21st October 2019

UNIT - 4

Part A:

COMPLEX (CANTOUR) INTEGRATION

- Simply and Multiply connected Domains (Definitions)
- Cauchy's integral theorem
- Cauchy's integral formula
- Cauchy's generalized Integral Formula

POWER SERIES

- Taylor's theorem
- Laurent's theorem (without proofs),
- Classification of singular points.

Let $w = f(z)$ be a complex function defined on a region Ω. Let c be an oriented simple piecewise smooth curve (open/closed) in the region Ω then integration of $f(z)$ along the curve c is called complex integration.

It is denoted by $\int_c f(z)\,dz$

Note: (1) If c is open which is oriented from one end point z_1 to another end point z_2 of the curve c then

$$\int_c f(z)\,dz = \int_{z_1}^{z_2} f(z)\,dz$$

(2) If c is simple closed, $\int_c f(z)\,dz = \oint_c f(z)\,dz$ [+ve orientation by default]

[cantour integral]

Complex integration over a simple closed curve can also be called as cantour integration.

Evaluation of Complex Integrals:

Consider $\int_c f(z)\,dz$

Let $f(z) = u(x,y) + i\,v(x,y)$

$z = x + iy$

$\Rightarrow dz = dx + i\,dy$

then $\int_C f(z)\,dz$

$$= \int_C (u+iv)(dx+idy)$$

$$= \int_C [udx - vdy] + i[udy + vdx]$$

$$= \int_C [udx - vdy] + i\int_C [vdx + udy]$$

Problems

(1) Evaluate $\int_C \bar{z}\,dz$ along (i) a straight line from $z=0$ to $z=2+4i$
(ii) a parabola $y=x^2$ from $z=0$ to $z=2+4i$

consider $\int \bar{z}\,dz$

$z = x+iy$ $\qquad dz = dx + idy$

$\bar{z} = x - iy$

$$\int_C (x-iy)\,dz$$

(i) $\int_{C_1} (x-iy)(dx+idy)$

$$= \int_{C_1} (xdx + ydy) + \int_{C_1} i(xdy - ydx)$$

Along C_1:

$z=0$ to $z=2+4i$ $\qquad m = \frac{y_2-y_1}{x_2-x_1} = \frac{4}{2} = 2$

$y = mx + c$

$dy = mdx = 2dx.$ $\qquad y = 2x.$

x varies from 0 to 2

$$\therefore \int_{C_1} (xdx + ydy) + i\int_{C_1} (xdy - ydx)$$

$$= \int_{C_1} xdx + 2x(2dx) + i\int_{C_1} x(2dx) - 2xdx$$

$$= \int_{C_1} xdx + 4xdx +$$

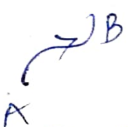
$$= \int_{C_1} 5xdx = 5\left(\frac{x^2}{2}\right)_0^2 = \frac{5}{2}(4-0) = \frac{5}{2}\times 4^2 = \underline{\underline{10}}$$

Recap

$\int_C \bar{F}\cdot d\bar{r}$

$\bar{F}$ is conservative $[\nabla \times \bar{F} = \bar{0}]$

- C is open



$$\int_C \bar{F}\cdot d\bar{r} = \int_A^B \bar{F}\cdot d\bar{r}$$
$$= \phi(B) - \phi(A)$$
where $F = \nabla\phi$

- C is closed

$$\oint \bar{F}\cdot d\bar{r} = 0$$

F is non-conservative $[\nabla \times \bar{F} \neq \bar{0}]$

- C is space curve

$$C: \bar{r} = x(t)\hat{i} + y(t)\hat{j} + z\hat{k}$$
$$t_1 \le t \le t_2$$
$$\int_C \bar{F}\cdot d\bar{r} = \int_{t_1}^{t_2} \left(\bar{F}\cdot \frac{d\bar{r}}{dt}\right) dt$$

- C is a plane curve
  - C closed

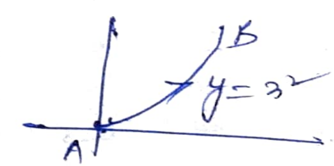
$$\oint \bar{F}\cdot d\bar{r} = \oint F_1 dx + F_2 dy$$
$$= \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA$$

↓ Green's theorem

  - C open

$$\int_C \bar{F}\cdot d\bar{r}$$
$$= \int_C F_1 dx + F_2 dy$$

Eliminate one variable



consider $\int_{C_2} \bar{z}\, dz$

$= \int_{C_2} (x-iy)(dx+idy)$

$= \int_{C_2} (xdx+ydy) + i(xdy-ydx)$

$= \int_{C_2} xdx+ydy + i\int_{C_2} xdy-ydx$

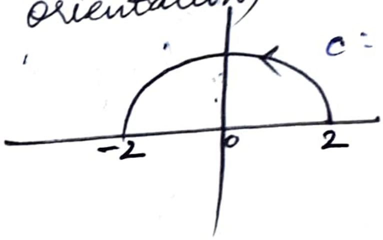
Along C_2: $y=x^2 \Rightarrow dy = 2xdx$

x varies from 0 to 2

$\therefore \int_{C_2} \bar{z}\, dz = \int_0^2 xdx + x^2(2xdx) + i\int_0^2 x(2xdx) - x^2dx$

$\int_0^2 (2x^3+x)dx + i\int_0^2 x^2 dx = \left[\frac{x^4}{2}+\frac{x^2}{2}\right]_0^2 + i\left(\frac{x^3}{3}\right)_0^2 = 10 + i\frac{8}{3}$ //.

Evaluate $\int_C z^2 dz$ along the upper semicircle centred at origin and radius 2 (in anticlock wise orientation)



$c: z = 2e^{i\theta}$
$0 \le \theta \le \pi$

$c: z = 2e^{i\theta}$; $0 \le \theta \le \pi$

$dz = 2e^{i\theta}(i)d\theta$

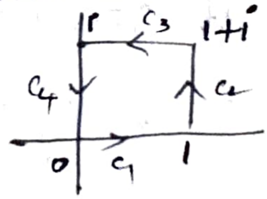
$\int_C z^2 dz = \int_0^{\pi} (2e^{i\theta})^2(2ie^{i\theta}d\theta)$

$= 8i\int_0^{\pi} e^{3i\theta}d\theta$

$= 8i\left[\frac{e^{3i\theta}}{3i}\right]_0^{\pi} = \frac{8}{3}\left[e^{3i\pi} - e^0\right] = \frac{8}{3}[-1-1] = \frac{-16}{3}$ //.

For analytic function, complex integral may be path independent

Evaluate $\oint_C |z|^2 dz$ over c where c is the square with vertices $0, 1, 1+i, i$



$\int |z|^2 dz = \int_{C = C_1 \cup C_2 \cup C_3 \cup C_4} |z|^2 dz$

$= \int_{C_1} |z|^2 dz + \int_{C_2} |z|^2 dz + \int_{C_3} |z|^2 dz + \int_{C_4} |z|^2 dz$

$\int |z|^2 dz = \int_C (x^2+y^2)(dx+idy) = \int_C (x^2+y^2)dx + i\int_C (x^2+y^2)dy$

(i) Along c_1: $y=0 \Rightarrow dy=0$
x varies from 0 to 1

$$\int_{c_1}|z|^2 dz = \int_0^1 x^2 dx + i\int_0^1 x^2(0) = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

(ii) Along c_2: $x=1 \Rightarrow dx=0$
y varies from 0 to 1

$$\int_{c_2}|z|^2 dz = \int_0^1(1+y^2)(0) + i\int_0^1(1+y^2)dy = \left[y+\frac{y^3}{3}\right]_0^1 = 1+\frac{1}{3} = \frac{4i}{3}$$

(iii) Along c_3: $y=1 \Rightarrow dy=0$
x varies from 1 to 0

$$\int_{c_3}|z|^2 dz = \int_1^0 (x^2+1)dx + i\int_1^0(x^2+1)0 = \left(\frac{x^3}{3}+x\right)_1^0 = -\frac{1}{3}-1$$

(iv) Along c_4: $x=0 \Rightarrow dx=0$
y varies from 1 to 0

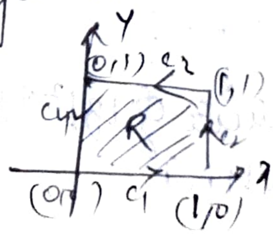
$$\int_{c_4}|z|^2 dz = \int_1^0 (0+y^2)0 + i\int_1^0(0+y^2)dy = i\left[\frac{y^3}{3}\right]_1^0 = -\frac{i}{3}$$

$$\int_c |z|^2 dz = \int_{c_1}|z|^2dz + \int_{c_2}|z|^2dz + \int_{c_3}|z|^2dz + \int_{c_4}|z|^2dz$$

$$= \frac{1}{3}+\frac{4i}{3}-\frac{4}{3}-\frac{i}{3} = \frac{1}{3}(1-i)+\frac{4}{3}(-1+i)$$

$$= \frac{1}{3}(1-i-4+4i) = \frac{1}{3}(3i-3) = \underline{\underline{i-1}} = -1+i$$

(05) Evaluate $\oint_C z^2 dz$ where C is a rectangle with vertices at $z=0$, $z=1$, $z=1+i$, $z=i$.



$$= \oint_c (x+iy)^2(dx+idy)$$

$$= \oint_c [(x^2-y^2)+i\,2xy][dx+idy]$$

$$= \oint_c \underbrace{(x^2-y^2)}_{F_1}dx \underbrace{-2xy}_{F_2}dy + i\oint_c (\underbrace{2xy}_{F_1}dx + \underbrace{x^2-y^2}_{F_2})dy$$

$$= \iint_R (-2y+2y)\,dy\,dx + i\iint_R (2x-2x)\,dy\,dx$$ [By Green's theorem

$$= 0+0 = 0$$

Homework

Evaluate $\int_C (z-z^2)\,dz$ where $c: |z-2|=3$

$\int_C Re(z)\,dz$ c: st line from $z=0$ to $z=1+2i$

$z = 2+3e^{i\theta}$
$\theta = 0$ to 2π

$z = z_0 + re^{i\theta}$

$\theta = 0$ to 2π.

c: $|z-2|=3$

$z = z_0 + re^{i\theta}$

$z = 2+3e^{i\theta}$ $\qquad z^2 = (2+3e^{i\theta})^2 = 4+9e^{2i\theta}+12e^{i\theta}$

$dz = 3e^{i\theta}(i)\,d\theta$

$\int_C (2+3e^{i\theta} - 4 - 9e^{2i\theta} - 12e^{i\theta})(3ie^{i\theta}\,d\theta)$

$\int_C (-2-9e^{i\theta}-9e^{2i\theta})\,3ie^{i\theta}\,d\theta$.

$= \int_0^{2\pi} (-2-9e^{i\theta}-9e^{2i\theta}) \ldots$

$= 3i\int_0^{2\pi} (-2e^{i\theta} - 9e^{2i\theta} - 9e^{3i\theta})\,d\theta$

$\cos\pi = (-1)^n$

$= 3i\left[-\frac{2(e^{i\theta})_0^{2\pi}}{i} - \frac{9}{2i}(e^{2i\theta})_0^{2\pi} - \frac{9}{3i}(e^{3i\theta})_0^{2\pi}\right]$

$= 3i\left[-\frac{2}{i}[e^{i2\pi}-e^0] - \frac{9}{2i}[e^{i4\pi}-e^0] - \frac{9}{3i}[e^{i6\pi}-e^{i0}]\right]$

$= 3i\left[-\frac{2}{i}[1-1] - \frac{9}{2i}[1-1] - \frac{9}{3i}[1-1]\right]$

$= 0$

2) $z = x+iy$ $\qquad dz = dx+idy$. $\qquad y = 2x$.
$dy = 2dx$.

