Conformal Prediction in the Loop: Risk-Aware Control Barrier Functions for Stochastic Systems with Data-Driven State Estimators

Junhui Zhang, *Member, IEEE*, Bardh Hoxha, *Member, IEEE*, Georgios Fainekos, *Member, IEEE*, and Dimitra Panagou, *Senior Member, IEEE*

***Abstract*—This paper proposes a sampled-data, measurement-robust, risk-aware control barrier function (RA-CBF) framework for stochastic systems with measurement uncertainty. In this framework, what is available for control design are measurements of the system states, which are subject to unknown noise. First, in order to estimate the system states from these measurements, an offline-trained neural network is employed as a state estimator. Next, to quantify the performance of the state estimator, the state space is discretized, and calibration datasets are sampled from the grid points. Conformal prediction is then implemented, providing the estimation error bound with user-defined probability. In addition, we leverage the estimation error bound into sampled-data robust RA-CBF design, such that the probability that the state of the system enters the unsafe set during a finite time horizon is bounded by a desired threshold. Various case studies demonstrate the effectiveness of the proposed method.**

***Index Terms*—Stochastic differential equation, measurement uncertainty, neural network, conformal prediction, risk-aware control barrier function.**

I. Introduction

ENSURING safety is a critical area of research in autonomous systems, with applications such as collision avoidance in self-driving vehicles and robotics [1]. Control barrier functions (CBFs) have proven to be highly effective and have attracted widespread attention for the design of safe controllers [2]–[4]. CBFs have been applied primarily to deterministic systems, while recent work has also been extended to stochastic systems [5], [6]. However, CBF-based controllers rely on full state feedback. In real-world applications, the actual state of the system is usually not directly accessible and can only be perceived through noisy sensors. As noted in [7], a state-based controller may fail to guarantee performance when the true states are replaced by state estimates. Thus, it is of great significance to address the challenge of measurement-robust CBF controller design for stochastic systems.

There are some preliminary results related to the problems in the existing literature. In [8], a measurement-robust CBF controller is designed for deterministic systems, where the authors assume that the sensor map is invertible and free of noise. In [9], a robust CBF is proposed for stochastic systems, yielding risk-aware safety. The authors are using the Extended Kalman Filter (EKF) as the state estimator, which relies on the assumption of a known linear measurement sensor map and additive Gaussian disturbances. Moreover, under known bounds of disturbances on the system dynamics and sensor measurements, a safe and robust observer-controller is proposed in [7]. To overcome the limitations of model-based state estimators, we employ a neural network (NN) to estimate the state from noisy measurements. The NN enables handling unknown and complex sensor mappings directly from data, thereby accommodating a wider class of measurement noise. Conformal prediction, introduced in [10], is then used to quantify the uncertainty of the NN estimator in a distribution-free manner. Especially, in [11], [12], a neural network-based state estimator is combined with conformal prediction to quantify estimation errors over a discretized state grid, and an event-triggered controller is proposed for deterministic systems. This approach eliminates the need for explicit sensor models or noise assumptions and is applicable to various real-world scenarios. However, the authors in [11], [12] did not consider uncertainty in the system dynamics. Extending learning-based estimation with conformal prediction to stochastic systems introduces significant challenges due to the randomness in system evolution. As a result, the problem of designing measurement-robust control barrier functions under stochastic dynamics remains open and technically demanding.

On the other hand, especially for stochastic systems, it is common to design control strategies with a probability of constraint violation. When the probability is known and designable, this is referred to as risk-aware control [13], [14]. Considering a finite-time horizon, a risk-aware CBF is constructed based on a stochastic level-crossing technique [15], and has been shown to be less conservative than stochastic CBF in [5]. Note that the existing results in [5], [15] on risk-aware safe control for stochastic systems assume perfect state measurement. However, when noise is present in the measurements, these methods fail to offer theoretical risk-aware safety guarantees. Our aim is to address this limitation and develop a framework applicable to stochastic systems with measurements subject to unknown noise.

To this end, we propose a sampled-data, measurement-robust, risk-aware CBF for stochastic systems with mea-

Junhui Zhang is with the School of Robotics and Automation at Nanjing University, Suzhou, Jiangsu 215163, China and was with the Department of Robotics at University of Michigan, Ann Arbor, MI 48109, USA (e-mail: junhuiz@umich.edu)

Bardh Hoxha and Georgios Fainekos are with the Toyota North America Research & Development, 1555 Woodridge Ave, Ann Arbor, MI 48105, USA (e-mail: firstname.lastname@toyota.com).

Dimitra Panagou is with the Department of Robotics and the Department of Aerospace Engineering at University of Michigan, Ann Arbor, MI 48109, USA (e-mail: dpanagou@umich.edu).

surement uncertainty. By utilizing a learning-based (neural network) state estimator and conformal prediction, we first obtain probabilistic estimation error bounds at state-space grid points, and subsequently derive a probabilistic estimation error bound for any point within the state space. To deal with the stochastic system dynamics, we employ the Doob's martingale inequality [16] and further derive the estimation error bound between estimated and real system trajectories of stochastic systems over finite-time horizon with a desired probability. Moreover, we incorporate the estimation error into the risk-aware CBF [15] such that the probability that the system state enters the unsafe set during a finite-time horizon is bounded by a desired threshold. Compared to the existing results on safe control with specific measurement noises assumptions [7]–[9], our method does not require any assumptions on noise. Additionally, unlike the deterministic system considered in [11], our method extends to stochastic systems. The contributions are summarized as follows:

- We propose a sampled-data, risk-aware control barrier function (RA-CBF) framework for stochastic systems that is robust to measurements subject to unknown noise.
- We develop a learning-based state estimator for stochastic systems combined with conformal prediction, and derive an estimation error bound between the estimated and actual stochastic system trajectories over a finite-time horizon with a desired probability.
- We leverage the estimation error bound into a sampled-data risk-aware CBF framework, ensuring that the risk of unsafe behavior over a finite-time horizon is bounded by a desired threshold.

