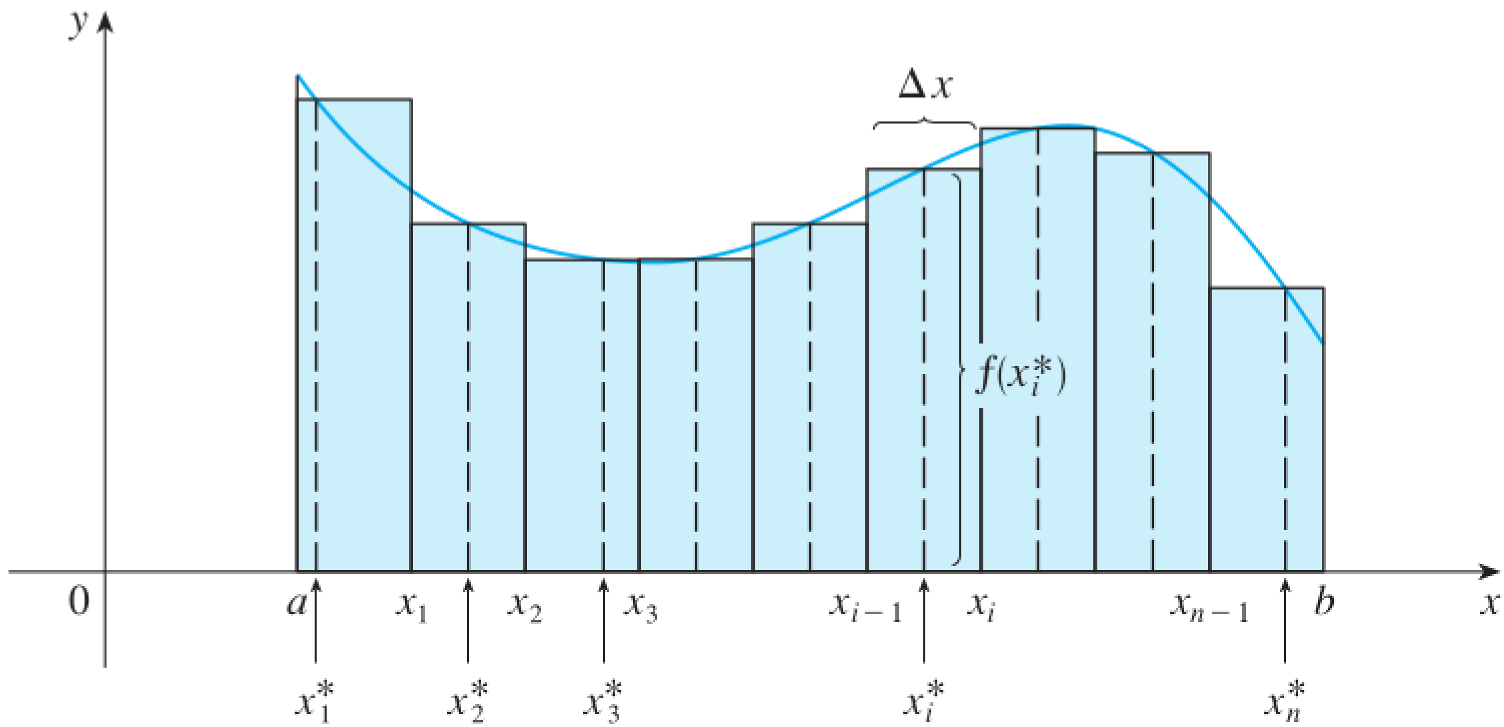
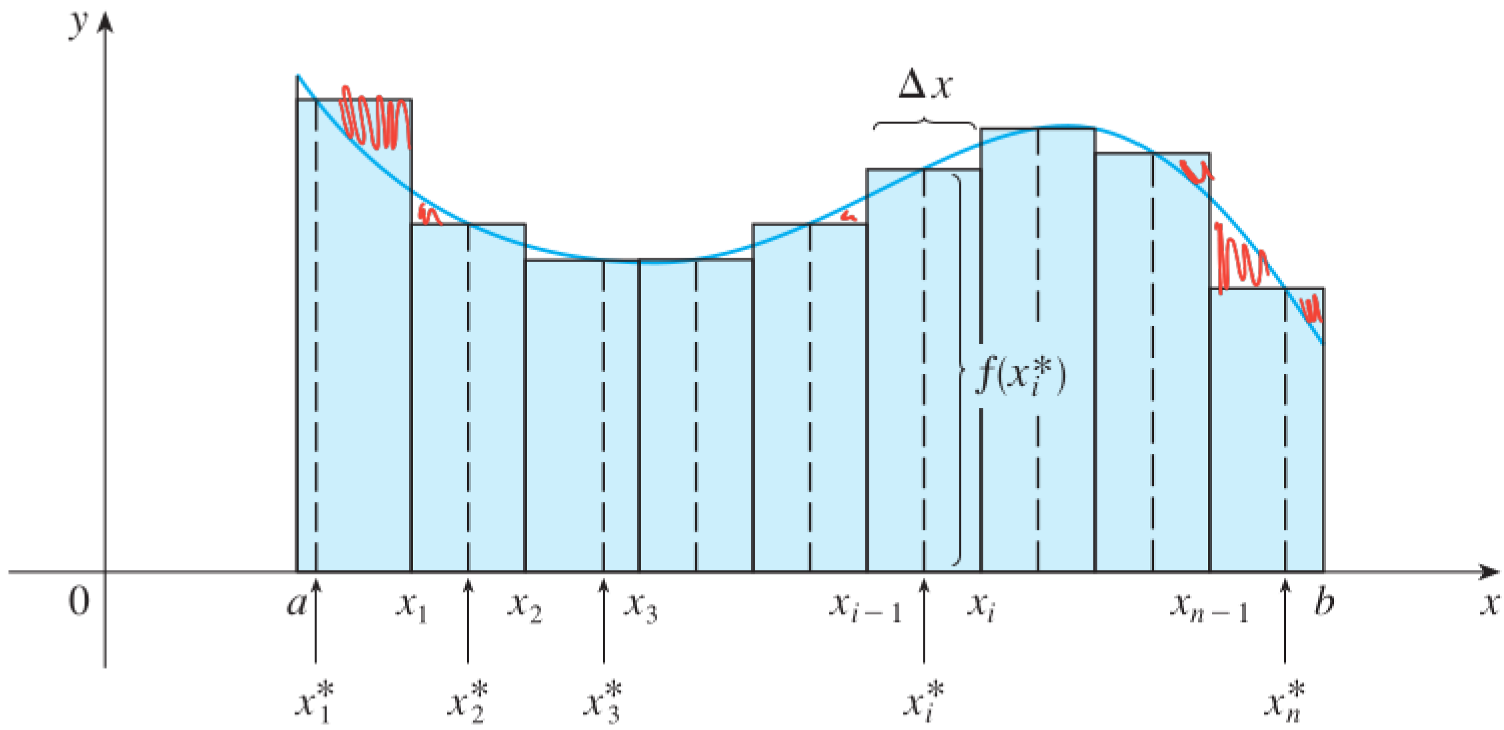
Cálculo III

