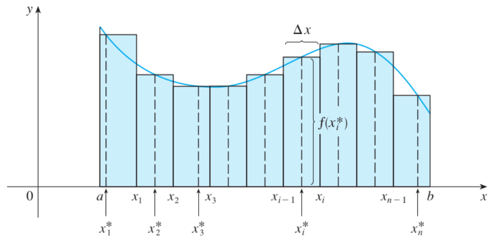
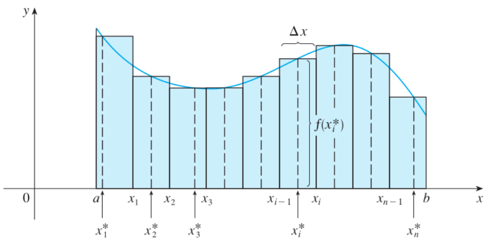
Cálculo III

Integral dupla

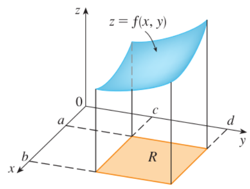
Prof. Adriano Barbosa



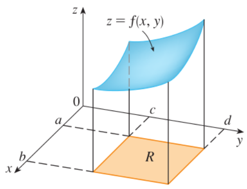


$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$



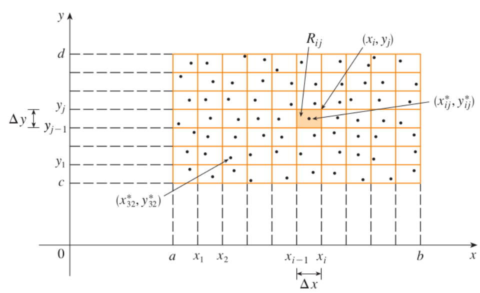
$f(x, y) \geq 0$



$f(x, y) \geq 0$

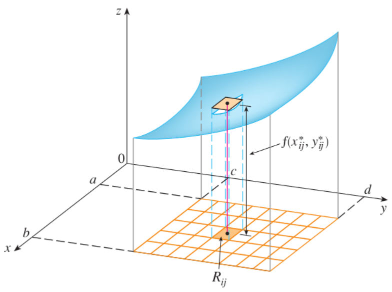
$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b,\ c \leq y \leq d\}$$

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq z \leq f(x, y),\ (x, y) \in R\}$$



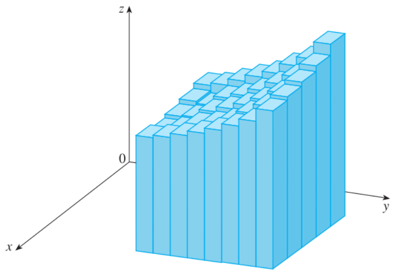
$$\Delta x = (b - a)/m \qquad \Delta y = (d - c)/n$$

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x, y) \mid x_{i-1} \leq x \leq x_i,\ y_{j-1} \leq y \leq y_j\}$$



$$\Delta A = \Delta x \, \Delta y$$

$$f(x_{ij}^*, y_{ij}^*) \, \Delta A$$



$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \, \Delta A$$

$$V = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \, \Delta A$$

Integral dupla:

$$\iint_R f(x, y) \, dA = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \, \Delta A$$

Se $f(x, y) \geqslant 0$, o volume do sólido acima da região R e abaixo gráfico da função é dado por:

$$V = \iint_R f(x, y) \, dA$$

Exemplo: Estime a área do sólido acima da região R e abaixo do gráfico da função abaixo

$R = [0, 2] \times [0, 2]$ $\qquad f(x, y) = 16 - x^2 - 2y^2$

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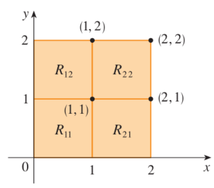

Exemplo: Estime a área do sólido acima da região *R* e abaixo do gráfico da função abaixo

