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Nota	

UNIVERSIDADE FEDERAL DA GRANDE DOURADOS
Cálculo Diferencial e Integral III — Avaliação PS
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Engenharia de Alimentos 26/06/2019

Aluno(a): ....

Todas as respostas devem ser justificadas.
Resolva apenas a avaliação referente a sua menor nota.

Avaliação P1:

1. Calcule as derivadas parciais de $f(x,y) = \dfrac{\sqrt{y-x^2}}{1-x^2}$.

2. Dada $z = e^r \cos\theta$, onde $r = st$ e $\theta = \sqrt{s^2+t^2}$, calcule $\dfrac{\partial z}{\partial s}$ e $\dfrac{\partial z}{\partial t}$.

3. Dados $f(x,y,z) = x^2yz - xyz^3$, $P = (1,2,1)$ e $u = \left(\frac{4}{5}, 0, -\frac{3}{5}\right)$
   (a) Calcule o gradiente de f.
   (b) Calcule a taxa de variação de f em P na direção de u.

4. Encontre os pontos de máximo local, mínimo local e de sela de $f(x,y) = x^4 + y^4 - 4xy + 1$.

5. Determine a menor distância entre o ponto $(2,0,-3)$ e o plano $x+y+z=1$.

Avaliação P2:

1. Calcule a integral dupla $\iint_R x \operatorname{sen}(x+y)\, dA$, onde $R = [0, \frac{\pi}{6}] \times [0, \frac{\pi}{3}]$.

2. Descreva o sólido cujo volume é dado pela integral $\displaystyle\int_0^1 \int_0^\pi \int_0^1 r\, dr\, d\theta\, dz$ e determine o valor dessa integral.

3. Calcule a integral $\displaystyle\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\, dV$, onde $B = \{(x,y,z) \mid x^2+y^2+z^2 \le 1\}$.

4. Dada $F(x,y) = (1 - ye^{-x}, e^{-x})$
   (a) Determine se F é conservativo. Caso positivo, determine a função potencial de F.
   (b) Calcule a integral $\int_C F \cdot\, dr$, onde C é o caminho $r(t) = (t\sqrt{t}, \sqrt{2t^2+2t})$, $0 \le t \le 1$.

5. Calcule a integral de linha $\int_C \ln(1+y)\, dx - \frac{xy}{1+y}\, dy$, onde C é o círculo $x^2+y^2 = \frac{1}{4}$.

Boa Prova!

Avaliação P1

① $f(x,y) = \dfrac{\sqrt{y-x^2}}{1-x^2}$

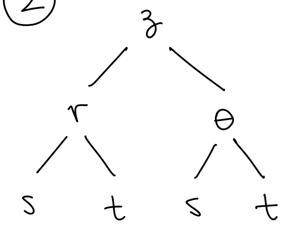
$$\frac{\partial f}{\partial x} = \frac{\frac{1}{2\sqrt{y-x^2}}\cdot(-2x)\cdot(1-x^2) - (-2x)\sqrt{y-x^2}}{(1-x^2)^2} = \frac{\frac{-x(1-x^2)}{\sqrt{y-x^2}} + 2x\sqrt{y-x^2}}{(1-x^2)^2}$$

$$= \frac{-x(1-x^2) + 2x(y-x^2)}{(1-x^2)^2\sqrt{y-x^2}} = \frac{-x+x^3+2xy-2x^3}{(1-x^2)^2\sqrt{y-x^2}}$$

$$= \frac{-x^3-x+2xy}{(1-x^2)^2\sqrt{y-x^2}}$$

$$\frac{\partial f}{\partial y} = \frac{1}{1-x^2}\cdot\frac{1}{2\sqrt{y-x^2}} = \frac{1}{2(1-x^2)\sqrt{y-x^2}}$$

