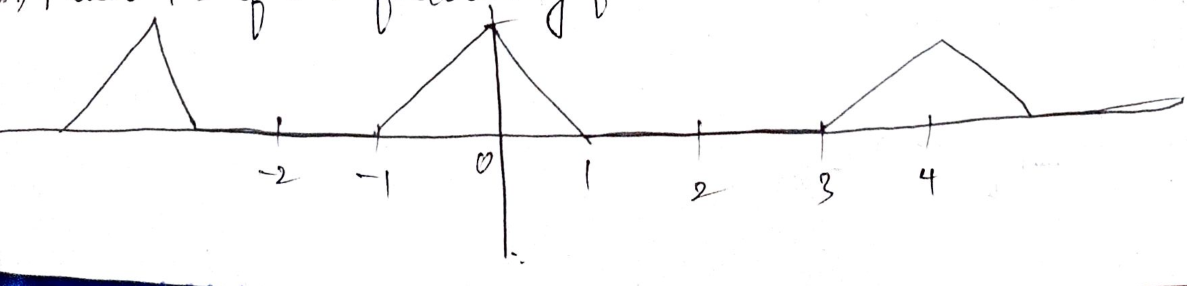
FCS is given by $\frac{a_0}{2} + \sum a_n \cos\left(\frac{n\pi}{L}x\right)$

$a_0 = \frac{2}{L}\int_0^L f(x)\,dx$ $\quad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx$

Special case-2 $f(x)$ is odd periodic on $(-L, L)$

$FSS = \sum b_n \sin\left(\frac{n\pi x}{L}\right)$ where $b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)$.

1) Find FS of the following functions:



From the graph, the function is even periodic defined on $(-2,2)$, therefore, Fourier series of $f(x)$ becomes Fourier Cosine series given by. $\frac{a_0}{2}+\sum a_n \cos\left(\frac{n\pi x}{L}\right)$

$$a_0=\frac{2}{L}\int_0^L f(x)dx \quad a_n=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx.$$

$$f(x+h)=f(x) \quad \forall x$$

$$T=2L=4 \Rightarrow L=2$$

$$f(x)=\begin{cases} 0 & -2<x<-1 \\ 1+x & -1<x\le 0 \\ 1-x. & 0<x\le 1 \\ 0 & 1<x\le 2 \end{cases}$$

$$a_0=\frac{2}{L}\int_0^L f(x)dx=\frac{2}{2}\int_0^2 f(x)dx=\int_0^1 f(x)dx+\int_1^2 f(x)dx$$

$$=\int_0^1(1-x)\,dx=\left(x-\frac{x^2}{2}\right)_0^1=1-\frac{1}{2}=\frac{1}{2} \quad \boxed{a_0=\frac{1}{2}}$$

$$a_n=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx=\int_0^2 f(x)\cos\left(\frac{n\pi}{2}x\right)dx=$$

$$=\int_0^1 \underset{u}{(1-x)}\underset{v}{\cos\left(\frac{n\pi}{2}x\right)}dx+\int_1^2(0)\cos\left(\frac{n\pi}{2}x\right)dx.$$

$$=\left[(1-x)\left[\frac{\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)}\right]-(-1)\left[\frac{-\cos\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^2}\right]\right]_0^1$$

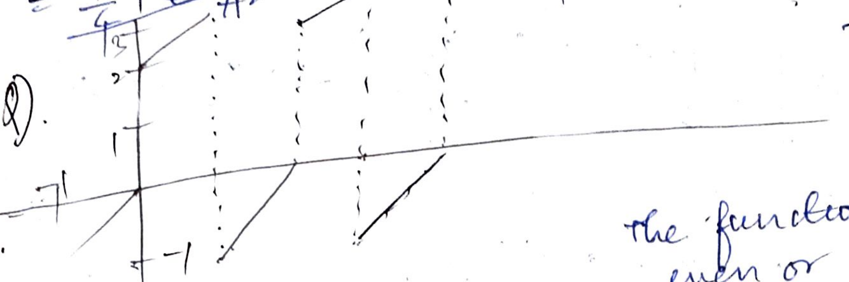
$$=\left[\left(0-\frac{4}{n^2\pi^2}\cos\frac{n\pi}{2}\right)-\left(0-\frac{4}{n^2\pi^2}\right)\right].$$

$$a_n=\frac{4}{n^2\pi^2}\left[1-\cos\frac{n\pi}{2}\right]$$

Therefore the fourier cosine series of $f(x)$ is given by $\frac{a_0}{2} + \sum a_n \cos\left(\frac{n\pi x}{L}\right)$

$= \frac{1}{4} + \frac{4}{\pi^2} \sum \frac{1}{n^2} \cos = \frac{1}{4} + \frac{4}{\pi^2} \sum \frac{1}{n^2}\left[1 - \cos\frac{n\pi}{2}\right]\left(\cos\frac{n\pi x}{L}\right).$

Q).



