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LTCS–Report

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Franz Baader Oliver Fernández Gil

LTCS-Report 24-01

This is an extended version of an article accepted at the 12th International Joint Conference on Automated Reasoning (IJCAR 2024).

Postal Address:
Lehrstuhl für Automatentheorie
Institut für Theoretische Informatik
TU Dresden
01062 Dresden

http://lat.inf.tu-dresden.de

Visiting Address:
Nöthnitzer Str. 46
Dresden

Unification in the Description Logic $\mathcal{ELH}_{\mathcal{R}^+}$ without the Top Concept modulo Cycle-Restricted Ontologies (Extended Version)

Franz Baader[1,2] and Oliver Fernández Gil[1,2]

[1]Theoretical Computer Science, TU Dresden, Germany
[2]Center for Scalable Data Analytics and Artificial Intelligence (ScaDS.AI), Dresden/Leipzig, Germany

Abstract

Unification has been introduced in Description Logic (DL) as a means to detect redundancies in ontologies. In particular, it was shown that testing unifiability in the DL $\mathcal{EL}$ is an NP-complete problem, and this result has been extended in several directions. Surprisingly, it turned out that the complexity increases to PSpace if one disallows the use of the top concept in concept descriptions. Motivated by features of the medical ontology SNOMED CT, we extend this result to a setting where the top concept is disallowed, but there is a background ontology consisting of restricted forms of concept and role inclusion axioms. We are able to show that the presence of such axioms does not increase the complexity of unification without top, i.e., testing for unifiability remains a PSpace-complete problem.

Description Logics (DLs) [10] are a prominent family of logic-based knowledge representation languages, which offer their users a good compromise between expressiveness and complexity of reasoning, and constitute the formal and algorithmic foundation of the standard Web Ontology Language OWL 2.[1] The DL $\mathcal{EL}$, which provides the concept constructors conjunction ($\sqcap$), existential restriction ($\exists r.C$), and top concept ($\top$), is a rather inexpressive, but nevertheless very useful member of this family. On the one hand, the important reasoning problems, such as the subsumption and the equivalence problem, in $\mathcal{EL}$ and some of its extensions are decidable in polynomial time [22, 8]. On the other hand, $\mathcal{EL}$ and its tractable extensions are frequently used to define biomedical ontologies, such as the large medical ontology SNOMED CT.[2] To illustrate the use of the top concept, whose absence plays an important rôle in this paper, consider the $\mathcal{EL}$ concept descriptions $\mathit{Man} \sqcap \exists \mathit{child}.\top$ and $\mathit{Man} \sqcap \exists \mathit{child}.\mathit{Female}$ of the concepts *Father* and *Father of a daughter*, respectively. In the former description, the top concept is used since no further properties of the child are to be required.

Unification in DLs has been introduced in [17] as a new inference service, motivated by the need for detecting redundancies in ontologies, in a setting where different ontology engineers (OEs) constructing the ontology may model the same concepts on different levels of granularity. For example, assume that (using the style of SNOMED CT definitions) one OE models the concept of a *viral infection of the lung* as $\mathit{ViralInfection} \sqcap \exists \mathit{findingSite}.\mathit{LungStructure}$ whereas another one models it as $\mathit{LungInfection} \sqcap \exists \mathit{causativeAgent}.\mathit{Virus}$. Here *ViralInfection* and *LungInfection* are used as atomic concepts without further defining them, i.e., the two OEs made different

[1] `https://www.w3.org/TR/owl2-overview/`
[2] `https://www.ihtsdo.org/snomed-ct/`

decisions when to stop the modelling process. The resulting concept descriptions are not equivalent, but they are nevertheless meant to represent the same concept. They can be made equivalent by treating the concept names *ViralInfection* and *LungInfection* as variables, and then substituting the first one by $\mathit{Infection} \sqcap \exists \mathit{causativeAgent}.\mathit{Virus}$ and the second one by $\mathit{Infection} \sqcap \exists \mathit{findingSite}.\mathit{LungStructure}$. In this case, we say that the descriptions are unifiable, and call the substitution that makes them equivalent a *unifier*. Intuitively, such a unifier proposes definitions for the concept names that are used as variables. In [7], unification and its extension to disunification are used to construct new medical concepts from SNOMED CT.

Unification in $\mathcal{EL}$ was first investigated in [14], where it was proved that deciding unifiability is an NP-complete problem. The NP upper bound was shown in that paper using a brute-force "guess and then test" NP algorithm. More practical algorithms for solving this problem and for computing unifiers were presented in [16] and [15], where the former describes a goal-oriented transformation-based algorithm and the latter is based on a translation to SAT. Implementations of these two algorithms are provided by the system UEL[3] [13], which is also available as a plug-in for the ontology editor Protégé. At the time these algorithms were developed, SNOMED CT was an $\mathcal{EL}$ ontology consisting of acyclic concept definitions. Since such definitions can be encoded into the unification problem (see Section 2.3 in [16]), algorithms for unification of $\mathcal{EL}$ concept descriptions (without background ontology) could be applied to SNOMED CT.

There was, however, one problem with using these algorithms in the context of SNOMED CT: the top concept is not used in SNOMED CT, but the concepts generated by $\mathcal{EL}$ unification might contain $\top$, even if applied to concept descriptions not containing $\top$. Thus, the concept descriptions produced by the unifier are not necessarily in the style of SNOMED CT. For example, assume that we are looking for a unifier satisfying the two subsumption constraints[4] $\exists \mathit{findingSite}.\mathit{LungStructure} \sqsubseteq^? \exists \mathit{findingSite}.X$, $\exists \mathit{findingSite}.\mathit{HeartStructure} \sqsubseteq^? \exists \mathit{findingSite}.X$. It is easy to see that there is only one unifier of these two constraints, which replaces X with $\top$. Unification in $\mathcal{EL}^{-\top}$, i.e., the fragment of $\mathcal{EL}$ in which the top constructor is disallowed, was investigated in [1, 18]. Surprisingly, it turned out that the absence of $\top$ makes unification considerably harder, both from a conceptual and a computational complexity point of view. In fact, the complexity of deciding unifiability increases from NP-complete for $\mathcal{EL}$ to PSpace-complete for $\mathcal{EL}^{-\top}$. The unification algorithm for $\mathcal{EL}^{-\top}$ introduced in [1, 18] basically proceeds as follows. It first applies the unification algorithm for $\mathcal{EL}$ to compute so-called local unifiers. If none of them is an $\mathcal{EL}^{-\top}$-unifier, then it tries to pad the images of the variables by conjoining concept descriptions called particles. The task of finding appropriate particles is reduced to solving certain systems of linear language inclusions, which can be realized in PSpace using an automata-based approach.

The current version of SNOMED CT consists not only of acyclic concept definitions, but also contains more general concept inclusions (GCIs). In addition, properties of the part-of relation are no longer encoded using the so-called SEP-triplet encoding [27], but are directly expressed via role axioms [29], which can, for instance, be used to state that the part-of relation is transitive and that proper-part-of is a subrole of part-of. Decidability of unification in $\mathcal{EL}$ w.r.t. a background ontology consisting of GCIs is still an open problem. In [2], it is shown that the problem remains in NP if the ontology is cycle-restricted, which is a condition that the current version of SNOMED CT satisfies. Extensions of this result to the DL $\mathcal{ELH}_{\mathcal{R}^+}$, which additionally allows for transitive roles and role inclusion axioms, were presented in [5] and [3], where the former introduces a SAT-based algorithm and the latter a transformation-based one. However, in all these algorithms, unifiers may introduce concept descriptions containing $\top$. In

[3] `https://sourceforge.net/projects/uel/`

[4] Instead of equivalence constraints, as in our above example and in early work on unification in DLs, we consider here a set of subsumption constraints as unification problem. It is easy to see that these two kinds of unification problems can be reduced to each other [2].

our example with the different finding site, however, the presence of the GCIs *LungStructure* $\sqsubseteq$ *UpperBodyStructure* and *HeartStructure* $\sqsubseteq$ *UpperBodyStructure* would yield a unifier not using $\top$, namely the one that replaces X with *UpperBodyStructure*.

The purpose of this paper is to combine the approach for unification in $\mathcal{EL}^{-\top}$ [1, 18] with the one for unification in $\mathcal{ELH}_{\mathcal{R}^+}$ w.r.t. cycle-restricted ontologies [2, 5, 3], to obtain a unification algorithm for $\mathcal{ELH}_{\mathcal{R}^+}^{-\top}$ w.r.t. cycle-restricted ontologies. This algorithm follows the line of the one for $\mathcal{EL}^{-\top}$ in that it basically first generates $\mathcal{ELH}_{\mathcal{R}^+}$-unifiers, which it then tries to pad with particles. Appropriate particles are found as solutions of certain linear language inclusions. However, due to the presence of GCIs and role axioms, quite a number of non-trivial changes and additions are required. In particular, the solutions of the systems of linear language inclusions as constructed in [1, 18] cannot capture particles that are appropriate due to the presence of an ontology. For instance, in our example, *UpperBodyStructure* would be such a particle. To repair this problem, we first need to show that, in $\mathcal{ELH}_{\mathcal{R}^+}^{-\top}$, unifiability w.r.t. a cycle-restricted ontology can be characterized by the existence of a special type of unifiers. Afterwards, we exploit the properties of this kind of unifiers to define more sophisticated systems of language inclusions, which encode the semantics of GCIs and role axioms occurring in a background ontology. The solutions of such systems then yield also particles that are appropriate only due to the presence of this ontology.

While the unification problem investigated in this paper is motivated by an application in ontology engineering, it is also of interest for unification theory [19], which is concerned with unification-related properties of equational theories. In fact, unification in DLs can be seen as a special case of unification modulo equational theories, where the respective equational theory axiomatizes equivalence in the DL under consideration. For $\mathcal{EL}$ and $\mathcal{ELH}_{\mathcal{R}^+}$, the corresponding equational theories can be found in [28]. The ones for the case without top can be obtained from them by removing the constant 1 from the signature, and all identities containing it from the axiomatization. The results in [1, 18] and in the present paper show that the seemingly harmless removal of a constant from the equational theory may increase the complexity of the unification problem considerably. Considering unification w.r.t. a background ontology corresponds to adding a finite set of ground identities to the corresponding equational theory. For the word problem, it was shown that decidability is stable under adding finite sets of ground identities to theories such as commutativity or associativity-commutativity [25, 20, 11, 24]. For unification, it was shown in [12] that adding finite sets of ground identities to the theory $ACUI$ of an associativity-commutativity-idempotent symbol with a unit leaves the unification problem decidable. The results in [2, 5, 3] can be seen as such transfer results, but they require a restriction on the ground identities corresponding to cycle-restrictedness.

In the next section, we introduce the DLs $\mathcal{ELH}_{\mathcal{R}^+}$ and $\mathcal{ELH}_{\mathcal{R}^+}^{-\top}$ and the ontologies they can be used to construct, define some important notions such as particles, and recall the recursive characterization of subsumption from [3]. Section 2 introduces unification in $\mathcal{ELH}_{\mathcal{R}^+}$ and $\mathcal{ELH}_{\mathcal{R}^+}^{-\top}$, shows that one can without loss of generality restrict the attention to flat ontologies and unification problems, and defines the notion of cycle-restricted ontologies. In Section 3, we recall the known approaches for unification in $\mathcal{ELH}_{\mathcal{R}^+}$ w.r.t. cycle-restricted ontologies [2, 5] and unification in $\mathcal{EL}^{-\top}$ [1, 18]. Section 4 is devoted to demonstrating our new results. It is divided into two subsections. The first introduces so-called subsumption mappings, and shows how they can be used to reduce unification of $\mathcal{ELH}_{\mathcal{R}^+}^{-\top}$ w.r.t. cycle-restricted ontologies to solving a simpler kind of unification problem. The second subsection then in turn reduces solving this simpler problem to solving certain linear language inclusions. Overall, this yields a PSpace-algorithm for testing unifiability in $\mathcal{ELH}_{\mathcal{R}^+}$ w.r.t. cycle-restricted ontologies. In Section 5, we briefly summarize the obtained results and describe ideas for future research.

1 The Description Logics $\mathcal{ELH}_{\mathcal{R}^+}$ and $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$

In this section, we first define the syntax and semantics of the DLs investigated in this paper. Then, we introduce the notions of atoms and particles, which will play an important rôle in our unification algorithm. To conclude, we recall a useful characterization of subsumption for the two investigated DLs.

1.1 Syntax and Semantics

Let $\mathsf{N_C}$ and $\mathsf{N_R}$ be countably infinite sets of concept names and role names. The set of $\mathcal{ELH}_{\mathcal{R}^+}$-*concept descriptions* (for short, *concepts*) over $\mathsf{N_C}$ and $\mathsf{N_R}$ is inductively defined by using the concept constructors *conjunction* ($\sqcap$), *existential restriction* ($\exists r.C$), and *top* ($\top$) in the following way:

$$C ::= \top \mid A \mid C \sqcap C \mid \exists r.C,$$

where $A \in \mathsf{N_C}$, $r \in \mathsf{N_R}$ and C is an $\mathcal{ELH}_{\mathcal{R}^+}$-concept. The subset of $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-concepts consists of all $\mathcal{ELH}_{\mathcal{R}^+}$-concepts defined without using $\top$, i.e., only conjunction and existential restriction can be used as concept constructors. Most of the definitions and results provided in the rest of this section transfer from $\mathcal{ELH}_{\mathcal{R}^+}$ to $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$. Therefore, we will only formulate them for $\mathcal{ELH}_{\mathcal{R}^+}$. For those where this is not the case, we will explicitly make the distinction.

A *general concept inclusion (GCI)* is an expression of the form $C \sqsubseteq D$ where C and D are $\mathcal{ELH}_{\mathcal{R}^+}$-concepts, a *role hierarchy axiom* is of the form $r \sqsubseteq s$ for role names r and s, and a *transitivity axiom* is of the form $r \circ r \sqsubseteq r$ for a role name r. An $\mathcal{ELH}_{\mathcal{R}^+}$*-ontology* is a finite set $\mathcal{O}$ of GCIs, role hierarchy axioms and transitivity axioms. In an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-ontology, the concepts occurring in GCIs must be $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-concepts. An $\mathcal{EL}$*-ontology* $\mathcal{O}$ contains only GCIs. If the concepts occurring in such GCIs are constructed without using $\top$, then $\mathcal{O}$ is an $\mathcal{EL}^{-\top}$*-ontology*.

The semantics of $\mathcal{ELH}_{\mathcal{R}^+}$-concepts is defined by using standard first-order logic interpretations. An *interpretation* $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ of the symbols in $\mathsf{N_C}$ and $\mathsf{N_R}$ consists of a non-empty domain $\Delta^{\mathcal{I}}$ and an interpretation function $\cdot^{\mathcal{I}}$ that assigns subsets $A^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$ to each concept name $A \in \mathsf{N_C}$, and binary relations $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ to role names $r \in \mathsf{N_R}$. The function $\cdot^{\mathcal{I}}$ is inductively extended to interpret arbitrary $\mathcal{ELH}_{\mathcal{R}^+}$-concepts as follows:

$$\begin{aligned}
\top^{\mathcal{I}} &:= \Delta^{\mathcal{I}},\\
(C \sqcap D)^{\mathcal{I}} &:= C^{\mathcal{I}} \cap D^{\mathcal{I}}, \text{ and}\\
(\exists r.C)^{\mathcal{I}} &:= \{d \in \Delta^{\mathcal{I}} \mid \exists e.((d,e) \in r^{\mathcal{I}} \wedge e \in C^{\mathcal{I}})\}.
\end{aligned}$$

An interpretation $\mathcal{I}$ is a *model* of an $\mathcal{ELH}_{\mathcal{R}^+}$-ontology $\mathcal{O}$ (written $\mathcal{I} \models \mathcal{O}$) if $C \sqsubseteq D \in \mathcal{O}$ implies $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$, $r \sqsubseteq s \in \mathcal{O}$ implies $r^{\mathcal{I}} \subseteq s^{\mathcal{I}}$, and $r \circ r \sqsubseteq r \in \mathcal{O}$ implies that $r^{\mathcal{I}}$ is transitive.

1.2 Atoms and Particles

An $\mathcal{ELH}_{\mathcal{R}^+}$*-atom* is either a concept name or an existential restriction. Every $\mathcal{ELH}_{\mathcal{R}^+}$-concept C consists of a conjunction of $\mathcal{ELH}_{\mathcal{R}^+}$-atoms, where the empty conjunction corresponds to $\top$. These conjuncts are called the *top-level atoms* of C. Note that no $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-concept corresponds to the empty conjunction. Given an $\mathcal{ELH}_{\mathcal{R}^+}$-concept C, we use $\mathit{Ats}(C)$ to denote the set of all atoms (not just top-level ones) occurring in C. Further, given an ontology $\mathcal{O}$, we write $\mathit{Ats}(\mathcal{O})$ to denote the set of atoms of all concepts occurring in $\mathcal{O}$. For example, if $C = \exists r.(\exists s.A \sqcap \exists r.B)$, then $\mathit{Ats}(C) = \{C, \exists s.A, \exists r.B, A, B\}$, where C is the only top-level atom.

A *particle* is an atom of the form $\exists r_1.\exists r_2.\cdots\exists r_n.A$, where $n \geq 0$, $r_1, \ldots, r_n$ are role names and $A \in \mathsf{N_C}$. If $n = 0$ then the particle is just the concept name A. We will often write $\exists w.A$, where $w = r_1 r_2 \ldots r_n$ is viewed as a word over the alphabet $\mathsf{N_R}$, as an abbreviation for $\exists r_1.\exists r_2.\cdots\exists r_n.A$. The set of particles $Part(C)$ of an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-concept C is recursively defined as follows:

$$Part(C) := \begin{cases} C, & \text{if } C \in \mathsf{N_C}, \\ \{\exists r.P \mid P \in Part(D)\}, & \text{if } C = \exists r.D, \\ Part(C_1) \cup Part(C_2), & \text{if } C = C_1 \sqcap C_2. \end{cases}$$

For instance, if $C = \exists r.(\exists s.A \sqcap \exists r.B)$, then $Part(C) = \{\exists rs.A, \exists rr.B\}$.

1.3 Subsumption in $\mathcal{ELH}_{\mathcal{R}^+}$ and $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$

Given an $\mathcal{ELH}_{\mathcal{R}^+}$-ontology $\mathcal{O}$ and $\mathcal{ELH}_{\mathcal{R}^+}$-concepts C, D, we say that C is *subsumed* by D w.r.t. $\mathcal{O}$ (written as $C \sqsubseteq_{\mathcal{O}} D$) if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ for all models $\mathcal{I}$ of $\mathcal{O}$. These concepts are *equivalent* w.r.t. $\mathcal{O}$ (written as $C \equiv_{\mathcal{O}} D$) if $C \sqsubseteq_{\mathcal{O}} D$ and $D \sqsubseteq_{\mathcal{O}} C$. If $\mathcal{O}$ is empty, we often write $C \sqsubseteq D$ and $C \equiv D$ instead of $C \sqsubseteq_{\emptyset} D$ and $C \equiv_{\emptyset} D$, respectively.

Subsumption (and thus also equivalence) between $\mathcal{ELH}_{\mathcal{R}^+}$-concepts w.r.t. arbitrary $\mathcal{ELH}_{\mathcal{R}^+}$-ontologies can be decided in polynomial time [8]. In the context of unification, a recursive characterization of subsumption turns out to be useful. For the case of the empty ontology, the following characterization for subsumption between $\mathcal{ELH}_{\mathcal{R}^+}$-concepts was provided in [16].

Lemma 1.1. *Let $C_1, \ldots, C_n, D_1, \ldots, D_m$ be $\mathcal{ELH}_{\mathcal{R}^+}$-atoms. Then, $C_1 \sqcap \cdots \sqcap C_n \sqsubseteq D_1 \sqcap \cdots \sqcap D_m$ iff for every $j \in \{1, \ldots, m\}$ there is an index $i \in \{1, \ldots, n\}$ such that:*

- *$C_i = D_j$ is a concept name, or*
- *$C_i = \exists r.E$, $D_j = \exists r.F$, and $E \sqsubseteq F$.*

The following result, shown in [18], is an easy consequence of this characterization. It states that the particles subsuming an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-concept C are exactly the particles of C.

Lemma 1.2. *If C is an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-concept and $\exists w.A$ a particle, then $C \sqsubseteq \exists w.A$ iff $\exists w.A \in Part(C)$.*

For arbitrary $\mathcal{ELH}_{\mathcal{R}^+}$-ontologies, a recursive characterization of subsumption was first given in [5], and later reformulated in [3]. In this paper we use the one given in [3], but before we can formulate this characterization, we must introduce the *role hierarchy* induced by an $\mathcal{ELH}_{\mathcal{R}^+}$-ontology $\mathcal{O}$:

- given role names r, s, we say that r is a *subrole* of s (written $r \trianglelefteq_{\mathcal{O}} s$) if $r^{\mathcal{I}} \subseteq s^{\mathcal{I}}$ holds for all models $\mathcal{I}$ of $\mathcal{O}$. We call a role name r *transitive* if $r \circ r \sqsubseteq r \in \mathcal{O}$.

It is easy to see that the relation $\trianglelefteq_{\mathcal{O}}$ is the reflexive-transitive closure of the explicitly stated subrole relationships $\{(r, s) \mid r \sqsubseteq s \in \mathcal{O}\}$. The role hierarchy $\trianglelefteq_{\mathcal{O}}$ can thus be computed in polynomial time in the size of $\mathcal{O}$, by using standard reachability algorithms.

The characterization of subsumption in [3] uses the notion of *structural subsumption*. More precisely, as defined in [5], given atoms C, D, we say that C *is structurally subsumed by* D w.r.t. an $\mathcal{ELH}_{\mathcal{R}^+}$-ontology $\mathcal{O}$ (written $C \sqsubseteq^{s}_{\mathcal{O}} D$) if one of the following cases applies:

1. $C = D$ is a concept name.

2. $C = \exists r.C'$, $D = \exists s.D'$, $r \trianglelefteq_{\mathcal{O}} s$, and $C' \sqsubseteq_{\mathcal{O}} D'$.

3. $C = \exists r.C'$, $D = \exists s.D'$, and $C' \sqsubseteq_{\mathcal{O}} \exists t.D'$ for some transitive role name t satisfying $r \trianglelefteq_{\mathcal{O}} t \trianglelefteq_{\mathcal{O}} s$.

It is easy to see that $C \sqsubseteq D$ implies $C \sqsubseteq^s_{\mathcal{O}} D$, which in turn implies that $C \sqsubseteq_{\mathcal{O}} D$.

Lemma 1.3. *Let $\mathcal{O}$ be an $\mathcal{ELH}_{\mathcal{R}^+}$-ontology and $C_1, \dots, C_n, D_1, \dots, D_m$ $\mathcal{ELH}_{\mathcal{R}^+}$-atoms. Then, $C_1 \sqcap \dots \sqcap C_n \sqsubseteq_{\mathcal{O}} D_1 \sqcap \dots \sqcap D_m$ iff for every $j \in \{1, \dots, m\}$:*

1. *there is an index $i \in \{1, \dots, n\}$ such that $C_i \sqsubseteq^s_{\mathcal{O}} D_j$, or*
2. *there are atoms $At_1, \dots, At_k, At'$ of $\mathcal{O}$ ($k \geq 0$) such that:*
   (a) *$At_1 \sqcap \dots \sqcap At_k \sqsubseteq_{\mathcal{O}} At'$,*
   (b) *for every $\ell \in \{1, \dots, k\}$ there exists $i \in \{1, \dots, n\}$ with $C_i \sqsubseteq^s_{\mathcal{O}} At_\ell$, and*
   (c) *$At' \sqsubseteq^s_{\mathcal{O}} D_j$.*

If $\mathcal{O}$ is empty, then the second case in the definition of structural subsumption can be modified to require that $r = s$ and $C' \sqsubseteq D'$, whereas the third case in the same definition as well as the second case in Lemma 1.3 can be removed. This then yields the characterization of subsumption w.r.t. the empty ontology from Lemma 1.1. Since $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$ is a fragment of $\mathcal{ELH}_{\mathcal{R}^+}$, this characterization also applies to subsumption between $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-concepts w.r.t. $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-ontologies. However, in this setting, the case $k = 0$ in 2. cannot occur. This is a direct consequence of the following result.

Lemma 1.4. *Let $\mathcal{O}$ be an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-ontology and let At be an atom of $\mathcal{O}$. Then, $\top \sqsubseteq_{\mathcal{O}} At$ does not hold.*

Proof. Let $\mathcal{I}$ be an arbitrary model of $\mathcal{O}$. We extend $\mathcal{I}$ by adding a new element d to $\Delta^{\mathcal{I}}$ without changing the interpretation function $\cdot^{\mathcal{I}}$. This means that

- $d \notin A^{\mathcal{I}}$ for all $A \in \mathsf{N_C}$, and
- $r^{\mathcal{I}}$ does not contain a pair of the form (d, e) nor (e, d) for all $r \in \mathsf{N_R}$.

This implies that $d \notin At^{\mathcal{I}}$. Moreover, since $\mathcal{O}$ contains no occurrence of $\top$, d trivially satisfies all GCIs in $\mathcal{O}$. Hence, $\mathcal{I}$ is still a model of $\mathcal{O}$. Thus, $\top \sqsubseteq_{\mathcal{O}} At$ cannot not hold since $d \in \top^{\mathcal{I}}$. $\square$

2 Unification in $\mathcal{ELH}_{\mathcal{R}^+}$ and $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$

We now define the unification problem for the DLs $\mathcal{ELH}_{\mathcal{R}^+}$ and $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$. After providing the formal definition of the problem, we recall the notions of flat ontologies and flat unification problems. We conclude the section with the definition of cycle-restricted ontologies, which are the type of ontologies we investigate in this paper.

