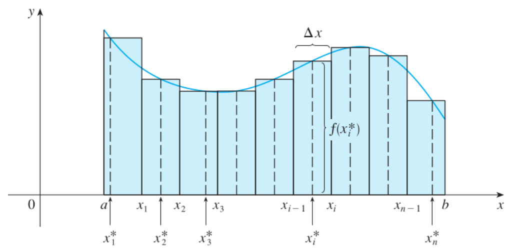
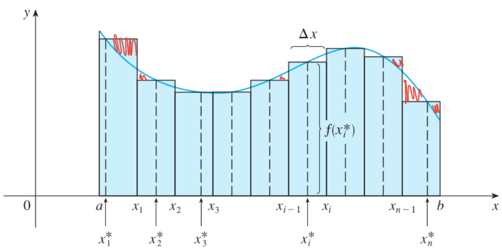
Cálculo III

Integral dupla

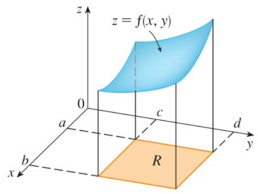
Prof. Adriano Barbosa



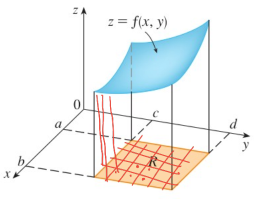


$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$



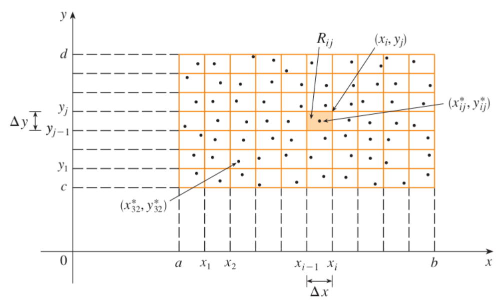
$f(x, y) \geqslant 0$



$f(x, y) \geqslant 0$

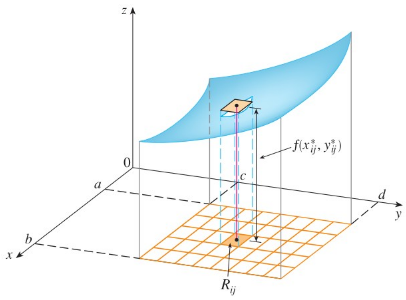
$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \mid a \leqslant x \leqslant b,\ c \leqslant y \leqslant d\}$$

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leqslant z \leqslant f(x, y),\ (x, y) \in R\}$$



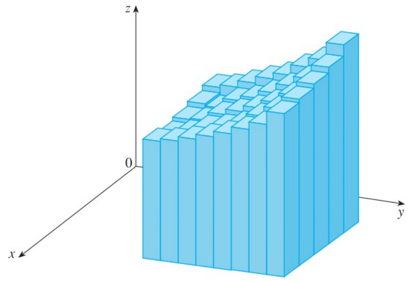
$\Delta x = (b - a)/m$ $\qquad$ $\Delta y = (d - c)/n$

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x, y) \mid x_{i-1} \leqslant x \leqslant x_i,\ y_{j-1} \leqslant y \leqslant y_j\}$$



$$\Delta A = \Delta x \, \Delta y$$

$$f(x_{ij}^*, y_{ij}^*) \, \Delta A$$



$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \, \Delta A$$

$$V = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \, \Delta A$$

Integral dupla:

$$\iint_R f(x, y) \, dA = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \, \Delta A$$

Se $f(x, y) \geqslant 0$, o volume do sólido acima da região *R* e abaixo gráfico da função é dado por:

$$V = \iint_R f(x, y) \, dA$$

Exemplo: Estime a área do sólido acima da região *R* e abaixo do gráfico da função abaixo

$R = [0, 2] \times [0, 2]$ $\quad$ $f(x, y) = 16 - x^2 - 2y^2$

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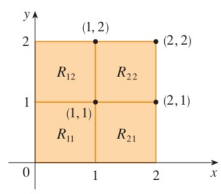

