06th November, 2019

UNIT 5: RESIDUE CALCULUS

* Calculus of Residues
* Residue theorem (without proof).
* Evaluation of Real integrals of type
  - a) $\int_{-\infty}^{\infty} f(x)\,dx$
  - b) $\int_{0}^{C+2\pi} f(\cos\theta, \sin\theta)\,d\theta$

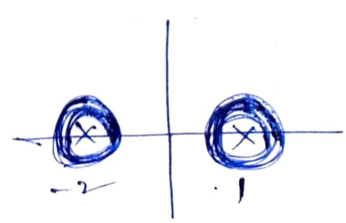
CLASSIFICATION OF SINGULARITIES : A singularity z_0 is

Isolated singularity:

A singularity z_0 is said to be an isolated singularity of $f(z)$ if there exists a neighbourhood of z_0 such that $f(z)$ is analytic at all points in that nbd of z_0 except at z_0.

Ex: $f(z) = \dfrac{1}{(z-1)(z+2)}$ has singularities at $z = 1$ & -2

Here, the singularities 1 & −2 are isolated singularities



NOTE: Any singularity which is not isolated is obviously is a non-isolated singularity.

$$f(z) = \frac{1}{\sin\left(\frac{\pi}{z}\right)}$$

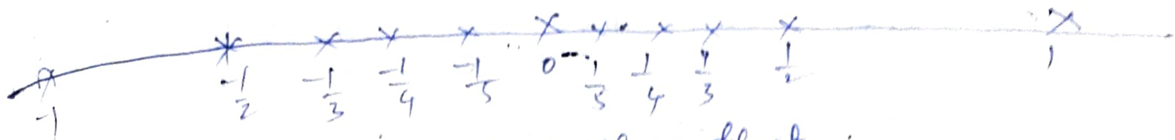
$$\sin\left(\frac{\pi}{z}\right) = 0$$

$$\frac{\pi}{z} = \pm n\pi$$

$$z = \pm\frac{1}{n} \qquad n = \pm 1, \pm 2, \ldots$$

$\therefore z = 1, -1, \frac{1}{2}, \frac{-1}{2}, \frac{1}{3}, \frac{-1}{3} \ldots$ are the singularities

Also $z=0$ is another singularity of $f(z)$



From the above diagram it is clear that

$z=\pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4} \ldots \pm\frac{1}{n}\ldots$ are all isolated singularities where $z=0$ is a non-isolated singularity since $\frac{1}{n}\to 0$ as $n\to\infty$. For every neighbourhood of zero, there will be an infinitely many singularities of $f(z)$

<u>Classification of Isolated Singularities:</u>

Let $f(z)$ be analytic in a region $R: 0<|z-z_0|<R$. and z_0 be an <u>isolated</u> <u>singularity</u> of <u>$f(z)$</u>

Let $f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$ be the Laurent's series of $f(z)$ in the region $0<|z-z_0|<R$ then z_0 can be classified as follows:

1) <u>Removable singularity</u> if $b_n=0 \ \forall n$.

Ex: $f(z) = \frac{\sin z}{z} = \frac{1}{z}\left[z - \cancel{\frac{z^3}{3!}} + \cancel{\frac{z^5}{5!}} - \ldots\right]$

Here $z=0$ is a isolated singularity of $f(z)$, then $= 1 - \frac{z^2}{3!} + \cancel{\frac{z^5}{5!}} \ldots$

Laurent's series of $f(z)$ is given by

$$\frac{\sin z}{z} = \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\right]$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots]$$

Since there is no principal path, [no -ve powers of z]

$z=0$ is a removable singularity.

(2) Pole of order 'm': if $b_n = 0$ for all $n > m$ and $b_m \neq 0$

~~If~~ Ex: $f(z) = \dfrac{\sin z}{z^5}$ has isolated singularity at $z=0$, therefore and the Laurent series of $f(z)$ about $z=0$ is given by

$$\frac{\sin z}{z^5} = \frac{1}{z^5}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} + \cdots\right]$$

$$= \left[\frac{1}{z^4} - \frac{1}{3!}\frac{1}{z^2}\right] + \left[\frac{1}{5!} - \frac{z^2}{7!} + \frac{z^4}{9!} + \cdots\right]$$

Here, $b_4 = 1 \neq 0$ and $b_n = 0 \ \forall n > 4$

$\therefore z=0$ is a pole of order 4

(3) Essential Singularity

If the principle path contains infinite number of terms.

Ex: $e^{1/z}$ has isolated singularity at $z=0$.

$$e^{1/z} = 1 + \left[\left(\frac{1}{z}\right) + \frac{\left(\frac{1}{z}\right)^2}{2!} + \frac{\left(\frac{1}{z}\right)^3}{3!} + \cdots\right]$$

Here principle part contains infinitely many -ve powers of z.

$\therefore z=0$ is an essential singularity.

- z_0
  - Regular point
  - Singular point
    - Isolated
      - Removable: $\lim_{z \to z_0} f(z)$ exists
      - Pole of order m
      - Essential
    - Non Isolated

~~RESIDUE THEOREM~~

Let z_0 be an isolated singularity of $f(z)$ then the coefficient of $\frac{1}{z-z_0}$ in the Laurent's series of $f(z)$ which is existing in the region $0<|z-z_0|<R$ is called residue of $f(z)$. It is denoted by $\underset{z=z_0}{\text{Res}} f(z) =$ coefficient of $\frac{1}{z-z_0}$
$= b_1$.

7/11/19

Note:

1) If z_0 is a removable singularity then $b_n=0$ for all n.
Therefore $b_1=0$ i.e., Residue at $z=z_0$ $f(z)=0$.

$$\underset{z=z_0}{\text{Res}} f(z)=0 \text{ if } z_0 \text{ is a removable singularity.}$$

2) Let $z=z_0$ be a pole of order 'm'

Case-1 $\boxed{m=1}$ z_0 is a simple pole

Then $f(z) = \frac{b_1}{z-z_0} + [a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots]$

$$(z-z_0) f(z) = b_1 + [a_0(z-z_0) + a_1(z-z_0)^2 + \dots]$$

$$\lim_{z\to z_0} [(z-z_0) f(z)] = \lim_{z\to z_0} b_1 + \lim_{z\to z_0} a_0(z-z_0) + \lim_{z\to z_0} a_1(z-z_0)^2 + \dots$$

$$= b_1$$

$$\therefore b_1 = \underset{z=z_0}{\text{Res}} f(z) = \lim_{z\to z_0} [(z-z_0) f(z)].$$

