Concepts in the Physical Sciences: Theory and Applications

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Abstract

A document of useful concepts and formulae aimed towards problem solving, intended for various courses in physics, mathematics, and computer science.

Last compiled: 28th July 2022 8:49pm GMT

Contents

Chapter 1

MATH 325: Honours ODEs

1.1 Methods of solution

In this section, we attempt to combine and formalise treatments for solving different types of problems in the subject of ordinary differential equations. We will start with the constant coefficient case, and then move on to the case of variable coefficients.

1.1.1 Constant coefficient case

As suggested, we start with the constant coefficient case; specifically, we mean that the differential equation would be in the form:

$$u^{(n)} + c_n u^{(n-1)} + \cdots + c_1 = f, \; c_i \in \mathbb{C}.$$

We proceed to the methods of solutions, primarily for first and second order equations since they are the most common situations that we would encounter especially in physics and applied mathematics.

First order homogeneous: if we consider that the ODEs are given in the form, alongside an initial condition:

$$\begin{cases} u' = lu, \\ u(0) = \alpha, \end{cases} \tag{1.1}$$

then the solution to Eq. (1.1) would be in the form:

$$\boxed{u(t) = \alpha e^{lt}}.$$

First order inhomogeneous: if a forcing term is added to the system; in other words:

$$\begin{cases} u' = lu + f, \\ u(0) = \alpha, \end{cases} \tag{1.2}$$

we would solve this using the method of variation of parameters. Instead of guessing a constant, such as:

$$u(t) = Ae^{lt},$$

we 'vary' the constant and force it to be in the form $A(t)$. After derivation, we would arrive at the following formula for the solution of Eq. (1.2):

Theorem 1.1 (Duhamel's formula)**.** The general solution to a first order, constant coefficient inhomogeneous IVP in the form of Eq. (1.2) is given by:

$$\boxed{u(t) = \alpha e^{lt} + \int_0^t e^{l(t-s)} f(s)\,\mathrm{d}s}.$$

■

As we can see, the solving methods for the first order cases are taking *Ansatz* and using variation of parameters, respectively for the homogeneous and inhomogeneous cases. As we carry on with exploring the higher order cases, we will see that those two methods can still be generalised effectively.

Second order homogeneous: we are mostly concerned with solving an ODE with two initial conditions, in the form of:

$$\begin{cases} u'' + pu' + qu = 0, \\ u(s) = \alpha, \\ u'(s) = \beta. \end{cases} \tag{1.3}$$

If we take the Ansatz $u(t) = e^{\lambda t}$, we would obtain the *characteristic polynomial* $p(\lambda)$ of the ODE. We then solve for the roots of $p(\lambda)$; if the roots are distinct, we have that:

$$u(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t},$$

which we then use the initial conditions in Eq. (1.3) to solve for the constants. If the roots are repeated[1], in order to maintain the two-dimensional solution space, we have that:

$$u(t) = Ae^{\lambda t} + Bte^{\lambda t}. \tag{1.4}$$

Another angle of approach to this problem would involve the use of *Wronskians* and the notion of the *fundamental set*. In general, for an initial value problem described in the form of Eq. (1.3) and a fundamental set given by $\{\phi_1(t), \phi_2(t)\}$, we have to solve the system of equations:

$$\begin{cases} A\phi_1(s) + B\phi_2(s) = \alpha, \\ A\phi_1'(s) + B\phi_2'(s) = \beta. \end{cases}$$

[1] For even higher-ordered cases, Eq. (1.4) can be generalised into the form $u_k(t) = c_1 e^{\lambda t} + \cdots + c_k t^{k-1} e^{\lambda t}$, where k is the multiplicity of the root.

Obviously, as we know from linear algebra, in order for the above system to have a solution, the determinant (which is defined as the *Wronskian* of the fundamental set at s):

$$W[\phi_1, \phi_2](s) = \begin{vmatrix} \phi_1(s) & \phi_2(s) \\ \phi_1'(s) & \phi_2'(s) \end{vmatrix} \neq 0.$$

Second order inhomogeneous: similar to the difference between first order homogeneous and inhomogeneous, we are adding a forcing term f to the right hand side of the second order ODE:

$$\begin{cases} u'' + pu' + qu = f, \\ u(0) = \alpha, \\ u'(0) = \beta. \end{cases} \tag{1.5}$$

The general solution for Eq. (1.5) comprises of two main parts: the (complementary) solution to the homogeneous equation, $u_c(t)$, as well as the particular solution $u_p(c)$:

$$u(t) = u_c(t) + u_p(t).$$

The complementary solution is obtainable via methods outlined in the homogeneous case; however, the method of obtaining $u_p(t)$ it is slightly more involved. We primarily deal with three methods that could solve for the particular solution:

1. **(Undetermined coefficient).** This method is the easiest, although it is not one-hundred per cent reliable. Basically, the gist of it is to observe the forcing term $f(t)$, and try to guess a solution involving undetermined coefficient terms of f and its derivatives. It is perhaps the most expedient to illustrate with an example:

   Example 1.1. *Consider the following ODE:*

   $$u'' + u' + u = te^{-t}.$$

   it is easy to solve for $u_c(t)$, hence we will focus on applying the method of undetermined coefficient. We take the derivatives of f:

   $$\begin{aligned} f' &= -te^{-t} + e^{-t}, \\ f'' &= te^{-t}. \end{aligned}$$

   As we can see, we can make a guess in the form of $Ae^{-t} + Bte^{-t}$. We could then obtain that $A = B = 1$ via comparison of coefficients. ■

   Remark. Often times, when part of the particular solution is of the form of part of the complementary solution, then we need to multiply a t in front of the 'guess' that we perform. †

   As we have foreshadowed, this method is not completely fool-proof, hence we introduce some other methods of solving for the particular solution.

2. **(Variation of parameters).** Similar to the idea of Thm. 1.1, given the fundamental set $\{\phi_1(t), \phi_2(t)\}$, we can 'guess' $u_p(t)$, with varying parameters:

$$u_p(t) = A(t)\phi_1(t) + B(t)\phi_2(t),$$

which we can then compute the derivative twice and plug into Eq. (1.5). In the end, we are left with a system of equations consisting the derivative of A and B:

$$\begin{cases} A'\phi_1 + B'\phi_2 = 0, \\ A'\phi_1' + B'\phi_2' = f, \end{cases}$$

which can then be used to solve A and B for. In the end, we have that the two coefficients A and B can be obtained via:

$$A(t) = -\int_0^t \frac{\phi_2(s)f(s)}{W(s)}\,\mathrm{d}s; \quad B(t) = \int_0^t \frac{\phi_1(s)f(s)}{W(s)}\,\mathrm{d}s,$$

where $W(s)$ is the canonical definition of the Wronskian. Putting it all together, we have the formula for the solution to Eq. (1.5):

Theorem 1.2 (Duhamel's formula (2^{nd} order))**.** The general solution to a second order, constant coefficient inhomogeneous IVP in the form of Eq. (1.5) is given by:

$$\boxed{u(t) = u_c(t) + \int_0^t \frac{\phi_2(t)\phi_1(s) - \phi_1(t)\phi_2(s)}{W(s)} f(s)\,\mathrm{d}s}.$$

Or equivalently, we can write the particular solution as a determinant:

$$u_p(t) = \int_0^t \frac{f(s)}{W(s)} \begin{vmatrix} \phi_1(s) & \phi_2(s) \\ \phi_1(t) & \phi_2(t) \end{vmatrix} \mathrm{d}s.$$

This is also known as the **representation formula** for this type of problems.

