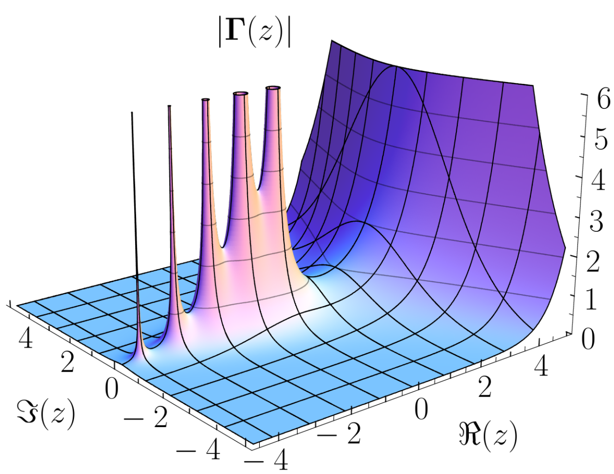
MATH 249

Honours Complex Variables

Winter 2022



Ningyuan (Leo) Li

Preface

It feels kind of like a dream now, how one second I was struggling in middle school with basic trigonometry and functions, and the next second I'm studying complex analysis in university, with so much more maths already on my belt.

Anyhow. I by no means claim the content of this note-package original; rather, the majority of the content comes from the in-class notes of prof. Anush Tserunyan. The original contributions from myself are:

- (most) of the figures in this document;
- the change in wording of certain parts of the notes in attempt to make the narrative more coherent;
- the **Preliminaries** chapter, where the main goal is to try to remedy prerequisite knowledge for people who have not taken analysis before (like myself);
- some of the examples given.

All mathematical errors are most likely mine; I'll try my best to check for errors as well.

Originally, I have envisioned this note-package to be something like a basic theorem-lemma-proof document, like normal course notes for a proof-based mathematics class. As you can see now, however, the content has evolved quite a bit more than merely that, with all the figures and explanations that should help somewhat with the comprehension of the (still central) theorems. Hopefully, as time goes on, I can continue to flesh out the content, to the point that it can fill a niche - a pamphlet that is not too lengthy and full of jargon like a proper textbook, but at the same time not as simple and one-dimensional as simple notes that is taken from the lectures.

Releases that include every week's new notes and other changes are (hopefully) weekly on Github. I aim to make a new release every Sunday, although schedule may not permit.

Contents

Chapter 1

Preliminaries

1.1 Real analysis

Complex analysis, as the name suggests, is heavily reliant upon basic techniques that root from real analysis. As a consequence, before getting into complex analysis, a few results from real analysis may be helpful to develop the content further down the notes; nevertheless, the rigour of the given topics in real analysis would not be displayed fully, as the definition and intuition are all that is needed for the subsequent chapters.

We start with the definition of a bound:

Definition 1.1.1: Bound

A set $S \subseteq \mathbb{R}$ is **bounded** from above if $\exists u \in \mathbb{R} : \forall x \in S,\ x \leq u$; u is then an upper bound of S. Likewise, v is a lower bound of S if $\exists v \in \mathbb{R} : \forall x \in S,\ x \geq v$.

With this definition in mind, we can continue to define the infimum and supremum of a set:

Definition 1.1.2: Infimum and Supremum

Let $S \subseteq \mathbb{R} : S \neq \emptyset$ be a set. We say that $s \in \mathbb{R}$ is the **supremum** or least upper bound of S if:

i) s is an upper bound of S;

ii) $s \leq u$ for all upper bounds u of S.

Likewise, the **infimum** is defined similarly, but for a greatest lower bound.

As the definitions suggest, the intuition for those concepts are not particularly difficult to grasp. Namely, the definition of a bound somewhat resembles the asymptotic behaviours that we have studies in calculus[1]. On a similar vein, the intuition behind infimum and supremum are quite similar to the definition of maximum and minimum, with a major caveat that will be explored in the following example.

