

Modeling, Simulation and Control

Finn Aakre Haugen

20 January 2022

Contents

I CONTROL CONCEPTS	27
1 Introduction to automatic control	28
1.1 Why automatic control?	28
1.1.1 What is the problem?	28
1.1.2 The solution!	29
1.1.3 What can be obtained with control?	30
1.2 The principle of feedback control	32
1.2.1 Manual feedback control	32
1.2.2 Automatic feedback control with technical components	35
1.2.3 Documentation with Piping & Instrumentation Diagram – P&ID	36
1.2.4 Documentation with block diagram	37
1.2.5 Control application: Temperature control of a liquid tank	38
1.2.6 Control application: Level control of a tank with conveyor belt	40
1.2.7 Control application: Averaging level control of buffer tanks	41
1.2.8 Control application: Position control of a ship (dynamic positioning) .	44
1.3 The principle of feedforward control	46
1.4 <i>Problems for Chapter 1</i>	50
1.5 <i>Solutions to problems for Chapter 1</i>	53
2 Piping & Instrumentation Diagrams	55

2.1	Instrument codes	55
2.1.1	Letter codes	55
2.1.2	Numeric codes	57
2.1.3	Localization	58
2.2	Signals	58
2.3	Material flows	58
2.4	Process equipment	59
2.4.1	Codes of process equipment	59
2.4.2	Valves	60
2.4.3	Rotational flow components	61
2.4.4	Heat exchangers	61
2.4.5	Vessels	61
2.4.6	Mathematical functions	61
2.4.7	Logical functions	62
2.5	<i>Problems for Chapter 2</i>	63
2.6	<i>Solutions to problems for Chapter 2</i>	64
3	Components of control systems	66
3.1	Introduction	66
3.2	Controllers	66
3.2.1	Introduction	66
3.2.2	Process controllers	66
3.2.3	Programmable logic controllers (PLCs)	66
3.2.4	Programmable automation controllers (PACs)	66
3.2.5	Plantwide control systems	66
3.2.6	Simple PC-based control systems	66
3.3	Actuators	66

3.3.1	Introduction	67
3.3.2	Valves	67
3.3.3	Pumps	67
3.3.4	Motors	67
3.3.5	Heaters	67
3.3.6	Signal conditioning of control signals	67
3.3.6.1	Scaling	67
3.3.6.2	Digital-analog (DA) conversion	67
3.3.6.3	Pulse-width modulation	67
3.3.6.4	Current to voltage conversion	67
3.4	Sensors	67
3.4.1	Introduction	68
3.4.2	Temperature sensors	68
3.4.3	Level sensors	68
3.4.4	Pressure sensors	68
3.4.5	Flow sensors	68
3.4.6	Position sensors	68
3.4.7	Speed sensors	68
3.4.8	Signal conditioning of measurement signals	68
3.4.8.1	Alternative forms of measurement signals	68
3.4.8.1.1	Current signals	68
3.4.8.1.2	Voltage signals	68
3.4.8.1.3	Current loop	68
3.4.8.1.4	Pneumatic signals	68
3.4.8.1.5	Digital signals and analog-digital (AD) conversion	68
3.4.8.2	Scaling	69
3.4.8.3	Accuracy of measurement signals	69

3.4.8.4	Adjustment of sensors	69
3.4.8.5	Calibration of sensors	69
3.4.8.6	Measurement filters	69
3.4.8.6.1	Introduction	69
3.4.8.6.2	Continuous-time filter function	70
3.4.8.6.3	Analog RC circuit filter	70
3.4.8.6.4	Discrete-time filter algorithm	70
3.4.8.6.5	Moving averaging measurement filter	71
3.4.8.6.5.1	Batch MA filter algorithm	71
3.4.8.6.5.2	Recursive (online) MA filter algorithm	71
3.4.8.6.6	Comparing moving average filter with time constant filter	71
3.5	<i>Problems for Chapter 3</i>	71
3.6	<i>Solutions to problems for Chapter 3</i>	71
II	MODELING and SIMULATION OF DYNAMIC SYSTEMS	72
4	Mechanistic modeling	73
4.1	Introduction	73
4.2	What is a dynamic system?	73
4.3	A procedure for mathematical modeling	75
4.4	Mathematical modeling of material systems	77
4.5	Mathematical modeling of thermal systems	78
4.6	Mathematical modeling of kinetic systems	81
4.6.1	Systems with linear motion	81
4.6.2	Systems with rotational motion	83
4.6.2.1	Momentum balance	83
4.6.2.2	Relations between rotational and linear motion	84

4.6.2.3	Coupled mechanical systems	84
4.7	Mathematical modeling of electrical systems	86
4.7.1	Kirchhoff's law	86
4.7.1.1	Kirchhoff's Current Law	86
4.7.1.2	Kirchhoff's Voltage Law	87
4.7.2	Resulting resistance	87
4.7.2.1	Resistors in series	87
4.7.2.2	Resistors in parallel	87
4.7.3	Models of resistor, capacitor, and inductor	88
4.7.4	Power	90
4.7.4.1	Instantaneous power	90
4.7.4.2	Mean power	90
4.8	Physical component based simulators	90
4.8.1	OpenModelica	90
4.8.2	Aspentech Hysys	90
4.8.3	Simscape	90
4.9	<i>Problems for Chapter 4</i>	91
4.10	<i>Solutions to problems for Chapter 4</i>	97
5	State space models	102
5.1	Introduction	102
5.2	The state space model	102
5.3	The response of a state space model	105
5.3.1	Dynamic response	105
5.3.2	Static response	106
5.4	Linear state space models	107
5.4.1	Standard model form of linear state space models	107

5.4.2	Linearization of non-linear models	108
5.4.2.1	When do we have to linearize?	108
5.4.2.2	Deriving the linearization formulas	109
5.5	Simulation tools for state space models (differential equation solvers)	112
5.5.1	Introduction	112
5.5.2	Python	112
5.5.3	Matlab	112
5.5.4	LabVIEW	112
5.6	<i>Problems for Chapter 5</i>	113
5.7	<i>Solutions to problems for Chapter 5</i>	115
6	Simulation algorithms of state space models	117
6.1	Why simulate?	117
6.2	Simulation algorithms	118
6.2.1	Introduction	118
6.2.2	Simulation algorithm for state space models	118
6.2.2.1	Introduction	118
6.2.2.2	The simulation algorithm	119
6.2.2.3	How to choose the simulation time step, T_s ?	122
6.2.2.4	Implementation of the simulator in Python	126
6.2.2.5	Implementation of the simulator in LabVIEW	129
6.2.2.6	Why predict?	131
6.2.2.7	Euler Forward vs. Euler Backward	131
6.2.2.8	Simulation of second order differential equation models	132
6.2.3	Simulation algorithm of time delays	136
6.3	<i>Problems for Chapter 6</i>	141
6.4	<i>Solutions to problems for Chapter 6</i>	143

7 Block diagram models and simulators	157
7.1 Introduction	157
7.2 How to draw block diagrams	157
7.3 Simulation tools for block diagrams	161
7.3.1 Introduction	161
7.3.2 OpenModelica	162
7.3.3 Simulink	162
7.3.4 LabVIEW	162
7.4 <i>Problems for Chapter 7</i>	163
7.5 <i>Solutions to problems for Chapter 7</i>	163
8 The Laplace transform	166
8.1 Introduction	166
8.2 Definition of the Laplace transform	166
8.3 Laplace transform pairs	168
8.4 Laplace transform properties	168
8.5 <i>Problems for Chapter 8</i>	171
8.6 <i>Solutions to problems for Chapter 8</i>	172
9 Transfer functions	175
9.1 Introduction	175
9.2 Definition of the transfer function	176
9.3 Characteristics of transfer functions	178
9.4 Combining transfer functions blocks in block diagrams	178
9.5 How to calculate responses from transfer function models	178
9.6 Static transfer function and static response	180
9.7 From transfer function to differential equation	181
9.8 From transfer function to state space model	182

9.9	From state space model to transfer function	183
9.10	<i>Problems for Chapter 9</i>	185
9.11	<i>Solutions to problems for Chapter 9</i>	188
10	Process dynamics	191
10.1	Introduction	191
10.2	Integrators	191
10.3	Time-constant systems	193
10.4	Second order systems	196
10.4.1	Transfer function model	196
10.4.2	Classification of second order systems	197
10.4.2.1	Overdamped systems	198
10.4.2.2	Underdamped system	199
10.4.2.3	Undamped system	201
10.5	Time delays	205
10.5.1	Approximation of time delay by Padé approximation	206
10.6	Higher order systems	206
10.7	<i>Problems for Chapter 10</i>	208
10.8	<i>Solutions to problems for Chapter 10</i>	211
11	Adaptation of models to data	215
11.1	Introduction	215
11.2	Model adaptation as an optimization problem	216
11.2.1	How to find the best model	216
11.2.2	Good excitation is necessary!	218
11.3	Adaptation of static models to data	219
11.3.1	Adaptation using grid optimization	219
11.3.1.1	Introduction	219

11.3.1.2	Model adaptation of static models using native grid optimization	219
11.3.1.3	Model adaptation of static models with Python's brute() function	225
11.3.2	Adaptation of static models using nonlinear programming (NLP)	229
11.3.3	Adaptation of static models using standard least squares method	231
11.3.3.1	The standard regression model	231
11.3.3.2	The LS problem	232
11.3.3.3	The LS solution	233
11.3.3.4	Properties of the LS estimate	237
11.3.3.5	Criterion for convergence of estimate towards the true value	237
11.4	Adaptation of dynamic models to data	238
11.4.1	Adaptation of dynamic models using grid optimization	238
11.4.1.1	Introduction	238
11.4.1.2	Adaptation of dynamic models using grid optimization	238
11.4.1.3	Model adaptation of dynamic models with Python's brute() function	243
11.4.2	Adaptation of dynamic models using nonlinear programming (NLP)	246
11.4.3	Adaptation of dynamic models using the least squares method	249
11.5	Recursive (real-time) model adaptation	250
11.6	<i>Problems for Chapter 11</i>	252
11.7	<i>Solutions to problems for Chapter 11</i>	257
III	BASIC CONTROL METHODS	265
12	PID controller	266
12.1	Introduction	266
12.2	Computer-based PID control loop	266
12.3	Continuous-time PID controller	267

