Cálculo III

Integral tripla

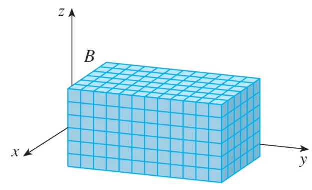
Prof. Adriano Barbosa

Integral tripla

$f: D \subset \mathbb{R}^3 \to \mathbb{R}$

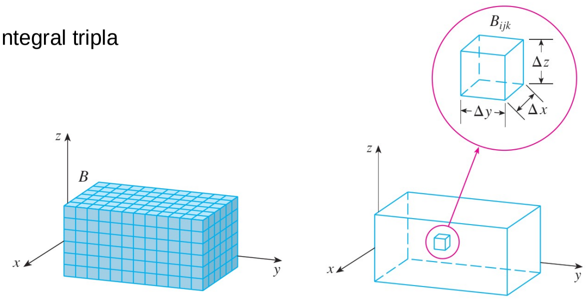
$t = f(x,y,z) \in \mathbb{R}$

O gráfico de f está $\mathbb{R}^4$.

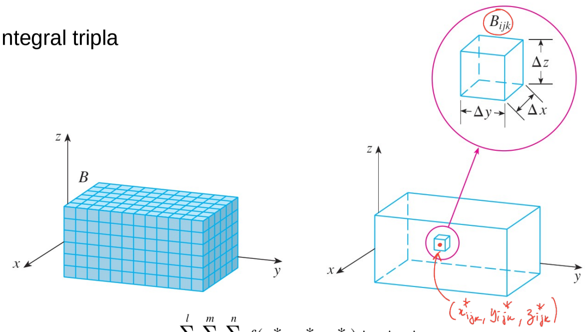


$$B = \{(x, y, z) \mid a \leqslant x \leqslant b,\ c \leqslant y \leqslant d,\ r \leqslant z \leqslant s\}$$

Integral tripla



Integral tripla



$$\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \underbrace{\Delta x\, \Delta y\, \Delta z}_{\Delta V}$$

Integral tripla

A **integral tripla** de f na caixa B é

$$\iiint_B f(x, y, z)\, dV = \lim_{l,\, m,\, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\, \Delta V$$

se esse limite existir.

$$\iint_R f(x,y)\, dA$$

$\underbrace{\Delta x \Delta y}_{\Delta A}$

Teorema de Fubini

Se f é contínua em uma caixa retangular $B = [a, b] \times [c, d] \times [r, s]$, então

$$\iiint_B f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz$$

Teorema de Fubini

Se f é contínua em uma caixa retangular $B = [a, b] \times [c, d] \times [r, s]$, então

$$\iiint_B f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz$$

Existem cinco outras ordens possíveis de integração

Exemplo

Calcule a integral tripla $\iiint_B xyz^2\, dV$, onde B é a caixa retangular dada por

$$B = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ -1 \leqslant y \leqslant 2,\ 0 \leqslant z \leqslant 3\}$$

$$\int_0^3 \int_{-1}^2 \int_0^1 xyz^2\, dx\, dy\, dz$$

$$\|$$

$$\int_{-1}^2 \int_0^1 \int_0^3 xyz^2\, dz\, dx\, dy$$

$dz\, dy\, dx$

$dx\, dz\, dy$

$dy\, dz\, dx$

$dy\, dx\, dz$

Exemplo

Calcule a integral tripla $\iiint_B xyz^2\, dV$, onde B é a caixa retangular dada por

$$B = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ -1 \leqslant y \leqslant 2,\ 0 \leqslant z \leqslant 3\}$$

$$\iiint_B xyz^2\, dV = \int_0^3 \int_{-1}^2 \left(\int_0^1 xyz^2\, dx \right) dy\, dz$$

Exemplo

Calcule a integral tripla $\iiint_B xyz^2\, dV$, onde B é a caixa retangular dada por

