Cálculo III

Coordenadas polares

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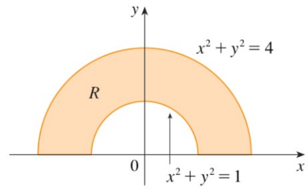
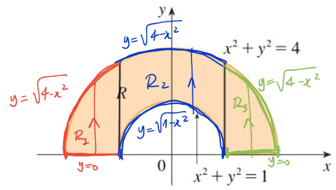
Coordenadas polares



Coordenadas polares

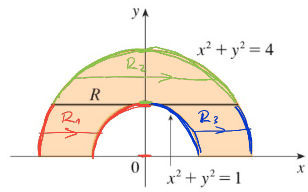


$R_1 = \left\{(x, y); -2 \leq x \leq -1 \text{ e } 0 \leq y \leq \sqrt{4 - x^2}\right\}$

$R_2 = \left\{(x, y); -1 \leq x \leq 1 \text{ e } \sqrt{1 - x^2} \leq y \leq \sqrt{4 - x^2}\right\}$

$R_3 = \left\{(x, y); 1 \leq x \leq 2 \text{ e } 0 \leq y \leq \sqrt{4 - x^2}\right\}$

Coordenadas polares

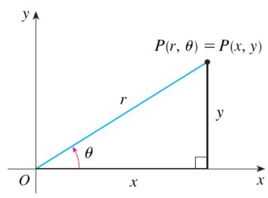


$R_1 = \left\{(x, y); -\sqrt{4 - y^2} \leq x \leq -\sqrt{1 - y^2} \text{ e } 0 \leq y \leq 1\right\}$

$R_2 = \left\{(x, y); -\sqrt{4 - y^2} \leq x \leq \sqrt{4 - y^2} \text{ e } 1 \leq y \leq 2\right\}$

$R_3 = \left\{(x, y); \sqrt{1 - y^2} \leq x \leq \sqrt{4 - y^2} \text{ e } 0 \leq y \leq 1\right\}$

Coordenadas polares



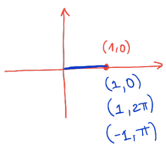
$\cos\theta = \dfrac{x}{r}$

$\operatorname{sen}\theta = \dfrac{y}{r}$

$$r^2 = x^2 + y^2$$
$$x = r\cos\theta$$
$$y = r \operatorname{sen}\theta$$

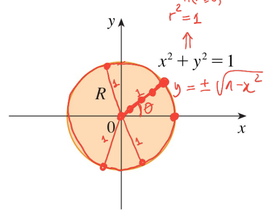
$r \geq 0$
$0 \leq \theta < 2\pi$

$r \in \mathbb{R}$
$0 \leq \theta < \pi$



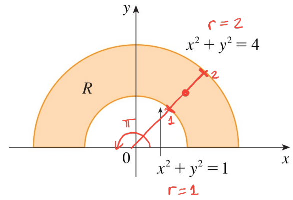
Coordenadas polares

$r = 1$
$\Uparrow (r \geq 0)$
$r^2 = 1$
$\Uparrow$



(a) $R = \{(r, \theta) \mid 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi\}$

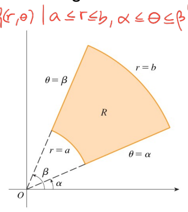
$R = \{(x, y) \mid -1 \leq x \leq 1, -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}\}$

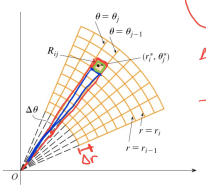


(b) $R = \{(r, \theta) \mid 1 \leqslant r \leqslant 2, 0 \leqslant \theta \leqslant \pi\}$

Integração sobre regiões circulares

$R = \{(r,\theta) \mid a \leq r \leq b, \alpha \leq \theta \leq \beta\}$





$A_{círc.} = \pi r^2$

$\frac{\pi r^2}{2\pi} = \frac{A}{\theta}$

$A = \frac{\theta r^2}{2}$

$R_{ij} = \{(r,\theta) \mid r_{i-1} \leq r \leq r_i,\ \theta_{j-1} \leq \theta \leq \theta_j\}$