II. Preliminaries and Problem Formulation

Notations: The set of real numbers is denoted by $\mathbb{R}$. A^T is the transpose of the matrix A. $Tr(A)$ is the trace of matrix A. The Gauss error function is $erf(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt$ and its inverse is denoted by $erf^{-1}(\cdot)$. $\mathbb{P}(e)$ is the probability of an event e. e^c denotes the complement of event e. $L_fB(x)$ is the Lie derivative of B along f at x. A continuous function $\alpha : (-b, a) \to (-\infty, +\infty)$ is an extended class $\mathcal{K}$ function for $a, b \in \mathbb{R}^+$, denoted by $\alpha \in \mathcal{K}$, if it is strictly increasing and $\alpha(0) = 0$. $\|\cdot\|$ represents 2-norm.

In this work, we follow [9], [15] and define a stochastic system via the stochastic differential equation (SDE):

$$d(x(t)) = (f(x(t)) + g(x(t))u(t))dt + G(x(t))dw(t), \quad (1)$$

where $x(t) \in \mathcal{X} \subset \mathbb{R}^n$ and $u \in \mathcal{U} \subset \mathbb{R}^m$ are the state and control input of the system, respectively. $f : \mathcal{X} \to \mathbb{R}^n$ and $g : \mathcal{X} \to \mathbb{R}^{n\times m}$ are Lipschitz-continuous functions. $G : \mathcal{X} \to \mathbb{R}^{n\times q}$ is a continuous function. $w \in \mathbb{R}^q$ is a standard Wiener process, defined over the probability space $(\Omega, \mathcal{F}, \mathcal{P})$. We assume that we can not directly access the state vector, but only an output (measurement) vector defined as:

$$y(t) = p_s(x(t), \delta(x(t))), \quad (2)$$

where $y(t) \in \mathbb{R}^l$ is the measurement vector or system output, p_s is an unknown sensor map. $\delta(x(t))$ is the noise modeled by a random variable drawn from an unknown distribution D_δ.

A safe set $\mathcal{S}$ is defined by a twice-differentiable function $B : \mathbb{R}^n \to \mathbb{R}$ as: $\mathcal{S} = \{x \in \mathcal{X} : 0 \le B(x) < 1\}$.

Assumption 1. *The initial state of the system* (1) *is in the safety set, $x(t_0) \in \mathcal{S}$. There exists a known constant γ such that $B(x(t_0)) \le \gamma$.*

To guarantee safety with a CBF-based controller, we need to estimate the state $x(t)$ from the measurement $y(t)$. Thus, we consider the following problems.

Problem 1. *How to obtain an estimate $\hat{x}$ of the state x using the measurements y? How to quantify the estimation error $\tilde{x} = x - \hat{x}$ with conformal prediction?*

Consider a finite-time horizon $[0, T)$, we define p_u as the probability that the stochastic process $x(t)$ enters the unsafe set during the horizon, namely,

$$p_u = \mathbb{P}\{x(\tau) \notin \mathcal{S}, \text{for some } \tau \in [0\ T)\}. \quad (3)$$

Problem 2. *: How to design a CBF-based controller for stochastic system* (1) *that incorporates the state estimate $\hat{x}$ and the estimation error $\tilde{x}$ quantified by conformal prediction, so that the probability of the system state x entering the unsafe set during $[0, T)$ is bounded by a desired threshold ρ, that is, $p_u \le \rho$?*

III. Methodology

A. Neural Network State Estimator and Conformal Prediction

To solve **Problem 1**, we employ a neural network to learn an inverse mapping from noisy measurements to system states, thereby enabling the handling of general and unknown measurement noise. Conformal prediction, introduced in [10], is subsequently utilized to quantify the uncertainty of the learned estimator in a distribution-free manner, providing probabilistic guarantees on the estimation error.

State Estimator: To estimate the state from measurements, we use a neural network as the state estimator, i.e.,

$$\hat{x}(t) = NN(y(t)), \quad (4)$$

where $\hat{x}(t)$ is the state estimate from measurement $y(t)$. NN is a neural network that is trained offline using training datasets with measurement-state pairs.