$f(x) = f(x+2) \; \forall x.$

period, $T = 2L = 2$

$L = 1$

the function given is neither even or not odd.

$f(x) = \begin{cases} x & -1 < x < 0 \\ x+2 & 0 < x < 1 \end{cases}$

Therefore, the F.S is given by $\frac{a_0}{2} + \sum a_n \cos\left(\frac{n\pi x}{L}\right) + \sum b_n \sin\left(\frac{n\pi x}{L}\right)$

$a_0 = \frac{1}{L}\int_{\alpha}^{\alpha+2L} f(x)\,dx, \quad a_n = \frac{1}{L}\int_{\alpha}^{\alpha+2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad b_n = \frac{1}{L}\int_{\alpha}^{\alpha+2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$

$a_0 = \frac{1}{1}\int_{\alpha}^{\alpha+2L} f(x)\,dx = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} x\,dx + \int_{0}^{1}(x+2)\,dx = \left(\frac{x^2}{2}\right)_{-1}^{0} + \left(\frac{x^2}{2} + 2x\right)_0^1$

$-1(-1)(-1))$

$= -\frac{1}{2} + \left(\frac{1}{2} + 2\right) = 2 \qquad \boxed{a_0 = 2}$

$a_n = \int_{-1}^{+1} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \int_{-1}^{0} x\cos\left(\frac{n\pi x}{L}\right)dx + \int_{0}^{1}(x+2)\cos\left(\frac{n\pi x}{L}\right)dx$

$= \left[x\left[\frac{\sin\left(\frac{n\pi x}{1}\right)}{\left(\frac{n\pi}{1}\right)}\right] + \left(\frac{\cos\left(\frac{n\pi x}{1}\right)}{\left(\frac{n\pi}{1}\right)^2}\right)\right]_{-1}^{0} + \left[(x+2)\left[\frac{\sin\frac{n\pi x}{1}}{\left(\frac{n\pi}{1}\right)}\right] + \frac{\cos\left(\frac{n\pi x}{1}\right)}{\left(\frac{n\pi}{1}\right)^2}\right]_0^1$

$= \left[\left(\frac{1}{n^2\pi^2}\right) - \left[(-1)(\sin(-0)) + \frac{\cos n\pi}{(n\pi)^2}\right]\right] + \left[\left(3(0) + \frac{(-1)^n}{n^2\pi^2}\right) - \left(2(0) + \frac{1}{n^2\pi^2}\right)\right]$

$= \frac{1}{n^2\pi^2} - \frac{(-1)^n}{n^2\pi^2} + \frac{(-1)^n}{n^2\pi^2} - \frac{1}{n^2\pi^2} = 0 \Rightarrow \boxed{a_n = 0}$

$b_n = \int_{-1}^{1} f(x)\sin\left(\frac{n\pi x}{L}\right)dx =$

$$= \int_{-1}^{0} x \sin\left(\frac{n\pi x}{L}\right) dx + \int_{0}^{1} (x+2) \sin\left(\frac{n\pi x}{L}\right) dx.$$

$$= \left[x\left(\frac{-\cos n\pi x}{n\pi}\right) + 1 \frac{\sin n\pi x}{n^2\pi^2}\right]_{-1}^{0} + \left[(x+2)\left(\frac{-\cos(n\pi x)}{n\pi}\right) + 1\left(\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_{0}^{1}$$

$$= \left(0 - \left[\frac{(-1)(-1)^n}{n\pi} + 0\right]\right) + \left[\left(3\left(\frac{-(-1)^n}{n\pi}\right) + 0\right) - \left(2\left(\frac{-1}{n\pi}\right) + 0\right)\right]$$

$$= \frac{(-1)^n}{n\pi} + \frac{3(-1)^{n+1}}{n\pi} + \frac{2}{n\pi} = \frac{1}{n\pi}\left[2 + (-1)^n + 3(-1)^{n+1}\right]$$

Fourier series for $f(x)$ is given by

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{1}\right) + \sum b_n \sin\left(n\pi x\right)$$

$$= 1 + 0 + \sum \frac{1}{n\pi}\left[2 + (-1)^n + 3(-1)^{n+1}\right]$$

$$= 1 + \frac{1}{\pi} \sum \frac{2 + (-1)^n + 3(-1)^{n+1}}{n}.$$

$\boxed{\text{series with sine terms is called FSS}}$

Note:- The above is Fourier Sine series but the function is not odd periodic function.