12.4 Transfer function of the PID controller	269
12.4.1 Ideal PID controller	269
12.4.2 Practical PID controller with filter in the derivative term	270
12.5 Discrete-time PID algorithm	270
12.6 How does the PID controller work?	272
12.7 Practical aspects of the PID controller	274
12.7.1 Reverse or direct controller action?	274
12.7.1.1 What is meant by reverse action and direct action?	274
12.7.1.2 How to select between reverse action and direct action modes?	276
12.7.2 Lowpass filter in the D-term	278
12.7.3 Reducing P-kick and D-kick caused by setpoint changes	278
12.7.4 Integrator anti wind-up	279
12.7.5 Bumpless transfer between manual/auto mode	283
12.8 <i>Problems for Chapter 12</i>	284
12.9 <i>Solutions to problems for Chapter 12</i>	288
 13 On-off controller	 293
13.1 The ideal On-off controller	293
13.2 The practical On-off controller	295
13.3 Programming the On-off controller	296
13.4 <i>Problems for Chapter 13</i>	297
13.5 <i>Solutions to problems for Chapter 13</i>	298
 14 Transfer functions of feedback control systems	 300
14.1 Introduction	300
14.2 Definition of setpoint tracking and disturbance compensation	300
14.3 The Sensitivity transfer function	301
14.4 The Tracking transfer function	303

14.5 Analytical calculation of responses with transfer functions	304
14.6 <i>Problems for Chapter 14</i>	305
14.7 <i>Solutions to problems for Chapter 14</i>	306
15 Simulation of control systems	308
15.1 Introduction	308
15.2 Simulation with block-diagram simulation tools (OpenModelica)	308
15.3 Simulation with programmed simulation algorithms (Python)	308
15.4 Simulation with transfer functions (Python Control package)	308
16 Tuning of PID controllers	309
16.1 Introduction	309
16.2 The Ziegler-Nichols closed loop method	311
16.3 Relaxed Ziegler-Nichols PI settings	315
16.4 Repeated Ziegler-Nichols tuning	316
16.5 The Åström-Hägglund Relay tuning method	318
16.6 Auto-tuning	320
16.7 The Good Gain method	321
16.8 The Skogestad controller tuning method	324
16.8.1 Background of the Skogestad method	324
16.8.2 Controller tuning for “integrator with time delay” processes	326
16.8.2.1 Mathematical model and dynamics	326
16.8.2.2 Controller settings	327
16.8.2.3 PI tuning for pretended “integrator with time delay” processes	329
16.8.2.4 Tuning for integrator <i>without</i> time delay	333
16.8.3 Controller tuning for “time constant with time delay” processes	334
16.8.3.1 Mathematical model and dynamics	334
16.8.3.2 Controller settings	334

16.8.4 Controller tuning for “double integrator” processes	336
16.8.4.1 Mathematical model	336
16.8.4.2 Controller settings	337
16.9 The Ziegler-Nichols open loop method	339
16.10 PID tuning when process dynamics varies	340
16.10.1 Introduction	340
16.10.2 PID parameter adjustment based on the Skogestad PID tuning method	340
16.10.3 Gain scheduling of PID parameters	341
16.10.4 Adaptive controller	346
16.11 <i>Problems for Chapter 16</i>	348
16.12 <i>Solutions to problems for Chapter 16</i>	353
17 Control loop stability	358
17.1 Heuristic stability analysis	358
17.2 Experimental gain margin (GM) and phase margin (PM)	363
17.3 <i>Problems for Chapter 17</i>	367
17.4 <i>Solutions to problems for Chapter 17</i>	369
18 Control structures based on the PID control loop	370
18.1 Cascade control	370
18.1.1 The principle of cascade control	370
18.1.2 Benefits of cascade control	371
18.1.3 Controller selection and controller tuning	372
18.1.4 Cascade control and state feedback	373
18.2 Ratio control	379
18.3 Split-range control	379
18.4 Averaging level control	380
18.5 Plantwide control	386

18.6 <i>Problems for Chapter 18</i>	393
18.7 <i>Solutions to problems for Chapter 18</i>	398
19 Feedforward control	405
19.1 Introduction	405
19.2 Designing feedforward control from differential equation models	406
19.3 Designing feedforward control from experimental data	410
19.4 <i>Problems for Chapter 19</i>	412
19.5 <i>Solutions to problems for Chapter 19</i>	415
20 Sequential control	417
20.1 <i>Problems for Chapter 20</i>	420
20.2 <i>Solutions to problems for Chapter 20</i>	421
IV ANALYSIS OF CONTINUOUS-TIME FEEDBACK SYSTEMS	423
21 Stability analysis based on poles	424
21.1 Introduction	424
21.2 Stability properties and impulse response	424
21.3 Stability properties and poles	426
21.4 Stability analysis of state space models	431
21.5 Stability analysis of feedback systems	432
21.6 <i>Problems for Chapter 21</i>	435
21.7 <i>Solutions to problems for Chapter 21</i>	438
22 Frequency response	442
22.1 Introduction	442
22.2 How to calculate frequency response from sinusoidal input and output	443
22.3 Bode diagram	444

22.4 How to calculate frequency response from transfer functions	445
22.5 Application of frequency response: Signal filters	448
22.5.1 Introduction	448
22.5.2 First order lowpass filters	449
22.6 <i>Problems for Chapter 22</i>	453
22.7 <i>Solutions to problems for Chapter 22</i>	454
23 Frequency response analysis of feedback control systems	457
23.1 Introduction	457
23.2 Frequency response analysis of setpoint tracking and disturbance compensation	458
23.2.1 Introduction	458
23.2.2 Frequency response analysis of setpoint tracking	459
23.2.3 Frequency response analysis of disturbance compensation	462
23.3 Stability analysis of feedback systems	465
23.3.1 Introduction	465
23.3.2 Nyquist's stability criterion	466
23.3.3 Stability margins	471
23.3.3.1 Stability margins in terms of gain margin and phase margin	471
23.3.3.2 Stability margins in terms of maximum sensitivity amplitude	473
23.3.4 Stability analysis in a Bode diagram	473
23.3.5 Robustness in term of stability margins	476
23.4 <i>Problems for Chapter 23</i>	478
23.5 <i>Solutions to problems for Chapter 23</i>	484
V ANALYSIS OF DISCRETE-TIME FEEDBACK SYSTEMS	490
24 Discrete-time signals	491
24.1 <i>Problems for Chapter 24</i>	493

24.2 <i>Solutions to problems for Chapter 24</i>	494
25 Difference equations	496
25.1 Difference equation models	496
25.2 Calculating responses from difference equation models	497
25.3 <i>Problems for Chapter 25</i>	498
25.4 <i>Solutions to problems for Chapter 25</i>	500
26 Discrete-time state space models	502
26.1 General form of discrete-time state space models	502
26.2 Linear discrete-time state space models	502
26.3 Discretization of continuous-time state space models	503
26.4 <i>Problems for Chapter 26</i>	504
26.5 <i>Solutions to problems for Chapter 26</i>	505
27 The z-transform	506
27.1 Definition of the z -transform	506
27.2 Properties of the z -transform	506
27.3 z -transform pairs	507
27.4 Inverse z -transform	507
27.5 <i>Problems for Chapter 27</i>	509
27.6 <i>Solutions to problems for Chapter 27</i>	510
28 Discrete-time (or z-) transfer functions	511
28.1 Introduction	511
28.2 From difference equation to transfer function	511
28.3 From transfer function to difference equation	512
28.4 Calculating time responses for discrete-time transfer functions	513
28.5 Static transfer function and static response	513

28.6 Poles and zeros	514
28.7 From s -transfer functions to z -transfer functions	515
28.8 <i>Problems for Chapter 28</i>	518
28.9 <i>Solutions to problems for Chapter 28</i>	520
29 Frequency response of discrete-time systems	522
29.1 <i>Problems for Chapter 29</i>	525
29.2 <i>Solutions to problems for Chapter 29</i>	527
30 Stability analysis of discrete-time dynamic systems	529
30.1 Definition of stability properties	529
30.2 Stability analysis of transfer function models	529
30.3 Stability analysis of state space models	533
30.4 <i>Problems for Chapter 30</i>	535
30.5 <i>Solutions to problems for Chapter 30</i>	537
31 Stability analysis of discrete-time feedback systems	540
31.1 <i>Problems for Chapter 31</i>	544
31.2 <i>Solutions to problems for Chapter 31</i>	545
32 Subspace identification of blackbox discrete-time models	547
32.1 <i>Problems for Chapter 32</i>	552
32.2 <i>Solutions to problems for Chapter 32</i>	553
VI STATE ESTIMATION	554
33 Stochastic signals	555
33.1 Introduction	555
33.2 How to characterize stochastic signals	555

33.2.1	Realizations of stochastic processes	555
33.2.2	Probability distribution of a stochastic variable	556
33.2.3	The expectation value and the mean value	556
33.2.4	Variance. Standard deviation	557
33.2.5	Auto-covariance. Cross-covariance	558
33.3	White and coloured noise	560
33.3.1	White noise	560
33.3.2	Coloured noise	561
33.4	Propagation of mean value and co-variance through static systems	564
33.5	<i>Problems for Chapter 33</i>	566
33.6	<i>Solutions to problems for Chapter 33</i>	568
34	State estimation with Kalman Filter	574
34.1	Introduction	574
34.2	Observability of discrete-time systems	576
34.3	The Kalman Filter algorithm	579
34.3.1	The assumed process model	579
34.3.2	The result of Kalman Filtering: an optimal state estimate	582
34.3.3	The Kalman Filter algorithm – step by step	582
34.3.4	Features of the Kalman Filter	587
34.3.4.1	The error-model	587
34.3.4.2	The dynamics of the Kalman Filter	587
34.3.4.3	The stability of the Kalman Filter	588
34.4	Tuning the Kalman Filter	588
34.5	Estimating parameters and disturbances with Kalman Filter	589
34.5.1	Introduction	589
34.5.2	The augmentative state (x_a) is constant	589

34.5.3 The augmentative state (x_a) has constant rate	590
34.6 Kalman Filtering when process measurement is absent	596
34.7 <i>Problems for Chapter 34</i>	599
34.8 <i>Solutions to problems for Chapter 34</i>	600
35 Moving Horizon Estimation	603
35.1 Introduction	603
35.2 The assumed process model	603
35.3 The optimization problem of MHE	605
35.3.0.0.1 About norms.	606
35.3.0.0.2 The optimization function.	606
35.3.0.0.3 Constraints.	607
35.3.0.0.4 Guessed value of X	608
35.3.0.0.5 No linearization.	608
35.4 Estimation of model parameters and state disturbances	608
35.5 Tuning factors of MHE	609
35.6 <i>Problems for Chapter 35</i>	611
35.7 <i>Solutions to problems for Chapter 35</i>	612
VII MODEL-BASED CONTROL	614
36 How to test robustness with simulations	615
36.1 <i>Problems for Chapter 36</i>	617
36.2 <i>Solutions to problems for Chapter 36</i>	619
37 Linear Quadratic (LQ) optimal control	620
37.1 Introduction	620
37.2 The basic LQ controller	621