■

3. **(Laplace transform).** Laplace transform is sort of the odd method here, in that it does not really seek to solve the problem in the standard variable space (say, t-space) of the problem. Rather, it 'transforms' the variable from t-space to the Laplace transformed space (say, s-space), solve the problem in s-space, then inverse-transform back to t-space. We can show this relationship easily with a commutative diagram:

$$\begin{tikzcd}
t\text{-Space} \arrow[r, "\text{Directly Solve IVP}"] \arrow[d, "\mathscr{L}"] & u(t) \\
s\text{-Space} \arrow[r, "\text{Algebraic Solving}"] & U(s) \arrow[u, "\mathscr{L}^{-1}"']
\end{tikzcd}$$

Figure 1.1: Commutative diagram for Laplace transform.

We first give the formal definition:

Definition 1.3 (Laplace transform)**.** The **Laplace transform** of a function $f(t) : t \in [0, \infty)$ is given by:

$$\boxed{F(s) = \int_0^\infty e^{-st} f(t) \, \mathrm{d}t}.$$

■

Really, though, unless we are trying to compute the Laplace transforms for individual functions, we would not have to use this definition very often. Before coming up with the table of Laplace transforms, though, we shall go through a few basic properties of the Laplace transform:

- **(Linearity).** The Laplace transform is linear. In other words,

$$\mathscr{L}\left[\alpha f(t) + \beta g(t)\right] = \alpha \mathscr{L}\left[f(t)\right] + \beta \mathscr{L}\left[g(t)\right].$$

- **(Differentiation in t-space).** The Laplace transform turns differentiation into multiplication. In other words,

$$\begin{aligned}
\mathscr{L}\left[f'(t)\right] &= \int_0^\infty f'(t) e^{-st} \, \mathrm{d}t, \\
\xRightarrow{\text{chain rule}} &= f(t)e^{-st}\Big|_0^\infty - \int_0^\infty f(t)(-s)e^{-st} \, \mathrm{d}t, \\
&= s\mathscr{L}\left[f(t)\right] - f(0).
\end{aligned}$$

  On a similar vein, we can compute the Laplace transform of the second derivative as well:

$$\begin{aligned}\mathscr{L}\left[f''(t)\right] &= s\mathscr{L}\left[f'(t)\right] - f'(0),\\ &= s^2\mathscr{L}\left[f(t)\right] - sf(0) - f'(0).\end{aligned}$$

- **(Scaling).** The Laplace transform of a function subjected to scaling is easily computable:

$$\begin{aligned}\mathscr{L}\left[f(nt)\right] &= \int_0^\infty f(nt)e^{-st}\,\mathrm{d}t,\\ &= \frac{1}{n}\int_0^\infty f(x)e^{-\frac{sx}{n}}\,\mathrm{d}x,\\ &= \frac{F(\frac{s}{n})}{n}.\end{aligned}$$

- **(Multiplication by t/differentiation in s-space).** The Laplace transform of a function $f(t)$, multiplied by a factor of t, can be derived by taking the derivative w.r.t. s in the s-space:

$$\begin{aligned}F'(s) &= \int_0^\infty f(t)(-t)e^{-st}\,\mathrm{d}t,\\ \Rightarrow \mathscr{L}\left[tf(t)\right] &= -F'(s).\end{aligned}$$

- **(Multiplication by $e^{\alpha t}$/Shifting in s-space).** A shift in s-space is equivalent to multiplication by $e^{\alpha t}$ in t-space:

$$\begin{aligned}F(s-\alpha) &= \int_0^\infty f(t)e^{-(s-\alpha)t}\,\mathrm{d}t,\\ &= e^{\alpha t}\int_0^\infty f(t)e^{-st}\,\mathrm{d}t,\\ &= \mathscr{L}\left[e^{\alpha t}f(t)\right].\end{aligned}$$

- **(Discontinuous functions).** Before demonstrating Laplace transform with discontinuous source terms, we first need a way to formalise the effect of 'turning on' a function at a certain t. The following definition is useful in this regard:

  Definition 1.4 (Heaviside theta (step) function)**.** The **Heaviside theta function** is defined as:

$$\boxed{\theta(t) = \begin{cases}1, & t \in (0,\infty),\\ 0, & t \in (-\infty, 0].\end{cases}}$$

Graphically, it looks something like this:

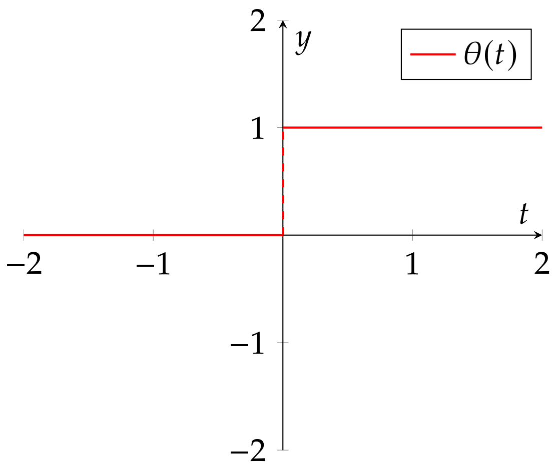

Figure 1.2: The Heaviside theta function.

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Since $\theta(t)$ at the end of the day evaluates to one in the domain that we are interested to, we can safely assume that $\mathscr{L}[\theta(t)] = \mathscr{L}[1] = \frac{1}{s}$[2]. It is terribly boring if we only use the theta function as a mere zero-to-one switch; as we have alluded to before, $\theta(t)$ can actually 'turn on' certain functions. We will explain what this means next.

Say, we would want to 'turn on' a function $g(t)$ at $t = a$; in other words, we would like to define:

$$g^*(t) := \begin{cases} 0, & t \in (-\infty, a], \\ g(t), & t \in (a, \infty). \end{cases}$$

We can illustrate this graphically:

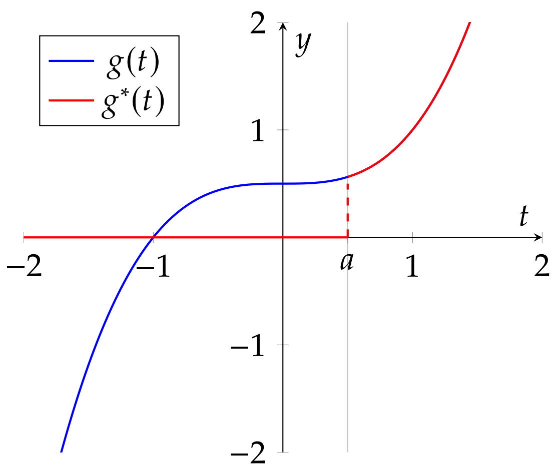

Figure 1.3: $g^*(t)$, being 'turned on' at $t = a$.

with the theta function, we can represent $g^*(t)$ alternatively as $g(t)\theta(t -$

[2]This fact will be established later in our table of Laplace transforms.

a).

So how do we compute the Laplace transform for $g^*(t)$? We can simply utilise Def. 1.3, but instead of integrating from zero, we integrate from a:

$$\begin{aligned}\mathscr{L}\left[g(t)\theta(t-a)\right] &= \int_a^\infty g(t)e^{-st}\,\mathrm{d}t, \\ &= \int_0^\infty g(a+t)e^{-st-sa}\,\mathrm{d}t, \\ &= e^{-sa}\mathscr{L}\left[g(a+t)\right].\end{aligned}$$

Using the same treatment, we can also derive the expression for a function $f(t)$ shifted in t-space by a, and also 'turned-on' at that point:

$$\mathscr{L}\left[\theta(t-a)f(t-a)\right] = e^{-sa}\mathscr{L}\left[f(t)\right].$$

In the end, for IVPs of the same form as Eq. (1.5), we can simply express the ODE in the following form (with $U \equiv \mathscr{L}\left[u(t)\right],\ F \equiv \mathscr{L}\left[f(t)\right]$):

$$s^2U - su(0) - u'(0) + p[sU - u(0)] + qU = F.$$

1.1.2 Variable coefficient case

It is indicated that this will not be on the exam; nevertheless, the following need to be recapped:

- Frobenius method. In particular, also review the Taylor and Maclaurin series of common functions;
- Bessel, Legendre, and Laguerre polynomials;
- reduction of order, but not confirmed yet.