05th August 2019

$A \Rightarrow \frac{a_0}{2} + \sum a_n \cos\left(\frac{n\pi x}{L}\right) + \sum b_n . \sin\left(\frac{n\pi x}{L}\right)$

$a_0 = \frac{1}{L}\int_{\alpha}^{\alpha+2L} f(x)\,dx \quad a_n = \frac{1}{L}\int_{\alpha}^{\alpha+2L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx.$

$b_n = \frac{1}{L}\int_{\alpha}^{\alpha+2L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx$

$B \Rightarrow \frac{a_0}{2} + \sum a_n \cos\left(\frac{n\pi x}{L}\right) \quad a_0 = \frac{2}{L}\int_0^L f(x)\,dx, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) dx.$

$C \Rightarrow \sum b_n \sin\left(\frac{n\pi x}{L}\right) \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx.$

$f(x)$

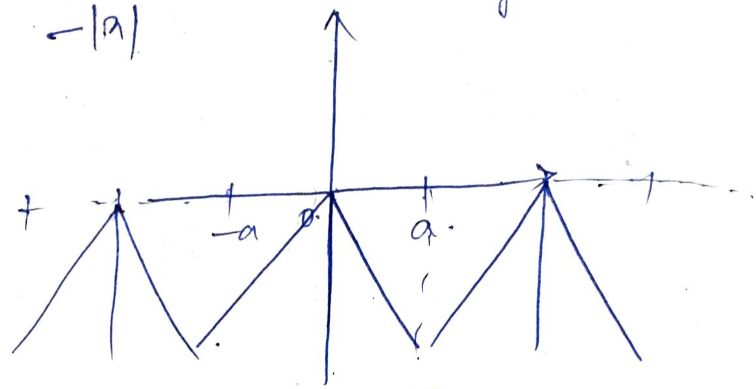
- PERIODIC
  - $T = 2\pi$
    - $(-\pi, \pi)$
      - even: (B) with $L = \pi$, FCS
      - odd: (C) with $L = \pi$, FSS
      - Neither even nor odd: A with $\alpha = -\pi$, $L = \pi$, FS
    - $(0, 2\pi)$ → (A) with $\alpha = 0$, $L = \pi$, FS
  - $T = 2L$
    - $(-L, L)$
      - even: (B) FCS
      - odd: (C) FSS
      - Neither even nor odd: (A) FS
    - $(\alpha, \alpha + 2L)$ → (A) FS
- NON PERIODIC
  - $(0, L)$
    - HFCS (B) $\boxed{\text{with } \alpha = 0}$
    - HFSS (C) $\boxed{\text{with } \alpha = 0}$

PROBLEMS

Find Fourier series of $f(x) = \begin{cases} x & -a < x < 0 \\ -x & 0 < x < a \end{cases}$

H.W

$-|x|$



Fourier series of $f(x)$ is

$f(x)$ is even periodic function on $(-a, a)$

Fourier series becomes Fourier Cosine series.

$$\text{Fourier Cosine Series} = \frac{a_0}{2} + \Sigma a_n \cos\left(\frac{n\pi x}{L}\right) \qquad L = a.$$

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{2}{a}\int_0^a (-x)\,dx = \frac{-2}{a}\left(\frac{x^2}{2}\right)_0^a = -\frac{1}{a}(a^2) = -a \quad \boxed{a_0 = -a}$$

$$a_n = \frac{2}{a}\int_0^a (-x)\cos\left(\frac{n\pi x}{a}\right) = -\frac{2}{a}\left[x\left(\frac{\sin\frac{n\pi x}{a}}{\frac{n\pi}{a}}\right) + 1 + \left(\frac{\cos\frac{n\pi x}{a}}{\frac{n^2\pi^2}{a^2}}\right)\right]_0^a$$

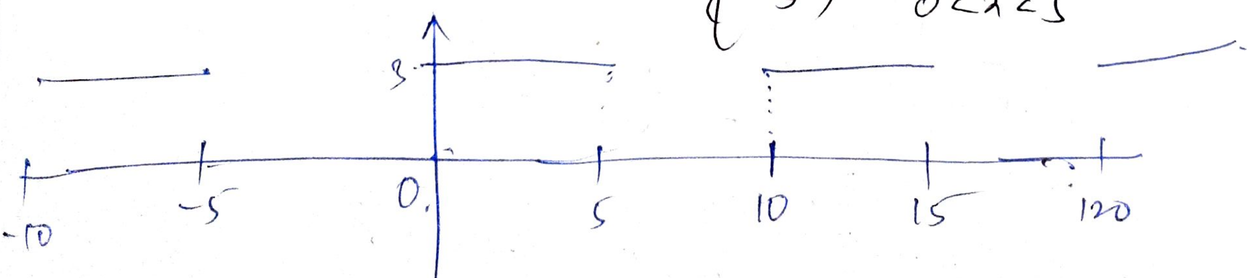
$$= -\frac{2}{a}\left[\left(a(0) + \frac{(-1)^n}{\frac{n^2\pi^2}{a^2}}\right) - \left((0) + \frac{1}{\frac{n^2\pi^2}{a^2}}\right)\right]$$