$Re(z) = x$.

x varies from 0 to 1.

$\int_C x\cdot(dx+idy)$

(0,0)
(1,2)

$\int_C x\,dx + i\int_C x\,dy$

$\int_C x\,dx + i\int x\,2dx$. $\quad = \frac{1}{2} + i$

$\cancel{3\int_C x\,dx = 3\int_0^1 x\,dx = 3\left[\frac{x^2}{2}\right]_0^1 = 3\times\frac{1}{2} = \frac{3}{2}}$

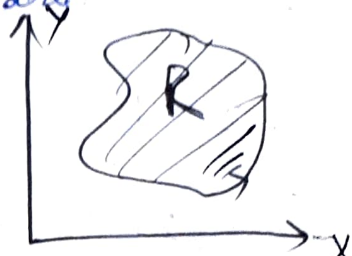
23|10|2019

★ CAUCHY'S INTEGRAL THEOREM (CIT):

If $f(z)$ is analytic and $f'(z)$ is continuous on and inside a simple closed curve C then $\oint_C f(z)\,dz = 0$

Proof. Suppose $f(z) = u(x,y)+iv(x,y)$ is analytic and $f'(z)$ is continuous on and inside a simple closed curve C lying on the complex plane.

$\therefore u \& v$ satisfy CR equations, and all the 1st order PD are continuous.



Consider $\oint_C f(z)\,dz$

$$= \oint_C (u+iv)(dx+idy)$$

$$= \oint_C (\overset{F_1}{u}dx - \overset{F_2}{v}dy) + i\oint_C (vdx+udy)$$

$$= \iint_R \left[\frac{\partial}{\partial x}(-v) - \frac{\partial u}{\partial y}\right] dy\,dx + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dy\,dx$$

(By Green's theorem)

$$= \iint_R \left(\frac{-\partial v}{\partial x} + \frac{\partial v}{\partial x}\right) dy\,dx + i\iint_R \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right) dy\,dx$$

[By CR equations]

$$= 0 + i0 = 0$$

Note:-
The above theorem is also valid even if we dont assume that $f'(z)$ is continuous.

Cauchy–Goursat Theorem:

If $f(z)$ is analytic on and inside a simple closed curve C then $\oint_C f(z)\,dz = 0$

- If $f(z)$ is analytic at a point z_0 then z_0 is said to be a regular point.

- If $f(z)$ is not analytic at z_0, then z_0 is said to be a singular point.

Ex: (i) $f(z) = \frac{1}{z^2+4}$ has two singular points which are $z = \pm 2i$ and all other points are regular points.

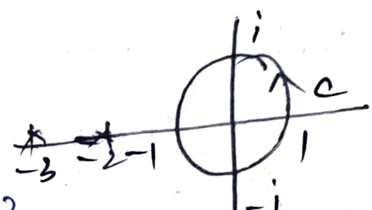
(ii) $f(z) = z^2$ has no singular point (Entire function $\downarrow$ Analytic everywhere)

Evaluate the Following Integrals

(1) $\oint_C \frac{z^2+1}{z^2+5z+6}\, dz$ (i) where $C_1 : |z| = 1$

(ii) $C_2 : |z+1| = 3/2$

(iii) $C_3 : |z-i| = 1$



Sol (i) Let $f(z) = \frac{z^2+1}{z^2+5z+6} = \frac{z^2+1}{(z+2)(z+3)}$

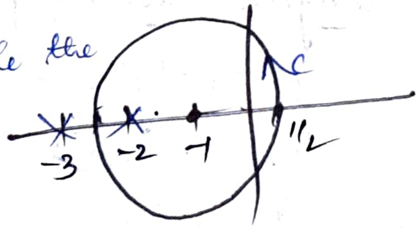
$f(z)$ has singularities at -2 and -3.

Therefore, there are no singularities of $f(z)$ on and inside the curve $|z| = 1$

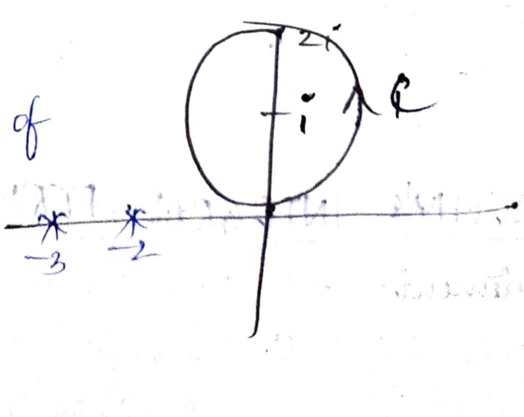
$\therefore$ By CIT, $\oint_{C_1} f(z)\, dz = 0$.

(ii) Since, there is a singularity inside the curve ($z=-2$)

Hence, CIT cannot be applied.



(iii) Since, there are no singularities of $f(z)$ on and inside the curve $|z-i| = 1$



$\therefore$ By CIT, $\oint_{C_3} f(z)\, dz = 0$

Consequences of Cauchy's Integral theorem

(01) Suppose $f(z)$ is analytic in a simply connected region R, then for any two distinct points z_1, z_2 in R $\int_{z_1}^{z_2} f(z)\,dz$ is path independent. In that case,

$$\int_{z_1}^{z_2} f(z)\,dz = F(z_2) - F(z_1)$$

$F(z)$ - primitive

where $F'(z) = f(z)\ \forall z$.

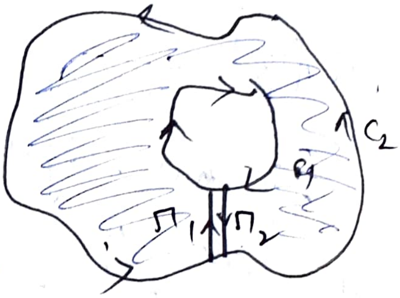
Cauchy's Integral Theorem for a doubly connected region

Let $f(z)$ be analytic on a doubly connected region R. Let C_1 and C_2 be inner and outer boundaries of R which are positively oriented wrt the region R, then

~~$C = C_1 \cup C_2$~~

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$

where the orientations of C_1 and C_2 are given wrt their interior regions.



24/10/2019

Cauchy's Integral Theorem for a multiply connected region

Let $f(z)$ be analytic in a multiply connected region bounded by a simple closed curve C (outer boundary) and n non overlapping simple closed curves $C_1, C_2, C_3, \ldots C_n$ (inner boundaries) then

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \ldots + \oint_{C_n} f(z)\,dz$$

where $C, C_1, C_2, \ldots C_n$ are oriented wrt their interior regions.

CAUCHY'S INTEGRAL FORMULA:

Statement: Let $f(z)$ is analytic on and inside a simple closed curve C then for any point z_0 inside C,

$$\oint_C \frac{f(z)}{z - z_0}\,dz = 2\pi i f(z_0)$$

Proof: Suppose $f(z)$ is analytic on and inside a simple closed curve C. Let z_0 be a point inside C.

Define $g(z) = \dfrac{f(z)}{z-z_0}$

Clearly, $g(z)$ is analytic at all points on and inside except at z_0.

Construct a circle Γ centred at z_0 with some radius $\epsilon > 0$, such that Γ is completely inside C then $g(z)$ ~~has no singularity~~ is analytic at all points on C, on Γ and in between C & Γ. Therefore by the principle of path deformation

$$\oint_C g(z)\,dz = \oint_\Gamma g(z)\,dz.$$

consider $\oint_C g(z)\,dz$

$$= \oint_\Gamma g(z)\,dz$$

$$= \oint_\Gamma \frac{f(z)}{z-z_0}\,dz = I$$

On $\Gamma \Rightarrow z = z_0 + \epsilon e^{i\theta}, \quad 0 \le \theta \le 2\pi$

$$dz = i\epsilon e^{i\theta}\,d\theta$$

$$I = \int_0^{2\pi} \frac{f(z_0 + \epsilon e^{i\theta})}{\epsilon e^{i\theta}}\, i\epsilon e^{i\theta}\,d\theta.$$

$$I = i\int_0^{2\pi} f(z_0 + \epsilon e^{i\theta})\,d\theta$$

$$\lim_{\epsilon\to 0} I = \lim_{\epsilon\to 0}\left[i\int_0^{2\pi} f(z_0 + \epsilon e^{i\theta})\,d\theta\right].$$

$$= i\int_0^{2\pi}\left[\lim_{\epsilon\to 0} f(z_0 + \epsilon e^{i\theta})\right]$$

$$= i\int_0^{2\pi} f\left(\lim_{\epsilon\to 0}(z_0 + \epsilon e^{i\theta})\right)d\theta$$

$$= i\int_0^{2\pi} f(z_0)\,d\theta$$

$$= f(z_0)\, i(\theta)_0^{2\pi}$$

$$= 2\pi i f(z_0)$$

$$\boxed{\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0)}$$

Generalised Cauchy's Integral Formula:

Statement: If $f(z)$ is analytic on and inside a simple closed curve c, then for any point z_0 inside c,

$$\oint_c \frac{f(z)}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} f^{(n)}(z_0) \quad ; n = 0, 1, 2, \ldots$$

Proof:

Suppose $f(z)$ is analytic on and inside a simple closed curve c. Let z_0 be any point inside c, then by Cauchy's Integral Formula

$$\oint_c \frac{f(z)}{z-z_0} dz = 2\pi i f(z_0)$$

$$\Rightarrow \frac{d}{dz_0} \oint_c \frac{f(z)}{(z-z_0)} dz = 2\pi i f'(z_0)$$

$$\oint_c \frac{\partial}{\partial z_0}\left[\frac{f(z)}{(z-z_0)}\right] dz = 2\pi i f'(z_0)$$

$$\oint_c f(z)\left[+\frac{1}{(z-z_0)^2}\right] dz = 2\pi i f'(z_0)$$

Differentiate both sides wrt z_0

$$\oint_c f(z)\left[\frac{+2}{(z-z_0)^3}\right] dz = 2\pi i f''(z_0)$$

Differentiate again wrt z_0 on both sides.

$$\oint_c f(z) \frac{3!}{(z-z_0)^4} dz = 2\pi i f'''(z_0)$$

In general, we have

$$\oint_c f(z) \frac{n!}{(z-z_0)^{n+1}} dz = 2\pi i f^{(n)}(z_0)$$

$$\oint_c \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(z_0) \qquad n = 0, 1, 2, 3 \ldots$$

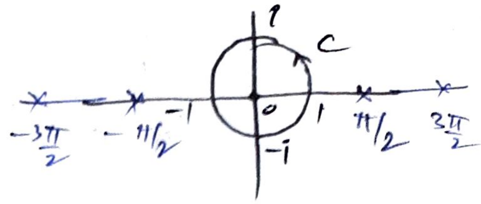
26/10/2019

Problems

Evaluate the following integrals

(i) $\oint \operatorname{Tan} z \, dz$.

$|z|=1$



Let $f(z) = \operatorname{Tan} z$

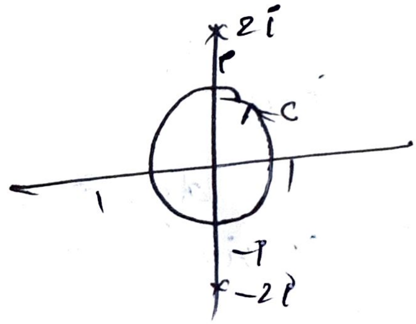
$= \frac{\sin z}{\cos z}$

$\cos z = 0 \Rightarrow z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$ $\therefore f(z)$ has singularities at $z = (2n+1)\pi/2$, $n \in \mathbb{Z}$ which are outside the curve $c: |z| = 1$,

$= (2n+1)\frac{\pi}{2} \; (n \in \mathbb{Z})$

Since there are no singularities of $f(z)$ on inside c, by Cauchy's Integral Theorem, $\oint_{|z|=1} \operatorname{Tan} z \, dz = 0$.