②



$z = e^r \cos\theta$

$r = st$

$\theta = \sqrt{s^2+t^2}$

Pela regra da cadeia:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial s} + \frac{\partial z}{\partial \theta}\cdot\frac{\partial \theta}{\partial s} = e^r\cos\theta\cdot t + (-e^r \operatorname{sen}\theta)\cdot\frac{1}{2\sqrt{s^2+t^2}}\cdot 2s$$

$$= t e^r\cos\theta - \frac{s e^r \operatorname{sen}\theta}{\sqrt{s^2+t^2}}$$

e $$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial t} + \frac{\partial z}{\partial \theta}\cdot\frac{\partial \theta}{\partial t} = e^r\cos\theta\cdot s + (-e^r \operatorname{sen}\theta)\cdot\frac{1}{2\sqrt{s^2+t^2}}\cdot 2t$$

$$= s e^r\cos\theta - \frac{t e^r \operatorname{sen}\theta}{\sqrt{s^2+t^2}}$$

③ $f(x,y,z) = x^2yz - xyz^3$

a) $\nabla f = (2xyz - yz^3,\ x^2z - xz^3,\ x^2y - 3xyz^2)$

b) $\|u\| = \sqrt{\left(\frac{4}{5}\right)^2 + 0^2 + \left(-\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25} + \frac{9}{25}} = 1 \Rightarrow u$ é unitário. Logo,

$\frac{\partial f}{\partial u}(1,2,1) = \nabla f(1,2,1) \cdot u = (2,0,-4) \cdot \left(\frac{4}{5}, 0, -\frac{3}{5}\right) = \frac{8}{5} + \frac{12}{5} = \frac{20}{5} = 4.$

④ Calculando os pontos críticos de f:

$\frac{\partial f}{\partial x} = 4x^3 - 4y$ e $\frac{\partial f}{\partial y} = 4y^3 - 4x$ estão def. p/ todo $(x,y) \in \mathbb{R}^2$.

Assim, os ptos críticos de f são as sol. do sistema:

$$\begin{cases} 4x^3 - 4y = 0 \\ 4y^3 - 4x = 0 \end{cases} \overset{(\div 4)}{\Rightarrow} \begin{cases} x^3 - y = 0 \\ y^3 - x = 0 \end{cases} \Rightarrow \begin{cases} y = x^3 & (1) \\ y^3 - x = 0 & (2) \end{cases}$$

Substituindo (1) em (2):

$(x^3)^3 - x = 0 \Rightarrow x^9 - x = 0 \Rightarrow x(x^8 - 1) = 0 \Rightarrow x = 0$ ou $x^8 - 1 = 0$

$\Rightarrow x = 0$ ou $x^8 = 1 \Rightarrow x = 0$ ou $x = 1$ ou $x = -1$.

$\therefore\ x = 0 \Rightarrow y = 0$, $x = 1 \Rightarrow y = 1$ e $x = -1 \Rightarrow y = -1$.

Portanto, os ptos críticos de f são $(0,0)$, $(1,1)$ e $(-1,-1)$.

Aplicando o teste da 2ª derivada:

$\frac{\partial^2 f}{\partial x^2} = 12x^2$, $\frac{\partial^2 f}{\partial y \partial x} = -4$, $\frac{\partial^2 f}{\partial x \partial y} = -4$ e $\frac{\partial^2 f}{\partial y^2} = 12y^2$

$$\Rightarrow D(x,y) = \begin{vmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{vmatrix} = 144x^2y^2 - 16$$

$\therefore\ D(0,0) = -16 < 0 \Rightarrow (0,0)$ é pto de sela

$D(1,1) = 128 > 0$ e $\frac{\partial^2 f}{\partial x^2}(1,1) = 12 > 0 \Rightarrow (1,1)$ é pto de mín. local.

$D(-1,-1) = 128 > 0$ e $\frac{\partial^2 f}{\partial x^2}(-1,-1) = 12 > 0 \Rightarrow (-1,-1)$ é pto de mín. local.