1.2 Comparison and separation

The next three theorems due to Sturm are the most intrinsic to our discussions.

Theorem 1.5 (Sturm separation)**.** If $f(x)$ and $g(x)$ are linearly independent solutions of a second-order homogeneous ODE, then $f(x)$ must vanish at one point between any two successive zeros of $g(x)$. In other words, the zeros of $f(x)$ and $g(x)$ occur alternately. ■

Somehow more importantly, primarily due to the parallel to an oscillator, we have the following theorem:

Theorem 1.6 (Sturm comparison)**.** Let $f(x)$ and $g(x)$ be non-trivial solutions of the ODEs $u'' + p(x)u = 0$ and $v'' + q(x)v = 0$, respectively, where $p(x) \geq q(x)$. Then, $f(x)$ vanishes at least once between any two zeros of $g(x)$, unless $p(x) \equiv q(x)$, in which case f is a constant multiple of g. ■

We also have the following important corollary:

Corollary 1.7. If $q(x) \leq 0$, then no non-trivial solution of $u'' + q(x)u = 0$ can have more than one zero. ■

1.3 Nonlinear equations

One important aside is the Leibniz formula for product rule:

$$\boxed{(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}}.$$

1.3.1 Exact equations

They are in this form:

$$\boxed{M\mathrm{d}x + N\mathrm{d}y = 0}.$$

Solutions are most notably level curves of the **potential** Φ, cf. vector calculus. The existence of this Φ is guaranteed by the mixed-partial test, which means that:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \implies \Phi \text{ exists.} \tag{1.6}$$

As a result, we simply partially-integrate M (w.r.t. x) and N (w.r.t. y) to find desired solutions.

But what if Eq. (1.6) is not exact? Then we would need **integrating factors**.

Case 1: Say, we try to find $\mu = \mu(x)$. Then, we have that:

$$\mu M\mathrm{d}x + \mu N\mathrm{d}y = 0.$$

To guarantee exactness, $\frac{\partial \mu N}{\partial x} = \frac{\partial \mu M}{\partial y}$. This implies that:

$$\mu' + \underbrace{\left(\frac{N_x - M_y}{N}\right)}_{\text{depends only on } x} \mu = 0.$$

Case 2: Finding $\mu = \mu(y)$ is also straightforward; we simply see if

$$\frac{M_y - N_x}{M}$$

depends only on y.

1.4 Dynamical systems

The topics need to be reviewed are:

- scalar autonomous equations:
  - stability criteria at equilibrium points;
  - linearisation at a point;
- autonomous systems. In particular, for a system in the following form:

$$\begin{cases} x' = f(x, y), \\ y' = g(x, y), \end{cases} \tag{1.7}$$

  we can achieve a linearisation at a critical/equilibrium point (x_0, y_0), i.e., fulfilling that:

$$\begin{cases} f(x_0, y_0) = 0, \\ g(x_0, y_0) = 0. \end{cases}$$

  Close to this critical point, we can use the **Jacobian** matrix to linearise the system. Particularly, Eq. (1.7) would be approximated as such:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{J}|_{(x_0, y_0)}\begin{bmatrix} x \\ y \end{bmatrix},$$

  where $\mathbf{J}$ is the Jacobian matrix given by:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}.$$

- constant coefficient systems:
  - phase plane analysis, with corresponding eigenvalues;
  - linear equivalence. In particular, similarity transformations ($A = QBQ^{-1}$), which means that B is diagonal and Q consists of corresponding eigenvectors in the columns;
  - solving linear systems via diagonalisation. Emphasis on complex eigenvalues, in which case the imaginary and real parts are both solutions;

  * non-diagonalisable case; generalised eigenvectors; Jordan forms; Cayley-Hamilton;

- matrix exponentials:
  - if we are given a system:

$$\begin{cases} u' = Au, \\ u(0) = \alpha, \end{cases} \tag{1.8}$$

    then we have that $u(t) = e^{tA}\alpha$, where e^{tA} is defined by:

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}.$$

    This formula converges for all $t \in \mathbb{R}$. The derivative is also readily given by:

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{tA} = Ae^{tA}.$$

    Proof. Differentiating the series and pull out one A. □

  - computations:

    diagonal(isable): if A is diagonal, then the matrix exponential is trivial to compute:

$$A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \implies e^{tA} = \begin{pmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{pmatrix}.$$

    The diagonalisable case is also not much harder. After factoring, we can clearly see that for $A = S\Lambda S^{-1}$ diagonalisable, we have that:

$$e^{tA} = Se^{t\Lambda}S^{-1},$$

    and we can compute as usual.

    non-diagonalisable: we would use the **binomial theorem**:

$$\boxed{(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}},$$

    where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. This is useful, because we can decompose matrices into sums of diagonal and nilpotent matrices.

- fundamental matrices: it is trivial to show that $E(t) = e^{tA}$ solves the matrix IVP $E' = AE, E(0) = \text{id}$. $E(t)$ is called the **principal fundamental matrix** of Eq. (1.8), i.e., $u(t) = E(t)\alpha$.

More generally, a **fundamental matrix** of Eq. (1.8) is a matrix function $\Phi(t)$ satisfying:

$$\Phi' = A\Phi; \; \det \Phi(0) \neq 0.$$

Chapter 2

COMP 252: Hons. Algos. & Data Structs.

2.1 Computational models

RAM model: all computations take constant time - 'uniform cost measure'.

Bit model: 'logarithmic cost measure' - every bit operation costs one time unit.

2.2 Trees and traversals

For a complete k-ary tree, $h = \lfloor \log_k n \rfloor$.

Binary trees can be easily represented by an array, where the i-th node is stored in position $[i]$ in the array:

1. left child of the i-th node is stored in position $[2i]$;
2. right child of the i-th node is stored in position $[2i + 1]$;
3. parent of the i-th node is stored in position $\lfloor \frac{i}{2} \rfloor$;
4. for the i-th node to be a leaf, we just have to check that it satisfies $2i > n$.

Numbering the nodes from top to bottom and left to right, we could also obtain the path from root by representing the node number in binary and delete the left most bit. Then, from left to right, 0 is right child and 1 is left child.

Some important properties;

1. every tree on n nodes has $n - 1$ edges;
2. in a binary tree, let n_i denote the number of nodes with i children, then:

$$n = n_0 + n_1 + n_2 \ \& \ n - 1 = n_1 + 2n_2 \implies n_0 = n_2 + 1.$$

Chapter 3

MATH 249: Hons. Complex Vars.

3.1 Basics of a complex variable

Chapter 4

PHYS 241: Signal Processing

4.1 Circuit pretext

Before going into actual circuit analysis, we should first spend some time defining several concepts that are pertinent for our further studies. One important theorem here is the principal of superposition:

Theorem 4.1 (Superposition (circuits))**.** Linear circuit elements obey the **principle of superposition** when comparing the input and output signals. ■

We proceed to define a few relationships between certain quantities in electricity:

Definition 4.2 (Current, voltage, and resistance)**.** The **current** (I), **voltage** (V), and **resistance** (R) in a circuit can be loosely governed by the following few relationships:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = I, \quad V = IR, \quad P = IV \underbrace{= I^2R = \frac{V^2}{R}}_{\text{when Ohmic}}.$$

■

Strictly under an AC context, there is another way to quantify the strength of an AC signal, called the *rms current*:

Definition 4.3 (Root-mean-squared current)**.** The **RMS current** is defined by:

$$I_{\mathrm{rms}} = \sqrt{\frac{1}{T}\int_0^T I^2(t)\,\mathrm{d}t},$$

where T denotes the period of AC signal. ■

Here is a simple example of this concept using a simple sine-wave AC signal:

Example 4.1. *Construct if time permits.* ■

4.2 Simple circuits

In this section, we primarily deal with DC circuits with simple elements, such as a complex resistor circuit. Of course, in order to actually have signals in our circuit, we need some sort of voltage source to drive our circuit, and this is precisely what an *EMF* is.