Remark 1. *The proposed method assumes that the system is partially observable in the state space, such that the states critical for safety can be estimated. Full state observability is not required.*

Conformal prediction: We assume that the state space $\mathcal{X} \in \mathbb{R}^n$ is a compact set. Given $\epsilon > 0$, we construct a grid of $\mathcal{X}$ such that for each $x \in \mathcal{X}$, there exists a grid point x_j satisfying $\|x - x_j\| \le \epsilon$. The set of all grid points is denoted by $\bar{\mathcal{X}}$. We assume that we have measurement-state pairs for each $x_j \in \bar{\mathcal{X}}$ as the calibration dataset. To collect it, we can measure N times for each state $x_j \in \bar{\mathcal{X}}$ with sensor map (2) independently, and thus we obtain the calibration dataset $\{y_j^{(i)}, x_j\}, i = 1, 2, \cdots, N$, where $y_j^{(i)}$ is the ith measurement for grid point x_j. Then for each grid point $x_j \in \bar{\mathcal{X}}$ we can calculate the nonconformity scores, i.e., the estimation errors, wrt the points in the calibration dataset as

$$R_j^{(i)} = \|NN(y_j^{(i)}) - x_j\|, \ i \in \{1, \cdots, N\}. \tag{5}$$

Thus, for a new test measurement y_j at x_j, with a target failure probability $\alpha_p \in (0,1)$, the estimation error with (4) can be quantified by the $(1-\alpha_p)$th quantile of the sequence of $R_j^{(1)}, \cdots, R_j^{(N)}, \infty$, and calculated by:

$$\begin{aligned} R_j = &\text{the } \lceil (1-\alpha_p)(N+1) \rceil \text{th smallest value in} \\ &\{R_j^{(i)}\}_{i=1}^N \cup \{\infty\}. \end{aligned} \tag{6}$$

Thus, we have that $\mathbb{P}(\|NN(y_j) - x_j\| \leq R_j) \geq 1-\alpha_p$. Furthermore, if the sensor map p_s and the neural network have Lipschitz constants L_p and L_{NN}, respectively, from [11, Proposition 1], it holds that for each $x \in \mathcal{X}$ and its measurement y from (2),

$$\mathbb{P}(\|NN(y) - x\| \leq \sup_j R_j + (L_p L_{NN} + 1)\epsilon) \geq 1 - \alpha_p,$$

that is,

$$\mathbb{P}(\|\hat{x} - x\| \leq \sup_j R_j + (L_p L_{NN} + 1)\epsilon) \geq 1 - \alpha_p. \tag{7}$$

Next, we incorporate (7) into a risk-aware CBF (RA-CBF) design.

B. Sampled-data measurement-robust CBF design

Definition 1. *[17] The (infinitesimal) generator of stochastic process $x(t)$ is defined by*

$$\mathcal{L}B(x_0) = \lim_{t \downarrow 0} \frac{\mathbb{E}[B(x(t))|x(0) = x_0] - B(x_0)}{t} \tag{8}$$

for functions $B : \mathbb{R}^n \to \mathbb{R}$, such that the limit exists for all $x_0 \in \mathbb{R}^n$.

For a stochastic process $x(t)$ satisfying (1), the generator of a twice continuously differentiable function B is [17]

$$\mathcal{L}B(x) = L_f B(x) + L_g B(x) u + \frac{1}{2} Tr(G(x)^T \frac{\partial^2 B}{\partial x^2} G(x)).$$

Assumption 2. *The functions $F(x,u)$, $L_f B(x)$, $L_G B(x)$, $G(x)$ and $q(x)$ are bounded with known bounds $\bar{F}$, ϵ_f, η, M and ϵ_q, where $F(x,u) = f(x) + g(x)u$, and $q(x) = Tr(G^T(x) \frac{\partial^2 B}{\partial x^2} G(x))$.*

Assumption 3. *The functions $L_f B(x)$, $L_g B(x)$ and $q(x)$ have Lipchitz constants $\mathcal{L}_{L_f B}$, $\mathcal{L}_{L_g B}$ and $\mathcal{L}_q$.*

Consider the time interval $[0,T)$ with sampling time instants $t_i = i\Delta t$, $i = 0,1,2,\cdots,r-1$, where $(r-1)\Delta t \leq T \leq r\Delta t$. The estimation error bound for the stochastic state trajectories is then provided by the following lemma,

Lemma 1. *Given assumption 2, consider the sampling time instants $\{t_i\}$, then*

$$\begin{aligned} &\mathbb{P}(\|x(t) - \hat{x}(t_i)\| \leq \beta, \forall t \in [t_i, t_{i+1}), \forall i \in \{0,1,\cdots,r-1\}) \\ &\quad \geq (1-\alpha_p)^r (1 - \frac{M^2 \Delta t}{\lambda^2})^r, \end{aligned}$$

where $\beta = \bar{F}\Delta t + \sup_j R_j + (L_p L_{NN} + 1)\epsilon + \lambda$, and $\lambda > 0$ is a design parameter.