Integral dupla

Prof. Adriano Barbosa

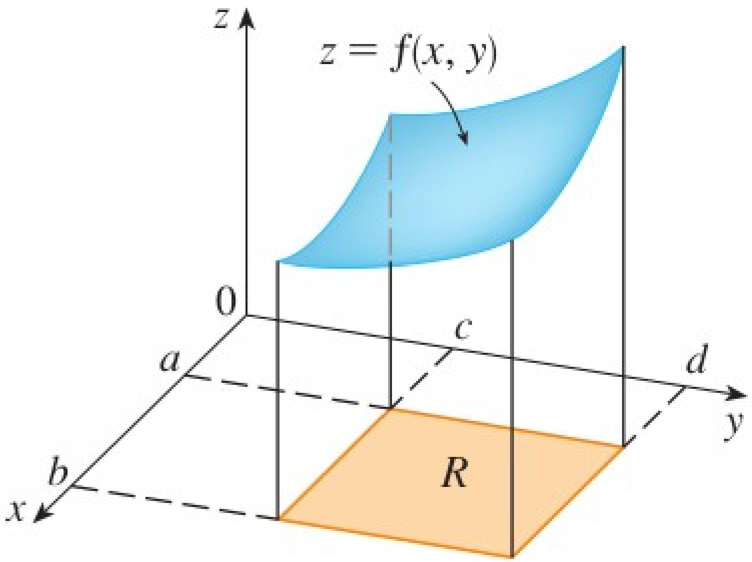


y
Δx
f(x_i*)
0
a
x_1
x_2
x_3
x_{i-1}
x_i
x_{n-1}
b
x
x_1*
x_2*
x_3*
x_i*
x_n*

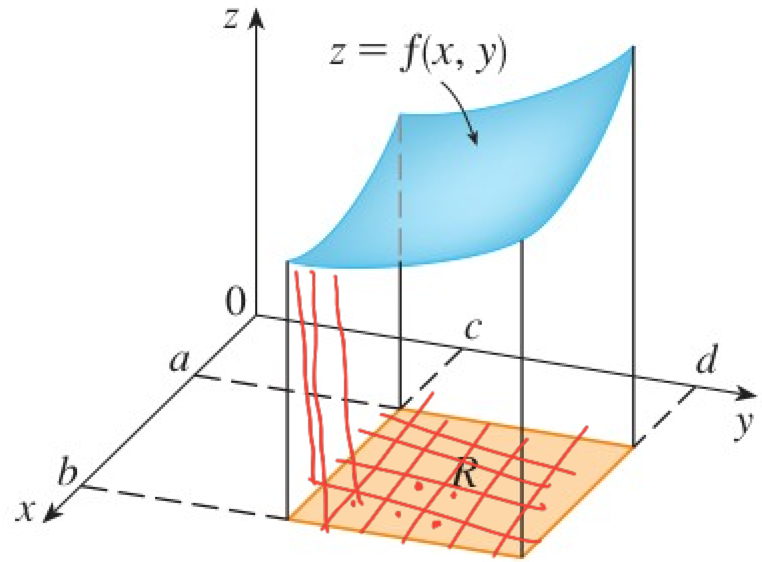


$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$



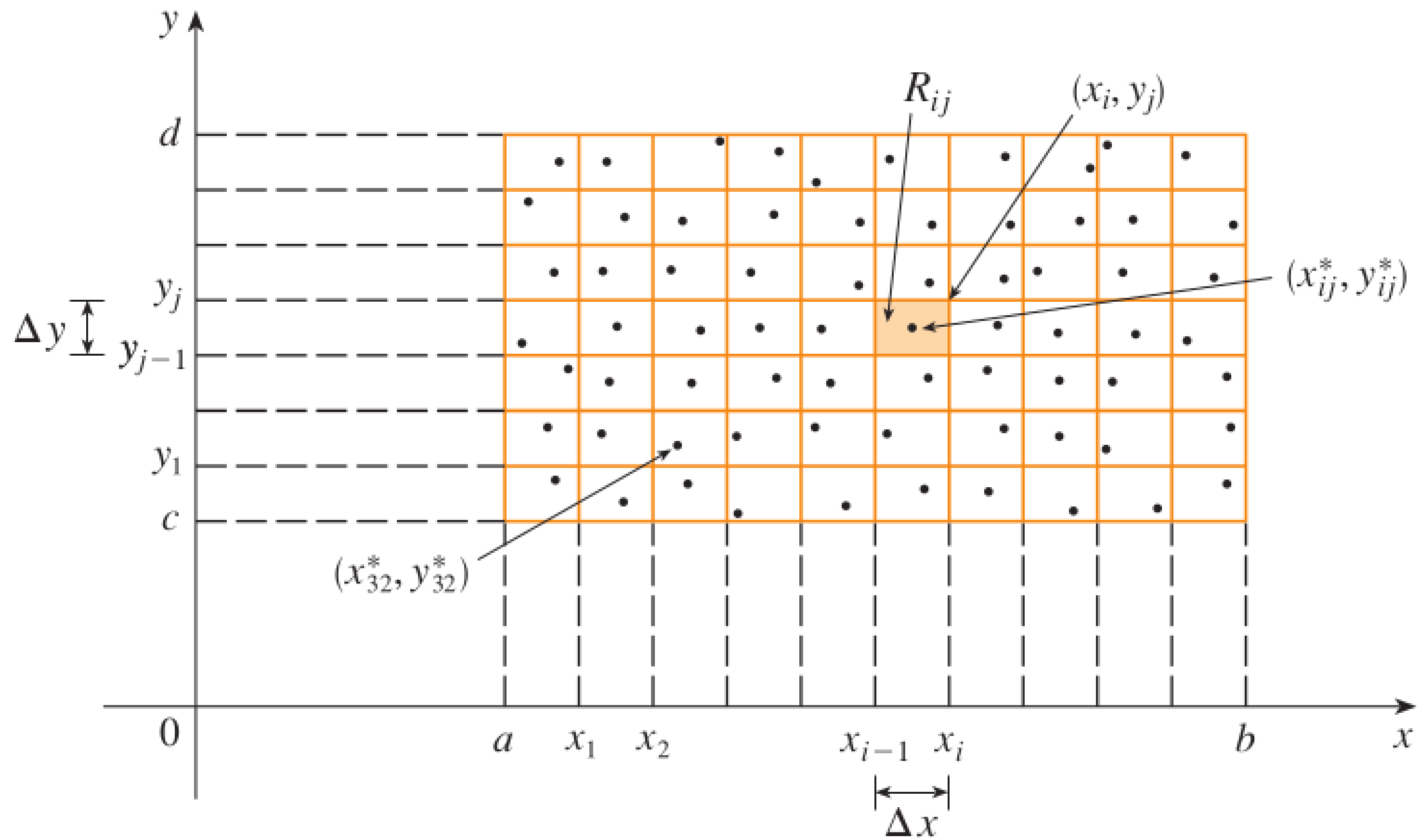
$f(x, y) \geqslant 0$



$f(x, y) \geqslant 0$

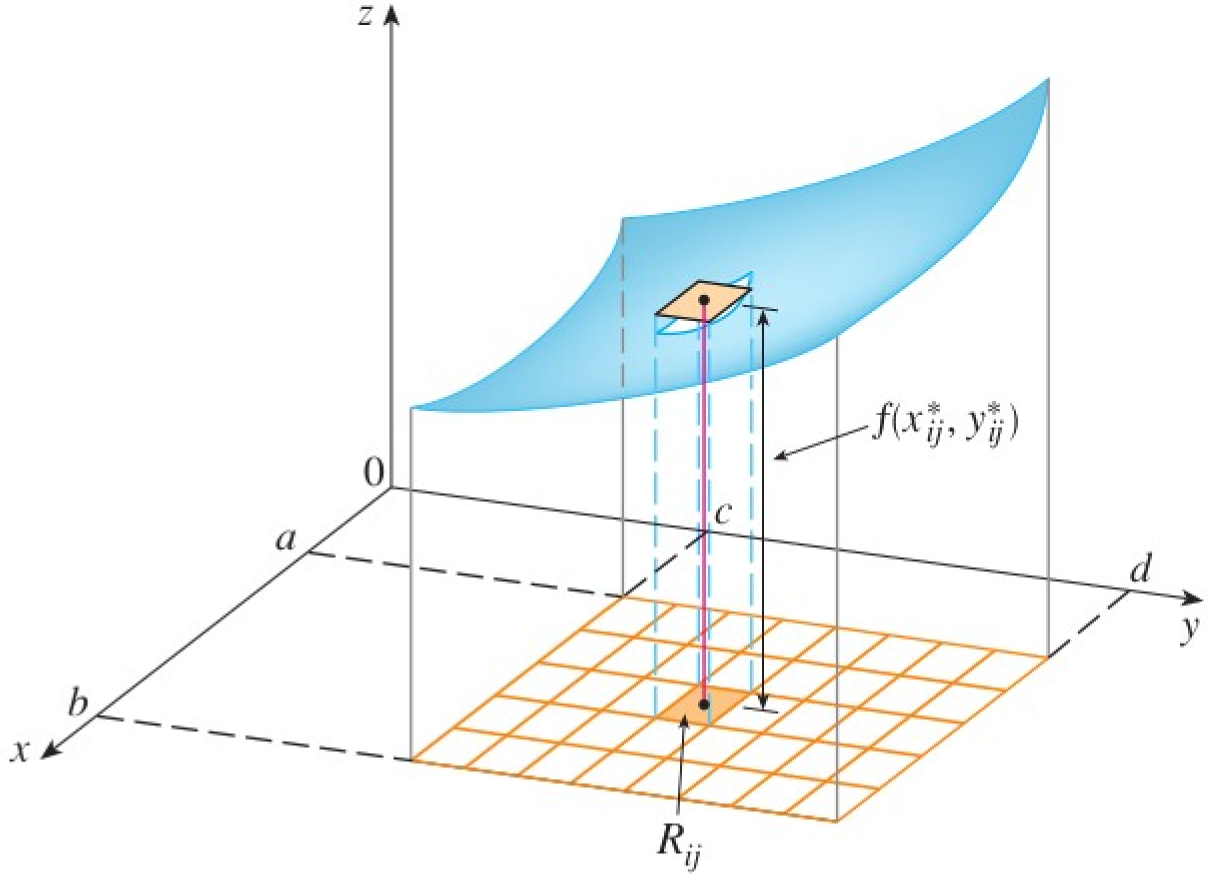
$$R = [a, b] \times [c, d] = \left\{(x, y) \in \mathbb{R}^2 \mid a \leqslant x \leqslant b,\ c \leqslant y \leqslant d\right\}$$

$$S = \left\{(x, y, z) \in \mathbb{R}^3 \mid 0 \leqslant z \leqslant f(x, y),\ (x, y) \in R\right\}$$



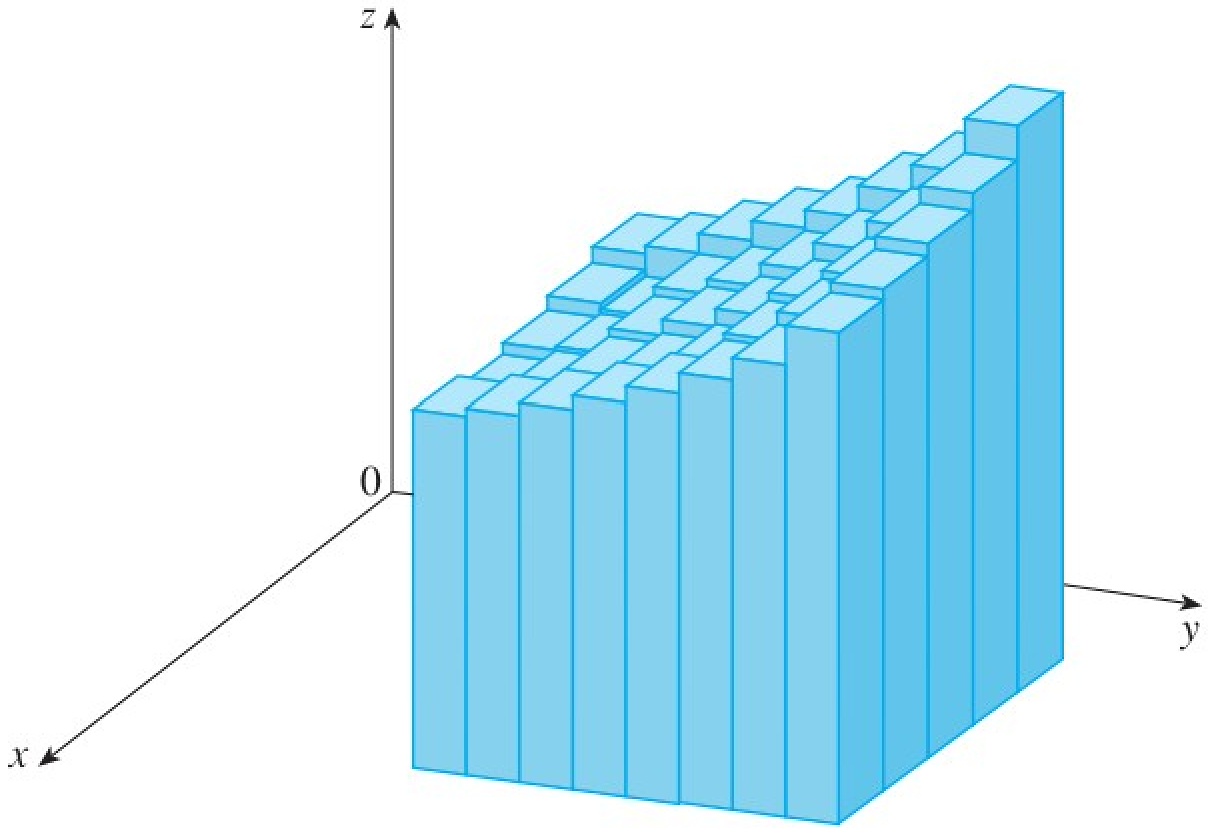
$$\Delta x = (b - a)/m \qquad \Delta y = (d - c)/n$$