$$= -\frac{2}{a}\left[\frac{a^2}{n^2\pi^2}[(-1)^n - 1]\right]$$

$$= \frac{-2a}{n^2\pi^2}[(-1)^n - 1]$$

$$\therefore \text{Fourier Cosine Series of } f(x) = \frac{-a}{2} + \Sigma \frac{-2a}{n^2\pi^2}[(-1)^n - 1]\cos\left(\frac{n\pi x}{a}\right)$$

Find Fourier series of $f(x) = \begin{cases} 0, & -5 < x < 0 \\ 3, & 0 < x < 5 \end{cases}$



~~Since~~ Fourier Series of $f(x)$ is given by $\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{L}+\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{L}$.

$\cancel{\alpha=}$

$\alpha+2L=$

$\cancel{a_0=\frac{1}{L}\int_{\alpha}^{\alpha+2L} f(x)\,dx}$

$$a_0=\frac{1}{L}\int_{-L}^{L} f(x)\,dx=\frac{1}{5}\int_{-5}^{5} f(x)\,dx=\frac{1}{5}\int_{0}^{5} 3\,dx=\frac{3(5)}{5}=3 \quad \alpha+2L$$

$$\boxed{a_0=3}$$

$$a_n=\frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx=\frac{1}{5}\int_{-5}^{5} f(x)\cos\left(\frac{n\pi x}{5}\right)dx$$

$$=\frac{1}{5}\int_{0}^{5} 3\cos\left(\frac{n\pi x}{5}\right)dx=\frac{3}{5}\left[\frac{\sin\frac{n\pi x}{5}}{\left(\frac{n\pi}{5}\right)}\right]_0^5=\frac{3}{5}\times\frac{5}{n\pi}\left[\sin n\pi-0\right]$$

$$=0.$$

$$b_n=\frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)=\frac{1}{5}\int_{-5}^{5} f(x)\sin\left(\frac{n\pi x}{5}\right)dx.$$

$$=\frac{1}{5}\int_{0}^{5} 3\sin\left(\frac{n\pi x}{5}\right)dx=\frac{3}{5}\left[\frac{-\left(\cos\frac{n\pi x}{5}\right)}{\left(\frac{n\pi}{5}\right)}\right]_0^5=\frac{-3}{5}\times\frac{5}{n\pi}\left[(-1)^n-1\right]$$

$$b_n=\frac{-3}{n\pi}\left[(-1)^n-1\right]$$

Fourier series $\Rightarrow \frac{3}{2}+\sum_{n=1}\frac{-3}{n\pi}\left[(-1)^n-1\right]$

(13) $f(x)=\begin{cases}-a, & -c<x<0 \\ 0, & a<x<c\end{cases}$ where $a>0$.

(14) $f(x)=4-x^2$ in $(-2,2)$

$f(-x)=4-(-x)^2=4-x^2=f(x)\ \forall x$

$f(-x)=f(x)\ \forall x \quad f(x+4)=f(x)\ \forall x.$

$f(x)$ is even periodic on $(-2,2)$ with period $T=2L=4$

$\Rightarrow L=2$

$\therefore f(x)$ has FCS.

Fourier Cosine Series:

$\frac{a_0}{2} + \sum a_n \cos\left(\frac{n\pi x}{L}\right) dx$ $\quad L=2$

$a_0 = \frac{2}{L}\int_0^L f(x)\,dx \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$

$= \frac{2}{2}\int_0^2 (4-x^2)\,dx = \left(4x - \frac{x^3}{3}\right)_0^2 = 8 - \frac{8}{3} = \frac{8\times 2}{3} = \frac{16}{3}$

$a_0 = \frac{16}{3}$

$a_n = \int_0^2 (4-x^2)\left(\cos\frac{n\pi x}{2}\right) = \left[(4-x^2)\left(\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) - 2x\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n^2\pi^2}{2^2}}\right) - 2\left(\frac{-\sin\frac{n\pi x}{2}}{\frac{n^3\pi^3}{8}}\right)\right]_0^2$

$= \left(\left[0 + 4\left(\frac{(-1)^n}{\frac{n^2\pi^2}{4}}\right) + 2(0)\right] - \left(4(0) - 2(0) - 2(0)\right]\right)$

$a_n = \frac{16}{n^2\pi^2}(-1)^n$

Fourier Cosine Series $= \frac{8}{3} + \sum \frac{16}{n^2\pi}(-1)^n$