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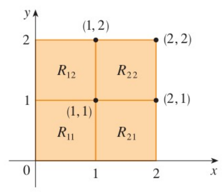

$$V \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_i, y_j)\, \Delta A$$

Exemplo: Estime a área do sólido acima da região R e abaixo do gráfico da função abaixo

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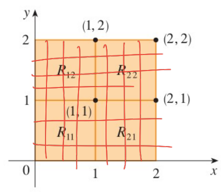
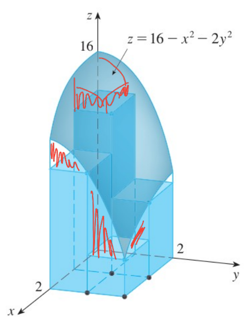
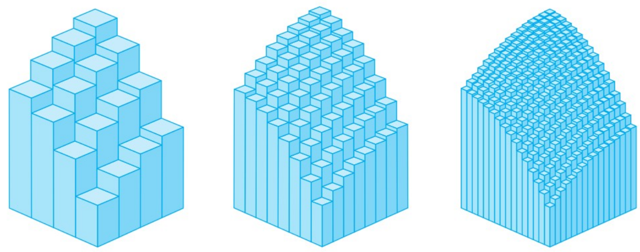


$$V \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_i, y_j)\, \Delta A = f(1, 1)\, \Delta A + f(1, 2)\, \Delta A + f(2, 1)\, \Delta A + f(2, 2)\, \Delta A$$
$$= 13(1) + 7(1) + 10(1) + 4(1) = 34$$



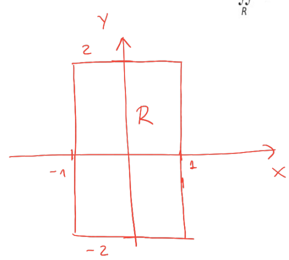


(a) $m = n = 4$, $V \approx 41.5$ (b) $m = n = 8$, $V \approx 44.875$ (c) $m = n = 16$, $V \approx 46.46875$

Exemplo: $R = \{(x, y) \mid -1 \leqslant x \leqslant 1, -2 \leqslant y \leqslant 2\}$

$$\iint_R \sqrt{1 - x^2}\, dA$$

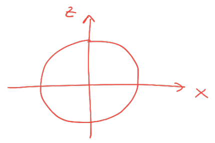
$\sqrt{z^2} = \sqrt{1-x^2} \Rightarrow |z| = \sqrt{1-x^2}$
$\Rightarrow z = \pm\sqrt{1-x^2}$



$z = \sqrt{1-x^2} = f(x,y)$

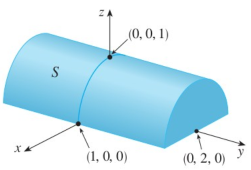
$\Rightarrow z^2 = 1 - x^2$

$\Leftrightarrow x^2 + z^2 = 1$



Exemplo: $R = \{(x, y) \mid -1 \leqslant x \leqslant 1, -2 \leqslant y \leqslant 2\}$

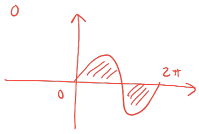
$$V = \iint_R \sqrt{1 - x^2}\, dA \, , \; z = \sqrt{1-x^2} \geqslant 0$$

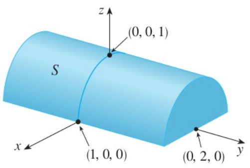


Exemplo: $R = \{(x, y) \mid -1 \leq x \leq 1, -2 \leq y \leq 2\}$

$$\iint_R \sqrt{1 - x^2}\, dA$$

$$\int_0^{2\pi} \operatorname{sen} x \, dx = 0$$





$$\iint_R \sqrt{1 - x^2}\, dA = \underbrace{\tfrac{1}{2}\pi(1)^2 \times 4}_{\text{metade do vol. do cilindro}} = 2\pi$$