2.1 The Unification Problem

To define the unification problem, we assume that the set of concept names is partitioned into a set $\mathsf{N_C}$ of concept constants and a set $\mathsf{N_V}$ of concept variables. Given a DL $\mathcal{L} \in$

$\{\mathcal{ELH}_{\mathcal{R}^+}, \mathcal{ELH}^{-\top}_{\mathcal{R}^+}\}$, an *$\mathcal{L}$-substitution* σ is a mapping from a finite subset of $\mathsf{N_V}$ to the set of $\mathcal{L}$-concepts. The application of σ to an arbitrary $\mathcal{L}$-concept over $\mathsf{N_R}$ and $\mathsf{N_C} \cup \mathsf{N_V}$ is defined inductively in the usual way. A concept (ontology) is *ground* if it does not contain variables. A substitution σ is ground if $\sigma(X)$ is ground for all variables X that have an image under σ.

Definition 2.1. Let $\mathcal{O}$ be a ground ontology. An *$\mathcal{L}$-unification problem* w.r.t. $\mathcal{O}$ is of the form $\Gamma = \{C_1 \sqsubseteq^? D_1, \dots, C_n \sqsubseteq^? D_n\}$, where $C_1, D_1, \dots, C_n, D_n$ are $\mathcal{L}$-concepts. An $\mathcal{L}$-substitution σ is an *$\mathcal{L}$-unifier* of Γ w.r.t. $\mathcal{O}$ iff $\sigma(C_i) \sqsubseteq_{\mathcal{O}} \sigma(D_i)$ for all $i \in \{1, \dots, n\}$. The unification problem Γ is called *$\mathcal{L}$-unifiable* w.r.t. $\mathcal{O}$ if it has an $\mathcal{L}$-unifier w.r.t. $\mathcal{O}$.

The use of *subsumption constraints* $C \sqsubseteq^? D$ instead of equations of the form $C \equiv^? D$ (which are the ones usually employed to define unification in DLs) is without loss of generality, since

$$\begin{aligned} C \equiv_{\mathcal{O}} D \quad &\text{iff} \quad C \sqsubseteq_{\mathcal{O}} D \text{ and } D \sqsubseteq_{\mathcal{O}} C, \text{ and} \\ C \sqsubseteq_{\mathcal{O}} D \quad &\text{iff} \quad C \equiv_{\mathcal{O}} C \sqcap D. \end{aligned}$$

The restriction to ground ontologies is not without loss of generality. A discussion about why this restriction is appropriate can be found in [2].

The following example illustrates that unifiability of a given unification problem may depend on the considered DL $\mathcal{L}$ and on the presence of a non-empty ontology.

Example 2.2. Let $\mathcal{O} = \emptyset$ and consider the following unification problem:

$$\Gamma_1 := \{\exists r.A \sqsubseteq^? X,\ \ \exists u.B \sqsubseteq^? Y,\ \ \exists s.X \sqcap A \sqsubseteq^? Y\}.$$

Viewed as an $\mathcal{ELH}_{\mathcal{R}^+}$-unification problem, it has the unifier σ with $\sigma(X) = \sigma(Y) = \top$. However, Γ_1 does not have an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unifier w.r.t. $\mathcal{O} = \emptyset$. To see this, suppose that δ is such a unifier. Using Lemma 1.1, we can deduce from $\exists u.B \sqsubseteq \delta(Y)$ that every top-level atom of $\delta(Y)$ is an existential restriction for the role u. However, we can also deduce from $\exists s.\delta(X) \sqcap A \sqsubseteq \delta(Y)$ that every top-level atom of $\delta(Y)$ is either A or an existential restriction for the role s. Since not both is possible, $\delta(Y)$ cannot have any top-level atoms, and thus must be $\top$, contradicting our assumption that δ is an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unifier.

The unifiability status of Γ_1 can change in the presence of an ontology. For instance, Γ_1 does have an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unifier w.r.t. the following $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-ontology:

$$\mathcal{O}' := \{B \sqsubseteq \exists r.A,\ u \sqsubseteq s\}.$$

It is not hard to verify that the following $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-substitution δ is a unifier of Γ_1 w.r.t. $\mathcal{O}'$:

$$\delta := \{X \mapsto \exists r.A,\ \ Y \mapsto \exists s.\exists r.A\}.$$

In particular, the ontology ensures that δ solves the *second* constraint. In fact, although $\exists u.B \sqsubseteq \exists s.\exists r.A$ does not hold (i.e., w.r.t. the empty ontology), the axioms $u \sqsubseteq s$ and $B \sqsubseteq \exists r.A$ in $\mathcal{O}'$ imply that $u \trianglelefteq_{\mathcal{O}'} s$ and $B \sqsubseteq_{\mathcal{O}'} \exists r.A$. Thus, it follows that $\exists u.B \sqsubseteq_{\mathcal{O}'} \exists s.B \sqsubseteq_{\mathcal{O}'} \exists s.\exists r.A = \delta(Y)$. $\triangle$

The *$\mathcal{L}$-unification decision problem* asks, given an $\mathcal{L}$-unification problem Γ and an ontology $\mathcal{O}$, whether Γ has an $\mathcal{L}$-unifier w.r.t. $\mathcal{O}$. There are two assumptions one can make regarding the form of the input and solutions of this decision problem. The first assumption tells us that a decision procedure for unifiability needs to search only for unifiers of a particular form:

- it is enough to consider ground $\mathcal{L}$-substitutions σ defined over the concept names and role names occurring in Γ or $\mathcal{O}$. In fact, as mentioned in the introduction, unification in

$\mathcal{ELH}_{\mathcal{R}^+}$ and $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$ can be seen as unification modulo an equational theory, where the corresponding equational theory consists of a finite set axiomatizing equivalence in these DLs (as defined in [28]) plus a finite set of ground identities representing the GCIs in $\mathcal{O}$. It then follows from well-known results in unification theory [19] that, if Γ is $\mathcal{L}$-unifiable w.r.t. $\mathcal{O}$, then it has an $\mathcal{L}$-unifier of the aforementioned form.[5]

Based on this, we assume in the following that $\mathsf{N_C}$ is the set of concept constants occurring in Γ or $\mathcal{O}$, and $\mathsf{N_R}$ is the set of role names occurring in Γ or $\mathcal{O}$, where we can assume without loss of generality that there is at least one concept constant. To simplify the technical details and development of our unification algorithm, we can without loss of generality also assume that the input ontology and unification problem are *flat*.

2.2 Flat Ontologies and Unification Problems

An $\mathcal{ELH}_{\mathcal{R}^+}$-atom is *flat* if it is a concept name or of the form $\exists r.C$, where C is a concept name or $\top$. This notion adapts to $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$ by restricting C (in $\exists r.C$) to be a concept name. An $\mathcal{ELH}_{\mathcal{R}^+}$-ontology $\mathcal{O}$ is called flat, if it only contains GCIs of the form $C_1 \sqcap \cdots \sqcap C_n \sqsubseteq D$, where $C_1, \ldots, C_n$ are flat $\mathcal{ELH}_{\mathcal{R}^+}$-atoms or $\top$ and D is a flat $\mathcal{ELH}_{\mathcal{R}^+}$-atom. This notion can naturally be adapted to $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-ontologies by requiring that $C_1, \ldots, C_n$ and D are flat $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-atoms.

As shown in [5, 6], by using the normalization procedure described in [9], every $\mathcal{ELH}_{\mathcal{R}^+}$-ontology can be transformed in polynomial time into an ontology in flat form. The role axioms in the resulting $\mathcal{ELH}_{\mathcal{R}^+}$-ontology remain unchanged, whereas the GCIs in the normalized ontology are of the form:

$$A \sqsubseteq B, \quad A_1 \sqcap A_2 \sqsubseteq B, \quad \exists r.A \sqsubseteq B, \quad B \sqsubseteq \exists r.A, \tag{1}$$

where A, A_1, A_2 and B are concept names or $\top$. Furthermore, unless $\top$ occurs in the given ontology, no rule application of this normalization procedure generates a GCI containing $\top$. Thus, the application of this procedure to an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-ontology takes polynomial time, and yields a flat $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-ontology consisting of a set of GCIs of the form in (1), where A, A_1, A_2 and B are concept names.

Given a DL $\mathcal{L} \in \{\mathcal{ELH}_{\mathcal{R}^+}, \mathcal{ELH}^{-\top}_{\mathcal{R}^+}\}$, an $\mathcal{L}$-unification problem is called *flat*, if it consists of subsumption constraints of the form $C_1 \sqcap \cdots \sqcap C_n \sqsubseteq^? D$, where $C_1, \ldots, C_n$ and D are flat $\mathcal{L}$-atoms. By introducing new concept variables, every $\mathcal{ELH}_{\mathcal{R}^+}$-unification problem Γ can be flatten in polynomial time, and this transformation stays within $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$ if applied to an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unification problem (see [14, 18]).

As shown in [5] for $\mathcal{ELH}_{\mathcal{R}^+}$, given an $\mathcal{ELH}_{\mathcal{R}^+}$-unification problem Γ and an $\mathcal{ELH}_{\mathcal{R}^+}$-ontology $\mathcal{O}$, the unification problem Γ' and the ontology $\mathcal{O}'$ obtained by flattening Γ and $\mathcal{O}$ are such that Γ is unifiable w.r.t. $\mathcal{O}$ iff Γ' is unifiable w.r.t. $\mathcal{O}'$. This result also applies to $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$. More precisely,

- as explained above, if Γ and $\mathcal{O}$ are formulated in $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$, the applications of the corresponding flattening procedures yield (in polynomial time) a flat $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unification problem Γ' and a flat $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-ontology $\mathcal{O}'$;
- as shown in the technical report [6], an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unifier σ of Γ w.r.t. $\mathcal{O}$ can be extended into an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unifier of Γ' w.r.t. $\mathcal{O}'$, by extending σ with appropriate definitions for the auxiliary variables introduced to obtain Γ'. Conversely, an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unifier σ' of Γ' w.r.t.

[5] In case of $\mathcal{L} = \mathcal{ELH}^{-\top}_{\mathcal{R}^+}$, we may need an additional concept constant A when $\mathcal{O}$ is empty and Γ does not contain any occurrence of a concept constant. However, we can assume without loss of generality that Γ contains a concept constant by adding the trivial subsumption constraint $A \sqsubseteq^? A$ to it if it does not.

$\mathcal{O}'$ can always be transformed into an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unifier of Γ w.r.t. $\mathcal{O}$ (this can be concluded from the proof of Lemma 17 in [6]).

Overall, we can without loss of generality assume that the given ontology and unification problem are both flat.

2.3 Cycle-Restricted Ontologies

In this paper, we investigate unification w.r.t. cycle-restricted ontologies, which were first introduced in [2]. This is motivated by the fact that, without this restriction, it is not even clear whether unification is decidable.

Definition 2.3. An $\mathcal{ELH}_{\mathcal{R}^+}$-ontology $\mathcal{O}$ is called *cycle-restricted* if there is no sequence of $n > 0$ role names $r_1, \dots, r_n \in \mathsf{N_R}$ and $\mathcal{ELH}_{\mathcal{R}^+}$-concept C such that $C \sqsubseteq_{\mathcal{O}} \exists r_1.\exists r_2.\cdots\exists r_n.C$.

As stated in [5] (and proved in the corresponding technical report [6]), one can test in polynomial time whether a given $\mathcal{ELH}_{\mathcal{R}^+}$-ontology is cycle-restricted or not. In addition, if the input ontology is cycle-restricted, then so is the flat ontology obtained by applying the normalization procedure mentioned above to it (see the proof of Lemma 21 in [6]).

The following result for flat, cycle-restricted $\mathcal{ELH}_{\mathcal{R}^+}$-ontologies will turn out to be quite useful later on to obtain our results. It basically follows from the proof of Lemma 8 in [4].

Lemma 2.4. *Let $\mathcal{O}$ be a flat, cycle-restricted $\mathcal{ELH}_{\mathcal{R}^+}$-ontology, $A \in \mathsf{N_C}$ and $\exists r.C$ an $\mathcal{ELH}_{\mathcal{R}^+}$-atom. Then, $A \sqsubseteq_{\mathcal{O}} \exists r.C$ iff there exists $\exists u.B \in Ats(\mathcal{O})$ such that $B \sqsubseteq_{\mathcal{O}} C$, and*

- *$A \sqsubseteq_{\mathcal{O}} \exists u.B$ and $u \trianglelefteq_{\mathcal{O}} r$, or*
- *$A \sqsubseteq_{\mathcal{O}} \exists t.B$ for a transitive role t with $u \trianglelefteq_{\mathcal{O}} t \trianglelefteq_{\mathcal{O}} r$.*

In Section 4, we will show how to decide unifiability of an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unification problem w.r.t. a cycle-restricted $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-ontology. Before doing that, we recall (in the next section) the existing results on unification in $\mathcal{ELH}_{\mathcal{R}^+}$ and $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$, and briefly describe the techniques employed to obtain these results.

3 Known results for unification in $\mathcal{ELH}_{\mathcal{R}^+}$ and $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$

The decision problem for unification in $\mathcal{ELH}_{\mathcal{R}^+}$ has been investigated in [14, 16, 2, 5, 3]. It was first established that unification in the DL $\mathcal{EL}$ w.r.t. the empty ontology is NP-complete [14, 16]. This result was later extended to unification w.r.t. cycle-restricted $\mathcal{EL}$-ontologies [2], and then further to unification w.r.t. cycle-restricted $\mathcal{ELH}_{\mathcal{R}^+}$-ontologies [5, 3]. For the DL $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$, unification has only been studied in the context of $\mathcal{EL}^{-\top}$ for the case of the empty ontology. Compared to $\mathcal{EL}$, the complexity of the decision problem increases: deciding unifiability in $\mathcal{EL}^{-\top}$ is PSpace-complete [18] rather than NP-complete. We continue with briefly describing the techniques used to obtain these results.

3.1 Deciding Unification in $\mathcal{ELH}_{\mathcal{R}^+}$

The main idea employed to obtain the "in NP" result is to show that, in $\mathcal{ELH}_{\mathcal{R}^+}$, unifiability w.r.t. cycle-restricted ontologies implies the existence of *local unifiers*. Based on this, a corresponding NP-procedure guesses an appropriate representation of a local substitution, and then checks by $\mathcal{ELH}_{\mathcal{R}^+}$ reasoning whether it really is a unifier. Basically, to guess a local substitution σ, one must guess for every variable X and non-variable atom C of Γ and $\mathcal{O}$ whether $\sigma(X) \sqsubseteq_{\mathcal{O}} \sigma(C)$ is supposed to hold. In the following, we introduce these ideas in a more formal way.

Given a unification problem Γ, we write $\mathit{Vars}(\Gamma)$ to denote the set of variables occurring in Γ. The atoms of Γ are the atoms of the concepts occurring in Γ. We denote the set of all such atoms as $\mathit{Ats}(\Gamma)$. For simplicity, given an ontology $\mathcal{O}$, we will write $\mathit{Ats}(\Gamma, \mathcal{O})$ to denote the set $\mathit{Ats}(\Gamma) \cup \mathit{Ats}(\mathcal{O})$. Furthermore, due to the third case in the definition of structural subsumption, we also need to consider certain atoms that are not explicitly present in the input of an $\mathcal{ELH}_{\mathcal{R}^+}$-unification problem:

$$\mathit{Ats}_{tr}(\Gamma, \mathcal{O}) := \mathit{Ats}(\Gamma, \mathcal{O}) \cup \{\exists t.C \mid \exists s.C \in \mathit{Ats}(\Gamma, \mathcal{O}),\ t \trianglelefteq_{\mathcal{O}} s,\ t \text{ is a transitive role}\}.$$

A *non-variable atom* is an atom in $\mathit{Ats}_{tr}(\Gamma, \mathcal{O})$ that is not a variable. We denote the set of all such atoms as $\mathit{At}_{nv}(\Gamma, \mathcal{O})$, i.e.,

$$\mathit{At}_{nv}(\Gamma, \mathcal{O}) := \mathit{Ats}_{tr}(\Gamma, \mathcal{O}) \setminus \mathit{Vars}(\Gamma).$$

Let S be an assignment mapping each variable in Γ to a set of non-variable atoms from $\mathit{Ats}_{tr}(\Gamma, \mathcal{O})$. The assignment S induces the following binary relation:

$$>_S := \{(X, Y) \in \mathit{Vars}(\Gamma) \times \mathit{Vars}(\Gamma) \mid Y \text{ occurs in an atom of } S(X)\}$$

Let $>_S^+$ be the *transitive closure* of $>_S$. We say that S is *acyclic* if $>_S^+$ is *irreflexive*, and thus a strict partial order. If S is acyclic, then it induces a substitution σ_S, defined by induction on $>_S^+$ as follows:

- If X is minimal w.r.t. $>_S^+$, then $\sigma_S(X) := \bigsqcap_{D \in S(X)} D$.
- Otherwise, assuming that $\sigma_S(Y)$ has already been defined for all Y such that $X >_S^+ Y$, one defines $\sigma_S(X) := \bigsqcap_{D \in S(X)} \sigma_S(D)$.

A substitution σ is called *local*, if there exists an acyclic assignment S such that $\sigma = \sigma_S$. A unifier σ of Γ w.r.t. an ontology $\mathcal{O}$ is called a *local unifier* if it is a local substitution.

Example 3.1. Let $\mathcal{O} = \emptyset$ and consider the following unification problem:

$$\Gamma_2 := \{X \sqsubseteq^? A,\ \ X \sqsubseteq^? \exists r.Z,\ \ \exists r.B \sqcap \exists s.X \sqsubseteq^? Y,\ \ \exists s.A \sqcap \exists r.Z \sqsubseteq^? Y,\ \ \exists r.A \sqcap \exists r.B \sqsubseteq^? \exists r.Z\}.$$

It is easy to see that the substitution $\sigma := \{X \mapsto A \sqcap \exists r.\top,\ Y \mapsto \top,\ Z \mapsto \top\}$ is a unifier of Γ_2 w.r.t. $\mathcal{O}$. Moreover, σ is also a local unifier. In fact, the assignment S defined as

$$S(X) := \{\exists r.Z, A\},\quad S(Y) := \emptyset,\quad S(Z) := \emptyset$$

is acyclic and induces the substitution $\sigma_S = \sigma$. $\triangle$

Theorem 3.2 ([5])**.** *Let Γ be a flat $\mathcal{ELH}_{\mathcal{R}^+}$-unification problem and $\mathcal{O}$ a flat cycle-restricted $\mathcal{ELH}_{\mathcal{R}^+}$-ontology. If Γ is unifiable w.r.t. $\mathcal{O}$, then it has a local unifier w.r.t. $\mathcal{O}$.*

Thus, the NP-decision procedure for unification in $\mathcal{ELH}_{\mathcal{R}^+}$ w.r.t. cycle-restricted ontologies works as follows. It first guesses an assignment S. If S is not acyclic, then the procedure fails. Otherwise, it checks whether the induced substitution σ_S is a unifier of Γ w.r.t. $\mathcal{O}$. Testing whether σ_S is a unifier in polynomial time requires some care, since $\sigma_S(X)$ may be a concept of exponential size. The main idea is to represent these concepts using a polynomial-size ontology (for details see [5]).

3.2 Deciding Unification in $\mathcal{EL}^{-\top}$

We now describe the idea underlying the PSpace algorithm devised in [18] for unification in $\mathcal{EL}^{-\top}$ w.r.t. the empty ontology. Given an $\mathcal{EL}^{-\top}$-unification problem Γ, the overall idea is to

- guess a local $\mathcal{EL}$-unifier σ of Γ, and
- if σ is not already an $\mathcal{EL}^{-\top}$-substitution, then try to extend σ into an $\mathcal{EL}^{-\top}$-unifier δ of Γ, by conjoining particles to the concepts $\sigma(X)$, where $X \in \mathit{Vars}(\Gamma)$.

To be more precise, instead of specifically guessing a local substitution, the algorithm guesses a more general ***subsumption mapping for*** Γ. This is a mapping of the form $\tau : \mathit{Ats}(\Gamma) \times \mathit{Ats}(\Gamma) \mapsto \{0, 1\}$, which is required to satisfy a series of properties. This mapping plays the following rôles:

- It describes a local $\mathcal{EL}$-unifier of Γ, i.e., each such mapping τ induces the assignment:

$$S^\tau(X) := \{D \in \mathit{At}_{nv}(\Gamma) \mid \tau(X, D) = 1\} \quad (\text{for all } X \in \mathit{Vars}(\Gamma)), \tag{2}$$

  and this assignment is required to be acyclic. Acyclicity of S^τ is defined as in Section 3.1, i.e., the assignment S^τ induces the binary relation $>_{S^\tau}$ between the variables of Γ (as described above to introduce local unifiers), whose transitive closure we denote as $>_\tau$. We say that S^τ is *acyclic* in case $>_\tau$ is irreflexive. If S^τ is acyclic, then it induces a substitution σ_{S^τ}, defined by induction on $>_\tau$ as described above. For simplicity, we will denote σ_{S^τ} as σ^τ. The other properties required of τ ensure that σ^τ is indeed an $\mathcal{EL}$-unifier of Γ.[6]

- It specifies other subsumption relations between atoms of Γ that should hold for the $\mathcal{EL}^{-\top}$-unifier δ the algorithm tries to generate from σ^τ. This means that if $\tau(D_1, D_2) = 1$ for some $D_1, D_2 \in \mathit{Ats}(\Gamma)$, then the search for δ can be restricted to substitutions satisfying $\delta(D_1) \sqsubseteq \delta(D_2)$.

As already mentioned, the local unifier obtained from a subsumption mapping τ need not be an $\mathcal{EL}^{-\top}$-unifier. To test for the existence of an $\mathcal{EL}^{-\top}$-unifier related to τ, the subsumption mapping τ together with the original unification problem Γ is then used to construct a new unification problem $\Delta_{\Gamma,\tau}$, in which only variables can occur on the right-hand side of subsumption constraints. This set is defined as $\Delta_{\Gamma,\tau} := \Delta_\Gamma \cup \Delta_\tau$, where

$$\Delta_\Gamma := \{C_1 \sqcap \cdots \sqcap C_n \sqsubseteq^? X \in \Gamma\} \quad \text{and} \quad \Delta_\tau := \{C \sqsubseteq^? X \mid \tau(C, X) = 1\}. \tag{3}$$

Example 3.3. Recall the unification problem Γ_2 and the local $\mathcal{EL}$-unifier σ from Example 3.1. Following (2) and the assignment S from Example 3.1, a subsumption mapping τ representing σ must satisfy:

$$\tau(X, A) = 1, \quad \tau(X, \exists r.Z) = 1 \ \text{ and } \ \tau(Y, At) = \tau(Z, At) = 0 \ (\text{for all } At \in \mathit{At}_{nv}(\Gamma)).$$

In addition, suppose that τ satisfies $\tau(\exists r.B, \exists r.Z) = 1$. This tells the algorithm, for instance, how the unifier δ should solve the subsumption constraint $\exists r.A \sqcap \exists r.B \sqsubseteq^? \exists r.Z \in \Gamma_2$, i.e., the algorithm will search for an $\mathcal{EL}^{-\top}$-substitution δ such that $\exists r.B \sqsubseteq \exists r.\delta(Z)$. Moreover, since τ captures the properties of the subsumption relation, it must set $\tau(B, Z) = 1$ since

[6] These properties are based on the properties of subsumption w.r.t. the empty ontology. Their precise definitions can be found in [18].

$\exists r.B \sqsubseteq \exists r.\delta(Z)$ implies $B \sqsubseteq \delta(Z)$. Consequently, τ transforms Γ_2 into the unification problem $\Delta_{\Gamma_2,\tau} := \Delta_{\Gamma_2} \cup \Delta_\tau$, where

$$\Delta_{\Gamma_2} := \{\exists r.B \sqcap \exists s.X \sqsubseteq^? Y,\ \exists s.A \sqcap \exists r.Z \sqsubseteq^? Y\} \quad \text{and} \quad \Delta_\tau := \{B \sqsubseteq^? Z\}.$$

The next task is then to solve $\Delta_{\Gamma_2,\tau}$ in a way that is compatible with τ. △

As shown in [18], Γ has an $\mathcal{EL}^{-\top}$-unifier iff there exists a subsumption mapping τ such that $\Delta_{\Gamma,\tau}$ has an $\mathcal{EL}^{-\top}$-unifier that is compatible with τ (we will formally define in Definition 4.2 what "compatible with" means). Thus, one can restrict the attention to solving unification problems where only concept variables occur on the right-hand side of the constraints. The advantage of this is that checking existence of an $\mathcal{EL}^{-\top}$-unifier of $\Delta_{\Gamma,\tau}$ that is compatible with τ can be reduced to checking existence of an admissible solution of a corresponding set $\mathfrak{I}_{\Gamma,\tau}$ of linear language inclusions. In the following, we briefly describe this reduction.

Definition 3.4. Let $X_1, \ldots, X_n$ be a finite set of *indeterminates*. A *linear language inclusion* over this set of indeterminates and the alphabet $\mathsf{N_R}$ is an expression of the form

$$X_i \subseteq L_0 \cup L_1 X_1 \cup \cdots \cup L_n X_n,$$

where $i \in \{1, \ldots, n\}$ and $L_j \subseteq \{\varepsilon\} \cup \mathsf{N_R}$ for all $j, 0 \le j \le n$. As usual, the symbol ε denotes the empty word. A solution θ of such an inclusion assigns sets of words $\theta(X_i) \subseteq \mathsf{N_R}^*$ to each indeterminate X_i such that $\theta(X_i) \subseteq L_0 \cup L_1{\cdot}\theta(X_1) \cup \cdots \cup L_n{\cdot}\theta(X_n)$, where "$\cdot$" denotes the concatenation of languages. The solution θ is *finite* if $\theta(X_i)$ is a finite set for all $i \in \{1, \ldots, n\}$.