Definition 4.4 (Electromotive force)**.** An **electromotive force (EMF)**, denoted by ε_{EMF}, is the voltage source of a given circuit, such a battery. It gives the circuit the electric potential V it needs to input and output signals. ■

In a way, this whole section is some sort of elaborate application of Def. 4.2, in that all we do is to rearrange resistors and analyse how the voltage and current characteristics behave. As we know from before, resistor combinations follow the following two behaviours:

In series: for two resistors R_1 and R_2, their resistance in series is given by:

$$\sum R = R_1 + R_2.$$

In particular, circuit of in-series resistor combination is known as a *voltage divider*.

In parallel: for the same two resistors, their resistance in parallel is given by:

$$\sum R = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-1}.$$

Similarly, circuit of in-parallel resistor combination is known as a *current divider*.

Proof. How they divide current or voltage follows directly from Def. 4.2. □

Until now, we have made the assumption that batteries are perfect sources of ε_{EMF} without any internal resistance. However, this is not the case in general. For a real battery, it is sometimes better to view it as an in-series combination of an ideal battery and a resistor of finite resistance r. As a result, for a simple resistor circuit with a resistor R, instead of:

$$\varepsilon_{EMF} = IR,$$

we would analyse the circuit with an additional resistor:

$$V_T = \varepsilon_{EMF} - Ir = IR,$$

where V_T is the true terminal voltage of the battery.

4.3 RC and LR circuits

4.3.1 DC case

It would be terribly boring if a circuit only contains resistors and nothing else. Often times, we would want some other circuit elements to be connected in combination with the resistor, to fulfil different needs. We introduced the two following circuit elements:

Capacitor: a capacitor is a linear, passive circuit element, used for storing charge like a temporary battery. The charge build-up on the capacitor is linearly related to the voltage differential across the capacitor. In other words, it obeys the following relation:

$$Q = CV_C. \tag{4.1}$$

For a parallel plate capacitor, the capacitance C is given by:

$$C = \varepsilon_0 \varepsilon_r \frac{A}{d},$$

where ε_0 is the permittivity[1], ε_r is the dielectric's relative permittivity, A is the surface area of the plate, and d is the plate-to-plate separation. Moreover, the energy-stored in a capacitor is given by:

$$U_C = \frac{1}{2} C V_C^2.$$

Similar to the case with resistors, we can also combine capacitors and form networks of them.

In series: for two capacitors C_1 and C_2, their capacitance in series is given by:

$$\sum C = \left(\frac{1}{C_1} + \frac{1}{C_2} \right)^{-1}.$$

In parallel: for the same two capacitors, their capacitance in parallel is given by:

$$\sum C = C_1 + C_2.$$

Note that capacitors adds the opposite way compared to resistors!

Inductor: an inductor is also a linear passive element, used to regulate sudden changes in current by creating a back-ε_{EMF}. This back-ε_{EMF} is also equivalent to the voltage drop across the inductor with an inductance L, given by:

$$-V_{drop} = \varepsilon_{EMF} = -L \frac{\mathrm{d}I}{\mathrm{d}t}.$$

[1] $\varepsilon_0 \approx 8.854 \times 10^{-12}\,\mathrm{F\,m^{-1}}$.

Note how the signs are opposite for the back-ε_{EMF} and the V_{drop}. This is to account for the direction that we loop through the inductor. The inductance of a solenoidal inductor is calculated by the following formula:

$$L = \mu_0 A \frac{N^2}{l},$$

where μ_0 is the permeability of free-space[2], A is the area of a solenoid, N is the number of wire windings, and l is the length of the solenoid. Moreover, the energy-stored in an inductor is given by:

$$U_L = \frac{1}{2} L I^2.$$

Similar to the case with resistors, we can also combine inductors and form networks of them.

In series: for two inductors L_1 and L_2, their inductance in series is given by:

$$\sum L = L_1 + L_2.$$

In parallel: for the same two inductors, their inductance in parallel is given by:

$$\sum L = \left(\frac{1}{L_1} + \frac{1}{L_2} \right)^{-1}.$$

Note that inductors combine like resistors!

Proof. The proof of the above properties will be given in the future if time permits. □

Now, we would place the capacitor/inductor inside a simple DC circuit, and investigate the characteristics displayed by this circuit. We start with the RC case, then move on to the LR case.

RC circuit: an RC circuit, evidenced by the name, is built with a resistor of resistance R in series with a capacitor of capacitance C, driven by a DC battery of voltage V_0. We first examine the charging case; using the loop rule, we can arrive easily at the following equation:

$$V_0 = V_C + V_R = \frac{Q}{C} + IR; \tag{4.2}$$

differentiating, we obtain:

$$\frac{\mathrm{d}I}{\mathrm{d}t} - \frac{I}{RC} = 0.$$

[2] $\mu_0 = 4\pi \times 10^7 \ \mathrm{H\,m^{-1}}$.

Solving the ODE, we would obtain that the current characteristic of this circuit is given by:

$$I(t) = \left[I(0) \equiv \frac{V_0}{R}\right] e^{-\frac{t}{RC}}. \tag{4.3}$$

With the aid of Eq. (4.3), we can naturally obtain the expression for V_C:

$$V_C = V_0 \left(1 - e^{-\frac{t}{RC}}\right),$$

since $V_C(0) = 0$. The charge comes naturally then, simply by applying Eq. (4.1).

The discharging scenario is also quite straightforward to analyse. We are using roughly the same relations compared to the charging case, simply with different initial conditions, as the battery is out of the equation. Then, Eq. (4.2) becomes:

$$IR + \frac{Q}{C} = 0,$$

which implies that:

$$Q(t) = Ae^{-\frac{t}{RC}},$$

where A is a constant of integration. Without loss of generality, assume that the capacitor starts discharging at $t = t_s$. As such, we get the following series of equations, characterising the discharge of an RC circuit:

$$\begin{aligned} Q(t) &= (V_0 C)e^{-\frac{t-t_s}{\tau}}, \\ V(t) &= V_0 e^{-\frac{t-t_s}{\tau}}, \\ I(t) &= -\frac{V_0}{R} e^{-\frac{t-t_s}{\tau}}. \end{aligned}$$

Here, we used the notation $\tau = RC$. This quantity is also known as the *time constant* of this RC circuit.

LR circuit: an LR circuit consists of a resistor of resistance R in series with a inductor of inductance L, driven by a DC battery of voltage V_0. We analyse the circuit in much of the same way compared to the RC case. The equation is given by:

$$V_0 = IR + L\frac{\mathrm{d}I}{\mathrm{d}t}.$$

This is a separable equation, with the initial condition that $I(0) = 0$. We thus integrate, and obtain that:

$$I(t) = \frac{V_0}{R}\left(1 - e^{-\frac{R}{L}t}\right).$$

Note that the quantity $\tau = \frac{L}{R}$ is known as the time constant of the LR circuit.