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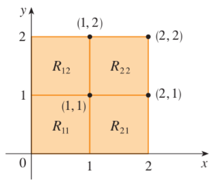

$$V \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_i, y_j)\,\Delta A$$

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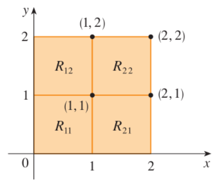
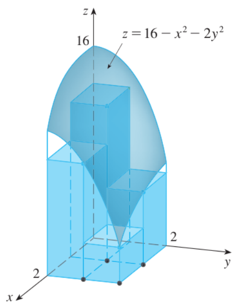
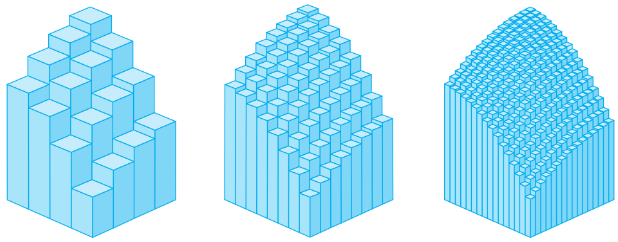


$$V \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_i, y_j)\,\Delta A = f(1, 1)\,\Delta A + f(1, 2)\,\Delta A + f(2, 1)\,\Delta A + f(2, 2)\,\Delta A$$
$$= 13(1) + 7(1) + 10(1) + 4(1) = 34$$





(a) $m = n = 4$, $V \approx 41.5$ (b) $m = n = 8$, $V \approx 44.875$ (c) $m = n = 16$, $V \approx 46.46875$

Exemplo: $R = \{(x, y) \mid -1 \leqslant x \leqslant 1, -2 \leqslant y \leqslant 2\}$

$$\iint_R \sqrt{1 - x^2}\, dA$$

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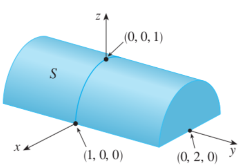

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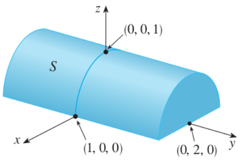
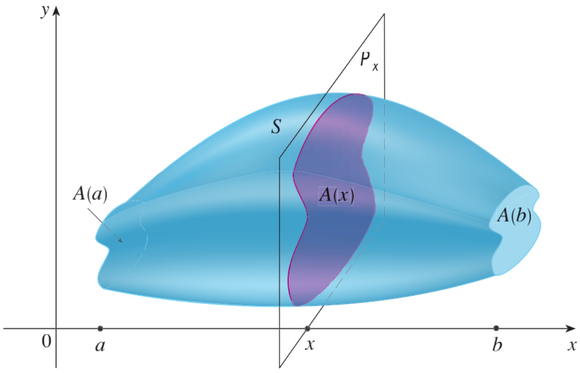


$$\iint_R \sqrt{1 - x^2}\, dA = \tfrac{1}{2}\pi(1)^2 \times 4 = 2\pi$$

Propriedades

- $$\iint_R [f(x, y) + g(x, y)]\, dA = \iint_R f(x, y)\, dA + \iint_R g(x, y)\, dA$$
- $$\iint_R c\, f(x, y)\, dA = c \iint_R f(x, y)\, dA$$
- $f(x, y) \geqslant g(x, y)$
$$\iint_R f(x, y)\, dA \geqslant \iint_R g(x, y)\, dA$$

Integral Iterada



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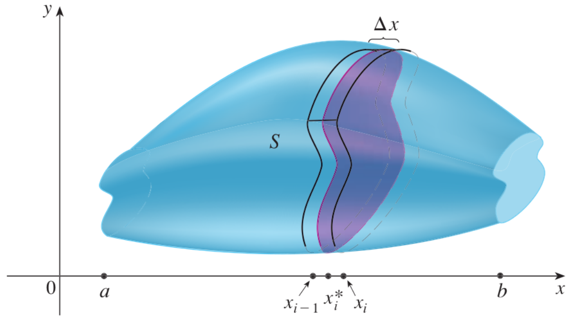

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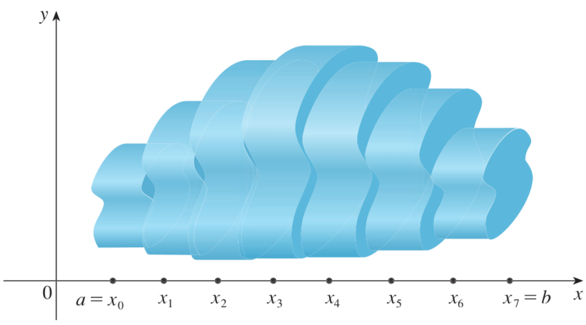
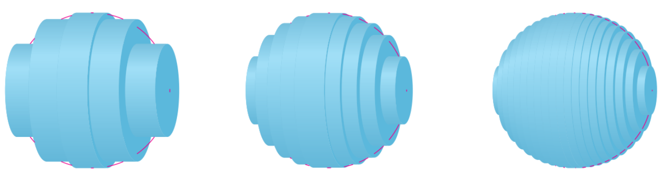
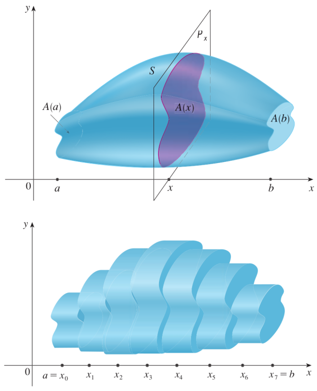