(ii) $\oint_{|z|=1} \frac{e^{2z}}{z^2+4} \, dz$.



Let $f(z) = \frac{e^{2z}}{z^2+4}$

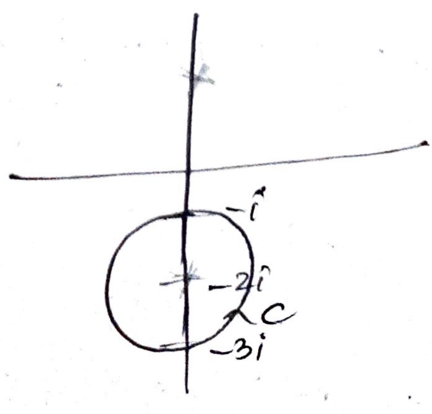
$z^2+4=0$

$\Rightarrow z = 2i, -2i$

$\therefore f(z)$ has singularities at $z = 2i, -2i$ which are outside the curve $c: |z|=1$

$\therefore$ By CIT, $\oint_c f(z)\, dz = \oint_c \frac{e^{2z}}{z^2+4} \, dz = 0$

(iii) $\oint_{|z+2i|=1} \frac{e^{2z}}{z^2+4} \, dz$



$= \oint_c \frac{e^{2z}}{(z-2i)(z+2i)} \, dz$

$= \oint_c \frac{(e^{2z}/z-2i)}{z-(-2i)} \, dz$

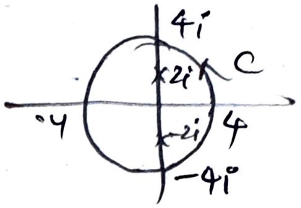
$= 2\pi i f(-2i)$ where $f(z) = \frac{e^{2z}}{z-2i}$

$$= 2\pi i \frac{e^{2(-2i)}}{-2i-2i} \quad \text{where } f(z) = \frac{e^{2z}}{z-2i}$$

$$= \frac{2\pi i}{-4i} e^{-4i}$$

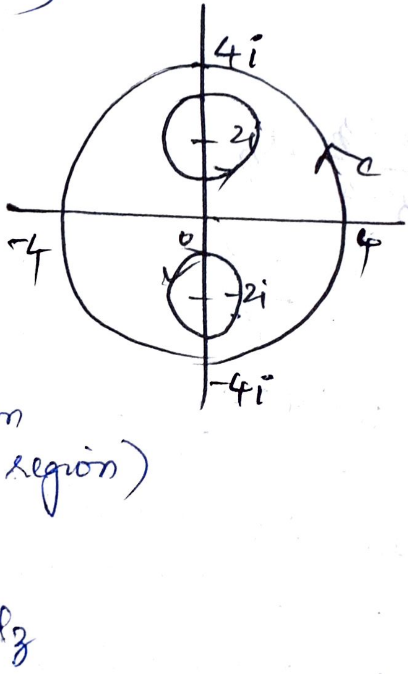
$$= -\frac{\pi}{2} e^{-4i}$$

(iv) $\oint_{|z|=4} \frac{e^{2z}}{z^2+4} dz$



$$= \oint_C \frac{e^{2z}}{(z-2i)(z+2i)} dz$$

$$= \oint_C \frac{\left(\frac{e^{2z}}{z+2i}\right)}{z-(-2i)} dz.$$ is not analytic



$$= \oint_{C_1} \frac{e^{2z}}{(z-2i)(z+2i)} dz$$

$$= \oint_{C_1} \frac{e^{2z}}{(z-2i)(z+2i)} dz + \oint_{C_2} \frac{e^{2z}}{(z-2i)(z+2i)} dz$$

By Cauchy's Integral Theorem on multiply connected region)

$$= \oint_{C_1} \frac{\left(\frac{e^{2z}}{z-2i}\right)}{z+2i} dz + \int_{C_2} \frac{\left(\frac{e^{2z}}{z+2i}\right)}{z-2i} dz$$

$$= 2\pi i\, g(-2i) + 2\pi i\, h(2i) \quad \text{where } g(z) = \frac{e^{2z}}{z-2i}$$

(by CIF)

$$h(z) = \frac{e^{2z}}{z+2i}$$

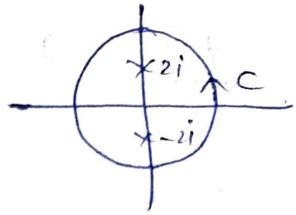
$$= 2\pi i \left[\frac{e^{2(-2i)}}{-4i} + \frac{e^{2(2i)}}{4i}\right]$$

$$= 2\pi i \left[\frac{-e^{-4i}}{4i} + \frac{e^{+4i}}{4i}\right] = \frac{2\pi i}{2i}\left[\frac{e^{4i}-e^{-4i}}{2i}\right]$$

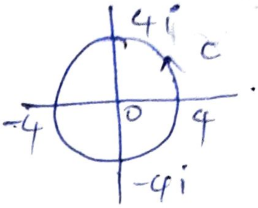
$$= \pi i \sin(4)$$

Other method :-

$$\oint_C e^{2z}\left[\frac{1}{(z-2i)(z+2i)}\right]dz$$

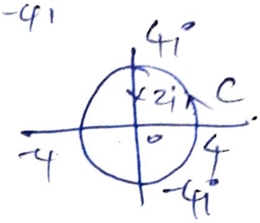


$$=-\frac{1}{4i}\oint_C e^{2z}\left[\frac{1}{z+2i}-\frac{1}{z-2i}\right]dz$$



$$=-\frac{1}{4i}\left[\oint_C\frac{e^{2z}}{z+2i}-\oint_C\frac{e^{2z}}{z-2i}dz\right]$$

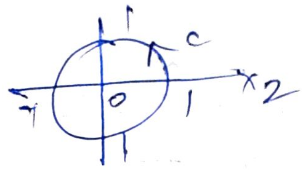
$$=-\frac{1}{4i}\left[2\pi i e^{2(-2i)}-2\pi i e^{2(2i)}\right]$$



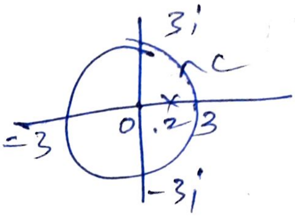
$$=\pi i\sin(4)$$

(v) $$\oint_{|z|=1}\frac{e^{2z}}{(z-2)^4}dz=0$$

[By CIT]



(vi) $$\oint_{|z|=3}\frac{e^{2z}}{(z-2)^4}dz.$$



$$=\oint_{|z|=3}\frac{e^{2z}}{(z-2)^{3+1}}dz=\frac{2\pi i}{3!}f^{(3)}(2)$$

$$=\frac{2\pi i}{6_3}[8e^4]$$

$$=\frac{8}{3}\pi i e^{2(4)}$$

$$=\frac{8\pi i e^4}{3}$$

$f(z)=e^{2z}$

$f'(z)=2e^{2z}$

$f''(z)=4e^{2z}$

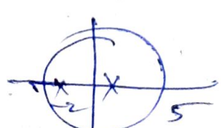
$f'''(z)=8e^{2z(4)}$

$f'''(2)=8e^{2(2)}$

$=8e^4$

GATE Q

$$\int_{z=5}\frac{z^3+2z^2+8}{z+2}dz=\frac{8\pi i}{=2\pi i f(-2)}$$



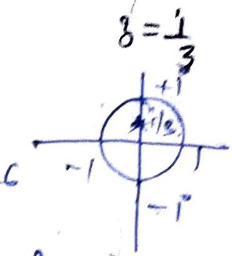
$$\begin{array}{r} z^2-z \\ z+2\overline{)z^3+2z^2+8} \\ z^3+2z^2 \\ \hline -z^2+8 \\ -z^2-2z \\ \hline \end{array}$$

$$\int_{|z|=3}\frac{z^3-2z+3}{z-2}dz=2\pi i f(2)$$

$$=2\pi i(8-4+3)$$

$$=14\pi i$$

$$\int_C \frac{z^2-6}{3z-i}\,dz \quad \text{where } c \text{ is unit circle}$$

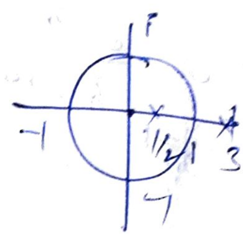


$$\int_{\substack{C\\|z|=1}} \frac{z^2-6}{3z-i} = \frac{1}{3}\oint_c \frac{z^2-6}{z-\frac{i}{3}} = \frac{1}{3}2\pi i f\left(\frac{i}{3}\right)$$

$$= \frac{1}{3}2\pi i \left(\left(\frac{i}{3}\right)^3-6\right)$$

$$= \frac{2\pi i}{3}\left[\frac{-i}{3^3}-6\right]$$

$$= \frac{2\pi}{81} - 4\pi i$$



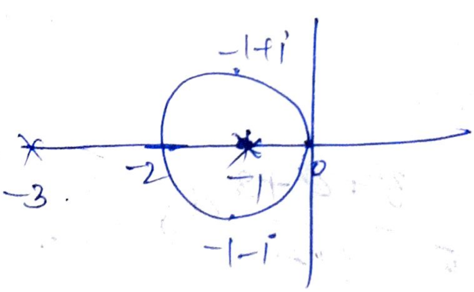
$$\oint_{\substack{C\\|z|=1}} \frac{\cos 2\pi z}{(2z-1)(z-3)}\,dz = \frac{1}{2}\int_{|z|=1} \frac{\frac{\cos 2\pi z}{z-3}}{\left(z-\frac{1}{2}\right)}$$

$$= \cancel{\frac{1}{2}\oint_{|z|=1}} \frac{1}{2}\left[2\pi i f\left(\frac{1}{2}\right)\right]$$

$$= \frac{2\pi i}{2}\left[\frac{\cos\frac{2\pi}{2}}{\frac{1}{2}-3}\right]$$

$$= \frac{2\pi i}{5}$$

$$\frac{1}{2\pi i}\oint_C \left[\frac{1}{z+1} - \frac{2}{z+3}\right]dz \quad C: |z+1| = 1$$



$$\frac{1}{2\pi i}\left[\int_{|z+1|=1} \frac{1}{z+1}\,dz - \int_{|z+1|=1} \frac{2}{z+3}\,dz\right]$$

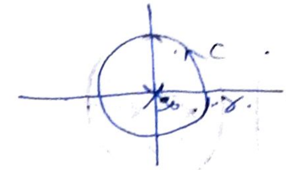
(By CIT)

$$\frac{1}{2\pi i}\left[2\pi i f(-1) - 0\right]$$

$$= \frac{1}{2\pi i}\left[2\pi i(1) - 0\right] = 1 - 0 = 1$$

$$\oint_C \frac{1}{(z-z_0)^{n+1}}\,dz = \underline{\frac{2\pi i}{n!} f^{(n)}(z_0) = 0} \quad c : |z-z_0| = r$$

$f^{(n)}(z_0) = 0$

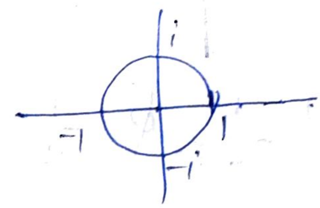


$$\oint_{|z-z_0|=r} \frac{1}{(z-z_0)^{n+1}}\,dz$$

$2\pi i f(z_0)$

$$\frac{1}{2\pi i}\int_{\substack{C\\|z|=1}} Re(z)\,dz = \underline{\;1/2\;} \qquad c : |z| = 1$$

not analytic anywhere



$$= \frac{1}{2\pi i}\int_{|z|=1} x\,(dx + i\,dy)$$

$$= \frac{1}{2\pi i}\int x\,dx + i\,x\,dy \quad (\text{or}) \quad \frac{1}{2\pi i}\int_0^{2\pi} \cos\theta \, i e^{i\theta}\,d\theta$$