As described in [18], the unification problem $\Delta_{\Gamma,\tau}$ can be translated into a system of linear language inclusions as follows. For each concept constant $A \in \mathsf{N_C}$ and each subsumption constraint $\mathfrak{s} = C_1 \sqcap \cdots \sqcap C_n \sqsubseteq^? X$ in $\Delta_{\Gamma,\tau}$, a linear language inclusion $\mathfrak{i}_A(\mathfrak{s})$ is defined as:

$$X_A \subseteq f_A(C_1) \cup \cdots \cup f_A(C_n), \quad \text{where } f_A(C) := \begin{cases} \{r\} f_A(C') & \text{if } C = \exists r.C', \\ Y_A & \text{if } C = Y \in \mathsf{N_V}, \\ \{\varepsilon\} & \text{if } C = A, \\ \emptyset & \text{if } C \in \mathsf{N_C} \setminus \{A\}. \end{cases}$$

The set of language inclusions $\mathfrak{I}_{\Gamma,\tau}$ consists of all the inclusions obtained from $\Delta_{\Gamma,\tau}$ in this way. Note that the flat form of $\Delta_{\Gamma,\tau}$ and the fact that the right-hand side of each constraint in $\Delta_{\Gamma,\tau}$ is a variable ensure that the application of this translation indeed yields a set of linear language inclusions. A solution θ of $\mathfrak{I}_{\Gamma,\tau}$ is called *admissible* if for each variable X in $\Delta_{\Gamma,\tau}$ there exists $A \in \mathsf{N_C}$ such that $\theta(X_A) \neq \emptyset$.

Example 3.5. Let us come back to the set $\Delta_{\Gamma_2,\tau}$ obtained in Example 3.3. The translation described above yields the following set of linear language inclusions:

$$\mathfrak{I}_{\Gamma_2,\tau} = \left\{ \begin{array}{ll} Y_A \subseteq \{r\}\emptyset \cup \{s\}X_A, & Y_B \subseteq \{r\}\{\varepsilon\} \cup \{s\}X_B, \\ Y_A \subseteq \{s\}\{\varepsilon\} \cup \{r\}Z_A, & Y_B \subseteq \{s\}\emptyset \cup \{r\}Z_B, \\ Z_A \subseteq \emptyset, \quad Z_B \subseteq \{\varepsilon\} & \end{array} \right\}$$

The following assignment θ is a finite, admissible solution of $\mathfrak{I}_{\Gamma_2,\tau}$:

$$\theta(X_A) := \{\varepsilon\},\ \ \theta(Y_A) := \{s\},\ \ \theta(Y_B) := \{r\},\ \ \theta(Z_B) := \{\varepsilon\},\ \ \theta(X_B) = \theta(Z_A) := \emptyset.$$

From θ, one can obtain an $\mathcal{EL}^{-\top}$-unifier γ of $\Delta_{\Gamma_2,\tau}$, by defining $\gamma(U)$ as the following conjunction of particles (for all $U \in \mathit{Vars}(\Gamma)$):

$$\gamma(U) := \bigsqcap_{A \in \mathsf{N_C}} \bigsqcap_{w \in \theta(U_A)} \exists w.A.$$

Hence, θ yields the $\mathcal{EL}^{-\top}$-substitution $\gamma := \{X \mapsto A,\ Y \mapsto \exists s.A \sqcap \exists r.B,\ Z \mapsto B\}$. One can easily verify that γ is a unifier $\Delta_{\Gamma_2,\tau}$ w.r.t. $\mathcal{O} = \emptyset$.

However, γ is not yet a unifier of the original problem Γ_2 from Example 3.1, since it does not satisfy the constraint $X \sqsubseteq^? \exists r.Z$. Basically, the set $\mathfrak{I}_{\Gamma_2,\tau}$ is agnostic of the constraints dropped when translating Γ_2 into $\Delta_{\Gamma_2,\tau}$. Nevertheless, the subsumption mapping τ stores the information on how to solve such constraints, e.g., $\tau(X, \exists r.Z) = 1$. Hence, an $\mathcal{EL}^{-\top}$-unifier δ of Γ_2 can be constructed by extending the local $\mathcal{EL}$-unifier σ induced by S^τ with the particles in γ. In our example, S^τ corresponds to the assignment S from Example 3.1. Thus, δ is defined as:

$$\delta(X) := \delta(\exists r.Z) \sqcap A = (\exists r.B) \sqcap A,\quad \delta(Y) := \exists s.A \sqcap \exists r.B,\quad \delta(Z) := B.$$

One can easily verify that δ is an $\mathcal{EL}^{-\top}$-unifier of Γ_2. △

The following theorem summarizes one of the main results obtained in [18]. It shows that deciding unification in $\mathcal{EL}^{-\top}$ can be reduced to solving linear language inclusions.

Theorem 3.6. *Let Γ be a flat $\mathcal{EL}^{-\top}$-unification problem. Then, Γ has an $\mathcal{EL}^{-\top}$-unifier iff there exists a subsumption mapping τ for Γ such that $\mathfrak{I}_{\Gamma,\tau}$ has a finite, admissible solution.*

It is also shown in [18] that deciding the existence of a finite, admissible solution of a set of linear language inclusions can be reduced in polynomial time to checking emptiness of *alternating finite automata with ε-transitions*. The emptiness problem for this class of automata is a PSpace-complete problem [23]. This, together with Theorem 3.6, yields a polynomial space decision procedure for unification in $\mathcal{EL}^{-\top}$, since a subsumption mapping can be guessed in polynomial time and the size of $\mathfrak{I}_{\Gamma,\tau}$ is polynomial in the size of Γ.

4 The Unification Algorithm for $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$

In the following, we assume that $\mathcal{O}$ is a flat and cycle-restricted $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-ontology, and Γ is a flat $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unification problem. We introduce an algorithm that can test whether Γ has an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unifier and needs only polynomial space for this task. This algorithm follows the approach described in the previous section for unification in $\mathcal{EL}^{-\top}$, but must take the ontology into account. There are two main obstacles that need to be overcome when doing this:

1. The notion of a subsumption mapping from [18] is *not complete* for subsumption w.r.t. a non-empty ontology. The reason is that its definition is based on the characterization of subsumption w.r.t. the empty ontology, as stated in Lemma 1.1, and thus does not take the additional cases in the definition of structural subsumption and in Lemma 1.3 into account, which are required to capture subsumption w.r.t. an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-ontology.

2. The second, and more challenging, obstacle to overcome is that the reduction to solving language inclusions described in Subsection 3.2 only yields a *sound* (but *not complete*) procedure to decide unifiability of $\Delta_{\Gamma,\tau}$ w.r.t. a cycle-restricted ontology. In fact, Example 2.2 provides an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unification problem Γ_1 and a cycle-restricted $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-ontology $\mathcal{O}'$ such that:

    - the subsumption constraints of Γ_1 have the same form as the ones in problems of the form $\Delta_{\Gamma,\tau}$,
    - Γ_1 has an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unifier w.r.t. $\mathcal{O}'$, but it does not have $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unifiers w.r.t. the empty ontology.

The latter implies that applying the translation from [18] to Γ_1 yields a set of language inclusions that has no finite, admissible solution.

To regain completeness we proceed as follows. In Subsection 4.1, we introduce the notion of subsumption mapping w.r.t. a cycle-restricted $\mathcal{ELH}_{\mathcal{R}^+}$-ontology $\mathcal{O}$, and show that (similarly to the case of the empty ontology) checking $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unifiability of Γ w.r.t. $\mathcal{O}$ can be reduced to deciding whether there is a subsumption mapping τ such that $\Delta_{\Gamma,\tau}$ has an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unifier w.r.t. $\mathcal{O}$ that is compatible with τ. Afterwards, we define in Subsection 4.2 a new translation of $\Delta_{\Gamma,\tau}$ into linear language inclusions, which takes into account the axioms in a cycle-restricted $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-ontology.

4.1 The New Subsumption Mapping

A mapping of the form $\tau : Ats_{tr}(\Gamma,\mathcal{O}) \times Ats_{tr}(\Gamma,\mathcal{O}) \to \{0,1\}$ induces an assignment S^τ that (in the current setting) maps variables in Γ to subsets of non-variable atoms in $Ats_{tr}(\Gamma,\mathcal{O})$:

$$S^\tau(X) := \{D \in At_{nv}(\Gamma,\mathcal{O}) \mid \tau(X,D) = 1\}.$$

Differently to the notion introduced in [18], a subsumption mapping τ must now capture the properties of $\sqsubseteq_{\mathcal{O}}$, where $\mathcal{O}$ is a cycle-restricted $\mathcal{ELH}_{\mathcal{R}^+}$-ontology (instead of just the *empty one*). To achieve this, the conditions imposed on our new (extended) notion of subsumption mapping are based on the characterization of subsumption from Lemma 1.3. In order to simplify the definition of these conditions, we define the following set (for atoms $\exists r.C, \exists s.D \in Ats_{tr}(\Gamma,\mathcal{O})$):

$$\mathcal{S}(\exists r.C, \exists s.D) := \{D \mid \text{if } r \trianglelefteq_{\mathcal{O}} s\} \cup \{\exists t.D \mid r \trianglelefteq_{\mathcal{O}} t \trianglelefteq_{\mathcal{O}} s,\ t \text{ is transitive}\}.$$

Basically, this set collects all concepts F such that $C \sqsubseteq_{\mathcal{O}} F$ implies $\exists r.C \sqsubseteq^s_{\mathcal{O}} \exists s.D$ (see the second and third case in the definition of $\sqsubseteq^s_{\mathcal{O}}$).

Definition 4.1. The mapping $\tau : Ats_{tr}(\Gamma,\mathcal{O}) \times Ats_{tr}(\Gamma,\mathcal{O}) \to \{0,1\}$ is called a *subsumption mapping for* Γ *w.r.t.* $\mathcal{O}$ if it satisfies the following conditions:

1. It respect the properties of subsumption w.r.t. $\mathcal{O}$:
   (a) $\tau(D,D) = 1$, for each $D \in Ats_{tr}(\Gamma,\mathcal{O})$.
   (b) For all $D_1, D_2, D_3 \in Ats_{tr}(\Gamma,\mathcal{O})$, if $\tau(D_1,D_2) = \tau(D_2,D_3) = 1$ then $\tau(D_1,D_3) = 1$.
   (c) $\tau(C,D) = 1$ iff $C \sqsubseteq_{\mathcal{O}} D$, for all ground atoms $C, D \in Ats_{tr}(\Gamma,\mathcal{O})$.
   (d) For each concept constant $A \in Ats(\Gamma,\mathcal{O})$, role name r, and variable X with $\exists r.X \in Ats_{tr}(\Gamma)$:
      i. $\tau(A, \exists r.X) = 1$ iff [7] there exists an atom $\exists u.B$ of $\mathcal{O}$ such that $\tau(B,X) = 1$, and
         - $A \sqsubseteq_{\mathcal{O}} \exists u.B$ and $u \trianglelefteq_{\mathcal{O}} r$, or
         - $A \sqsubseteq_{\mathcal{O}} \exists t.B$ for a transitive role t with $u \trianglelefteq_{\mathcal{O}} t \trianglelefteq_{\mathcal{O}} r$.
      ii. $\tau(\exists r.X, A) = 1$ iff
         - there are atoms $\exists r_1.A_1, \ldots, \exists r_k.A_k$ of $\mathcal{O}$ ($k \geq 0$) and atoms $F_\ell \in \mathcal{S}(\exists r.X, \exists r_\ell.A_\ell)$ ($1 \leq \ell \leq k$), such that:
$$\tau(X, F_\ell) = 1 \ (1 \leq \ell \leq k) \quad \text{and} \quad \exists r_1.A_1 \sqcap \cdots \sqcap \exists r_k.A_k \sqsubseteq_{\mathcal{O}} A. \qquad (4)$$
   (e) For all role names $r, s \in \mathsf{N_R}$, variable X and atoms $\exists r.C, \exists s.D \in Ats_{tr}(\Gamma)$ with $C = X$ or $D = X$: $\tau(\exists r.C, \exists s.D) = 1$ iff

[7] The right-hand side of this equivalence is based on Lemma 2.4.

- there exists $F \in \mathcal{S}(\exists r.C, \exists s.D)$ such that $\tau(C, F) = 1$, or
- there are atoms $\exists r_1.A_1, \ldots, \exists r_k.A_k, \exists u.B$ of $\mathcal{O}$ ($k \geq 0$), atoms $F_\ell \in \mathcal{S}(\exists r.C, \exists r_\ell.A_\ell)$ ($1 \leq \ell \leq k$), and an atom $F \in \mathcal{S}(\exists u.B, \exists s.D)$, such that:

$$\tau(C, F_\ell) = 1 \; (1 \leq \ell \leq k), \quad \exists r_1.A_1 \sqcap \cdots \sqcap \exists r_k.A_k \sqsubseteq_{\mathcal{O}} \exists u.B, \quad \tau(B, F) = 1. \quad (5)$$

2. The assignment S^τ is acyclic. Note that this means that the mapping τ induces the $\mathcal{ELH}_{\mathcal{R}^+}$-substitution σ^τ.

3. The substitution σ^τ is an $\mathcal{ELH}_{\mathcal{R}^+}$-unifier of Γ w.r.t. $\mathcal{O}$. In combination with the conditions already introduced, this is expressed by the following conditions for each subsumption constraint $C_1 \sqcap \cdots \sqcap C_n \sqsubseteq^? D \in \Gamma$:

   (a) If D is a non-variable atom, then either $\tau(C_i, D) = 1$ for some $i \in \{1, \ldots, n\}$, or there exist atoms $At_1, \ldots, At_k, At'$ of $\mathcal{O}$ ($k \geq 0$) such that:
   - $At_1 \sqcap \cdots \sqcap At_k \sqsubseteq_{\mathcal{O}} At'$,
   - for each $\ell \in \{1, \ldots, k\}$ there exists $i \in \{1, \ldots, n\}$ such that $\tau(C_i, At_\ell) = 1$, and
   - $\tau(At', D) = 1$.

   (b) If D is a variable and $\tau(D, C) = 1$ for a non-variable atom $C \in At_{nv}(\Gamma, \mathcal{O})$, then $C_1 \sqcap \cdots \sqcap C_n \sqsubseteq^? C$ satisfies the previous case.

By using the close relationship between this definition and the characterization of subsumption in Lemma 1.3, one can show that Γ has an $\mathcal{ELH}_{\mathcal{R}^+}$-unifier w.r.t. $\mathcal{O}$ iff there is a subsumption mapping for Γ w.r.t. $\mathcal{O}$. In the proof of the if-direction, one shows that the substitution induced by the subsumption mapping is indeed a unifier. For the other direction, one takes an $\mathcal{ELH}_{\mathcal{R}^+}$-unifier σ and shows that the mapping τ satisfying $\tau(C, D) = 1$ iff $\sigma(C) \sqsubseteq_{\mathcal{O}} \sigma(D)$ is a subsumption mapping for Γ w.r.t. $\mathcal{O}$.

However, using subsumption mappings to characterize unifiability in $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$ requires more effort. As defined in (3) for the case of the empty ontology, together with the unification problem Γ, a subsumption mapping τ yields a simpler unification problem $\Delta_{\Gamma,\tau} = \Delta_\Gamma \cup \Delta_\tau$. We can here re-use this definition without change. Before we can formulate the main result of this subsection, we need to define the notion of compatibility of a substitution with a subsumption mapping.

Definition 4.2. Any substitution σ induces an assignment S^σ of the form

$$S^\sigma(X) := \{D \in At_{nv}(\Gamma, \mathcal{O}) \mid \sigma(X) \sqsubseteq_{\mathcal{O}} \sigma(D)\}.$$

We write $S^\tau \leq S^\sigma$ if $S^\tau(X) \subseteq S^\sigma(X)$ holds for all variables X. In this case we say that σ is *compatible* with τ.

The following result gives a characterization of the existence of an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unifier w.r.t. an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-ontology.

Proposition 4.3. *Let $\mathcal{O}$ be a flat and cycle-restricted $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-ontology and Γ a flat $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unification problem. Then, Γ has an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unifier w.r.t. $\mathcal{O}$ iff there exists a subsumption mapping τ for Γ w.r.t. $\mathcal{O}$ such that $\Delta_{\Gamma,\tau}$ has an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unifier γ w.r.t. $\mathcal{O}$ that is compatible with τ.*

Before we can prove this proposition, we first need to show two lemmas.

Lemma 4.4. *Let $\exists r.C, \exists s.D$ be atoms in $Ats_{tr}(\Gamma, \mathcal{O})$ and σ an $\mathcal{ELH}_{\mathcal{R}^+}$-substitution. Further, let $F \in \mathcal{S}(\exists r.C, \exists s.D)$ be such that $\sigma(C) \sqsubseteq_{\mathcal{O}} \sigma(F)$. Then, $\exists r.\sigma(C) \sqsubseteq^s_{\mathcal{O}} \exists s.\sigma(D)$.*

Proof. By definition of $\mathcal{S}(\cdot,\cdot)$, there are two possibilities for having F in $\mathcal{S}(\exists r.C, \exists s.D)$:

- $F = D$ and $r \trianglelefteq_{\mathcal{O}} s$, or
- $F = \exists t.D$, where t is a transitive role such that $r \trianglelefteq_{\mathcal{O}} t \trianglelefteq_{\mathcal{O}} s$.

Then $\sigma(C) \sqsubseteq_{\mathcal{O}} \sigma(F)$ implies $\exists r.\sigma(C) \sqsubseteq^s_{\mathcal{O}} \exists s.\sigma(D)$, by the definition of $\sqsubseteq^s_{\mathcal{O}}$. □

The next lemma shows that, given an $\mathcal{ELH}_{\mathcal{R}^+}$-substitution σ that is compatible with τ, we have that $\tau(At_1, At_2) = 1$ implies $\sigma(At_1) \sqsubseteq_{\mathcal{O}} \sigma(At_2)$, where $At_1 \in Ats_{tr}(\Gamma, \mathcal{O})$ and $At_2 \in At_{nv}(\Gamma, \mathcal{O})$.

Lemma 4.5. *Let τ be a subsumption mapping for Γ w.r.t. $\mathcal{O}$ and σ an $\mathcal{ELH}_{\mathcal{R}^+}$-substitution that is compatible with τ. For all atoms $C \in Ats_{tr}(\Gamma, \mathcal{O})$ and $D \in At_{nv}(\Gamma, \mathcal{O})$ we have that:*

1. *If D is a ground atom, then $\tau(C, D) = 1$ implies $\sigma(C) \sqsubseteq_{\mathcal{O}} \sigma(D)$.*
2. *If D is of the form $\exists r.Y$ for some variable Y and σ satisfies all subsumption constraints $C' \sqsubseteq^? Y \in \Delta_\tau$, then $\tau(C, D) = 1$ implies $\sigma(C) \sqsubseteq_{\mathcal{O}} \sigma(D)$.*

Proof. If C is a variable X, then $\tau(X, D) = 1$ implies $D \in S^\tau(X)$. Since σ is compatible with τ, we know that $S^\tau(X) \subseteq S^\sigma(X)$ holds, and thus $D \in S^\tau(X)$ implies $D \in S^\sigma(X)$. Hence, the definition of S^σ yields $\sigma(X) \sqsubseteq_{\mathcal{O}} \sigma(D)$. Thus, both cases hold regardless of the form of D. The rest of the proof consists of proving 1 and 2 for the remaining possible forms of C and D.

1. Assume that D is ground and $\tau(C, D) = 1$. If C is also ground, then Condition (1c) of Definition 4.1 implies that $\sigma(C) \sqsubseteq_{\mathcal{O}} \sigma(D)$. Otherwise, $C = \exists r.X$ for some variable X. Let us distinguish between the two possible forms of D:

   - $D = A$. By the second case in Condition (1d) of Definition 4.1, $\tau(\exists r.X, A) = 1$ implies that there are atoms $\exists r_1.A_1, \ldots, \exists r_k.A_k$ of $\mathcal{O}$ ($k \geq 0$) and atoms $F_\ell \in \mathcal{S}(\exists r.X, \exists r_\ell.A_\ell)$ ($1 \leq \ell \leq k$) satisfying (4). Hence, we know that $\tau(X, F_\ell) = 1$ for all $\ell \in \{1, \ldots, k\}$. Since $\exists r_\ell.A_\ell$ is ground, the definition of $\mathcal{S}(\cdot,\cdot)$ yields that F_ℓ is also a ground atom. Therefore, as shown above, $\tau(X, F_\ell) = 1$ implies $\sigma(X) \sqsubseteq_{\mathcal{O}} \sigma(F_\ell)$. We can then apply Lemma 4.4 to $\exists r.X$, $\exists r_\ell.A_\ell$ and F_ℓ to obtain that $\exists r.\sigma(X) \sqsubseteq^s_{\mathcal{O}} \exists r_\ell.A_\ell$. This, together with $\exists r_1.A_1 \sqcap \cdots \sqcap \exists r_k.A_k \sqsubseteq_{\mathcal{O}} A$, yields the following subsumption relations:
   $$\sigma(\exists r.X) \sqsubseteq_{\mathcal{O}} \exists r_1.A_1 \sqcap \cdots \sqcap \exists r_k.A_k \sqsubseteq_{\mathcal{O}} A.$$
   Thus, $\sigma(\exists r.X) \sqsubseteq_{\mathcal{O}} A$.
   - $D = \exists s.A$. Since $C = \exists r.X$, we can apply Condition (1e) of Definition 4.1. This yields two possibilities for having $\tau(\exists r.X, \exists s.A) = 1$. The first one tells us that there exists $F \in \mathcal{S}(\exists r.X, \exists s.A)$ such that $\tau(X, F) = 1$. Since $\exists s.A$ is ground, the definition of $\mathcal{S}(\cdot,\cdot)$ yields that F is also a ground atom. Hence, as in the previous case, we know that $\tau(X, F)$=1 implies $\sigma(X) \sqsubseteq_{\mathcal{O}} \sigma(F)$. This means that we can apply Lemma 4.4 to $\exists r.X$, $\exists s.A$ and F to obtain that $\exists r.\sigma(X) \sqsubseteq^s_{\mathcal{O}} \exists s.A$. Consequently, $\sigma(\exists r.X) \sqsubseteq_{\mathcal{O}} D$. The second case yields atoms $\exists r_1.A_1, \ldots, \exists r_k.A_k, \exists u.B$ of $\mathcal{O}$ ($k \geq 0$), atoms $F_\ell \in \mathcal{S}(\exists r.X, \exists r_\ell.A_\ell)$ ($1 \leq \ell \leq k$), and an atom $F \in \mathcal{S}(\exists u.B, \exists s.A)$ satisfying (5). As in the case with $D = A$, we know that $\tau(X, F_\ell) = 1$ ($1 \leq \ell \leq k$). Hence, the same arguments can be applied to obtain $\exists r.\sigma(X) \sqsubseteq^s_{\mathcal{O}} \exists r_\ell.A_\ell$ ($1 \leq \ell \leq k$). Furthermore, (5) also tells us that $\tau(B, F) = 1$. In addition, since $\exists s.A$ is ground, $F \in \mathcal{S}(\exists u.B, \exists s.A)$ implies that F is also ground. Hence, by Condition (1c), $\tau(B, F) = 1$ implies that $B \sqsubseteq_{\mathcal{O}} F$. We can then apply Lemma 4.4 to $\exists u.B$, $\exists s.A$ and F to obtain that

$\exists u.B \sqsubseteq^s_{\mathcal{O}} \exists s.A$. This, together with $\exists r_1.A_1 \sqcap \cdots \sqcap \exists r_k.A_k \sqsubseteq_{\mathcal{O}} \exists u.B$, yields the following subsumption relations:

$$\sigma(\exists r.X) \sqsubseteq_{\mathcal{O}} \exists r_1.A_1 \sqcap \cdots \sqcap \exists r_k.A_k \sqsubseteq_{\mathcal{O}} \exists u.B \sqsubseteq_{\mathcal{O}} \exists s.A.$$

Thus, $\sigma(\exists r.X) \sqsubseteq_{\mathcal{O}} \exists s.A$.

2. Assume that $D = \exists r.Y$ for some variable Y, σ satisfies all constraints $C' \sqsubseteq^? Y \in \Delta_\tau$, and $\tau(C, D) = 1$. We consider the possible forms of C:

   - $C = A$. Hence, since $\tau(C, \exists r.Y) = 1$, the first case in Condition (1d) of Definition 4.1 yields an atom $\exists u.B$ of $\mathcal{O}$ such that $\tau(B, Y) = 1$, and
     - $A \sqsubseteq_{\mathcal{O}} \exists u.B$ and $u \trianglelefteq_{\mathcal{O}} r$, or
     - $A \sqsubseteq_{\mathcal{O}} \exists t.B$ for a transitive role t with $u \trianglelefteq_{\mathcal{O}} t \trianglelefteq_{\mathcal{O}} r$.

