

# Análise Numérica

## Aula 11 — Métodos de Runge-Kutta

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## Difference equations

Classe de métodos da forma:

- ▶  $w_0 = y_0$
- ▶  $w_{i+1} = w_i + h\phi(t_i, w_i)$

### Exemplos:

- ▶ Método de Euler:  $\phi(t_i, w_i) = f(t_i, w_i)$
- ▶ Método de Taylor:  $\phi(t_i, w_i) = T^{(n)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2}f'(t_i, w_i) + \cdots + \frac{h^{(n-1)}}{n!}f^{(n-1)}(t_i, w_i)$

## Métodos de Runge-Kutta de ordem 2

Os métodos de Runge-Kutta são precisos (erro de truncamento local) sem a necessidade do cálculo de derivadas como nos métodos de Taylor de ordem superior.

## Teorema de Taylor em duas variáveis

Suponha  $f(t, y)$  e suas derivadas parciais contínuas em  $D = [a, b] \times [c, d]$  e  $(t_0, y_0) \in D$

$$\begin{aligned} f(t, y) = & f(t_0, y_0) + \left[ (t - t_0) \frac{\partial f}{\partial t}(t_0, y_0) + (y - y_0) \frac{\partial f}{\partial y}(t_0, y_0) \right] \\ & + \frac{1}{2!} \left[ (t - t_0)^2 \frac{\partial^2 f}{\partial t^2}(t_0, y_0) + 2(t - t_0)(y - y_0) \frac{\partial^2 f}{\partial t \partial y}(t_0, y_0) + (y - y_0)^2 \frac{\partial^2 f}{\partial y^2}(t_0, y_0) \right] \\ & + \cdots + \frac{1}{n!} \sum_{i=0}^n \binom{n}{i} (t - t_0)^{n-i} (y - y_0)^i \frac{\partial^n f}{\partial t^{n-i} \partial y^i}(t_0, y_0) + R(t, y) \end{aligned}$$

para algum  $\xi$  entre  $t$  e  $t_0$  e  $\mu$  entre  $y$  e  $y_0$  e

$$R(t, y) = \frac{1}{(n+1)!} \sum_{i=0}^{n+1} \binom{n+1}{i} (t - t_0)^{n+1-i} (y - y_0)^i \frac{\partial^{n+1} f}{\partial t^{n+1-i} \partial y^i}(\xi, \mu)$$

## Derivando um método

Queremos resolver o PVI  $y' = f(t, y)$ ,  $y(a) = y_0$ ,  $t \in [a, b]$ . Sejam  $y(t)$  sua solução e  $t_i = a + hi$  os pontos da malha. Denotando  $y_i = y(t_i)$ , temos:

$$y_{i+1} = y_i + hy'_i + \frac{h^2}{2}y''_i + R_{i+1}$$

$$\Rightarrow y_{i+1} = y_i + hf(t_i, y_i) + \frac{h^2}{2} \frac{df}{dt}(t_i, y_i) + R_{i+1}$$

$$\Rightarrow y_{i+1} \approx y_i + hf(t_i, y_i) + \frac{h^2}{2} \frac{df}{dt}(t_i, y_i)$$

## Derivando um método

$$y_{i+1} \approx y_i + hf(t_i, y_i) + \frac{h^2}{2} \frac{df}{dt}(t_i, y_i)$$

onde

$$\frac{df}{dt}(t_i, y_i) = \frac{\partial f}{\partial t}(t_i, y_i) + \frac{\partial f}{\partial y}(t_i, y_i)y'_i$$

$$\Rightarrow \frac{df}{dt}(t_i, y_i) = \frac{\partial f}{\partial t}(t_i, y_i) + \frac{\partial f}{\partial y}(t_i, y_i)f(t_i, y_i)$$

logo,

$$y_{i+1} \approx y_i + hf(t_i, y_i) + \frac{h^2}{2} \left[ \frac{\partial f}{\partial t}(t_i, y_i) + \frac{\partial f}{\partial y}(t_i, y_i)f(t_i, y_i) \right]$$

## Derivando um método

$$\begin{aligned}y_{i+1} &\approx y_i + hf(t_i, y_i) + \frac{h^2}{2} \left[ \frac{\partial f}{\partial t}(t_i, y_i) + \frac{\partial f}{\partial y}(t_i, y_i)f(t_i, y_i) \right] (\star) \\&= y_i + \frac{h}{2}f(t_i, y_i) + \frac{h}{2}f(t_i, y_i) + \frac{h^2}{2} \left[ \frac{\partial f}{\partial t}(t_i, y_i) + \frac{\partial f}{\partial y}(t_i, y_i)f(t_i, y_i) \right] \\&= y_i + \frac{h}{2}f(t_i, y_i) + \frac{h}{2} \left[ f(t_i, y_i) + h \frac{\partial f}{\partial t}(t_i, y_i) + h \frac{\partial f}{\partial y}(t_i, y_i)f(t_i, y_i) \right] \\&\approx y_i + \frac{h}{2}f(t_i, y_i) + \frac{h}{2}f(t_i + h, y_i + hf(t_i, y_i)) \\&= y_i + \frac{h}{2}f(t_i, y_i) + \frac{h}{2}f(t_{i+1}, y_i + hf(t_i, y_i))\end{aligned}$$

## Método de Euler Modificado

É um método de Runge-Kutta de ordem 2.