$r_i^* = \frac{1}{2}(r_{i-1} + r_i) \qquad \theta_j^* = \frac{1}{2}(\theta_{j-1} + \theta_j)$

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=0}^{m} \sum_{j=0}^{n} f(x_i^*, y_i^*)\,\Delta A$$

$$\Delta A_i = \frac{1}{2} r_i^2 \Delta\theta - \frac{1}{2} r_{i-1}^2 \Delta\theta = \frac{1}{2}(r_i^2 - r_{i-1}^2)\,\Delta\theta$$

$$= \underbrace{\frac{1}{2}(r_i + r_{i-1})}_{r_i^*}\underbrace{(r_i - r_{i-1})}_{\Delta r}\,\Delta\theta = r_i^*\,\Delta r\,\Delta\theta$$

Integração sobre regiões circulares

Se f é contínua

R dado por $0 \leq a \leq r \leq b,\ \alpha \leq \theta \leq \beta$, onde $0 \leq \beta - \alpha \leq 2\pi$, então

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\,\mathrm{sen}\,\theta)\,r\,dr\,d\theta$$

Integração sobre regiões circulares

- Trocamos a variação do x e do y pela variação do raio e do ângulo;
- Trocamos na regra da função:

  x por $r\cos\theta$ e y por $r\mathrm{sen}\theta$.

- Trocamos a variação dos retângulos cartesianos $dA = dx\,dy$ pela variação dos retângulos polares $r\,dr\,d\theta$.

Exemplo

Calcule $\iint_R (3x + 4y^2)\,dA$, onde R é a região no semiplano superior limitada pelos círculos $x^2 + y^2 = 1$ e $x^2 + y^2 = 4$.

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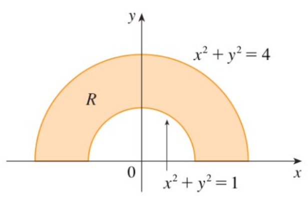
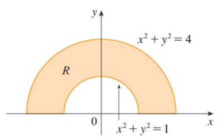
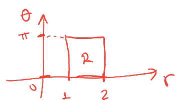


Exemplo

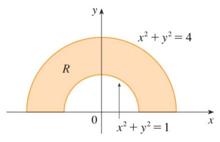
$$\iint_R (3x + 4y^2)\,dA = \int_0^\pi \int_1^2 (3r\cos\theta + 4r^2\,\mathrm{sen}^2\theta)\,r\,dr\,d\theta$$



$R = \{(r,\theta) \mid 1 \leq r \leq 2,\ 0 \leq \theta \leq \pi\}$



Exemplo



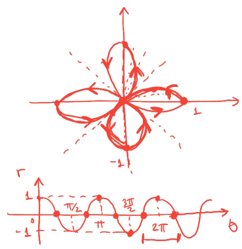
$R = \{(r, \theta) \mid 1 \leqslant r \leqslant 2, 0 \leqslant \theta \leqslant \pi\}$

$\operatorname{sen}^2\theta = \dfrac{1 - \cos(2\theta)}{2}$

$$\iint_R (3x + 4y^2)\, dA = \int_0^\pi \int_1^2 (3r\cos\theta + 4r^2 \operatorname{sen}^2\theta)\, r\, dr\, d\theta$$
$$= \int_0^\pi \int_1^2 (3r^2\cos\theta + 4r^3 \operatorname{sen}^2\theta)\, dr\, d\theta$$
$$= \int_0^\pi \left[r^3\cos\theta + r^4 \operatorname{sen}^2\theta\right]_{r=1}^{r=2} d\theta$$
$$= \int_0^\pi (7\cos\theta + 15\operatorname{sen}^2\theta)\, d\theta$$
$$= \int_0^\pi \left[7\cos\theta + \tfrac{15}{2}(1 - \cos 2\theta)\right] d\theta$$
$$= 7\operatorname{sen}\theta + \frac{15\theta}{2} - \frac{15}{4}\operatorname{sen} 2\theta\Big]_0^\pi = \frac{15\pi}{2}$$

Exemplo

Use a integral dupla para determinar a área contida em um laço da rosácea de quatro pétalas $r = \cos 2\theta$.