     It is easy to see that $A \sqsubseteq_{\mathcal{O}} \exists r.B$ is a consequence of both cases. Furthermore, $\tau(B, Y) = 1$ implies that $B \sqsubseteq^? Y \in \Delta_\tau$ (see the definition of $\Delta_{\Gamma,\tau}$). Hence, our assumption about σ yields that $B \sqsubseteq_{\mathcal{O}} \sigma(Y)$, which implies that $\exists r.B \sqsubseteq_{\mathcal{O}} \exists r.\sigma(Y)$. Thus, $A \sqsubseteq_{\mathcal{O}} \exists r.B$ and $\exists r.B \sqsubseteq_{\mathcal{O}} \exists r.\sigma(Y)$ imply $A \sqsubseteq_{\mathcal{O}} \sigma(\exists r.Y)$.
   - $C = \exists s.C'$. By Condition (1e) of Definition 4.1 there are two possibilities for having $\tau(\exists s.C', \exists r.Y) = 1$:
     - $\tau(C', F) = 1$ for some $F \in \mathcal{S}(\exists s.C', \exists r.Y)$. Since $\exists s.C'$ is a flat atom, C' is either a constant or a variable. In addition, by definition of $\mathcal{S}(\cdot, \cdot)$, we have that $F = Y$ or $F = \exists t.Y$. If $F = Y$, then $\tau(C', Y) = 1$ implies that $C' \sqsubseteq^? Y \in \Delta_\tau$. Hence, our assumption about σ yields $\sigma(C') \sqsubseteq_{\mathcal{O}} \sigma(Y)$. Otherwise, one of the previous cases applies, and $\tau(C', F) = 1$ implies $\sigma(C') \sqsubseteq_{\mathcal{O}} \sigma(F)$. Thus, we can apply Lemma 4.4 to $\exists s.C'$, $\exists r.Y$ and F to obtain that $\sigma(\exists s.C') \sqsubseteq_{\mathcal{O}} \sigma(\exists r.Y)$.
     - There are atoms $\exists s_1.A_1, \ldots, \exists s_k.A_k$ (for some $k \geq 0$) and $\exists u.B$ of $\mathcal{O}$, atoms $F_\ell \in \mathcal{S}(\exists s.C', \exists s_\ell.A_\ell)$ $(1 \leq \ell \leq k)$, and an atom $F \in \mathcal{S}(\exists u.B, \exists r.Y)$ satisfying (5). If C' is a variable then the same arguments used for the case $C = \exists r.X$ and $D = \exists s.A$ above, together with the assumptions made for σ, can be applied to obtain that $\sigma(\exists s.C') \sqsubseteq_{\mathcal{O}} \sigma(\exists r.Y)$. Otherwise, C' must be a concept name since $\exists s.C'$ is a flat atom. Therefore, by Condition (1c), the mappings $\tau(C', F_\ell) = 1$ derived from (5) imply that $\sigma(C') \sqsubseteq_{\mathcal{O}} F_\ell$ $(1 \leq \ell \leq k)$. Hence, we can again re-use the aforementioned arguments to show that $\sigma(\exists s.C') \sqsubseteq_{\mathcal{O}} \sigma(\exists r.Y)$.

     Thus, we have shown that $\sigma(\exists s.C') \sqsubseteq_{\mathcal{O}} \sigma(\exists r.Y)$ holds in both cases.

This concludes the proof of the lemma. □

We are now ready to prove the correspondence stated in Proposition 4.3.[8]

Proof of Proposition 4.3.

($\Rightarrow$) Assume that Γ has an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unifier δ w.r.t. $\mathcal{O}$. We define the assignment τ as follows:

$$\tau(D_1, D_2) = 1 \text{ iff } \delta(D_1) \sqsubseteq_{\mathcal{O}} \delta(D_2), \text{ for all } D_1, D_2 \in Ats_{tr}(\Gamma, \mathcal{O}).$$

It is an immediate consequence of this definition and the fact that δ is an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unifier of Γ w.r.t. $\mathcal{O}$ that δ is an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unifier of $\Delta_{\Gamma,\tau}$ w.r.t. $\mathcal{O}$ that is compatible with τ. Therefore, it remains to show that τ is a subsumption mapping for Γ w.r.t. $\mathcal{O}$. To this end, we consider the conditions required in Definition 4.1, and show that τ satisfies them.

[8] Lemma 4.5 and Proposition 4.3 are the analoga of Lemmas 10 and 11 in [18], respectively.

1. We show that τ satisfies (1a)–(1e). Conditions (1a), (1b) and (1c) follow directly from the properties of $\sqsubseteq_{\mathcal{O}}$ and the definition of τ. The remaining cases are more involved and require the application of Lemma 1.3. We consider them separately.

   - First case in (1d). The definition of τ yields

     $$\tau(A, \exists r.X) = 1 \text{ iff } A \sqsubseteq_{\mathcal{O}} \delta(\exists r.X).$$

     By Lemma 2.4, $A \sqsubseteq_{\mathcal{O}} \delta(\exists r.X)$ holds iff there is $\exists u.B \in Ats(\mathcal{O})$ such that $B \sqsubseteq_{\mathcal{O}} \delta(X)$ and the conditions required for A and $\exists u.B$ in the first case of (1d) are satisfied. Furthermore, by the definition of τ, we know that $B \sqsubseteq_{\mathcal{O}} \delta(X)$ iff $\tau(B, X) = 1$. Thus, we have shown that τ satisfies the first case of Condition (1d).
   - Second case in (1d). The definition of τ yields

     $$\tau(\exists r.X, A) = 1 \text{ iff } \delta(\exists r.X) \sqsubseteq_{\mathcal{O}} A.$$

     Since $\delta(\exists r.X) \not\sqsubseteq^s_{\mathcal{O}} A$, Lemma 1.3 tells us that $\delta(\exists r.X) \sqsubseteq_{\mathcal{O}} A$ iff there are atoms $\exists r_1.A_1, \ldots, \exists r_k.A_k, B$ of $\mathcal{O}$ ($k \geq 0$) such that:

     $$\delta(\exists r.X) \sqsubseteq^s_{\mathcal{O}} \exists r_\ell.A_\ell \ (1 \leq \ell \leq k), \quad \exists r_1.A_1 \sqcap \cdots \sqcap \exists r_k.A_k \sqsubseteq_{\mathcal{O}} B, \quad B \sqsubseteq^s_{\mathcal{O}} A.$$

     By definition of $\sqsubseteq^s_{\mathcal{O}}$, the first group of k subsumption relations holds iff there are $F_\ell \in \mathcal{S}(\exists r.X, \exists r_\ell.A_\ell)$ such that $\delta(X) \sqsubseteq_{\mathcal{O}} F_\ell$ ($1 \leq \ell \leq k$). Moreover, by definition of τ, we know that $\delta(X) \sqsubseteq_{\mathcal{O}} F_\ell$ iff $\tau(X, F_\ell) = 1$ ($1 \leq \ell \leq k$). Thus, since $B \sqsubseteq^s_{\mathcal{O}} A$ implies that $B = A$, we have shown that τ satisfies the second condition in (1d).
   - Case (1e). By definition of τ, we have

     $$\tau(\exists r.C, \exists s.D) = 1 \text{ iff } \delta(\exists r.C) \sqsubseteq_{\mathcal{O}} \delta(\exists s.D).$$

     An application of Lemma 1.3 yields that $\delta(\exists r.C) \sqsubseteq_{\mathcal{O}} \delta(\exists s.D)$ iff
     - $\delta(\exists r.C) \sqsubseteq^s_{\mathcal{O}} \delta(\exists s.D)$, or
     - there are atoms $\exists r_1.A_1, \ldots, \exists r_k.A_k, \exists u.B$ of $\mathcal{O}$ ($k \geq 0$) such that:

     $$\delta(\exists r.C) \sqsubseteq^s_{\mathcal{O}} \exists r_\ell.A_\ell \ (1 \leq \ell \leq k), \quad \exists r_1.A_1 \sqcap \cdots \sqcap \exists r_k.A_k \sqsubseteq_{\mathcal{O}} \exists u.B, \quad \exists u.B \sqsubseteq^s_{\mathcal{O}} \delta(\exists s.D).$$

     In the first case, the definition of $\sqsubseteq^s_{\mathcal{O}}$ tells us that $\delta(\exists r.C) \sqsubseteq^s_{\mathcal{O}} \delta(\exists s.D)$ iff

     $$\text{there exists } F \in \mathcal{S}(\exists r.C, \exists s.D) \text{ such that } \delta(C) \sqsubseteq_{\mathcal{O}} \delta(F).$$

     Furthermore, by definition of τ, $\delta(C) \sqsubseteq_{\mathcal{O}} \delta(F)$ iff $\tau(C, F) = 1$. Thus, it follows that $\delta(\exists r.C) \sqsubseteq^s_{\mathcal{O}} \delta(\exists s.D)$ iff the first case of (1e) is true.
     Regarding the second case, several applications of $\sqsubseteq^s_{\mathcal{O}}$ yield that $\delta(\exists r.C) \sqsubseteq^s_{\mathcal{O}} \exists r_\ell.A_\ell$ ($1 \leq \ell \leq k$) and $\exists u.B \sqsubseteq^s_{\mathcal{O}} \delta(\exists s.D)$ iff there are $F_\ell \in \mathcal{S}(\exists r.C, \exists r_\ell.A_\ell)$ ($1 \leq \ell \leq k$) and $F \in \mathcal{S}(\exists u.B, \exists s.D)$ such that:

     $$\delta(C) \sqsubseteq_{\mathcal{O}} F_\ell \ (1 \leq \ell \leq k) \quad \text{and} \quad B \sqsubseteq_{\mathcal{O}} \delta(F).$$

     By definition of τ, these subsumption relations hold iff

     $$\tau(C, F_\ell) = 1 \ (1 \leq \ell \leq k) \quad \text{and} \quad \tau(B, F) = 1.$$

     Hence, the second case holds iff the statements in (5) hold. Overall, we have thus shown that τ satisfies (1e).

2. To show that τ satisfies Condition 2, assume that S^τ is not acyclic. Then, there is a sequence of variables $X_1, \ldots, X_{n+1}$ and role names $r_1, \ldots, r_n$ such that $X_1 = X_{n+1}$, $\exists r_i.X_{i+1} \in At_{nv}(\Gamma)$ and $\tau(X_i, \exists r_i.X_{i+1}) = 1$ ($1 \leq i \leq n$). The definition of τ yields $\delta(X_i) \sqsubseteq_{\mathcal{O}} \exists r_i.\delta(X_{i+1})$ for all $i \in \{1, \ldots, n\}$, which implies $\delta(X_1) \sqsubseteq_{\mathcal{O}} \exists r_1. \cdots \exists r_n.\delta(X_1)$. However, this contradict our assumption that $\mathcal{O}$ is cycle-restricted. Thus, we can conclude that τ is acyclic.

3. To show that τ satisfies Condition 3, we fix a subsumption constraint $C_1 \sqcap \cdots \sqcap C_n \sqsubseteq^? D \in \Gamma$. If D is a non-variable atom, then we must show that Condition (3a) holds. Note that $\delta(C_1) \sqcap \cdots \sqcap \delta(C_n) \sqsubseteq_{\mathcal{O}} \delta(D)$ holds because δ is a unifier of Γ. Hence, since $\delta(D)$ is an atom, the application of Lemma 1.3 yields two possibilities:

   - there exists $i \in \{1, \ldots, n\}$ and a top-level atom C of $\delta(C_i)$ such that $C \sqsubseteq^s_{\mathcal{O}} \delta(D)$. Since $\delta(C_i) \sqsubseteq_{\mathcal{O}} C$, this yields $\delta(C_i) \sqsubseteq_{\mathcal{O}} \delta(D)$, which by the definition of τ implies $\tau(C_i, D) = 1$, as required in Condition (3a).
   - $\delta(C_1) \sqcap \cdots \sqcap \delta(C_n) \sqsubseteq_{\mathcal{O}} \delta(D)$ satisfies case 2 of Lemma 1.3. This yields the existence of atoms $At_1, \ldots, At_k, At'$ of $\mathcal{O}$ ($k \geq 0$) such that: (a) $At_1 \sqcap \cdots \sqcap At_k \sqsubseteq_{\mathcal{O}} At'$, (b) for each $\ell \in \{1, \ldots, k\}$, there exists $i \in \{1, \ldots, n\}$ and a top-level atom D_ℓ of $\delta(C_i)$ such that $\delta(C_i) \sqsubseteq_{\mathcal{O}} D_\ell \sqsubseteq^s_{\mathcal{O}} At_\ell$, and (c) $At' \sqsubseteq^s_{\mathcal{O}} \delta(D)$.

     Hence, by definition of τ, we have that $\tau(C_i, At_\ell) = 1$ (for each respective pair) and $\tau(At', D) = 1$. Thus, the second case in Condition (3a) is satisfied.

   It remains to show that, if D is a variable, then Condition (3b) holds. Let $C \in At_{nv}(\Gamma, \mathcal{O})$ such that $\tau(D, C) = 1$. The definition of τ yields $\delta(D) \sqsubseteq_{\mathcal{O}} \delta(C)$, which then implies that $\delta(C_1) \sqcap \cdots \sqcap \delta(C_n) \sqsubseteq_{\mathcal{O}} \delta(C)$. Thus, since $\delta(C)$ is a non-variable atom, we can show as above that $C_1 \sqcap \cdots \sqcap C_n$ and C satisfy Condition (3a).

($\Leftarrow$) Let τ be a subsumption mapping for Γ w.r.t. $\mathcal{O}$. Furthermore, let γ be an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unifier of $\Delta_{\Gamma,\tau}$ w.r.t. $\mathcal{O}$ such that γ is compatible with τ. We show that γ is also a unifier of Γ w.r.t. $\mathcal{O}$. It suffices to consider subsumption constraints in $\Gamma \setminus \Delta_\Gamma$. These constraints are of the form $C_1 \sqcap \cdots \sqcap C_n \sqsubseteq^? D$ where D is not a variable. Since τ is a subsumption mapping for Γ w.r.t. $\mathcal{O}$, it satisfies Condition (3a) of Definition 4.1. Let us consider the two possible cases.

- There is $i \in \{1, \ldots, n\}$ such that $\tau(C_i, D) = 1$. Note that $D \in At_{nv}(\Gamma, \mathcal{O})$ and γ solves all constraints in Δ_τ. Hence, we can apply Lemma 4.5 to obtain that $\gamma(C_i) \sqsubseteq_{\mathcal{O}} \gamma(D)$. Thus, $\gamma(C_1) \sqcap \cdots \sqcap \gamma(C_n) \sqsubseteq_{\mathcal{O}} \gamma(D)$ holds.
- There are atoms $At_1, \ldots, At_k, At'$ of $\mathcal{O}$ ($k \geq 0$) satisfying the conditions required in the second case of (3a) w.r.t. $C_1 \sqcap \cdots \sqcap C_n$ and D. As above, we can apply Lemma 4.5 to each pair (C_i, At_ℓ) and to (At', D), to obtain that $\gamma(C_i) \sqsubseteq_{\mathcal{O}} At_\ell$ and $At' \sqsubseteq_{\mathcal{O}} \gamma(D)$. Overall, we have that:

$$\gamma(C_1) \sqcap \cdots \sqcap \gamma(C_n) \sqsubseteq_{\mathcal{O}} At_1 \sqcap \cdots \sqcap At_k \sqsubseteq_{\mathcal{T}} At' \sqsubseteq_{\mathcal{O}} \gamma(D).$$

  Hence, it follows that $\gamma(C_1) \sqcap \cdots \sqcap \gamma(C_n) \sqsubseteq_{\mathcal{O}} \gamma(D)$.

Thus, we have shown that γ is a an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unifier of Γ w.r.t. $\mathcal{O}$. □

Example 4.6. Let $\mathcal{O} = \emptyset$ and consider the following unification problem:

$$\Gamma_3 := \{\exists r.B \sqsubseteq^? \exists r.Y, \;\; \exists s.X \sqcap \exists r.A \sqsubseteq^? Y\}.$$

Due to Condition 3 in Definition 4.1 and the fact that $\mathcal{O}$ is empty, any subsumption mapping τ must satisfy $\tau(\exists r.B, \exists r.Y) = 1$. Condition (1e) then implies that $\tau(B, Y) = 1$ must hold as well. Regarding the second subsumption constraint in Γ_3, Condition (3b) does not apply as long as there is no non-variable atom C with $\tau(Y, C) = 1$. We can conclude that, for any subsumption mapping τ, the set $\Delta_{\Gamma_3,\tau}$ contains at least the subsumption constraints $B \sqsubseteq^? Y$ and $\exists s.X \sqcap \exists r.A \sqsubseteq^? Y$. Using an argument similar to the one employed in Example 2.2, one can show that such a set $\Delta_{\Gamma_3,\tau}$ cannot have an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unifier w.r.t. $\mathcal{O}$.

It is easy to see that there also is a subsumption mapping τ that has only these two constraints in $\Delta_{\Gamma_3,\tau}$ since the only other mandatory values 1 are the ones required by (1a). For the ontology $\mathcal{O}'' = \{B \sqsubseteq \exists r.A\}$, the set $\Delta_{\Gamma_3,\tau}$ then has an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unifier w.r.t. $\mathcal{O}''$. This unifier is compatible with τ since the subsumption mapping τ that yields value 1 only if required satisfies $S^\tau(X) = S^\tau(Y) = \emptyset$. Thus, by Proposition 4.3, Γ_3 has an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unifier w.r.t. $\mathcal{O}''$. Note that this unifier is not σ_τ since σ_τ in this case assigns $\top$ to X and Y. $\triangle$

4.2 The New Translation into Linear Language Inclusions

We have already pointed out at the beginning of this section that the translation from $\Delta_{\Gamma,\tau}$ into $\mathfrak{I}_{\Gamma,\tau}$ for the case of an empty ontology (as sketched in Section 3.2) is not suitable for unification w.r.t. non-empty cycle-restricted ontologies. Let us illustrate this with two concrete examples.

Example 4.7. Consider the system $\Delta_{\Gamma_3,\tau} = \{B \sqsubseteq^? Y, \exists s.X \sqcap \exists r.A \sqsubseteq^? Y\}$ from Example 4.6. The first subsumption constraint yields the language inclusions

$$Y_A \subseteq \emptyset \quad \text{and} \quad Y_B \subseteq \{\varepsilon\},$$

and the second yields

$$Y_A \subseteq \{s\}X_A \cup \{r\}\{\varepsilon\} \quad \text{and} \quad Y_B \subseteq \{s\}X_B \cup \{r\}\emptyset.$$

There are no language inclusions constraining X_A or X_B. Obviously, any solution θ of $\mathfrak{I}_{\Gamma_3,\tau}$ must satisfy $\theta(Y_A) = \emptyset$. This means that, if θ is admissible, then $\theta(Y_B)$ must be non-empty. The first inclusion for Y_B says that $\theta(Y_B)$ consists of the empty word, whereas the second says that every element of $\theta(Y_B)$ must start with the letter s. Thus, $\mathfrak{I}_{\Gamma_3,\tau}$ cannot have an admissible solution.

However, it is easy to see that $\Delta_{\Gamma_3,\tau}$ has an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unifier w.r.t. the ontology $\mathcal{O}'' = \{B \sqsubseteq \exists r.A\}$ from Example 4.6. For instance, the substitution $\gamma := \{X \mapsto B, Y \mapsto \exists r.A\}$ is such an unifier. Note that $B \sqsubseteq_{\mathcal{O}''} \gamma(Y) = \exists r.A$ holds because $B \sqsubseteq \exists r.A$ is a GCI in $\mathcal{O}''$. This subsumption relationship can also be explained by applying the characterization of subsumption in Lemma 1.3, i.e., an application of the second case in Lemma 1.3 yields $B \sqsubseteq_{\mathcal{O}''} \exists r.A$.

Summing up, this example demonstrates that the translation of the unification problem $\Delta_{\Gamma_3,\tau}$ into a system of linear language inclusions must be augmented to take subsumption relationships induced by GCIs into account. $\triangle$

The following example is more involved. It also considers the effect of role inclusion axioms.

Example 4.8. Recall the unification problem Γ_1 and ontology $\mathcal{O}'$ from Example 2.2. We have seen that Γ_1 has $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unifiers w.r.t. $\mathcal{O}'$, e.g., the substitution δ defined in Example 2.2.

However, the system $\mathfrak{I}_{\Gamma_1,\tau}$ obtained by applying the translation described in Section 3.2 to Γ_1 does not have any finite, admissible solution. To see why, note that translating the *first* and *third* subsumption constraints in Γ_1 w.r.t. the concept constant B yields the following inclusions in $\mathfrak{I}_{\Gamma_1,\tau}$:

$$X_B \subseteq \{r\}\emptyset, \quad \text{and} \quad Y_B \subseteq \{s\}X_B \cup \emptyset.$$

This means that any solution θ of $\mathfrak{I}_{\Gamma_1,\tau}$ must satisfy $\theta(X_B) = \emptyset$, and hence, $\theta(Y_B) = \emptyset$. Thus, in an admissible solution of $\mathfrak{I}_{\Gamma_1,\tau}$, the set $\theta(Y_A)$ must be non-empty. The problem is, however, that translating the *second* constraint in Γ_1 w.r.t. A yields the inclusion:

$$Y_A \subseteq \{u\}\emptyset,$$

which forces $\theta(Y_A)$ to be empty. Thus, the system cannot have an admissible solution.

To see the contrast with the unifiability of Γ_1 w.r.t. $\mathcal{O}'$, consider first the subsumption constraint $\exists u.B \sqsubseteq^? Y \in \Gamma_1$. We have seen in Example 2.2 that $\exists u.B \sqsubseteq_{\mathcal{O}'} \delta(Y) = \exists s.\exists r.A$. This can be explained by applying the characterization of subsumption in Lemma 1.3, i.e.,

- the application of the second case in Lemma 1.3 yields $B \sqsubseteq_{\mathcal{O}'} \exists r.A$, and
- $u \trianglelefteq_{\mathcal{O}'} s$ and $B \sqsubseteq_{\mathcal{O}'} \exists r.A$ can be used to apply $\sqsubseteq^s_{\mathcal{O}'}$ and conclude that $\exists u.B \sqsubseteq_{\mathcal{O}'} \exists s.\exists r.A$.

Nevertheless, as illustrated above, the inclusion $Y_A \subseteq \{u\}\emptyset$ does not admit a solution θ with $sr \in \theta(Y_A)$. Consequently, the translation must be modified such that a solution θ with $sr \in \theta(Y_A)$ is possible. $\triangle$

To summarize, the problem is that the language inclusions generated by the existing translation are not equipped to recognize the sequence of steps that leads, for instance, to inferring $B \sqsubseteq_{\mathcal{O}''} \exists r.A$ in Example 4.7 and $\exists u.B \sqsubseteq_{\mathcal{O}'} \exists s.\exists r.A$ in Example 4.8. They are only appropriate to "simulate" consecutive applications of the subsumption relation $\sqsubseteq^s_{\emptyset}$.

Our new translation is designed to overcome these limitations. It constructs, given τ, $\Delta_{\Gamma,\tau}$, and $\mathcal{O}$, a new set of inclusions $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$ such that the following holds:

- if γ is an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unifier of $\Delta_{\Gamma,\tau}$ compatible with τ, then there is an assignment θ_γ of sets of words over $\mathsf{N_R}$ to the indeterminates[9] in $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$ satisfying

$$\theta_\gamma(X_A) = \{w \mid \exists w.A \in \mathit{Part}(\gamma(X))\}$$

  that is a finite, admissible solution of the system $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$.

Conversely, finite, admissible solutions of $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$ yield an appropriate unifier of $\Delta_{\Gamma,\tau}$:

- if $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$ has a finite, admissible solution, then it has such a solution θ that yields an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unifier γ_θ of $\Delta_{\Gamma,\tau}$ that is compatible with τ. This unifier is defined similarly to σ_τ, but using particles provided by θ for padding:
  - if X is minimal w.r.t. $>_\tau$, then

$$\gamma_\theta(X) := \bigsqcap_{D \in S^\tau(X)} D \sqcap \bigsqcap_{A \in \mathsf{N_C}} \bigsqcap_{w \in \theta(X_A)} \exists w.A$$

  - if $\gamma_\theta(Y)$ has already been defined for all Y such that $X >_\tau Y$, then

$$\gamma_\theta(X) := \bigsqcap_{D \in S^\tau(X)} \gamma_\theta(D) \sqcap \bigsqcap_{A \in \mathsf{N_C}} \bigsqcap_{w \in \theta(X_A)} \exists w.A.$$

To achieve this, we exploit the characterization of subsumption in Lemma 1.3. Basically, given a particle $\exists w.A \in \mathit{Part}(\gamma(X))$ and a constraint $C_1 \sqcap \cdots \sqcap C_n \sqsubseteq^? X \in \Delta_{\Gamma,\tau}$, we know that $\gamma(C_1) \sqcap \cdots \sqcap \gamma(C_n) \sqsubseteq_{\mathcal{O}} \exists w.A$ holds. Hence, the idea is to encode within the inclusions in $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$, whether a conjunction of atoms and a particle satisfy the characterization of subsumption in Lemma 1.3. It is not clear to us how to simulate the satisfaction of the conditions required in Lemma 1.3 for arbitrary conjunctions and particles. Nevertheless, as we will next show, it is possible to do that for conjunctions $\sigma(C_1) \sqcap \cdots \sqcap \sigma(C_n)$ and particles $\exists w.A \in \mathit{Part}(\sigma(X))$, where σ is a special kind of unifier, which we call *simple*.

The rest of this subsection is structured as follows. We continue by formally defining the notion of a simple unifier. We then show that unifiability can be characterized by whether a simple unifier exists or not. Afterwards, we show how to exploit the properties of these unifiers to define the new set of linear inclusions $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$. We finish by proving the correctness of the new translation.

[9] In contrast to the system obtained by the old translation, the system $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$ uses additional auxiliary indeterminates that are not of the form X_A for a variable of Γ and a concept name A.

4.2.1 Simple Unifiers for $\Delta_{\Gamma,\tau}$

Let us start with the definition of simple unifiers for $\Delta_{\Gamma,\tau}$ w.r.t. $\mathcal{O}$.