Example 1.1.1 (Infimum of $\mathbb{R}^+$)**.** *The set of all positive real numbers $\mathbb{R}^+$ does not have a minimum, as any real number $r_0 > 0$ can simply be divided by 2 and achieve a smaller $r_1 > 0$. However, the set* does *have an infimum, which is precisely 0, as it is smaller than all positive real numbers, yet greater than any other (negative) real numbers that could be used as a lower bound.*

As illustrated, the infimum/supremum of a set does not have to be a member of the same set, which is the major difference that sets them apart from minimum/maximum.

[1] Note that unlike asymptotes in calculus, **no** value in the set (range, in the sense of a function) is allowed to be smaller/bigger than the lower/upper bound

We now proceed to some further definitions:

Definition 1.1.3: Epsilon Neighbourhood

Let $a \in \mathbb{R}$. The **epsilon(ϵ) neighbourhood**, centred around a, is defined as follows:

$$V_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\}.$$

With this in mind, the definition of a boundary point comes rather naturally:

Definition 1.1.4: Boundary Point

Let $A \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is called a **boundary point** of A if:

$$\forall \epsilon > 0,\ V_\epsilon(x) \cap A \neq \emptyset \wedge V_\epsilon(x) \cap A^c \neq \emptyset.$$

If we generalise this definition to $\mathbb{R}^2$, which is the set in which complex analysis is most concerned about, the following illustration might help with building an intuition:

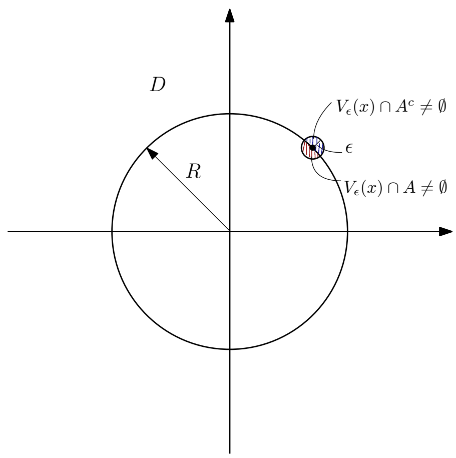


Figure 1.1: Graphical representation of a boundary points in $\mathbb{R}^2$

Coincidentally, the above figure illustrates a disk:

Definition 1.1.5: Disk

In $\mathbb{R}^2$, a **disk** is the region bounded by a circle.

The following corollary roots upon Def. 1.1.4. Consider fig. 1.1; the boundary of this disk would contain all elements of the geometric boundary line of the circle.

Corollary 1.1.1: Boundary Set

The set of all points satisfying the condition for a boundary point is called the **boundary (set)** of A and is denoted ∂A.

An interesting point that arises from the preceding definitions is the rather philosophical discussion of whether the boundary of a set is actually contained in the set. As a resolution, the following definition is proposed:

Definition 1.1.6: Open and Closed Sets

A set $S \subseteq \mathbb{R}$ is **open** if it does not contain any of its boundary points; in other words,

$$\partial S \cap S = \emptyset,$$

and a set $A \subseteq \mathbb{R}$ is **closed** if and only if its complement is open. An alternative representation of the same idea would be the following:

$$\partial A \subseteq A.$$

Example 1.1.2 (Open and closed disk)**.** *Consider, once again, the disk D in fig. 1.1. Then, the* open *disk would be the set that does not include the boundary circle of the disk, and the* closed *disk would be the set that does include the boundary circle. More formally, the open disk (centred around (a, b)) is defined as follows:*

$$D = \{(x, y) \in \mathbb{R}^2 : (x - a)^2 + (y - b)^2 < R^2\},$$

and the closed disk (centred around (a, b)) is defined similarly:

$$\overline{D} = \{(x, y) \in \mathbb{R}^2 : (x - a)^2 + (y - b)^2 \leq R^2\}.$$

With the previous definitions, we can now define the last concept that we need in this complex analysis course:

Definition 1.1.7: Compactness

A set is said to be **compact** if it is both closed and bounded.

1.2 Point-set topology

Chapter 2

Complex numbers

2.1 Definitions and elementary properties

We start with the definition of the complex numbers:

Definition 2.1.1: Complex Numbers

A **complex number** is a point (a, b) in $\mathbb{R}^2$, which we write as $a + ib$, where i is the so-called *imaginary unit* such that $i^2 = -1$. The set of complex numbers is denoted by $\mathbb{C}$.