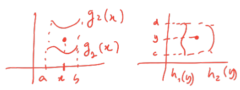
$$B = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ -1 \leqslant y \leqslant 2,\ 0 \leqslant z \leqslant 3\}$$

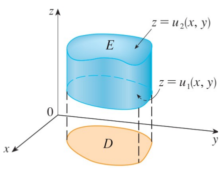
$$\iiint_B xyz^2\, dV = \int_0^3 \int_{-1}^2 \int_0^1 xyz^2\, dx\, dy\, dz = \int_0^3 \int_{-1}^2 \left[\frac{x^2yz^2}{2} \right]_{x=0}^{x=1} dy\, dz$$

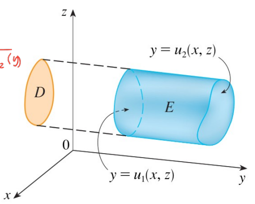
$$= \int_0^3 \left(\int_{-1}^2 \frac{yz^2}{2}\, dy \right) dz = \int_0^3 \left[\frac{y^2z^2}{4} \right]_{y=-1}^{y=2} dz$$

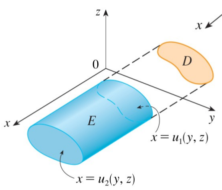
$$= \int_0^3 \frac{3z^2}{4}\, dz = \frac{z^3}{4} \Big]_0^3 = \frac{27}{4}$$

Regiões gerais





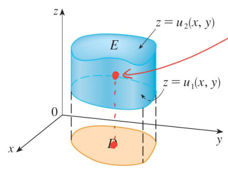




Regiões gerais: tipo I

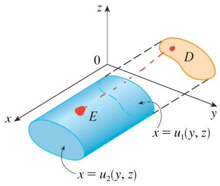
$(x, y, z) \in E$ se

$(x, y) \in D$ e $u_1(x, y) \leq z \leq u_2(x, y)$



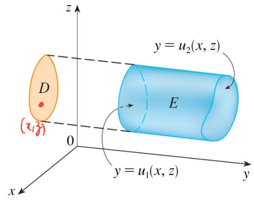
$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz \right] dA$$

Regiões gerais: tipo II



$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)\, dx \right] dA$$

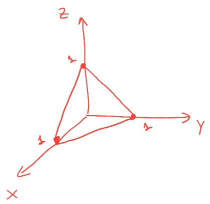
Regiões gerais: tipo III



$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)\, dy \right] dA$$

Exemplo

Calcule $\iiint_E z\, dV$, onde E é o tetraedro sólido limitado pelos quatro planos $x = 0, y = 0, z = 0$ e $x + y + z = 1$.



$x = 0$: plano YZ

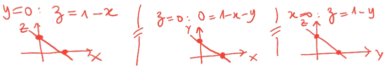
$y = 0$: plano XZ

$z = 0$; plano XY

$x + y + z = 1 \Leftrightarrow \underset{\uparrow}{\underline{z = 1 - x - y}} \Leftrightarrow y = 1 - x - z$

$\Leftrightarrow x = 1 - y - z$

$y = 0$: $z = 1 - x$ | $z = 0$: $0 = 1 - x - y$ | $x = 0$: $z = 1 - y$



Exemplo

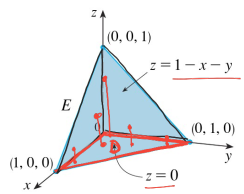
Calcule $\iiint_E \underset{f(x,y,z)}{z}\, dV$, onde E é o tetraedro sólido limitado pelos quatro planos $x = 0, y = 0, z = 0$ e $x + y + z = 1$.