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x, y) \mid x_{i-1} \leqslant x \leqslant x_i,\ y_{j-1} \leqslant y \leqslant y_j\}$$



$$\Delta A = \Delta x \, \Delta y$$

$$f(x_{ij}^*, y_{ij}^*) \, \Delta A$$



$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$$

$$V = \lim_{m,\,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$$

Integral dupla:

$$\iint_R f(x, y)\, dA = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

Se $f(x, y) \geqslant 0$, o volume do sólido acima da região *R* e abaixo gráfico da função é dado por:

$$V = \iint_R f(x, y)\, dA$$

Exemplo: Estime a área do sólido acima da região R e abaixo do gráfico da função abaixo

$$R = [0, 2] \times [0, 2] \qquad f(x, y) = 16 - x^2 - 2y^2$$

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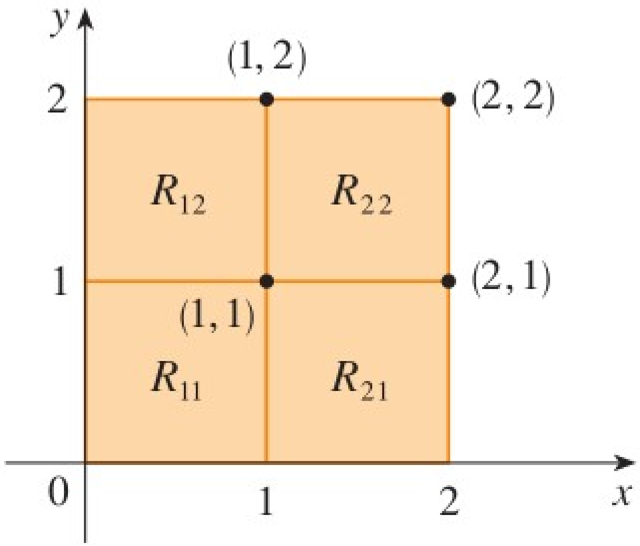

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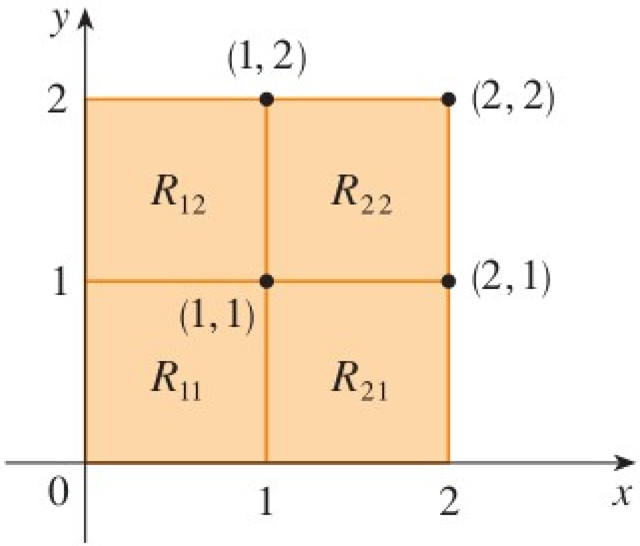

$$V \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_i, y_j)\, \Delta A$$

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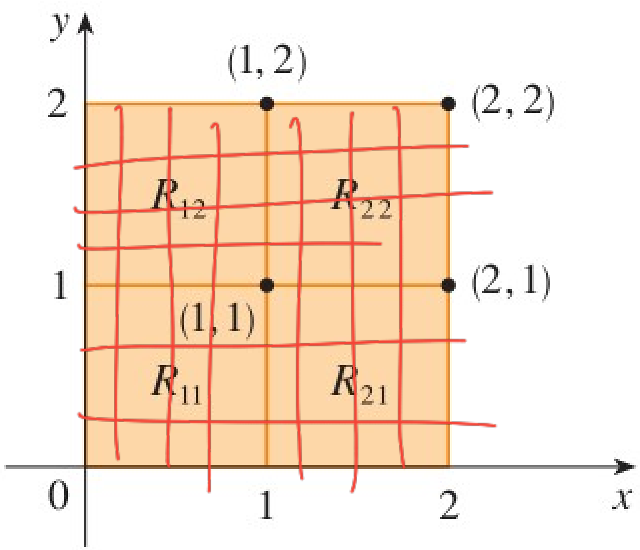
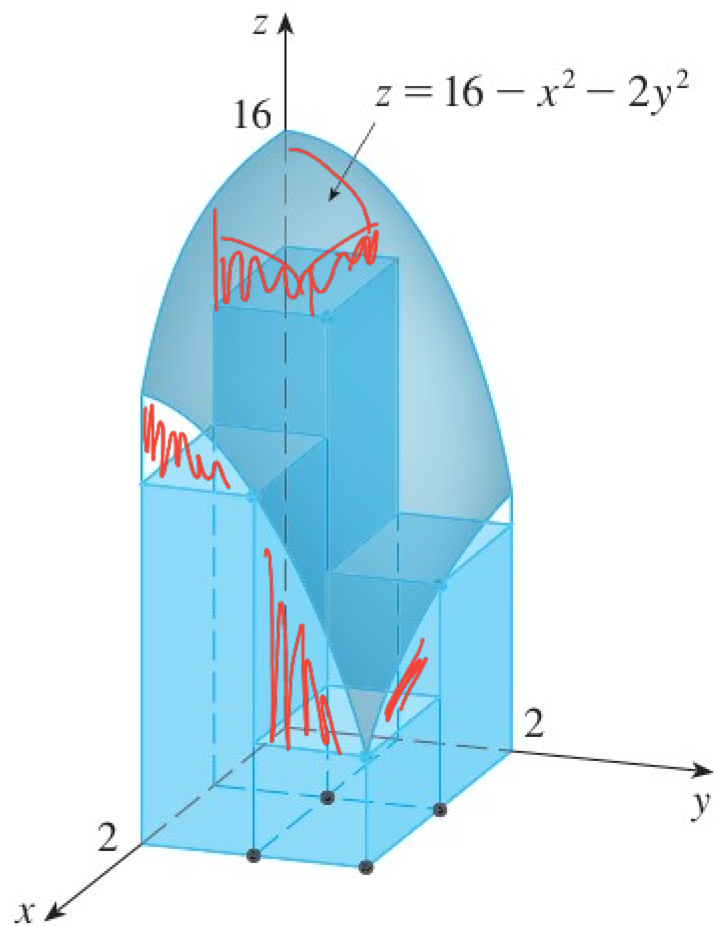
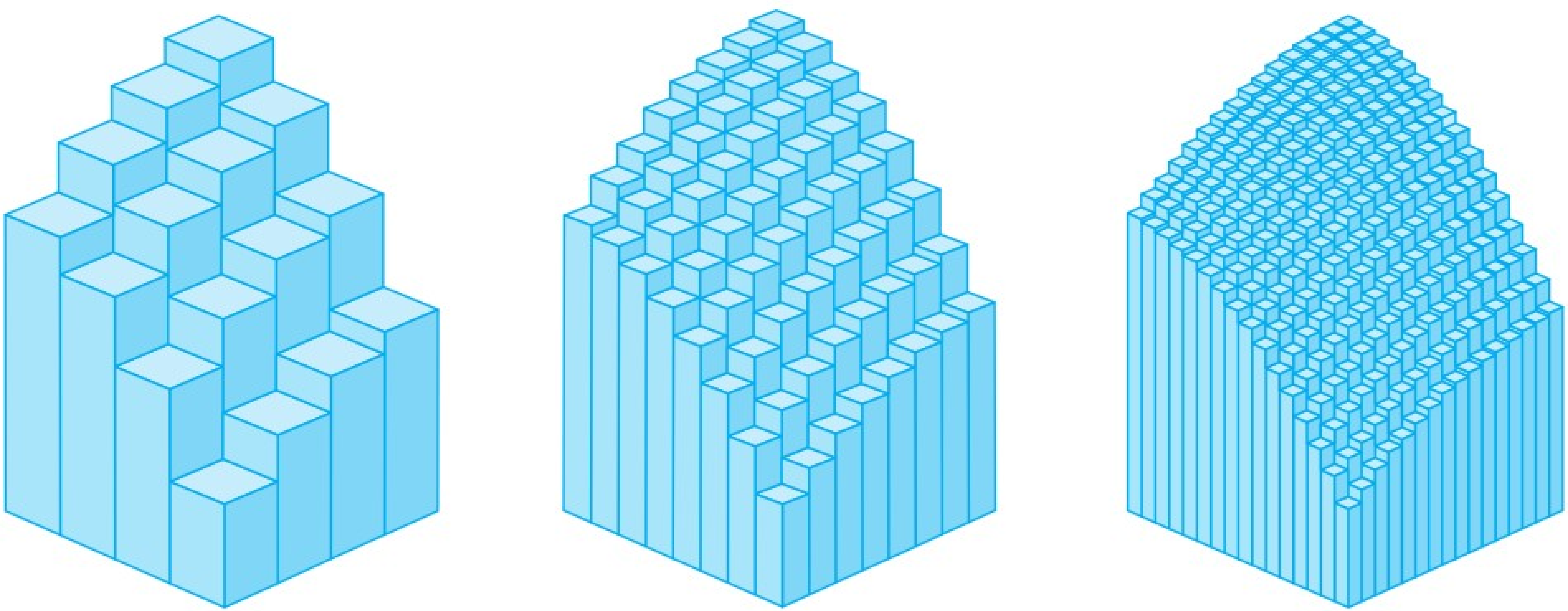


$$V \approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(x_i, y_j)\,\Delta A = f(1, 1)\,\Delta A + f(1, 2)\,\Delta A + f(2, 1)\,\Delta A + f(2, 2)\,\Delta A$$
$$= 13(1) + 7(1) + 10(1) + 4(1) = 34$$



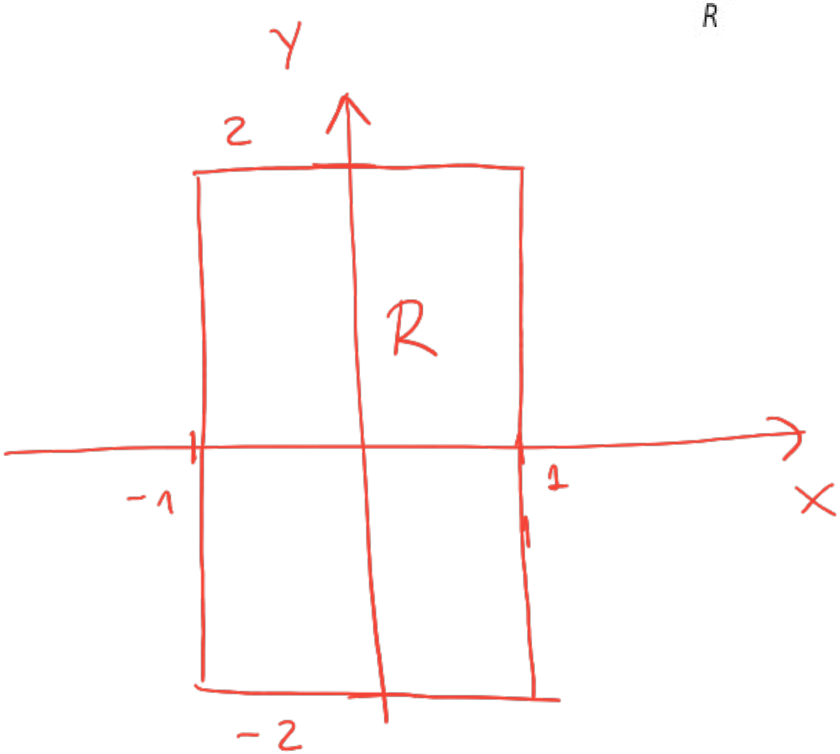
z
16
$z = 16 - x^2 - 2y^2$
2
y
2
x



(a) $m = n = 4$, $V \approx 41.5$ (b) $m = n = 8$, $V \approx 44.875$ (c) $m = n = 16$, $V \approx 46.46875$