Propriedades

➡ $$\iint_R [f(x, y) + g(x, y)]\, dA = \iint_R f(x, y)\, dA + \iint_R g(x, y)\, dA$$

➡ $$\iint_R c\, f(x, y)\, dA = c \iint_R f(x, y)\, dA$$

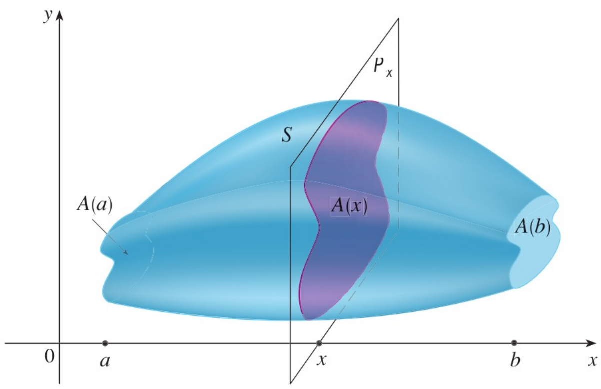
➡ $f(x, y) \geq g(x, y)$

$$\iint_R f(x, y)\, dA \geq \iint_R g(x, y)\, dA$$

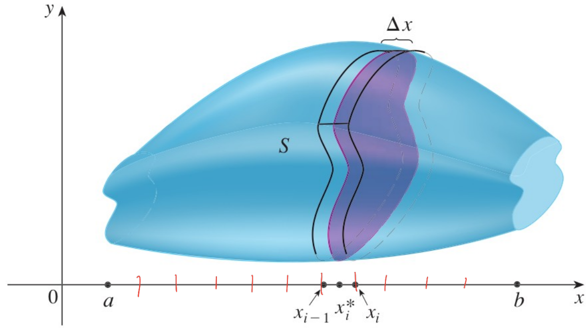
De modo geral:

$$\iint_R f(x,y) \cdot g(x,y)\, dA \neq \iint_R f(x,y)\, dA \cdot \iint_R g(x,y)\, dA$$

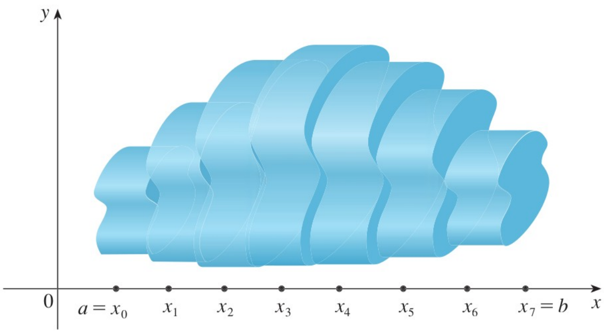
Integral Iterada



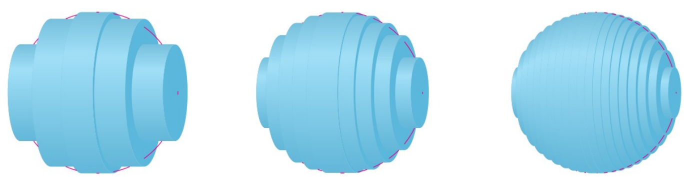
Integral Iterada



Integral Iterada

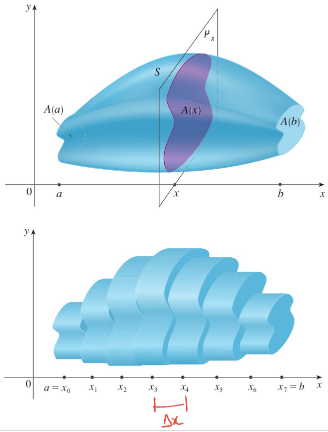


Integral Iterada



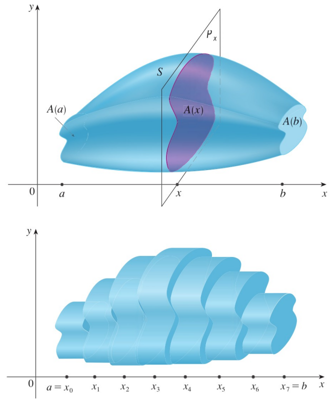
(a) Using 5 disks, $V \approx 4.2726$ (b) Using 10 disks, $V \approx 4.2097$ (c) Using 20 disks, $V \approx 4.1940$