Case-2 Let z_0 be a pole of order 'm', $m>1$

Then

$$f(z) = \left[\frac{b_m}{(z-z_0)^m} + \frac{b_{m-1}}{(z-z_0)^{m-1}} + \dots + \frac{b_1}{z-z_0}\right].$$

$$+ [a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots]$$

$\Rightarrow (z-z_0)^m f(z) = [b_m + b_{m-1}(z-z_0) + b_{m-2}(z-z_0)^2 + \dots + b_1 (z-z_0)^{m-1}] + [a_0 (z-z_0)^m + a_1 (z-z_0)^{m+1} + \dots]$

diff both sides wrt z successively $(m-1)$ times and take limit $z \to z_0$ then

$$\lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z) \right] = b_1 (m-1)!$$

$$\Rightarrow \boxed{b_1 = \underset{z=z_0}{\text{Res}}\, f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z) \right]}$$

Note: For a simple pole z_0, and $f(z) = \frac{p(z)}{q(z)}$, $p(z_0) \neq 0$

$$b_1 = \underset{z=z_0}{\text{Res}}\, f(z)$$

$$= \lim_{z \to z_0} \left[(z-z_0) f(z) \right]$$

$$= \underset{z=z_0}{\text{Res}}\, \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

CAUCHY RESIDUE THEOREM:

Let $f(z)$ be analytic at all points on and inside a simple closed curve 'C' except, at the points $z_1, z_2, \dots z_n$ which are inside C

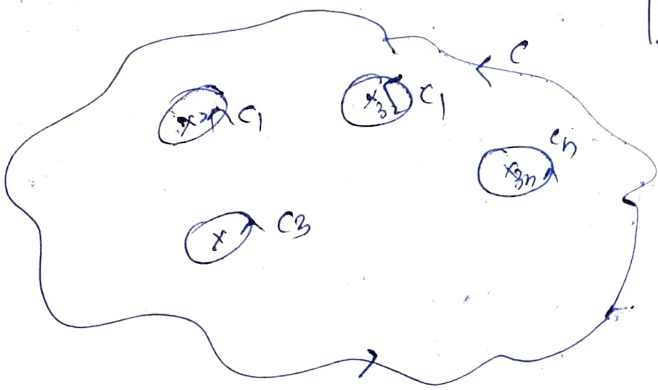
$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \dots + \oint_{C_n} f(z)\,dz$$

$$= 2\pi i R_1 + 2\pi i R_2 + \dots + 2\pi i R_n$$

$$= 2\pi i \, [R_1 + R_2 + \cdots + R_n]$$

$$= \cancel{2\pi i \, \{R_1\}} = 2\pi i \sum_{k=1}^{n} R_k,$$

$$\boxed{R_k = \operatorname*{Res}_{z=z_k} f(z)} \quad 1 \le k \le n$$



$$b_1 = C_{-1} = \frac{1}{2\pi i} \oint \frac{f(z)}{(z-z_0)^{-1+1}}\, dz$$

$$b_1 = \frac{1}{2\pi i} \oint_{C_1} f(z)\, dz.$$

$$\oint_C f(z)\, dz = 2\pi i \, [R_1 + R_2 + \cdots + R_n].$$

Q) Determine residues of the following functions at all its singularities.

1) $f(z) = \dfrac{\sin z}{z}$

Here, $z = 0$ is a removable singularity

$\therefore \operatorname*{Res}_{z=0} f(z) = 0.$

2) $f(z) = e^{\frac{1}{z-2}}$

$$= 1 + \left[\frac{1}{z-2} + \frac{1}{2!\,(z-2)^2} + \frac{1}{3!\,(z-2)^3} + \cdots\right].$$

$\therefore z = 2$ is an essential singularity.

$\therefore \operatorname*{Res}_{z=2} e^{\frac{1}{z-2}} = b_1 = \text{Coeff of } \dfrac{1}{z-2} = 1.$

Q3) $f(z) = \dfrac{z^2}{(z-1)^2(z+2)}$

Here $z = 1, -2$ are is isolated singularities

$z = 1$ is a pole of order 2 (double pole)

and $z = -2$ is a pole of order 1 (simple pole)

(1) $\underset{z=1}{\text{Res}}\, f(z) = \dfrac{1}{(2-1)!} \underset{z \to 1}{\text{Lt}} \dfrac{d^{2-1}}{dz^{2-1}} \left[(z-1)^2 \cdot \dfrac{z^2}{(z-1)^2(z+2)} \right]$

$= \underset{z \to 1}{\text{Lt}} \dfrac{d}{dz} \left[\dfrac{z^2}{z+2} \right]$

$= \underset{z \to 1}{\text{Lt}} \left[\dfrac{(z+2)(2z) - z^2(1)}{(z+2)^2} \right]$

$= 5/9 \to R_1$

(2) $\underset{z=-2}{\text{Res}}\, f(z)$

$= \underset{z \to -2}{\text{Lt}} [z-(-2)] f(z)]$

$= \underset{z \to -2}{\text{Lt}} (z+2) \dfrac{z^2}{(z-1)^2(z+2)}$

$= \underset{z \to -2}{\text{Lt}} \dfrac{z^2}{(z-1)^2} = \dfrac{(-2)^2}{(-2-1)^2} = \dfrac{4}{9} = R_2$

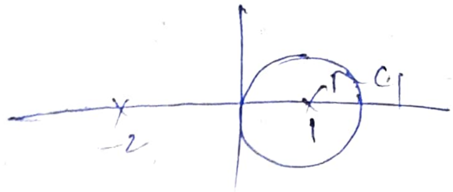
Q) Evaluate $\oint_c \dfrac{z^2}{(z-1)^2(z+2)} dz$

where c is (i) $|z-1| = 1$

(ii) $|z+2| = 1$

(iii) $|z| = 4$

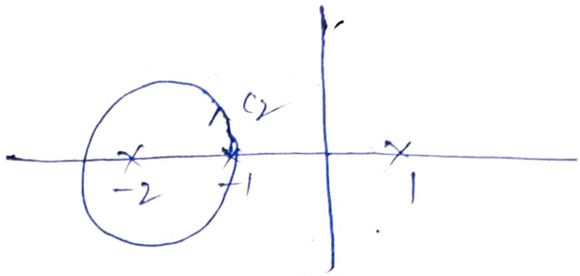
(i) $C_1 : |z-1| = 1$



$$\int_{C_1} f(z)\,dz = 2\pi i \left[\operatorname*{Res}_{z=1} f(z)\right]$$

$$= 2\pi i \left[\frac{5}{9}\right] = \frac{10\pi i}{9}$$

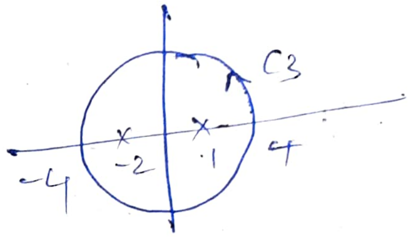
(ii) $C_2 : |z+2| = 1$



$$\int_{C_2} f(z)\,dz = 2\pi i \left[\operatorname*{Res}_{z=-2} f(z)\right]$$

$$= 2\pi i \left(\frac{4}{9}\right) = \frac{8\pi i}{9}$$

(iii) $C_3 : |z| = 4$



$$\int_{C_3} f(z)\,dz = 2\pi i\,(R_1 + R_2)$$

$$= 2\pi i$$

11/11/2019

Note:

If z_0 is

z_0

Removable | Pole of order m | Essential.

