Why PFB Works?

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We established that what the convolution does is to combine every r frequency bin in a staggered manner – as to say, take $r = 4$ as an example, we have the result of the convolution ($*$) between the DFT of the input signal (X_k) and the 4-wide boxcar (H_k) as $C := X * H =$ `sum(0,1,2,3),sum(1,2,3,4),...`[1]. As a result, we now simply need to find a way to take every fourth element from C, and compute the DFT of the sequence.

Say, we have that originally, the signal is of length N, and we wish to compute an r-tap PFB for the signal. This means that the output of the PFB (disregarding the Nyquist criterion) would be of length $M = \frac{N}{r}$. We want to compute the DFT of the convolved signal C, and take every r-th element of it. In other words, we want:

$$C_k = \sum_{n=0}^{N-1} c_n e^{-i\frac{2\pi}{N}kn},$$

and we take every r-th element of the sequence $\{C_k\}, k \in (0, 1, \ldots, N-1)$. We can achieve the same effect by re-indexing k into k':

$$C_{k'} = \sum_{n=0}^{N-1} c_n e^{-i\frac{2\pi}{N}rk'n}, \tag{1}$$

where k' goes from 0 to $M-1$[2]. We can rewrite Eq. (1) into a double sum of r segments:

$$C_{k'} = \sum_{m=0}^{r-1}\sum_{n=0}^{M-1} c_{n+mM} e^{-i\frac{2\pi}{N}rk'(n+mM)}. \tag{2}$$

[1] I still need to straighten out the maths about how convolution works, but I'll do it in my own time.

[2] Recall that $r = \frac{N}{M}$.

Look deeper in the exponential;

$$\begin{aligned}
\exp\left(-i\frac{2\pi}{N}rk'(n+mM)\right) &= \exp\left(-i\frac{2\pi}{N}\frac{N}{M}k'n\right)\cdot\exp\left(-i\frac{2\pi}{N}\frac{N}{M}k'mM\right),\\
&= \exp\left(-i\frac{2\pi}{M}k'n\right)\cdot\exp\left(-i\frac{2\pi}{\cancel{N}}\frac{\cancel{N}}{\cancel{M}}k'm\cancel{M}\right),\\
&= \exp\left(-i\frac{2\pi}{M}k'n\right)\cdot 1.
\end{aligned}$$

Remarkably, after the cancellation, Eq. (2) is actually in the form of r DFT's, each of length M:

$$C_{k'} = \sum_{m=0}^{r-1}\sum_{n=0}^{M-1} c_{n+mM}e^{-i\frac{2\pi}{M}k'n}. \tag{3}$$

The final step is to exchange the two sums in Eq. (3). This is valid, because there is no m index in the exponential. We thus get our final form:

$$C_{k'} = \sum_{n=0}^{M-1}\left(\sum_{m=0}^{r-1} c_{n+mM}\right)e^{-i\frac{2\pi}{M}k'n}.$$

This indicates that the act of splitting and summing actually does achieve the purpose of picking out every r-th element!