37.3 LQ controller with integral action	628
37.3.1 Introduction	628
37.3.2 Including integrators in the controller	628
37.3.3 Discrete-time implementation of the LQ controller	630
37.4 <i>Problems for Chapter 37</i>	631
37.5 <i>Solutions to problems for Chapter 37</i>	633
38 Model Predictive Control (MPC)	635
38.1 Introduction	635
38.2 The MPC method	636
38.2.1 The principle of MPC	636
38.2.2 The optimization objective function of MPC	637
38.2.3 Control signal blocking	639
38.2.4 Tuning factors of MPC	639
38.2.5 The need for a state estimator	641
38.3 <i>Problems for Chapter 38</i>	644
38.4 <i>Solutions to problems for Chapter 38</i>	645
39 Inverse dynamics control	646
39.1 Introduction	646
39.2 Inverse dynamics control of first order processes	647
39.3 Inverse dynamics control of second order processes	652
39.4 <i>Problems for Chapter 39</i>	657
39.5 <i>Solutions to problems for Chapter 39</i>	661
VIII APPENDICES	664
40 Some good control questions	665

41 Process models	668
41.1 Wood chips tank	668
41.1.1 System description	668
41.1.2 Variables and parameters	669
41.1.3 Overall block diagram	669
41.1.4 Mathematical model	669
41.2 Ship	670
41.2.1 System description	670
41.2.2 Variables and parameters	671
41.2.3 Overall block diagram	672
41.2.4 Mathematical model	672
41.3 Buffer tank	673
41.3.1 System description	673
41.3.2 Variables and parameters	673
41.3.3 Overall block diagram	674
41.3.4 Mathematical model	674
41.4 Heated liquid tank	674
41.4.1 System description	674
41.4.2 Variables and parameters	674
41.4.3 Overall block diagram	674
41.4.4 Mathematical model	675
41.5 Air heater	676
41.5.1 System description	676
41.5.2 Variables and parameters	676
41.5.3 Overall block diagram	676
41.5.4 Mathematical model	676
41.5.5 Data file	677

41.6 DC-motor	678
41.6.1 System description	678
41.6.2 Overall block diagram	678
41.6.3 Variables and parameters	679
41.6.4 Mathematical model	679
41.6.5 Datafile	679
41.7 Biogas reactor	681
41.7.1 System description	681
41.7.2 Variables and parameters	681
41.7.3 Overall block diagram	681
41.7.4 Mathematical model	682
41.7.5 Operating point	685
41.8 Pendulum on cart	686
41.8.1 System description	686
41.8.2 Variables and parameters	686
41.8.3 Overall block diagram	687
41.8.4 Mathematical model	687
42 Optimization	690
42.1 The optimization problem	690
42.1.1 Introduction	690
42.1.2 Mathematical formulation of the optimization problem	691
42.1.3 Feasibility region	692
42.1.4 Some characteristics of the optimal solution	692
42.1.5 What about maximization problems?	694
42.2 How to solve optimization problems	694
42.2.1 Introduction	694

42.2.2 Analytical solution	697
42.2.3 The grid or brute force method of optimization	698
42.2.4 Testing: Have you actually found the minimum?	713
42.2.5 Steepest decent method	716
42.2.5.1 Introduction	716
42.2.5.2 Scalar x	716
42.2.5.3 Vectorial x	716
42.2.5.4 Numerical calculation of the derivative	717
42.2.6 The Newton search method	723
42.2.6.1 Introduction	723
42.2.6.2 Scalar x	723
42.2.6.3 An explanation of why the Newton search is fast	725
42.2.6.4 For a quadratic f , the optimum is found in one search iteration!	726
42.2.6.5 Vectorial x	726
42.2.6.6 Avoiding the inversion of the Hessian	727
42.2.6.7 Numerical calculation of the gradient and the Hessian	728
42.2.7 Combining steepest descent with Newton for robust search	734
42.2.8 Some professional NLP optimizers	737
42.2.8.1 Introduction	737
42.2.8.2 slsqp (Python)	738
42.2.8.3 fmincon (Matlab)	738
42.2.9 Global optimization	742
43 A guide to the Python Control package	743
43.1 Introduction	743
43.1.1 About this guide	743
43.1.2 Information about Python Control package on the Internet	743

43.1.3	Installing the Python Control package	744
43.1.4	Importing the Python Control package into Python	744
43.1.5	Using arrays for numerical data	744
43.2	Transfer functions	745
43.2.1	How to create transfer functions	745
43.2.2	Combinations of transfer functions	746
43.2.2.1	Series combination	746
43.2.2.2	Parallel combination	748
43.2.2.3	Feedback combination	749
43.2.3	How to get the numerator and denominator of a transfer function	751
43.2.4	Simulation with transfer functions	753
43.2.5	Poles and zeros of transfer functions	756
43.2.6	The Padé-approximation of a time delay	757
43.3	Frequency response	758
43.3.1	Frequency response of transfer functions	758
43.3.2	Frequency response and stability analysis of feedback loops	760
43.4	State space models	763
43.4.1	How to create state space models	763
43.4.2	How to get the model matrices of a state space model	766
43.4.3	Simulation with state space models	767
43.4.4	From state space model to transfer function	769
43.5	Discrete-time models	772
43.5.1	Transfer functions	772
43.5.1.1	Introduction	772
43.5.1.2	How to create transfer functions	772
43.5.1.3	Discretizing an s -transfer function	773
43.5.1.4	Exact representation of a time delay with a z -transfer function	775

43.5.2 Frequency response	777
43.5.3 State space models	777
44 Selected mathematical formulas	778
44.1 Differentiation of vector functions	778
45 Abbreviations	779
Bibliography	779
Index	781

Preface

Automatic control is about using a computer to automatically manipulate physical processes – mechanical, thermal, chemical, eletrical, etc – so that they behave as we want. Obviously, automatic control is of crucial importance in industrial and other kinds of technical systems.

The main aim of this book is to describe methods to design and implement automatic control systems. It covers both theoretical and practical topics, and both basic and advanced, model-based methods. It may serve as a textbook in bachelor courses and in master courses, and as a reference book for professionals.

This book is actually a draft of a coming book which I plan to name Modeling, Simulation and Control. Some chapter will certainly be modified, and some new material will be included.

Problems with detailed solutions are at the end of (most of) the chapters.

Many simulations in the book are run with simulators in the SimView library, which is available for free on <http://techteach.no/simview>. The Python programming language is used in several examples and problems.

A few words about my background: I have a MSc degree from former Norwegian Institute of Technology (Norwegian: Norges tekniske høgskole) and a PhD degree from former Telemark University College (Norwegian: Høgskolen i Telemark). I have experience as university teacher and researcher, textbook author, and participant in industrial and research projects about modeling, simulation and control. At present, I am employed as professor¹ at the University of South-Eastern Norway, and I have my one-person firm TechTeach. I am also teaching at OsloMet (Oslo Metropolitan University) and Norwegian University of Life Sciences.

I enjoy the field of modeling, simulation and control, and the programming to obtain theoretical and practical results. Without that enjoyment, there would not be a book.

If you see errors or have suggestions or other comments about the book, you are welcome to send them to me in email.

Finn Aakre Haugen

¹Norwegian title: dosent

CONTENTS

<http://techteach.no/fh>
finn@techteach.no

Part I

CONTROL CONCEPTS

Chapter 1

Introduction to automatic control

1.1 Why automatic control?

1.1.1 What is the problem?

Figure 1.1 illustrates in general terms the problem that automatic control aims at solving: Given a physical process, e.g. a water tank, a ship, a biogas reactor, with a process output

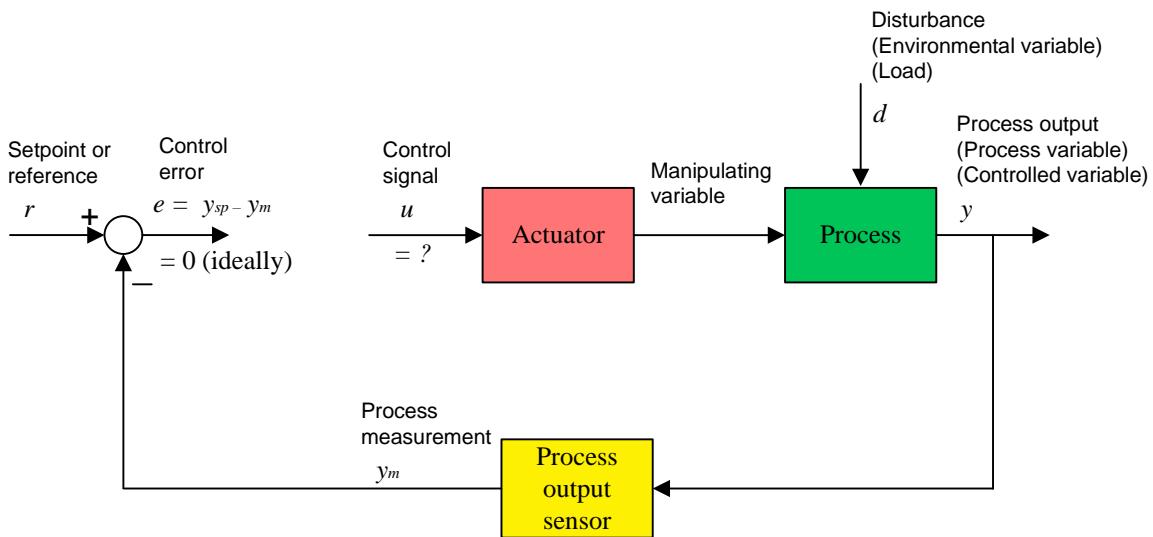


Figure 1.1: Given a process. Aim: The control error is sufficiently small.

which is a process variable that we want to maintain at or sufficiently close to a setpoint, or reference – a desired value. The process output is influenced by some disturbances, e.g. consumption of liquid from a tank, wind acting on the ship, ambient temperature to the reactor temperature. The process output can be manipulated by manipulating the control signal, u , acting on the process. We assume that the process output is measured. The

control error is defined as the difference between the setpoint and the process measurement:

$$e = r - y_m \quad (1.1)$$

The aim of a control system for the process is to ensure that the control error is sufficiently small – ideally: zero. *This book shows how to reach that aim with automatic control implemented in a computer.*

Control can be done also by humans – manual control. However, in industrial and technical system it is often preferred that the control is automatic rather than manual.