Definition 4.9. The $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unifier γ of $\Delta_{\Gamma,\tau}$ w.r.t. $\mathcal{O}$ is called *simple* if, for all $C_1 \sqcap \cdots \sqcap C_n \sqsubseteq^? X \in \Delta_{\Gamma,\tau}$ and $\exists w.A \in Part(\gamma(X))$ the following holds:

1. there exists $i, 1 \leq i \leq n$ such that
   (a) C_i is a ground atom and $C_i \sqsubseteq^s_{\mathcal{O}} \exists w.A$, or
   (b) $C_i = Y$ is a variable and $\exists w.A \in Part(\gamma(C_i))$, or
   (c) $C_i = \exists r.Y$ for a variable Y, $w = sw'$ for some $s \in \mathsf{N_R}$ and $w' \in {\mathsf{N_R}}^*$, and
      - $\exists w'.A \in Part(\gamma(Y))$ and $r \trianglelefteq_{\mathcal{O}} s$, or
      - $\exists t.\exists w'.A \in Part(\gamma(Y))$ for a transitive role t such that $r \trianglelefteq_{\mathcal{O}} t \trianglelefteq_{\mathcal{O}} s$; or
2. There are atoms $At_1, \ldots, At_k, At'$ of $\mathcal{O}$ ($k \geq 0$) such that:
   (a) $At_1 \sqcap \cdots \sqcap At_k \sqsubseteq_{\mathcal{O}} At'$,
   (b) for all $\ell \in \{1, \ldots, k\}$, there exists $i \in \{1, \ldots, n\}$ such that $\tau(C_i, At_\ell) = 1$, and
   (c) $At' \sqsubseteq^s_{\mathcal{O}} \exists w.A$.

Intuitively, the first condition in this definition strengthens the first condition of Lemma 1.3. It tells us that, if $\gamma(C_i) \sqsubseteq^s_{\mathcal{O}} \exists w.A$ and C_i is not ground, then depending on the case, either

$$\gamma(Y) \sqsubseteq^s_{\emptyset} \exists w.A \quad \text{or} \quad \gamma(Y) \sqsubseteq^s_{\emptyset} \exists w'.A \quad \text{or} \quad \gamma(Y) \sqsubseteq^s_{\emptyset} \exists t.\exists w'.A.$$

This basically means that the existing translation can be re-used to simulate these structural subsumption relations. Hence, one can restrict the attention to finding linear inclusions that can capture the relation $C_i \sqsubseteq^s_{\mathcal{O}} \exists w.A$, where C_i is a ground atom of Γ and $\exists w.A$ an arbitrary particle. Regarding the second condition, it rephrases item (b) in Condition 2 of Lemma 1.3 in terms of the subsumption mapping τ. Similarly to the first condition, this will prove to be convenient to handle the relation $C_i \sqsubseteq^s_{\mathcal{O}} At_\ell$ when C_i is a non-ground atom.

The following lemma strengthens the correspondence established in Proposition 4.3, in terms of the existence of simple unifiers.

Lemma 4.10. *If Γ is an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unification problem that is unifiable w.r.t. $\mathcal{O}$, then there exists a subsumption mapping τ for Γ w.r.t. $\mathcal{O}$ such that*

- *$\Delta_{\Gamma,\tau}$ has a* simple *$\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unifier σ w.r.t. $\mathcal{O}$ that is compatible with τ.*

Proof. Assume that Γ has an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unifier δ w.r.t. $\mathcal{O}$. We define the assignment τ as

$$\tau(D_1, D_2) = 1 \text{ iff } \delta(D_1) \sqsubseteq_{\mathcal{O}} \delta(D_2), \text{ for all } D_1, D_2 \in Ats_{tr}(\Gamma, \mathcal{O}).$$

As shown in Proposition 4.3, the assignment τ is a subsumption mapping for Γ w.r.t. $\mathcal{O}$, and δ is a unifier of $\Delta_{\Gamma,\tau}$ w.r.t. $\mathcal{O}$ that is compatible with τ.

We use δ to define a substitution σ satisfying the properties required in our claim. Let us define the set of particles of δ as

$$Part(\delta) := \bigcup_{X \in dom(\delta)} Part(\delta(X)).$$

Furthermore, we denote as $CPart(\delta)$ the closure of $Part(\delta)$ under *building subconcepts* and *left concatenation of subsumed transitive roles*, i.e.,

$$\begin{aligned} CPart(\delta) := \{\exists u.A \mid w = vu \ \wedge \ \{v,u\} \subseteq \mathsf{N_R}^* \ \wedge \ \exists w.A \in Part(\delta)\} \cup \\ \{\exists t.\exists u.A \mid w = vsu \ \wedge \ s \in \mathsf{N_R} \ \wedge \ \{v,u\} \subseteq \mathsf{N_R}^* \ \wedge \ t \text{ is a transitive role} \ \wedge \\ t \trianglelefteq_{\mathcal{O}} s \ \wedge \ \exists w.A \in Part(\delta)\}. \end{aligned}$$

Since $Part(\delta)$ is a finite set and $\mathcal{O}$ contains finitely many transitive roles, it follows that $CPart(\delta)$ is also a finite set. Hence, we can extend δ into a substitution σ with $dom(\sigma) = dom(\delta)$ as follows: for all $X \in dom(\delta)$, we define:

$$\sigma(X) := \delta(X) \ \sqcap \ \bigsqcap \{\exists w.A \in CPart(\delta) \mid \delta(X) \sqsubseteq_{\mathcal{O}} \exists w.A\}.$$

Since δ is an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-substitution, this means that σ is also an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-substitution. In addition, the following observations follow from the definition of σ (for all $X \in dom(\sigma)$):

- $\sigma(X)$ is obtained from $\delta(X)$ by possibly adding new particles to the top-level conjunction of $\delta(X)$. Hence, $\sigma(X) \sqsubseteq \delta(X)$ holds.
- Every new particle $\exists w.A$ added to obtain $\sigma(X)$ is such that $\delta(X) \sqsubseteq_{\mathcal{O}} \exists w.A$. Hence, it follows that $\delta(X) \sqsubseteq_{\mathcal{O}} \sigma(X)$.

Therefore, $\sigma(X) \equiv_{\mathcal{O}} \delta(X)$ holds for all variables X, and thus $\sigma(D) \equiv_{\mathcal{O}} \delta(D)$ holds for all concept descriptions D. A direct consequence of this is that σ is a unifier of $\Delta_{\Gamma,\tau}$ w.r.t. $\mathcal{O}$ that is compatible with τ.

It remains to prove that σ is *simple*. Let $\mathfrak{s}$ be a subsumption constraint $C_1 \sqcap \cdots \sqcap C_n \sqsubseteq^? X \in \Delta_{\Gamma,\tau}$ and $\exists w.A \in Part(\sigma(X))$. We need to show that $\mathfrak{s}$ and $\exists w.A$ satisfy one of the conditions in Definition 4.9. Since $\exists w.A \in Part(\sigma(X))$, the definition of σ implies that $\exists w.A \in CPart(\delta)$, and an application of Lemma 1.2 yields $\sigma(X) \sqsubseteq_{\mathcal{O}} \exists w.A$. Moreover, since σ is a unifier of $\Delta_{\Gamma,\tau}$, we know that $\sigma(C_1) \sqcap \cdots \sqcap \sigma(C_n) \sqsubseteq_{\mathcal{O}} \sigma(X) \sqsubseteq_{\mathcal{O}} \exists w.A$. Hence, one of the two cases of Lemma 1.3 applies to this subsumption relation:

- The first case holds. Then, there is an index $i \in \{1,\ldots,n\}$ and a top-level atom D of $\sigma(C_i)$ such that $D \sqsubseteq^s_{\mathcal{O}} \exists w.A$. If C_i is a ground atom, then $C_i = D$ since C_i is flat. Hence, (1a) in Definition 4.9 immediately holds. Otherwise, $C_i = Y$ or $C_i = \exists r.Y$ for some variable Y. If $C_i = Y$, then $\delta(Y) \sqsubseteq_{\mathcal{O}} \sigma(Y) \sqsubseteq D$ and $D \sqsubseteq^s_{\mathcal{O}} \exists w.A$ imply that $\delta(Y) \sqsubseteq_{\mathcal{O}} \exists w.A$. Hence, since $\exists w.A \in CPart(\delta)$, the definition of σ yields that $\exists w.A$ is a top-level atom of $\sigma(Y)$. Thus, $\exists w.A \in Part(\sigma(Y))$ and (1b) in Definition 4.9 holds.

  It remains to look at the case where $C_i = \exists r.Y$. This means that $\sigma(C_i) = \exists r.\sigma(Y) = D$. Since $\sigma(C_i) \sqsubseteq^s_{\mathcal{O}} \exists w.A$ and $\sigma(C_i)$ is not a concept name, the definition of structural subsumption implies that $w = sw'$ for some $s \in \mathsf{N_R}$ and $w' \in \mathsf{N_R}^*$. Moreover, the definition of $\sqsubseteq^s_{\mathcal{O}}$ gives us two possibilities for having $\exists r.\sigma(Y) \sqsubseteq^s_{\mathcal{O}} \exists s.\exists w'.A$. We distinguish between these two cases, and show that each of them implies that (1c) in Definition 4.9 holds:

  - $\sigma(Y) \sqsubseteq_{\mathcal{O}} \exists w'.A$ and $r \trianglelefteq_{\mathcal{O}} s$. Hence, $\sigma(Y) \equiv_{\mathcal{O}} \delta(Y)$ implies that $\delta(Y) \sqsubseteq_{\mathcal{O}} \exists w'.A$. In addition, we know that $\exists s.\exists w'.A \in CPart(\delta)$, which means that there exists a particle $\exists w''.A \in Part(\delta)$ such that:
    - $\exists s.\exists w'.A$ is a sub-concept of $\exists w''.A$, or
    - $\exists w''.A$ has a sub-concept of the form $\exists s'.\exists w'.A$ such that $s \trianglelefteq_{\mathcal{O}} s'$.

These two cases have in common that $\exists w'.A$ is a sub-concept of $\exists w''.A$, which is a particle in $\mathit{Part}(\delta)$. Hence, since $\mathit{CPart}(\delta)$ contains the closure of $\mathit{Part}(\delta)$ under sub-concepts, we know that $\exists w'.A \in \mathit{CPart}(\delta)$. The latter, together with $\sigma(Y) \sqsubseteq_{\mathcal{O}} \exists w'.A$, yields that $\exists w'.A$ is a top-level atom of $\sigma(Y)$ (see the definition of σ). Thus, $\exists w'.A \in \mathit{Part}(\sigma(Y))$, and the first case in (1c) is true.

  - $\sigma(Y) \sqsubseteq_{\mathcal{O}} \exists t.\exists w'.A$ for a transitive role t such that $r \trianglelefteq_{\mathcal{O}} t \trianglelefteq_{\mathcal{O}} s$. As in the previous case, we can infer that $\delta(Y) \sqsubseteq_{\mathcal{O}} \exists t.\exists w'.A$. Furthermore, since $\exists s.\exists w'.A \in \mathit{CPart}(\delta)$, we know from the two cases considered above that $\exists s.\exists w'.A$ is a sub-concept of $\exists w''.A$ or $\exists s'.\exists w'.A$ is a sub-concept of $\exists w''.A$ for a particle $\exists w''.A \in \mathit{Part}(\delta)$ and a role s' with $s \trianglelefteq_{\mathcal{O}} s'$. In addition, $t \trianglelefteq_{\mathcal{O}} s \trianglelefteq_{\mathcal{O}} s'$ implies $t \trianglelefteq_{\mathcal{O}} s'$. Hence, since t is a transitive role, the definition of $\mathit{CPart}(\delta)$ yields $\exists t.\exists w'.A \in \mathit{CPart}(\delta)$. Therefore, as in the previous case, we can conclude that $\exists t.\exists w'.A$ is a top-level atom of $\sigma(Y)$. Thus, $\exists t'.\exists w'.A \in \mathit{Part}(\sigma(Y))$, and the second case in (1c) is true.

Summing up, we have shown that case 1) in Definition 4.9 holds.

- The second case of Lemma 1.3 holds. Hence, there are atoms $At_1, \dots, At_k, At'$ of $\mathcal{O}$ ($k \geq 0$) such that:
  - $At_1 \sqcap \cdots \sqcap At_k \sqsubseteq_{\mathcal{O}} At'$,
  - for every $\ell \in \{1, \dots, k\}$, there is an index $i \in \{1, \dots, n\}$ and a top-level atom D of $\sigma(C_i)$ such that $D \sqsubseteq^s_{\mathcal{O}} At_\ell$, and
  - $At' \sqsubseteq^s_{\mathcal{O}} \exists w.A$.

  We only need to show that (2b) in Definition 4.9 holds. Suppose that $k > 0$ and let $\ell \in \{1, \dots, k\}$. Then, there exists $i \in \{1, \dots, n\}$ and a top-level atom D of $\sigma(C_i)$ such that $D \sqsubseteq^s_{\mathcal{O}} At_\ell$. This implies that $\sigma(C_i) \sqsubseteq_{\mathcal{O}} At_\ell$ since $\sigma(C_i) \sqsubseteq D$. Hence, since $\sigma(C_i) \equiv_{\mathcal{O}} \delta(C_i)$, it follows that $\delta(C_i) \sqsubseteq_{\mathcal{O}} At_\ell$. The latter implies that $\tau(C_i, At_\ell) = 1$, by construction of τ. Thus, we have shown that case 2) in Definition 4.9 holds.

Overall, we have shown that σ is a *simple* $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unifier of $\Delta_{\Gamma,\tau}$ w.r.t. $\mathcal{O}$ that is compatible with τ. This concludes the proof. □

4.2.2 The Set of Linear Inclusions $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$

The inclusions in the set $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$ must take into account a non-empty ontology $\mathcal{O}$. To this end, the right-hand sides of the original language inclusions in $\mathfrak{I}_{\Gamma,\tau}$ must be extended. Our new translation yields, for each $\mathfrak{s} = C_1 \sqcap \cdots \sqcap C_n \sqsubseteq^? X \in \Delta_{\Gamma,\tau}$ and each concept constant A, a linear language inclusion $\mathfrak{i}^*_A(\mathfrak{s})$ of the form

$$X_A \subseteq f^*_A(C_1) \cup \cdots \cup f^*_A(C_n) \cup \mathcal{U}_A(\mathfrak{s}), \tag{6}$$

where $f^*_A(C)$ differs from $f_A(C)$ in the way existential restrictions are treated:

$$f^*_A(\exists r.C') := L_r f_A(C'), \text{ where } L_r := \{s \in \mathsf{N_R} \mid r \trianglelefteq_{\mathcal{O}} s\}.$$

The intention is that the right-hand side of the inclusion $\mathfrak{i}^*_A(\mathfrak{s})$ should capture words $w \in {\mathsf{N_R}}^*$ satisfying that

$$\gamma(C_1) \sqcap \cdots \sqcap \gamma(C_n) \sqsubseteq_{\mathcal{O}} \exists w.A, \tag{7}$$

where γ is a *simple* $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unifier of $\Delta_{\Gamma,\tau}$ w.r.t. $\mathcal{O}$ and $\exists w.A \in \mathit{Part}(\gamma(X))$. The fact that γ can be assumed to be a simple unifier tells us that one of the cases in Definition 4.9 applies whenever the subsumption in (7) holds. Thus, our idea to define the right-hand side of $\mathfrak{i}^*_A(\mathfrak{s})$ is to include terms that can encode the cases stated in Definition 4.9. Among theses cases, the one stated in (1b) is already taken care of since it is covered by the original translation. More precisely,

- this case states that $\exists w.A \in Part(\gamma(C_i))$, where C_i is a variable Y. By Lemma 1.2, this means that $\gamma(C_i) \sqsubseteq \exists w.A$. Since this subsumption does not require the ontology, we can just re-use the term $f_A^*(Y) = f_A(Y) = Y_A$.

The other cases are not that trivial since they depend on the axioms in the ontology. A first step towards encoding them is already implemented by the modification of f_A to f_A^*, which takes care of the role hierarchy induced by $\mathcal{O}$.

Example 4.11. For instance, if in the system of Example 4.7 we replace $B \sqsubseteq^? Y$ with $\exists u.X \sqsubseteq^? Y$, then the language inclusions corresponding to this constraint are

$$Y_A \subseteq \{u\}X_A \ \text{ and } \ Y_B \subseteq \{u\}X_B.$$

The new system again does not have an admissible solution. However, if we consider an ontology $\mathcal{O}$ containing $u \sqsubseteq s$, then the application of the new translation to this constraint yields the language inclusions

$$Y_A \subseteq \{u, s\}X_A \ \text{ and } \ Y_B \subseteq \{u, s\}X_B.$$

Consequently, the new system of language inclusions has a finite, admissible solution. For instance, since the other inclusion for Y_A is $Y_A \subseteq \{s\}X_A \cup \{r\}\{\varepsilon\}$ and there are no language inclusions constraining X_A or X_B, the following assignment is such a solution:

$$\theta(X_A) := \{\varepsilon\}, \ \ \theta(Y_A) := \{s\}, \ \ \theta(X_B) = \theta(Y_B) := \emptyset.$$

This reflects the fact that the substitution $\gamma := \{X \mapsto A, Y \mapsto \exists s.A\}$ is an $\mathcal{ELH}_{\mathcal{R}^+}^{-\top}$-unifier of the modified system of subsumption constraints w.r.t. $\mathcal{O}$. △

The scenario illustrated in this example is an instance of the general case where (7) follows from the first part of (1c) in Definition 4.9. In fact, with the simple modification of f_A to f_A^* we can already simulate the general case. Let us briefly explain why this is true.

- The first part of (1c) considers the situation where $C_i = \exists r.Y$ for some variable Y, $w = sw'$ for some $s \in \mathsf{N_R}$ and $w' \in \mathsf{N_R}^*$, and the following holds:

  - $r \trianglelefteq_{\mathcal{O}} s$ and $\exists w'.A \in Part(\gamma(Y))$.

  An application of Lemma 1.2 yields that $\gamma(Y) \sqsubseteq \exists w'.A$. Hence, we can use the term $f_A^*(\exists r.Y) = L_r Y_A$ in $\mathfrak{i}_A^*(\mathfrak{s})$, since the prefix set L_r contains all role names s that satisfy $r \trianglelefteq_{\mathcal{O}} s$, and $\gamma(Y) \sqsubseteq \exists w'.A$ holds w.r.t. the empty ontology.

  Coming back to Example 4.11, note that $u \trianglelefteq_{\mathcal{O}} s$ and $A \in Part(\gamma(X))$, since $u \sqsubseteq s \in \mathcal{O}$ and $\gamma(X) = A$, respectively. This implies that $\exists u.\gamma(X) \sqsubseteq_{\mathcal{O}} \gamma(Y) = \exists s.A$. It also implies that $\exists u.X$ and $\exists s.A$ satisfy the first case of (1c). This is captured by the term $f_A^*(\exists u.X) = L_u X_A = \{u, s\}X_A$ in the linear inclusion obtained from $\exists u.X \sqsubseteq^? Y$.

The remaining cases from Definition 4.9 depend on the GCIs and transitivity axioms of the ontology. They are taken care of by the additional term $\mathcal{U}_A(\mathfrak{s})$ in (6). This term uses additional types of indeterminates whose meaning is encoded using additional language inclusions. Let us first consider the cases (1a) and (2) of Definition 4.9, which describe scenarios that depend on the GCIs of the ontology. In what follows, we first introduce the indeterminates and language inclusions used to take care of these cases. Then we explain how they can be used to encode the aforementioned cases into the definition of $\mathcal{U}_A(\mathfrak{s})$.

For all concept constants A and B occurring in Γ and $\mathcal{O}$, we introduce an indeterminate of the form $Z_{B\to A}$. The purpose of these indeterminates is to represent languages containing only words w such that $B \sqsubseteq_{\mathcal{O}} \exists w.A$. This intuition is formalized by a corresponding set of linear language inclusions $\mathfrak{I}_{\mathcal{O}}$ that we will shortly introduce. Its definition is inspired by the following result, which is an easy consequence of Lemma 2.4.

Proposition 4.12. *Let $\mathcal{O}$ be a flat and cycle-restricted $\mathcal{ELH}_{\mathcal{R}^+}$-ontology, $A, B \in \mathsf{N_C}$, $r \in \mathsf{N_R}$ and $w' \in \mathsf{N_R}^*$. Then, $B \sqsubseteq_{\mathcal{O}} \exists r.\exists w'.A$ iff there exists $B' \in Ats(\mathcal{O}) \cap \mathsf{N_C}$ such that:*

- *$B \sqsubseteq_{\mathcal{O}} \exists r.B'$ and $B' \sqsubseteq_{\mathcal{O}} \exists w'.A$.*

Proof. The implication from *right-to-left* is obvious. For the other direction, assume that $B \sqsubseteq_{\mathcal{O}} \exists r.\exists w'.A$. By Lemma 2.4, there exists $\exists u.B' \in Ats(\mathcal{O})$ such that $B' \sqsubseteq_{\mathcal{O}} \exists w'.A$, and

- $B \sqsubseteq_{\mathcal{O}} \exists u.B'$ and $u \trianglelefteq_{\mathcal{O}} r$, or
- $B \sqsubseteq_{\mathcal{O}} \exists t.B'$ for a transitive role t with $u \trianglelefteq_{\mathcal{O}} t \trianglelefteq_{\mathcal{O}} r$.

Both cases have in common that $B \sqsubseteq_{\mathcal{O}} \exists r.B'$. Thus, the implication from *left-to-right* holds. □

Intuitively, this tells us that a subsumption relationship $B \sqsubseteq_{\mathcal{O}} \exists w.A$, with $w = r_1 r_2 \ldots r_n$ $(n > 0)$, can be explained by a finite sequence of subsumption relationships

$$B \sqsubseteq_{\mathcal{O}} \exists r_1.B_1,\ B_1 \sqsubseteq_{\mathcal{O}} \exists r_2.B_2,\ \ldots,\ B_{n-1} \sqsubseteq_{\mathcal{O}} \exists r_n.B_n,\ B_n \sqsubseteq_{\mathcal{O}} A,$$

where each B_i is a concept name occurring in $\mathcal{O}$ $(1 \leq i \leq n)$. Based on this, we define $\mathfrak{I}_{\mathcal{O}}$ in the following way. For each concept name $B \in \mathsf{N_C}$, we define $I(B)$ as the following set:

$$I(B) := \{(r, B') \in \mathsf{N_R} \times (Ats(\mathcal{O}) \cap \mathsf{N_C}) \mid B \sqsubseteq_{\mathcal{O}} \exists r.B'\}.$$

Then, the set of linear inclusions $\mathfrak{I}_{\mathcal{O}}$ consists of one language inclusion for each indeterminate $Z_{B \to A}$ having the following form:

$$Z_{B \to A} \subseteq L \cup \bigcup_{(r,B') \in I(B)} \{r\} Z_{B' \to A}, \tag{8}$$

where $L := \{\varepsilon\}$ if $B \sqsubseteq_{\mathcal{O}} A$, and $L := \emptyset$ otherwise. The system of linear inclusions $\mathfrak{I}_{\mathcal{O}}$ captures subsumptions of the form $B \sqsubseteq_{\mathcal{O}} \exists w.A$ in the following sense.

Lemma 4.13. *Let $\mathcal{O}$ be a flat, cycle-restricted $\mathcal{ELH}_{\mathcal{R}^+}$-ontology.*

1. *If θ is a solution of $\mathfrak{I}_{\mathcal{O}}$, then $w \in \theta(Z_{B \to A})$ implies $B \sqsubseteq_{\mathcal{O}} \exists w.A$.*
2. *If we define $\theta(Z_{B \to A}) := \{w \in \mathsf{N_R}^* \mid B \sqsubseteq_{\mathcal{O}} \exists w.A\}$, then θ is a finite solution of $\mathfrak{I}_{\mathcal{O}}$.*

The proof of this lemma requires several steps. We defer it to the next subsection. Let us see how the indeterminates $Z_{B \to A}$ help in defining $\mathcal{U}_A(\mathfrak{s})$. The following example gives a glimpse of the intuition behind this.

Example 4.14. Consider again the system $\Delta_{\Gamma_3,\tau}$ of Example 4.7, but replace $B \sqsubseteq^? Y$ with $\exists r.B \sqsubseteq^? Y$. The language inclusions corresponding to this constraint are

$$Y_A \subseteq \{r\}\emptyset \ \text{ and } \ Y_B \subseteq \{r\}\{\varepsilon\}.$$

The new system again does not have an admissible solution. However, if we consider the ontology $\mathcal{O} = \{B \sqsubseteq A\}$, then there are solutions θ of $\mathfrak{I}_{\mathcal{O}}$ that satisfy $\varepsilon \in \theta(Z_{B \to A})$. Thus, if we extend the language inclusion $Y_A \subseteq \{r\}\emptyset$ obtained from $\exists r.B \sqsubseteq^? Y$ to

$$Y_A \subseteq \{r\}\emptyset \cup \{r\} Z_{B \to A},$$

then the new system of language inclusions has a solution θ such that $r \in \theta(Y_A)$ since the other inclusion for Y_A is $Y_A \subseteq \{s\}X_A \cup \{r\}\{\varepsilon\}$. Hence, it follows that there is an admissible solution since there are no language inclusions constraining X_A or X_B.

This reflects the fact that the modified set of subsumption constraints has an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unifier w.r.t. $\mathcal{O}$. For instance, the substitution $\gamma := \{X \mapsto B, Y \mapsto \exists r.A\}$ is one such unifier. Note that $\exists r.B \sqsubseteq_{\mathcal{O}} \gamma(Y) = \exists r.A$ because $\mathcal{O}$ contains the GCI $B \sqsubseteq A$. This is captured in the extended language inclusion $Y_A \subseteq \emptyset \cup \{r\}Z_{B\to A}$ by the term $\{r\}Z_{B\to A}$. △

This example illustrates an instance of the more general case where (7) follows from case (1a) in Definition 4.9. This simple instance would be encoded by including $\{r\}Z_{B\to A}$ into the term $\mathcal{U}_A(\mathfrak{s})$ of the language inclusion obtained from $\mathfrak{s} = \exists r.B \sqsubseteq^? Y$. We generalize this intuition to cover the scenarios described in case (1a) and case (2) of Definition 4.9.