Exemplo: $R = \{(x, y) \mid -1 \leqslant x \leqslant 1, -2 \leqslant y \leqslant 2\}$

$$\iint_R \sqrt{1 - x^2}\, dA$$



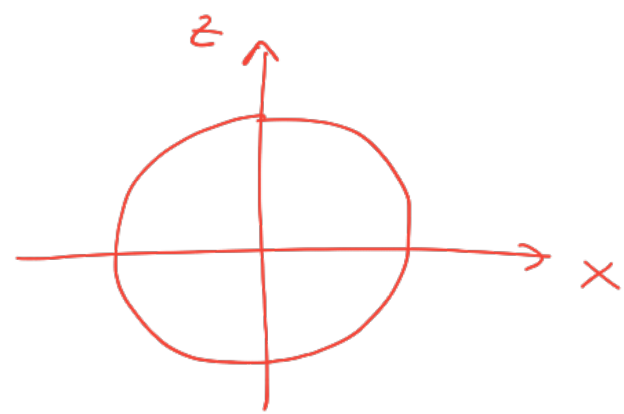
$$z = \sqrt{1-x^2} = f(x,y)$$

$$\Rightarrow z^2 = 1 - x^2$$

$$\Leftrightarrow x^2 + z^2 = 1$$

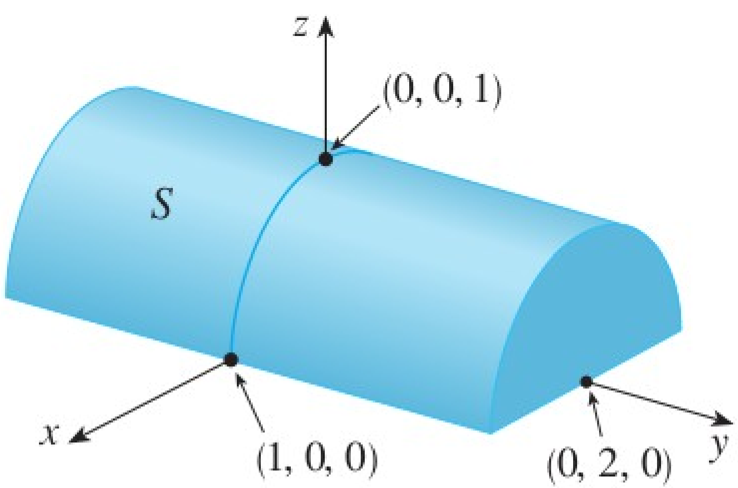
$$\sqrt{z^2} = \sqrt{1-x^2} \Rightarrow |z| = \sqrt{1-x^2}$$

$$\Rightarrow z = \pm\sqrt{1-x^2}$$



Exemplo: $R = \{(x, y) \mid -1 \leqslant x \leqslant 1, -2 \leqslant y \leqslant 2\}$

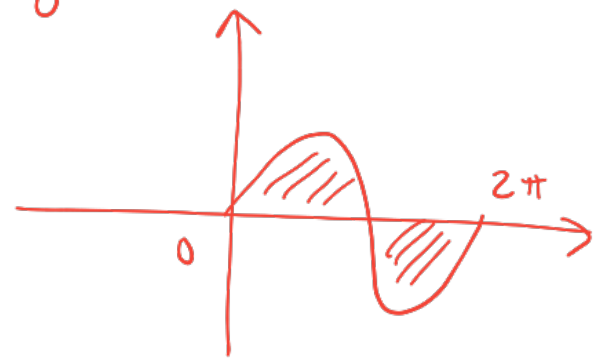
$$V = \iint_R \sqrt{1 - x^2}\, dA \quad , \; z = \sqrt{1-x^2} \geqslant 0$$

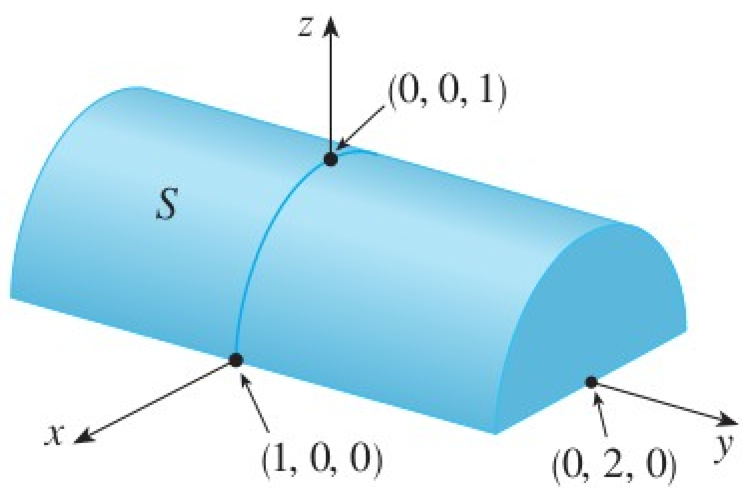


Exemplo: $R = \{(x, y) \mid -1 \leqslant x \leqslant 1, -2 \leqslant y \leqslant 2\}$

$$\int_0^{2\pi} \operatorname{sen} x \, dx = 0$$

$$\iint_R \sqrt{1 - x^2}\, dA$$





$$\iint_R \sqrt{1 - x^2}\, dA = \underbrace{\tfrac{1}{2}\pi(1)^2 \times 4}_{\text{metade do vol. do cilindro}} = 2\pi$$