In figure 1.1, the process output to be controlled to follow their setpoints are typically among the following:

- **Level** (in a storage tank)
- **Temperature** (in a room; in the fluid passing a heat exchanger; in a reactor; in a greenhouse)
- **Flow** (of feeds into a reactor)
- **Pressure** (in a chemical reactor)
- **pH** (in a reactor)
- **Chemical composition** (of nitric acid; fertilizers, polypropylene)
- **Position** (of a ship; a painting robot arm; the tool of a cutting machine; a rocket)
- **Speed** (of a motor; a car; a fan)

1.1.2 The solution!

Figure 1.2 shows the principal solution of the control problem. The control system consists of two controllers:

- **Feedback controller** generating the control signal u_{fb} as a function of the control error. The standard feedback controller is the PID controller (proportional + integral + derivative). The term “feedback” is because the process output is connected back (via the sensor and the controller) to the control signal which a process input.
- **Feedforward controller** generating the control signal u_{ff} as a function of the setpoint and measured process disturbances. The feedforward controller is typically based on a mathematical process model, and is therefore dependent on the given process. The term “feedforward” is because the process disturbance(s) and the setpoint, which are input signal, are connected to the process input (the control signal).

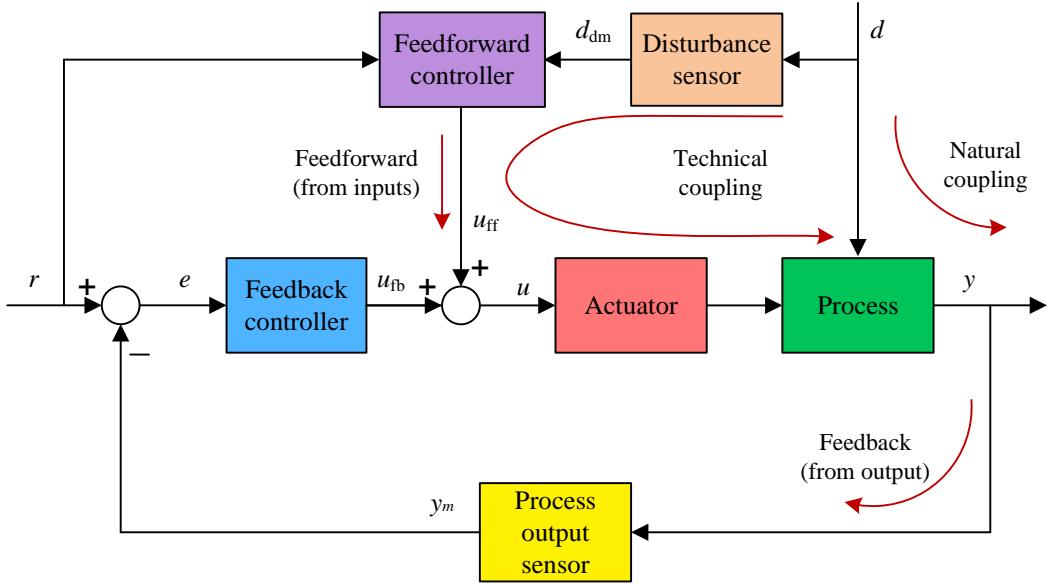


Figure 1.2: Solution of the control problem: Feedback control and feedforward control.

The total control signal is the sum of u_{fb} and u_{ff} :

$$u = u_{fb} + u_{ff} \quad (1.2)$$

In practice, feedback control is “compulsory”, while feedforward control is not so common; Although the feedforward controller can improve the control substantially, it may be challenging to implement because it relies on a mathematical process model which may be difficult to derive.

We will see examples of both feedback control and feedforward control later in this chapter.

1.1.3 What can be obtained with control?

Figure 1.3 compares typical results of “no active control” (i.e. using a fixed or constant control signal) with “active control” (continuously adjusting the control signal). The control error may be kept sufficiently small with active control, while it may be too large with no active control. Usually, we want the active control to be accomplished *automatically* (by a computer) - not by a human.

Automatic control is important in a large number of practical industrial and technical systems. With automatic control we may obtain:

- **Good product quality:** A product will have acceptable quality only if certain process variables are sufficiently close to their setpoints. One example: In artificial (chemical) fertilizers the pH value and the composition of Nitrogen, Phosphate and Potassium are factors which express the quality of the fertilizer (for example, too low

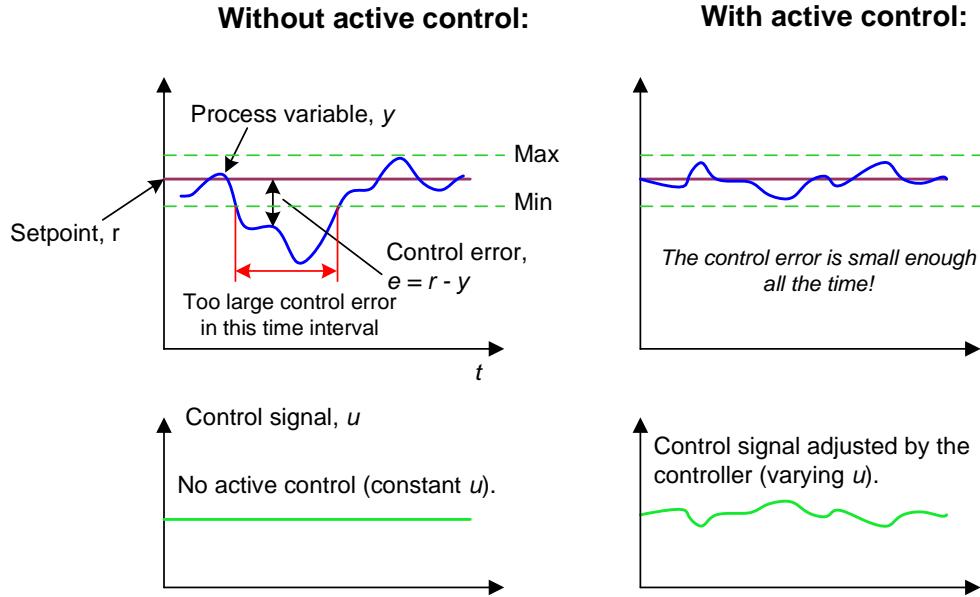


Figure 1.3: Active control can ensure that the control error is small enough all the time!

pH value is not good for the soil). Therefore the pH value and the compositions must be controlled.

- **Good production economy:** The production economy will become worse if part of the products has unacceptable quality so that it can not be sold. Good control may maintain the good product quality, and hence, contribute to good production economy. Further, by good control it may be possible to tighten the limits of the quality so that a higher price may be taken for the product!
- **Safety:** To guarantee the security both for humans and equipment, it may be required to keep variables like pressure, temperature, level, and others within certain limits— that is, these variables must be controlled. Some examples:
 - An aircraft with an autopilot (an autopilot is a positional control system).
 - A chemical reactor where pressure and temperature must be controlled.
- **Environmental care:** The amount of poisons to be emitted from a factory is regulated through laws and directions. The application of control engineering may help to keep the limits. Some examples:
 - In a wood chip tank in a paper pulp factory, hydrogen sulfide gas from the pulp process is used to preheat the wood chip. If the chip level in the tank is too low, too much (stinking) gas is emitted to the atmosphere, causing pollution. With level control the level is kept close to a desired value (set-point) at which only a small amount of gas is expired.
 - In the so-called washing tower nitric acid is added to the intermediate product to neutralize exhaust gases from the production. This is accomplished by controlling the pH value of the product by means of a pH control system. Hence, the pH control system ensures that the amount of emitted ammonia is between specified limits.

- Automatically controlled spray painting robots avoid humans working in dangerous areas. See Figure 1.4.



Figure 1.4: Spray painting robot (IRB580, ABB)

- **Comfort:**

- The automatic positional control which is performed by the autopilot of an aircraft to keep a steady course contributes to the comfort of the journey.
- Automatic control of indoor temperature may give better comfort.

- **Feasibility:** Numerous technical systems could not work or would even not be possible without the use of control engineering. Some examples:

- An exothermal reactor operating in an unstable (but optimal) operating point
- Launching a space vessel (the course is stabilized)
- A dynamic positioning system holds a ship at a given position without an anchor despite the influence of waves, wind and current on the ship. The heart of a dynamic positioning system is the positional control system which controls the thrusters which are capable of moving the ship in all directions.

- **Automation:** Computers and other kinds of hardware and software implementing control solutions can accomplish tedious and dangerous operations for the benefit of human operators. Also, automation may reduce costs in a factory, thereby indirectly reducing product prices, to customer's benefit.

1.2 The principle of feedback control

1.2.1 Manual feedback control

I will explain the principle of feedback control with a shower.

Imagine that you are to take a shower, and you want to have the temperature of the shower water as you desire, see Figure 1.5. The figure also shows responses, as explained below.

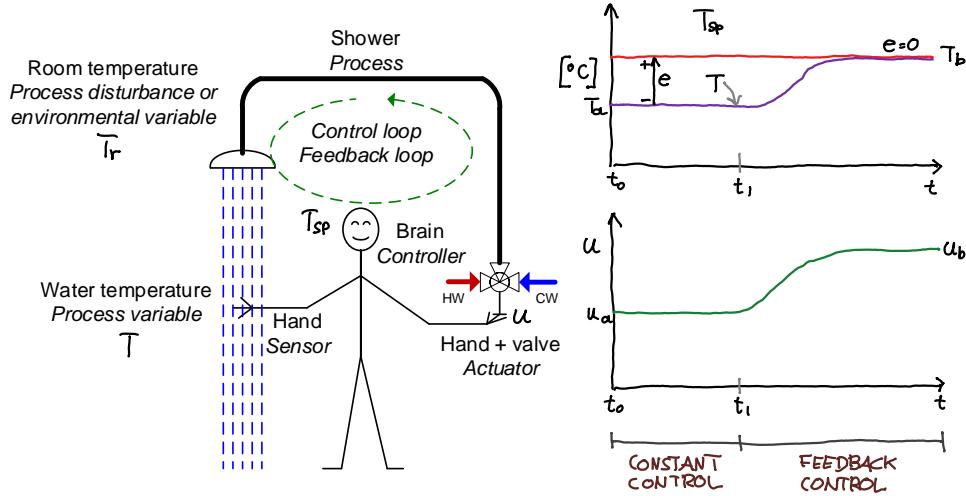


Figure 1.5: Controlling the water temperature of a shower. (HW = hot water. CW = cold water.)

In control terminology, the shower is here the *process*. The water temperature, T , is the *process variable* which we want to follow its desired temperature – the temperature *setpoint*, r_T . The room temperature, T_r , makes an impact on T , and we can therefore say that T_r is a *process disturbance*. To control T , you can use your hand to manipulate a mixing valve which sets the ratio between the hot and cold water flows. The hand and the valve constitute the *actuator*. Let u be the setting of the mixing value.