Removable: $\lim_{z \to z_0} f(z)$ exists

Pole of order m: $\lim_{z \to z_0} |f(z)| = \infty$

Essential: $\lim_{z \to z_0} f(z)$ does not exists

Ex: $\lim_{z \to 0} \frac{\sin z}{z} = 1$

$z=0$ is a removable singularity

Ex: $f(z) = \frac{z+2}{(z-1)^2} \to \infty$ as $z \to 1$

Here $z=1$ is a pole of order 2

Ex: $\lim_{z \to 0} e^{1/z}$ does not exist

Here $z=0$ is an essential singularity

Zero of order m:

Let z_0 be a regular point of $f(z)$ then z_0 is said to be a zero of order m if $f(z_0) = f^{(1)}(z_0) = f^{(2)}(z_0) = \cdots = f^{(m-1)}(z_0) = 0$ and $f^{(m)}(z_0) \neq 0$.

$f(z) = (z-2)^2$

$f(2) = 0 \Rightarrow z = 2$ is a zero of $f(z)$.

$f'(z) = 2(z-2) \Rightarrow f'(2) = 0$

$f''(2) = 2 \neq 0$

$\therefore z = 2$ is a zero of order $\underline{2}$.

Result:

z_0 is a ~~pole~~ zero of order m of $f(z)$ then z_0 is a pole of order m for $\frac{1}{f(z)}$

Ex: $f(z) = (z-2)^2$ has zero at $z = 2$ which is of order 2

$\frac{1}{f(z)} = \frac{1}{(z-2)^2}$ has pole at $z = 2$ which is of order 2

Zero / pole at $z = \infty$

The function $f(z)$ is said to have zero or singularity at $z = \infty$ if $f(\frac{1}{z})$ is having zero or singularity.

Ex: $f(z) = z^2$

$f(\frac{1}{z}) = \frac{1}{z^2}$

$z = 0$ is a pole of order 2 for $\frac{1}{z^2} = f(\frac{1}{z})$

$z = \infty$ is a pole of order 2 for $1/z^2 = f(z)$.

(2)** $f(z) = \frac{1}{z - z^3} = \frac{1}{z(1-z^2)} = \frac{1}{z(1+z)(1-z)}$

Here $z = 0, 1, -1$ are simple poles

$f(\frac{1}{z}) = \frac{1}{1/z - \frac{1}{z^3}} = \frac{z^3}{z^2 - 1}$

$z=0$ is a zero of $f(\frac{1}{z})$ of order 3
$z=\infty$ is a zero of $f(z)$ of order 3

① Determine and classify all the singularities of the following functions:

$$f(z)=\frac{z^4}{1+z^4}$$

Sol. Reason: ② Part ②.

Let $g(z)=\frac{1}{f(z)}=0$

$$g(z)=\frac{1+z^4}{z^4}=0$$

$$\sqrt[n]{z}=r^{1/n}e^{i\left(\theta+\frac{2k\pi}{n}\right)}$$

$k=0,1,2,\ldots n-1$

$$1+z^4=0$$

$$z^4=-1$$

$z=\sqrt[4]{-1}=z_1, z_2, z_3, z_n$ (say) $k=0,1,2,3$

$$y'(z)=\frac{z^4(4z^3)-(1+z^4)4z^3}{z^8}$$

$$=\frac{-4z^3}{z^8}=\frac{-4}{z^5}\neq 0.$$

$g'(z_k)\neq 0$ for $k=1,2,3,4$

$\therefore z_k$ is a zero of order 1 for $\frac{1}{f(z)}=g(z)$

$\Rightarrow z_k$ is a pole of order 1 for $f(z)$

Part I ①

$$f(z)=\frac{z^4}{1+z^4}$$

$$1+z^4=0$$

$\Rightarrow z^4=-1 \to z=\sqrt[n]{-1}$

$= 1^{1/4} \cdot e^{i\left(\frac{\pi+2k\pi}{4}\right)}, k=0,1,2,3$

are poles of order 1.

(2) $f(z) = \dfrac{z^5}{(z-2)^3(z^2+4)^3(z-5)^4}$

$z=2$ is a pole of order 3
$z=5$ " " " 4
$z=-2i$ " " " 3
$z=2i$ " " " 3

(3) $f(z) = \dfrac{e^z}{1+z^2}$

(a) $1+z^2=0 \Rightarrow z=-i, i$

$\dfrac{1}{f(z)} = g(z) = \dfrac{1+z^2}{e^z}$.

$g'(z) = \dfrac{e^z(2z)-(1+z^2)e^z}{e^z} = \dfrac{2ze^z - e^z - z^2e^z}{e^z}$

$\neq 0$

$g'(i) \neq 0 \Rightarrow z=i$ is a simple zero of $g(z)$
$\Rightarrow z=i$ is a simple pole of $f(z)$
$g'(i) \neq 0 \Rightarrow z=-i$ is a simple zero of $g(z)$
$\Rightarrow z=-i$ is a simple pole of $f(z)$

(b) $f\left(\frac{1}{z}\right) = \dfrac{e^{1/z}}{1+\frac{1}{z^2}} = \dfrac{z^2 e^{1/z}}{z^2+1}$

$= \dfrac{z^2}{z^2+1}\left[1+\frac{1}{z}+\frac{\left(\frac{1}{z}\right)^2}{2!}+\frac{\left(\frac{1}{z}\right)^3}{3!}+\cdots\right]$

$= z^2(1+z^2)^{-1}\left[1+\frac{1}{z}+\frac{1}{2!}\frac{1}{z^2}+\frac{1}{3!}\frac{1}{z^3}+\cdots\right]$

$= z^2 [1 - z^2 + (z^2)^2 - (z^2)^3 + \cdots] [1 + \frac{1}{z} + \frac{1}{2!} \cdot \frac{1}{z^2} + \cdots]$

$= [z^2 - z^4 + z^6 - z^8 + \cdots] [1 + \frac{1}{z} + \frac{1}{2!} \frac{1}{z^2} + \cdots]$

The above series contains infinitely many negative powers of z.