Integral Iterada



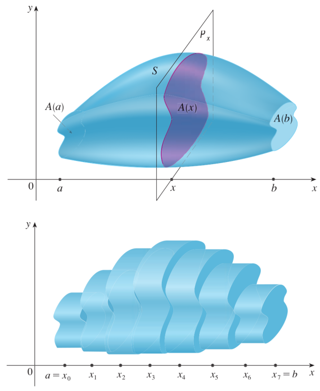
(a) Using 5 disks, $V \approx 4.2726$ (b) Using 10 disks, $V \approx 4.2097$ (c) Using 20 disks, $V \approx 4.1940$

Integral Iterada



$$V(S_i) \approx A(x_i^*)\,\Delta x$$

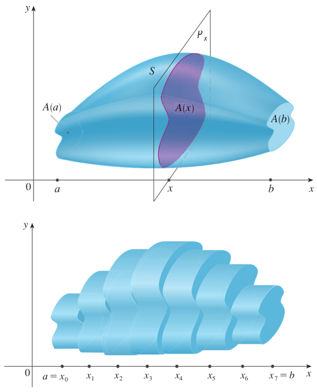
Integral Iterada



$$V(S_i) \approx A(x_i^*)\,\Delta x$$

$$V \approx \sum_{i=1}^{n} A(x_i^*)\,\Delta x$$

Integral Iterada

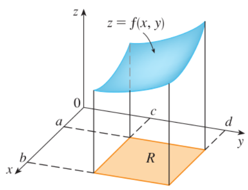


$$V(S_i) \approx A(x_i^*)\,\Delta x$$

$$V \approx \sum_{i=1}^{n} A(x_i^*)\,\Delta x$$

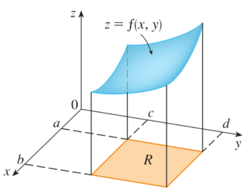
$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\,\Delta x = \int_a^b A(x)\,dx$$

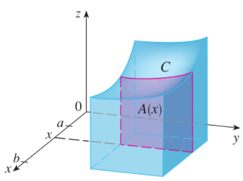
Integral Iterada



$$R = [a, b] \times [c, d]$$

Integral Iterada





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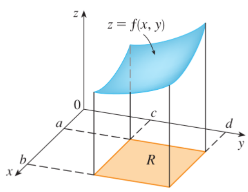

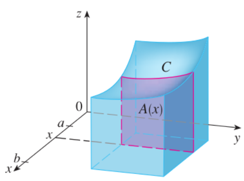

$$R = [a, b] \times [c, d]$$

$$A(x) = \int_c^d f(x, y)\,dy$$

Integral Iterada

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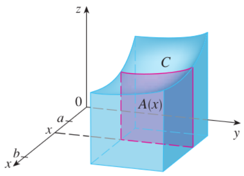
$$\int_a^b A(x)\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx$$



Integral Iterada

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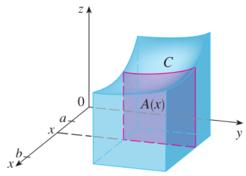
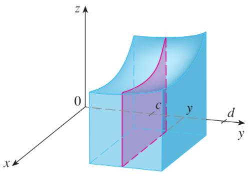
$$\int_a^b \int_c^d f(x, y)\, dy\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx$$



Integral Iterada

Analogamente

$$\int_c^d \int_a^b f(x, y)\, dx\, dy = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy$$



Exemplo: $\int_0^3 \int_1^2 x^2 y\, dy\, dx$

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$$\int_1^2 x^2 y\, dy = \left[x^2 \frac{y^2}{2} \right]_{y=1}^{y=2} = x^2\left(\frac{2^2}{2}\right) - x^2\left(\frac{1^2}{2}\right) = \tfrac{3}{2}x^2$$