In control terminology, the difference between the setpoint and the process variable, or its measurement, is denoted the *control error*, e

$$\text{control error } (e) = \text{setpoint} - \text{process variable} \quad (1.3)$$

In the shower example, we can write

$$e = r_T - T \quad (1.4)$$

Typically, the aim of control is $e = 0$, or $e \approx 0$ in practice as e will inevitably vary somewhat.¹

Let us consider the following two alternative temperature control strategies:

- **Constant control:** Consider the time interval between t_0 and t_1 in Figure 1.5. Assume you took a shower yesterday. The shower temperature was as you desired, i.e. $T = r_T$. Say that a valve setting of $u = u_a$ gave the desired temperature. At time t_0 (today) you enter the shower. Naturally, you try the successful setting $u = u_a$ also today. If T_r is the same as yesterday, using $u = u_a$ gives $T = r_T$ also today. But assume that T_r is actually lower today than yesterday, maybe because you take the shower with an open window today while the window was closed yesterday.

¹However, in so-called averaging level control of buffer tanks, we actually want a relatively large e for the level, as this makes the tank compliant to inflow variations. Averaging level control is described in Ch. 18.4.

Consequently, $u = u_a$ gives $T = T_a$ which is less than r_T , or in other terms: $e > 0$. Assume that keeping $u = u_a = \text{constant}$ makes you freeze, and that you conclude that constant control is not a good control strategy. This makes you step aside of the cold water for a moment to think about a better strategy.

- **Error-driven control, or feedback control:** At time t_1 , while freezing, you decide to improve the control to make T reach r_T , or in other terms, to obtain $e = 0$. How would you improve the control? I guess you decide to measure the temperature with say your right hand, which is a *sensor*. That measurement, also denoted T here as we assume it represents the actual temperature, is detected in your brain. Your brain – the *controller* – then adjust the nerve signal to the left hand to change the setting, u , of the mixing valve until $e \approx 0$, and then you are ready to take the shower – with the desired temperature. Since it the control error that “drives” the control, we can denote this control strategy as error-driven control. However, a more common term is feedback control, which is explained below. Also, the term closed loop control is used, and for constant control, open loop control is an alternative term.

This excellent temperature control system is a *manual* control system since *you* are the controller. In an *automatic* control system you are replaced by a computer which can generate e.g. an electrical control signal to the valve, and the sensor with an industrial temperature sensor, for example a Pt100 element which sends an electrical signal to the controller. More details about an automatic control system for the shower are given below.

In Figure 1.5, I have indicated the control loop. It consist of the following three main components needed to control the process. These are the main components also in purely technical control loops.

- Sensor
- Controller
- Actuator

Sometimes also a measurement filter, which attenuates measurement noise, is included as component (following the sensor) in the control loop. Alternatively, the filter can also be regarded as a part of (included in) the sensor.

As I mentioned above, control loops are also denoted feedback (control) loops. The reason for this name is that there is a connection from the process output, which is the water temperature, *back* to the process input, which is the control signal to the valve, via the sensor and the controller.

An alternative term to feedback control, is error-driven control as it is the control error that “drives” the control action.² It is tempting to use the term “measurement-driven control”. However, that term is ambiguous because also feedforward control, cf. Section 1.3, and not only feedback control is driven by measurements.

²Personally, I like the term error-driven control better than feedback control since it expresses better the principle of control systems.

1.2.2 Automatic feedback control with technical components

I guess you dream about a purely technical shower control system at home. Figure 1.6 shows a possible implementation using only technical components.

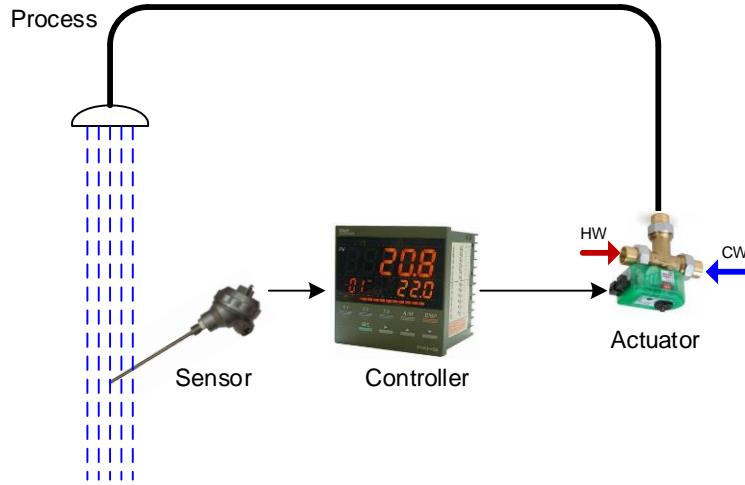


Figure 1.6: The shower water temperature control system implemented with only technical components. (Sensor: Autek. Controller: Fuji. Valve: Taco.)

The components used in Figure 1.6 are:

- The actuator is an electronically controlled mixing valve.
- The temperature sensor, which measures the water temperature out of the shower, is of the type Pt100 sensor, which is a sensor based on the principle that the resistance value of an electrical resistor inside the sensor varies in a known way with the temperature. The electronics in the sensor detect this resistance value and hence calculate the temperature.
- The controller is an industrial process controller containing a computer which executes a program implementing the control function is a standard PID controller function (proportional-integral-derivative). (The PID controller function is described in detail in Ch. 12.) The measurement signal from the sensor is connected to the controller. You can set the temperature setpoint using buttons on the front panel of the controller. The controller adjusts the control signal, u , to the valve automatically to make T become equal to T_{sp} – without any human interaction. Thus, the control system shown in Figure 1.6 is an automatic temperature control system.

Industrial controllers can be switched between *automatic mode* and *manual mode*:

- **Automatic mode:** The control signal is generated automatically by the controller function (algorithm) programmed in the built-in computer in the controller.

- **Manual mode:** The built-in controller function is deactivated, and the control signal can be adjusted by a human via the user interface of the controller.

The switching option between automatic and manual modes is illustrated in Figure 1.7.

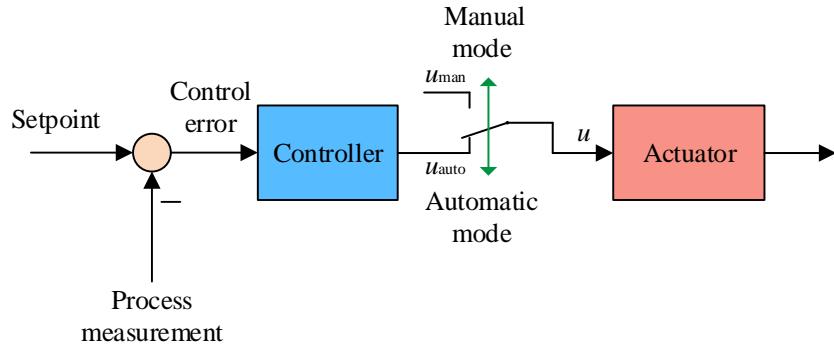


Figure 1.7: Industrial controllers can be switched between automatic and manual modes.

Industrial controllers implement a standard controller function denoted the PID controller. PID is abbreviation of proportional-integra-derivative, referring to the mathematical functions comprising the controller. We will study the PID controller in detail in Ch. 12. Still, you may want to know the names of the main parameters of the PID-controller. They are:

- K_c – the controller gain
- T_i – the integral time
- T_d – the derivative time

These parameters must be tuned to appropriate numerical values so that the controller behaves as desired. Controller tuning is the topic of Ch. 16.

1.2.3 Documentation with Piping & Instrumentation Diagram – P&ID

In the industry, it is common to document the structure of control systems with Piping & Instrumentation Diagrams (P&I Ds). Figure 1.8 shows a Piping & Instrumentation Diagram (P&I D) of the shower water temperature control system. P&I Ds are described in more detail in Appendix 2.

In Figure 1.8:

- TT = Temperature Transmitter, which is the standard letters of temperature sensors. A transmitter is actually not a sensor, but a device which sends the measurement signal, which in general may be electrical or pneumatic or hydraulic or digital, to the controller. Still, TT here represents the temperature sensor.

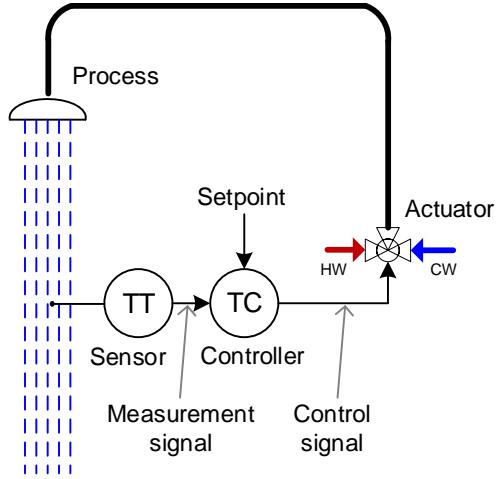


Figure 1.8: Piping & Instrumentation Diagram (P&I D) of the temperature control system of the shower.

- TC = Temperature Controller.
- The actuator – a mixing valve – is shown with a representative symbol. There is no general symbol for actuator. If the actuator were a pump, a pump symbols should be shown.
- Process flows in e.g. pipelines are drawn with relatively thick lines.
- Signals, as measurement signals and control signals, are drawn with relatively thin lines. If necessary, you can use special dashes to indicate the signal type, i.e. electrical, digital, etc. A line without dashes does not indicate any special signal type – it is just a signal.
- Strictly, signal lines are not drawn with arrow heads, just pure lines. I draw with arrow heads in this book because it makes the diagram easier to read, although it breaks with the standards of P&I Ds.

1.2.4 Documentation with block diagram

In many contexts, e.g. in teaching, it is useful to draw a block diagram of a control system. Block diagrams are useful for showing the various variables (signals) and the components of a control system. The level of detail of block diagrams may vary, depending on what to show in the diagram. There are no specific standards for block diagrams.

Figure 1.9 shows a block diagram of the temperature control system of the shower. This block diagram is representative for control systems in general, with the general terms shown in *italics* in the figure. Note the three components needed to implement any feedback control system: *Controller*, *actuator* and *sensor*.