Propriedades

$$\iint_R [f(x,y) + g(x,y)]\, dA = \iint_R f(x,y)\, dA + \iint_R g(x,y)\, dA$$

$$\iint_R c f(x,y)\, dA = c \iint_R f(x,y)\, dA$$

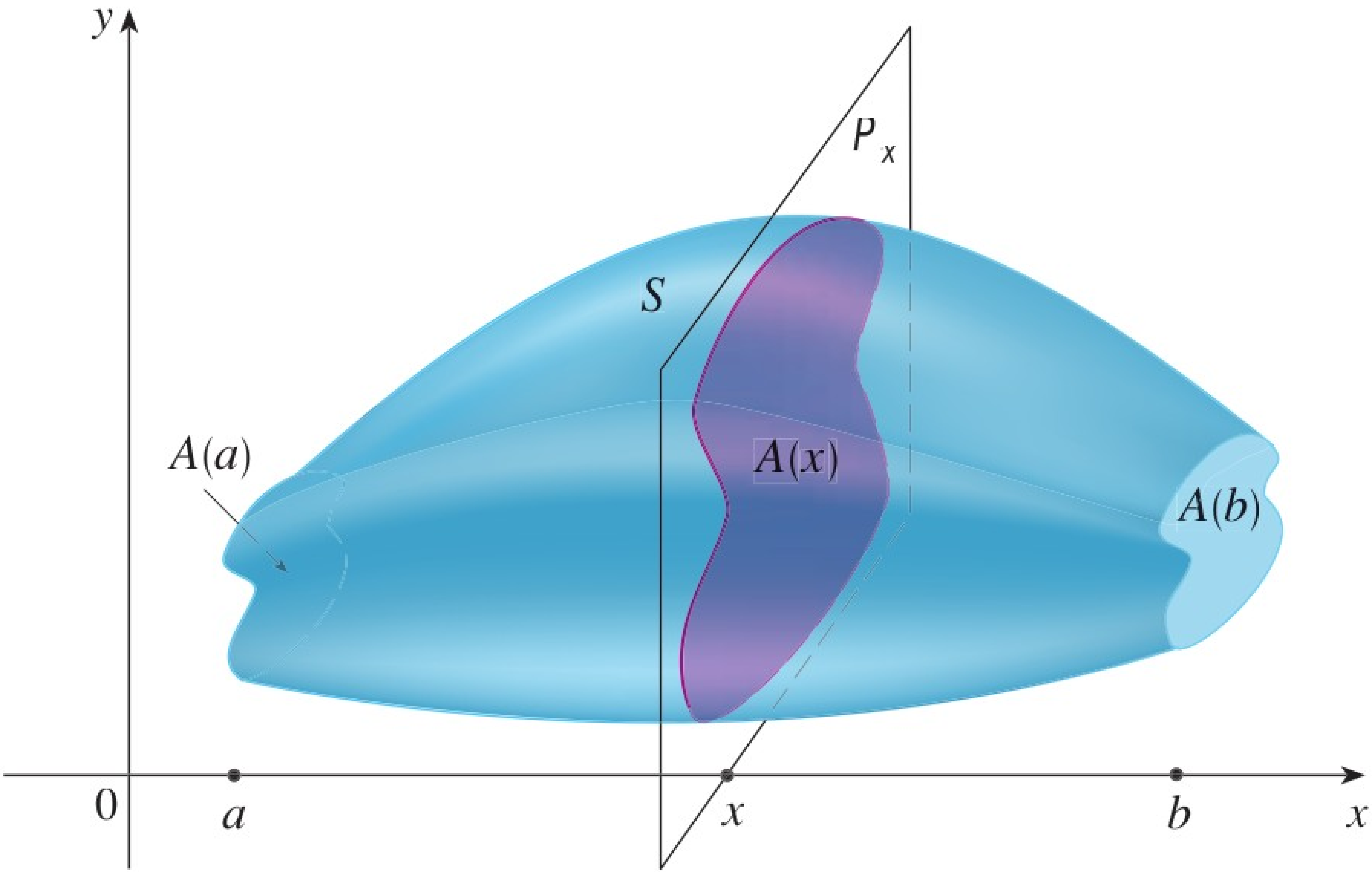
$$f(x,y) \geqslant g(x,y)$$

$$\iint_R f(x,y)\, dA \geqslant \iint_R g(x,y)\, dA$$

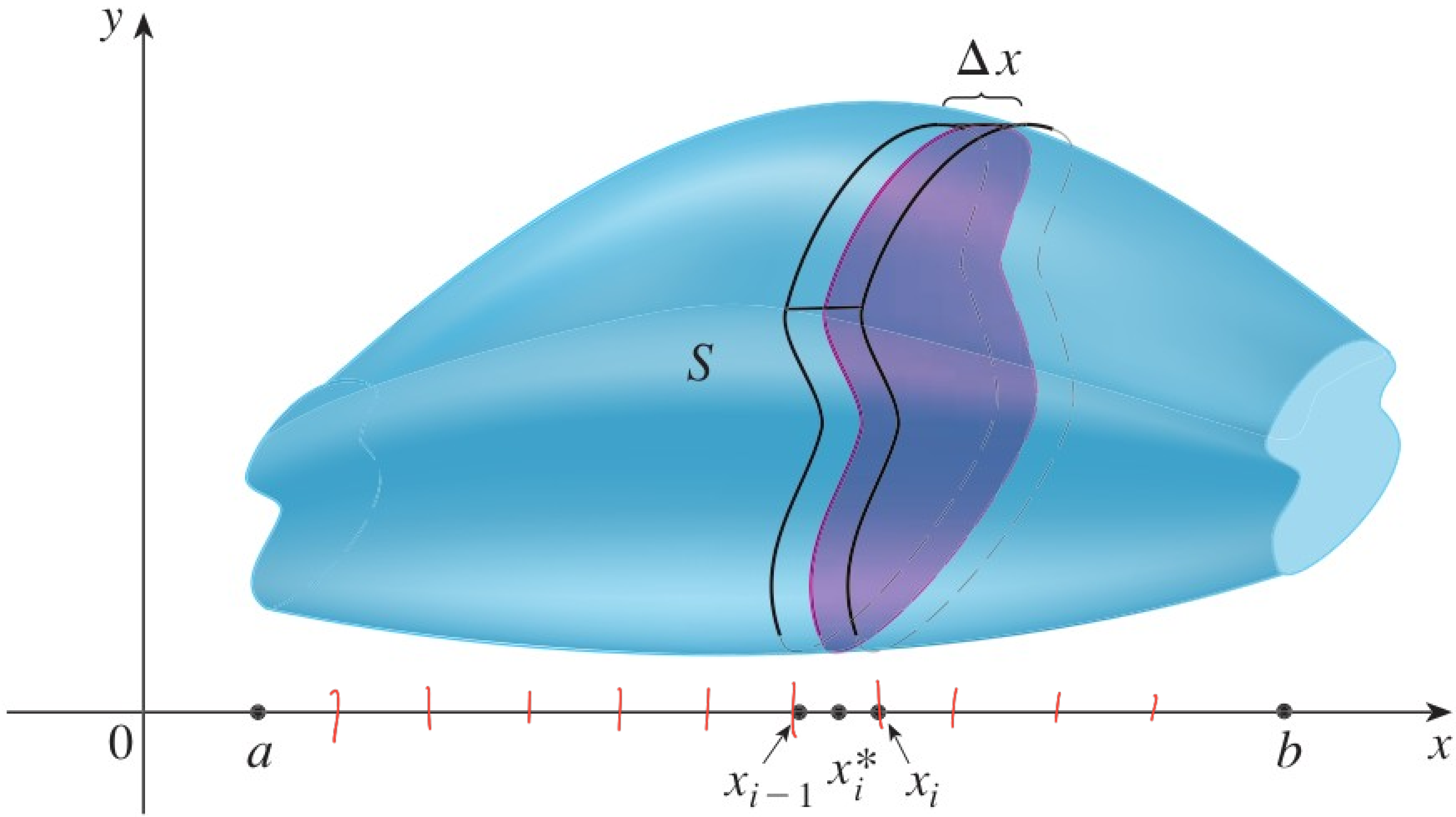
De modo geral:

$$\iint_R f(x,y) \cdot g(x,y)\, dA \neq \iint_R f(x,y)\, dA \cdot \iint_R g(x,y)\, dA$$

