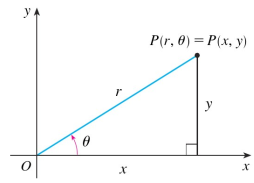
Cálculo III

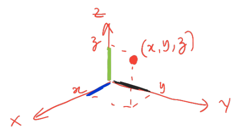
Coordenadas cilíndricas e esféricas

Prof. Adriano Barbosa

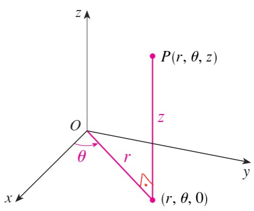
Coordenadas polares



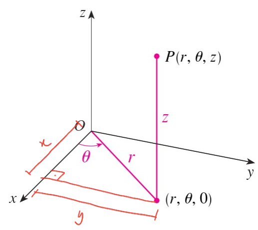
$$x = r\cos\theta$$
$$y = r \operatorname{sen}\theta$$
$$r^2 = x^2 + y^2$$
$$\operatorname{tg}\theta = \frac{y}{x}$$



Coordenadas cilíndricas



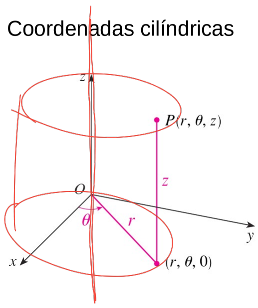
Coordenadas cilíndricas



$$x = r\cos\theta \qquad y = r\operatorname{sen}\theta \qquad z = z$$

$$r^2 = x^2 + y^2 \qquad \operatorname{tg}\theta = \frac{y}{x} \qquad z = z$$

Coordenadas cilíndricas



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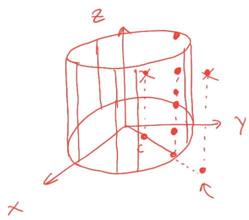
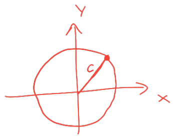
Coordenadas cilíndricas são úteis em problemas que envolvem simetria em torno de um eixo e o eixo z é escolhido de modo a coincidir com o eixo de simetria.

Exemplo

Descreva a superfície cuja equação em coordenadas cilíndricas é $r = c$.

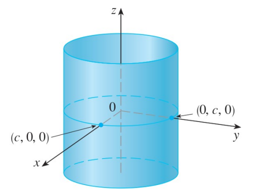


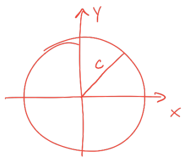


Exemplo

Descreva a superfície cuja equação em coordenadas cilíndricas é $r = c$.

$x^2 + y^2 = c^2$

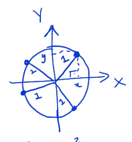




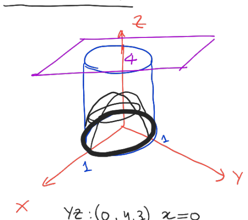
Exemplo

Um sólido E está contido no cilindro $x^2 + y^2 = 1$, abaixo do plano $z = 4$ e acima do paraboloide $z = 1 - x^2 - y^2$.

$(x, y, 4)$



$x^2 + y^2 = 1^2$



$z = 1 - x^2 - y^2$

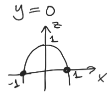
$XY: (x, y, 0) \quad z = 0$

$0 = 1 - x^2 - y^2 \Rightarrow x^2 + y^2 = 1$

$XZ: (x, 0, z) \quad y = 0$

$z = 1 - x^2$

$1 - x^2 = 0 \Rightarrow x^2 = 1$

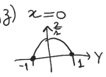


$z = \frac{1}{2}$:

$1 - x^2 - y^2 = \frac{1}{2} \Rightarrow x^2 + y^2 = \frac{1}{2}$

$YZ: (0, y, z) \quad x = 0$

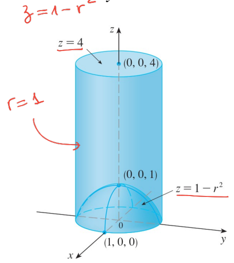
$z = 1 - y^2$



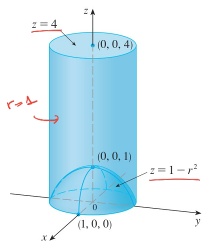
Exemplo

Um sólido E está contido no cilindro $x^2 + y^2 = 1$, abaixo do plano $z = 4$ e acima do paraboloide $z = 1 - x^2 - y^2$.