Some comments to Figure 1.9:

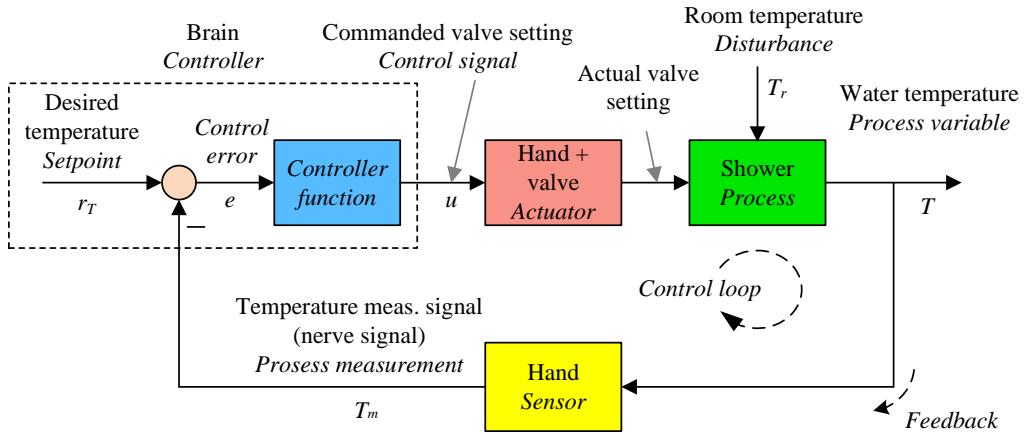


Figure 1.9: Block diagram of the temperature control system of the shower.

- The controller as a physical component is represented with a frame with a dashed line, while the controller function is represented with the blue block inside the dashed frame. Often the dashed frame representing the (physical) controller is not drawn in block diagrams.
- The circle to the left is an adder. The negative sign indicates the measurement enters the adder with a negative sign. Therefore, the output of the adder is setpoint minus measurement, i.e. the control error, e .
- The symbol T_m represents the temperature measurement signal. However, if it is assumed that the measurement gives a precise representation of the process variable, T , the same symbol, T , may be used also for the measurement.

1.2.5 Control application: Temperature control of a liquid tank

Figure 1.10 shows the front panel of a simulator of the temperature control system of a liquid tank. Figure 1.11 shows a block diagram of the temperature control system.

The tank can be e.g. a biogas reactor where the temperature in the tank is maintained at a set point favorable to the microorganisms that form biogas (mainly methane gas) of the biological raw material. The temperature control, which is made by the controller TC (Temperature Controller) is based on feedback from the measured temperature in the tank provided by the sensor TT (Temperatuere Transmitter). The controller manipulates the tank temperature by adjusting the control signal to the heating element in the tank. A significant process disturbance, or environmental variable, is the varying inlet temperature T_{in} . The temperature setpoint, r_T , i.e. the desired temperature, is 40°C.

The simulation (SimView simulator: http://techteach.no/simview/temp_control_pid_onoff/index.php) shown in Figure 1.10 is carried out as follows:

- At time $t = 60$ min, and before, there is only constant control, i.e. the controller is set

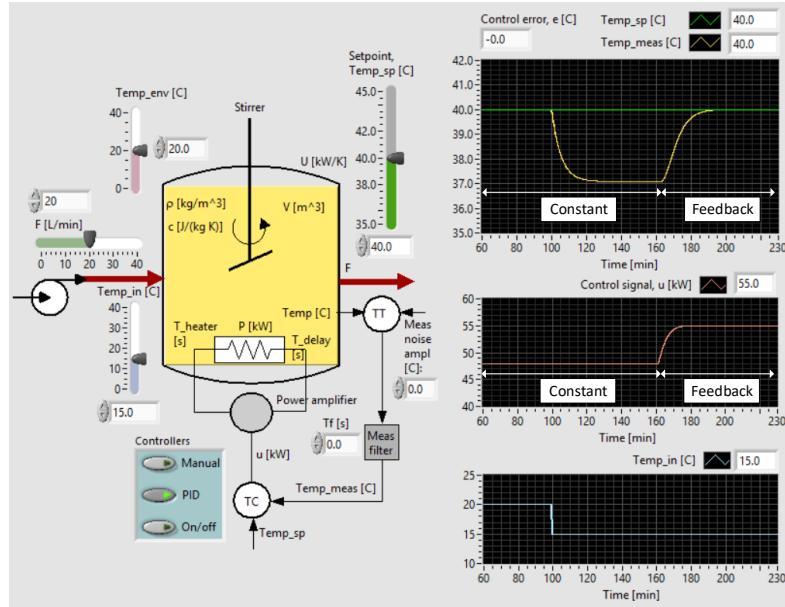


Figure 1.10: Temperature control system of a liquid tank with initially constant control and then feedback control.

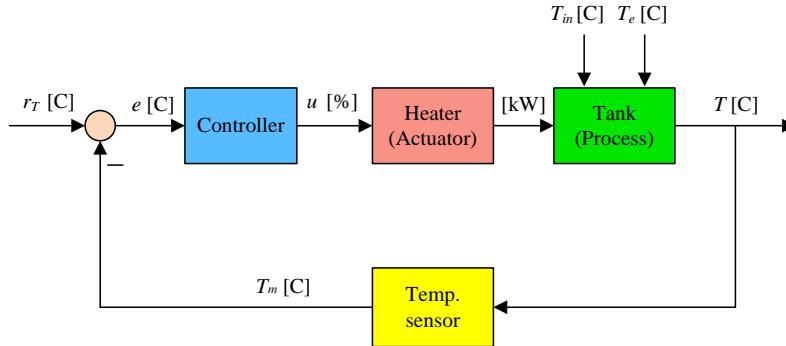


Figure 1.11: Block diagram of the temperature control system of the liquid tank

to manual mode with a fixed control signal equal to $u = 48 \text{ kW}$, which is the value of the control signal that makes the tank temperature become 40°C under the initial conditions.

- At time $t = 100 \text{ min}$, still with constant control, the disturbance T_{in} is changed as it is decreased from 20 to 15°C . We see that this change causes T to decrease to 37°C , i.e. the control error becomes $e = 40 - 37 = 3^\circ\text{C}$.
- At $t = 160 \text{ min}$, the controller is activated by setting it into automatic mode (automatic feedback mode). We see that the controller is able to compensate for the permanent change in the disturbance by increasing the control signal to the heating element, so that the tank temperature eventually reaches the setpoint, as we want. The control system has good stability. The stability depends on the settings of the controller parameters. Controller tuning is the topic of Ch. 16.

1.2.6 Control application: Level control of a tank with conveyor belt

Level control is prevalent in industry. Figure 1.12 shows a level control system for a wood chip tank with feed screw and conveyor belt which runs with constant speed. The tank is³

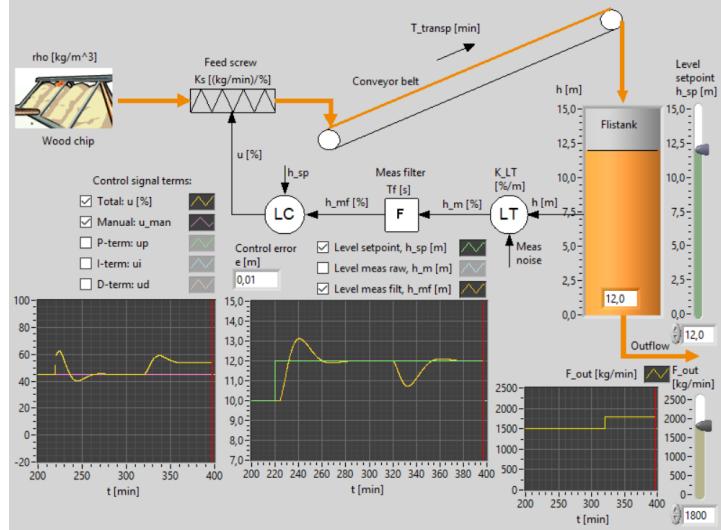


Figure 1.12: Level control of wood chip tank: Simulated responses due to a step change of the level setpoint and a step change of the outflow (disturbance).

in the production line of a paper pulp producing factory. There is a continuous outflow of wood-chip which constitutes a disturbance on the chip level in the tank. A level controller (LC) manipulates the feed screw. The conveyor belt makes up a transport delay of 250 sec of the chip from screw to tank. A model with parameter values is described in Appendix 41.1.

Figure 1.13 shows a block diagram of the level control system.

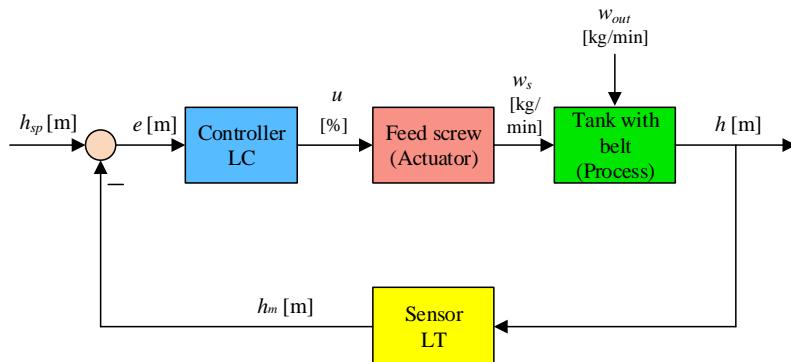


Figure 1.13: Block diagram of the temperature control system of the wood chip tank.

Figure 1.12 shows simulated responses in the level controlled tank. The level setpoint is changed as a step from 10 to 12 m at $t = 220$ min, and the outflow (level disturbance) is

³Actually: was, since the factory in Norway – Södra Cell at Tofte – has been shut down.

changed as a step from 1500 to 1800 kg/min at $t = 320$ min. We observe that the level control system is able to bring the level control error to zero in steady state after the setpoint change and after the disturbance change. (SimView simulator: http://techteach.no/simview/levelcontrol_chiptank/.)

1.2.7 Control application: Averaging level control of buffer tanks

Averaging level control is an important part of several process systems. Figure 1.14 shows a general buffer tank with a level control system aiming at averaging (or equalizing, or attenuating) inflow variations so that the outflow becomes smoother than the inflow. Figure

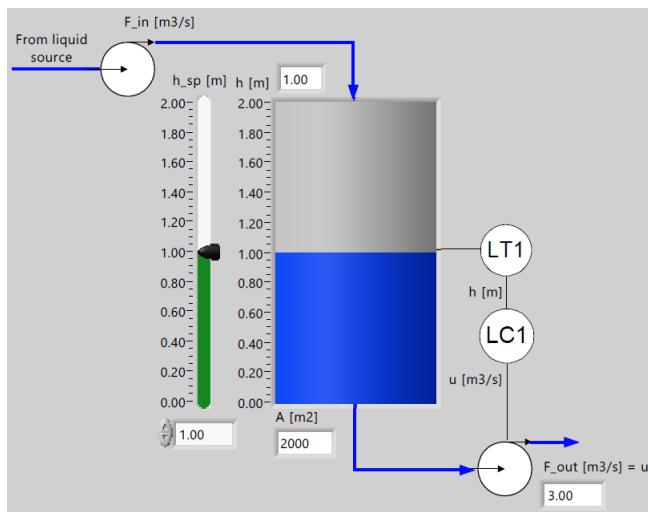


Figure 1.14: Averaging level control of a buffer tank

1.15 shows a block diagram of the level control system.