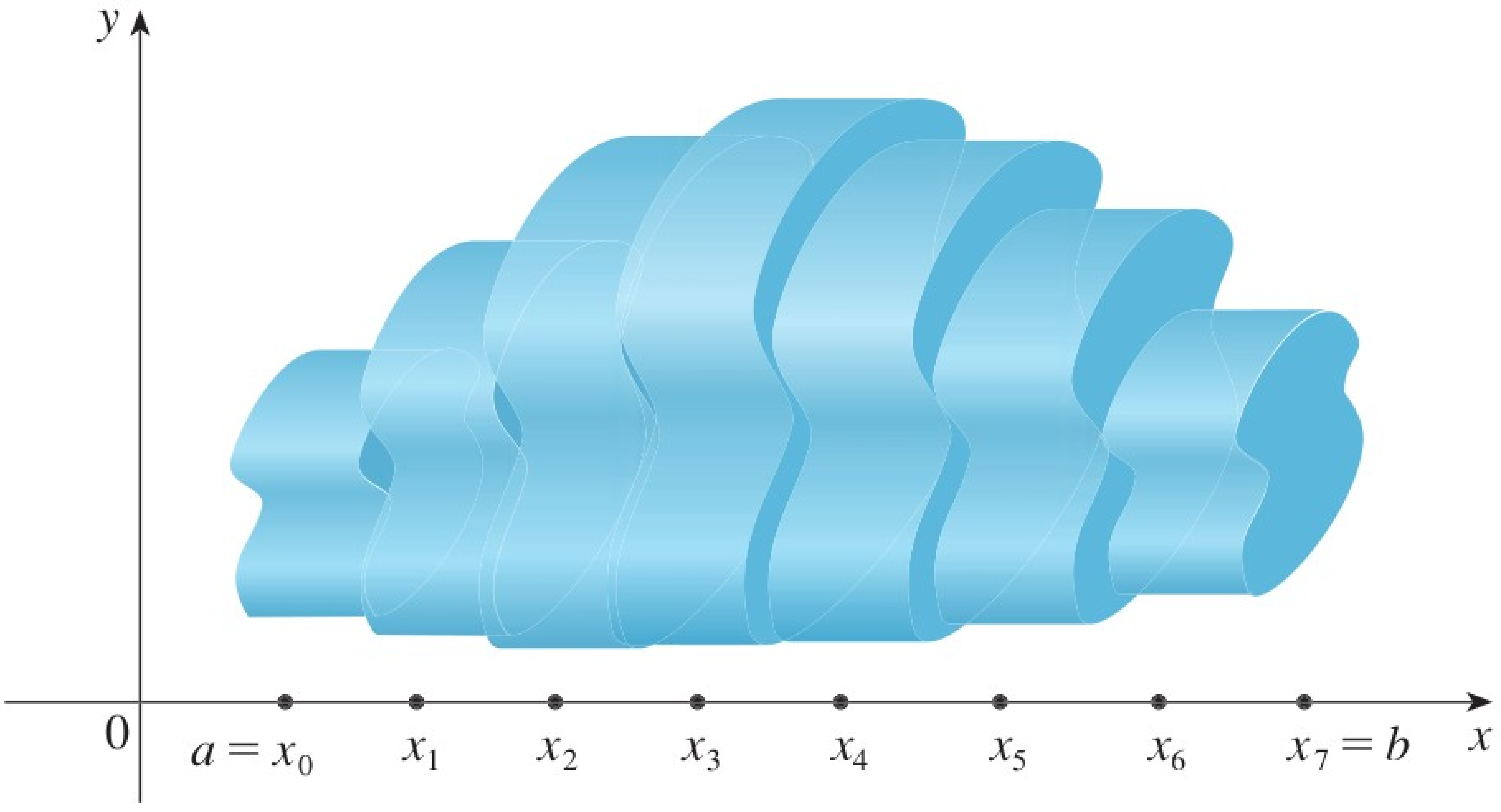
Integral Iterada



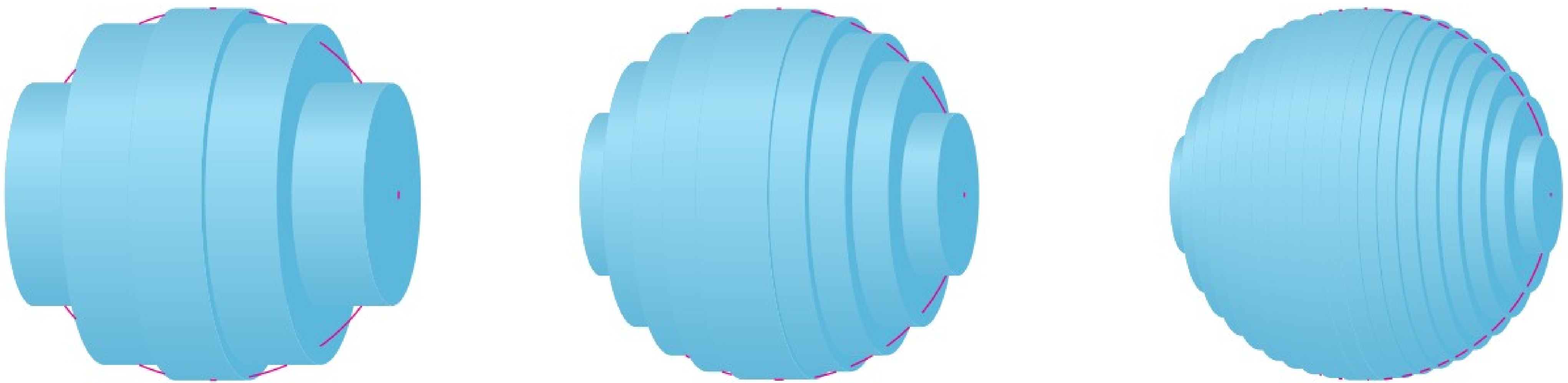
Integral Iterada



Integral Iterada

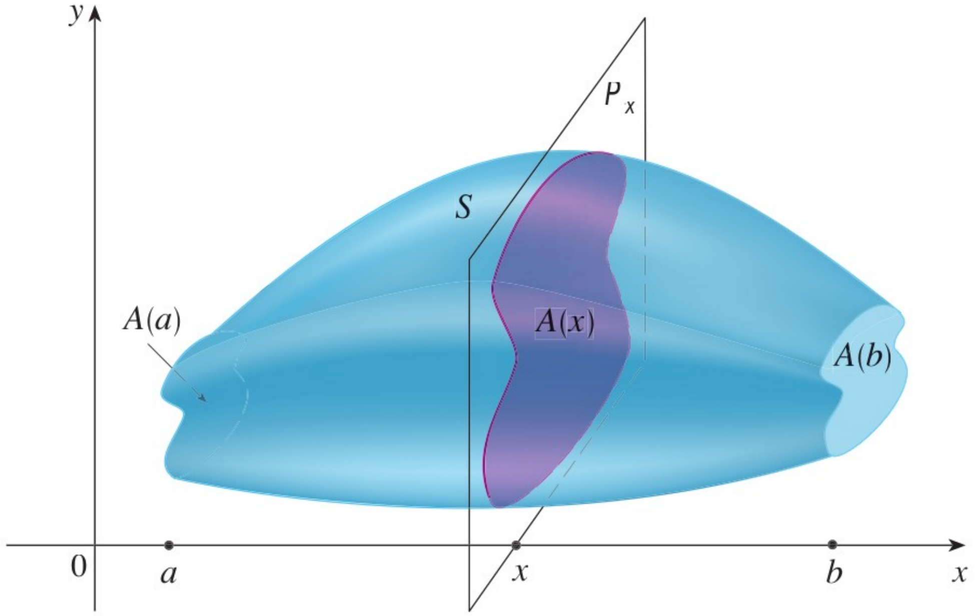


Integral Iterada

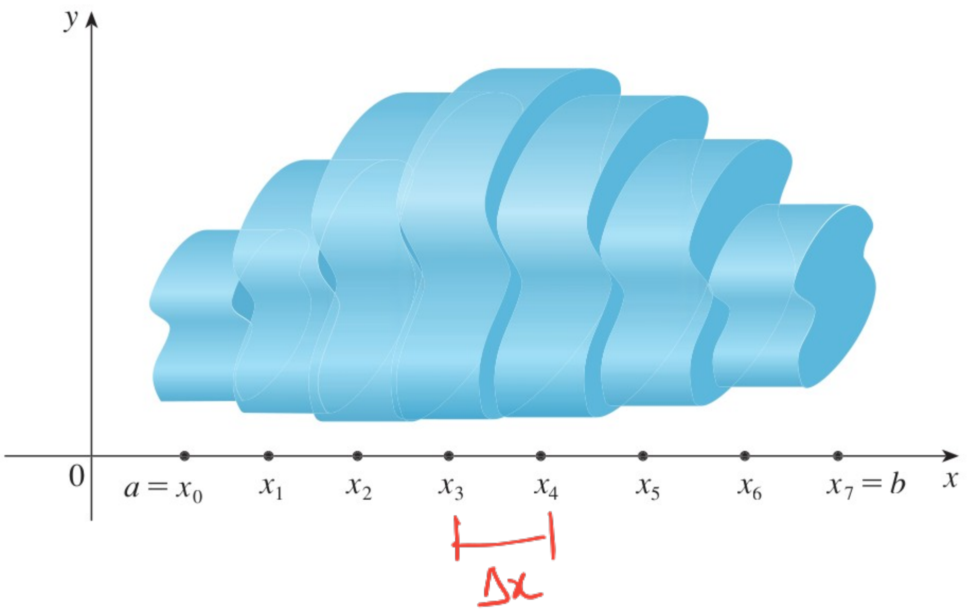


(a) Using 5 disks, $V \approx 4.2726$ (b) Using 10 disks, $V \approx 4.2097$ (c) Using 20 disks, $V \approx 4.1940$

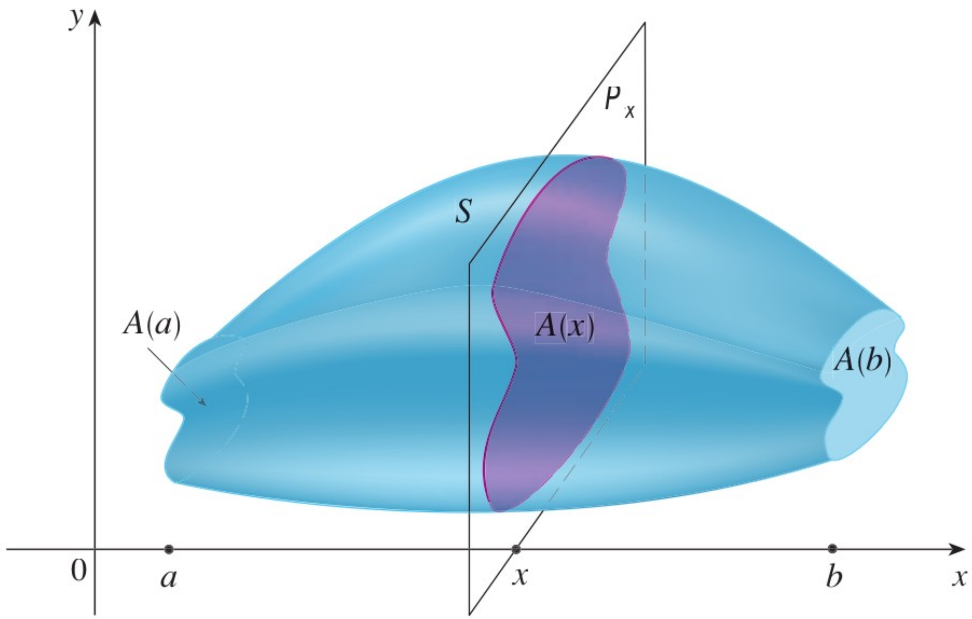
Integral Iterada

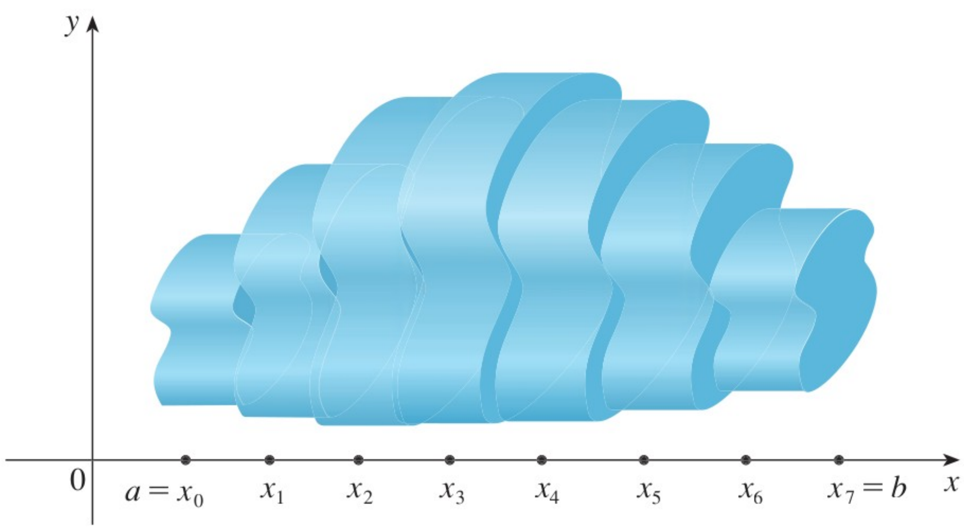


$$V(S_i) \approx A(x_i^*)\,\Delta x$$



Integral Iterada

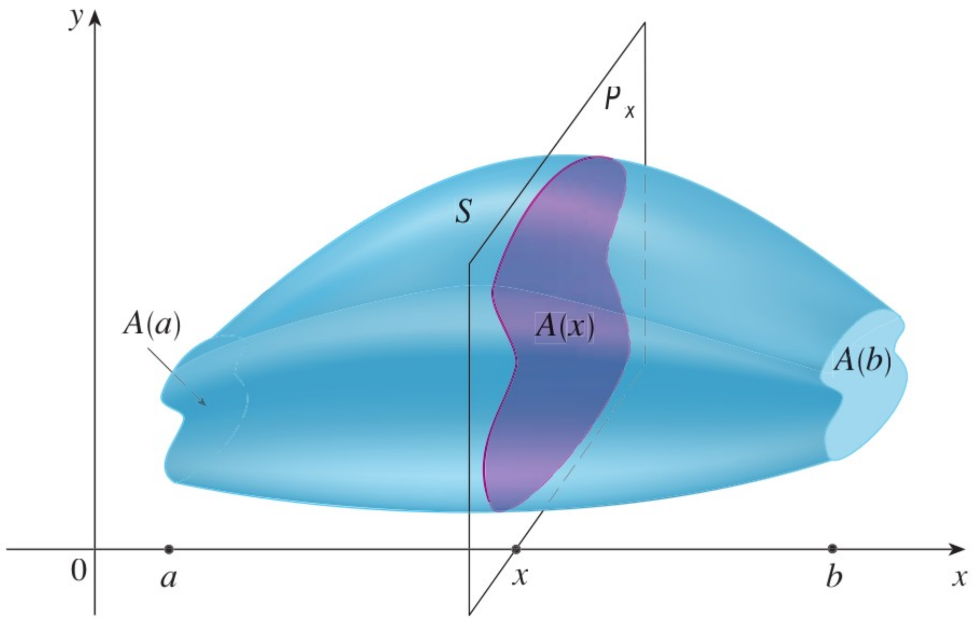


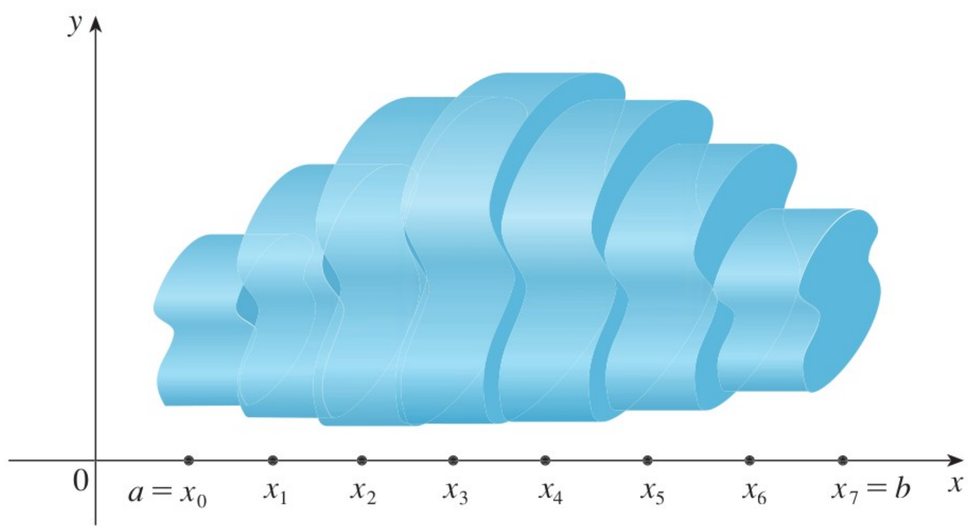


$$V(S_i) \approx A(x_i^*)\,\Delta x$$

$$V \approx \sum_{i=1}^{n} A(x_i^*)\,\Delta x$$

Integral Iterada



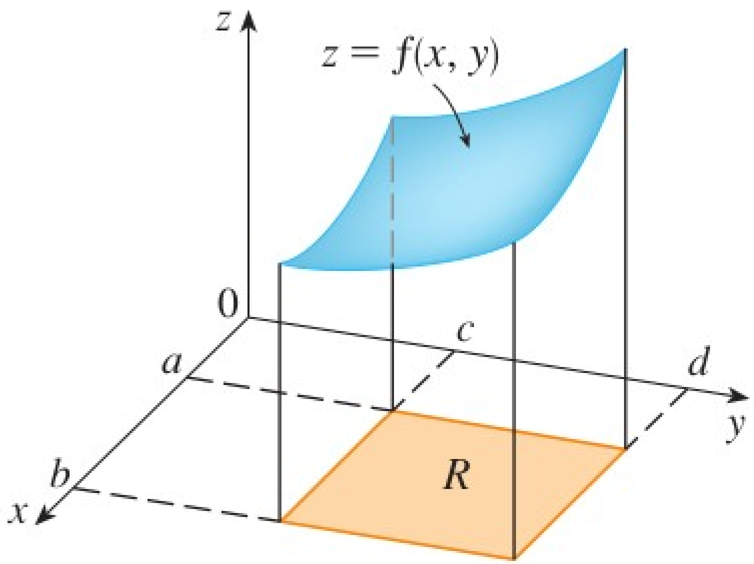


$$V(S_i) \approx A(x_i^*)\,\Delta x$$

$$V \approx \sum_{i=1}^{n} A(x_i^*)\,\Delta x$$

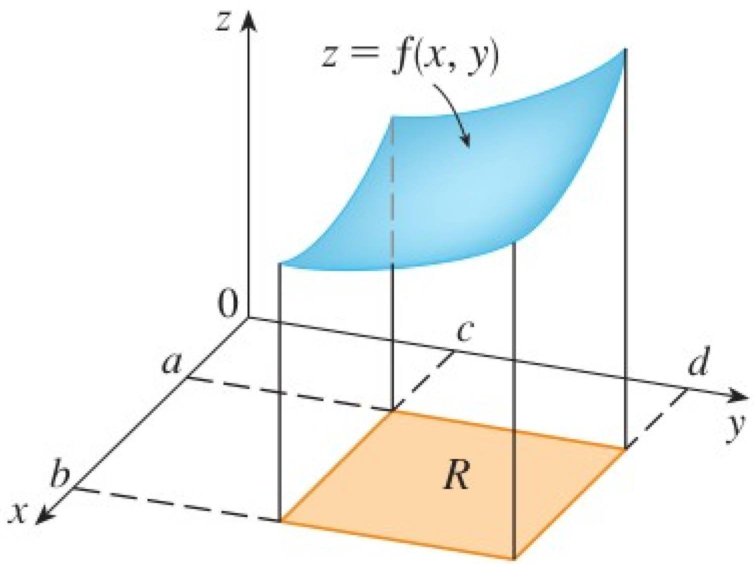
$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\,\Delta x = \int_a^b A(x)\,dx$$