$r = 1$ $\quad z = 4 \quad z = 1 - r^2$



Exemplo

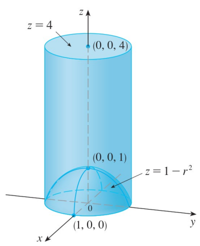


$E = \{(r, \theta, z) \mid 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi, 1 - r^2 \leq z \leq 4\}$

$$\iiint_E K\sqrt{x^2 + y^2}\, dV = \int_0^{2\pi} \int_0^1 \int_{1-r^2}^4 K\sqrt{r^2} \cdot r \; dz\, dr\, d\theta$$

Jacobiano

Exemplo



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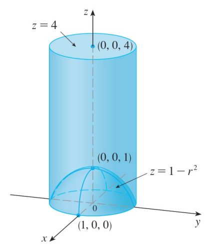

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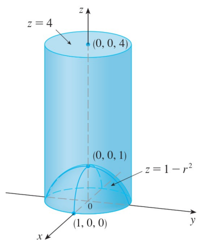

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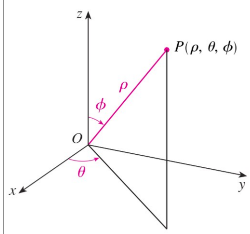
$$= \int_0^{2\pi}\int_0^1 Kr^2[4 - (1 - r^2)]\, dr\, d\theta$$

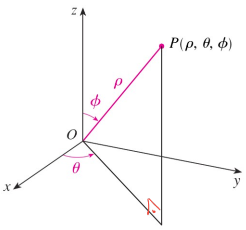
$$= K\int_0^{2\pi} d\theta \int_0^1 (3r^2 + r^4)\, dr$$

$$= 2\pi K\left[r^3 + \frac{r^5}{5}\right]_0^1 = \frac{12\pi K}{5}$$

Coordenadas esféricas



Coordenadas esféricas



$\rho = |OP|$ é a distância da origem a P

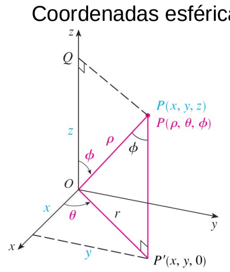
θ é o mesmo ângulo que nas coordenadas cilíndricas

ϕ é o ângulo entre o eixo z positivo e o segmento de reta OP

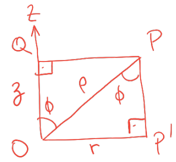
$\rho \geqslant 0 \qquad 0 \leqslant \phi \leqslant \pi$

$0 \leq \theta \leq 2\pi$

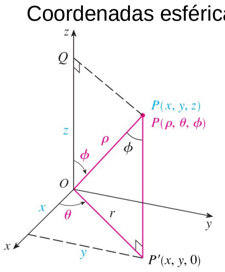
Coordenadas esféricas



$z = \rho \cos \phi \qquad r = \rho \operatorname{sen} \phi$



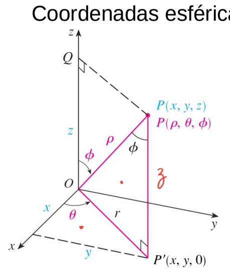
Coordenadas esféricas



$z = \rho \cos \phi \qquad r = \rho \operatorname{sen} \phi$

$x = r \cos \theta \qquad y = r \operatorname{sen} \theta$

Coordenadas esféricas



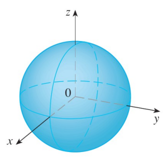
$\rho^2 = r^2 + z^2 = x^2 + y^2 + z^2$

$z = \rho \cos \phi \qquad r = \rho \operatorname{sen} \phi$

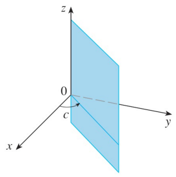
$x = r \cos \theta \qquad y = r \operatorname{sen} \theta$

$$\boxed{\begin{aligned} x &= \rho \operatorname{sen} \phi \cos \theta \\ y &= \rho \operatorname{sen} \phi \operatorname{sen} \theta \\ z &= \rho \cos \phi \\ \rho^2 &= x^2 + y^2 + z^2 \end{aligned}}$$

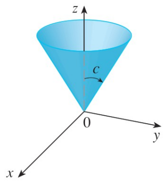
Exemplos



$\rho = c$, uma esfera



$\theta = c$, um semiplano



$0 < c < \pi/2$
$\phi = c$, um cone

Exemplo

Calcule $\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\, dV$, onde B é a bola unitária:

$$B = \{(x, y, z) \mid x^2 + y^2 + z^2 \leqslant 1\}$$

$0 \le \rho \le 1$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$

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$$\int_0^{2\pi}\int_0^{\pi}\int_0^{1} e^{(\rho^2)^{3/2}} \cdot \underbrace{\rho^2 \operatorname{sen}\phi}_{\text{Jacobiano}}\, d\rho\, d\phi\, d\theta$$

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$$= \int_0^{\pi} \operatorname{sen}\phi\, d\phi \int_0^{2\pi} d\theta \int_0^{1} \rho^2 e^{\rho^3}\, d\rho$$

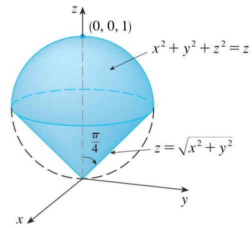
$u = \rho^3$
$du = 3\rho^2 d\rho$

$$= \left[-\cos\phi\right]_0^{\pi}(2\pi)\left[\tfrac{1}{3}e^{\rho^3}\right]_0^1 = \tfrac{4}{3}\pi(e-1)$$

Exercícios

Utilize coordenadas esféricas para determinar o volume do sólido que fica acima do cone $z = \sqrt{x^2 + y^2}$ e abaixo da esfera $x^2 + y^2 + z^2 = z$.

$\frac{\pi}{8}$



Calcule $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2} (x^2 + y^2)\, dz\, dy\, dx$.

$\frac{16}{5}\pi$