⑤ Temos que:

$$d\left[(x,y,z),(2,0,-3)\right] = \sqrt{(x-2)^2 + y^2 + (z+3)^2}$$

$$\Rightarrow d\left[(x,y,z),(2,0,-3)\right]^2 = (x-2)^2 + y^2 + (z+3)^2.$$

Aplicando o método dos multiplicadores de Lagrange, sejam

$$f(x,y,z) = (x-2)^2 + y^2 + (z+3)^2$$

$$g(x,y,z) = x + y + z - 1$$

$$\Rightarrow \quad \nabla f = \left(2(x-2),\ 2y,\ 2(z+3)\right)$$

$$\nabla g = (1,1,1)$$

$$\therefore \begin{cases} 2(x-2) = \lambda & (1) \\ 2y = \lambda & (2) \\ 2(z+3) = \lambda & (3) \\ x+y+z-1 = 0 & (4) \end{cases}$$

De (1) e (2), temos: $2(x-2) = 2y \Rightarrow x-2 = y \Rightarrow x = y+2.$

De (2) e (3), temos: $2y = 2(z+3) \Rightarrow y = z+3 \Rightarrow z = y-3.$

Substituindo em (4):

$$(y+2) + y + (y-3) - 1 = 0 \Rightarrow 3y = 2 \Rightarrow y = \frac{2}{3}$$

$$\Rightarrow x = \frac{2}{3} + 2 = \frac{8}{3}$$

$$\Rightarrow z = \frac{2}{3} - 3 = -\frac{7}{3}$$

Avaliação P2

① $\iint_R x \operatorname{sen}(x+y)\, dA = \int_0^{\pi/3}\int_0^{\pi/6} x \operatorname{sen}(x+y)\, dx\, dy$

$$\left(\begin{array}{ll} \text{Por partes:} & \\ u = x & \Rightarrow \quad du = dx \\ dv = \operatorname{sen}(x+y)\, dx & \quad\;\; v = -\cos(x+y) \end{array} \right)$$

$$= \int_0^{\pi/3} \left[-x\cos(x+y)\Big|_{x=0}^{x=\pi/6} + \int_0^{\pi/6} \cos(x+y)\, dx \right] dy$$

$$= \int_0^{\pi/3} \left[-\frac{\pi}{6}\cos\left(\frac{\pi}{6}+y\right) + 0\cdot\cos(0+y) + \operatorname{sen}(x+y)\Big|_{x=0}^{x=\pi/6} \right] dy$$

$$= \int_0^{\pi/3} -\frac{\pi}{6}\cos\left(\frac{\pi}{6}+y\right) + \operatorname{sen}\left(\frac{\pi}{6}+y\right) - \operatorname{sen} y \;\; dy$$

$$= -\frac{\pi}{6}\left[\operatorname{sen}\left(\frac{\pi}{6}+y\right)\Big|_0^{\pi/3} \right] - \left[\cos\left(\frac{\pi}{6}+y\right)\Big|_0^{\pi/3} \right] + \left[\cos y \Big|_0^{\pi/3} \right]$$

$$= -\frac{\pi}{6}\left[\operatorname{sen}\left(\frac{\pi}{6}+\frac{\pi}{3}\right) - \operatorname{sen}\left(\frac{\pi}{6}\right) \right] - \left[\cos\left(\frac{\pi}{6}+\frac{\pi}{3}\right) - \cos\left(\frac{\pi}{6}\right) \right] + \left[\cos\frac{\pi}{3} - \cos 0 \right]$$

$$= -\frac{\pi}{6}\left[\operatorname{sen}\frac{\pi}{2} - \operatorname{sen}\frac{\pi}{6} \right] - \left[\cos\frac{\pi}{2} - \cos\frac{\pi}{6} \right] + \left[\cos\frac{\pi}{3} - \cos 0 \right]$$

$$= -\frac{\pi}{6}\left[1 - \frac{1}{2} \right] - \left[0 - \frac{\sqrt{3}}{2} \right] + \left[\frac{1}{2} - 1 \right] = \frac{6\sqrt{3} - 6 - \pi}{12}$$

② O conjunto $E = \{ (r,\theta,z) \mid 0 \le r \le 1,\ 0 \le \theta \le \pi,\ 0 \le z \le 1 \}$ descreve em coord. cilíndricas a metade de um cilindro de raio 1 contido no semi-espaço onde $y \geq 0$ e limitado pelos planos $z=0$ e $z=1$.