$x^2+y^2=1$

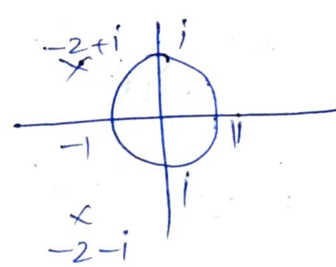
$2x\,dx + 2y\,dy = 0$

$dy =$

$$= \frac{i}{2\pi i}\int_0^{2\pi} e^{i\theta}\cos\theta\,d\theta$$

$$= \frac{i}{2\pi i}\left[\frac{e^{i\theta}}{i^2+1}\,[i\cos\theta - \sin\theta]\right]_0^{2\pi}$$

$$\oint_C \frac{4-3z}{z^2+4z+5}\,dz = \underline{\;0\;} \qquad c : |z| = 4$$



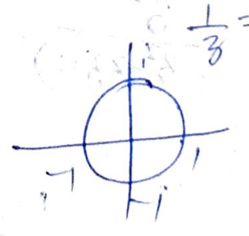
$$= \int_{|z|=1} \frac{4-3z}{(z-(-2+i))(z-(-2-i))}$$

$$\oint_C \left(z+\frac{1}{z}\right)^2 dz = \underline{\;0\;} \qquad |z| = 1$$

$z = x+iy$

$\frac{1}{z} = \frac{1}{x+iy}$

$= \frac{x-iy}{x^2+y^2}$



$$\oint_C \left(\frac{z^2+1}{z}\right)^2 dz = \oint_C \frac{(z^2+1)^2}{(z-0)^{1+1}}\,dz$$

$f' = 2(z^2+1)(2z)$

$= 4$

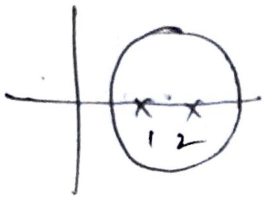
$$= \frac{2\pi i}{1!} f'(0) = 2\pi i(0)$$

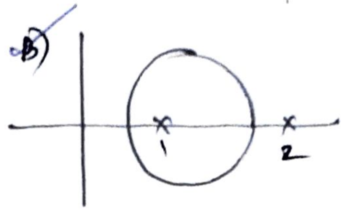
$$= 2\pi i(0)$$

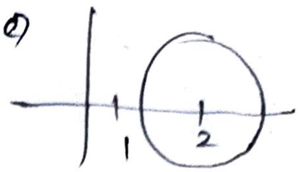
05) $\oint_{\Gamma} \frac{3z-5}{(z-1)(z-2)} = 4\pi i$ then Γ is .

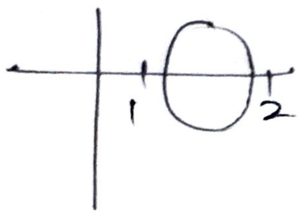
$2\pi i f(n) = 4\pi i$

$f(2) = 2$

A) 

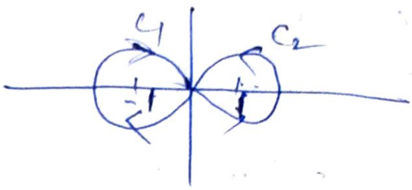
B) 

C) 

D) 

$\frac{\left(\frac{3z-5}{z-2}\right)}{z-1} = 2\pi i\left(\frac{-2}{-1}\right) = 4\pi i$

06) $\frac{1}{\pi i} \oint_{c} \frac{1}{z^2-1}\, dz$



$c = c_1 \cup c_2$

$\frac{1}{\pi i}\left[\int_{|z+1|=1} \frac{1}{(z+1)(z-1)} + \int_{|z-1|=1} \frac{1}{(z+1)(z-1)}\right]$

$\frac{1}{\pi i}\left[\int_{|z+1|=1} \frac{1}{z^2-1} + \int_{|z-1|=1} \frac{1}{z^2-1}\right] = \frac{1}{\pi i}\oint_{|z+1|=1} \frac{1}{(z-1)(z+1)}dz + \frac{1}{\pi i}\oint_{|z-1|=1} \frac{1}{(z-1)(z+1)}dz$

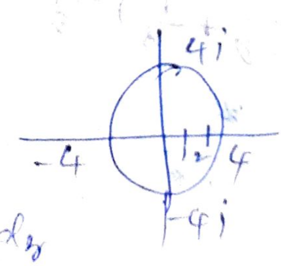
$= \frac{1}{\pi i} \int_{|z+1|=1} \frac{\left(\frac{1}{z-1}\right)}{z+1}\, dz + \frac{1}{\pi i} \int_{|z-1|=1} \frac{\left(\frac{1}{z+1}\right)}{z-1}\, dz$

$= \frac{1}{\pi i}\left[2\pi i\left[\frac{1}{-1-1}\right]\right] + \frac{1}{\pi i}\left[2\pi i\left(\frac{1}{1+1}\right)\right]$

$= 1 + 1 = 2$

07) $\int_{c} \frac{z^2}{(z^2-3z+2)^2}\, dz =$ ________

$c: |z| = 4$

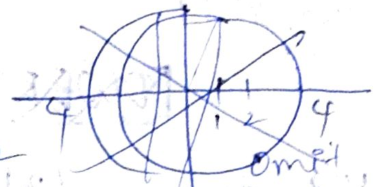


$\frac{(1+i)\cancel{(1-i)}}{(z-(1+i))^2(z-(1-i))^2}$

z^2

$\int_{c} \frac{z^2}{(z-1)^2(z-2)^2}\, dz$

$= \int_C \frac{z^2}{(z-1)^2(z-2)^2} dz$



$\int_{\Gamma_1} \frac{\frac{z^2}{(z-2)^2}}{(z-1)^2} dz = \int_{\Gamma_1} \frac{\left(\frac{z}{z-2}\right)^2}{(z-1)^2} + \int_{\Gamma_2} \frac{\left(\frac{z}{z-1}\right)^2}{(z-2)^2}$

$f(z) = \frac{z}{z-2}$

$= \frac{2\pi i}{1!} f'(1) + \frac{2\pi i}{1!} f'(2)$

$f'(z) = \frac{(z-2)(1) - z(1)}{(z-2)^2}$

$= \cancel{-4\pi i} \quad \cancel{8\pi i}$

$= \cancel{2\pi i(-2) + 2\pi i(-1)}$

$= \frac{z-2-z}{}$

$= -8\pi i + 8\pi i$

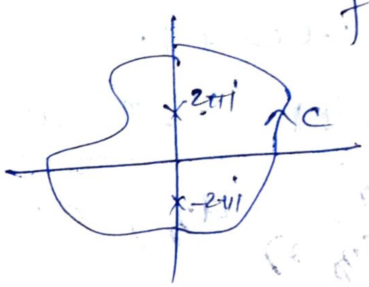
$= \frac{-2}{(z-2)^2}$

3) $\frac{-1}{2\pi} \int_C \frac{\sin z}{(z-2\pi i)^3} dz =$ ______

$f'(1) = \frac{-2}{(1-2)^2}$

$= -2$

$\frac{-1}{2\pi} \int_C \frac{\sin z}{(z-2\pi i)^3} dz =$ ______



$\frac{(z-1)(1) - z(1)}{(z-1)^2}$

$= \frac{-1}{2\pi}\left[\frac{2\pi i}{2!} \cancel{\cos(2\pi i)} \sin(2\pi i)\right]$

$= \frac{-1}{2\pi}\left[\pi i\left(-i\sinh 2\pi\right)\right]$

$\sin iz = i\sinh z$

$\cos iz = \cosh z$

$= \frac{-1}{(z-1)^2}$

$= -1$

$= \frac{-1}{2\pi}\left[\pi \sinh 2\pi\right]$

$= \frac{+1}{2} \sinh 2\pi$

$f(z) = \sin z$

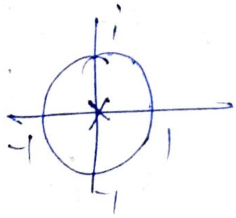
$f'(z) = \cos z$

$= 133.87$

$f''(z) = -\sin z$

$f''(2\pi i)$

4) $\int_{|z|=1} \frac{\cos z}{z} dz$



$2\pi i f(0) = 2\pi i \cos 0 = 2\pi i$

$= -\sin(2\pi i)$

$\mathrm{Arg}\left(\frac{1+i}{1-i}\right) = \mathrm{Arg}(i) = \frac{\pi}{2}$

Principal Argument

28th October, 2019

~~POWER SERIES~~

1Q) $\oint \frac{\tan z}{(z^2-1)} dz$, $c: |z| = \frac{3}{2}$

$= \frac{1}{2}\left[\int_{c_1} \frac{\left(\frac{\tan z}{z-1}\right)}{z+1} dz + \int_{c_2} \frac{\left(\frac{\tan z}{z+1}\right)}{z-1} dz\right]$

By CIF/

$= 2\pi i g(-1) + 2\pi i h(1)$

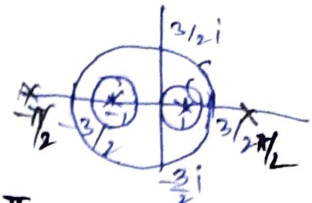
$= 2\pi i \frac{\tan(-1)}{-2} + 2\pi i \frac{\tan(1)}{2}$

$= \pi i [\tan(1) - \tan(-1)]$

$= \pi i [1.557 + 1.557]$

$= 3.114\pi i = 2\pi i \tan(1)$

Q. $\oint \frac{\sin z}{\cos z (z^2-1)}$



$\frac{\pi}{2}, \frac{3\pi}{2}$

$\cos z = 0, z^2 - 1 = 0$

$\Rightarrow z = (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}, z = \pm 1$

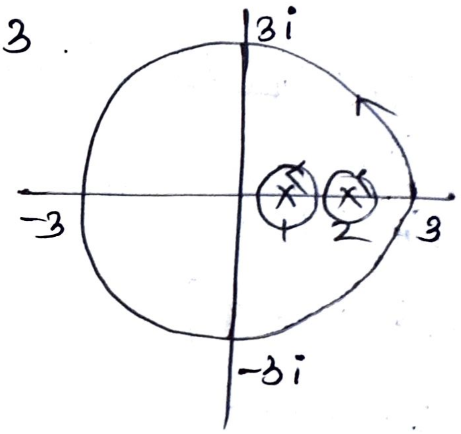
are the singularities of $f(z)$

But $z = -1, 1$ are the only singularities inside c

$\oint_c \frac{\tan z}{z^2-1} dz$

$= \frac{1}{2}\oint_c \tan z \left[\frac{1}{z-1} - \frac{1}{z+1}\right] dz$

② $\oint_c \frac{e^{2z}}{(z-1)(z-2)} dz$, $c: |z| = 3$.