$\Rightarrow z = 0$ is an essential singularity of $f(\frac{1}{z})$

$z = \infty$ is an essential singularity of $f(z)$

(4) $f(z) = \frac{1 - e^{2z}}{z^4}$

Here $z = 0$ is the singularity of $f(z)$

$$f(z) = \frac{1}{z^4} [1 - e^{2z}]$$

$$= \frac{1}{z^4} [1 - (1 + \frac{(2z)^1}{1!} + \frac{(2z)^2}{2!} + \frac{(2z)^3}{3!} + \frac{(2z)^4}{4!} + \frac{(2z)^5}{5!} + \frac{(2z)^6}{6!} + \cdots]$$

$$= -2 \cdot \frac{1}{z^3} - \frac{2^2}{2!} \frac{1}{z^2} - \frac{2^3}{3!} \frac{1}{z} - [\frac{2^4}{4!} + \frac{2^5}{5!} z + \cdots]$$

$\Rightarrow z = 0$ is a pole of order 3

(5) $f(z) = \frac{1 - \cos z}{z}$

Here $z = 0$ is the singularity of $f(z)$

$$f(z) = \frac{1}{z} [1 - \cos z]$$

$$= \frac{1}{z} [1 - [1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \frac{z^8}{8!} - \cdots]]$$

$$= + \frac{z}{2!} - \frac{z^3}{4!} + \frac{z^5}{6!}$$

$\Rightarrow z = 0$ is a removable singularity

($\because$ no negative powers, no principle parts)

(6) $f(z) = e^{\frac{z}{z-2}}$

Here $z=2$ is a singularity ($\because f(2)$ is not defined)

$$f(z) = e^{\frac{(z-2)+2}{z-2}} = e^{1+\frac{2}{z-2}} = e^{1} \cdot e^{\frac{2}{z-2}}$$

$$= e^{1}\left[1 + \frac{2}{z-2} + \frac{1}{2!}\left(\frac{2}{z-2}\right)^2 + \frac{1}{3!}\left(\frac{2}{z-2}\right)^3 + \cdots\right]$$

$$= e^{1} + \frac{2e}{z-2} + \frac{2^2 \cdot e}{2!} \times \frac{1}{(z-2)^2} + \cdots]$$

$\Rightarrow z=2$ is an essential singularity.

Recall:

Residue formulae

a) z_0 is a removable singularity:

$$\operatorname*{Res}_{z-z_0} f(z) = 0$$

b) z_0 is a pole of order m:

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z\to z_0} \left[\frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z)\right]\right]$$

Special case 1: $m=1$ (z_0 is a simple pole)

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0} (z-z_0) f(z)$$

Case-2: z_0 is a simple pole of $f(z) = \frac{p(z)}{q(z)}$,

$p(z_0) \neq 0$

$$\operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{z=z_0} \frac{p(z)}{Q(z)} = \frac{p(z_0)}{Q'(z_0)}$$

(ii) z_0 is an essential singularity:

$\operatorname{Res}_{z=z_0} f(z) = b_1$ = Coeff of $\frac{1}{z-z_0}$ is Laurents series of $f(z)$ in $0<|z-z_0|<R$

(1) $f(z) = \frac{e^{2z}}{(z-1)^3}$

$= \frac{e^{2[(z-1)+1]}}{(z-1)^3}$

$= \frac{e^2}{(z-1)^3}\left[e^{2(z-1)}\right]$

$= \frac{e^2}{(z-1)^3}\left[1 + \frac{2(z-1)}{1!} + \frac{2^2(z-1)^2}{2!} + \frac{2^3(z-1)^3}{3!} + \frac{2^4(z-1)^4}{4!} + \frac{2^5(z-1)^5}{5!} + \cdots\right]$

$= \left[\frac{e^2}{(z-1)^3} + \frac{2e^2}{1!}\left(\frac{1}{z-1}\right)^2 + \frac{2^2e^2}{2!}\frac{1}{(z-1)}\right] + \left[\frac{2^4e^2}{4!}(z-1) + \frac{2^5e^2}{5!}(z-1)^2 + \cdots\right]$

Here $z=1$ is a pole of order 3

$\operatorname{Res}_{z=1} f(z) = b_1 = \text{Coeff}\ \frac{1}{z-1} = \frac{2^2e^2}{2!} = \frac{2\times 2\times e^2}{2} = 2e^2$

(2) $f(z) = \frac{z-\sin z}{z^3}$

Sol Here $z=0$ is an isolated singularity of $f(z)$

$f(z) = \frac{1}{z^3}(z-\sin z)$

$= \frac{1}{z^3}\left[z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right)\right]$

$= \left[\frac{1}{3!} - \frac{1}{5!}z^2 + \frac{1}{7!}z^4 - \cdots\right]$

$z=0$ is a removable singularity

$\therefore \operatorname{Res}_{z=0} f(z)=0.$

(3) $f(z)=(z-3)\sin\left(\frac{1}{z+2}\right)$

Here $z=-2$ is an isolated singularity of $f(z)$

$f(z)=[(z+2)-5]\sin\left(\frac{1}{z+2}\right)$

$=[(z+2)-5]\left[\left(\frac{1}{z+2}\right)-\frac{\left(\frac{1}{z+2}\right)^3}{3!}+\frac{\left(\frac{1}{z+2}\right)^5}{5!}-\cdots\right]$

contains infinite powers of $\frac{1}{z+2}$

$\Rightarrow z=-2$ is as essential singularity

$\therefore \operatorname{Res}_{z=-2} f(z)=b_1=$ coeff of $\frac{1}{z+2}=-5$

(4) $f(z)=\frac{1}{\cos z-\sin z}$

$\cos z-\sin z=0 \Rightarrow \cos z=\sin z$

$\Rightarrow z=\frac{\pi}{4}+2n\pi,\ n\in Z$ is an isolated singularity of $f(z)$

$\frac{1}{f(z)}=g(z)=\cos z-\sin z$

$g'(z)=-\sin z-\cos z$

$g'\left(\frac{\pi}{4}+2n\pi\right)\neq 0$

$\Rightarrow z=\frac{\pi}{4}+2n\pi$ are simple zeroes of $g(z)=\frac{1}{f(z)}$

$\Rightarrow z=\frac{\pi}{4}+2n\pi$ are simple poles of $f(z)$

$\operatorname{Res}_{z=\pi/4} f(z)=\lim_{z\to\frac{\pi}{4}}\left(z-\frac{\pi}{4}\right)f(z)=\lim_{z\to\frac{\pi}{4}}\frac{(z-\pi/4)}{(\cos z-\sin z)}$