4.3.2 AC: basics and square-wave drive

We proceed with analysing the same circuits, but with an AC driving signal. Here, Thm. 4.1 comes in handy. AC signals often exhibit changes in signs; by shifting the AC signal up the y-axis and superimpose the shift with a negative DC signal of the same magnitude, we can perform circuit analysis in a simpler manner. This principle is illustrated in the following image:

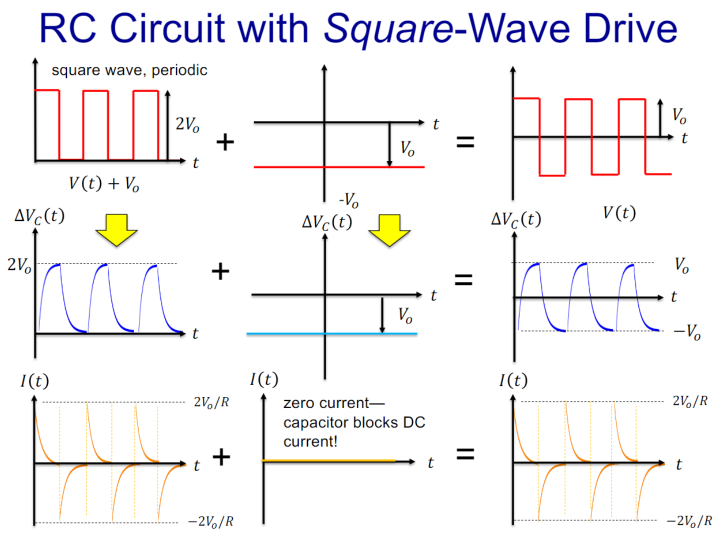

Figure 4.1: Illustration of the technique where superposition can be used to simplify AC signals.

Here, note that the highest amplitude of the current is $2\frac{V_0}{R}$, instead of $\frac{V_0}{R}$. This is due to the fact that for a square-wave drive, at the start of each pulse, the discharging of the capacitor would produce the same magnitude of current as the power source, thereby producing a current of twice the magnitude.

We start by analysing circuit behaviours with a square-wave drive, especially at different frequency limits. We would also derive the differentiator versus integrator behaviours of the capacitor and inductor at those limits. We would mainly deal with RC circuits here, since the results for LR circuits are somewhat analogous; for an identical driving voltage $V(t)$, the current $I(t)$ in LR behaves the same way as $Q(t)$ in RC. As a result, the qualitative observations are simply a flip for LR instead of RC circuit.

Low frequency ($\frac{T}{\tau} >> 1$): We first examine the qualitative characteristics:

- voltage dropped almost completely across capacitor;
- current mostly zero, except from spikes when charging/discharging.

As we can see, the capacitor acts approximately like a break in the circuit at this limit.

Quantitatively, we notice that in an RC circuit, since $V_C \approx V(t)$, then:

$$I = \frac{\mathrm{d}Q}{\mathrm{d}t} \approx C\frac{\mathrm{d}V(t)}{\mathrm{d}t}.$$

As a result,

$$V_R = IR \approx \tau\frac{\mathrm{d}V(t)}{\mathrm{d}t}.$$

For an LR circuit, the same principle applies, but across the inductor:

$$V_L = L\frac{\mathrm{d}I}{\mathrm{d}t} \approx \frac{L}{R}\frac{\mathrm{d}V(t)}{\mathrm{d}t} = \tau\frac{\mathrm{d}V(t)}{\mathrm{d}t}.$$

This is the differentiator behaviour that we have mentioned.

High frequency ($\frac{T}{\tau} << 1$): qualitatively for the RC circuit, we have:

- very little voltage drop across capacitor;
- voltage displays a triangular wave-like profile;
- current roughly aligns with a pure resistor circuit.

As we can see, the capacitor acts approximately like a short at this limit.

Quantitatively, we notice that $V_R \approx V(t)$. As a result,

$$I(t) = \frac{V_R}{R} \approx \frac{V(t)}{R}.$$

Using the capacitor equation, we have:

$$\begin{aligned} V_C = \frac{Q}{C} &= \frac{1}{C}\int_0^t I(s)\,\mathrm{d}s, \\ &= \frac{1}{\tau}\int_0^t V(s)\mathrm{d}s. \end{aligned} \tag{4.4}$$

Similarly for an LR circuit, $V_L \approx V(t)$. As a result,

$$\frac{\mathrm{d}I}{\mathrm{d}t} \approx \frac{V(t)}{L}.$$

Integrate both sides and multiply by R, we have that:

$$V_R = IR \approx \frac{R}{L}\int_0^t V(s)\,\mathrm{d}s = \frac{1}{\tau}\int_0^t V(s)\,\mathrm{d}s.$$

This is the desired integrator behaviour at high frequency limit.

4.3.3 AC: sine-wave drive

Under a sine-wave driving conditions, say, if we are given $V(t) = V_0 \cos(\omega t)$ as the driving voltage, we would have the ODE for the circuit as:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{Q}{\tau} = \frac{V_0}{R}\cos(\omega t).$$

This is an annoying equation to solve, hence we will simply present the final solution for the voltage across capacitor:

$$V_C(t) = V_0 \left[\underbrace{\frac{\cos(\omega t + \varphi)}{\sqrt{1+(\tau\omega)^2}}}_{\text{long-term}} - \underbrace{\frac{e^{-\frac{t}{\tau}}}{1+(\tau\omega)^2}}_{\text{transient}} \right], \tag{4.5}$$

where $\varphi = \arccos \frac{1}{\sqrt{1+(\tau\omega)^2}} = \arctan -\tau\omega$. More specifically, we investigate the frequency limits of Eq. (4.5). Clearly, in the high frequency limit of $\omega \to \infty$, we have that $\varphi \to -\frac{\pi}{2}$, and $V_C = \frac{V_0}{\tau}\frac{\sin(\omega t)}{\omega}$, which aligns with the integrator behaviour that we had in Eq. (4.4).

Moreover, for a capacitor or inductor, no power is dissipated with an AC input signal.

Proof. Proof will be given if time permits. □

To analyse this type of problems in a easier way, we would try to simplify our discussion with the notion of the *impedance* of a circuit element, which we shall discuss immediately after.

4.3.4 Impedance

For any sinusoidal wave-form, we could use our familiar Euler's relation to transform it into complex-exponential form. For example,

$$V(t) = V_0 \cos(\varphi(t)) = \mathrm{Re}(V_0 e^{i\varphi(t)}) \equiv \mathrm{Re}(\tilde{V}(t)).$$

We have the following impedance table for the common circuit elements:

	Impedance (Z)	Phase offset of $I(t)$ relative to $V(t)$ input
Resistor	R	$\varphi = 0$
Capacitor	$\frac{1}{i\omega C}$	Leads by $\frac{\pi}{2}$
Inductor	$i\omega L$	Lags by $\frac{\pi}{2}$

Table 4.1: Impedance table for common circuit elements.

The use of impedance is somewhat analogous to Ohm's law; in fact, we can draw a parallel to that here:

Theorem 4.5 (Complex Ohm's law)**.** The **complex Ohm's law** for a circuit with a given impedance Z is given by:

$$\boxed{\tilde{V} = \tilde{I}Z},$$

where the '$\sim$' represents complex-embedded quantities. ■

The particular draw of the complex impedance approach is that we can both capture the amplitude as well as the phase information in our computations.

We would illustrate the uses of complex impedance with an example; before which, however, we introduce an important theorem, which is a somewhat generalised version of the resistor divider formula. We consider a circuit in the form described in Fig. 4.2, and state the following theorem:

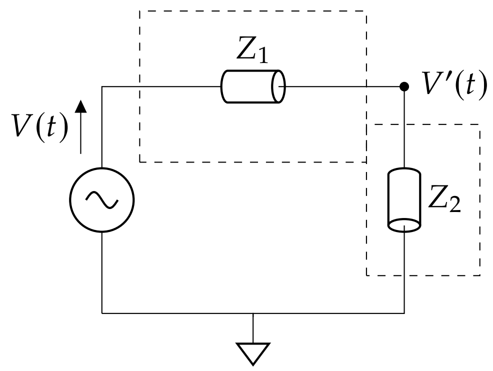

Figure 4.2: A circuit with complex impedance blocks Z_1 and Z_2.