Integral Iterada



$$R = [a, b] \times [c, d]$$

Integral Iterada



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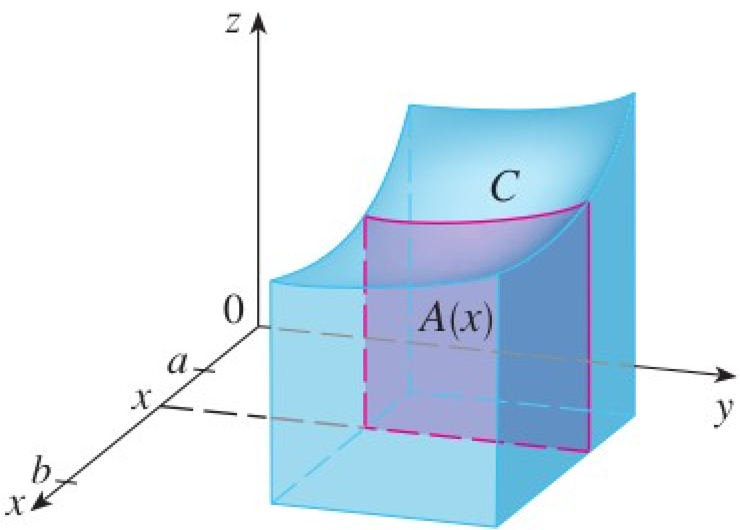
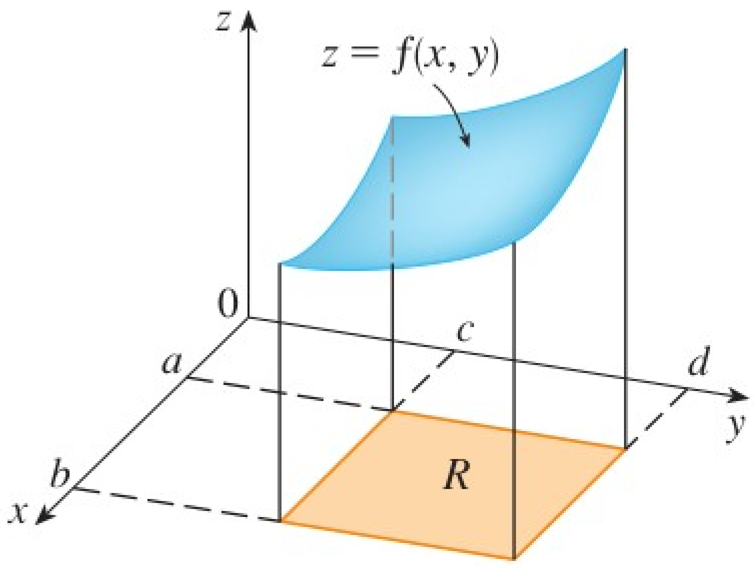


Integral Iterada



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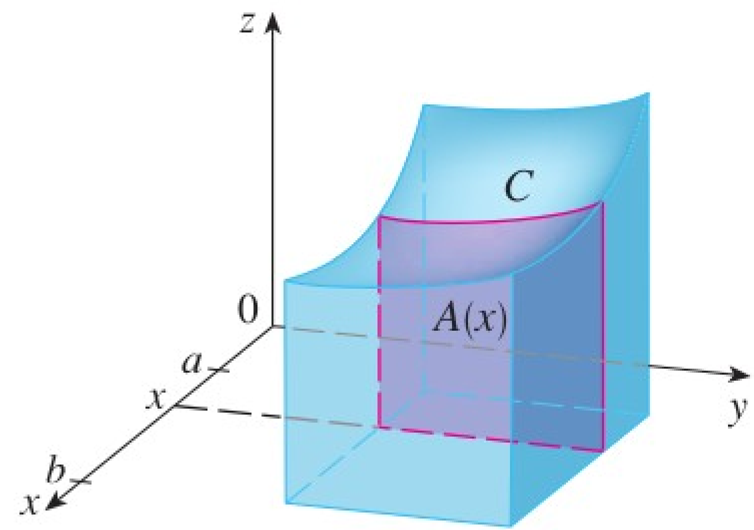
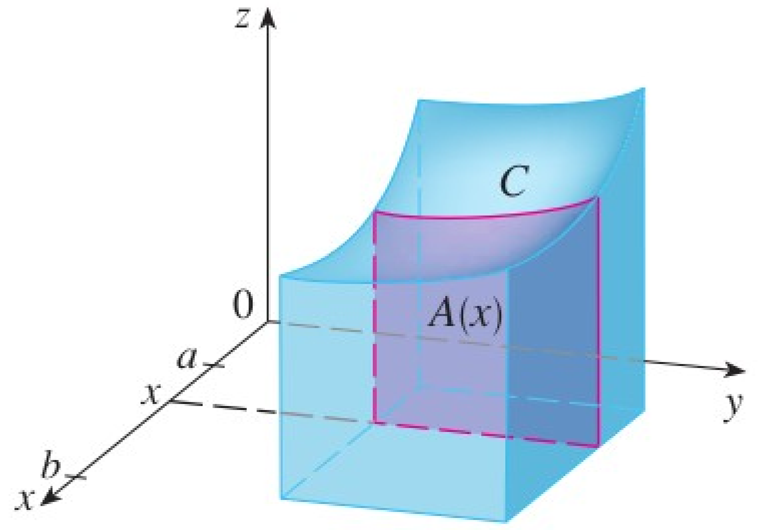


$$A(x) = \int_c^d f(x, y)\, dy$$

Integral Iterada

$$A(x) = \int_c^d f(x, y)\, dy$$

$$\int_a^b A(x)\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx$$

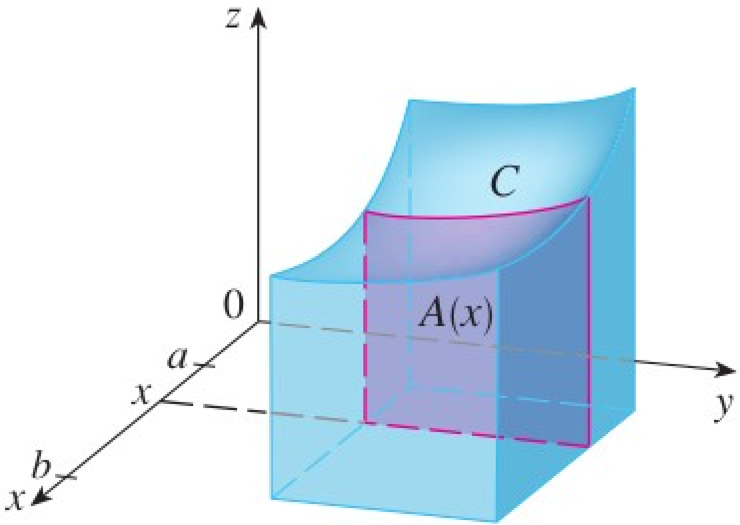


Integral Iterada

$$A(x) = \int_c^d f(x, y)\, dy$$

$$\int_a^b A(x)\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx$$

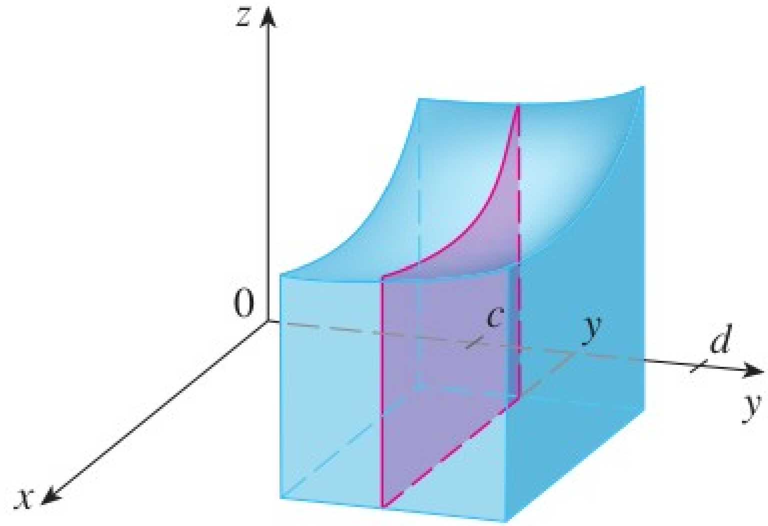
$$\int_a^b \int_c^d f(x, y)\, dy\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx$$



Integral Iterada

Analogamente

$$\int_c^d \int_a^b f(x, y)\, dx\, dy = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy$$



Exemplo: $\int_0^3 \int_1^2 x^2 y \, dy \, dx$

Exemplo: $\displaystyle \int_0^3 \left(\int_1^2 x^2 y \, dy \right) dx$

$$\int_1^2 x^2 y \, dy = \left[x^2 \frac{y^2}{2} \right]_{y=1}^{y=2} = x^2 \left(\frac{2^2}{2} \right) - x^2 \left(\frac{1^2}{2} \right) = \tfrac{3}{2} x^2 = A(x)$$