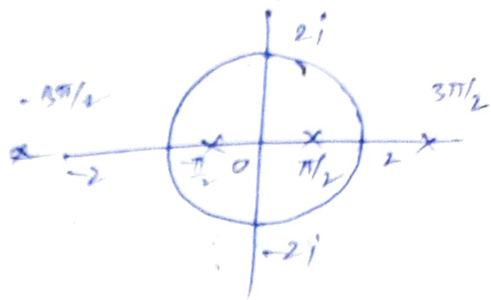
$\frac{0}{0}$ form $\Rightarrow$ L-Hospital's rule

$$\lim_{z \to \pi/4} \frac{1}{-\sin z - \cos z} = \frac{1}{-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}} = \frac{\sqrt{2}}{-2}$$

$$= -\frac{1}{\sqrt{2}}$$

Use Residue theorem to evaluate the following integrals

1) $\int_C \tan z \, dz$ where $c : |z| = 2$



Sol Let $f(z) = \tan z = \frac{\sin z}{\cos z}$

$\cos z = 0$
$\Rightarrow z = (2n+1)\frac{\pi}{2}$, $n \in \mathbb{Z}$

$\frac{1}{f(z)} = g(z) = \frac{1}{\tan z} = \cot z$.

$g'(z) = -\operatorname{cosec}^2 z \Rightarrow g'((2n+1)\pi/2) \neq 0$.

$\therefore z = (2n+1)\frac{\pi}{2}$ are simple zeros of $g(z) = \frac{1}{f(z)}$

$\Rightarrow z = (2n+1)\pi/2$ are simple poles of $f(z)$

But only $z = -\frac{\pi}{2}, \frac{\pi}{2}$ are inside c

$\therefore$ By residue theorem,

$$\int_C f(z)\, dz = 2\pi i [R_1 + R_2]$$

$$R_1 = \operatorname*{Res}_{z=-\pi/2} f(z) = \operatorname*{Res}_{z=-\frac{\pi}{2}} \frac{\sin z}{\cos z} = \frac{P(z_0)}{Q'(z_0)} = \frac{\sin(-\pi/2)}{-\sin(-\frac{\pi}{2})}$$

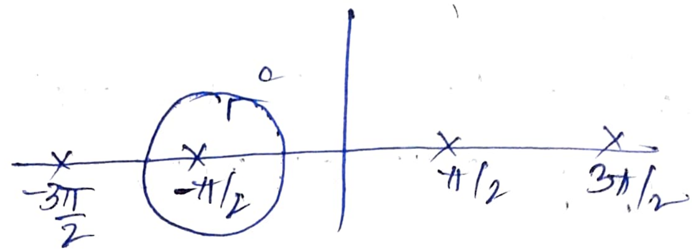
$$= -1$$

$$R_2 = \operatorname*{Res}_{z=\frac{\pi}{2}} f(z) = \operatorname*{Res}_{z=\pi/2} \frac{\sin z}{\cos z} = \frac{\sin\frac{\pi}{2}}{-\sin\pi/2} = -1$$

$$\Rightarrow \int_C f(z)\, dz = 2\pi i(-1-1) = -4\pi i$$

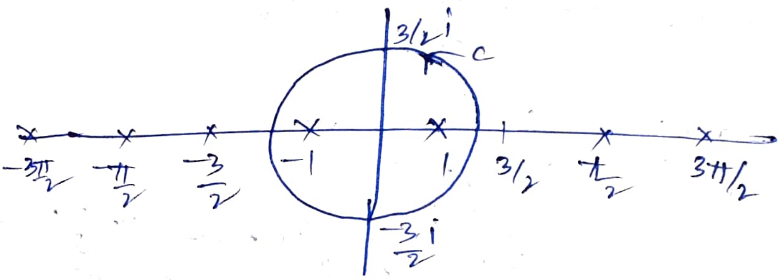
Note: $\oint\limits_{|z|=2} \tan z\, dz = 2\pi i [R_1 + R_2] = -4\pi i$

$\oint\limits_{|z+\frac{\pi}{2}|=1} \tan z\, dz = 2\pi i [R_1] = 2\pi i(-1) = -2\pi i$



$\oint\limits_{|z|=1} \tan z = 0$ [By CIT]

Q2) $\oint\limits_{C} \frac{\tan z}{z^2-1}\, dz$ where $C: |z| = \frac{3}{2}$



Let $f(z) = \frac{\tan z}{z^2-1}$

$= \frac{\sin z}{\cos z (z-1)(z+1)}$

has singularities at $z = -1, 1, (2n+1)\frac{\pi}{2}$ where $n \in \mathbb{Z}$

which are simple poles (of order 1)

But only $z = -1, 1$ are inside C; $|z| = \frac{3}{2}$

By residue theorem,

$\oint\limits_{C} \frac{\tan z}{z^2-1}\, dz = 2\pi i [R_1 + R_2]$

$R_1 = \operatorname{Res}_{z=-1} f(z) = \lim_{z\to -1} (z-(-1)) f(z) = \lim_{z\to -1} (z+1) \frac{\tan z}{(z-1)(z+1)}$

$= \frac{\tan(-1)}{-1-1} = \frac{\tan 1}{2}$

$R_2 = \operatorname{Res}_{z=1} f(z) = \lim_{z\to 1} (z-1) f(z) = \lim_{z\to 1} \frac{(z-1)\tan z}{(z-1)(z+1)}$

$= \frac{\tan(1)}{1+1} = \frac{\tan 1}{2}$

$\therefore \int_C \frac{\tan z}{z^2-1}\,dz = 2\pi i\left[\frac{\tan 1}{2} + \frac{\tan 1}{2}\right] = 2\pi \tan 1$ //.