- Case (1a). This requires the left-hand side of (7) to have a ground top-level atom C_i such that $C_i \sqsubseteq^s_{\mathcal{O}} \exists w.A$. The flat form of $\Delta_{\Gamma,\tau}$ implies that C_i is of the form $\exists\alpha.B$ where $\alpha \in \{\varepsilon\} \cup \mathsf{N_R}$ and $B \in \mathsf{N_C}$. By definition of structural subsumption, $C_i \sqsubseteq^s_{\mathcal{O}} \exists w.A$ holds iff one of the following is the case:

  - $C_i = \exists\varepsilon.B = B = A$ and $w = \varepsilon$.
  - $C_i = \exists r.B$, $w = sw'$ for some $s \in \mathsf{N_R}$ satisfying $r \trianglelefteq_{\mathcal{O}} s$, and $B \sqsubseteq_{\mathcal{O}} \exists w'.A$.
  - $C_i = \exists r.B$, $w = sw'$, and $B \sqsubseteq_{\mathcal{O}} \exists t.\exists w'.A$ for a transitive role t such that $r \trianglelefteq_{\mathcal{O}} t \trianglelefteq_{\mathcal{O}} s$.

  The first possibility is already covered since $f^*_A(A) = \{\varepsilon\}$, which means that the right-hand side of $\mathfrak{i}^*_A(\mathfrak{s})$ already has a term that matches $w = \varepsilon$. Let us continue with the second one.

  - We simulate this case in $\mathfrak{i}^*_A(\mathfrak{s})$ by including in $\mathcal{U}_A(\mathfrak{s})$ a term of the form:

    $$L_r Z_{B\to A}, \quad \text{where } L_r := \{s \in \mathsf{N_R} \mid r \trianglelefteq_{\mathcal{O}} s\}.$$

    In this way, L_r matches any role name s such that $r \trianglelefteq_{\mathcal{O}} s$, whereas $Z_{B\to A}$ takes care of recognizing w'. The use of $Z_{B\to A}$ is based on the idea that this indeterminate is meant to represent the set of all words $w' \in \mathsf{N_R}^*$ such that $B \sqsubseteq_{\mathcal{O}} \exists w'.A$, as explained above.

  It remains to consider the third case.

  - In this case, an application of Proposition 4.12 yields that $B \sqsubseteq_{\mathcal{O}} \exists t.\exists w'.A$ iff there is $B' \in \mathit{Ats}(\mathcal{O}) \cap \mathsf{N_C}$ such that:

    $$B \sqsubseteq_{\mathcal{O}} \exists t.B' \quad \text{and} \quad B' \sqsubseteq_{\mathcal{O}} \exists w'.A.$$

    Hence, to recognize the words $w = sw'$ such that $\exists r.B \sqsubseteq^s_{\mathcal{O}} \exists s.\exists w'.A$ satisfies the third case, we can use a term of the form $L_t Z_{B'\to A}$ for each suitable t and B'. This way, the prefix set L_t matches any role name s such that $r \trianglelefteq_{\mathcal{O}} t \trianglelefteq_{\mathcal{O}} s$, whereas $Z_{B'\to A}$ recognizes the words w' such that $B' \sqsubseteq_{\mathcal{O}} \exists w'.A$. We denote as $Tr(\exists r.B)$ the set of such suitable pairs (t, B'), i.e.,

    $$Tr(\exists r.B) := \{(t, B') \mid B' \in \mathit{Ats}(\mathcal{O}) \cap \mathsf{N_C},\ t \text{ is transitive},\ r \trianglelefteq_{\mathcal{O}} t, \text{ and } B \sqsubseteq_{\mathcal{O}} \exists t.B'\}.$$

    Thus, we include in $\mathcal{U}_A(\mathfrak{s})$ a term of the form $L_t Z_{B'\to A}$ for each $(t, B') \in Tr(\exists r.B)$.

- Case (2). This case is defined in terms of atoms At of $\mathcal{O}$ such that $\tau(C_i, At) = 1$ for some $i \in \{1, \ldots, n\}$, as required by Condition (2b). We denote by $C_{\mathfrak{s},\tau}$ the conjunction of such atoms, i.e.,

  $$C_{\mathfrak{s},\tau} := \bigsqcap\{At \in \mathit{Ats}(\mathcal{O}) \mid \tau(C_i, At) = 1 \text{ for some } i \in \{1, \ldots, n\}\}. \tag{9}$$

  Then, (7) satisfies the second case in Definition 4.9 iff there exists $At \in \mathit{Ats}(\mathcal{O})$ such that:

$$C_{\mathfrak{s},\tau} \sqsubseteq_{\mathcal{O}} At \sqsubseteq^{s}_{\mathcal{O}} \exists w.A.$$

These conditions can also be simulated by using the set $\{\varepsilon\}$ and the new indeterminates $Z_{B\to A}$. Note that $At \sqsubseteq^{s}_{\mathcal{O}} \exists w.A$ is similar to the case (1a) considered above. Basically, for each $At \in Ats(\mathcal{O})$ such that $C_{\mathfrak{s},\tau} \sqsubseteq_{\mathcal{O}} At$, we include in $\mathcal{U}_A(\mathfrak{s})$:

- a term of the form $\{\varepsilon\}$, if $At = A$, and
- in case $At = \exists r.B$, one of the form $L_r Z_{B\to A}$, as well as all terms obtained from $Tr(\exists r.B)$.

It only remains to consider the second part of (1c) in Definition 4.9. This case depends on the transitivity axioms in the ontology. To deal with these axioms, additional indeterminates and linear language inclusions are needed. We continue by introducing them. Afterwards, we will explain how they are used to encode the second part of (1c) into $\mathcal{U}_A(\mathfrak{s})$.

We introduce additional indeterminates of the form $X_{A,t}$, which are constrained by the following linear language inclusions:

$$\mathfrak{i}_{A,t}(\mathfrak{s}) = X_{A,t} \subseteq f_{A,t}(C_1) \cup \cdots \cup f_{A,t}(C_n) \cup \mathcal{U}_{A,t}(\mathfrak{s}), \text{ where} \tag{10}$$

$$f_{A,t}(C) := \begin{cases} f_A(C') & \text{if } C = \exists r.C' \wedge r \trianglelefteq_{\mathcal{O}} t, \\ Y_{A,t} & \text{if } C = Y \in \mathsf{N_V}, \\ \emptyset & \text{otherwise.} \end{cases}$$

Intuitively, the difference between $\mathfrak{i}^*_A(\mathfrak{s})$ and $\mathfrak{i}_{A,t}(\mathfrak{s})$ is that, given a particle $\exists t.\exists w.A$ satisfying (7), the right-hand side of $\mathfrak{i}_{A,t}(\mathfrak{s})$ is designed to recognize w instead of tw. This can already be seen with the use of $f_{A,t}$ instead of f^*_A in the definition of $\mathfrak{i}_{A,t}(\mathfrak{s})$.

Example 4.15. Suppose that a particle $\exists t.\exists w.A$ satisfies (7) w.r.t. some ontology $\mathcal{O}$ because there is an $i, 1 \leq i \leq n$, such that

$$C_i = \exists r.Y, \quad r \trianglelefteq_{\mathcal{O}} t, \quad \gamma(Y) \sqsubseteq \exists w.A.$$

The right-hand side of the inclusion $\mathfrak{i}^*_A(\mathfrak{s})$ contains the term $f^*_A(\exists r.Y) = L_r Y_A$. This term matches t through L_r and recognizes w via Y_A. In contrast, the right-hand side of $\mathfrak{i}_{A,t}(\mathfrak{s})$ contains the term $f_{A,t}(\exists r.Y) = Y_A$, provided that $r \trianglelefteq_{\mathcal{O}} t$ holds. The condition $r \trianglelefteq_{\mathcal{O}} t$ ensures that t is implicitly taken into account, while using Y_A instead of $L_r Y_A$ is in line with the idea of recognizing just w.

Consider now the scenario in which $\exists t.\exists w.A$ satisfies (7) for the following reasons:

$$C_i = Y \quad \text{and} \quad \gamma(Y) \sqsubseteq \exists t.\exists w.A.$$

To treat this case, differently from $f^*_A(Y) = Y_A$, the function $f_{A,t}$ defines $f_{A,t}(Y) = Y_{A,t}$. The reason is that including Y_A in $\mathfrak{i}_{A,t}(\mathfrak{s})$ to recognize w would forget the fact that t still needs to be taken into account. △

The term $\mathcal{U}_{A,t}(\mathfrak{s})$ in (10) has the same purpose as $\mathcal{U}_A(\mathfrak{s})$ has for $\mathfrak{i}^*_A(\mathfrak{s})$. It consists of a slight variant of $\mathcal{U}_A(\mathfrak{s})$ that fits with the intended meaning of $\mathfrak{i}_{A,t}(\mathfrak{s})$, i.e., its right-hand is supposed to recognize w instead of tw. The definition of $\mathcal{U}_{A,t}(\mathfrak{s})$ will become clear later on, after we fully define $\mathcal{U}_A(\mathfrak{s})$.

Let us continue by explaining how the indeterminates $X_{A,t}$ contribute to the definition of $\mathcal{U}_A(\mathfrak{s})$ in $\mathfrak{i}^*_A(\mathfrak{s})$. The following example illustrates the intuition behind this.

Example 4.16. Assume that

$$\Delta_{\Gamma,\tau} = \{\exists r.B \sqsubseteq^? Y, \exists s.X \sqcap \exists r.A \sqsubseteq^? Y, \exists t.B \sqsubseteq^? X\}.$$

In addition, consider the ontology $\mathcal{O} = \{s \sqsubseteq t, t \sqsubseteq r\}$. Since $\exists r.B \sqsubseteq^? Y$ yields the language inclusion $Y_A \subseteq \{r\}\emptyset$, any solution θ of $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$ must satisfy $\theta(Y_A) = \emptyset$. Hence, if θ is admissible, then $\theta(Y_B) \neq \emptyset$. In the presence of $\mathcal{O}$, the new translation also yields the inclusions:

$$Y_B \subseteq \{r\}\{\varepsilon\}, \quad Y_B \subseteq \{s,t,r\}X_B \cup \{r\}\emptyset \;\text{ and }\; X_B \subseteq \{t,r\}\{\varepsilon\}.$$

Together with $\theta(Y_B) \neq \emptyset$, the first of these inclusions yields $\theta(Y_B) = \{r\}$. Thus, the second inclusion implies that $\varepsilon \in \theta(X_B)$, and thus θ does not solve the third inclusion. Consequently, $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$ cannot have an admissible solution, corresponding to the fact that $\Delta_{\Gamma,\tau}$ does not have an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unifier w.r.t. $\mathcal{O}$.

However, if we add the transitivity axiom $t \circ t \sqsubseteq t$ to $\mathcal{O}$, then $\Delta_{\Gamma,\tau}$ has an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unifier γ with $\gamma(X) = \exists t.B$ and $\gamma(Y) = \exists r.B$ w.r.t. this ontology. The inclusion $\mathfrak{i}_{B,t}(\mathfrak{s}) = X_{B,t} \subseteq \{\varepsilon\}$, obtained from $\mathfrak{s} = \exists t.B \sqsubseteq^? X$, admits solutions θ with $\theta(X_{B,t}) = \{\varepsilon\}$. Hence, if we extend the language inclusion $Y_B \subseteq \{s,t,r\}X_B \cup \emptyset$ to the new one

$$Y_B \subseteq \{s,t,r\}X_B \cup \{r\}\emptyset \cup \{r\}X_{B,t}$$

that takes transitivity of t into account, then the new system of language inclusions has an admissible solution with $\theta(Y_B) = \{r\}$ and $\theta(X_B) = \{t\}$, which corresponds to the unifier γ. $\triangle$

In this example, the role inclusions and the transitivity axiom in $\mathcal{O}$ ensure that $\exists s.\exists t.B \sqsubseteq_{\mathcal{O}} \exists r.B$ holds, which implies that $\exists s.\gamma(X) \sqcap \exists r.A \sqsubseteq_{\mathcal{O}} \gamma(Y) = \exists r.B$ holds. It is not hard to see that this represents an instance of the more general case where (7) follows from the second part of (1c) in Definition 4.9. We now explain how to generalize these ideas to encode such a case into $\mathcal{U}_A(\mathfrak{s})$.

- The second part of (1c) requires that $C_i = \exists r.Y$ for some variable Y, $w = sw'$ for some $s \in \mathsf{N_R}$ and $w' \in \mathsf{N_R}^*$, and the following holds:
  - there is a transitive role t such that $r \trianglelefteq_{\mathcal{O}} t \trianglelefteq_{\mathcal{O}} s$ and $\exists t.\exists w'.A \in \mathit{Part}(\gamma(Y))$.

  Since $C_i = \exists r.Y$, the right-hand side of the inclusion $\mathfrak{i}^*_A(\mathfrak{s})$ contains a term of the form $f^*_A(\exists r.Y) = L_r Y_A$. Hence, in order to admit sw', one could in principle try to use L_r to match the admissible role names s since $r \trianglelefteq_{\mathcal{O}} s$. However, using $L_r Y_A$ to match the whole word sw' would not be correct, since this case requires $\exists t.\exists w'.A$ to be a particle of $\gamma(Y)$ instead of simply $\exists w'.A$.

  Basically, this scenario requires us to simulate the introduction of t. This is where the new indeterminates $Y_{A,t}$ and the inclusions of the form $\mathfrak{i}_{A,t}(\mathfrak{s})$ are needed. More precisely, for each transitive role t such that $r \trianglelefteq_{\mathcal{O}} t$, we include in $\mathcal{U}_A(\mathfrak{s})$ a term of the form:

  $$L_t Y_{A,t}.$$

  In this way, the prefix L_t matches the appropriate role names s, while $Y_{A,t}$ recognizes w' but remembers that t needs to be taken into account.

Finally, we are ready to provide the formal definitions of $\mathcal{U}_A(\mathfrak{s})$ and $\mathcal{U}_{A,t}(\mathfrak{s})$, which together with (6) and (10) then completes the definition of the language inclusions $\mathfrak{i}^*_A(\mathfrak{s})$ and $\mathfrak{i}_{A,t}(\mathfrak{s})$.

We start by defining the following subset of $\mathit{Ats}(\Gamma, \mathcal{O})$:

$$R_{\mathfrak{s},\tau} := \{\exists r.B \mid (C_i = \exists r.B \;\text{ for some } i, 1 \leq i \leq n) \text{ or } (C_{\mathfrak{s},\tau} \sqsubseteq_{\mathcal{O}} \exists r.B \wedge \exists r.B \in \mathit{Ats}(\mathcal{O}))\}.$$

These atoms are the ones that generate the terms of the form $L_r Z_{B\to A}$ and $L_t Z_{B'\to A}$ (where the latter are obtained from $Tr(\exists r.B)$) that are relevant to define $\mathcal{U}_A(\mathfrak{s})$, as discussed above in the analysis of the cases (1a) and (2) from Definition 4.9. As for the terms of the form $L_t Y_{A,t}$, they are derived from pairs $(Y,t) \in \mathit{Vars}(\Delta_{\Gamma,\tau}) \times \mathsf{N_R}$ satisfying the conditions described above when analyzing the second part of (1c). We collect all admissible such pairs in the set

$$V_{\mathfrak{s}} := \{(Y,t) \mid C_i = \exists r.Y \text{ for some } i, 1 \le i \le n,\ r \trianglelefteq_{\mathcal{O}} t,\ t \text{ is a transitive role}\}.$$

Hence, taking into account the previous analysis concerning the cases (1a), second part of (1c), and (2) from Definition 4.9, we define $\mathcal{U}_A(\mathfrak{s})$ as

$$\mathcal{U}_A(\mathfrak{s}) := L_{\mathfrak{s},\tau} \cup \bigcup_{\exists r.B \in R_{\mathfrak{s},\tau}} \left(L_r Z_{B\to A} \cup \bigcup_{(t,B')\in Tr(\exists r.B)} L_t Z_{B'\to A} \right) \cup \bigcup_{(Y,t)\in V_{\mathfrak{s}}} L_t Y_{A,t}, \quad (11)$$

where $L_{\mathfrak{s},\tau} := \{\varepsilon\}$ if $C_{\mathfrak{s},\tau} \sqsubseteq_{\mathcal{O}} A$ and $A \in \mathit{Ats}(\mathcal{O})$, and $L_{\mathfrak{s},\tau} := \emptyset$ otherwise.

We now modify the definition of $\mathcal{U}_A(\mathfrak{s})$ to formally define $\mathcal{U}_{A,t}(\mathfrak{s})$. More precisely, based on the discussions about $\mathfrak{i}_{A,t}(\mathfrak{s})$, the terms occurring in $\mathcal{U}_A(\mathfrak{s})$ that are relevant for $\mathcal{U}_{A,t}(\mathfrak{s})$ are those whose prefix set L_r contains t. Hence, $\mathcal{U}_{A,t}(\mathfrak{s})$ is defined by dispensing with those prefix sets, as well as the term $L_{\mathfrak{s},\tau}$:

$$\mathcal{U}_{A,t}(\mathfrak{s}) := \bigcup_{\substack{\exists r.B \in R_{\mathfrak{s},\tau} \\ t \in L_r}} \left(Z_{B\to A} \cup \bigcup_{\substack{(t',B')\in Tr(\exists r.B) \\ t\in L_{t'}}} Z_{B'\to A} \right) \cup \bigcup_{\substack{(Y,t')\in V_{\mathfrak{s}} \\ t\in L_{t'}}} Y_{A,t'}. \quad (12)$$

Definition 4.17. The system of linear language inclusions $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$ consists of $\mathfrak{I}_{\mathcal{O}}$ and the inclusions $\mathfrak{i}^*_A(\mathfrak{s})$ and $\mathfrak{i}_{A,t}(\mathfrak{s})$ for every subsumption constraint $\mathfrak{s}$ in $\Delta_{\Gamma,\tau}$, as defined in (6), (11) and (10), (12), respectively. We call a solution θ of $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$ *admissible* if for each variable X in $\Delta_{\Gamma,\tau}$ there exists $A \in \mathsf{N_C}$ such that $\theta(X_A) \neq \emptyset$.

The next step is to prove that the new translation is correct, i.e., to show the following proposition.

Proposition 4.18. *Let τ be a subsumption mapping for Γ w.r.t. $\mathcal{O}$. The unification problem $\Delta_{\Gamma,\tau}$ has an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unifier γ w.r.t. $\mathcal{O}$ that is compatible with τ iff the system of linear language inclusions $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$ has a finite, admissible solution.*

We proceed in two steps. First, we show the properties stated in Lemma 4.13 about $\mathfrak{I}_{\mathcal{O}}$. Once we have done this, we can prove both directions of Proposition 4.18.

4.2.3 Proof of Lemma 4.13

We start by proving the first of the statements claimed in Lemma 4.13.

Lemma 4.19. *Let $\mathcal{O}$ be a flat and cycle-restricted $\mathcal{ELH}_{\mathcal{R}^+}$-ontology. If θ is a solution of $\mathfrak{I}_{\mathcal{O}}$, then for all $w \in \mathsf{N_R}^*$ and $A, B \in \mathsf{N_C}$ the following holds:*

$$w \in \theta(Z_{B\to A}) \text{ implies } B \sqsubseteq_{\mathcal{O}} \exists w.A.$$

Proof. Let $w \in \mathsf{N_R}^*$ and $A, B \in \mathsf{N_C}$ such that $w \in \theta(Z_{B\to A})$. We prove $B \sqsubseteq_{\mathcal{O}} \exists w.A$ by induction on the length of w.

- $|w| = 0$. This means that $w = \varepsilon$, and thus $\varepsilon \in \theta(Z_{B\to A})$ and $\exists w.A = A$. Since θ is a solution of $\mathfrak{I}_\mathcal{O}$, ε must belong to $\theta(\mathfrak{t})$ for some term $\mathfrak{t}$ on the right-hand side of (8). This can only be the case if $L = \{\varepsilon\}$, which in turn is only the case if $B \sqsubseteq_\mathcal{O} A$.

- $|w| > 0$. Then, $w = rw'$ for some $r \in \mathsf{N_R}$ and $w' \in \mathsf{N_R}^*$. In this case, $w \in \theta(Z_{B\to A})$ implies the existence of a term of the form $\{r\}Z_{B'\to A}$ on the right-hand side of (8) such that $w' \in \theta(Z_{B'\to A})$. Since $|w'| < |w|$, the induction hypothesis applied to $w' \in \theta(Z_{B'\to A})$ yields $B' \sqsubseteq_\mathcal{O} \exists w'.A$. Furthermore, by definition of (8), the presence of the term $\{r\}Z_{B'\to A}$ in the right-hand side of the inclusion means that $(r, B') \in I(B)$. Hence, by definition of $I(B)$, we know that $B \sqsubseteq_\mathcal{O} \exists r.B'$. Together with $B' \sqsubseteq_\mathcal{O} \exists w'.A$, this subsumption relationship implies $B \sqsubseteq_\mathcal{O} \exists w.A$.

This concludes the proof since we have shown that $B \sqsubseteq_\mathcal{O} \exists w.A$ holds in both cases. □

The proof of the second statement of Lemma 4.13 is given in two steps. The first one proves that the set of all valid subsumption relationships $B \sqsubseteq_\mathcal{O} \exists w.A$ induces a solution of $\mathfrak{I}_\mathcal{O}$.

Lemma 4.20. *Let $\mathcal{O}$ be a flat and cycle-restricted $\mathcal{ELH}_{\mathcal{R}^+}$-ontology. Define θ as the following assignment of subsets of $\mathsf{N_R}^*$ to the indeterminates $Z_{B\to A}$:*

$$\theta(Z_{B\to A}) := \{w \in \mathsf{N_R}^* \mid B \sqsubseteq_\mathcal{O} \exists w.A\}.$$

Then, θ is a solution of $\mathfrak{I}_\mathcal{O}$.

Proof. Consider an indeterminate $Z_{B\to A}$ and the corresponding inclusion (8) in $\mathfrak{I}_\mathcal{O}$, i.e.,

$$Z_{B\to A} \subseteq L \cup \bigcup_{(r,B')\in I(B)} \{r\}Z_{B'\to A} \quad \text{where } L := \{\varepsilon\} \text{ if } B \sqsubseteq_\mathcal{O} A, \text{ and } L := \emptyset \text{ otherwise.}$$

Let $w \in \theta(Z_{B\to A})$. We need to show that $w \in \theta(\mathfrak{t})$ for some term $\mathfrak{t}$ on the right-hand side of this inclusion. To this end, we make the following case distinction:

- $w = \varepsilon$. By definition of $\theta(Z_{B\to A})$, this means that $B \sqsubseteq_\mathcal{O} A$. Hence, as defined in (8), the set L is equal to $\{\varepsilon\}$. Thus, $w = \varepsilon \in \theta(L)$.

- $w = rw'$ for some $r \in \mathsf{N_R}$ and $w' \in \mathsf{N_R}^*$. This means that $B \sqsubseteq_\mathcal{O} \exists r.\exists w'.A$. An application of Proposition 4.12 yields a concept name $B' \in \mathit{Ats}(\mathcal{O})$ such that $B \sqsubseteq_\mathcal{O} \exists r.B'$ and $B' \sqsubseteq_\mathcal{O} \exists w'.A$. This implies that $(r, B') \in I(B)$. Consequently, the right-hand side of the linear inclusion must contain a term of the form $\{r\}Z_{B'\to A}$. Moreover, since $B' \sqsubseteq_\mathcal{O} \exists w'.A$, the definition of θ implies that $w' \in \theta(Z_{B'\to A})$. Thus, we have that $w \in \{r\}\cdot\theta(Z_{B'\to A})$.

Overall, we have shown that, for all words $w \in \theta(Z_{B\to A})$, there is a term $\mathfrak{t}$ on the right-hand side of the linear inclusion (8) introduced for $Z_{B\to A}$ such that $w \in \theta(\mathfrak{t})$. Thus, we can conclude that θ is a solution of $\mathfrak{I}_\mathcal{O}$. □

It remains to show that the solution introduced in this lemma is finite. The next lemma does this by establishing a bound on the length of words in solutions of $\mathfrak{I}_\mathcal{O}$, when $\mathcal{O}$ is cycle-restricted.

Lemma 4.21. *Let $\mathcal{O}$ be a cycle-restricted $\mathcal{ELH}_{\mathcal{R}^+}$-ontology, θ be a solution of $\mathfrak{I}_\mathcal{O}$, and k be the number of distinct concept names occurring in $\mathcal{O}$. Then $|w| \leq k$ holds for all indeterminates $Z_{B\to A}$ and words $w \in \theta(Z_{B\to A})$.*

Proof. Let $Z_{B\to A}$ be an indeterminate such that $w \in \theta(Z_{B\to A})$ for some $w \in \mathsf{N_R}^*$. The case where $w = \varepsilon$ is trivial. Hence, we now assume that $w = r_1 r_2 \ldots r_n$ with $n > 0$.

Since θ is a solution of $\mathfrak{I}_\mathcal{O}$ and $w \in \theta(Z_{B\to A})$, the inclusion corresponding to $Z_{B\to A}$ in $\mathfrak{I}_\mathcal{O}$ must contain a term of the form $\{r_1\}Z_{B_1\to A}$ on its right-hand side, such that $r_2 \ldots r_n \in \theta(Z_{B_1\to A})$. By repeatedly applying this argument until we are left with the suffix ε, we can infer that there are concept names $B_0, B_1, \ldots, B_n$ such that $B_0 = B$, $B_1, \ldots, B_n$ occur in $\mathcal{O}$, and the following holds for all $j, 1 \leq j \leq n$:

- $\{r_j\}Z_{B_j\to A}$ is a term on the right-hand side of the inclusion corresponding to $Z_{B_{j-1}\to A}$.