REMARK. We consider $\mathbb{R}$ as a subset of $\mathbb{C}$ by identifying a real number $r \in \mathbb{R}$ with the complex number $r + 0i$. △

REMARK. The complex numbers can also be seen as a field, cf. MATH 235. In particular, by using the Euclidean norm, we can define the absolute value of a complex number as follows:

Definition 2.1.2: Absolute Value of a Complex Number

The **absolute value** of a complex number is computed via:

$$|z| = |a + ib| := \sqrt{a^2 + b^2}.$$

As we know, since the definition already precludes the presence of zero divisors, the only thing we need to worry about is to check the existence of inverse. To this end, we need another piece of definition:

Definition 2.1.3: Complex Conjugate

The **complex conjugate** of z is defined as $\overline{z} := a - ib$.

Finally, we arrive at the following corollary, which hammers home the reason why $\mathbb{C}$ is a field:

Corollary 2.1.1: Inverse of a Complex Number

$\forall z \in \mathbb{C}$, $z \cdot \frac{\overline{z}}{|z|^2} = 1$, so every (non-zero) $z \in \mathbb{C}$ has a multiplicative inverse.

△

There are a few useful notions about complex numbers that we should know:

- For $z \in \mathbb{C}$ where $z = a + ib$, $\mathrm{Re}(z) := a$ and $\mathrm{Im}(z) := b$.
- $\mathbb{C}$ can be viewed as $\mathbb{R}^2$, so it is a 2-dimensional vector space over the reals. Consequently, addition and scalar multiplication on $\mathbb{C}$ is defined just like in $\mathbb{R}^2$
- Unlike in $\mathbb{R}^2$, however, we can also multiply two complex numbers like binomials.

The calculus rules mentioned above are put down concretely in the following definition:

Definition 2.1.4: Complex Calculus

- (Addition) $(a + ib) + (c + id) := (a + c) + i(b + d)$;
- (Scalar mult.) For $\lambda \in \mathbb{R}$, $\lambda \cdot (a + ib) := (\lambda a) + i(\lambda b)$;
- (Multiplication) $(a + ib) \times (c + id) := (ac - bd) + i(ad + bc)$.

Aside 2.1.1: Motivation of Complex Numbers

The significance of complex numbers are perhaps subtle: they were introduced by Gauss as an extension of the field of the reals so that every polynomial of real or complex coefficients would have a root. As it turns out, simply by adding a root of $x^2 + 1$ was enough to guarantee the existence of roots. As we know now, the statement is known as the fundamental theorem of algebra, which we will state and prove later. Although the theorem is central in our studies, the proof of which cannot be developed until a few more ideas are explained.

A natural consequence of the vector space definition of $\mathbb{C}$ is to view $\mathbb{C}$ in polar coordinates in $\mathbb{R}^2$. For $z := a + ib$, by putting $r := |z|$ and $\alpha := \arctan\left(\frac{b}{a}\right)$, we may write $(r, \alpha)_p$ to represent z. In particular, α is denoted as $\arg(z)$.

Corollary 2.1.2: Complex Number in Polar

For $z \in \mathbb{C}$, we have that $z = |z|[\cos(\arg(z)) + i \sin(\arg(z))]$.

Proof. The proof roots from basic geometry:

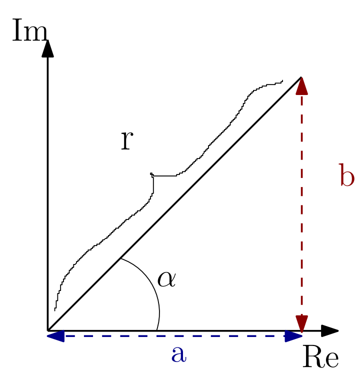


Figure 2.1: Geometric representation of a complex number

□

By geometry, we can also arrive at the following two lemmata:

Lemma 2.1.1: Multiplication of Complex Numbers in Polar

$\forall z_1, z_2 \in \mathbb{C}$, $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$ and $\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$.