Theorem 4.6 (Complex impedance divider)**.** For a circuit with two complex impedance blocks Z_1 and Z_2 and an output voltage $V'(t)$ in between (such as in Fig. 4.2), the (complex) ratio between the input voltage $\tilde{V}(t)$ and the output voltage $\tilde{V}'(t)$ is given by:

$$\boxed{\frac{\tilde{V}'}{\tilde{V}} = \frac{Z_2}{Z_1 + Z_2}}.$$

■

With its help, we can illustrate the power of impedance using the following example:

Example 4.2. *Will add in future if time permits. First part of lecture 12.* ■

4.4 RLC circuits

An RLC circuit, as suggested by its name, is one step more complicated than the previously mentioned circuits, in that it contains all three linear passive circuit elements: a resistor, an inductor, and a capacitor. As we shall see in the following discourse, such circuits exhibit resonance behaviours, not dissimilar at all to the behaviour of harmonic oscillators. And just like how we studied harmonic oscillators, we shall discuss different options of driving voltages and how they affect the response of the circuit.

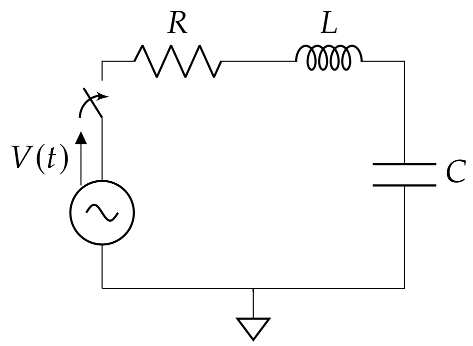

Figure 4.3: An example of an RLC circuit.

4.4.1 Driven by step impulse

The first case that we shall treat is when the driving voltage $V(t) \equiv V_0$ a constant, in Fig. 4.3. We sum all voltage drops across the circuit:

$$V_0 = RI + L\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{Q}{C}, \tag{4.6}$$

differentiate and multiply through by C,

$$\underbrace{LC}_{\equiv \frac{1}{\omega_0^2}} \frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + \underbrace{RC}_{\equiv \tau} \frac{\mathrm{d}I}{\mathrm{d}t} + I = 0,$$

and multiply through by ω_0^2,

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} + \underbrace{\Gamma}_{\equiv \tau\omega_0^2} \frac{\mathrm{d}I}{\mathrm{d}t} + \omega_0^2 I = 0. \tag{4.7}$$

Eq. (4.7) is clearly a second order homogeneous ODE, which means that we could directly utilise methods outlined in Chapter 1. We use the *Ansatz* that $I(t) = ce^{pt}$, and obtain the characterist polynomial:

$$p^2 + \Gamma p + \omega_0^2 = 0,$$

which we can then immediately solve for p:

$$p = -\frac{\Gamma}{2} \pm \sqrt{\underbrace{\left(\frac{\Gamma}{2}\right)^2 - \omega_0^2}_{\equiv\Delta}}.$$

Clearly, we need to treat the cases when the discriminant $\Delta < 0, > 0, = 0$ separately, and we do just that:

(Case 1: $\Delta < 0$). This case indicates a complex component, which implies that:

$$p = -\frac{\Gamma}{2} \pm \underbrace{\sqrt{\left(\frac{\Gamma}{2}\right)^2 - \omega_0^2}}_{\equiv i\omega_f},$$

where i is the imaginary unit, as usual (we are not using filth j notation here). This leads to the complex exponential, or trigonometric solution:

$$\begin{aligned} I(t) &= C_1 e^{\left(-\frac{\Gamma}{2} - i\omega_f\right)t} + C_2 e^{\left(-\frac{\Gamma}{2} + i\omega_f\right)t}, \\ \xRightarrow{\text{Euler's}} &= e^{-\frac{\Gamma}{2}t}[A\cos(\omega_f t) + B\sin(\omega_f t)]. \end{aligned}$$

(Case 2: $\Delta > 0$). This case prevents a complex exponent, which implies that:

$$p = -\frac{\Gamma}{2} \pm \underbrace{\sqrt{\left(\frac{\Gamma}{2}\right)^2 - \omega_0^2}}_{\equiv\omega'_f},$$

and this leads to the real exponential, or equivalently the hyperbolic solution:

$$\begin{aligned} I(t) &= C_1 e^{\left(-\frac{\Gamma}{2} - \omega'_f\right)t} + C_2 e^{\left(-\frac{\Gamma}{2} + \omega'_f\right)t}, \\ &= e^{-\frac{\Gamma}{2}t}[A'\cosh(\omega'_f t) + B'\sinh(\omega'_f t)]. \end{aligned}$$

(Case 3: $\Delta = 0$). This yields identical roots of $p = -\omega_0$. By ODEs, we obtain the following solution:

$$I(t) = e^{-\omega_0 t}(A'' + B''t).$$

Since each case requires separate treatment, we would solve for the constants separately as well. We know that the initial condition is $I(0) = 0$; as such, all the constants A equal to zero (check?). As such, we only need to treat the B's in each case.

We check back to Eq. (4.6), and notice that the terms that contain I and Q both vanish.

As a result, for the first case, we can compute that:

$$V_0 = L \left.\frac{\mathrm{d}I}{\mathrm{d}t}\right|_{t=0} = BL\omega_f,$$
$$\implies B = \frac{V_0}{L\omega_f}.$$

The other two cases can be treated similarly; as a result, we end up obtaining that for the second case,

$$B = \frac{V_0}{L\omega'_f},$$

and for the third case,

$$B = \frac{V_0}{L}.$$

Recall from classical mechanics and ODEs that the three cases are significant in the context of harmonic motion; in particular, we ascribe different 'extents' of damping to the different constants. We restate the three cases and their final solutions in the following table:

Cases	Solution	Damping
1 ($\Gamma < 2\omega_0$)	$I(t) = \frac{V_0}{L\omega_f}\sin(\omega_f t)e^{-\frac{\Gamma}{2}t}$	underdamped
2 ($\Gamma > 2\omega_0$)	$I(t) = \frac{V_0}{L\omega'_f}\sinh(\omega'_f t)e^{-\frac{\Gamma}{2}t}$	overdamped
3 ($\Gamma = 2\omega_0$)	$I(t) = \frac{V_0}{L}te^{-\omega_0 t}$	critically damped

Table 4.2: Three solutions for RLC circuit with step-impulse charging.

We notice that, if the resistance R in the circuit becomes very small, the circuit simply oscillates at the natural frequency ω_0.

For the physical origin of oscillation, consult slide 12–13 of lecture 12. For a graphical illustration of the damping cases, confer to slide 9 of the same lecture.

4.4.2 RLC circuits and spring oscillators

Going through the analysis in the previous section certainly seems like a déjà vu of some sort, and that should not come as a surprise, and we will explain why in this part.

There exists a very precise parallel between an RLC circuit in the form Fig. 4.3 and a harmonic oscillator with a driving force $F(t)$, spring constant k, and a damping coefficient γ. In fact, if we notice the parallel between the charge Q in the RLC case and the position x in the oscillator case, we see that the two differential equations are exactly the same:

$$\begin{cases} L\dfrac{\mathrm{d}^2Q}{\mathrm{d}t^2} + R\dfrac{\mathrm{d}Q}{\mathrm{d}t} + \dfrac{Q}{C} = V(t), \\ m\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} + \gamma\dfrac{\mathrm{d}x}{\mathrm{d}t} + kx = F(t). \end{cases}$$

As a result, we see that the current I is like the velocity v, the inductance L forms some sort of parallel with the mass (and hence inertia) m, and the spring coefficient k is analogous to the quantity $\frac{1}{C}$.