This means that $(r_j, B_j) \in I(B_{j-1})$, which by definition of $I(B_{j-1})$ yields that $B_{j-1} \sqsubseteq_\mathcal{O} \exists r_j.B_j$ for all $j, 1 \leq j \leq n$. Consequently, the following subsumption relationships hold:

$$B_{j-1} \sqsubseteq_\mathcal{O} \exists r_j.\exists r_{j+1}.\cdots\exists r_{j+m}.B_{j+m} \quad (1 \leq j \leq n, \quad 0 \leq m \leq n-j). \tag{13}$$

Suppose now that $|w| > k$. Since $B_1, \ldots, B_n$ occur in $\mathcal{O}$, there must exist two indices $1 \leq i < j \leq n$ such that $B_i = B_j$. Hence, (13) implies that $B_i \sqsubseteq_\mathcal{O} \exists r_{i+1}.\cdots\exists r_{i+m}.B_i$ with $m \geq 1$. Since this contradicts our assumption that $\mathcal{O}$ is cycle-restricted, we can conclude that $|w| \leq k$. □

Recall that $\mathsf{N_R}$ is assumed to be the set of role names occurring in Γ or $\mathcal{O}$, which is a finite set. Hence, this lemma implies that, if $\mathcal{O}$ is cycle-restricted, then all solutions of $\mathfrak{I}_\mathcal{O}$ are finite. Thus, the previous three lemmas provide us with a proof of Lemma 4.13.

4.2.4 Proof of Proposition 4.18

We are now ready to show the correctness of our new translation. Let us start by proving the *left-to-right* implication in Proposition 4.18. By Lemma 4.10, it is enough to show this implication for simple $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unifiers of $\Delta_{\Gamma,\tau}$ w.r.t. $\mathcal{O}$. This is done in Lemma 4.23 below, with the help of the result shown in the following lemma.

Lemma 4.22. *Let $\mathfrak{s} \in \Delta_{\Gamma,\tau}$, $A \in \mathsf{N_C}$, and $\mathfrak{i}^*_A(\mathfrak{s})$ be the language inclusion of $\mathfrak{I}^\mathcal{O}_{\Gamma,\tau}$ obtained from $\mathfrak{s}$ and A. In addition, let $L_r T$ be a term on the right-hand side of $\mathfrak{i}^*_A(\mathfrak{s})$, where $r \in \mathsf{N_R}$. Then, for all transitive roles t of $\mathcal{O}$ such that $r \trianglelefteq_\mathcal{O} t$, the following holds:*

- *the right-hand side of the inclusion $\mathfrak{i}_{A,t}(\mathfrak{s})$ of $\mathfrak{I}^\mathcal{O}_{\Gamma,\tau}$ (obtained from $\mathfrak{s}$, A and t) contains a term of the form T.*

Proof. Let $L_r T$ be a term on the right-hand side of $\mathfrak{i}^*_A(\mathfrak{s})$ such that $r \in \mathsf{N_R}$. Assume $\mathfrak{s}$ is of the form $C_1 \sqcap \cdots \sqcap C_n \sqsubseteq^? X$. We consider the two possible ways such a term can occur in $\mathfrak{i}^*_A(\mathfrak{s})$.

- $f^*_A(C_i) = L_r T$, where C_i is of the form $\exists r.C'$ $(1 \leq i \leq n)$. This means that $f^*_A(C') = T$. Since $\Delta_{\Gamma,\tau}$ is in flat form, C' is a concept name. Hence, $f^*_A(C') = f_A(C') = T$. Now, the definition of $\mathfrak{i}_{A,t}(\mathfrak{s})$ tells us that $f_{A,t}(\exists r.C')$ is a term on the right-hand side of this linear inclusion. Hence, since $r \trianglelefteq_\mathcal{O} t$, we have that $f_{A,t}(\exists r.C') = f_A(C') = T$.
- $L_r T$ is a term of the union $\mathcal{U}_A(\mathfrak{s})$. By looking at the definitions of $\mathcal{U}_A(\mathfrak{s})$ and $\mathcal{U}_{A,t}(\mathfrak{s})$ in (11) and (12), it is easy to see that $r \trianglelefteq_\mathcal{O} t$ implies that T is a term of $\mathcal{U}_{A,t}(\mathfrak{s})$.

Thus, in both cases we can conclude that the right-hand side of $\mathfrak{i}_{A,t}(\mathfrak{s})$ contains T. □

Lemma 4.23. *Let τ be a subsumption mapping for Γ w.r.t. $\mathcal{O}$. If $\Delta_{\Gamma,\tau}$ has a* simple *$\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unifier γ w.r.t. $\mathcal{O}$ that is compatible with τ, then $\mathfrak{I}^\mathcal{O}_{\Gamma,\tau}$ has a finite, admissible solution.*

Proof. Let γ be a *simple* $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unifier of $\Delta_{\Gamma,\tau}$ w.r.t. $\mathcal{O}$ that is compatible with the subsumption mapping τ. As explained in Section 2, we can without loss of generality assume that γ is a ground substitution. We define a solution θ_γ of $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$ as follows:

- For each $X \in \mathit{Vars}(\Delta_{\Gamma,\tau})$, concept constant A, and transitive role t, we define:

$$\theta_\gamma(X_A) := \{w \in \mathsf{N_R}^* \mid \exists w.A \in \mathit{Part}(\gamma(X))\},$$
$$\theta_\gamma(X_{A,t}) := \{w \in \mathsf{N_R}^* \mid \exists t.\exists w.A \in \mathit{Part}(\gamma(X))\}.$$

- For each indeterminate $Z_{B\to A}$, we define:

$$\theta_\gamma(Z_{B\to A}) := \{w \in \mathsf{N_R}^* \mid B \sqsubseteq_{\mathcal{O}} \exists w.A\}.$$

For all indeterminates $Z_{B\to A}$, we can infer from Lemma 4.13 that $\theta_\gamma(Z_{B\to A})$ is a finite set. This is also the case for all indeterminates of the form X_A and $X_{A,t}$, since $\mathit{Part}(\gamma(X))$ is a finite set and $\mathcal{O}$ has finitely many transitive roles. Hence, θ_γ is finite. The admissibility of θ_γ is a direct consequence of γ being a ground $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-substitution. The reason is that $\gamma(X)$ is mapped to an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-concept for all $X \in \mathit{Vars}(\Delta_{\Gamma,\tau})$. Hence, $\mathit{Part}(\gamma(X)) \neq \emptyset$, which by definition of θ_γ implies that $\theta_\gamma(X_A) \neq \emptyset$ for some concept constant A.

It remains to show that θ_γ is indeed a solution of $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$. To start with, the definition of $\theta_\gamma(Z_{B\to A})$ and the application of Lemma 4.13 immediately yields that θ_γ solves all inclusions in $\mathfrak{I}_{\mathcal{O}}$. To show that θ_γ also solves the other inclusions in $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$, we first show that it solves all inclusions of the form $\mathfrak{i}^*_A(\mathfrak{s})$. Then, we will use the result in Lemma 4.22 to prove that it also solves the ones of the form $\mathfrak{i}_{A,t}(\mathfrak{s})$.

Let $\mathfrak{i}^*_A(\mathfrak{s}) \in \mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$, where $\mathfrak{s} \in \Delta_{\Gamma,\tau}$ is of the form $C_1 \sqcap \cdots \sqcap C_n \sqsubseteq^? X$. Then, by its definition in (6), the language inclusion $\mathfrak{i}^*_A(\mathfrak{s})$ has the following form:

$$X_A \subseteq f^*_A(C_1) \cup \cdots \cup f^*_A(C_n) \cup \mathcal{U}_A(\mathfrak{s}).$$

If $\theta_\gamma(X_A) = \emptyset$, then θ_γ trivially solves $\mathfrak{i}^*_A(\mathfrak{s})$. Otherwise, let $w \in \theta_\gamma(X_A)$. By definition of θ_γ, we know that $\exists w.A \in \mathit{Part}(\gamma(X))$. Since γ is a *simple* unifier, one of the cases in Definition 4.9 holds for $\mathfrak{s}$ and $\exists w.A$. For each such case, we show that $w \in \theta_\gamma(R(\mathfrak{i}^*_A(\mathfrak{s})))$, where $R(\mathfrak{i}^*_A(\mathfrak{s}))$ denotes the right-hand side of $\mathfrak{i}^*_A(\mathfrak{s})$:

- There is $i \in \{1,\ldots,n\}$ such that C_i is a ground atom and $C_i \sqsubseteq^s_{\mathcal{O}} \exists w.A$. This structural subsumption relationship holds due to one of the following cases:
  - $C_i = \exists w.A \in \mathsf{N_C}$. This means that $w = \varepsilon$ and $C_i = \exists w.A = A$. Hence, in this case we have $f^*_A(C_i) = \{\varepsilon\}$. Therefore, $\{\varepsilon\}$ is a term on the right-hand side of $\mathfrak{i}^*_A(\mathfrak{s})$ and $\varepsilon \in \theta_\gamma(\{\varepsilon\})$. Thus, $w \in \theta_\gamma(R(\mathfrak{i}^*_A(\mathfrak{s})))$.
  - $C_i = \exists r.B$, $w = sw'$, $r \trianglelefteq_{\mathcal{O}} s$, and $B \sqsubseteq_{\mathcal{O}} \exists w'.A$. We know that $B \in \mathsf{N_C}$ because Γ is in flat form. In addition, by definition of $R_{\mathfrak{s},\tau}$, we know that $\exists r.B \in R_{\mathfrak{s},\tau}$. Hence, $L_r Z_{B\to A}$ is a term in $\mathcal{U}_A(\mathfrak{s})$. Finally, $r \trianglelefteq_{\mathcal{O}} s$ yields that $s \in L_r$, whereas $B \sqsubseteq_{\mathcal{O}} \exists w'.A$ and the definition of θ_γ imply that $w' \in \theta_\gamma(Z_{B\to A})$. Thus, $w \in L_r{\cdot}\theta_\gamma(Z_{B\to A})$, which implies that $w \in \theta_\gamma(R(\mathfrak{i}^*_A(\mathfrak{s})))$.
  - $C_i = \exists r.B$, $w = sw'$, and $B \sqsubseteq_{\mathcal{O}} \exists t.\exists w'.A$ for a transitive role t such that $r \trianglelefteq_{\mathcal{O}} t \trianglelefteq_{\mathcal{O}} s$. By Proposition 4.12, there exists $B' \in \mathit{Ats}(\mathcal{O}) \cap \mathsf{N_C}$ such that

$$B \sqsubseteq_{\mathcal{O}} \exists t.B' \text{ and } B' \sqsubseteq_{\mathcal{O}} \exists w'.A.$$

    Hence, since $r \trianglelefteq_{\mathcal{O}} t$ and t is transitive, we know that $(t, B') \in \mathit{Tr}(\exists r.B)$. Furthermore, $C_i = \exists r.B$ implies that $\exists r.B \in R_{\mathfrak{s},\tau}$. Therefore, $L_t Z_{B'\to A}$ is a term in $\mathcal{U}_A(\mathfrak{s})$. Finally, as in the previous case, we have that $s \in L_t$ and $w' \in \theta_\gamma(Z_{B'\to A})$. Thus, $w \in L_t{\cdot}\theta_\gamma(Z_{B'\to A})$, which yields $w \in \theta_\gamma(R(\mathfrak{i}^*_A(\mathfrak{s})))$.

- There is $i \in \{1, \dots, n\}$ such that $C_i = Y$ for a variable Y and $\exists w.A \in \mathit{Part}(\gamma(C_i))$. The definition of $\mathfrak{i}_A^*(\mathfrak{s})$ yields $f_A^*(C_i) = Y_A$ and the definition of θ_γ yields $w \in \theta_\gamma(Y_A)$. Thus, $w \in \theta_\gamma(R(\mathfrak{i}_A^*(\mathfrak{s})))$.

- There is $i \in \{1, \dots, n\}$ such that $C_i = \exists r.Y$ for a variable Y, $w = sw'$, and one of the two cases in (1c) holds. In the first case, $\exists w'.A \in \mathit{Part}(\gamma(Y))$ and $r \trianglelefteq_{\mathcal{O}} s$. By definition of θ_γ and L_r, it follows that $w' \in \theta_\gamma(Y_A)$ and $s \in L_r$, and thus $w \in L_r{\cdot}\theta_\gamma(Y_A)$. Since $f_A^*(\exists r.Y) = L_r Y_A$, we can conclude that $w \in \theta_\gamma(R(\mathfrak{i}_A^*(\mathfrak{s})))$.

  In the second case, $\exists t.\exists w'.A \in \mathit{Part}(\gamma(Y))$ for a transitive role t such that $r \trianglelefteq_{\mathcal{O}} t \trianglelefteq_{\mathcal{O}} s$. By definition of θ_γ, we have that $w' \in \theta_\gamma(Y_{A,t})$. Moreover, since t is transitive and $r \trianglelefteq_{\mathcal{O}} t$, the pair (Y, t) belongs to the set $V_\mathfrak{s}$. This implies that $\mathcal{U}_A(\mathfrak{s})$ contains a term of the form $L_t Y_{A,t}$. In addition, $t \trianglelefteq_{\mathcal{O}} s$ implies that $s \in L_t$. Thus, $w \in L_t{\cdot}\theta_\gamma(Y_{A,t})$, which yields $w \in \theta_\gamma(R(\mathfrak{i}_A^*(\mathfrak{s})))$.

- There are atoms $At_1, \dots, At_k, At'$ of $\mathcal{O}$ ($k \geq 0$) satisfying the three conditions in the second case of Definition 4.9. In case $k > 0$, all atoms $At_1, \dots, At_k$ are conjuncts of $C_{\mathfrak{s},\tau}$ (recall the definition in (9)), since they satisfy Condition (2b). Hence, together with Conditions (2a) and (2c) of Definition 4.9, this yields:

$$C_{\mathfrak{s},\tau} \sqsubseteq At_1 \sqcap \dots \sqcap At_k \sqsubseteq_{\mathcal{O}} At' \sqsubseteq_{\mathcal{O}}^s \exists w.A.$$

  The same holds for $k = 0$, since the empty conjunction corresponds to $\top$. Let us continue by considering the possible forms of At'.

  - $At' = B$ for some $B \in \mathsf{N_C}$. In this case, $At' \sqsubseteq_{\mathcal{O}}^s \exists w.A$ implies $w = \varepsilon$ and $B = A$. Since $C_{\mathfrak{s},\tau} \sqsubseteq_{\mathcal{O}} At'$, it follows that $L_{\mathfrak{s},\tau} = \{\varepsilon\}$. Hence, the right-hand side of $\mathfrak{i}_A^*(\mathfrak{s})$ contains a term of the form $\{\varepsilon\}$. Thus, $w \in \theta_\gamma(R(\mathfrak{i}_A^*(\mathfrak{s})))$.
  - $At' = \exists r.B$ for some $r \in \mathsf{N_R}$ and $B \in \mathsf{N_C}$. Since $C_{\mathfrak{s},\tau} \sqsubseteq_{\mathcal{O}} At'$, we know that $\exists r.B \in R_{\mathfrak{s},\tau}$. This implies that $L_r Z_{B \to A}$ is a term in $\mathcal{U}_A(\mathfrak{s})$, as well as all terms $L_t Z_{B' \to A}$ with $(t, B') \in \mathit{Tr}(\exists r.B)$. To conclude the proof, recall that $At' = \exists r.B \sqsubseteq_{\mathcal{O}}^s \exists w.A$. Then, by employing the arguments used in the last two cases considering a ground atom C_i, we can show that $w \in L_r{\cdot}\theta_\gamma(Z_{B \to A})$, or $w \in L_t{\cdot}\theta_\gamma(Z_{B' \to A})$ for some $(t, B') \in \mathit{Tr}(\exists r.B)$. Thus, we again obtain $w \in \theta_\gamma(R(\mathfrak{i}_A^*(\mathfrak{s})))$.

Thus, we have now shown that θ_γ also solves all inclusions of the form $\mathfrak{i}_A^*(\mathfrak{s})$. It remains to deal with the language inclusions of the form $\mathfrak{i}_{A,t}(\mathfrak{s})$ for a transitive role t. By (10), such an inclusion has the following form:

$$X_{A,t} \subseteq f_{A,t}(C_1) \cup \dots \cup f_{A,t}(C_n) \cup \mathcal{U}_{A,t}(\mathfrak{s}).$$

Assume that $w \in \theta_\gamma(X_{A,t})$ for some $w \in \mathsf{N_R}^*$. By definition of θ_γ, this means that $\exists t.\exists w.A \in \mathit{Part}(\gamma(X))$, and thus $tw \in \theta_\gamma(X_A)$. We have already proved that θ_γ solves the language inclusion $\mathfrak{i}_A^*(\mathfrak{s})$. Hence, the right-hand side of $\mathfrak{i}_A^*(\mathfrak{s})$ contains a term $\mathfrak{t}$ such that $tw \in \theta_\gamma(\mathfrak{t})$. This term must be of one of the following forms:

- $\mathfrak{t} = L_r T$, where $r \in \mathsf{N_R}$ such that $r \trianglelefteq_{\mathcal{O}} t$. This means that $w \in \theta_\gamma(T)$. An application of Lemma 4.22 yields that the right-hand side of $\mathfrak{i}_{A,t}(\mathfrak{s})$ contains the term T. Hence, $w \in \theta_\gamma(R(\mathfrak{i}_{A,t}(\mathfrak{s})))$.

- $\mathfrak{t} = Y_A$ for some variable Y. This means that $\mathfrak{s}$ contains an atom $C_i = Y$ (for some $i, 1 \leq i \leq n$). Hence, the right-hand side of $\mathfrak{i}_{A,t}(\mathfrak{s})$ contains the term $f_{A,t}(Y) = Y_{A,t}$. Furthermore, $tw \in \theta_\gamma(\mathfrak{t})$ yields $tw \in \theta_\gamma(Y_A)$. Hence, by definition of θ_γ, it follows that $\exists t.\exists w.A \in \mathit{Part}(\gamma(Y))$, which then implies that $w \in \theta_\gamma(Y_{A,t})$. Thus, we can conclude that $w \in \theta_\gamma(R(\mathfrak{i}_{A,t}(\mathfrak{s})))$.

Overall, we have shown that θ_γ solves all language inclusions in $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$. Thus, $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$ has a finite, admissible solution. □

We continue by proving the *right-to-left* direction of Proposition 4.18. But first, we must show an auxiliary result, which states that solvability of $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$ implies the existence of a special kind of solutions.

Lemma 4.24. *If $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$ has a finite, admissible solution, then it has a finite, admissible solution θ such that the following holds for all $X \in Vars(\Delta_{\Gamma,\tau})$, $A \in \mathsf{N_C}$, $w \in \mathsf{N_R}^*$, and transitive roles t:*

$$w \in \theta(X_{A,t}) \text{ implies } tw \in \theta(X_A). \tag{14}$$

Proof. Let θ be a solution of $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$. We extend θ to an assignment θ' as follows. For each indeterminate of the form X_A in $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$, we define:

$$\theta'(X_A) := \theta(X_A) \cup \{tw \mid w \in \theta(X_{A,t}) \text{ for a transitive role } t\}.$$

For any other indeterminate W, we define $\theta'(W) := \theta(W)$.

Since θ is finite and $\mathcal{O}$ has finitely many transitive roles, θ' is also finite. Hence, since $\theta(W) \subseteq \theta'(W)$ for all indeterminates W of $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$ and θ is admissible, we can conclude that θ' is a finite, admissible assignment. It remains to show that θ' is also a solution of $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$.

Since θ is a solution of $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$, and the indeterminates of the form X_A whose assignment may differ between θ and θ' occur only in language inclusion of the form $\mathfrak{i}^*_A(\mathfrak{s})$, it is sufficient to check these inclusions. In case θ' does not solve $\mathfrak{i}^*_A(\mathfrak{s})$, then this can only be caused by a word tw assigned to $\theta'(X_A)$ for some $w \in \theta(X_{A,t})$. Hence, to see that θ' is really a solution of $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$, it is enough to prove that $w \in \theta(X_{A,t})$ implies that the right-hand side of $\mathfrak{i}^*_A(\mathfrak{s})$ contains a term $\mathfrak{t}$ such that $tw \in \theta'(\mathfrak{t})$. To this end, consider the language inclusion $\mathfrak{i}_{A,t}(\mathfrak{s})$. Since θ is a solution of $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$ and $w \in \theta(X_{A,t})$, the right-hand side of $\mathfrak{i}_{A,t}(\mathfrak{s})$ contains a term $\mathfrak{t}'$ such that $w \in \theta(\mathfrak{t}')$. Let us look at the possible forms of the term $\mathfrak{t}'$:

- $\mathfrak{t}' = f_{A,t}(C)$ for some top-level atom C of $\mathfrak{s}$. This means that $C = \exists r.C'$ and $r \trianglelefteq_{\mathcal{O}} t$, or $C = Y$ for some variable Y. In the first case, we know that $f_{A,t}(C) = f_A(C')$ and $w \in \theta(f_A(C'))$. Moreover, by definition of $\mathfrak{i}^*_A(\mathfrak{s})$, its right-hand side contains a term $\mathfrak{t} = f^*_A(\exists r.C') = L_r f_A(C')$. Hence, since $r \trianglelefteq_{\mathcal{O}} t$ implies $t \in L_r$, it follows that $tw \in \theta(\mathfrak{t})$. This implies that $tw \in \theta'(\mathfrak{t})$ since θ is contained in θ'.

  Regarding the second case, we know that $f_{A,t}(Y) = Y_{A,t}$ and $w \in \theta(Y_{A,t})$. The definition of θ' then yields $tw \in \theta'(Y_A)$. Thus, since $\mathfrak{i}^*_A(\mathfrak{s})$ contains the term $f^*_A(Y) = Y_A$, this case also satisfies the claim.

- $\mathfrak{t}'$ is a term of $\mathcal{U}_{A,t}(\mathfrak{s})$. By comparing the definitions of $\mathcal{U}_{A,t}(\mathfrak{s})$ and $\mathcal{U}_A(\mathfrak{s})$, it is not hard to see that $\mathfrak{t}'$ corresponds to a term $\mathfrak{t}$ of $\mathcal{U}_A(\mathfrak{s})$ such that $tw \in \theta'(\mathfrak{t})$.

This concludes the proof of the lemma. □

In addition to the previous lemma, we will also use the binary relation $>_\tau \subseteq Vars(\Gamma) \times Vars(\Gamma)$ induced by the assignment S^τ. Note that $>_\tau$ is a well-founded strict order, since S^τ is acyclic and $Vars(\Gamma)$ is a finite set. Based on this, the proof of the next lemma shows how to use S^τ and a finite, admissible solution of $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$, to construct an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unifier of $\Delta_{\Gamma,\tau}$ w.r.t. $\mathcal{O}$ that is compatible with τ.

Lemma 4.25. *Let τ be a subsumption mapping for Γ w.r.t. $\mathcal{O}$. If $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$ has a finite, admissible solution, then $\Delta_{\Gamma,\tau}$ has an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unifier w.r.t. $\mathcal{O}$ that is compatible with τ.*

Proof. Let θ be a finite, admissible solution of $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$. By Lemma 4.24, we can assume that θ satisfies the implication in (14). We use θ and the relation $>_\tau$ induced by S^τ to define the substitution γ_θ. More precisely, we define γ_θ by well-founded induction on $>_\tau$ (for all $X \in \mathit{Vars}(\Delta_{\Gamma,\tau})$):

- If X is minimal w.r.t. $>_\tau$, then

$$\gamma_\theta(X) := \bigsqcap_{D \in S^\tau(X)} D \sqcap \bigsqcap_{A \in \mathsf{N_C}} \bigsqcap_{w \in \theta(X_A)} \exists w.A.$$

- If $\gamma_\theta(X)$ has already been defined for all variables Y with $X >_\tau Y$, then

$$\gamma_\theta(X) := \bigsqcap_{D \in S^\tau(X)} \gamma_\theta(D) \sqcap \bigsqcap_{A \in \mathsf{N_C}} \bigsqcap_{w \in \theta(X_A)} \exists w.A.$$

Since θ is finite and admissible, we have the following consequences:

- θ assigns finite subsets of $\mathsf{N_R}^*$ to each indeterminate in $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$.
- For each $X \in \mathit{Vars}(\Delta_{\Gamma,\tau})$, there exists at least one indeterminate X_A such that $\theta(X_A) \neq \emptyset$.

Hence, it is easy to see that γ_θ really is an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-substitution. Moreover, by definition of γ_θ, we know that $D \in S^\tau(X)$ implies $\gamma_\theta(X) \sqsubseteq \gamma_\theta(D)$ for all variables $X \in \mathsf{N_V}$. Consequently, $S^\tau(X) \subseteq S^{\gamma_\theta}(X)$ holds for all $X \in \mathsf{N_V}$. Thus, γ_θ is compatible with τ.

It remains to show that γ_θ is a unifier of $\Delta_{\Gamma,\tau}$ w.r.t. $\mathcal{O}$. To this end, we show that all $X \in \mathit{Vars}(\Delta_{\Gamma,\tau})$ satisfy the following property:

$$\text{If } \; C_1 \sqcap \cdots \sqcap C_n \sqsubseteq^? X \in \Delta_{\Gamma,\tau} \; \text{ then } \; \gamma_\theta(C_1) \sqcap \cdots \sqcap \gamma_\theta(C_n) \sqsubseteq_{\mathcal{O}} \gamma_\theta(X). \tag{15}$$

Since all subsumption constraints in $\Delta_{\Gamma,\tau}$ are of the form $C_1 \sqcap \cdots \sqcap C_n \sqsubseteq^? X$ for some $X \in \mathit{Vars}(\Delta_{\Gamma,\tau})$, showing (15) is sufficient to prove that γ_θ solves all constraints in $\Delta_{\Gamma,\tau}$.