Proof. Let $z_j = r_j(\cos\alpha_j + i\sin\alpha_j)$, $j = 1, 2$, where $r_j = |z_j|$ and $\alpha_j = \arg(z_j)$. Then, we have that:

$$\begin{aligned} z_1 \cdot z_2 &= (r_1\cos\alpha_1 r_2\cos\alpha_2 - r_1\sin\alpha_1 r_2\sin\alpha_2) + i(r_1\sin\alpha_1 r_2\cos\alpha_2 + r_1\cos\alpha_1 r_2\sin\alpha_2) \\ &= r_1r_2[\cos(\alpha_1+\alpha_2) + i\sin(\alpha_1+\alpha_2)] \text{ (using double-angle identities)} \end{aligned}$$

This concludes the proof. □

REMARK. Graphically, the addition and multiplication of complex numbers look something like this:

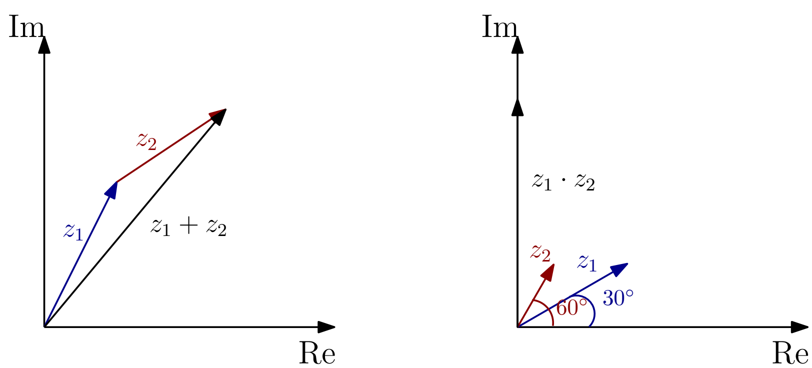


Figure 2.2: Geometric representation of addition and multiplication of complex numbers

△

Lemma 2.1.2: Triangle Inequality

For any $z_1,\ z_2 \in \mathbb{C}$,

a) $|z_1 + z_2| \le |z_1| + |z_2|$,

b) (Reverse triangle inequality) $|z_1 - z_2| \ge \big||z_1| - |z_2|\big|$.

Proof. Part a) is the usual triangle inequality for vectors in any inner product space, which can be proven using Cauchy-Schwartz. Or, in $\mathbb{R}^2$, it is also possible to prove it using basic geometry:

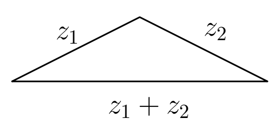


Figure 2.3: Geometric proof of triangle inequality

Part b) of the proof follows immediately from part a):

$$\begin{aligned} &|z_1| = |z_1 - z_2 + z_2| \le |z_1 - z_2| + |z_2| \\ \Rightarrow &|z_1 - z_2| \ge |z_1| - |z_2| \end{aligned}$$

□

An interesting point also arises with the observation that adding the angles results in multiplying the numbers - this is the behaviour that an exponential exhibits. As a result, it is perceivable that a connection between complex numbers and exponentials could be observed. To this end, we use the Taylor series of e^x and view it over $\mathbb{C}$, such that we can define exponentials over the field of complex numbers.

Lemma 2.1.3: Absolute Convergence of e^x

The Taylor series $e^z = \displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges absolutely.

Proof. Proven via ratio test (cf. *Calculus II*). □

This lemma builds up the underlying principle for the following theorem:

Theorem 2.1.1: Euler's Formula

For any $t \in [-\pi, \pi)$,

$$\boxed{e^{it} = \cos t + i \sin t}$$

Thus, $\forall z \in \mathbb{C}$, $z = |z|e^{i\alpha}$, where $\alpha = \arg(z)$.