Proof: Recall $d(x(t)) = (f(x(t) + g(x(t))u(t))dt + G(x(t))dw(t)$. By integrating the above SDE, one has

$$x(t) = x(t_i) + \int_{t_i}^t F(x(s), u(s)) ds + \int_{t_i}^t G(x(t)) dw(t). \tag{9}$$

Then, for any $t \in [t_i, t_{i+1})$,

$$\begin{aligned} \|x(t) - \hat{x}(t_i)\| \leq &\| \int_{t_i}^t F(x(s), u(s)) ds\| + \|x(t_i) - \hat{x}(t_i)\| \\ &+ \sup_{t_i \leq t \leq t_{i+1}} \| \int_{t_i}^t G(x(s)) dw(s)\|. \end{aligned} \tag{10}$$

It holds that

$$\| \int_{t_i}^t F(x(s), u(s)) ds\| \leq \bar{F}(t_{i+1} - t_i) = \bar{F}\Delta t, \tag{11}$$

and it follows from (7) that the following holds with probability $1-\alpha_p$

$$\|x(t_i) - \hat{x}(t_i)\| \leq \sup_j R_j + (L_p L_{NN} + 1)\epsilon. \tag{12}$$

By use of Doob's martingale inequality [16, Theorem 1.7], one has

$$\begin{aligned} &\mathbb{P}(\sup_{t_i \leq t \leq t_{i+1}} \| \int_{t_i}^t G(x(s)) dw(s)\| \geq \lambda) \\ &\leq \frac{\mathbb{E}[\| \int_{t_i}^{t_{i+1}} G(x(s)) dw(s)\|^2]}{\lambda^2}, \end{aligned} \tag{13}$$

on the other hand, applying Itô isometry and $\|G(x(t))\| \leq M$, we obtain

$$\begin{aligned} &\mathbb{E}[\| \int_{t_i}^{t_{i+1}} G(x(s)) dw(s)\|^2] = \mathbb{E}[\int_{t_i}^{t_{i+1}} \|G(x(s))\|^2 ds] \\ &\leq M^2(t_{i+1} - t_i) = M^2 \Delta t. \end{aligned} \tag{14}$$

Substituting (14) into (13), one has

$$\mathbb{P}(\sup_{t_i \leq t \leq t_{i+1}} \| \int_{t_i}^t G(x(s)) dw(s)\| \geq \lambda) \leq \frac{M^2 \Delta t}{\lambda^2}.$$

Using the complement of an event, one further has

$$\mathbb{P}(\sup_{t_i \leq t \leq t_{i+1}} \| \int_{t_i}^t G(x(s)) dw(s)\| < \lambda) \geq 1 - \frac{M^2 \Delta t}{\lambda^2}. \tag{15}$$

Substituting (11),(12) and (15) into (10), it holds that

$$\mathbb{P}(\|x(t) - \hat{x}(t_i)\| \leq \beta, \forall t \in [t_i, t_{i+1})) \geq (1-\alpha_p)(1 - \frac{M^2 \Delta t}{\lambda^2})$$

where $\beta = \bar{F}\Delta t + \sup_j R_j + (L_p L_{NN} + 1)\epsilon + \lambda$. Then it holds

$$\begin{aligned} &\mathbb{P}(\|x(t) - \hat{x}(t_i)\| \leq \beta, \forall t \in [t_i, t_{i+1}), \forall i \in \{0,1,\cdots,r-1\}) \\ &\geq (1-\alpha_p)^r (1 - \frac{M^2 \Delta t}{\lambda^2})^r. \end{aligned} \tag{16}$$

The Proof is then completed. ■

Using the estimation error between the estimated and actual stochastic system trajectories provided in Lemma 1, we incorporate it into the RA-CBF framework and propose sampled-data, measurement-robust, risk-aware CBF.

Theorem 1. ***(Sampled-data measurement-robust risk-aware CBF):*** *Given Assumptions 1 to 3, the function $B(x)$ is a sampled-data measurement-robust risk-aware control barrier function if there exist a $\alpha > 0$, a designed parameter $\rho_d \in [1-erf(\frac{1-\gamma}{\sqrt{2}\eta T}), (1-\alpha_p)^r(1-\frac{M^2\Delta t}{\lambda^2})^r]$ such that zero-order hold (ZOH) control for stochastic system* (1) *satisfies the following condition at all sampling time instants $t_i, i = 0, 1, 2, \cdots, r-1$ on $[0, T)$,*

$$
\begin{aligned}
&L_f B(\hat{x}(t_i)) + (\mu_a(\hat{x}(t_i)) + L_g B(\hat{x}(t_i)))u(t_i) + \frac{1}{2}q(\hat{x}(t_i)) \\
&+ \mu_b(\hat{x}(t_i))\|u(t_i)\| + \mu_c(\hat{x}(t_i)) \\
&\leq \begin{cases} \alpha\varepsilon, & \text{if } i = 0, \\ \alpha(\varepsilon - \sum_{k=0}^{i-1}\xi_k\Delta t), & \text{if } i = 1, 2, \ldots, r-1, \end{cases}
\end{aligned} \tag{17}
$$

where

$$
\begin{aligned}
\mu_a(\hat{x}(t_i)) =& \alpha\Delta t L_g B(\hat{x}(t_i)), \mu_b = (\alpha\Delta t + 1)\mathcal{L}_{L_g B}\beta, \\
\mu_c(\hat{x}(t_i)) =& \alpha\Delta t(L_f B(\hat{x}(t_i)) + \mathcal{L}_{L_f B}\beta + \frac{1}{2}q(\hat{x}(t_i)) \\
&+ \frac{1}{2}\mathcal{L}_q\beta) + (\mathcal{L}_{L_f B} + \frac{1}{2}\mathcal{L}_q)\beta, \\
\varepsilon =& 1 - \gamma - (\sqrt{2}\eta T)erf^{-1}(1-\rho_d), \\
\xi_k =& \max_{x\in\delta(\hat{x}(t_k))} L_f B(x) + L_g B(x)u(t_k) + \frac{1}{2}q(x),
\end{aligned}
$$

$\quad with\ \delta(\hat{x}(t_k)) = \{x : \|x - \hat{x}(t_k)\| \leq \beta\}$.