13/11/2019

Q. Evaluate $\int_C \frac{z-3}{z^2+2z+5}\,dz$ where C is (i) $|z|=1$
(ii) $|z+1-i|=2$
(iii) $|z+1+i|=2$
(iv) $|z|=5$

Let $f(z) = \frac{z-3}{z^2+2z+5}$

$z^2+2z+5=0$

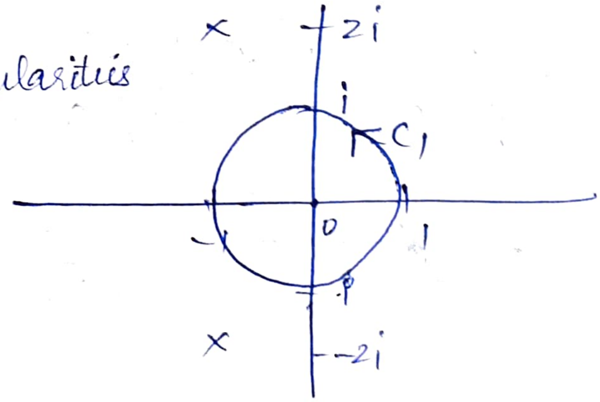
$z = \frac{-2\pm\sqrt{4-4\times5}}{2} = \frac{-2\pm4i}{2} = \frac{-2+4i}{2}, \frac{-2-4i}{2}$

$z = -1\pm2i$

$\therefore$ f(z) has singularities at $\alpha = -1-2i$, $\beta = -1+2i$

$\therefore f(z) = \frac{z-3}{(z-\alpha)(z-\beta)}$

$\Rightarrow z=\alpha,\beta$ are ~~poles~~ simple poles (poles of order - 1)

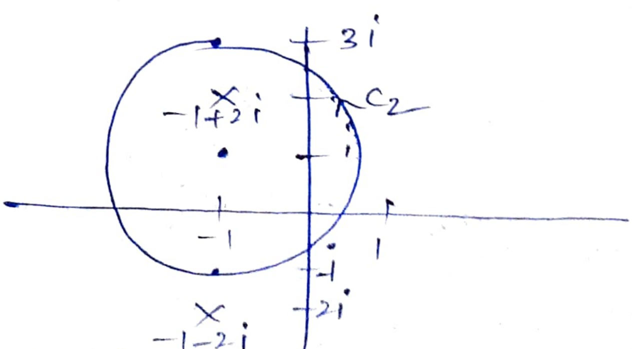


(i) Clearly, there are no singularities inside or on C_1

$\oint_{C_1} f(z)\,dz = 0$ (by Cauchy's integral theorem)

(ii) $|z+1-i|=2$

$z_0 = -1+i$



From the figure,

$\oint_{C_2} f(z)\,dz = 2\pi i\left[\operatorname{Res}_{z=\beta} f(z)\right]$ (By Cauchy's Residue theorem)

$= 2\pi i$

$\operatorname{Res}_{z=\beta} f(z) = \lim_{z\to\beta}(z-\beta)f(z) = \lim_{z\to\beta}(z-\beta)\frac{(z-3)}{(z-\alpha)(z-\beta)}$

$$= \frac{\beta-3}{\beta-\alpha} = \frac{-1+2i-3}{-1+2i+1+2i} = \frac{-4+2i}{4i}$$

$$= \frac{-4i-2}{-4} = i+\frac{1}{2}$$

$$\oint_{C_2} f(z)dz = 2\pi i\left(i+\frac{1}{2}\right)$$

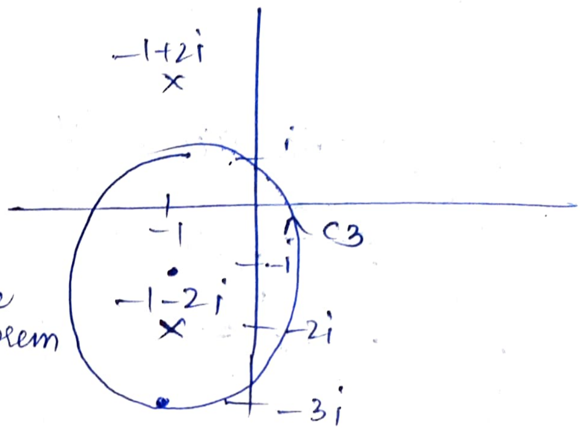
$$= 2\pi\left(-1+\frac{i}{2}\right)$$

$$= \pi(-2+i)$$

$$\oint_{C_2} f(z)dz = \pi(i-2)$$

(iii) $|z+1+i|=2$

From the figure,



$$\oint_{C_3} f(z)dz = 2\pi i\left[\underset{z=\alpha}{\text{Res}}\, f(z)\right]$$

By residue theorem

$$\underset{z=\alpha}{\text{Res}}\, f(z) = \lim_{z\to\alpha} \frac{\cancel{(z-\alpha)}(z-3)}{\cancel{(z-\alpha)}(z-\beta)}$$

$$= \frac{\alpha-3}{\alpha-\beta} = \frac{-1-2i-3}{-1-2i+1-2i} = \frac{-4-2i}{-4i}$$

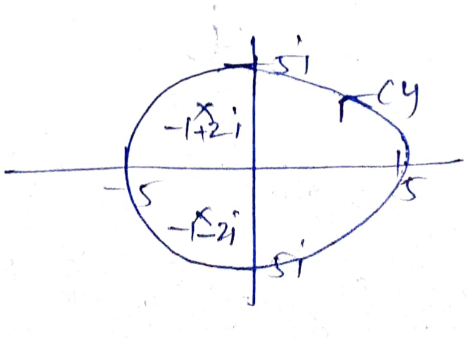
$$= \frac{4+2i}{4i} = \frac{2+i}{2i}$$

$$\oint_{C_3} f(z)dz = 2\pi i\left[\frac{2+i}{2i}\right]$$

$$\oint_{C_3} f(z)dz = \pi(2+i)$$

(iv) $|z|=5$

Clearly, both the singularities lie inside C_4



From the figure, we have

$$\oint_{C_4} f(z)\,dz = 2\pi i\left[\operatorname*{Res}_{z=\alpha} f(z) + \operatorname*{Res}_{z=\beta} f(z)\right]$$

$$= 2\pi i\left[\lim_{z\to\alpha}\frac{\cancel{(z-\alpha)}(z-3)}{\cancel{(z-\alpha)}(z-\beta)} + \lim_{z\to\beta}\frac{\cancel{(z-\beta)}(z-3)}{(z-\alpha)\cancel{(z-\beta)}}\right]$$

$$= 2\pi i\left[\frac{\alpha-3}{\alpha-\beta} + \frac{\beta-3}{\beta-\alpha}\right]$$

$$= 2\pi i\left[\frac{2+i}{2i} + \frac{-2i-1}{2i}\right] = 2\pi i\left[\frac{-4+2i+4+2i}{4i}\right]$$

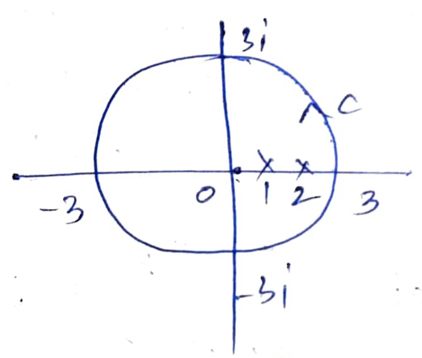
$$= \frac{2\pi i}{2i}\left[2+i-2i-1\right] \qquad = \frac{\pi}{2}\left[4i\right]$$

$$= 2\pi i$$

$$= \pi\,[$$

Q) Evaluate $\int_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)^2(z-2)}\,dz$ where $C: |z|=3$