Proof. Using the Taylor series expansion for e^{it}:

$$\begin{aligned}
e^{it} &= 1 + it + \frac{i^2t^2}{2} + \frac{i^3t^3}{3!} + \frac{i^4t^4}{4!} + \cdots \\
&= 1 + it - \frac{t^2}{2} - \frac{it^3}{3!} + \frac{t^4}{4!} + \frac{it^5}{5!} - \frac{t^6}{6!} + \cdots \\
&= (1 - \frac{t^2}{2} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots) + i(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots) \\
&= \cos t + i \sin t
\end{aligned}$$

□

A direct observation following Thm. 2.1.1 is the fact that $\left|e^{\alpha i}\right| = 1$. As a result, exponentiation and root-taking formulae can thus developed:

Corollary 2.1.3: Complex Exponentiation

For $z = re^{\alpha i} \in \mathbb{C}$, we have that:

a) $z^n = r^n e^{n\alpha i}$,

b) $z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{\frac{\alpha i}{n}}$.

Proof. The proof for part a) of this corollary is quite straightforward; it follows directly from Lem. 2.1.1. For part b), the proof involves knowledge of the *root of unity*, which will be explored in the following aside. □

Aside 2.1.2: Root of Unity

To be constructed.

2.2 Fundamental theorem of algebra

By now, we are finally well (enough)-equipped to state and prove the fundamental theorem of algebra, through the lens of rather basic complex analysis:

Theorem 2.2.1: Fundamental Theorem of Algebra

Every polynomial over $\mathbb{C}$ has a root in $\mathbb{C}$.

Proof. One proof has already been produced in the past, notably in MATH 247 via linear algebra. The proof via complex analysis is given as follows:

Let $p(z) = \sum_{i=0}^{n} a_i z^i$ be a non-constant polynomial with complex coefficients. We seek to prove the following statement:

$$z_0 \in \mathbb{C} \text{ is a root of } p(z) \iff |p(z_0)| = 0.$$

Thus, it suffices to show that $|p(z)|$:

i) achieves its infimum, ie. $\inf(|p(z)|) = p(z_0)$, for some $z_0 \in \mathbb{C}$,

ii) $|p(z)| = 0$.

To construct the proof fully, we first have to start with a few claims:

C1: $|p(z)| \to \infty$ as $|z| \to \infty$.

Proof. We have that:

$$\begin{aligned}|p(z)| = \sum_{i=0}^{n} a_i z^i &= |z|^n \left|a_n + \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + \frac{a_{n-1}}{z}\right| \\ &\geq |z|^n \left(|a_n| - \frac{|a_0|}{|z|^n} - \frac{|a_1|}{|z|^{n-1}} - \cdots - \frac{|a_{n-1}|}{|z|}\right).\end{aligned}$$

As $|z| \to \infty$, $\frac{|a_k|}{z^{n-k}} \to \infty$ for $k < n$, so $\exists R > 0 : \forall |z| \geq R$, $\frac{|a_0|}{|z|^n} + \cdots + \frac{|a_{n-1}|}{|z|} < \epsilon$. As a result, $|p(z)| > |z|^n(|a_n| - \epsilon) \to \infty$, which implies that $|p(z)| \to \infty$ as $|z| \to \infty$. □

Thus,

$$\exists R > 0 : \forall |z| \geq R, |p(z)| > |p(0)|. \tag{2.2.1}$$

Therefore, by constructing a compact disk centred around 0 of radius R, denoted by $\overline{D}_R$, we could then say that $\inf_{z\in\mathbb{C}}(|p(z)|) = \inf_{z\in\overline{D}_R}(|p(z)|)$. But $z \mapsto |p(z)|$ is a continuous function from $\mathbb{R}^2$ to $\mathbb{R}$, so it achieves its minimum on the compact set $\overline{D}_R$. Thus, $\exists z_0 \in \overline{D}_R$ such that $\inf(|p(z)|) = |p(z_0)|$.

Now, suppose towards a contradiction that $|p(z_0)| > 0$. This z_0 must be in the open disk D_R because if $z_0 \in \partial D_R$, then by eq. (2.2.1), $|p(z_0)| > |p(0)|$, contradicting the supposition that $|p(z_0)|$ being the minimum. We rewrite $p(z) = b_0 + b_1(z - z_0) + \cdots + b_n(z - z_0)^n$ and let $k \geq 1$ be the least k such that $b_k \neq 0$, so $p(z) = b_0 + b_k(z - z_0)^k + b_{k+1}(z - z_0)^{k+1} + \cdots + b_n(z - z_0)^n = b_0 + b_k(z - z_0)^k + (z - z_0)^k q(z)$, where[1] $q(z) := b_{k+1} + b_{k+2}z + \cdots + b_n z^{n-k-1}$.