$\oint_{c_1} \frac{\left(\frac{e^{2z}}{z-2}\right)}{z-1} + \oint_{c_2} \frac{\left(\frac{e^{2z}}{z-1}\right)}{z-2}$

$= 2\pi i f(1) + 2\pi i h(2)$

$= 2\pi i \frac{e^2}{1-2} + 2\pi i \frac{e^4}{2-1}$

$= -2\pi i e^2 + 2\pi i e^4$

$= 2\pi i [e^4 - e^2]$

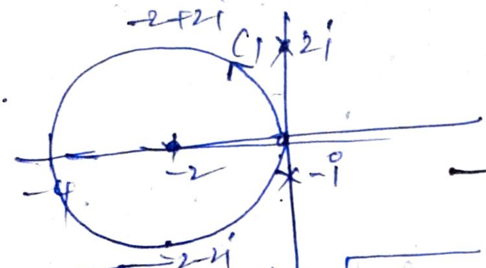
$= 2\pi i e^2 [e^2 - 1]$

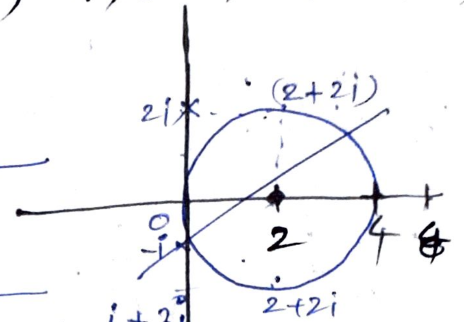
$\oint_C \frac{\sin z}{z^2 - iz + 2}$

(i) $C_1 : |z+2| = 2$

(ii) C_2: Rectangle with vertices at $(1,0), (1,3), (-1,3), (-1,0)$

(iii) C_3: Rectangle with vertices at $(2,3), (2,-3), (-2,3), (-2,-3)$





i) $\oint_C \frac{\sin z}{z^2 - iz + 2} dz$.

$\sin z$

$z^2 - iz + 2 = \frac{i \pm \sqrt{i^2 - 4(2)}}{2} = \frac{i \pm \sqrt{-9}}{2} = \frac{i \pm 3i}{2}$

$= \frac{4i}{2}, \frac{-2i}{2}$

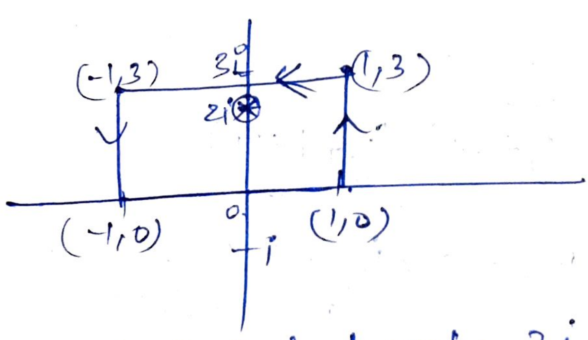
$= 2i, -i$

$\oint \frac{\sin z}{(z-2i)(z+i)} dz$ $\qquad z = 2i, -i$ are the singularities of $f(z)$

$(-2,0)\ (0,-1)$ $= \sqrt{(2)^2 + 1^2} = \sqrt{4+1} = \sqrt{5} = 2 \ldots$

$\therefore$ by CIT, $\oint_C \frac{\sin z}{z^2 - iz + 2} = 0$ [$\because f(z)$ is analytic on and inside C_1]

ii) $\oint_{C_2} \frac{\sin z}{z^2 - iz + 2}$.



$\oint_{C_2} \frac{\sin z}{(z-2i)(z+i)} dz$

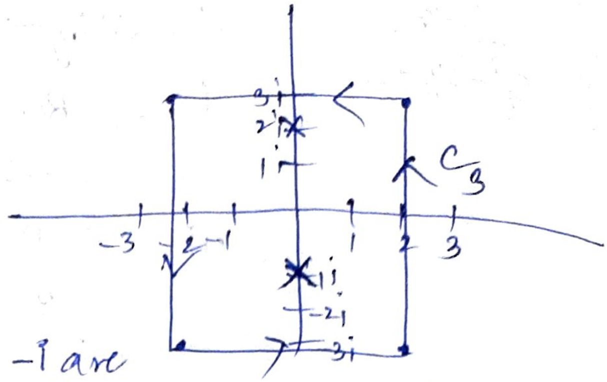
$z = 2i, -i$ are the singularities of $f(z)$ but only $2i$ is inside C_2

$\therefore \oint_{C_2} \frac{\left(\frac{\sin z}{z+i}\right)}{(z-2i)} dz$.

By CIF, $= 2\pi i \left(\frac{\sin 2i}{2i+i}\right) = \frac{2\pi i \sin 2i}{3i} = \frac{2\pi}{3} \sin 2i$

$= \frac{2\pi}{3} i \sinh 2$

(iii) $f(z) = \frac{\sin z}{z^2 - iz + 2}$



$f(z) = \frac{\sin z}{(z-2i)(z+i)}$

$\oint_{C_3} \frac{\sin z \, dz}{(z-2i)(z+i)}$ Both $z = 2i, -i$ are the singularities which are inside c

$= \oint_{C_3} \sin z \left[\frac{1}{z-2i} - \frac{1}{z+i}\right] = \frac{1}{3i}\left[\oint_{C_3} \frac{\sin z}{z-2i} dz - \oint_{C_3} \frac{\sin z}{z+i} dz\right]$

$= \frac{1}{3i}\left[2\pi i f(2i) - 2\pi i f(-i)\right]$

$= \frac{1}{3i}\left[2\pi i \sin(2i) - 2\pi i \sin(-i)\right]$

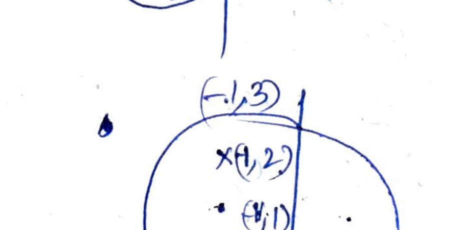
$= \frac{1}{3i}\left[2\pi i \, i \sinh 2 - 2\pi i \, i \sinh(-1)\right] =$

$= \frac{1}{3i}\left[-2\pi \sinh 2 + 2\pi \sinh(-1)\right] = \frac{2\pi i}{3}\left[\sinh 2 + \sinh 1\right]$

(4) $\oint_C \frac{z+4}{z^2+2z+5}$ $c : |z+1-i| = 2$

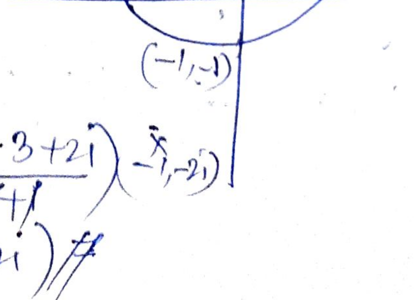
$c = -1+i$
$= (-1, 1)$

$\oint \frac{z+4}{(z+1-2i)(z+1+2i)}$



$z = -1+2i, -1-2i$ are the singularities

of which only $-1+2i$ is inside c



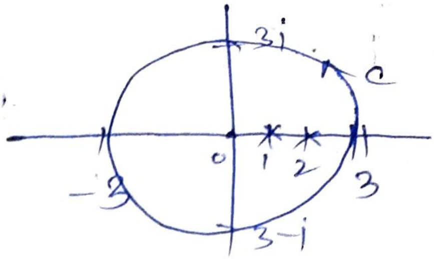
$\oint_C \frac{\left(\frac{z+4}{z+1+2i}\right)}{z+1-2i} dz$

$2\pi i f(-1+2i)$

$2\pi i \left(\frac{-1+2i+4}{-1+2i+1+2i}\right) = 2\pi i \left(\frac{3+2i}{4i}\right)$

$= \frac{\pi}{2}(3+2i)$

$\oint_{|z|=3} \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$



Let $f(z) = \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}$

has singularities at $z = 1, 2$
which are inside the circle $C: |z| = 3$

$\oint_C f(z)\, dz = \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$

$= \oint_C [\sin \pi z^2 + \cos \pi z^2] \left[\frac{1}{z-2} - \frac{1}{z-1}\right] dz$.

$= \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2} dz - \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-1} dz$.

$= 2\pi i (\sin 4\pi + \cos 4\pi) - 2\pi i (\sin \pi + \cos \pi)$

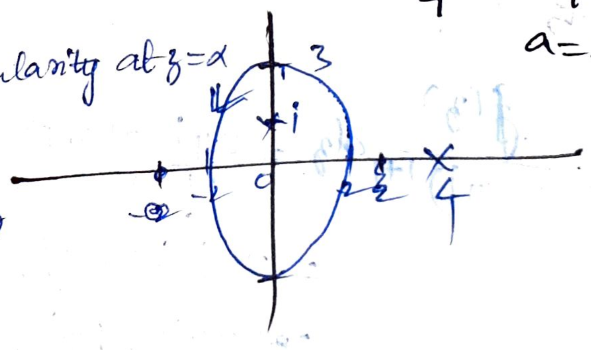
$= 2\pi i (1) - 2\pi i (-1) = 4\pi i$

(b) Determine $F(4)$, $F(i)$, $F'(-1)$, $F''(-i)$ if

$F(\alpha) = \oint_C \frac{4z^2 + z + 5}{z - \alpha} dz$ $\quad C: 9x^2 + 4y^2 = 36$.

$\frac{x^2}{4} + \frac{y^2}{9} = 1$

$a = 2, b = 3$.



Let $f(z) = \frac{4z^2 + z + 5}{z - \alpha}$ has singularity at $z = \alpha$

Define $F(\alpha) = \oint_C \frac{4z^2 + z + 5}{z - \alpha} dz$

(i) If $\alpha = 4$, then
$f(z)$ is analytic on and inside C

$\therefore$ By CIT, $F(4) = \oint_C \frac{4z^2 + z + 5}{z - 4} dz = 0$.

(ii) If $\alpha = i$, ~~there~~ which is inside

$\therefore$ By CIF, $F(i) = \oint_C \frac{4z^2 + z + 5}{z - i} dz = 2\pi i\, g(i)$

$= 2\pi i (4(i)^2 + i + 5) = 2\pi i (-4 + 5 + i)$

$= 2\pi i (1 + i)$.

(iii) If $\alpha = -1$ which is inside

$\therefore F(-1) = \oint_c \frac{4z^2+z+5}{z+1}$

$= 2\pi i f(-1) = 2\pi i\,(4(1) + -1 + 5)$

$= 2\pi i\,(8)$

$= 16\pi i$

If α is inside c, then $F(\alpha) = \oint_c \frac{4z^2+z+5}{z-\alpha}$

$= 2\pi i\, g(\alpha)$

$\Rightarrow F'(\alpha) = 2\pi i\, g'(\alpha)$

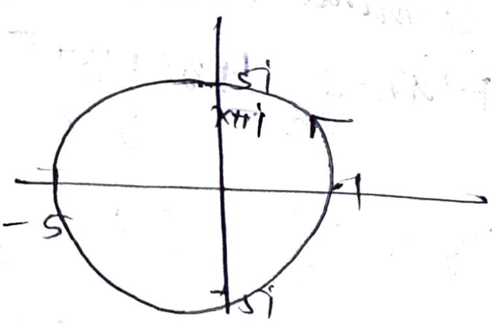
$\Rightarrow F''(\alpha) = 2\pi i\, g''(\alpha)$

(iii) $F'(-1) = 2\pi i\, g'(-1) = 2\pi i\,(8z+1)\; z=-1$

$= 2\pi i\,(-8+1) = -14\pi i$

$F''(-i) = 2\pi i\, g''(-i) = 2\pi i\,(8) = 16\pi i$

① $\oint_{|z|=5} \frac{\cos z}{(z-\pi i)^2}\, dz$



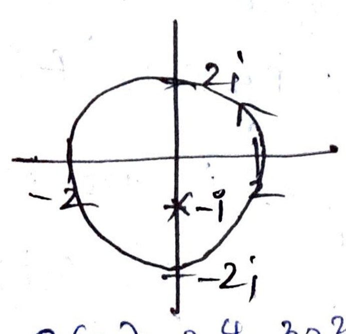
$\oint_{|z|=5} \frac{g(z)}{(z-\pi i)^{1+1}}\, dz = \frac{2\pi i}{1!}\, g'(\pi i)$

$= 2\pi i\,(-\sin \pi i)$

$= 2\pi(\sinh \pi)$

$\int \frac{g(z)}{(z-z_0)^{n+1}}\, dz = \frac{2\pi i}{n!}\, g^{(n)}(z_0)$

② $\oint_{|z|=2} \frac{z^4-3z^2+6}{(z+i)^3}\, dz$.