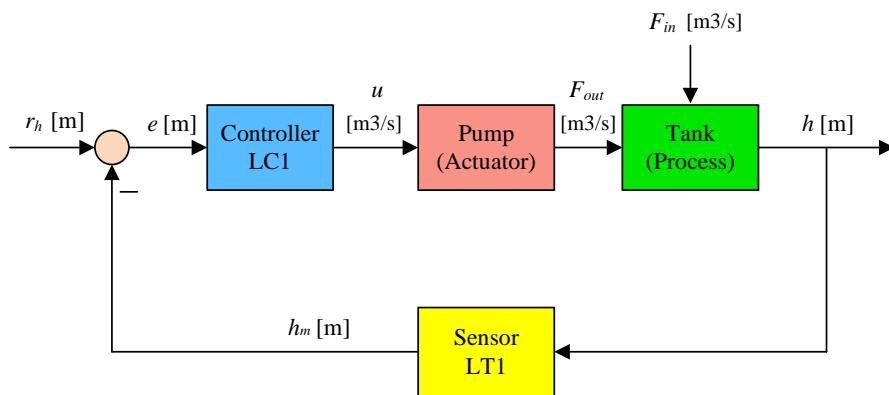


Figure 1.15: Block diagram of the temperature control system of the liquid tank.

The tank resembles the inlet magazine to the WWRF presented in the subsequent Example 1.1 and the oil/water separator presented in presented in the subsequent Example 1.2.

Based on the difference between the level setpoint and the level measurement provided by the sensor LT1, the level controller, LC1, manipulates the outflow through the pump to compensate for inflow variations.

Figure 1.16 shows a simulation with the level controller LC1 in automatic mode. (SimView simulator: http://techteach.no/simview/level_control_equalization_tank.) The inflow is sinusoidal, with amplitude of $1 \text{ m}^3/\text{s}$ and period of 1200 s (refering to the WRRF, this inflow variation may be washwater from the treatment processes subsequent to the inlet magazine in the WRRF). The bottom plot of Figure 1.16 shows that the level controlled buffer tank (magazine) attenuates the flow variations. The level is allowed to vary around the level setpoint. These variations are necessary to obtain the buffering or averaging of the varying inflow. It is actually a special challenge to tune the level controller in averaging level control systems, since the level control must be sufficiently compliant, not “stiff” is we typically want control system to be. Tuning for averaging control is described in Ch. 18.4.

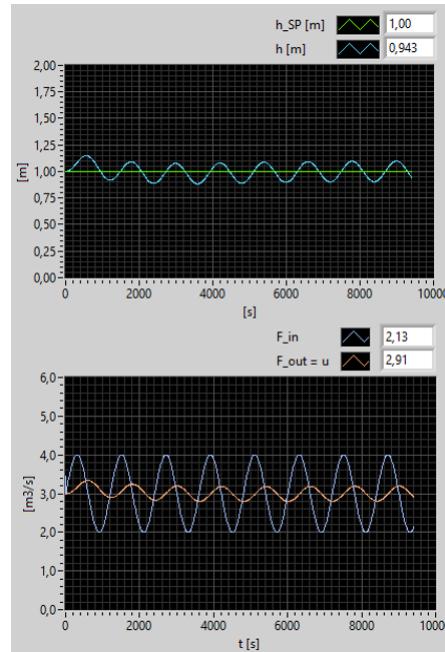


Figure 1.16: Simulation with the PI level controller in automatic mode. The controller is tuned with the Skogestad method. F_{in} has a sinusoidal variation.

Example 1.1 Equalization or buffer magazine at the inlet of a WRRF

Water resource recovery facilities (WRRF)⁴ are crucial facilities in modern societies. They recover resources from what has traditionally been regarded as “waste”, namely sewage and other forms of organic matter.

Figure 1.17 shows the equalization or buffer magazine at the inlet of a typical WRRF.

⁴The traditional term is wastewater treatment plants (WWTPs), but international organizations, like International water Association (IWA) encourages a change of terminology.

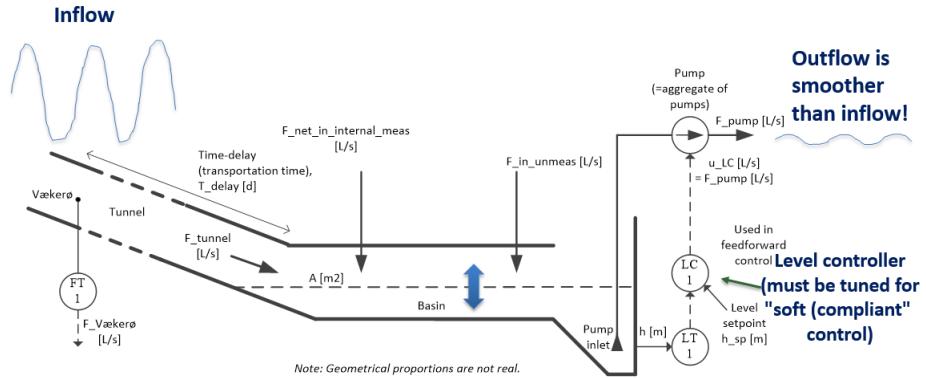


Figure 1.17: Level control of equalization magazine upstreams the VEAS WRRF, Slemmestad, Norway:

In the equalization magazine, the level should be compliant to flow variations so that variations in the inflow are attenuated through the magazine, making the outflow considerably smoother than the inflow. Smoother outflow is advantageous for the subsequent processes, e.g. for the biological treatment processes of the WRRF. The level controller must be tuned for compliant (or soft, or sluggish) level control so that the volume of the tank can absorb the inlet variations.

[End of Example 1.1]

Example 1.2 Control of a water and oil separation system

In the oil & gas production where separators are used to separate oil and gas and water from a three-phase oil/water/gas flow coming from reservoirs.⁵ Separators are level-controlled to maintain the mass balance in the separator. It is important that *slugs*, which are sudden changes in the inflow to the separators, are attenuated through the separator, otherwise the slug will propagate through the subsequent production line. This attenuation will not take place if the level control works too fast, because in a fast level control system, the outflow is nearly the same as the input! So, slugs require sluggish (slow) level control. On the other hand, the slugs must not cause too large level variations inside the separator, otherwise level alarms will be triggered.

Figure 1.18 shows the front panel of a simulator of a water, oil and gas separator system on the Vigdis plant, Norway. The main flow rate is given by the inflow from the reservoir. So, in this system there is no flow control loop. Instead, the flow is given by the flow coming from the reservoir. The level controllers must maintain the mass balances of the separators. (Mass of gas is maintained using pressure control, but the pressure control system is not shown in the figure.) In the first stage separator there is a level control system for the oil and another level control system for the water. In the second stage separator there is a level control system for the oil (it is assumed that all the water has been removed in the first stage separator).

⁵The residence in the separator causes the separation to take place due to the different densities of oil, water and gas.

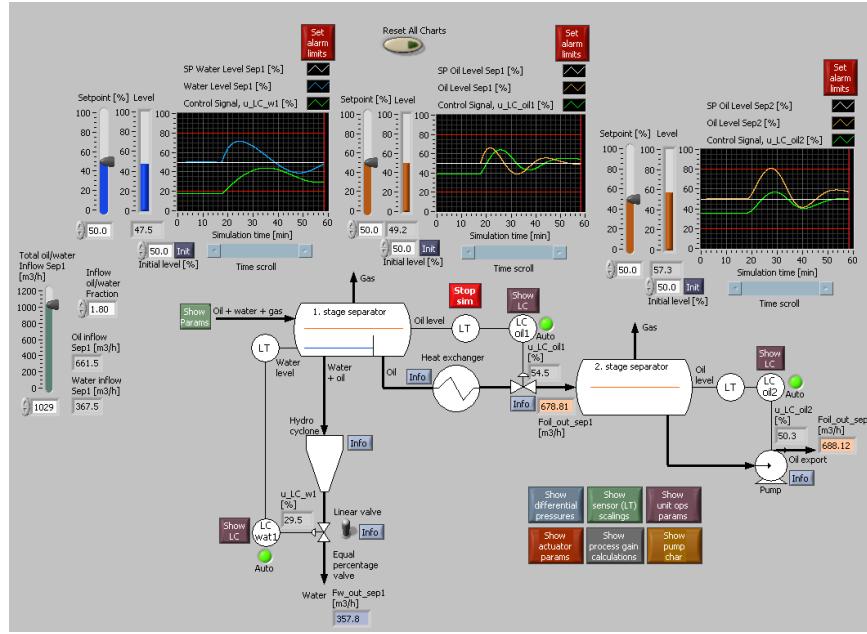


Figure 1.18: Control structure for a water, oil and gas separator system

[End of Example 1.2]

1.2.8 Control application: Position control of a ship (dynamic positioning)

Dynamic Positioning (DP) of ships is automatic position control through the manipulation of the actuators which are the main propeller, maneuvering thrusters, and the rudder. DP systems are very important in various marine operations. DP systems make ships stay sufficiently close to e.g. platforms and other ships, and make ships follow a given position trajectory accurately. Obviously, DP systems increase the level of safety largely.

Figure 1.19 shows the structure of a DP system. The position controller performs feedback control with a PID controller. The main position measurement is the GPS position measurement, but other position measurements are also used. The main disturbances are wind of speed V_w and water current of speed u_c .

Figure 1.20 shows a block diagram of the DP system.

I have programmed a simulator in Python of the longitudinal or surge motion of a ship, using model parameters provided by the company Kongsberg Maritime AS, Norway. The mathematical model is presented in Ch. 41.2. Figure 1.21 shows simulated responses where:

- The position reference (setpoint) y_r is changed softly (as a sinusoid) from 0 to 20 m from $t = 0$ s and 200 s, and is kept constant at 20 m between $t = 200$ s and 600 s, see the upper subplot of Figure 1.21.
- The speed, V_w , of the wind acting on the ship is changed from 0 to -30 m/s at time

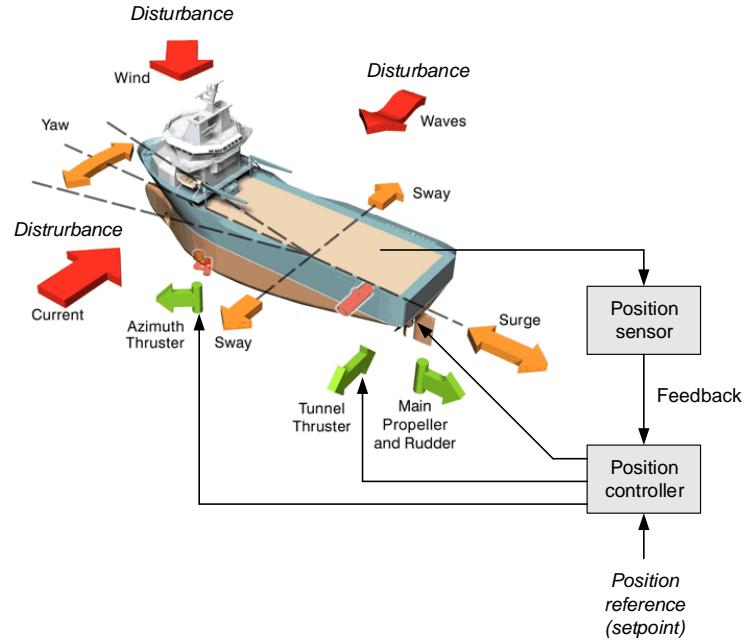


Figure 1.19: Dynamic positioning (DP) of ships is position control based on position sensors. (The drawing is based on a drawing originally made by Kongberg Maritime AS.)