C2: $|p(z)| \leq |b_0 + b_k(z - z_0)^k| + M|z - z_0|^{k+1}$, for some $M > 0$, $\forall z \in \overline{D}_1(z_0)$.

Proof. By triangle inequality, $|p(z)| \leq |b_0 + b_k(z - z_0)^k| + |z - z_0|^k |q(z)|$. But by the same principle, $|q(z)| \leq |b_{k+1}| + |b_{k+2}||z| + \cdots + |b_n||z|^{n-k-1}$, which is in turn $\leq |b_{k+1}| + |b_{k+2}| + \cdots + |b_n| =: M$. □

[1] Note that $|p(z_0)| = |b_0| > 0$.

To get a contradiction, we will choose z with $|z - z_0|$ small enough so that $|p(z)| < |b_0| = |p(z_0)|$.

C3: For any $0 \neq w_1, w_2 \in \mathbb{C}, \forall k \geq 1, \exists$ unit[2] $z \in \mathbb{C}$ such that for small enough $r > 0$, $|w_1 + w_2(rz)^k| = |w_1| - |w_2|r^k$.

Proof. Let z be the unique solution to $\frac{w_1}{|w_1|} + \frac{w_2}{|w_2|}z^k = 0$; in particular, $z^k = -\frac{w_1}{|w_1|}\frac{|w_2|}{w_2}$. Then, for any $r > 0$,

$$\begin{aligned}|w_1 + w_2(rz)^k| &= \left|w_1 + |w_2|\frac{w_2}{|w_2|}z^k r^k\right| \\ &= \left|w_1 - |w_2|\frac{w_1}{|w_1|}r^k\right| \\ &\underset{r \text{ small enough}}{=} |w_1|\left(1 - \frac{|w_2|}{|w_1|}r^k\right) \\ &= |w_1| - |w_2|r^k,\end{aligned}$$

for r small enough so that $\frac{|w_2|}{|w_1|}r < 1$. $\square$

The motivation for **C3** is perhaps subtle; however, it is arguably the most powerful part of this proof. In essence, we are constructing a vector $w_2(rz)^k$ such that it is colinear with w_1, and this is only possible in the complex numbers, in that we can choose a power to rz such that the angle between the two vectors are exactly $180°$, which means that $|w_1 + w_2(rz)^k| = |w_1| - |w_2|r^k$ would hold[3].

By **C3**, there exists a unit $z_1 : |b_0 + b_k(rz_1)^k| = |b_0| - |b_k|r^k$ for small enough $r > 0$. Consider[4]:

$$\begin{aligned}|p(z_0 + rz_1)| &\underset{\mathbf{C2}}{\leq} |b_0 + b_k(rz_1)^k| + M|rz_1|^{k+1} \\ &= |b_0| - |b_k|r^k + Mr^{k+1} \\ &\leq |b_0| - |b_k|r^k + \frac{1}{2}|b_k|r^k \\ &= |b_0| - \frac{1}{2}|b_k|r^k < |b_0|\end{aligned}$$

Thus, $|p(z_0 + rz_1)| < |b_0| = |p(z_0)|$, contradicting that $|p(z_0)| = \inf(|p(z)|) \leq |p(z_0 + rz_1)|$. This proves everything. $\square$

This theorem is fundamental, in the sense that guaranteeing and finding zeroes of a polynomial are of paramount importance in many fields of mathematics. For example, an important application of Thm. 2.2.1 in Eigenvalue theory is the guarantee of the existence of roots for the characteristic polynomial of a matrix; consequently, the significance permeates into various fields that have linear algebra as (one of) their intrinsic operations, such as quantum mechanics.