Then the probability that the stochastic process $x(t)$ enters the unsafe set during the horizon $[0, T)$ is bounded by

$$
\rho = \rho_d + 1 - (1-\alpha_p)^r(1 - \frac{M^2\Delta t}{\lambda^2})^r.
$$

Proof: Let

$$
\begin{aligned}
&c(\hat{x}(t_i), u(t_i)) = L_f B(\hat{x}(t_i)) + L_g B(\hat{x}(t_i))u(t_i) \\
&+ \frac{1}{2}q(\hat{x}(t_i)) - \alpha(\varepsilon - I_L(t_i)),
\end{aligned}
$$

and for any $t \in [t_i, t_{i+1})$,

$$
\begin{aligned}
&c(x(t), u(t)) = L_f B(x(t)) + L_g B(x(t))u(t) \\
&+ \frac{1}{2}q(x(t)) - \alpha(\varepsilon - I_L(t)),
\end{aligned}
$$

where $I_L(t) = \int_0^t \mathcal{L}B(x(s), u(s))ds$. If $\|x(t) - \hat{x}(t_i)\| \leq \beta, \forall t \in [t_i, t_{i+1}), \forall i \in \{0, 1, \cdots, r-1\}$, then we have

$$
\begin{aligned}
&c(x(t), u(t)) - c(\hat{x}(t_i), u(t_i)) \\
\leq& \mathcal{L}_{L_f B}\|x(t) - \hat{x}(t_i)\| + \mathcal{L}_{L_g B}\|u(t_i)\|\|\hat{x}(t_i) - x(t)\| \\
&+ \frac{1}{2}\mathcal{L}_q\|x(t) - \hat{x}(t_i)\| - \alpha(-I_L(t) + I_L(t_i)) \\
\leq& (\mathcal{L}_{L_f B} + \mathcal{L}_{L_g B}\|u(t_i)\| + \frac{1}{2}\mathcal{L}_q)\beta + \alpha\Delta t(L_f B(\hat{x}(t_i)) \\
&+ \mathcal{L}_{L_f B}\beta + L_g B(\hat{x}(t_i))u(t_i) + \mathcal{L}_{L_g B}\|u(t_i)\|\beta \\
&+ \frac{1}{2}q(\hat{x}(t_i)) + \frac{1}{2}\mathcal{L}_q\beta) \\
\leq& \mu_a(\hat{x}(t_i))u(t_i) + \mu_b\|u(t_i)\| + \mu_c(\hat{x}(t_i)).
\end{aligned}
$$

Then, one has

$$
\begin{aligned}
c(x(t), u(t)) \leq& c(\hat{x}(t_i), u(t_i)) + \mu_a(\hat{x}(t_i))u(t_i) + \mu_b\|u(t_i)\| \\
&+ \mu_c(\hat{x}(t_i)).
\end{aligned}
$$

Thus, under the control satisfying (17), if $\|x(t) - \hat{x}(t_i)\| \leq \beta, \forall t \in [t_i, t_{i+1}), \forall i \in \{0, 1, \cdots, r-1\}$, then we have

$$
c(x(t), u(t)) \leq 0,\ \forall t \in [0, T).
$$

Combing with Lemma1, it holds that

$$
\mathbb{P}(c(x(t), u(t)) \leq 0, \forall t \in [0, T)) \geq (1-\alpha_p)^r(1 - \frac{M^2\Delta t}{\lambda^2})^r. \tag{18}
$$

$c(x(t), u(t)) \leq 0, \forall t \in [0, T)$ implies the risk-aware CBF condition in [15, Definition 5] is satisfied, which is denoted by event e_1. The system state enters unsafe set during $[0, T)$ is denoted by event e_2. Thus, the (18) is re-written as

$$
\mathbb{P}(e_1) \geq (1-\alpha_p)^r(1 - \frac{M^2\Delta t}{\lambda^2})^r.
$$

and from [15, Theorem 1], one has $\mathbb{P}(e_2|e_1) \leq \rho_d$. Then, using the total probability theorem [18, section 1.4],

$$
\begin{aligned}
\mathbb{P}(e_2) =& \mathbb{P}(e_2|e_1)\mathbb{P}(e_1) + \mathbb{P}(e_2|e_1^c)\mathbb{P}(e_1^c) \\
\leq& \rho_d + (1 - (1-\alpha_p)^r(1 - \frac{M^2\Delta t}{\lambda^2})^r).
\end{aligned}
$$

The Proof is completed. ■

Remark 2. *In practice, a candidate CBF $B(x)$ for the condition* (17) *is constructed based on the desired safety specifications, where the safe set is characterized by $0 \leq B(x) < 1$. The parameter $\alpha > 0$ involved in condition* (17) *is typically tuned to ensure that the constraint* (17) *remains feasible while avoiding excessive conservativeness. The parameter ρ_d is specified to set the target risk level, such that the actual risk level of system ρ under the resulting control meets the desired safety requirements.*

Remark 3. *The state-space discretization required for conformal prediction increases offline computational cost as the state dimension grows. Although this step is performed offline, scalability remains a concern for high-dimensional systems. Future work will investigate randomized sampling and dimensionality reduction to mitigate this issue.*