$(x, y, z) \in E$

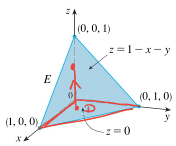
$\Updownarrow$

$(x, y) \in D$ e

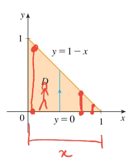
$0 \leq z \leq 1 - x - y$



Exemplo



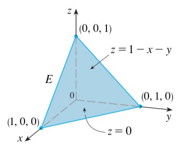
$$\iint_D \left(\int_0^{1-x-y} z\, dz \right) dA = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\, dz\, dy\, dx$$



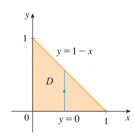
$D = \{(x, y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 1 - x\}$

$E = \{(x, y, z) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 1 - x,\ 0 \leq z \leq 1 - x - y\}$

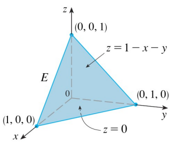
Exemplo

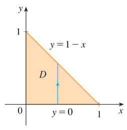


$$E = \{(x, y, z) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 1 - x,\ 0 \leq z \leq 1 - x - y\}$$



Exemplo

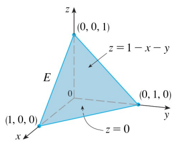


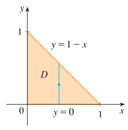


$$E = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1 - x,\ 0 \leqslant z \leqslant 1 - x - y\}$$

$$\iiint_E z\, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\, dz\, dy\, dx$$

Exemplo





$$E = \{(x, y, z) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1 - x,\ 0 \leqslant z \leqslant 1 - x - y\}$$

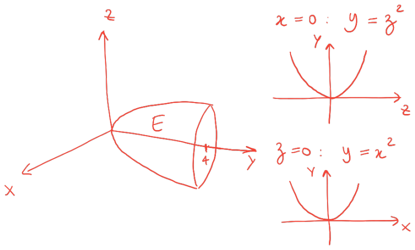
$$\iiint_E z\, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\, dz\, dy\, dx = \int_0^1 \int_0^{1-x} \left[\frac{z^2}{2}\right]_{z=0}^{z=1-x-y} dy\, dx$$

$$= \tfrac{1}{2}\int_0^1 \int_0^{1-x} (1 - x - y)^2\, dy\, dx = \tfrac{1}{2}\int_0^1 \left[-\frac{(1 - x - y)^3}{3}\right]_{y=0}^{y=1-x} dx$$

$$= \tfrac{1}{6}\int_0^1 (1 - x)^3\, dx = \frac{1}{6}\left[-\frac{(1 - x)^4}{4}\right]_0^1 = \frac{1}{24}$$

Exemplo

Calcule $\iiint_E \sqrt{x^2 + z^2}\, dV$, onde E é a região limitada pelo paraboloide $y = x^2 + z^2$ e pelo plano $y = 4$.



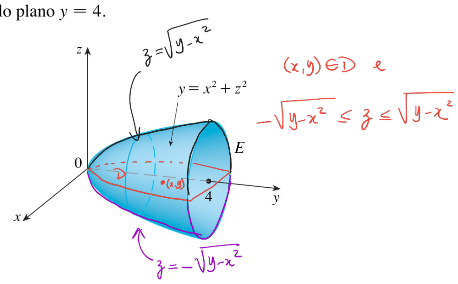
Exemplo

Calcule $\iiint_E \sqrt{x^2 + z^2}\, dV$, onde E é a região limitada pelo paraboloide $\underbrace{y = x^2 + z^2}_{\mathfrak{z}}$ e pelo plano $y = 4$.

$y = x^2 + z^2$

$\Rightarrow z^2 = y - x^2$

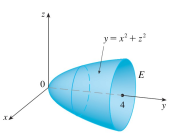
$\Rightarrow z = \pm\sqrt{y - x^2}$

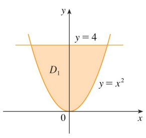


$(x,y) \in D$ e

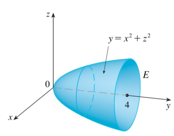
$-\sqrt{y-x^2} \leq z \leq \sqrt{y-x^2}$

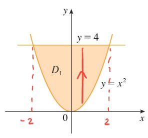
Exemplo (tipo I)