If we go one step further, we could also compare the conservation of energy between the two systems:

$$\begin{cases} E_{RLC} = \dfrac{1}{2}L[I(t)]^2 + \dfrac{1}{2C}Q^2, \\ E_{\text{spring}} = \dfrac{1}{2}m[v(t)]^2 + \dfrac{1}{2}kx^2, \end{cases}$$

and in a system without damping, both quantities are conserved.

4.4.3 Driven by sine-wave voltage

Now, we treat the case when $V(t) = V_0 \cos(\omega t)$ in Fig. 4.3, using the impedance method. Recall the impedance of each passive circuit element, cf. Table 4.1. Then, we use Thm. 4.5 to acquire the complex current:

$$\begin{aligned} \tilde{I} &= \frac{\tilde{V}}{Z_R + Z_L + Z_C}, \\ &= \frac{\tilde{V}}{R + i\omega L + \frac{1}{i\omega C}}, \\ &= \tilde{V}\frac{i\omega C}{1 - \omega^2 LC + i\omega RC}. \end{aligned}$$

Then, we perform the substitution that $\omega_0 \equiv \frac{1}{\sqrt{LC}}, \tau = RC$:

$$\tilde{I} = \frac{\tilde{V}}{R}\frac{i\omega\tau}{1 - \frac{\omega^2}{\omega_0^2} + i\omega\tau}. \tag{4.8}$$

Now, we can compute the magnitude and phase difference of the current, compared to the driving voltage:

$$\begin{aligned} |\tilde{I}| &= \sqrt{\left(\frac{V_0}{R}\right)^2 \frac{i\omega\tau}{1 - \frac{\omega^2}{\omega_0^2} + i\omega\tau}\frac{-i\omega\tau}{1 - \frac{\omega^2}{\omega_0^2} - i\omega\tau}}, \\ &= \frac{V_0}{R}\frac{\omega\tau}{\sqrt{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + (\omega\tau)^2}}, \end{aligned} \tag{4.9}$$

while the phase difference can be computed by rearranging the second fraction in Eq. (4.8):

$$\tilde{I} = \frac{\tilde{V}}{R}\frac{i\omega\tau}{1-\frac{\omega^2}{\omega_0^2}+i\omega\tau}\frac{1-\frac{\omega^2}{\omega_0^2}-i\omega\tau}{1-\frac{\omega^2}{\omega_0^2}-i\omega\tau},$$
$$= \frac{\tilde{V}}{R}\underbrace{\frac{(\omega\tau)^2+i\omega\tau\left(1-\frac{\omega^2}{\omega_0^2}\right)}{\left(1-\frac{\omega^2}{\omega_0^2}\right)^2+(\omega\tau)^2}}_{\equiv z},$$
$$\implies \varphi_I = \arctan\left(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right),$$
$$= \arctan\left(\frac{1-\frac{\omega^2}{\omega_0^2}}{\omega\tau}\right). \tag{4.10}$$

This yields very interesting results. We see that as evidenced by Fig. 4.4, the current in the circuit exhibits resonance behaviour, and at resonance the inductor and the capacitor in combination present no effective resistance to the current.

In the limit when the resistance goes to zero, the circuit becomes a pure LC resonator, which *only* supports the resonance frequency ω_0. This explains the narrowing of the resonance peaks.

We should also investigate the power input into the circuit. We have that the time-averaged power over a cycle is given by[3]:

$$\overline{P}_d = \frac{1}{T}\int_0^T V(t)I(t)\,\mathrm{d}t,$$
$$= \frac{1}{2}V_0 I_0\cos(\varphi_I). \tag{4.11}$$

It is clear that the phase offset introduced by the capacitor and inductor makes the driving less efficient. In a purely LC circuit, the phase offset is necessarily $\pm\frac{\pi}{2}$, and the system inputs *no* power whatsoever unless the circuit is driven at resonance, where the capacitive and inductive impedances *cancel exactly*.

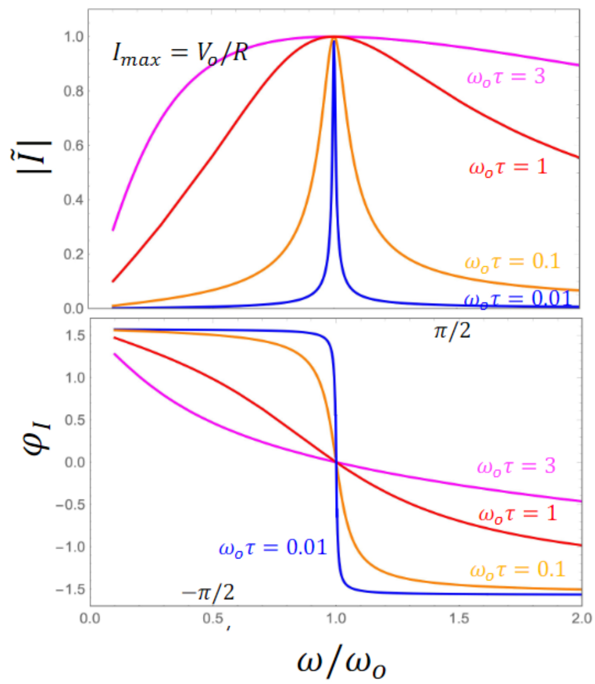

Figure 4.4: The magnitude and phase behaviour of current in an RLC circuit driven by sine-wave input.

[3]See lecture 10 for exact computations.

For phasor explanations and representations, cf. slide 8 and onwards of lecture 13.

We continue further with our exploration of driven RLC circuits, and now start to investigate voltages – specifically, the voltage drop across the capacitive element of the circuit. We first notice that the voltage can be given by:

$$\Delta V_C = \frac{Q}{C} = \frac{1}{C}\int_0^t I(s)\,\mathrm{d}s. \tag{4.12}$$

We integrate, writing $I(t)$ in complex form:

$$\begin{aligned}\Delta\tilde{V}_C &= \frac{|\tilde{I}|}{C}\int_0^t e^{i(\omega t+\varphi_I)}\,\mathrm{d}s,\\ &= \frac{|\tilde{I}|}{i\omega C}e^{i(\omega t+\varphi_I)},\\ &= \underbrace{\frac{V_0}{\sqrt{\left(1-\frac{\omega^2}{\omega_0^2}\right)^2+(\omega\tau)^2}}}_{\text{Voltage dropped across } C} e^{i(\omega t+\varphi_I-\frac{\pi}{2})},\end{aligned}$$

where φ_I is given directly by Eq. (4.10).