O método calcula  $w_i \approx y(t_i)$ :

- ▶  $w_0 = y_0$
- ▶  $w_{i+1} = w_i + \frac{h}{2} [f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))],$   
para  $i = 1, 2, \dots, N - 1$

## Derivando outro método

$$\begin{aligned}y_{i+1} &\approx y_i + hf(t_i, y_i) + \frac{h^2}{2} \left[ \frac{\partial f}{\partial t}(t_i, y_i) + \frac{\partial f}{\partial y}(t_i, y_i)f(t_i, y_i) \right] (*) \\&= y_i + hf(t_i, y_i) + h \left[ \frac{h}{2} \frac{\partial f}{\partial t}(t_i, y_i) + \frac{h}{2} \frac{\partial f}{\partial y}(t_i, y_i)f(t_i, y_i) \right] \\&= y_i + h \left[ f(t_i, y_i) + \frac{h}{2} \frac{\partial f}{\partial t}(t_i, y_i) + \frac{h}{2} \frac{\partial f}{\partial y}(t_i, y_i)f(t_i, y_i) \right] \\&\approx y_i + hf \left( t_i + \frac{h}{2}, y_i + \frac{h}{2}f(t_i, y_i) \right)\end{aligned}$$

## Método do ponto médio

Também é um método de Runge-Kutta de ordem 2.

O método calcula  $w_i \approx y(t_i)$ :

- ▶  $w_0 = y_0$
- ▶  $w_{i+1} = w_i + hf \left( t_i + \frac{h}{2}, w_i + \frac{h}{2}f(t_i, w_i) \right),$   
para  $i = 1, 2, \dots, N-1$

## Exemplo

$$y' = y - t^2 + 1, \quad t \in [0, 2], \quad y(0) = 0.5, \quad h = 0.2$$

$$w_0 = 0.5$$

$$w_{i+1} = 1.22w_i - 0.0088i^2 - 0.008i + 0.218 \quad (\text{Ponto médio})$$

$$w_{i+1} = 1.22w_i - 0.0088i^2 - 0.008i + 0.216 \quad (\text{Euler modificado})$$

$$w_1 = 1.22(0.5) - 0.0088(0)^2 - 0.008(0) + 0.218 = 0.828$$

$$w_1 = 1.22(0.5) - 0.0088(0)^2 - 0.008(0) + 0.216 = 0.826$$

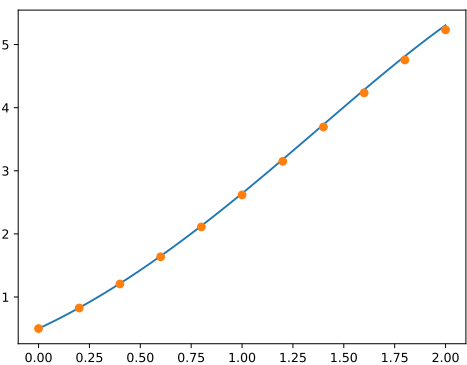
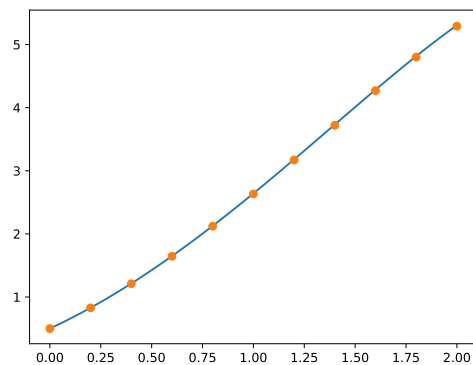
$$\begin{aligned} w_2 &= 1.22(0.828) - 0.0088(0.2)^2 - 0.008(0.2) + 0.218 \\ &= 1.21136 \end{aligned}$$

$$\begin{aligned} w_2 &= 1.22(0.826) - 0.0088(0.2)^2 - 0.008(0.2) + 0.216 \\ &= 1.20692 \end{aligned}$$

## Exemplo

$t_i$	$y(t_i)$	Midpoint Method	Error	Modified Euler Method	Error
0.0	0.5000000	0.5000000	0	0.5000000	0
0.2	0.8292986	0.8280000	0.0012986	0.8260000	0.0032986
0.4	1.2140877	1.2113600	0.0027277	1.2069200	0.0071677
0.6	1.6489406	1.6446592	0.0042814	1.6372424	0.0116982
0.8	2.1272295	2.1212842	0.0059453	2.1102357	0.0169938
1.0	2.6408591	2.6331668	0.0076923	2.6176876	0.0231715
1.2	3.1799415	3.1704634	0.0094781	3.1495789	0.0303627
1.4	3.7324000	3.7211654	0.0112346	3.6936862	0.0387138
1.6	4.2834838	4.2706218	0.0128620	4.2350972	0.0483866
1.8	4.8151763	4.8009586	0.0142177	4.7556185	0.0595577
2.0	5.3054720	5.2903695	0.0151025	5.2330546	0.0724173

## Exemplo



Ponto médio e Euler modificado

## Comparação