A integral calcula o volume de E, logo deve ser igual a

$$\frac{\pi r^2 \cdot h}{2} = \frac{\pi \cdot 1^2 \cdot 1}{2} = \frac{\pi}{2}.$$

Calculando a integral:

$$\int_0^1\int_0^{\pi}\int_0^1 r\, dr\, d\theta\, dz = \int_0^1 r\, dr \cdot \int_0^{\pi} d\theta \cdot \int_0^1 dz = \left(\frac{r^2}{2}\Big|_0^1 \right) \cdot \pi \cdot 1 = \frac{\pi}{2}.$$

③ Usando coord. esféricas, temos:

$$B = \{(\rho, \theta, \phi) \mid 0 \le \rho \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi\}$$

$$\therefore \iiint_B e^{(x^2+y^2+z^2)^{3/2}}\, dV = \int_0^1 \int_0^{2\pi} \int_0^{\pi} e^{(\rho^2)^{3/2}} \rho^2 \operatorname{sen}\phi \, d\rho\, d\theta\, d\phi$$

$$= \int_0^1 e^{\rho^3} \cdot \rho^2\, d\rho \cdot \int_0^{2\pi} d\theta \cdot \int_0^{\pi} \operatorname{sen}\phi\, d\phi \qquad \left(\begin{array}{l} u = \rho^3 \Rightarrow du = 3\rho^2 d\rho \\ \rho = 0 \Rightarrow u = 0 \ \text{e}\ \rho = 1 \Rightarrow u = 1 \end{array} \right)$$

$$= \frac{1}{3} \int_0^1 e^u\, du \cdot (2\pi) \cdot \left(-\cos\phi \Big|_0^{\pi}\right)$$

$$= \frac{2\pi}{3}\left(e^u \Big|_0^1\right) \cdot \left(-\cos\pi + \cos 0\right) = \frac{2\pi}{3}(e-1) \cdot 2 = \frac{4\pi}{3}(e-1).$$

④ a) Queremos f tal que $\nabla f = F$, logo

$$\begin{cases} \dfrac{\partial f}{\partial x} = 1 - y e^{-x} & (1) \\ \dfrac{\partial f}{\partial y} = e^{-x} & (2) \end{cases}$$

Integrando (2): $\displaystyle\int \frac{\partial f}{\partial y}\, dy = \int e^{-x} dy \Rightarrow f(x,y) = e^{-x} \cdot y + C(x).$

Substituindo em (1):

$$1 - y e^{-x} \overset{(1)}{=} \frac{\partial f}{\partial x} = -e^{-x} y + C'(x) \Rightarrow C'(x) = 1 \Rightarrow C(x) = x.$$

Portanto, $f(x,y) = x + y e^{-x}$ é uma função potencial de F e F é conservativo.

b) Pelo Teo. Fund. das Int. de Linha:

$$\int F \cdot dr = \int \nabla f \cdot dr = f(r(1)) - f(r(0)) = f(1,2) - f(0,0)$$

$$= (1 + 2e^{-1}) - (0 + 0 \cdot e^{0}) = 1 + 2e^{-1}.$$

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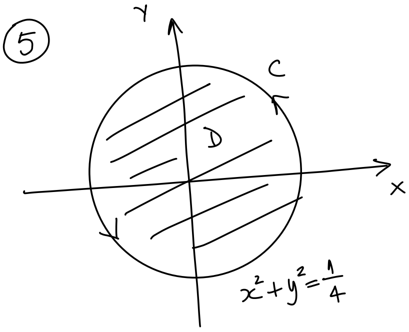

A curva C é simples, fechada, suave e pode ser orientada positivamente.

$$\frac{\partial P}{\partial y} = \frac{1}{1+y} \quad e \quad \frac{\partial Q}{\partial x} = -\frac{y}{1+y} \text{ são contínuas em } D,$$

Pelo Teo. de Green:

$$\int_C \ln(1+y)\,dx - \frac{xy}{1+y}\,dy = \iint_D -\frac{y}{1+y} - \frac{1}{1+y}\,dA = \iint_D -\frac{y+1}{1+y}\,dA$$

$$= \iint_D -1\,dA = -\iint_D dA = -\text{área}(D) = -\pi\left(\frac{1}{2}\right)^2 = -\frac{\pi}{4}.$$