We can inspect the behaviour of the voltage under different circuit parameters. We see that:

at low f: the capacitive impedance is large, which causes voltage drop predominantly across the capacitor; predictably, the phase offset is zero;

at resonance: the current is limited only by the resistor R; as a result, the current is in phase with the voltage. Since capacitor voltage lags current by $\frac{\pi}{2}$, capacitor voltage lags current by $\frac{\pi}{2}$;

at high f: voltage is dropped predominantly over the inductor,

$$\begin{aligned}\implies V(t) &\approx L\frac{\mathrm{d}I}{\mathrm{d}t},\\ \implies I(t) &\approx \frac{1}{L}\int_0^t V(s)\,\mathrm{d}s\\ &= \frac{V_0}{L\omega}\sin(\omega t),\end{aligned}$$

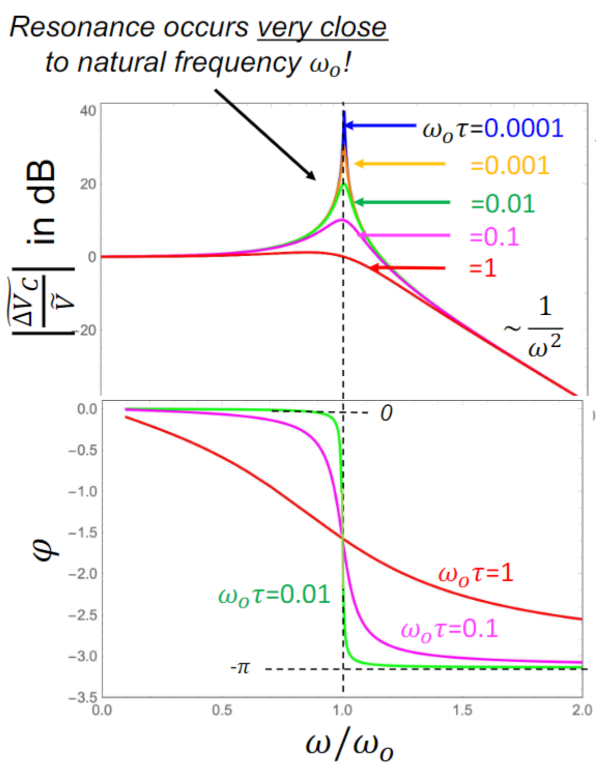

Figure 4.5: The magnitude and phase behaviour of voltage across capacitor in an RLC circuit driven by sine-wave input.

which means that we could use Eq. (4.12) to compute $V'(t)$:

$$\begin{aligned} \implies V'(t) &= \frac{1}{C}\int_0^t I(s)\,\mathrm{d}s, \\ &\approx -\frac{V_0}{LC\omega^2}\cos(\omega t), \\ &= -\left(\frac{\omega_0}{\omega}\right)^2 V_0\cos(\omega t), \\ &= -\left(\frac{\omega_0}{\omega}\right)^2 V(t). \end{aligned}$$

The current of an inductor lags voltage by $\frac{\pi}{2}$, and capacitor voltage lags current by $\frac{\pi}{2}$. As a result, capacitor voltage lags input voltage by π.

Comments on the position of resonance frequency given in slide 7 of lecture 14.

We can similarly derive the same quantities using the impedance method, but we need to fix the arctan *function there. See slides 8 and onwards in lecture 14.*

4.4.4 Quality factor of RLC circuits

We now arrive at the final topic surrounding RLC circuits: the quality factor of an RLC circuit. We first define this notion:

Definition 4.7 (Quality factor of a circuit)**.** The **quality factor** of a circuit, denoted by Q_f, is a measure of energy stored to energy lost per cycle. It is related to the rate at which energy is drained from the circuit. Symbolically, it is given by:

$$\boxed{Q_f = 2\pi\frac{W_S}{W_L}},$$

where W_S denotes energy stored and W_L denotes energy lost as heat per cycle. ■

We now apply this definition to our RLC circuit at resonance. Say, given the resonance frequency ω_0, the current is given by:

$$I(t) = I_0\cos(\omega_0 t),$$

and the energy stored would be equal to the maximum energy stored in an inductor:

$$W_S = \frac{1}{2}LI_0^2.$$

The energy lost is computed via the Joule heating law:

$$\begin{aligned} W_L &= R\int_0^{T_0} I_0^2 \cos^2(\omega_0 t)\,\mathrm{d}t, \\ &= \frac{RI_0^2\pi}{\omega_0}, \end{aligned}$$

since $T_0 = \frac{2\pi}{\omega_0}$. Putting them together, we obtain that:

$$Q_f = 2\pi\frac{1}{2}LI_0^2\frac{\omega_0}{RI_0^2\pi} = \frac{\omega_0 L}{R} = \frac{1}{\omega_0\tau}, \tag{4.13}$$

with the latter obtained via the substitution $\tau = RC$.

To see how the quality factor mathematically relates to the rate of energy drain, see slide 14 of lecture 14.

4.4.5 Power input into RLC series

The instantaneous power input into the RLC series circuit is give by:

$$\begin{aligned} P(t) &= V(t)I(t) \\ &= V_0\cos(\omega t)I_0(\omega)\cos(\omega t + \varphi(\omega)), \end{aligned}$$

where $I_0(\omega)$ and $\varphi(\omega)$ are given by Eq. (4.9) and Eq. (4.10), respectively. Averaging the power input over a single cycle T, we use a similar technique as in Eq. (4.11), and obtain that:

$$\overline{P}(\omega) = \frac{1}{2}V_0 I_0(\omega)\cos(\varphi(\omega)).$$

We now use the trig-identity $\cos\arctan x = \frac{1}{\sqrt{1+x^2}}$, and obtain through some algebra[4] that:

$$\overline{P}(\omega) = \frac{V_0^2}{2R}\frac{(\omega\tau)^2}{\left(1-\frac{\omega^2}{\omega_0^2}\right)^2 + (\omega\tau)^2}. \tag{4.14}$$

Notice that at the limit where $\frac{\omega}{\omega_0} = 1$, we have that $P_{max} = \frac{V_0^2}{2R}$. This is expected behaviour, since we are using a sine-wave input here, and recall the definition for RMS quantities (cf. Def. 4.3). This is where the factor of 2 in the denominator comes from.

Eq. (4.14) can also be written in terms of the quality factor of the circuit, which could be beneficial in certain ways. By substituting the forms in Eq. (4.13) to the equation,

[4] See slide 5, lecture 15.

we can get that:

$$\overline{P}\left(\frac{\omega}{\omega_0}\right) = \frac{V_0^2}{2R} \frac{\left(\frac{\omega}{\omega_0}\right)^2 \frac{1}{Q_f^2}}{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \left(\frac{\omega}{\omega_0}\right)^2 \frac{1}{Q_f^2}}. \tag{4.15}$$

This equation may seems cumbersome, and it indeed it. However, we shall now see that with the following definition, the algebra would work out nicely into something interesting.

The final piece of definition ties the resonance curve back to the quality factor of the circuit. We first define the resonance width:

Definition 4.8 (Resonance width)**.** The **resonance width** of the circuit, denoted by $\Delta\omega$, is given by:

$$\boxed{\Delta\omega = \omega_H - \omega_L},$$

where ω_H and ω_L denote the points on both sides of the peak, where the power is half of the peak power. ■

Immediately following from the definition, we state the corollary, which we will then prove:

Corollary 4.9 (Q_f expressed in $\Delta\omega$)**.** The quality factor Q_f can also be expressed in the following form:

$$\boxed{Q_f = \frac{\omega_0}{\Delta\omega}}.$$

One added benefit for this expression is that it is valid for all resonators, not just RLC circuits. ■

Proof. This corollary is proven on slide 8 of lecture 15. □

The final remark in this section is that an RLC circuit can also be used to amplify signals, by varying the capacitance of the tuning circuit to move the peak of the RLC circuit response curve.

See the first 9 slides of lecture 15 for graphics and more fleshed-out derivations.

4.5 Circuit networks

Previous to this point, we have been mainly dealing with circuits with simpler structure.

Chapter 5

MATH 576: Geometry & Topology I

5.1 Fundamentals of a topological space

The first important definition is the most central:

Definition 5.1 (Topology)**.** A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

1. $\emptyset$ and X are in $\mathcal{T}$.
2. The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$.
3. The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

A set X for which a topology $\mathcal{T}$ has been specified is called a **topological space**. ■

If $(X, \mathcal{T})$ is a topological space, then a subset U of X is an *open set* of X if U belongs to the collection $\mathcal{T}$. With this in mind, we can say that a topological space is a set X together with a collection of subsets of X, called open sets, such that $\emptyset$ and X are both open, and such that arbitrary unions and finite[1] intersections of open sets are open.

[1] *Infinite* intersections, however, do not need to fulfil this axiom.

Appendix A

Index of Reference

Theorems and definitions

Figures and plots