Let $f(z) = \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)^2(z-2)}$



$z=1$ is a pole of order 2

$z=2$ is a pole of order 1

Both $z=1$ and $z=2$ are inside C

$\therefore$ By Residue theorem, $\oint f(z)\,dz = 2\pi i\,[R_1+R_2]$

where

$$R_1 = \operatorname*{Res}_{z=1} f(z) = \frac{1}{(2-1)!}\lim_{z\to1}\frac{d^{2-1}}{dz^{2-1}}\left[(z-1)^2 f(z)\right]$$

$$= \lim_{z\to1}\frac{d}{dz}\left[\cancel{(z-1)^2}\times\frac{\sin\pi z^2+\cos\pi z^2}{\cancel{(z-1)^2}(z-2)}\right]$$

$$= \lim_{z\to1}\frac{d}{dz}\left[\frac{\sin\pi z^2+\cos\pi z^2}{(z-2)}\right]$$

$$\lim_{z\to1}\left[\frac{(z-2)\left[2\pi z\cos\pi z^2 - 2\pi z\sin\pi z^2\right] - (\sin\pi z^2+\cos\pi z^2)(1)}{(z-2)^2}\right]$$

$$= (-1)\frac{[2\pi(-1) - (0+(-1))]}{(-1)^2}$$

$$= \cancel{1(= 2\pi} \ \frac{2\pi+1}{1} = 2\pi+1 \ //$$

$$R_2 = \underset{z=2}{\text{Res}}\, f(z)$$

$$= \cancel{\frac{1}{(2-2)!} \ \underset{z\to 2}{lt} \ d^{2-}} = \underset{z\to 2}{lt} \ (z-2)f(z)$$

$$= \underset{z\to 2}{lt} \ \cancel{(z-2)} \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2 \cancel{(z-2)}}$$

$$= \frac{\sin 4\pi + \cos 4\pi}{(2-1)^2} = \frac{1}{1} = 1 \ //$$

$$\oint_c f(z)\,dz = 2\pi i(R_1 + R_2)$$

$$= 2\pi i(2\pi+1+1)$$

$$= 2\pi i(2\pi+2)$$

$$= 4\pi i(\pi+1) \ //$$

H.W

1) Evaluate $\int_c \left[\frac{z e^{\pi z}}{z^4-16} + z e^{\frac{\pi}{z}}\right] dz$ where c is the ellipse

$9x^2+y^2=9$

2) $\int_c \frac{4-3z}{z(z-1)(z-2)}\,dz \quad c: |z|=3$

EVALUATION OF REAL INTEGRALS USING RESIDUE THEOREM

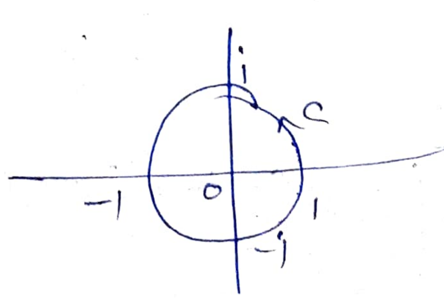
Type-1:

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)\,d\theta$$

Let $z = e^{i\theta}$

$dz = i e^{i\theta}\,d\theta$

$d\theta = \frac{1}{iz}\,dz.$



$$= \int_{|z|=1} F\left(\frac{z+\frac{1}{z}}{2}, \frac{z-\frac{1}{z}}{2i}\right)\frac{1}{iz}\,dz.$$

$$= 2\pi i\,[R_1 + R_2 + \cdots + R_k] \in \mathbb{R},$$

where $R_i = \operatorname{Res}_{z=z_i} f(z)$, $1 \le i \le k$.

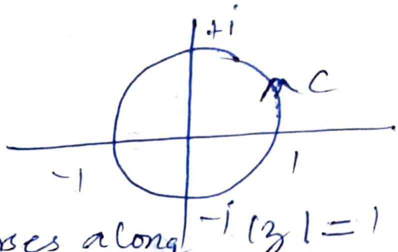
Here, $z_i \in |z| = 1$

Show that $\int_0^{2\pi} \frac{1}{a+b\sin\theta}\,d\theta = \frac{2\pi}{\sqrt{a^2-b^2}}$ where $a>b>0$

Let $z = e^{i\theta}$

$dz = i e^{i\theta}\,d\theta$

$d\theta = \frac{1}{iz}\,dz.$



As θ varies from 0 to 2π, z traverses along $|z|=1$ in counter clockwise sense.

$$\therefore I = \int_0^{2\pi} \frac{1}{a+b\sin\theta}\,d\theta$$

$$= \int_{|z|=1} \frac{1}{a+b\left(\frac{z-\frac{1}{z}}{2i}\right)}\,\frac{1}{iz}\,dz.$$

$$= \int_{|z|=1} \frac{2i}{2ai+b\left(z-\frac{1}{z}\right)} \times \frac{1}{iz}\,dz.$$

$$= \int_{|z|=1} \frac{2}{2ai+b\left(z-\frac{1}{z}\right)} \times \frac{1}{z}\,dz.$$

$$= \int_{|z|=1} \frac{2z}{2azi+b(z^2-1)} \times \frac{1}{z}\,dz$$

$$= \int_{|z|=1} \frac{2\,dz}{2azi+b(z^2-1)}\,dz = 2\oint_{|z|=1} \frac{dz}{2azi+b(z^2-1)}\,dz.$$

$$= 2\oint_{|z|=1} \frac{dz}{bz^2+2aiz-b}\,dz$$

Consider $bz^2 + 2iaz - b = 0$

$$\frac{-2ai \pm \sqrt{(2ia)^2 + 4 \times b \times b}}{2b}$$

$$\frac{-2ai \pm \sqrt{-4a^2 + 4b^2}}{2b}$$

$$\frac{-2ai \pm 2\sqrt{(b^2 - a^2)}}{2b}$$

$$z = \frac{-ai \pm \sqrt{b^2 - a^2}}{b}$$

$$\alpha = \frac{-ai - \sqrt{b^2 - a^2}}{b}, \quad \beta = \frac{-ai + \sqrt{b^2 - a^2}}{b}.$$