Exemplo (tipo I)

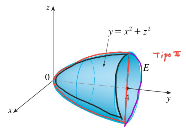


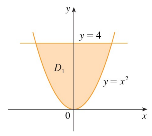


De $y = x^2 + z^2$ obtemos $z = \pm\sqrt{y - x^2}$, e então a superfície limite de baixo de E é $z = -\sqrt{y - x^2}$ e a superfície de cima é $z = \sqrt{y - x^2}$. Portanto, a descrição de E como região do tipo 1 é

$$E = \{(x, y, z) \mid -2 \leqslant x \leqslant 2,\ x^2 \leqslant y \leqslant 4,\ -\sqrt{y - x^2} \leqslant z \leqslant \sqrt{y - x^2}\}$$

Exemplo (tipo I)





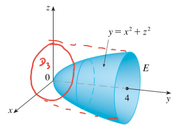
De $y = x^2 + z^2$ obtemos $z = \pm\sqrt{y - x^2}$, e então a superfície limite de baixo de E é $z = -\sqrt{y - x^2}$ e a superfície de cima é $z = \sqrt{y - x^2}$. Portanto, a descrição de E como região do tipo 1 é

$$E = \{(x, y, z) \mid -2 \le x \le 2,\ x^2 \le y \le 4,\ -\sqrt{y - x^2} \le z \le \sqrt{y - x^2}\}$$

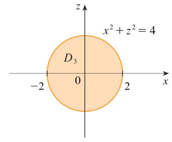
e obtemos

$$\iiint_E \sqrt{x^2 + z^2}\, dV = \int_{-2}^{2}\int_{x^2}^{4}\int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2 + z^2}\, dz\, dy\, dx$$

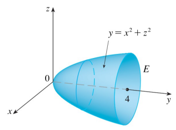
Exemplo (tipo III)



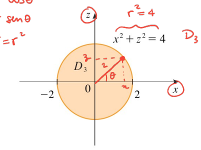
$x^2 + z^2 \le y \le 4$



Exemplo (tipo III)



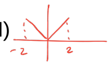
$x = r\cos\theta$
$y = r\,\text{sen}\,\theta$
$x^2 + y^2 = r^2$



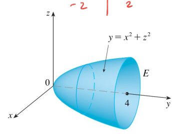
$D_3: \{(r,\theta) \mid 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$

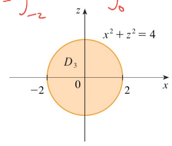
$$\iiint_E \sqrt{x^2 + z^2}\, dV = \iint_{D} \left[\int_{x^2+z^2}^{4} \sqrt{x^2 + z^2}\, dy\right] = \int_0^{2\pi}\int_0^{2} \dots\ r\,dr\,d\theta$$

Exemplo (tipo III)



$$\int_{-2}^{2} |r|\,dr = \int_{-2}^{0} -r\,dr + \int_0^2 r\,dr = 2\int_0^2 r\,d\theta$$





$$\iiint_E \sqrt{x^2 + z^2}\, dV = \iint_{D_3} \left[\int_{x^2+z^2}^{4} \sqrt{x^2 + z^2}\, dy\right] = \iint_{D_3} (4 \underbrace{- x^2 - z^2}_{-r^2})\underbrace{\sqrt{x^2 + z^2}}_{r^2}\, dA$$

$$= \int_0^{2\pi}\left(\int_0^{2} (4 - r^2) r\, r\, dr\right) d\theta = \int_0^{2\pi} d\theta \int_0^{2} (4r^2 - r^4)\, dr$$

$$= 2\pi\left[\frac{4r^3}{3} - \frac{r^5}{5}\right]_0^2 = \frac{128\pi}{15}$$

Exercício

Calcule $\iiint_E z\, dV$, onde E é o tetraedro sólido limitado pelos quatro planos $x = 0, y = 0, z = 0$ e $x + y + z = 1$.

Resolver como tipo II e tipo III.