$\oint_{|z|=2} \frac{g(z)}{(z+i)^{2+1}} = \frac{2\pi i}{2!}\, g^2(-i)$

$= \pi i\,(12(-1)-6)$

$= -18\pi i$

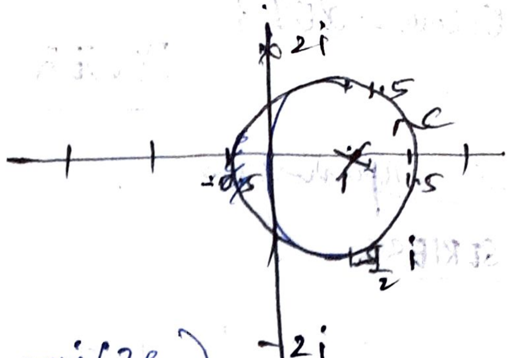
$g(z) = z^4 - 3z^2 + 6$

$g'(z) = 4z^3 - 6z$

$g''(z) = 12z^2 - 6$

$$\oint_{|z-1|=\frac{1}{2}} \frac{e^z}{(z-1)^2(z^2+4)} dz.$$

$z=1, \pm 2i$ of which only 1 is inside c



$$\oint_c \frac{\left(\frac{e^z}{z^2+4}\right)}{(z-1)^{1+1}} dz = \frac{2\pi i}{1!} g'(1) = 2\pi i\left(\frac{3e}{25}\right)$$

$$= \frac{6\pi i e}{25}$$

$g(z) = \frac{e^z}{z^2+4}$

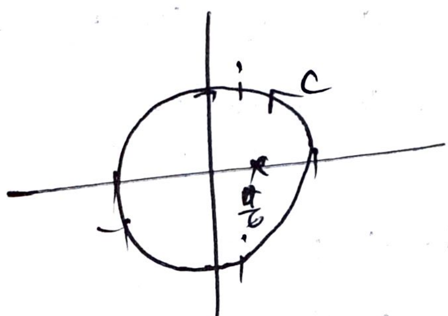
$g'(z) = \frac{(z^2+4)e^z - e^z(2z)}{(z^2+4)^2}$

$g'(1) = \frac{(1+4)e - e(2)}{(1+4)^2}$

$= \frac{5e-2e}{5^2}$

$= \frac{3e}{25}$

4) $$\oint_{|z|=1} \frac{\sin^2 z}{\left(z-\frac{\pi}{6}\right)^3} dz$$



$$\oint_c \frac{\sin^2 z}{\left(z-\frac{\pi}{6}\right)^{2+1}} dz$$

$$= \frac{2\pi i}{2!} g''\left(\frac{\pi}{6}\right)$$

$$= \pi i \cos \frac{2\pi}{6} \frac{\pi}{3}$$

$$= \pi i \, 2\left(\frac{1}{2}\right) = \pi i$$

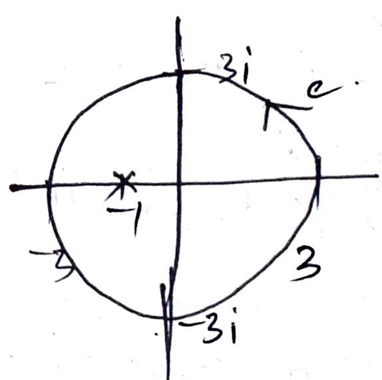
$g(z) = \sin^2 z$

$g'(z) = 2\sin z \cos z$

$= \sin 2z$

$g''(z) = \cos 2z \cdot$ $g(z$

5) $$\oint_{|z|=3} \frac{e^{2z}}{(z+1)^4} dz$$



By Generalised CIF,

$$\oint_c \frac{e^{2z}}{(z+1)^{3+1}} dz = \frac{2\pi i}{3!} g'''(-1)$$

$$= \frac{2\pi i}{6_3} \times 8 e^{-2}$$

$$= \frac{8}{3}\pi i e^{-2}$$

$g(z) = e^{2z}$

$g'(z) = 2e^{2z}$

$g''(z) = 4e^{2z}$

$g'''(z) = 8e^{2z}$

28th October 2019

POWER SERIES

Series expansions

POWER SERIES:

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + a_3(z-z_0)^3 \ldots$$

where $a_0, a_1, a_2, \ldots$ are the coefficients of the power series which are complex constants and z_0 also a complex constant called Centre of the series.

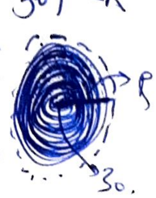
z is a complex variables.

Observations:

Properties of Power Series:

(i) Power series does not contain negative powers.

(ii) Power series always converges in the region $D : |z-z_0| < R$ where $R = \lim_{n\to\infty} \frac{a_n}{a_{n+1}}$. Here, R is called radius of convergence and the D region is called Region of convergence and z_0 is centre of convergence.



(iii) The equation $|z-z_0| = R$ is called circle of convergence.

(iv) Power series is divergent at all points outside its circle of convergence. $[|z-z_0| > R]$

(V) On the circle of convergence, the power series may or may not converge. (All / none / some of the points of circle of convergence)

(VI) We can differentiate the Power Series term by term also we can integrate the power series term by term.

30th October, 2019

Taylor's theorem:

If $f(z)$ is analytic on and inside a circle 'c' centred at z_0, $f(z)$ has Taylor's series about $z=z_0$ given by

$$f(z)=\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n = \sum a_n (z-z_0)^n$$

Note:

(i) Taylor series is a special case of Power Series

(ii) The region of convergence (ROC) for the Taylor series of $f(z)$ about $z=z_0$ is $|z-z_0|<R$ where 'R' is the distance from z_0 to nearest singularity of $f(z)$.

(iii) Taylor series of $f(z)$ about $z=z_0$ does not exist if z_0 is a singular point.

(iv) Taylor series exists about $z=z_0$ only when z_0 is a regular point.

(v) Taylor series about z_0 is Mc clawrin's series.

$$f(z)=\sum_{n=0}^{\infty}\frac{f^n(0)}{n!}z^n.$$

Standard Mc Claurin's series expansion:

1) $e^z = 1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+ \dots = \sum_{n=0}^{\infty}\frac{z^n}{n!}$

2) $\sin z = z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\dots = \sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)!}(-1)^n$

3) $\sinh z$ ~~$\cos z$~~ $= z+\frac{z^3}{3!}+\frac{z^5}{5!}+\dots = \sum\frac{z^{2n+1}}{(2n+1)!}$

(4) $\cos z = 1-\frac{z^2}{2!}+\frac{z^4}{4!}+\dots = \sum(-1)^n\frac{z^{2n}}{(2n)!}$

(5) $\cosh z = 1+\frac{z^2}{2!}+\frac{z^4}{4!}+\dots = \sum\frac{z^{2n}}{(2n)!}$

(6) $\ln(1+z) = z-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}+\dots = \sum(-1)^{n+1}\frac{z^n}{n}$

07) $\ln \frac{1}{1-z} = z + \frac{z^2}{2} + \frac{z^2}{3} + \cdots = \sum_{n=1}^{\infty} \frac{z^n}{n}$.

08) $(1-z)^{-1} = 1 + z + z^2 + \cdots\cdots = \sum z^n$

09) $(1+z)^{-1} = 1 - z + z^2 - z^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n z^n$

10) $(1-z)^{-2} = 1 + 2z + 3z^2 + 4z^3 + \cdots = \sum_{n=1}^{\infty} (n+1)z^n$

11) $(1+z)^{-2} = 1 - 2z + 3z^2 - 4z^3 + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1}(n+1)z^n$

Q) $f(z) = e^z$ about $z=1$; Find Taylor series.

Sol:

$$\boxed{f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n}$$

Method-1:

Let $f(z) = e^z$

We know $f^{(n)}(z) = e^z$, $n = 0, 1, 2 \ldots$

$f^{(n)}(i) = e^i$

TS at $f(z) = e^z$ about $z = i$ is

$$e^z = \sum_{n=0}^{\infty} \frac{f^{(n)} i}{n!} (z-i)^n$$

$$= \sum_{n=0}^{\infty} \frac{e^i}{n!} (z-i)^n$$

Method-2:

Let $z - z_0 = t$

Here $z_0 = i$

$z - i = t$

$z = i + t$

$f(z) = e^z = e^{i+t} = e^i \cdot e^t$

$= e^i \sum_{n=0}^{\infty} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{e^i}{n!} (z-i)^n$

Q) Find TS at $f(z) = \sin z$ about $z = \frac{-\pi}{4}$

Sol

$z - z_0 = t$

$z + \frac{\pi}{4} = t$

$z = t - \pi/4$

$f(z) = \sin z$

$f(z) = \sin(t - \pi/4) = \frac{\sin t}{\sqrt{2}} - \frac{\cos t}{\sqrt{2}} = \frac{\sin t - \cos t}{\sqrt{2}}$

$f(t) = \frac{1}{\sqrt{2}}\left[\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} - \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!}\right]$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{2}} \left[\frac{(z+\pi/4)^{2n+1}}{(2n+1)!} - \frac{(z+\pi/4)^{2n}}{(2n)!}\right]$

Q) TS of $f(z) = \frac{2z+1}{z^2-5z+6}$ about

(i) $z=0$

(ii) $z=1$

(iii) $z=2$

i) $\frac{2z+1}{(z-2)(z-3)}$

TS at $z=2, 3$ doesn't exist as function is not analytic at those pts

$\frac{A}{z-2} + \frac{B}{z-3}$

$A = -5, \; B = 7$

TS at $z=0$,

$f(z) = \frac{-5}{z-2} + \frac{7}{z-3}$

$= \frac{-5}{-2\left(1-\frac{z}{2}\right)} + \frac{7}{-3\left(1-\frac{z}{3}\right)}$

$= \frac{5}{2}\left[1-\frac{z}{2}\right]^{-1} - \frac{7}{3}\left(1-\frac{z}{3}\right)^{-1}$

$= \frac{5}{2}\sum\left[\frac{z}{2}\right]^{n} - \frac{7}{3}\sum\left(\frac{z}{3}\right)^{n}$

$= \sum_{n=0}^{\infty} z^{n}\left[\frac{5}{2^{n+1}} - \frac{7}{3^{n+1}}\right]$

(ii) at $z=1$

$z-1=t$

$z=t+1$

$f(z) = \frac{-5}{z-2} + \frac{7}{z-3}$

$f(z) = \frac{-5}{t+1-2} + \frac{7}{t+1-3}$

$f(t) = \frac{-5}{t-1} + \frac{7}{t-2}$

$f(t) = \frac{-5}{(z-1)-1} + \frac{7}{(z-1)-2}$

$= \frac{-5}{(-1)(1-(z-1))} + \frac{7}{-2\left(1-\frac{(z-1)}{2}\right)}$

$= \frac{5}{1-(z-1)} - \frac{7}{2\left(1-\left(\frac{z-1}{2}\right)\right)}$

$= 5[1-(z-1)]^{-1} - \frac{7}{2}\left[1-\frac{(z-1)}{2}\right]^{-1}$

$= 5\sum(z-1)^{n} - \frac{7}{2}\sum\left(\frac{z-1}{2}\right)^{n}$

$= \sum(z-1)^{n}\left[5 - \frac{7}{2^{n+1}}\right]$

(iii) $z=2$,

We cannot write TS expansion cuz $f(z)$ is not analytic at $z=2$

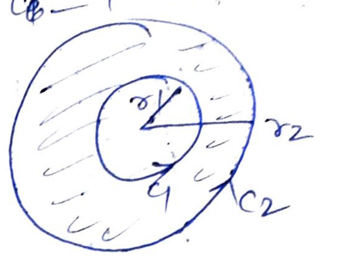
$\therefore$ TS doesn't exist

LAURENT'S THEOREM

Let $f(z)$ be analytic in an annulus region $r_1 \le |z-z_0| \le r_2$ then $f(z)$ Laurent series expansion about $z-z_0$ given by $f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$

$$= \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$$

where $c_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz$ where $C = C_1 \cup C_2$



$$c_n = \begin{cases} a_n, & n = 0, 1, 2, 3 \ldots \\ b_n, & n = -1, -2, -3 \ldots \end{cases}$$

~~$C = C_1 \cup C_2$~~

Note: $\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$ is called principle part.