The proof is by *well-founded induction* on $>_\tau$. More precisely, given $X \in \mathit{Vars}(\Delta_{\Gamma,\tau})$ and a subsumption constraint $\mathfrak{s} = C_1 \sqcap \cdots \sqcap C_n \sqsubseteq^? X$ in $\Delta_{\Gamma,\tau}$, we must show that γ_θ satisfies the right-hand side of (15) for $\mathfrak{s}$, under the assumption that (15) holds for all Y such that $X >_\tau Y$. To show this, it is enough to prove that $\gamma_\theta(C_1) \sqcap \cdots \sqcap \gamma_\theta(C_n) \sqsubseteq_{\mathcal{O}} At$ for each top-level atom At of $\gamma_\theta(X)$. We distinguish the two possible forms such a top-level atom At can have:

- $At = \gamma_\theta(C)$ for a non-variable atom $C \in S^\tau(X)$. This means that $\tau(X, C) = 1$. If $\mathfrak{s} \in \Delta_\Gamma \subseteq \Gamma$, we can apply (3b) in Definition 4.1 to obtain that $C_1 \sqcap \cdots \sqcap C_n \sqsubseteq^? C$ satisfies Condition (3a) in Definition 4.1, which yields two possibilities:
  - $\tau(C_i, C) = 1$ for some $i \in \{1, \ldots, n\}$. If C is a ground atom, the first case of Lemma 4.5 can be directly applied to obtain that $\gamma_\theta(C_i) \sqsubseteq_{\mathcal{O}} \gamma_\theta(C)$. Otherwise, C is of the form $\exists r.Y$ for some variable Y. Since $\tau(X, C) = 1$, this means that $\exists r.Y \in S^\tau(X)$. Hence, $X >_\tau Y$ and Y satisfies (15), i.e., γ_θ solves all subsumption constraints of the form $\cdots \sqcap \cdots \sqsubseteq^? Y$ in $\Delta_{\Gamma,\tau}$. Since $\Delta_\tau \subseteq \Delta_{\Gamma,\tau}$, the second case of Lemma 4.5 can be applied to obtain that $\gamma_\theta(C_i) \sqsubseteq_{\mathcal{O}} \gamma_\theta(\exists r.Y) = \gamma_\theta(C)$. Therefore, $\gamma_\theta(C_1) \sqcap \cdots \sqcap \gamma_\theta(C_n) \sqsubseteq_{\mathcal{O}} \gamma_\theta(C)$ holds.
  - There are atoms $At_1, \ldots, At_k, At'$ of $\mathcal{O}$ ($k \geq 0$) satisfying the properties listed in Condition (3a) of Definition 4.1. The first of these properties tells us that

$$At_1 \sqcap \cdots \sqcap At_k \sqsubseteq_{\mathcal{O}} At',$$

whereas the other two, combined with induction and an application of Lemma 4.5 (as in the previous case), yield that for each $\ell \in \{1, \ldots, k\}$ there exists $i \in \{1, \ldots, n\}$ such that $\gamma_\theta(C_i) \sqsubseteq_\mathcal{O} At_\ell$, and $At' \sqsubseteq_\mathcal{O} \gamma_\theta(C)$. Overall, we obtain the following subsumption relationships:

$$\gamma_\theta(C_1) \sqcap \cdots \sqcap \gamma_\theta(C_n) \sqsubseteq_\mathcal{O} At_1 \sqcap \cdots \sqcap At_k \sqsubseteq_\mathcal{O} At' \sqsubseteq_\mathcal{O} \gamma_\theta(C).$$

Therefore, $\gamma_\theta(C_1) \sqcap \cdots \sqcap \gamma_\theta(C_n) \sqsubseteq_\mathcal{O} \gamma_\theta(C)$ holds.

Finally, if $\mathfrak{s} \in \Delta_\tau$, then $\mathfrak{s}$ is of the form $C_1 \sqsubseteq^? X$ and $\tau(C_1, X) = 1$. Since $\tau(X, C) = 1$, an application of (1b) in Definition 4.1 yields that $\tau(C_1, C) = 1$. As shown for $\tau(C_i, C) = 1$ above, we obtain that $\gamma_\theta(C_1) \sqsubseteq_\mathcal{O} \gamma_\theta(C)$ holds.

Summing up, we have thus shown that $\gamma_\theta(C_1) \sqcap \cdots \sqcap \gamma_\theta(C_n) \sqsubseteq_\mathcal{O} \gamma_\theta(C)$ for all $C \in S^\tau(X)$.

- $At = \exists w.A$ for some $A \in \mathsf{N_C}$ and $w \in \theta(X_A)$. Let us consider the language inclusion $\mathfrak{i}^*_A(\mathfrak{s})$ in $\mathfrak{I}^\mathcal{O}_{\Gamma,\tau}$ obtained from $\mathfrak{s}$ and A, i.e.,

$$\mathfrak{i}^*_A(\mathfrak{s}) \quad = \quad X_A \subseteq f^*_A(C_1) \cup \cdots \cup f^*_A(C_n) \cup \mathcal{U}_A(\mathfrak{s}).$$

Since θ is a solution of $\mathfrak{I}^\mathcal{O}_{\Gamma,\tau}$ and $w \in \theta(X_A)$, there exists a term $\mathfrak{t}$ on the right-hand side of $\mathfrak{i}^*_A(\mathfrak{s})$ such that $w \in \theta(\mathfrak{t})$. Let us first look at the case where $\mathfrak{t} = f^*_A(C_i)$ for some $i \in \{1, \ldots, n\}$. We show that $\gamma_\theta(C_i) \sqsubseteq_\mathcal{O} \exists w.A$, by distinguishing between C_i being ground or not:

  - C_i is a ground atom. Then, either $C_i = A$ or $C_i = \exists r.A$, for otherwise $f^*_A(C_i) = \emptyset$ contradicting $w \in \theta(\mathfrak{t})$. This yields two possible forms for $\mathfrak{t}$:

  $$\mathfrak{t} = f^*_A(A) = \{\varepsilon\} \quad \text{or} \quad \mathfrak{t} = f^*_A(\exists r.A) = L_r\{\varepsilon\}.$$

  Since $w \in \theta(\mathfrak{t})$, this means that $w = \varepsilon$ or $w = s \in L_r$, respectively. By definition of L_r, the second case yields that $r \trianglelefteq_\mathcal{O} s$. Thus, in both cases we have that $\gamma_\theta(C_i) \sqsubseteq_\mathcal{O} \exists w.A$

  - C_i is not ground. In case $C_i = Y$ for some variable Y, we have $\mathfrak{t} = f^*_A(Y) = Y_A$ and $w \in \theta(Y_A)$. By definition of γ_θ, the latter implies that $\exists w.A$ is a top-level conjunct of $\gamma_\theta(Y)$. Hence, $\gamma_\theta(C_i) \sqsubseteq \exists w.A$.

  The other possible case is $C_i = \exists r.Y$. This means that $\mathfrak{t} = f^*_A(\exists r.Y) = L_r Y_A$ and $w \in L_r \cdot \theta(Y_A)$. Consequently, $w = sw'$ for some $s \in \mathsf{N_R}$ and $w' \in \mathsf{N_R}^*$ such that $r \trianglelefteq_\mathcal{O} s$ and $w' \in \theta(Y_A)$. As in the previous case, the latter implies that $\gamma_\theta(Y) \sqsubseteq \exists w'.A$. Thus, since $r \trianglelefteq_\mathcal{O} s$ and $C_i = \exists r.Y$, it follows that $\gamma_\theta(C_i) \sqsubseteq_\mathcal{O} \exists w.A$.

As a direct consequence of this case distinction, we obtain that $\gamma_\theta(C_1) \sqcap \cdots \sqcap \gamma_\theta(C_n) \sqsubseteq_\mathcal{O} \exists w.A$, whenever $\mathfrak{t}$ is of the form $f^*_A(C_i)$.

It remains to consider the case where $\mathfrak{t}$ is a term in $\mathcal{U}_A(\mathfrak{s})$. These terms are of the form $\{\varepsilon\}$, $L_r Z_{B \to A}$, $L_t Z_{B' \to A}$ or $L_t Y_{A,t}$. We distinguish between these cases:

  - $\mathfrak{t} = \{\varepsilon\}$. This means that $w = \varepsilon$ and $C_{\mathfrak{s},\tau} \sqsubseteq_\mathcal{O} A$. Recall the definition of $C_{\mathfrak{s},\tau}$ in (9):

  $$C_{\mathfrak{s},\tau} = \bigsqcap\{At \in Ats(\mathcal{O}) \mid \tau(C_i, At) = 1 \text{ for some } i \in \{1, \ldots, n\}\}.$$

  Note that each atom At in this conjunction is ground. Hence, since γ_θ is compatible with τ, we can apply Lemma 4.5 to obtain that $\gamma_\theta(C_i) \sqsubseteq_\mathcal{O} At$ for some $i \in \{1, \ldots, n\}$. As a consequence of this, we obtain

  $$\gamma_\theta(C_1) \sqcap \cdots \sqcap \gamma_\theta(C_n) \sqsubseteq_\mathcal{O} C_{\mathfrak{s},\tau} \sqsubseteq_\mathcal{O} A.$$

  Thus, since $A = \exists w.A$, it follows that $\gamma_\theta(C_1) \sqcap \cdots \sqcap \gamma_\theta(C_n) \sqsubseteq_\mathcal{O} \exists w.A$.

– $\mathfrak{t} = L_r Z_{B\to A}$. In this case, $w \in L_r{\cdot}\theta(Z_{B\to A})$ implies that $w = sw'$ for some $s \in \mathsf{N_R}$ and $w' \in \mathsf{N_R}^*$ such that

$$s \in L_r = \{s \in \mathsf{N_R} \mid r \trianglelefteq_{\mathcal{O}} s\} \quad \text{and} \quad w' \in \theta(Z_{B\to A}).$$

Since θ is a solution of $\mathfrak{I}_{\mathcal{O}}$, an application of Lemma 4.13 to $w' \in \theta(Z_{B\to A})$ yields that $B \sqsubseteq_{\mathcal{O}} \exists w'.A$. The definition of $\mathcal{U}_A(\mathfrak{s})$ yields $\exists r.B \in R_{\mathfrak{s},\tau}$, i.e.,

$$(C_i = \exists r.B \text{ for some } i, 1 \leq i \leq n) \ \text{ or } \ (C_{\mathfrak{s},\tau} \sqsubseteq_{\mathcal{O}} \exists r.B \text{ and } \exists r.B \in \mathit{Ats}(\Gamma, \mathcal{O})).$$

Since $\exists r.B$ is ground and $r \trianglelefteq_{\mathcal{O}} s$, we obtain $\gamma_\theta(\exists r.B) \sqsubseteq_{\mathcal{O}} \exists s.B$. Hence, if $C_i = \exists r.B$, then the following holds:

$$\gamma_\theta(C_1) \sqcap \cdots \sqcap \gamma_\theta(C_n) \sqsubseteq_{\mathcal{O}} \exists r.B \sqsubseteq_{\mathcal{O}} \exists s.B.$$

Otherwise, $C_{\mathfrak{s},\tau} \sqsubseteq_{\mathcal{O}} \exists r.B$. Hence, similarly to the case where $\mathfrak{t} = \{\varepsilon\}$, we have:

$$\gamma_\theta(C_1) \sqcap \cdots \sqcap \gamma_\theta(C_n) \sqsubseteq_{\mathcal{O}} C_{\mathfrak{s},\tau} \sqsubseteq_{\mathcal{O}} \exists r.B \sqsubseteq_{\mathcal{O}} \exists s.B.$$

Overall, we can infer that $\gamma_\theta(C_1) \sqcap \cdots \sqcap \gamma_\theta(C_n) \sqsubseteq_{\mathcal{O}} \exists s.B$. Thus, since $B \sqsubseteq_{\mathcal{O}} \exists w'.A$, it follows that $\gamma_\theta(C_1) \sqcap \cdots \sqcap \gamma_\theta(C_n) \sqsubseteq_{\mathcal{O}} \exists w.A$.

– $\mathfrak{t} = L_t Z_{B'\to A}$. This means that there is $\exists r.B \in R_{\mathfrak{s},\tau}$ such that (t, B') is a pair in $\mathit{Tr}(\exists r.B)$. Following the previous case, $w \in L_t{\cdot}\theta(Z_{B'\to A})$ implies that $w = sw'$ for some $s \in \mathsf{N_R}$ and $w' \in \mathsf{N_R}^*$ such that:

$$t \trianglelefteq_{\mathcal{O}} s, \quad \gamma_\theta(C_1) \sqcap \cdots \sqcap \gamma_\theta(C_n) \sqsubseteq_{\mathcal{O}} \exists r.B, \quad B' \sqsubseteq_{\mathcal{O}} \exists w'.A$$

In addition, $(t, B') \in \mathit{Tr}(\exists r.B)$ yields $r \trianglelefteq_{\mathcal{O}} t$, $B \sqsubseteq_{\mathcal{O}} \exists t.B'$, and t is a transitive role. This, together with $\gamma_\theta(C_1) \sqcap \cdots \sqcap \gamma_\theta(C_n) \sqsubseteq_{\mathcal{O}} \exists r.B$ yields

$$\gamma_\theta(C_1) \sqcap \cdots \sqcap \gamma_\theta(C_n) \sqsubseteq_{\mathcal{O}} \exists r.B \sqsubseteq_{\mathcal{O}} \exists r.\exists t.B' \sqsubseteq_{\mathcal{O}} \exists t.\exists t.B' \sqsubseteq_{\mathcal{O}} \exists t.B'.$$

Thus, since $t \trianglelefteq_{\mathcal{O}} s$ and $B' \sqsubseteq_{\mathcal{O}} \exists w'.A$, it follows that $\gamma_\theta(C_1) \sqcap \cdots \sqcap \gamma_\theta(C_n) \sqsubseteq_{\mathcal{O}} \exists w.A$.

– $\mathfrak{t} = L_t Y_{A,t}$. From $w \in L_t{\cdot}\theta(Y_{A,t})$, we obtain $w = sw'$ for some $s \in \mathsf{N_R}$ such that

$$t \trianglelefteq_{\mathcal{O}} s \quad \text{and} \quad w' \in \theta(Y_{A,t}).$$

In addition, by construction of $\mathcal{U}_A(\mathfrak{s})$, we know that $(Y, t) \in V_{\mathfrak{s}}$, which means that there is an index $i \in \{1, \ldots, n\}$ such that

$$C_i = \exists r.Y, \quad r \trianglelefteq_{\mathcal{O}} t, \quad \text{and} \quad t \text{ is a transitive role.}$$

Finally, since θ satisfies the implication in (14), we have that $tw' \in \theta(Y_A)$. Hence, by definition of γ_θ, it follows that $\gamma_\theta(Y) \sqsubseteq \exists t.\exists w'.A$. Hence, since $r \trianglelefteq_{\mathcal{O}} t$, $t \trianglelefteq_{\mathcal{O}} s$ and t is a transitive role, we obtain the following subsumption chain:

$$\exists r.\gamma_\theta(Y) \sqsubseteq \exists r.\exists t.\exists w'.A \sqsubseteq_{\mathcal{O}} \exists t.\exists w'.A \sqsubseteq_{\mathcal{O}} \exists s.\exists w'.A.$$

Thus, since $\gamma_\theta(C_i) = \exists r.\gamma_\theta(Y)$, we have shown that $\gamma_\theta(C_1) \sqcap \cdots \sqcap \gamma_\theta(C_n) \sqsubseteq_{\mathcal{O}} \exists w.A$.

The previous case distinction shows that $\gamma_\theta(C_1) \sqcap \cdots \sqcap \gamma_\theta(C_n) \sqsubseteq_{\mathcal{O}} \exists w.A$ also holds when $\mathfrak{t}$ is a term in $\mathcal{U}_A(\mathfrak{s})$.

Overall, we have thus shown that $\gamma_\theta(C_1) \sqcap \cdots \sqcap \gamma_\theta(C_n) \sqsubseteq_{\mathcal{O}} At$ holds for each top-level atom At of $\gamma_\theta(X)$. This implies that $\gamma_\theta(C_1) \sqcap \cdots \sqcap \gamma_\theta(C_n) \sqsubseteq_{\mathcal{O}} \gamma_\theta(X)$ holds, i.e., γ_θ satisfies the right-hand side of (15) for $\mathfrak{s}$. Therefore, we have shown by well-founded induction that γ_θ solves all subsumption constraints in $\Delta_{\Gamma,\tau}$. Thus, γ_θ is a unifier of $\Delta_{\Gamma,\tau}$ w.r.t. $\mathcal{O}$. As already explained immediately after the definition of γ_θ, this unifier is an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-substitution that is compatible with τ. This concludes the proof of the lemma. □

4.3 The PSpace Algorithm

Based on the results described in the previous two subsections, we can construct an NPSpace decision procedure for unification in $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$ w.r.t. cycle-restricted $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-ontologies. Due to Savitch's theorem [26], this implies that the problem is also in PSpace.

Given an input consisting of an $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unification problem and a cycle-restricted $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-ontology, the algorithm transforms the ontology and the unification problem into flat ones, which we denote as Γ and $\mathcal{O}$. It then proceeds as follows:

1. It guesses a subsumption mapping τ for Γ w.r.t. $\mathcal{O}$. If no such mapping exists, then it fails.

2. It transforms Γ into $\Delta_{\Gamma,\tau}$, and then translates the latter into the set of linear language inclusions $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$.

3. Finally, the algorithm answers "yes" iff $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$ has a finite, admissible solution.

Flattening can be done in polynomial time and preserves unifiability [5, 18]. A mapping $\tau : Ats_{tr}(\Gamma, \mathcal{O}) \times Ats_{tr}(\Gamma, \mathcal{O}) \to \{0, 1\}$ can be guessed in non-deterministic polynomial time, and checking whether it satisfies the properties of a subsumption mapping (see Definition 4.1) can be realized in polynomial time. In fact, since subsumption between $\mathcal{ELH}_{\mathcal{R}^+}$-concepts and $\trianglelefteq_{\mathcal{O}}$ can be decided in polynomial time w.r.t. $\mathcal{ELH}_{\mathcal{R}^+}$-ontologies, the only conditions in Definition 4.1 that might look problematic are those stated in terms of *the existence* of atoms $At_1, \ldots, At_k$ of $\mathcal{O}$. However, such existential tests can be decided in polynomial time, since the tested property holds iff it holds for the sequence of all atoms of $\mathcal{O}$ that have the required syntactic form and satisfy the sub-property about τ. To be more precise, let us illustrate this with the second case of Condition (1d). The same arguments can be applied for the other relevant cases.

- Suppose there are atoms $\exists r_1.A_1, \ldots, \exists r_k.A_k$ of $\mathcal{O}$ $(k \geq 0)$ and atoms $F_\ell \in \mathcal{S}(\exists r.X, \exists r_\ell.A_\ell)$ $(1 \leq \ell \leq k)$, such that:

  $$\tau(X, F_\ell) = 1 \; (1 \leq \ell \leq k) \quad \text{and} \quad \exists r_1.A_1 \sqcap \cdots \sqcap \exists r_k.A_k \sqsubseteq_{\mathcal{O}} A.$$

  Let M be the set of all atoms of $\mathcal{O}$ of the form $\exists u.B$ such that $\tau(X, F) = 1$ for some $F \in \mathcal{S}(\exists r.X, \exists u.B)$. Clearly, the set M contains the atoms $\exists r_1.A_1, \ldots, \exists r_k.A_k$, and hence

  $$\bigsqcap_{At \in M} At \quad \sqsubseteq_{\mathcal{O}} \quad \exists r_1.A_1 \sqcap \cdots \sqcap \exists r_k.A_k \quad \sqsubseteq_{\mathcal{O}} \quad A.$$

  Therefore, checking whether the sequence of atoms $\exists r_1.A_1, \ldots, \exists r_k.A_k$ exists is equivalent to compute M and check whether $\bigsqcap_{At \in M} At \sqsubseteq_{\mathcal{O}} A$ holds or not. The computation of M can be done in polynomial time in the size of $\mathcal{O}$, since $\mathcal{S}(\exists r.X, \exists u.B)$ can be computed in polynomial time in the size of $\mathcal{O}$.

Once a subsumption mapping τ is guessed, the set $\Delta_{\Gamma,\tau}$ can clearly be computed in polynomial time. Moreover, the translation from $\Delta_{\Gamma,\tau}$ into $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$ can also be carried out in polynomial time:

- The number of language inclusions in $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$ is polynomial in the size of the input. The set $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$ consists of the languages inclusions in $\mathfrak{I}_{\mathcal{O}}$, the language inclusions of the form $\mathfrak{i}^*_A(\mathfrak{s})$, and the language inclusions of the form $\mathfrak{i}_{A,t}(\mathfrak{s})$. The set $\mathfrak{I}_{\mathcal{O}}$ contains one inclusion for each indeterminate $Z_{B \to A}$, where A and B are concept constants occurring in Γ or $\mathcal{O}$. This implies that the number of language inclusions in $\mathfrak{I}_{\mathcal{O}}$ is polynomial in the size of Γ and $\mathcal{O}$. The system $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$ contains one inclusion of the form $\mathfrak{i}^*_A(\mathfrak{s})$ for each subsumption constraint $\mathfrak{s} \in \Delta_{\Gamma,\tau}$ and concept constant A. It also contains one language inclusion of the form $\mathfrak{i}_{A,t}(\mathfrak{s})$ for each transitive role t of $\mathcal{O}$. Hence, since the number of subsumption constraints in $\Delta_{\Gamma,\tau}$ is polynomial in the size of Γ and $\mathcal{O}$, then the number of inclusions in $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$ of the form $\mathfrak{i}^*_A(\mathfrak{s})$ and $\mathfrak{i}_{A,t}(\mathfrak{s})$ is polynomial in the size of Γ and $\mathcal{O}$. Overall, we can conclude that the cardinality of $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$ is polynomial in the size of the input Γ and $\mathcal{O}$.

- All the inclusions are of polynomial size. For $\mathfrak{I}_{\mathcal{O}}$, the number of terms on the right-hand side of an inclusion is bounded by the cardinality of $I(B)$, which consists of pairs of role names and concept constants occurring in Γ and $\mathcal{O}$. As for the inclusions of the form $\mathfrak{i}^*_A(\mathfrak{s})$ and $\mathfrak{i}_{A,t}(\mathfrak{s})$ in $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$, the number of additional terms in $\mathcal{U}_A(\mathfrak{s})$ and $\mathcal{U}_{A,t}(\mathfrak{s})$ is polynomial in the combined size of the sets $R_{\mathfrak{s},\tau}$, $Tr(\exists r.B)$ where $\exists r.B \in Ats(\Gamma,\mathcal{O})$, and $V_{\mathfrak{s}}$. Since the cardinality of these sets is polynomial in the size of the input, it follows that such inclusions are of polynomial size. Thus, every language inclusion contained in $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$ is of size polynomial in the size of the input Γ and $\mathcal{O}$.
- The set $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$ can be computed in polynomial time. This follows from the fact that the sets $I(B)$, $R_{\mathfrak{s},\tau}$, $Tr(\exists r.B)$ and $V_{\mathfrak{s}}$ can all be computed in polynomial time.

Finally, as shown in [18], testing for the existence of a finite, admissible solution of $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$ can be reduced in polynomial time to checking emptiness of alternating finite automata with ε-transitions, which is a PSpace-complete problem [23]. This shows that the introduced algorithm really is an NPSpace algorithm. Its correctness is an immediate consequence of Propositions 4.3 and 4.18. Since PSpace-hardness already holds for the special case of an empty ontology, we thus have shown the following main result of this paper.

Theorem 4.26. *Deciding unifiability of $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-unification problems w.r.t. cycle-restricted $\mathcal{ELH}^{-\top}_{\mathcal{R}^+}$-ontologies is PSpace-complete.*

5 Conclusion

We have shown that the approach for obtaining a PSpace decision procedure for $\mathcal{EL}^{-\top}$-unification without a background ontology [18] can be extended to unification w.r.t. a cycle-restricted $\mathcal{ELH}_{\mathcal{R}^+}$-ontology, i.e., an ontology that may contain general concept inclusions (GCIs) formulated in $\mathcal{EL}^{-\top}$ as well as role inclusion and transitivity axioms, but does not entail a cyclic subsumption of the form $C \sqsubseteq_{\mathcal{O}} \exists r_1.\exists r_2.\cdots\exists r_n.C$ $(n \geq 1)$. As explained in the introduction, both considering concept descriptions not containing the top concept $\top$ and considering GCIs and role axioms is motivated by the expressivity employed in the medical ontology SNOMED CT. Dealing with such a background ontology not only makes the approach more complicated due to the more involved characterization of subsumption (see Lemma 1.3 and Definition 4.1, compared to the much simpler versions in [18]). It also requires the development of new notions, such as simple unifiers and the extension of the system of linear language inclusions with new indeterminates and corresponding inclusions.

With SNOMED CT in mind, it would be interesting to see whether results on unification (with or without top) can be further extended to ontologies additionally containing so-called right-identity rules, i.e., role axioms of the form $r \circ s \sqsubseteq r$, since they are also needed to get rid of the SEP-triplet encoding mentioned in the introduction. However, extending the characterization of subsumption to this setting is probably a non-trivial problem. From a theoretical point of view, the big open problem is whether one can dispense with the requirement that the ontology must be cycle-restricted. Even for pure $\mathcal{EL}$, decidability of unification w.r.t. unrestricted ontologies is an open problem.

From a practical point of view, the next step is to develop an algorithm that replaces non-deterministic guessing by a more intelligent search procedure. Since the unification problem is

PSpace-complete, a polynomial translation of the whole problem into SAT is not possible (unless NP = PSpace). However, one could try to delegate the search for a subsumption mapping to a SAT solver, which interacts with a solver for the additional condition on such a mapping (existence of a finite, admissible solution of $\mathfrak{I}^{\mathcal{O}}_{\Gamma,\tau}$) in an SMT-like fashion [21].

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