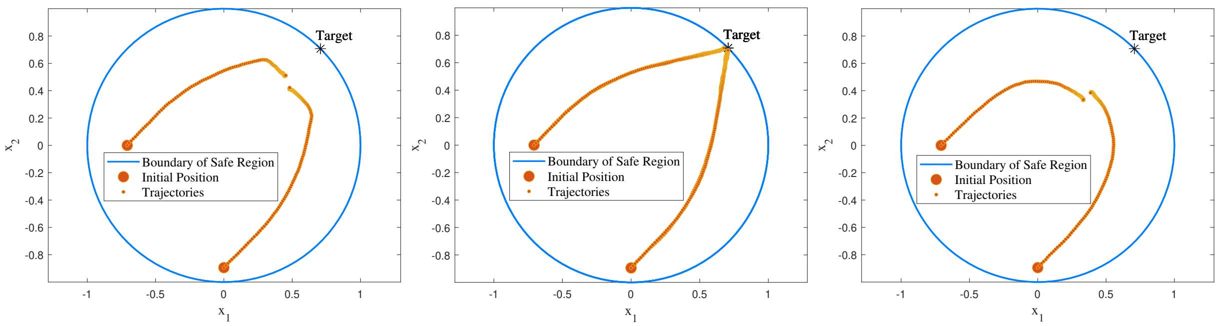
IV. Case Study

In this section, we verify the effectiveness of our proposed sampled-data measurement-robust risk-aware CBF controller through two examples: robot visitation and obstacle avoidance.

1) Robot Visitation: We consider a mobile robot which is modeled by a 2-D single-integrator stochastic model below,

$$
\begin{aligned}
dx_1 &= v_1 dt + G_1 dw, \\
dx_2 &= v_2 dt + G_2 dw,
\end{aligned}
$$

where $\boldsymbol{x} = [x_1, x_2]^T \in \mathbb{R}^2$ is the system state, which represents position of the robot, $\boldsymbol{u} = [v_1, v_2]^T \in \mathbb{R}^2$ is the control input, which consists of velocities. The sensor measures the state with $\boldsymbol{y} = \boldsymbol{x} + \delta$, where $\boldsymbol{y}$ is the measurement vector and δ represents the measurement noise, which follows an exponential distribution with a mean of 1/200. The mobile robot aims to reach the target $x_g = [1/\sqrt{2}, 1/\sqrt{2}]^T$, while remaining within a safe set defined by a circular region with



(a) Ours (b) RA-CBF (Perfect measurement) (c) S-CBF (Perfect measurement)

Fig. 1: The trajectories under our method, compared to the RA-CBF and S-CBF: the mobile robot moves starting from initial positions and aims to reach the target, while remaining within a safe set marked by the blue circular region.

radius $R = 1$, centered on $s_o = [0, 0]^T$. In this case, we employ a three-layer feedforward neural network as the state estimator, and use 1.02×10^6 state-measurement pairs to train the neural network offline with the MATLAB R2023b Deep Learning Toolbox. To quantify the estimation error of the NN, we discretize the state space as a grid over the range $[-1, 1] \times [-1, 1]$ and the distance between every two adjacent sampling grid points, either horizontally or vertically, is 0.02.

We measure 1000 times for each grid point independently and obtain state-measurement pairs as calibration datasets. We then set the failure probability α_p as 0.005, and implement conformal prediction at each grid point, to eventually get $\mathbb{P}(\|\hat{x} - x\| \leq 0.068) \geq 0.995$. To test our proposed sampled-data measurement-robust risk-aware CBF incorporating the estimation error quantified by conformal prediction, we consider the time horizon $[0, 0.6)$ and set the sampling time interval as $0.01s$. We construct $B(\boldsymbol{x}) = x_1^2 + x_2^2$ such that the safe region is characterized by $0 \leq B(\boldsymbol{x}) < 1$, employ $-k[\hat{x} - s_g]$ with $k = 10$, as nominal control and construct an optimization-based controller to minimize the difference between control signal and nominal control, while satisfying the constraint (17). We evaluate our method with two cases: (i) initial position $[-1/\sqrt{2}, 0]^T$ and risk $\rho = 0.5$; and (ii) initial position $[0, -2/\sqrt{5}]^T$ and risk $\rho = 0.8$, and compare with the following existing methods: risk-aware CBF (RA-CBF) in [15], and stochastic CBF (S-CBF) in [5]. The comparative trajectories are shown in Fig 1. In addition, we test 100 trajectories for each case and conclude the results in the table.