The series $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ is called analytic part.

$f(z)$ has no singularity inside inner circle C_1 Laurent series becomes TS.

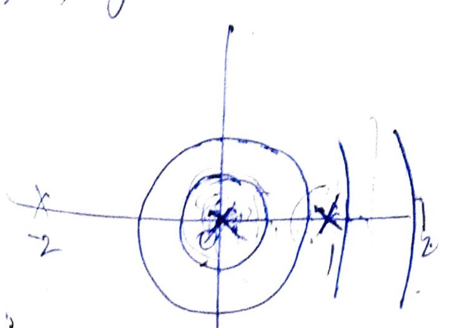
1) Identify all possible regions for the existance of Laurent's series about a given pt z_0 of $f(z)$ where $f(z)$ is $f(z) = \frac{2z-1}{z(z-1)(z+2)}$ and z_0 is (i) 0; (ii) 1; (iii) -2 (iv) 2

Sol i) $z_0 = 0$

a) $0 < |z| < 1$

b) $1 < |z| < 2$

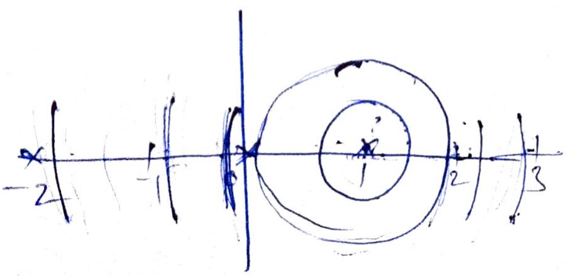
c) $|z| > 2$



ii) $z_0 = 1$ a) $0 < |z-1| < 1$

b) $1 < |z-1| < 3$

c) $|z-1| > 3$



~~(ii) Find the Laurent series of~~

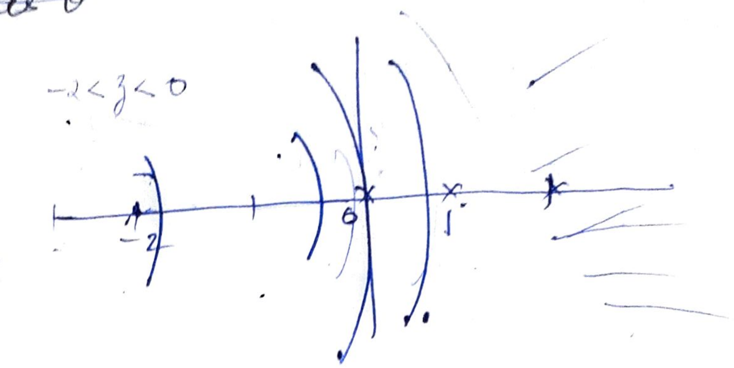
(iii) $z_0 = -2$

$-2 < z < 0$

a) $0 < |z+2| < 2$

b) $2 < |z+2| < 3$

c) $|z+1| > 3$



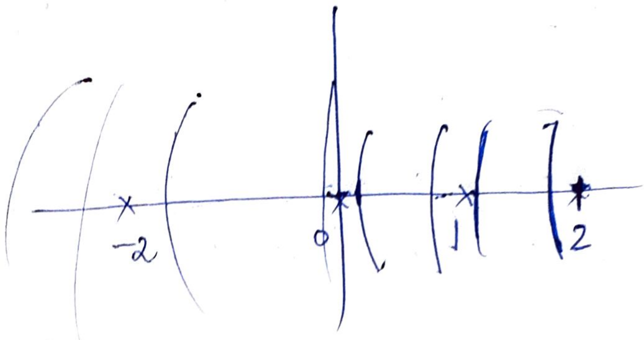
(iv) $z_0 = 2$

a) $|z-2| < 1$

b) $1 < |z-2| < 2$

c) $2 < |z-2| < 4$

d) $|z-2| > 4$



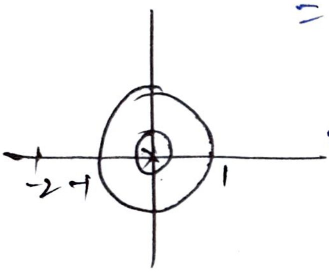
(ii) Find the Laurent series of $f(z)$ in all possible regions where $f(z) = \frac{2z-1}{z(z-1)(z+2)}$ and z_0 is (i) 0, (ii) 2

i) $z_0 = 0$

a) $0 < |z| < 1$

$$f(z) = \frac{(+\frac{1}{2})}{z} + \frac{(\frac{1}{3})}{z-1} + \frac{(\frac{-5}{6})}{z+2}$$

$$\Rightarrow f(z) = \frac{1}{2}\frac{1}{z} + \frac{1}{3}\frac{1}{(-1)(1-z)} - \frac{5}{6}\frac{1}{2(1+\frac{z}{2})}$$



$$= \frac{1}{2}\frac{1}{z} - \frac{1}{3}(1-z)^{-1} - \frac{5}{12}\left(1+\frac{z}{2}\right)^{-1}$$

$$= \frac{1}{2z} - \frac{1}{3}\sum z^n - \frac{5}{12}\sum (-1)^n \left(\frac{z}{2}\right)^n$$

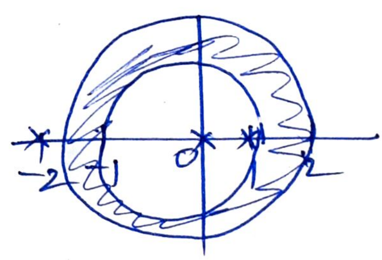
$$= \frac{1}{2z} + \sum_{n=0}^{\infty}\left(-\frac{1}{3} - \frac{(-1)^n}{12}\frac{5}{2^n}\right) z^n$$

$|z| > 1 \rightarrow$ var. common

$|z| < 2 \rightarrow Q \rightarrow$ cont common.

b) $1 < |z| < 2$

$\Rightarrow |z| > 1, \; |z| < 2$



$$f(z) = \frac{1}{2}\frac{1}{z} + \frac{1}{3}\frac{1}{z-1} - \frac{5}{6}\frac{1}{z+2}$$

$$= \frac{1}{2z} + \frac{1}{3z(1-\frac{1}{z})} - \frac{5}{6}\frac{1}{2(1+\frac{z}{2})}$$

Reason:

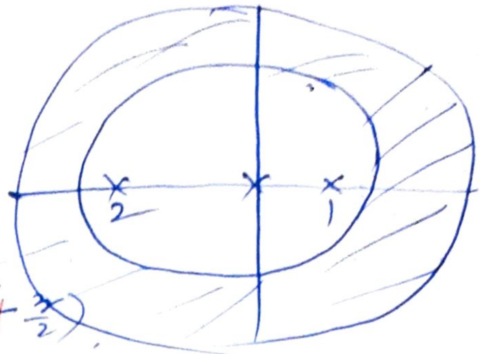
$$= \frac{1}{2z} + \frac{1}{3z}\left(1-\frac{1}{z}\right)^{-1} - \frac{5}{12}\left(1+\frac{z}{2}\right)^{-1}$$

$$= \frac{1}{2z} + \frac{1}{3z}\sum\left(\frac{1}{z}\right)^n - \frac{5}{12}\sum(-1)^n\left(\frac{z}{2}\right)^n$$

$$= \frac{1}{2z} + \sum_{n=0}^{\infty}\frac{1}{3\cdot z^{n+1}} - \sum_{n=0}^{\infty}\frac{5}{12}(-1)^n\frac{z^n}{2^n}$$

(iii) $|z| > 2$

$f(z) = \frac{\frac{1}{2}}{z} + \frac{(\frac{1}{3})}{z-1} + \frac{(-\frac{5}{6})}{z+2}$



$f(z) = \frac{1}{2z} + \frac{1}{3}\frac{1}{z-1} - \frac{5}{6}\frac{1}{z+2}$

$= \frac{1}{2z} + \frac{1}{3z(1-\frac{1}{z})} - \frac{5}{6\cdot 2(1+\frac{z}{2})}$

⇒ Reason why we are taking variable as common is that $\frac{1}{(1-x)}$ exists only for $|x|<1$

If we take const. common, we get $|z|<1$ which is opp to $|z|>1$

given condition ∴ $|\frac{1}{z}|<1$, $|z|>1$

$= \frac{1}{2z} + \frac{1}{3z}\left[1-\frac{1}{z}\right]^{-1} - \frac{5}{12}\left(1+\frac{z}{2}\right)^{-1}$

$= \frac{1}{2z} + \frac{1}{3z}\sum\left(\frac{1}{z}\right)^n - \frac{5}{12}\sum(-1)^n\left(\frac{z}{2}\right)^n$

$= \frac{1}{2z} + \sum_{n=0}^{\infty}\frac{1}{3z^{n+1}} - \sum\frac{5}{12}(-1)^n\frac{z^n}{2^n}$

$|z|>2$ we have only 1 inequality ∴ variable common.

$= \frac{1}{2z} + \frac{1}{3z(1-\frac{1}{z})} - \frac{5}{6}\frac{1}{z(1+\frac{2}{z})}$

$= \frac{1}{2z} + \frac{1}{3z}\left(1-\frac{1}{z}\right)^{-1} - \frac{5}{6z}\left(1+\frac{2}{z}\right)^{-1}$

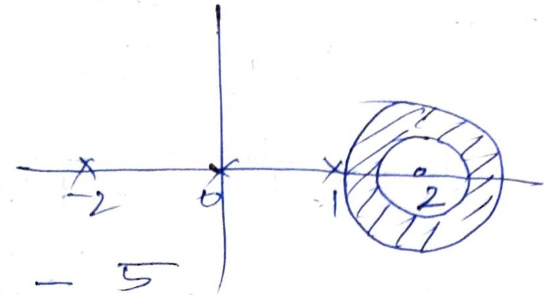
$= \frac{1}{2z} + \frac{1}{3z}\sum\left(\frac{1}{z}\right)^n - \frac{5}{6z}\sum(-1)^n\left(\frac{2}{z}\right)^n$

$= \frac{1}{2z} + \frac{1}{3}\sum_{n=0}^{\infty}\frac{1}{z^{n+1}} - \frac{5}{6}\sum_{n=0}^{\infty}\frac{(-1)^n 2^n}{z^{n+1}}$