[2] $|z| = 1$

[3] If it is not yet clear, the absolute value was reduced to subtraction of two simple scalars, precisely because we were able to manipulate the vectors such that they are colinear and pointing towards opposite directions.

[4] For r small enough so that $r^{k+1}M \leq \frac{1}{2}|b_k|r^k$

2.3 Riemann sphere and stereographic projection

On first sight, the Riemann Sphere (denoted by S^2) seems to just be a geometric object - a unit sphere in $\mathbb{R}^3$.

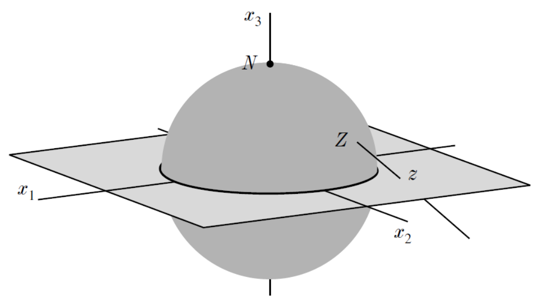


Figure 2.4: The Riemann Sphere

However, as we shall explore in this section, after being equipped with a powerful tool called the stereographic projection, we would be able to have an alternative definition of the complex numbers, with even an added element that was previously not present in $\mathbb{C}$.

We first provide the definition of the stereographic projection:

Theorem 2.3.1: Stereographic Projection

The **stereographic projection** is a function:

$$\text{sproj} : S^2 \setminus \{N\} \to \mathbb{C}$$

that is defined as follows:

$$\text{sproj}(Z := (x_1, x_2, x_3)) = \frac{x_1}{1 - x_3} + i\frac{x_2}{1 - x_3}.$$

The function is also a bijection.

Proof. The definition of the function roots from the following geometric definition: for each point $Z \in S^2$, we take the line from N to Z and define $\text{sproj}(Z)$ as the point where that line intersects the plane $\mathbb{R}^2 \cong \mathbb{C}$. If $Z := (x_1, x_2, x_3)$, then the parametric equations of the line is given by:

$$\begin{cases} a = x_1 t \\ b = x_2 t \\ c = 1 - t(1 - x_3), \end{cases}$$

so at $t = 0$, we get $N = (0, 0, 1)$, and at $t = 1$, we get $Z = (x_1, x_2, x_3)$. Thus, solving for $c = 0$, we get the value of $\text{sproj}(Z)$.

We show that the function is a bijection by showing the existence of an inverse. Let $w := a + ib \in \mathbb{C}$. The line going through N and w is given by[5]:

$$\begin{cases} x_1 = at \\ x_2 = bt \\ x_3 = 1 - t. \end{cases}$$

The point of intersection of this line with the sphere is the solution to $x_1^2 + x_2^2 + x_3^2 = 1$, so $a^2t^2 + b^2t^2 + (1 - t)^2 = 1$, solving for t yields us $t = \frac{2}{a^2+b^2+1} = \frac{2}{|w|^2+1}$. Thus, we have that $\text{sproj}^{-1}(w) = (\frac{2a}{|w|^2+1}, \frac{2b}{|w|^2+1}, 1 - \frac{2}{|w|^2+1})$. ☐

[5]Note that as before, $t = 0$ yields N and $t = 1$ yields w.

As we have already alluded to, the Riemann sphere definition of the complex numbers gives an extra member to the set. Notice, geometrically, the image under the stereographic projection of a neighbourhood around the north pole N corresponds to an unbounded annulus in $\mathbb{C}$:

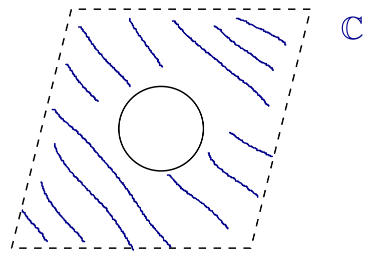


Figure 2.5: Compactification of $\mathbb{C}$

This allows us to think of the set of points in $\mathbb{C}$ outside of a disk as a neighbourhood of ∞. So, in other words, we take an artificial point ∞ - represented naturally by N on the Riemann sphere - and add it to $\mathbb{C}$ so that the convergence to ∞ is the same as convergence to N in S^2. In essence, we have arrived at the following definition:

Definition 2.3.1: Extended Complex Plane

If we compactify $\mathbb{C}$ by defining $\hat{\mathbb{C}} := S^2 \cong \mathbb{C} \cup \{\infty\}$, this definition is what is known as the **extended complex plane**.