TABLE I: Comparison of Average Stopping Distances to x_g for Different Methods

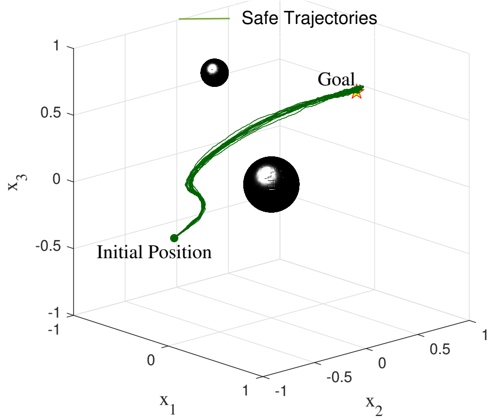
Cases	Methods	Measurement	Average Stopping Distance to x_g	Observed p_u
Case(i)	Ours	Noisy	0.3378	0.00
	RA-CBF	Perfect	0.0087	0.00
	S-CBF	Perfect	0.5239	0.00
	RA-CBF	Noisy	—	0.91
Case(ii)	Ours	Noisy	0.3733	0.00
	RA-CBF	Perfect	0.0113	0.00
	S-CBF	Perfect	0.4388	0.00
	RA-CBF	Noisy	—	0.98

From Fig. 1, we observe that the trajectories under our method and the S-CBF (perfect measurement) converge within some different distances to the target, which lies on the boundary of the circle. The trajectories under the RA-CBF (perfect measurement) consistently move towards the target. From TABLE I, we compare the average stopping distances to x_g, and the RA-CBF (perfect measurement) achieves the shortest distance. Note that the RA-CBF and S-CBF are using perfect measurements, whereas our method considers noisy measurement and NN-based state estimator. Hence, the quantification of the estimation error using conformal prediction and the robust design in our method introduce some level of conservativeness compared to the RA-CBF, which considers perfect measurement. However, notably our method is still less conservative than the S-CBF since the robot reaches closer to x_g under noisy measurements, compared to the S-CBF with perfect measurement. Moreover, we observe that the observed p_u exceeds the designed risk bound when RA-CBF is used to handle noisy measurements. This implies that RA-CBF fails to provide risk-aware safety guarantees in the presence of measurement noise. These results demonstrate the effectiveness of our method.

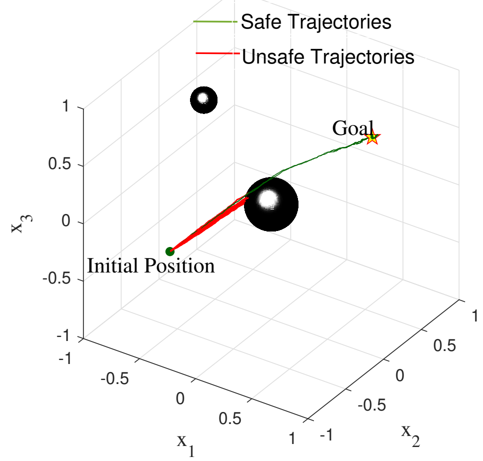
2) Obstacle Avoidance: We consider a robot with 3-D nonlinear stochastic model,

$$dx_1 = 0.2\tanh(x_1) + u_1 dt + G_1 dw,$$
$$dx_2 = 0.2\tanh(x_2) + u_2 dt + G_2 dw,$$
$$dx_3 = 0.2\tanh(x_3) + u_3 dt + G_3 dw,$$

where $\boldsymbol{x} = [x_1, x_2, x_3]^T \in \mathbb{R}^3$ is the system state. $\boldsymbol{u} = [u_1, u_2, u_3]^T \in \mathbb{R}^3$ is the control input vector. The sensor measures the state with $\boldsymbol{y} = \boldsymbol{x} + \delta$, where $\boldsymbol{y}$ is the measurement and δ is a gaussian noise. The robot aims to move from its initial position, $[-0.7, -0.3, -0.5]^T$ to the goal position $\boldsymbol{x}_g : [0.9, 0, 0.9]^T$. Along the way, it must avoid collision with two obstacles: obstacle 1: a sphere with radius $r_1 = 0.2$ and centered at $[0, 0, 0]^T$; obstacle 2: a sphere with radius $r_2 = 0.1$ and centered at $[-0.6, 0, 0.7]^T$. We use a feedforward neural network as the state estimator, discretize the 3D state space, and implement conformal prediction to quantify the estimation error. Then we get $\mathbb{P}(\|\hat{\boldsymbol{x}} - \boldsymbol{x}\| \leq 0.1653) \geq 0.995$. We consider a finite-time horizon $[0, 0.8s)$ and the sampling time interval $\Delta t = 0.01s$. We construct $B_1(\boldsymbol{x}) = e^{-c(x_1^2 + x_2^2 + x_3^2 - r_1^2)}$ and $B_2(\boldsymbol{x}) = e^{-c((x_1+0.6)^2 + x_2^2 + (x_3-0.7)^2 - r_2^2)}$ with $c = 0.8$ w.r.t. the safe region avoiding collision with two obstacles, respectively. We use $-k[\hat{\boldsymbol{x}} - \boldsymbol{x}_g]$ with $k = 10$, as the nominal control, and the two constraints derived from (17) as the safe filter. We set the risk bounds for collision with both obstacles to 0.6 and compare our method with a baseline where conformal prediction is not used to quantify estimation error. That is, in the baseline, the state estimate from the neural network is used directly in the risk-aware CBF. The comparative results are depicted in Fig. 2.



(a) Ours



(b) Baseline: a risk-aware CBF integrated with a NN state estimator without conformal prediction

Fig. 2: Comparison of trajectories under our method and baseline: the mobile robot moves starting from the initial position and aims to reach the goal, while avoiding collision with obstacles.