(1) $z_0 = 2$

(i) $|z-2|<1$

We want in powers of $z-2$



$$f(z) = \frac{1}{2((z-2)+2)} + \frac{1}{3(z-2+1)} - \frac{5}{6(z-2+4)}$$

$$= \frac{1}{4\left[1+\left(\frac{z-2}{2}\right)\right]} + \frac{1}{3(1+(z-2))} - \frac{5}{24\left(1+\left(\frac{z-2}{4}\right)\right)}$$

$$= \frac{1}{4}\left[1+\left(\frac{z-2}{2}\right)\right]^{-1} + \frac{1}{3}\left[1+(z-2)\right]^{-1} - \frac{5}{24}\left[1+\left(\frac{z-2}{4}\right)\right]^{-1}$$

$$= \frac{1}{4}\sum(-1)^n\left(\frac{z-2}{2}\right)^n + \frac{1}{3}\sum(-1)^n(z-2)^n - \frac{5}{24}\sum(-1)^n\left(\frac{z-2}{4}\right)^n$$

$$= \sum(-1)^n(z-2)^n\left[\frac{1}{4\cdot 2^n} + \frac{1}{3\cdot 1} - \frac{5}{24\cdot 4^n}\right]$$

Verification:

1st term exists in $\left|\frac{z-2}{2}\right|<1$

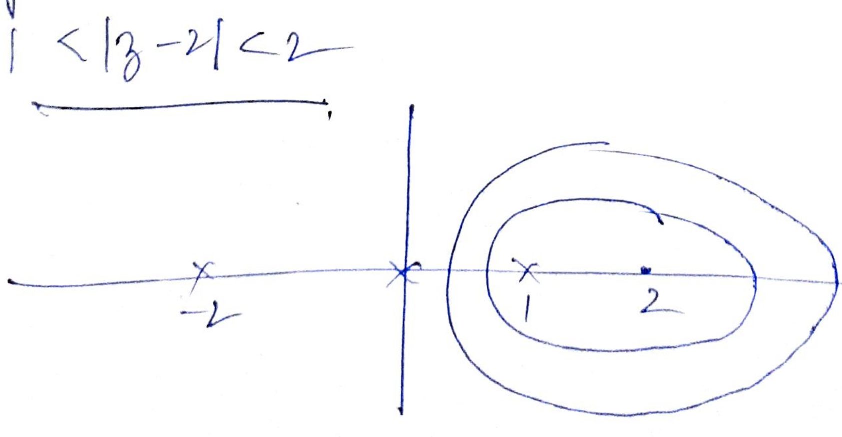
$|z-2|<2$

2nd $-$ $|z-2|<1$

3rd $-$ $|z-2|<4$

$\therefore$ Intersection of 3 regions is $|z-2|<1$ which is given Q.

(ii) $1<|z-2|<2$



$$f(z) = \frac{1}{2(z-2+2)} + \frac{1}{3(z-2+1)} - \frac{5}{6(z-2+4)}$$

$|z-2| < 2 \longrightarrow CC$

$|z-2| > 1 \longrightarrow VC$

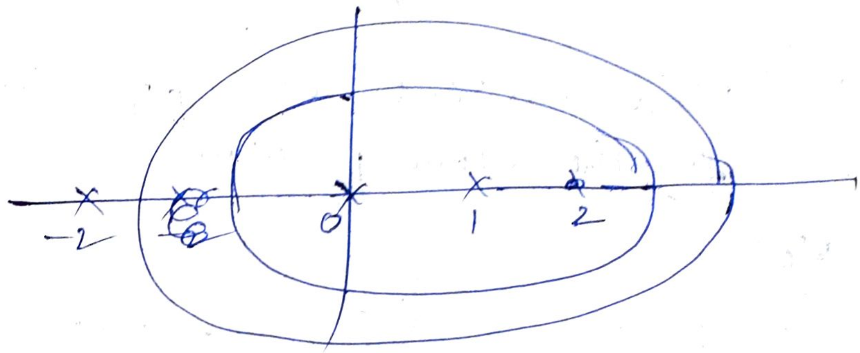
$$= \frac{1}{4\left(1+\frac{z-2}{2}\right)} + \frac{1}{3(z-2)\left(1+\frac{1}{z-2}\right)} - \frac{5}{6(4)\left(1+\frac{z-2}{4}\right)}$$

$$= \frac{1}{4}\Sigma(-1)^n\left(\frac{z-2}{2}\right)^n + \frac{1}{3}\,\frac{1}{z-2}\,\Sigma\left(\frac{1}{z-2}\right)^n(-1)^n - \frac{5}{24}\Sigma\left(\frac{z-2}{4}\right)^n(-1)^n$$

(ii) $\underline{2 < |z-2| < 4}$

$|z-2| > 2$

$|z-2| < 4$

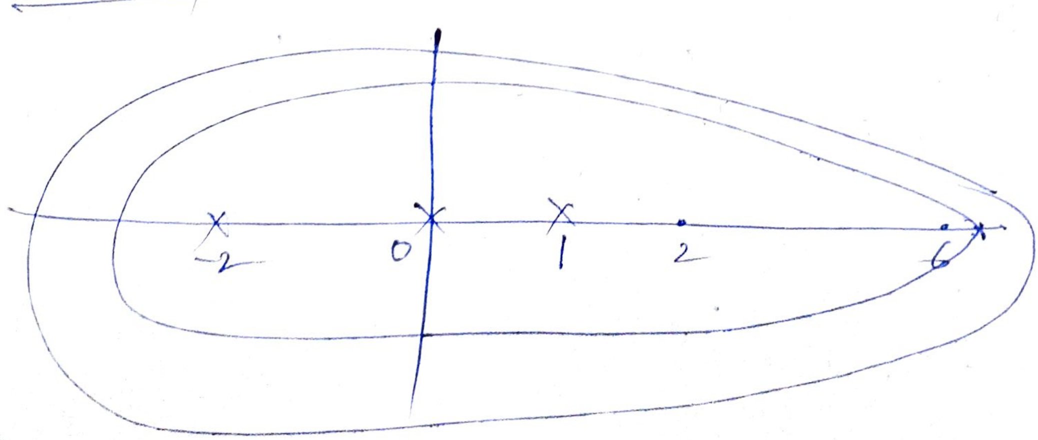


$$f(z) = \frac{1}{2(z-2+2)} + \frac{1}{3(z-2+1)} - \frac{5}{6(z-2+4)}$$

$$= \frac{1}{2(z-2)\left(1+\frac{2}{z-2}\right)} + \frac{1}{3(z-2)\left(1+\frac{1}{z-2}\right)} - \frac{5}{6(4)\left(1+\frac{z-2}{4}\right)}$$

$$= \frac{1}{2(z-2)}\Sigma(-1)^n\left(\frac{2}{z-2}\right)^n + \frac{1}{3(z-2)}\left(1+\frac{1}{z-2}\right)^n - \frac{5}{24}\Sigma(-1)^n\left(\frac{z-2}{4}\right)^n$$

(iii) $\underline{|z-2| > 4}$



$$f(z) = \frac{1}{2(z-2+2)} + \frac{1}{3(z-2+1)} - \frac{5}{6(z-2+4)}$$

$$= \frac{1}{2(z-2)\left(1+\frac{2}{z-2}\right)} + \frac{1}{3(z-2)\left(1+\frac{1}{z-2}\right)} - \frac{5}{6(z-2)\left(1+\frac{4}{z-2}\right)}$$

$$= \frac{1}{2(z-2)} \Sigma (-1)^n \left(\frac{2}{z-2}\right)^n + \frac{1}{3(z-2)} \Sigma (-1)^n \left(\frac{1}{z-2}\right)^n - \frac{5}{6(z-2)} \Sigma (-1)^n \left(\frac{4}{z-2}\right)^n$$

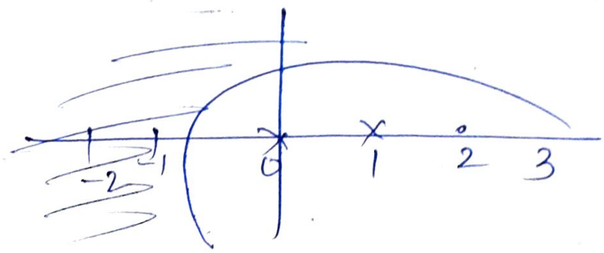
$$= \Sigma \frac{(-1)^n}{(z-2)^{n+1}} \left[2^{n-1} + \frac{1}{3} - \frac{5}{6} 4^n\right]$$

HW

Q) Find Laurent series of $f(z)$ about $z=1$ and $z=-2$ for the above problem

<u>Note:-</u>

If given $|z-1|>2$,
In the Annulus region,
we have singularity,
LT not possible.

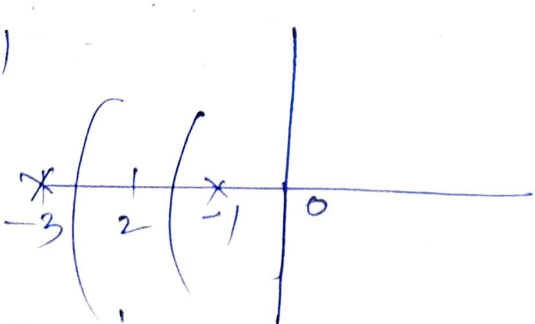


Q) Find LS of $f(z) = \frac{1}{(z+1)(z+3)}$ valid for $1<|z|<3$

Centre - $z_0 = 1$

$1<|z|$, $|z|<3$ $|z|>1$

$\therefore$ LS is possible



$1 = A(z+3) + B(z+1)$

$0 = 3A + 3B$ $\quad A+B=0$

$1 = 3A + 2B$ $\quad \boxed{B=-1,\ A=1}$

$-1 = B$

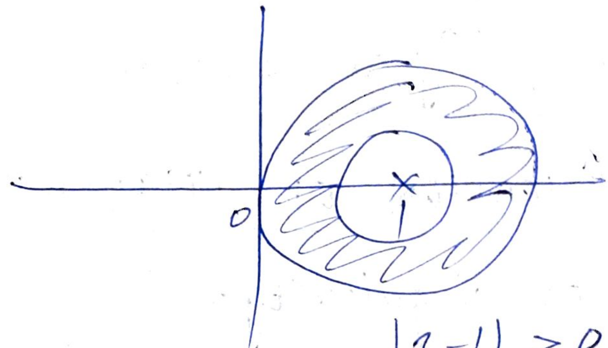
$$f(z) = \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{z+3}\right]$$

$f(z) = \frac{1}{2}\left[\frac{1}{z(1+\frac{1}{z})} - \frac{1}{3(1+\frac{z}{3})}\right]$

$= \frac{1}{2}\left[\frac{(1+\frac{1}{z})^{-1}}{z} - \frac{1}{3}(1+\frac{z}{3})^{-1}\right]$

$= \frac{1}{2z}\Sigma(-1)^n\left(\frac{1}{z}\right)^n - \frac{1}{z}\Sigma(-1)^n\left(\frac{z}{3}\right)^n$

$f(z) = \frac{e^{2z}}{(z-1)^3}$ about $z=1$



$|z-1| > 0$

$f(z) = \frac{1}{(z-1)^3} e^{2[(z-1)+1]}$

$= \frac{e^2}{(z-1)^3} e^{2(z-1)}$

$= \frac{e^2}{(z-1)^3} \Sigma \left[\frac{2(z-1)]^n}{n!}\right.$

$= \sum_{n=0}^{\infty} \frac{e^2 2^n (z-1)^{n-3}}{n!}$

$f(z) = \frac{e^{2z}}{(z-1)^2}$ about $z=0$

$|z|>1$, $|z|<1$

$|z|>1$

e^{+2z} $(z=1)$

$f(z) = \frac{1}{(z-1)^2} e^{2z}$

$= e^{2z} \cdot \frac{1}{z^2(1-\frac{1}{z})^2}$

$= \left(1+2z+\frac{(2z)^2}{2!}+\cdots\right)\frac{1}{z^2}\left(1+2\left(\frac{1}{z}\right)+3\left(\frac{1}{z}\right)^2+\cdots\right)$