Exemplo: $\int_0^3 \int_1^2 x^2 y \, dy \, dx$

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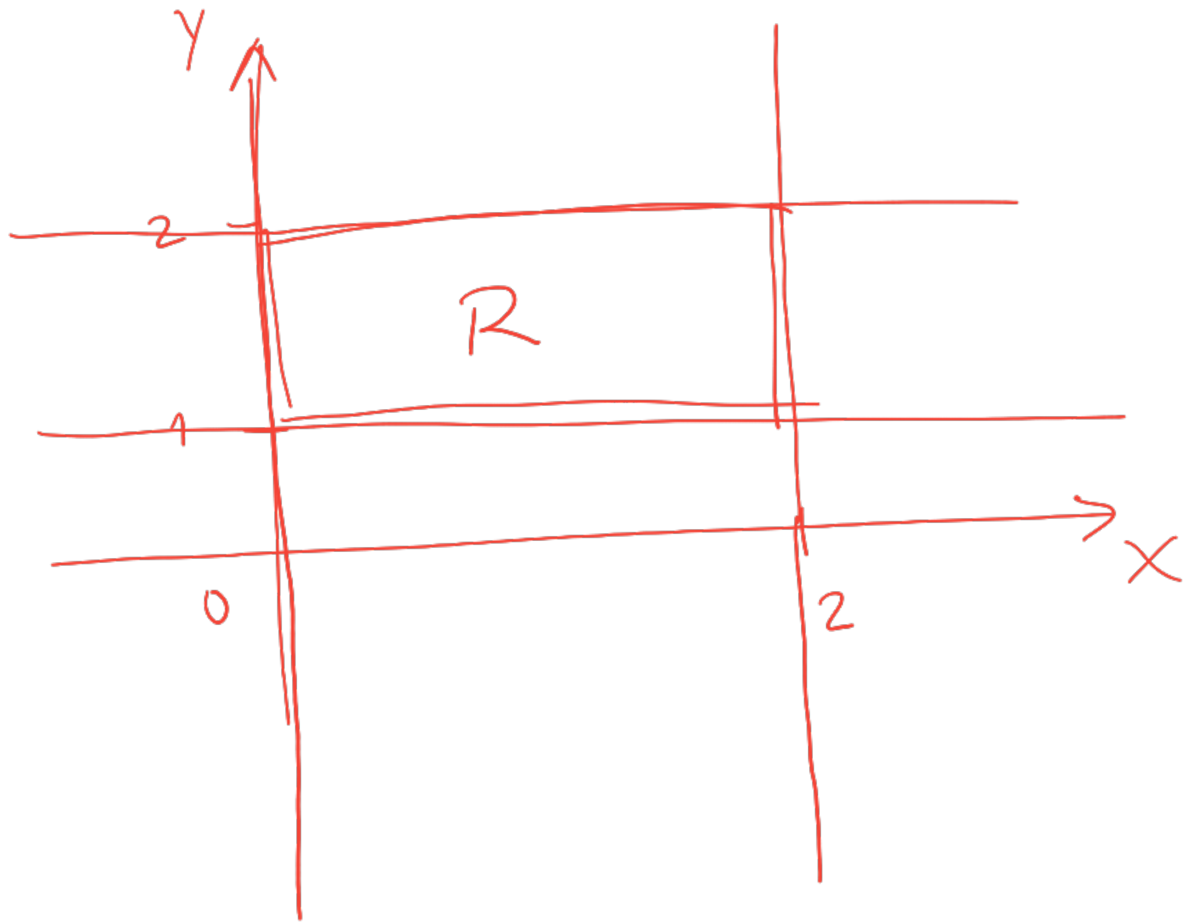
$$\int_0^3 \int_1^2 x^2 y \, dy \, dx = \int_0^3 \left[\int_1^2 x^2 y \, dy \right] dx$$

$$= \int_0^3 \tfrac{3}{2} x^2 \, dx = \frac{x^3}{2} \bigg]_0^3 = \frac{27}{2}$$

Teorema de Fubini: Se f é contínua em $R = \{(x, y) \mid a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$, então

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

Exemplo: $\iint_R (x - 3y^2)\, dA$, $R = \{(x, y) \mid 0 \leqslant x \leqslant 2, 1 \leqslant y \leqslant 2\}$



Exemplo: $\iint_R (x - 3y^2)\, dA,\ R = \{(x, y) \mid 0 \leqslant x \leqslant 2, 1 \leqslant y \leqslant 2\}$

$$\Rightarrow \iint\limits_R (x - 3y^2)\, dA = \int_0^2 \int_1^2 (x - 3y^2)\, dy\, dx = \int_0^2 \left[xy - y^3\right]_{y=1}^{y=2} dx$$

$$= \int_0^2 (x - 7)\, dx = \left.\frac{x^2}{2} - 7x\right]_0^2 = -12$$

Exemplo: $\iint_R (x - 3y^2)\, dA$, $R = \{(x, y) \mid 0 \leqslant x \leqslant 2,\ 1 \leqslant y \leqslant 2\}$

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$$= \int_0^2 (x - 7)\, dx = \frac{x^2}{2} - 7x \Big]_0^2 = -12$$

$$\iint\limits_R (x - 3y^2)\, dA = \int_1^2 \int_0^2 (x - 3y^2)\, dx\, dy$$

$$= \int_1^2 \left[\frac{x^2}{2} - 3xy^2\right]_{x=0}^{x=2} dy \quad = \int_1^2 \left(\frac{\cancel{4}^{\,2}}{\cancel{2}} - 6y^2\right) - \cancel{\left(\frac{0}{2} - 0\right)}\, dy$$

$$= \int_1^2 (2 - 6y^2)\, dy = 2y - 2y^3\Big]_1^2 = -12$$

$$2 \quad \frac{\cancel{6}\,y^3}{\cancel{3}} \qquad\qquad 4 - 16 - \cancel{2} + \cancel{2}$$

Exemplo: $\iint_R y \sin(xy)\, dA,\ R = [1, 2] \times [0, \pi]$

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$$\iint\limits_R y \sin(xy)\, dA = \int_1^2 \int_0^\pi y \sin(xy)\, dy\, dx$$

Exemplo: $\iint_R y \sin(xy)\, dA,\ R = [1,2] \times [0, \pi]$

$$\iint_R y \sin(xy)\, dA = \int_1^2 \int_0^\pi y \sin(xy)\, dy\, dx$$

$$u = y \qquad dv = \sin(xy)\, dy$$

$$du = dy \qquad v = -\frac{\cos(xy)}{x}$$

$$-\frac{y \cos(xy)}{x}\Big|_0^\pi + \int_0^\pi \frac{\cos(xy)}{x}\, dy$$

$$-\frac{y \cos(xy)}{x}\Big|_0^\pi + \frac{1}{x} \int_0^\pi \cos(xy)\, dy$$

Exemplo: $\iint_R y \sin(xy)\, dA,\ R = [1, 2] \times [0, \pi]$

$$\iint_R y \sin(xy)\, dA = \int_1^2 \int_0^\pi y \sin(xy)\, dy\, dx$$

$$u = y \qquad dv = \sin(xy)\, dy$$

$$du = dy \qquad v = -\frac{\cos(xy)}{x}$$

$$\int_0^\pi y \sin(xy)\, dy = -\frac{y \cos(xy)}{x}\Bigg]_{y=0}^{y=\pi} + \frac{1}{x}\int_0^\pi \cos(xy)\, dy$$

$$= -\frac{\pi \cos \pi x}{x} + \frac{1}{x^2}\left[\sin(xy)\right]_{y=0}^{y=\pi}$$

$$= -\frac{\pi \cos \pi x}{x} + \frac{\sin \pi x}{x^2}$$

Exemplo: $\iint_R y \sin(xy)\, dA,\ R = [1, 2] \times [0, \pi]$

$$\iint_R y \sin(xy)\, dA = \int_1^2 \int_0^{\pi} y \sin(xy)\, dy\, dx$$

$$u = y \qquad dv = \sin(xy)\, dy$$

$$du = dy \qquad v = -\frac{\cos(xy)}{x}$$

$$\int_0^{\pi} y \sin(xy)\, dy = -\frac{y \cos(xy)}{x}\Bigg]_{y=0}^{y=\pi} + \frac{1}{x}\int_0^{\pi} \cos(xy)\, dy$$

$$= -\frac{\pi \cos \pi x}{x} + \frac{1}{x^2}\left[\sin(xy)\right]_{y=0}^{y=\pi}$$

$$= -\frac{\pi \cos \pi x}{x} + \frac{\sin \pi x}{x^2}$$

$$\int \left(-\frac{\pi \cos \pi x}{x}\right) dx = -\frac{\sin \pi x}{x} - \int \frac{\sin \pi x}{x^2}\, dx$$

$$u = -1/x \qquad dv = \pi \cos \pi x\, dx$$

$$du = dx/x^2 \qquad v = \sin \pi x$$

$$\int \left(-\frac{\pi \cos \pi x}{x}\right) dx = -\frac{\sin \pi x}{x} - \int \frac{\sin \pi x}{x^2}\, dx$$

$$\int \left(-\frac{\pi \cos \pi x}{x} + \frac{\sin \pi x}{x^2}\right) dx = -\frac{\sin \pi x}{x}$$

$$\int \left(-\frac{\pi \cos \pi x}{x}\right) dx = -\frac{\sin \pi x}{x} - \int \frac{\sin \pi x}{x^2}\, dx$$

$$\int \left(-\frac{\pi \cos \pi x}{x} + \frac{\sin \pi x}{x^2}\right) dx = -\frac{\sin \pi x}{x}$$

$$\begin{aligned}\int_1^2 \int_0^\pi y \sin(xy)\, dy\, dx &= \left[-\frac{\sin \pi x}{x}\right]_1^2 \\ &= -\frac{\sin 2\pi}{2} + \sin \pi = 0\end{aligned}$$

Solução alternativa:

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$$\iint_R y \sin(xy)\, dA = \int_0^{\pi} \int_1^2 y \sin(xy)\, dx\, dy = \int_0^{\pi} \Big[-\cos(xy)\Big]_{x=1}^{x=2} dy$$

$$= \int_0^{\pi} (-\cos 2y + \cos y)\, dy$$

$$= -\tfrac{1}{2} \sin 2y + \sin y\Big]_0^{\pi} = 0$$

Suponha $f(x, y) = g(x)h(y)$

$f(x,y) = x^2 \cdot \operatorname{sen} y$

$\ln y \cdot e^x$

$\cos\left(\frac{y+y^2}{2}\right) \cdot \frac{x^2+2\sqrt{x}}{3x}$

$x^2 y + 2y = y(x^2+2)$

Suponha $f(x, y) = g(x)h(y)$

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b g(x)h(y)\, dx\, dy = \int_c^d \left[\int_a^b g(x)h(y)\, dx \right] dy$$

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$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b g(x)h(y)\, dx\, dy = \int_c^d \left[\int_a^b g(x)h(y)\, dx \right] dy$$

$$= \int_c^d \left[h(y) \left(\int_a^b g(x)\, dx \right) \right] dy = \int_a^b g(x)\, dx \int_c^d h(y)\, dy$$

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$$\iint_R g(x)h(y)\, dA = \int_a^b g(x)\, dx \int_c^d h(y)\, dy \qquad R = [a, b] \times [c, d]$$

Exemplo: $R = [0, \pi/2] \times [0, \pi/2]$

$$\iint_R \sin x \, \cos y \, dA = \int_0^{\pi/2} \sin x \, dx \int_0^{\pi/2} \cos y \, dy$$

$$= \Big[-\cos x\Big]_0^{\pi/2} \Big[\sin y\Big]_0^{\pi/2} = 1 \cdot 1 = 1$$