$\theta = 0° \Rightarrow r = 1$

$\theta = 15° \Rightarrow r = \frac{\sqrt{3}}{2} \approx 0{,}86$

$\theta = \frac{45°}{2} \Rightarrow r = \frac{\sqrt{2}}{2} \approx 0{,}7$

$\theta = 45° \Rightarrow r = 0$

⋮

Exemplo

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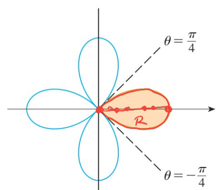
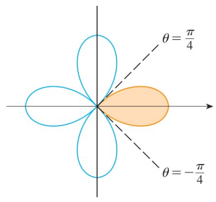


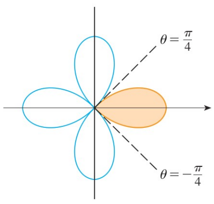
$R = \{(r,\theta) \mid -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4},\ 0 \leq r \leq \cos(2\theta)\}$

Exemplo



$D = \left\{(r, \theta) \mid -\pi/4 \leqslant \theta \leqslant \pi/4,\ 0 \leqslant r \leqslant \cos 2\theta\right\}$

Exemplo



$\cos^2\theta = \dfrac{1 + \cos(2\theta)}{2}$

$\cos^2(2\theta) = \dfrac{1 + \cos(4\theta)}{2}$

$D = \left\{(r, \theta) \mid -\pi/4 \leqslant \theta \leqslant \pi/4,\ 0 \leqslant r \leqslant \cos 2\theta\right\}$

$$A(D) = \iint_D dA = \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r\, dr\, d\theta$$
$$= \int_{-\pi/4}^{\pi/4} \left[\tfrac{1}{2}r^2\right]_0^{\cos 2\theta} d\theta = \tfrac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2 2\theta\, d\theta$$
$$= \tfrac{1}{4}\int_{-\pi/4}^{\pi/4} (1 + \cos 4\theta)\, d\theta = \tfrac{1}{4}\left[\theta + \tfrac{1}{4}\operatorname{sen} 4\theta\right]_{-\pi/4}^{\pi/4} = \frac{\pi}{8}$$

Exemplo

Determine o volume do sólido limitado pelo plano $z = 0$ e pelo paraboloide $z = 1 - x^2 - y^2$.

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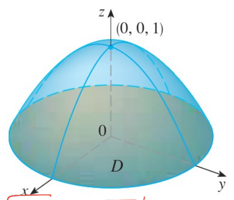

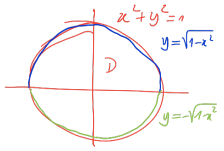

$D = \{(x,y) \mid -1 \le x \le 1, -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}\}$

$D = \{(r,\theta) \mid 0 \le r \le 1, 0 \le \theta \le 2\pi\}$

Exemplo

Se trabalhássemos com coordenadas retangulares

$$V = \iint_D (1 - x^2 - y^2)\, dA = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)\, dy\, dx$$

$$\int_{-1}^{1} y - x^2 y - \frac{y^3}{3} \Big|_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} dx$$

Exemplo

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Em coordenadas polares, D é dado por $0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi$.

$$V = \iint_D (1 - x^2 - y^2)\, dA = \int_0^{2\pi} \int_0^1 (1 - r^2)\, r\, dr\, d\theta$$

Exemplo

Se trabalhássemos com coordenadas retangulares

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$$V = \iint_D (1 - x^2 - y^2)\, dA = \int_0^{2\pi} \int_0^1 (1 - r^2)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 (r - r^3)\, dr = 2\pi \left[\frac{r^2}{2} - \frac{r^4}{4} \right]_0^1 = \frac{\pi}{2}$$

Exercícios

Esboce a região cuja área é dada pela integral e calcule-a.

$$\int_{\pi/4}^{3\pi/4} \int_1^2 r\, dr\, d\theta$$

$3\pi/4$

Calcule a integral dada, colocando-a em coordenadas polares.
$\iint_D x^2 y\, dA$, onde D é a metade superior do disco com centro na origem e raio 5

$\frac{1250}{3}$