2.4 Topology of the complex plane

So far, in section 1.1, we have gone over some terminology used in real analysis, valid for the set $\mathbb{R}$ and $\mathbb{R}^2$. In this section, we shall endeavour to expand those concepts such that they would also be valid for $\mathbb{C}$, which is the set we are interested in. It may also be interesting to note that topology, at the end of the day, still remains to be topology regardless of which topological space we are working with; as a result, many of the definitions that we shall see in this section would be quite analogous to the definitions given back in section 1.1. Nevertheless, there are still some definition that may seem a bit new, and some that are actually equivalent to our previous definition of the same terminology.

We start with the definition for the openness of a set:

Definition 2.4.1: Open (Set)

A set $U \subseteq \mathbb{C}$ is **open** if it is a union of open balls[a].

[a] Or equivalently, open disks, as defined in section 1.1.

REMARK. As we can see, the definition seems to be quite different compared to that in Def. 1.1.6. However, the two definitions are actually equivalent. Consider an open ball B_i; by definition, the boundary ∂B_i is not contained in B_i. If we take the union $\bigcup_i B_i$ of any amount of open balls, the boundary $\partial \bigcup_i B_i$ is still not contained in the union, as illustrated in the following figure.

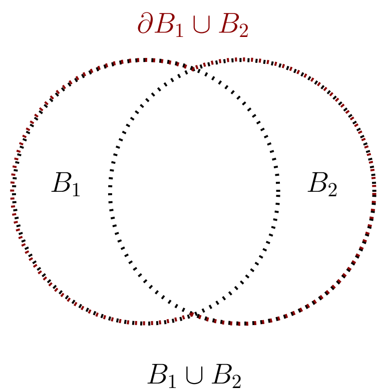


Figure 2.6: Union of open balls is open

△

The following lemma characterises the openness of a set:

Lemma 2.4.1: Characterisation of an Open Set

The following statements are equivalent:

i) A set $U \subseteq \mathbb{C}$ is open;
ii) $\forall x \in U,\ \exists$ an open ball $B \ni x : B \subseteq U$;
iii) $\forall x \in U, \exists r > 0 : B_r(x) \subseteq U$[a].

[a] Or put it colloquially, every $x \in U$ is there with a circle of friends.

Proof. We shall prove it implication-by-implication:

- i)⇒ii): If $U = \bigcup_{i \in I} B_i$, where each B_i is an open ball and I is some [POTENTIAL CORRECTION NEEDED] index set, then $\forall x \in U,\ \exists i \in I : x \in B_i$.

- ii)$\Rightarrow$iii): $\forall x \in U$, let $B \ni x$ be an open ball. Let r denote the radius of B and y be its centre, i.e. $B = B_r(y)$, then let $r' := r - |y - x|$, so $B_{r'}(x) \subseteq B_r(y)$.
- iii)$\Rightarrow$i): For a set U, if $\forall x \in U,\ \exists r_x : B_{r_x}(x) \subseteq U$, then $U = \bigcup_{x \in U} B_{r_x}(x)$.

$\square$

As we have defined the meaning for a set to be open, it is natural to also state the converse definition:

Definition 2.4.2: Closed (Set)

$K \subseteq \mathbb{C}$ is called **closed** if it is a complement of an open set; in other words, $K = \mathbb{C} \setminus U$ for some open set $U \subseteq \mathbb{C}$.

The notion of a neighbourhood in complex analysis is similar to that from real analysis, cf. Def. 1.1.3:

Definition 2.4.3: Neighbourhood

A **neighbourhood** of a point $z \in \mathbb{C}$ is a set $N \subseteq \mathbb{C}$ such that $B_r(z) \subseteq N$ for some $r > 0$.

Example 2.4.1. *The closed unit square is a neighbourhood of zero although it is not an open set.*