PFB inversion

Ningyuan Li

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1 Forward PFB's matrix form

We recap first the basic procedures of an r-tap PFB. Say, we want to have n_c channels after the transformation. In a naive FFT procedure, we would need a time stream of length $2n_c$, after taking account of the Nyquist criterion[1]. However, in the PFB:

1. We first take a time stream that is of length $2rn_c$.
2. We then apply a sinc function of appropriate shape to the original time stream. The sinc is also of length $2rn_c$.
3. The sinc-modified time stream is then split into r pieces of the same length (say, M[2]). The r pieces are then stacked together (term-wise added).
4. We finally take the M-point real-DFT on the stacked signal, which gives us the final answer.

We wish to enumerate some of these steps with linear algebra notation, which makes the inversion procedure more straightforward. Using the prof's notation, we have that:

$$D = FSWd, \tag{1}$$

where D is the transformed signal, d is the original time stream, and:

F: is the DFT matrix, of size $M \times M$. It is unitary and hence invertible, which is convenient. The matrix is used in step 4. (Question 1: we are taking the real-DFT instead of the full DFT, which changes the matrix from an invertible one to a non-invertible one (since it would not be square). Is it not a big issue, because the real-DFT is simply the DFT minus some complex conjugates, so we can easily reconstruct the original, and the maths will be the same?)

S: is the matrix that takes care of the split-and-stacking. It does what step 3 instructs.

W: is the matrix that applies the window. It takes care of step 2.

[1] In practice, we take the `rfft` function, which disregards the complex conjugates – and hence having half the number of channels than the original time stream.

[2] Notice that $M = 2n_c$.

Writing everything out explicitly, we have that[3]:

$$\underbrace{\begin{bmatrix} D_1 \\ \vdots \\ D_{n_c} \end{bmatrix}}_{D} = \underbrace{\begin{bmatrix} F_{1,1} & \cdots & F_{1,2n_c} \\ \vdots & \ddots & \vdots \\ F_{n_c,1} & \ldots & F_{n_c,2n_c} \end{bmatrix}}_{F} \underbrace{\begin{bmatrix} \underbrace{\mathrm{id}_{2n_c} \cdots \mathrm{id}_{2n_c}}_{r} \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} W_{1,1} & 0 & \cdots & 0 \\ 0 & W_{2,2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & W_{2rn_c,2rn_c} \end{bmatrix}}_{W} \underbrace{\begin{bmatrix} d_1 \\ \vdots \\ d_{2rn_c} \end{bmatrix}}_{d}$$

2 Inverting the matrix representation

After left-multiplying Eq. (1) with F^{-1}, the problem is reduced to inverting SW[4]. Notice that, since S is a size $2n_c \times 2rn_c$ matrix, the product would not actually be square, and hence not invertible. As a result, some sort of 'pseudo-inverse' needs to be taken in its stead. We first examine the actual form of the product SW:

$$SW = \begin{bmatrix} W_1 & \cdots & 0 & W_{2n_c+1} & \cdots & 0 & \cdots & W_{2(r-1)n_c+1} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & W_{2n_c} & 0 & \cdots & W_{4n_c} & \cdots & 0 & \cdots & W_{2rn_c} \end{bmatrix},$$

as we can see, SW is in the form of r size $2n_c$ diagonal matrices horizontally aligned.

(Question 2: in the first paragraph of section 3 in your notes, I'm not quite sure what the procedure is that gives a Toeplitz matrix in the end. Is it the result of taking the IDFT (so multiplying F^{-1} again on SW)? Or is it something else? Moreover, is the 'decoupled chunk' that you are referring to the horizontal, diagonal blocks in SW?)

[3] id_s stands for identity matrix of size s.

[4] In below discussion, elements $W_{i,j} \in W$ will be denoted more compactly as W_i, since $i = j \; \forall i, j$.