$$f(z) = 2 \oint_{|z|=1} \frac{1}{(z-\alpha)(z-\beta)}$$

$0 < b < a$

$\therefore a - b > 0 > -b$

$a > b > 0$

$\frac{a}{b} > 1$

$\frac{a}{b} i + \sqrt{b^2 - a^2}$

$\sqrt{(b^2 - a^2) + \frac{a^2}{b^2}}$

$a > b$

$\frac{a}{b} > 1$

$$\alpha = \left(-\frac{a}{b}\right) i - \sqrt{1 - \left(\frac{a}{b}\right)^2}$$

$$\cancel{|\alpha| = \sqrt{\left(\frac{a}{b}\right)^2 + 1 - \left(\frac{a}{b}\right)^2}}$$

$$|\alpha| = \left(-\frac{a}{b}\right) i - \frac{i\sqrt{a^2 - b^2}}{b}$$

$$= \left(\frac{-a - \sqrt{a^2 - b^2}}{b}\right) i$$

$$\alpha = \left[\left(-\frac{a}{b}\right) - \sqrt{\left(\frac{a}{b}\right)^2 - 1}\right] i$$

$$|\alpha| = \left| -\frac{a}{b} - \sqrt{\left(\frac{a}{b}\right)^2 - 1} \right|$$

$$|\alpha| = \frac{a}{b} + \sqrt{\left(\frac{a}{b}\right)^2 - 1}$$

$a > b$

$\frac{a}{b} > 1$

$|\alpha| > 1$

α lies outside as $|\alpha| > 1$

$\beta = \dfrac{-ai + \sqrt{b^2-a^2}}{b}$

$\beta = \left(\dfrac{-a}{b}\right) i + i \dfrac{\sqrt{a^2-b^2}}{b}$

$= \left[\left(\dfrac{-a}{b}\right) i + i\sqrt{\left(\dfrac{a}{b}\right)^2 - 1}\right] i$

$= \left[\dfrac{-a}{b} + \sqrt{\left(\dfrac{a}{b}\right)^2 - 1}\right] i$

$|\beta| = -\dfrac{a}{b} + \sqrt{\left(\dfrac{a}{b}\right)^2 - 1}$

$= -(>1) + \ldots$ cannot say

$\alpha\beta = \dfrac{-b}{b} = -1$

$|\alpha\beta| = 1$

$|\beta| = \dfrac{1}{|\alpha|} < 1 \quad (\because |\alpha| > 1)$

$\Rightarrow z = \beta$ is inside $|z| = 1$

By Residue theorem

$$2\oint_C \frac{1}{bz^2 + 2iaz - b}\,dz = 2\oint_C \frac{1}{b(z-\alpha)(z-\beta)}\,dz$$

$$= \frac{2}{b}\oint_C \frac{1}{(z-\alpha)(z-\beta)}\,dz$$

$$= \frac{2}{b}\left[2\pi i \operatorname*{Res}_{z=\beta} \frac{1}{(z-\alpha)(z-\beta)}\right]$$

$$= \frac{2}{b} \times 2\pi i \times \lim_{z\to\beta} \cancel{(z-\beta)} \times \frac{1}{(z-\alpha)\cancel{(z-\beta)}}$$

$$= \frac{2}{b} \times 2\pi i \times \frac{1}{\beta - \alpha}$$

$$= \frac{4\pi i}{b}\left[\frac{1}{\frac{2\sqrt{b^2-a^2}}{b}}\right] = \frac{4\pi i}{b} \times \frac{b}{2\sqrt{b^2-a^2}}$$

$$\cancel{= \frac{8\pi i}{b^2}\sqrt{b^2-a^2}}$$

$$\cancel{= 8\pi i\sqrt{1-\frac{a^2}{b^2}}}$$

$$= \frac{4\pi i}{2\sqrt{b^2-a^2}}$$

$$= \frac{2\pi i}{\sqrt{b^2-a^2}}$$

$$= \frac{2\pi i}{i\sqrt{a^2-b^2}}$$

$$\therefore \int_0^{2\pi} \frac{1}{a+b\sin\theta}\,d\theta = \frac{2\pi}{\sqrt{a^2-b^2}} \qquad = \frac{2\pi}{\sqrt{a^2-b^2}}$$

Hence, proved!

H.W

1) Check if the previous result is also valid if ~~$a < |b|$~~ $a > |b|$

2) Evaluate $\int_0^{2\pi} \frac{1}{a+b\cos\theta}\, d\theta$ where $a > b > 0$

3) $\int_0^{\pi} \frac{1}{5+3\cos\theta}\, d\theta = \frac{1}{2}\int_0^{2\pi} \frac{1}{5+3\cos\theta}\, d\theta = \frac{1}{2} \times \frac{2\times\pi}{\sqrt{25-9}} = \frac{\pi}{4}$

By Residue theorem

$$\int_0^{2\pi} \frac{1}{a+b\left(\frac{z+\frac{1}{z}}{2}\right)} \frac{dz}{iz}$$

$$\int_{|z|=1} \frac{2z}{2az+b(z^2+1)} \frac{dz}{iz}$$

$$\frac{2}{i}\int_{|z|=1} \frac{1}{bz^2+2az+b}\, dz$$

$$\frac{-2a \pm \sqrt{(2a)^2-4b^2}}{2b}$$

$$\frac{-a \pm \sqrt{a^2-b^2}}{b}$$

$$\alpha = \frac{-a+\sqrt{a^2-b^2}}{b}$$

$$= \frac{-a}{b} + \sqrt{\left(\frac{a}{b}\right)^2-1}$$

$$\beta = \frac{-a}{b} - \sqrt{\left(\frac{a}{b}\right)^2-1}$$

$|\beta| > 1$ $\qquad \frac{a}{b} > 1$

$|\alpha| < 1$

$\alpha\beta = 1$

$\alpha = \frac{1}{\beta}$

$$\frac{2}{i}\oint_C \frac{1}{b(z-\alpha)(z-\beta)}$$

$$= \frac{2}{bi}\left[2\pi i \operatorname*{Res}_{z=\alpha} \frac{1}{(z-\alpha)(z-\beta)}\right]$$

$$= \frac{4\pi}{b}\left[\lim_{z\to\alpha} \frac{(z-\alpha)}{(z-\alpha)(z-\beta)}\right]$$

$$= \frac{4\pi}{b} \times \frac{1}{(\alpha-\beta)}$$

$$= \frac{4\pi}{b} \times \frac{1}{2\sqrt{\left(\frac{a}{b}\right)^2-1}}$$

$$= \frac{4\pi}{b} \times \frac{b}{2\sqrt{a^2-b^2}}$$

$$= \frac{2\pi}{\sqrt{a^2-b^2}} \quad (|a| > |b|)$$

$$\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta}$$

singularity $\to \frac{-a+\sqrt{a^2-b^2}\, i}{b}$

Res $\frac{1}{\sqrt{a^2-b^2}}\, i$