Integral Iterada



$$V(S_i) \approx A(x_i^*)\,\Delta x$$

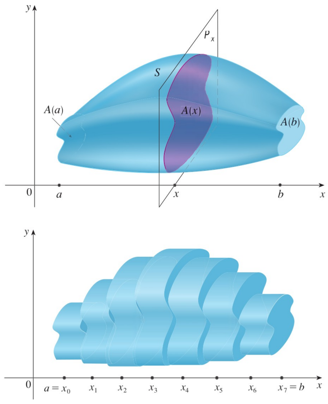
Integral Iterada



$$V(S_i) \approx A(x_i^*)\,\Delta x$$

$$V \approx \sum_{i=1}^{n} A(x_i^*)\,\Delta x$$

Integral Iterada

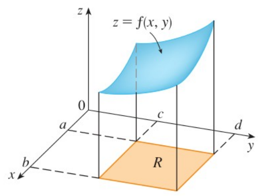


$$V(S_i) \approx A(x_i^*)\,\Delta x$$

$$V \approx \sum_{i=1}^{n} A(x_i^*)\,\Delta x$$

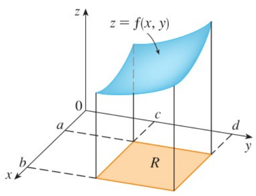
$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\,\Delta x = \int_a^b A(x)\,dx$$

Integral Iterada



$R = [a, b] \times [c, d]$

Integral Iterada



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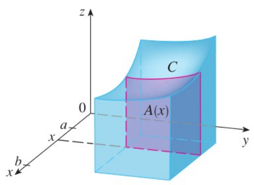
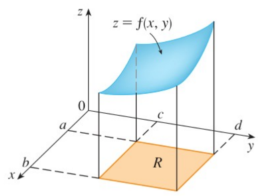
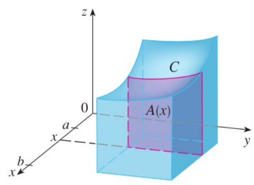


Integral Iterada



$R = [a, b] \times [c, d]$

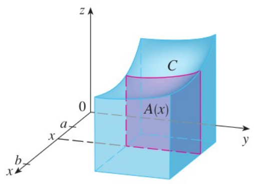


$$A(x) = \int_c^d f(x, y)\,dy$$

Integral Iterada

$$A(x) = \int_c^d f(x, y)\, dy$$

$$\int_a^b A(x)\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx$$

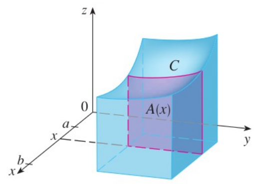


Integral Iterada

$$A(x) = \int_c^d f(x, y)\, dy$$

$$\int_a^b A(x)\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx$$

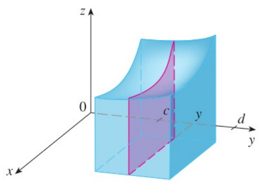
$$\int_a^b \int_c^d f(x, y)\, dy\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx$$



Integral Iterada

Analogamente

$$\int_c^d \int_a^b f(x, y)\, dx\, dy = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy$$



Exemplo: $\int_0^3 \int_1^2 x^2 y\, dy\, dx$

Exemplo: $\int_0^3 \left(\int_1^2 x^2 y\, dy \right) dx$

$$\int_1^2 x^2 y\, dy = \left[x^2 \frac{y^2}{2} \right]_{y=1}^{y=2} = x^2 \left(\frac{2^2}{2} \right) - x^2 \left(\frac{1^2}{2} \right) = \tfrac{3}{2} x^2 = A(x)$$