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$$\int_0^3 \int_1^2 x^2 y\, dy\, dx = \int_0^3 \left[\int_1^2 x^2 y\, dy \right] dx$$

$$= \int_0^3 \tfrac{3}{2}x^2\, dx = \frac{x^3}{2}\Bigg]_0^3 = \frac{27}{2}$$

Teorema de Fubini: Se f é contínua em $R = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$, então

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

Exemplo: $\iint_R (x - 3y^2)\, dA,\ R = \{(x, y) \mid 0 \leq x \leq 2, 1 \leq y \leq 2\}$

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$$\iint_R (x - 3y^2)\, dA = \int_0^2 \int_1^2 (x - 3y^2)\, dy\, dx = \int_0^2 \left[xy - y^3\right]_{y=1}^{y=2} dx$$

$$= \int_0^2 (x - 7)\, dx = \frac{x^2}{2} - 7x\Big]_0^2 = -12$$

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$$\iint_R (x - 3y^2)\, dA = \int_1^2 \int_0^2 (x - 3y^2)\, dx\, dy$$

$$= \int_1^2 \left[\frac{x^2}{2} - 3xy^2\right]_{x=0}^{x=2} dy$$

$$= \int_1^2 (2 - 6y^2)\, dy = 2y - 2y^3\Big]_1^2 = -12$$

Exemplo: $\iint_R y \sin(xy)\, dA,\ R = [1, 2] \times [0, \pi]$

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$$\iint_R y \sin(xy)\, dA = \int_1^2 \int_0^\pi y \sin(xy)\, dy\, dx$$

$$u = y \qquad dv = \sin(xy)\, dy$$
$$du = dy \qquad v = -\frac{\cos(xy)}{x}$$

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$$du = dy \qquad v = -\frac{\cos(xy)}{x}$$

$$\begin{aligned}\int_0^\pi y \sin(xy)\, dy &= -\frac{y\cos(xy)}{x}\Bigg]_{y=0}^{y=\pi} + \frac{1}{x}\int_0^\pi \cos(xy)\, dy \\ &= -\frac{\pi\cos \pi x}{x} + \frac{1}{x^2}\big[\sin(xy)\big]_{y=0}^{y=\pi} \\ &= -\frac{\pi\cos \pi x}{x} + \frac{\sin \pi x}{x^2}\end{aligned}$$

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$$\int \left(-\frac{\pi\cos \pi x}{x}\right) dx = -\frac{\sin \pi x}{x} - \int \frac{\sin \pi x}{x^2}\, dx$$

$$u = -1/x \qquad dv = \pi\cos \pi x\, dx$$
$$du = dx/x^2 \qquad v = \sin \pi x$$

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$$\int \left(-\frac{\pi\cos \pi x}{x} + \frac{\sin \pi x}{x^2}\right) dx = -\frac{\sin \pi x}{x}$$

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$$\int \left(-\frac{\pi\cos \pi x}{x} + \frac{\sin \pi x}{x^2}\right) dx = -\frac{\sin \pi x}{x}$$

$$\begin{aligned}\int_1^2 \int_0^\pi y \sin(xy)\, dy\, dx &= \left[-\frac{\sin \pi x}{x}\right]_1^2 \\ &= -\frac{\sin 2\pi}{2} + \sin \pi = 0\end{aligned}$$

Solução alternativa:

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$$\iint_R y \sin(xy)\, dA = \int_0^\pi \int_1^2 y \sin(xy)\, dx\, dy = \int_0^\pi \Big[-\cos(xy)\Big]_{x=1}^{x=2} dy$$

$$= \int_0^\pi (-\cos 2y + \cos y)\, dy$$

$$= -\tfrac{1}{2}\sin 2y + \sin y\Big]_0^\pi = 0$$

Suponha $f(x, y) = g(x)h(y)$

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$$= \int_c^d \left[h(y)\left(\int_a^b g(x)\, dx\right)\right] dy = \int_a^b g(x)\, dx \int_c^d h(y)\, dy$$

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$$\iint_R g(x)\, h(y)\, dA = \int_a^b g(x)\, dx \int_c^d h(y)\, dy \qquad R = [a, b] \times [c, d]$$

Exemplo: $R = [0, \pi/2] \times [0, \pi/2]$

$$\iint_R \sin x \cos y\, dA = \int_0^{\pi/2} \sin x\, dx \int_0^{\pi/2} \cos y\, dy$$

$$= \Big[-\cos x\Big]_0^{\pi/2} \Big[\sin y\Big]_0^{\pi/2} = 1 \cdot 1 = 1$$