From Fig. 2, we observe that the robot using our method consistently moves safely toward the goal, while most trajectories of the robot using the baseline method result in collisions with obstacles. In addition, we test 100 trajectories for the two methods. The robot under our method can always avoid collision with obstacles. However, the robot using the baseline method collides with the obstacles 94 times out of 100, resulting in an unsafe probability of 94% (which exceeds the set risk bound of 0.6). This demonstrates that the uncertainty in the neural network estimation significantly increases the risk of unsafe outcomes. These results highlight the importance of quantifying estimation error using conformal prediction and incorporating robust design, as done in our method [1].

[1]Codes: https://github.com/MrJUNHUIZHANG/Robust_Risk-aware-CBF.

V. Conclusions

This paper presented a sampled-data measurement-robust risk-aware CBF for stochastic systems. With only sensor measurement, a neural network is utilized as the state estimator, and conformal prediction is employed to quantify its estimation error. Subsequently, a sampled-data measurement-robust risk-aware CBF is proposed by incorporating the estimation error, so that the probability that the system state enters the unsafe set during a finite-time horizon is bounded by a desired threshold. Finally, two case studies are provided to demonstrate the effectiveness of the proposed method.

References

[1] J. Zhang, S. Z. Yong, and D. Panagou, "Safety-critical control with offline-online neural network inference," *arXiv preprint arXiv:2408.00918, to appear in ACC 2025*, 2024.

[2] A. D. Ames, X. Xu, J. W. Grizzle, and P. Tabuada, "Control barrier function based quadratic programs for safety critical systems," *IEEE Transactions on Automatic Control*, vol. 62, no. 8, pp. 3861–3876, 2016.

[3] K. Garg, J. Usevitch, J. Breeden, M. Black, D. Agrawal, H. Parwana, and D. Panagou, "Advances in the theory of control barrier functions: Addressing practical challenges in safe control synthesis for autonomous and robotic systems," *Annual Reviews in Control*, vol. 57, 2024.

[4] J. Zhang, L. Ding, X. Lu, and W. Tang, "A novel real-time control approach for sparse and safe frequency regulation in inverter intensive microgrids," *IEEE Transactions on Industry Applications*, vol. 59, no. 5, pp. 5550 – 5558, 2023.

[5] C. Santoyo, M. Dutreix, and S. Coogan, "A barrier function approach to finite-time stochastic system verification and control," *Automatica*, vol. 125, p. 109439, 2021.

[6] O. So, A. Clark, and C. Fan, "Almost-sure safety guarantees of stochastic zero-control barrier functions do not hold," *arXiv preprint arXiv:2312.02430*, 2023.

[7] D. R. Agrawal and D. Panagou, "Safe and robust observer-controller synthesis using control barrier functions," *IEEE Control Systems Letters*, vol. 7, pp. 127–132, 2022.

[8] S. Dean, A. Taylor, R. Cosner, B. Recht, and A. Ames, "Guaranteeing safety of learned perception modules via measurement-robust control barrier functions," in *Conference on Robot Learning*. PMLR, 2021, pp. 654–670.

[9] S. Yaghoubi, G. Fainekos, T. Yamaguchi, D. Prokhorov, and B. Hoxha, "Risk-bounded control with kalman filtering and stochastic barrier functions," in *2021 60th IEEE Conference on Decision and Control (CDC)*. IEEE, 2021, pp. 5213–5219.

[10] A. N. Angelopoulos, S. Bates *et al.*, "Conformal prediction: A gentle introduction," *Foundations and Trends® in Machine Learning*, vol. 16, no. 4, pp. 494–591, 2023.

[11] S. Yang, G. J. Pappas, R. Mangharam, and L. Lindemann, "Safe perception-based control under stochastic sensor uncertainty using conformal prediction," in *2023 62nd IEEE Conference on Decision and Control (CDC)*. IEEE, 2023, pp. 6072–6078.

[12] L. Lindemann, Y. Zhao, X. Yu, G. J. Pappas, and J. V. Deshmukh, "Formal verification and control with conformal prediction," *arXiv preprint arXiv:2409.00536*, 2024.

[13] P. Akella, A. Dixit, M. Ahmadi, L. Lindemann, M. P. Chapman, G. J. Pappas, A. D. Ames, and J. W. Burdick, "Risk-aware robotics: Tail risk measures in planning, control, and verification," *arXiv preprint arXiv:2403.18972*, 2024.

[14] M. Black, G. Fainekos, B. Hoxha, and D. Panagou, "Risk-aware fixed-time stabilization of stochastic systems under measurement uncertainty," in *2024 American Control Conference (ACC)*, 2024, pp. 3276–3283.

[15] M. Black, G. Fainekos, B. Hoxha, D. Prokhorov, and D. Panagou, "Safety under uncertainty: Tight bounds with risk-aware control barrier functions," in *2023 IEEE International Conference on Robotics and Automation (ICRA)*. IEEE, 2023, pp. 12 686–12 692.

[16] D. Revuz and M. Yor, *Continuous martingales and Brownian motion*. Springer Science & Business Media, 2013, vol. 293.

[17] S. Prajna, A. Jadbabaie, and G. J. Pappas, "A framework for worst-case and stochastic safety verification using barrier certificates," *IEEE Transactions on Automatic Control*, vol. 52, no. 8, pp. 1415–1428, 2007.

[18] D. Bertsekas and J. N. Tsitsiklis, *Introduction to probability, section Edition*. Athena Scientific, Belmont, Massachusetts, 2008.