Exemplo: $\int_0^3 \int_1^2 x^2 y\, dy\, dx$

$$\int_1^2 x^2 y\, dy = \left[x^2 \frac{y^2}{2} \right]_{y=1}^{y=2} = x^2 \left(\frac{2^2}{2} \right) - x^2 \left(\frac{1^2}{2} \right) = \tfrac{3}{2} x^2$$

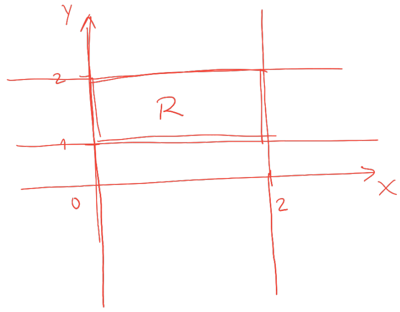
$$\int_0^3 \int_1^2 x^2 y\, dy\, dx = \int_0^3 \left[\int_1^2 x^2 y\, dy \right] dx$$

$$= \int_0^3 \tfrac{3}{2} x^2\, dx = \frac{x^3}{2} \Big]_0^3 = \frac{27}{2}$$

Teorema de Fubini: Se f é contínua em $R = \{(x,y) \mid a \leq x \leq b, c \leq y \leq d\}$, então

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Exemplo: $\iint_R (x - 3y^2)\,dA$, $R = \{(x,y) \mid 0 \leq x \leq 2, 1 \leq y \leq 2\}$



Exemplo: $\iint_R (x - 3y^2)\,dA$, $R = \{(x,y) \mid 0 \leq x \leq 2, 1 \leq y \leq 2\}$

$$\iint_R (x - 3y^2)\,dA = \int_0^2 \int_1^2 (x - 3y^2)\,dy\,dx = \int_0^2 \left[xy - y^3\right]_{y=1}^{y=2} dx$$

$$= \int_0^2 (x - 7)\,dx = \left.\frac{x^2}{2} - 7x\right]_0^2 = -12$$

Exemplo: $\iint_R (x - 3y^2)\,dA$, $R = \{(x,y) \mid 0 \leq x \leq 2, 1 \leq y \leq 2\}$

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$$= \int_0^2 (x - 7)\,dx = \left.\frac{x^2}{2} - 7x\right]_0^2 = -12$$

$$\iint_R (x - 3y^2)\,dA = \int_1^2 \int_0^2 (x - 3y^2)\,dx\,dy$$

$$= \int_1^2 \left[\frac{x^2}{2} - 3xy^2\right]_{x=0}^{x=2} dy = \int_1^2 \left(\frac{4}{2} - 6y^2\right) - \cancel{\left(\frac{0}{2} - 0\right)}\,dy$$

$$= \int_1^2 (2 - 6y^2)\,dy = 2y - 2y^3\Big]_1^2 = -12$$

$$\frac{2\,\cancel{6}y^3}{\cancel{3}} \qquad 4 - 16 - \cancel{2} + \cancel{2}$$

Exemplo: $\iint_R y\sin(xy)\,dA$, $R = [1,2] \times [0,\pi]$

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$$\iint_R y\sin(xy)\,dA = \int_1^2 \int_0^\pi y\sin(xy)\,dy\,dx$$

Exemplo: $\iint_R y\sin(xy)\,dA$, $R = [1,2]\times[0,\pi]$

$$\iint_R y\sin(xy)\,dA = \int_1^2\int_0^\pi y\sin(xy)\,dy\,dx$$

$u = y \qquad dv = \sin(xy)\,dy$

$du = dy \qquad v = -\dfrac{\cos(xy)}{x}$

$$-\frac{y\cos(xy)}{x}\Big|_0^\pi + \int_0^\pi \frac{\cos(xy)}{x}\,dy$$

$$-\frac{y\cos(xy)}{x}\Big|_0^\pi + \frac{1}{x}\int_0^\pi \cos(xy)\,dy$$

Exemplo: $\iint_R y\sin(xy)\,dA$, $R = [1,2]\times[0,\pi]$

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$du = dy \qquad v = -\dfrac{\cos(xy)}{x}$

$$\int_0^\pi y\sin(xy)\,dy = -\frac{y\cos(xy)}{x}\Bigg]_{y=0}^{y=\pi} + \frac{1}{x}\int_0^\pi \cos(xy)\,dy$$

$$= -\frac{\pi\cos\pi x}{x} + \frac{1}{x^2}\left[\sin(xy)\right]_{y=0}^{y=\pi}$$

$$= -\frac{\pi\cos\pi x}{x} + \frac{\sin\pi x}{x^2}$$

Exemplo: $\iint_R y\sin(xy)\,dA$, $R = [1,2]\times[0,\pi]$

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$$\int_0^\pi y\sin(xy)\,dy = -\frac{y\cos(xy)}{x}\Bigg]_{y=0}^{y=\pi} + \frac{1}{x}\int_0^\pi \cos(xy)\,dy$$

$$= -\frac{\pi\cos\pi x}{x} + \frac{1}{x^2}\left[\sin(xy)\right]_{y=0}^{y=\pi}$$

$$= -\frac{\pi\cos\pi x}{x} + \frac{\sin\pi x}{x^2}$$

$$\int\left(-\frac{\pi\cos\pi x}{x}\right)dx = -\frac{\sin\pi x}{x} - \int\frac{\sin\pi x}{x^2}\,dx$$

$u = -1/x \qquad dv = \pi\cos\pi x\,dx$

$du = dx/x^2 \qquad v = \sin\pi x$

$$\int\left(-\frac{\pi\cos\pi x}{x}\right)dx = -\frac{\sin\pi x}{x} - \int\frac{\sin\pi x}{x^2}\,dx$$

$$\int\left(-\frac{\pi\cos\pi x}{x} + \frac{\sin\pi x}{x^2}\right)dx = -\frac{\sin\pi x}{x}$$

$$\int\left(-\frac{\pi\cos\pi x}{x}\right)dx = -\frac{\sin\pi x}{x} - \int\frac{\sin\pi x}{x^2}\,dx$$

$$\int\left(-\frac{\pi\cos\pi x}{x} + \frac{\sin\pi x}{x^2}\right)dx = -\frac{\sin\pi x}{x}$$

$$\int_1^2\int_0^\pi y\sin(xy)\,dy\,dx = \left[-\frac{\sin\pi x}{x}\right]_1^2$$

$$= -\frac{\sin 2\pi}{2} + \sin\pi = 0$$

Solução alternativa:

Solução alternativa:

$$\iint_R y\sin(xy)\,dA = \int_0^\pi \int_1^2 y\sin(xy)\,dx\,dy = \int_0^\pi \left[-\cos(xy)\right]_{x=1}^{x=2} dy$$

$$= \int_0^\pi (-\cos 2y + \cos y)\,dy$$

$$= -\tfrac{1}{2}\sin 2y + \sin y\Big]_0^\pi = 0$$

Suponha $f(x, y) = g(x)h(y)$

$f(x,y) = x^2 \cdot \operatorname{sen} y$

$\ln y \cdot e^x$

$\cos\left(\frac{y+y^2}{2}\right) \cdot \frac{x^2+2\sqrt{x}}{3x}$

$x^2y + 2y = y\left(x^2+2\right)$

Suponha $f(x, y) = g(x)h(y)$

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b g(x)h(y)\,dx\,dy = \int_c^d \left[\int_a^b g(x)h(y)\,dx\right] dy$$

Suponha $f(x, y) = g(x)h(y)$

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Exemplo: $R = [0, \pi/2] \times [0, \pi/2]$

$$\iint_R \sin x \cos y\,dA = \int_0^{\pi/2} \sin x\,dx \int_0^{\pi/2} \cos y\,dy$$

$$= \left[-\cos x\right]_0^{\pi/2} \left[\sin y\right]_0^{\pi/2} = 1 \cdot 1 = 1$$