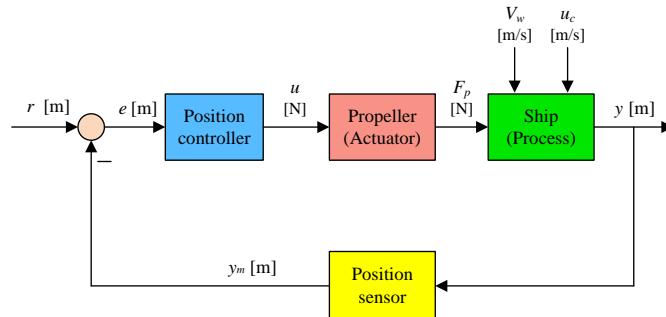


Figure 1.20: Block diagram of the position control system of the ship.

$t = 600 \text{ s}$.⁶ The wind speed causes a wind force, F_w , which is a disturbance on the DP system. F_w is plotted in the lower subplot of Figure 1.21.

Based on the control error, i.e. the difference between the position reference and the measured position, the controller generates a propeller force F_p in an attempt to keep the ship at the reference, see the lower plot in Figure 1.21. The ship tracks the varying setpoint with a maximum control error of a few meters, and the wind gust drives the ship off the setpoint with approximately 2.5 m. In steady state, the control error is zero.

If the control error is too large with PID feedback control, we must consider improved control. In Section 1.3 we shall see that *feedforward* control can reduce the control error

⁶You may say that this wind gust is unrealistically large. But keep in mind that this change may be due to the ship being in harsh weather with a strong side wind, and when the ship turns for some reason, the wind force suddenly changes direction and force relative to the ship.

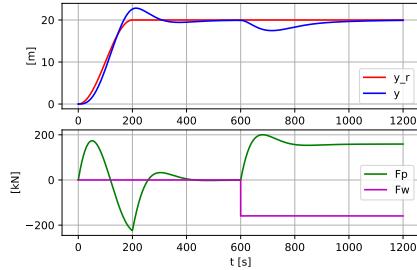


Figure 1.21: Simulation of position control of a ship

significantly!

1.3 The principle of feedforward control

By now, we know that feedback control – or error-driven control – can bring the process output variable to or close to the setpoint in steady state (strictly: when time goes to infinity). Feedback control is in most cases a sufficiently good control method. But sometimes you want more. The “problem” with feedback is that there *has to exist a control error different from zero* for any change of the control signal to take place, since the control variable is adjusted as a function of the control error. Here, *feedforward control* come to help. Feedforward control is based on the following information about the process to be controlled:

- ***Desired behaviour of the process in terms of the setpoint.*** This is always known, and is needed also in feedback control.
- ***Measurements of the process disturbances***
- ***A mathematical model of the process.*** A model is an abstract, mathematical representation of process, and expresses how the process behaves.

Using all this information to calculate the control signal, we may obtain excellent (accurate) control, i.e. very good setpoint tracking. However, the information is certainly more or less imprecise, causing the control to be more or less imperfect so that the control error becomes somewhat different from zero. But here comes feedback control to help: It reduces the control error which exist due to the inevitable imperfect feedforward control. Hence, when feedforward control is used, it is typically used together with feedback control.

Let us see how feedforward control can be used in the dynamic positioning system of a ship. The main disturbances are:

- Wind, represented with the wind speed, V_w [m/s], causing a disturbing force, F_w [N], on the ship.

- Water current, represented with the water current speed, u_c [m/s], causing a disturbing hydraulic force, F_h , on the ship.

Assume that we can measure or estimate V_w and u_c and calculate F_w and F_h , respectively. This can actually be done with a wind force model and a water current model. Such models are presented in Ch. 41.2, but we skip details about how to design the feedforward controller using these models until Ch. 19. Assuming that we know F_w and F_h from measurements of V_w and u_c and the mentioned models. To compensate for these disturbing forces, we can increase the propeller force, F_p , with an amount which is equal to the negative sum of F_w and F_h , thereby cancelling out their impacts on the ship motion! In other words, we establish a technical coupling – a feedforward controller – that cancels out the natural coupling that the disturbances has on the ship (the process). The feedforward controller also uses the position reference, y_r , to calculate the feedforward control signal.

The resulting control signal (which is a propeller force demand) is the sum of the feedback control signal and the feedforward control signal:

$$F_p = F_{p,fb} + F_{p,ff} \quad (1.5)$$

Figure 1.22 shows the dynamic positioning system with feedforward control, which is added to the feedback control based on position measurement.

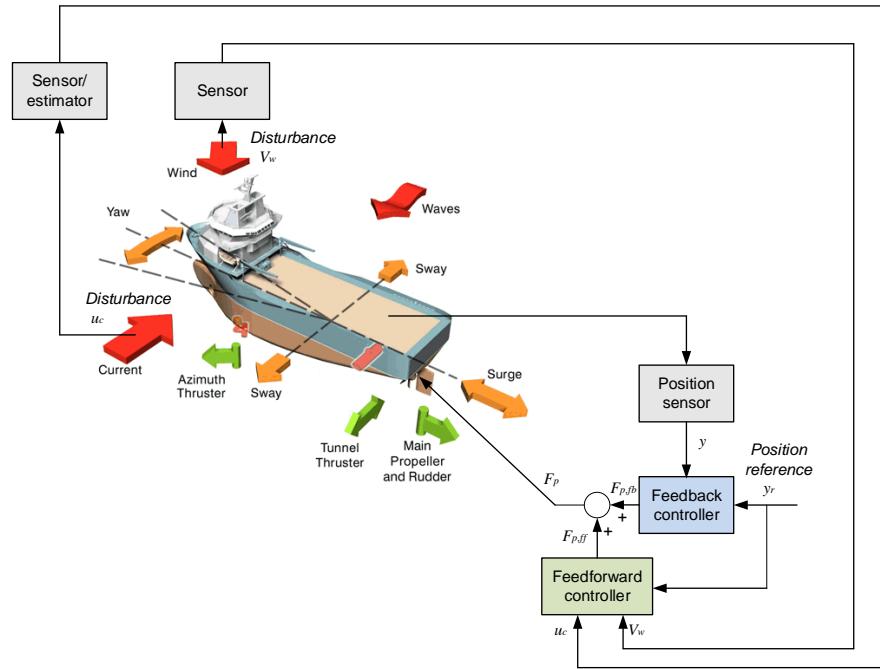


Figure 1.22: Dynamic positioning system with feedforward control added to the feedback control.

Figure 1.23 shows a block diagram of the dynamic positioning system. This block diagram does not convey much information which is not already given in Figure 1.22, but it has a more general form.

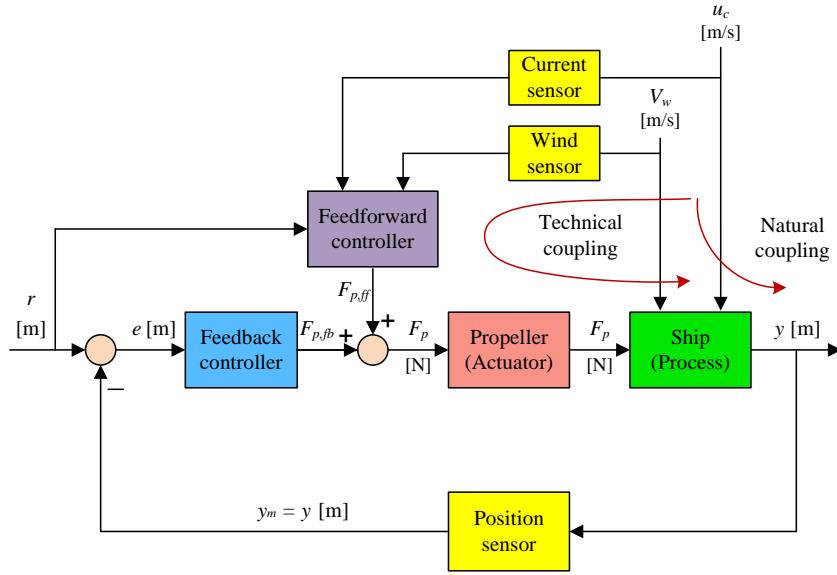


Figure 1.23: Dynamic positioning of a ship with both feedback control and feedforward control.

Figure 1.24 shows a simulation of the ship with position control based on feedforward control combined with feedback control. The control system behaves excellently:

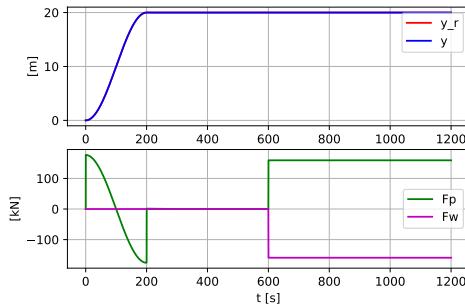


Figure 1.24: Simulation of the ship with position control based on feedforward control combined with feedback control.

- The position reference is tracked precisely (the position reference and the position measurement can hardly be distinguished).
- The wind disturbance (there is a wind gust at $t = 600$ s) is compensated for effectively. Notice that the propeller force counteracts exactly the wind force, causing the ship to stay still despite the heavy wind gust. This demonstrates very well the behaviour of the feedforward controller.

In practice, we can not expect such an excellent behaviour of the controlled ship because we can not (never) implement a perfect feedforward controller due to modeling and/or

measurement errors. However, we can expect a *large improvement* of the control with a well-designed feedforward controller.

1.4 Problems for Chapter 1

Problem 1.1 Components of a speed control system

Figure 1.25 shows the different components of a speed control system of an electric motor.

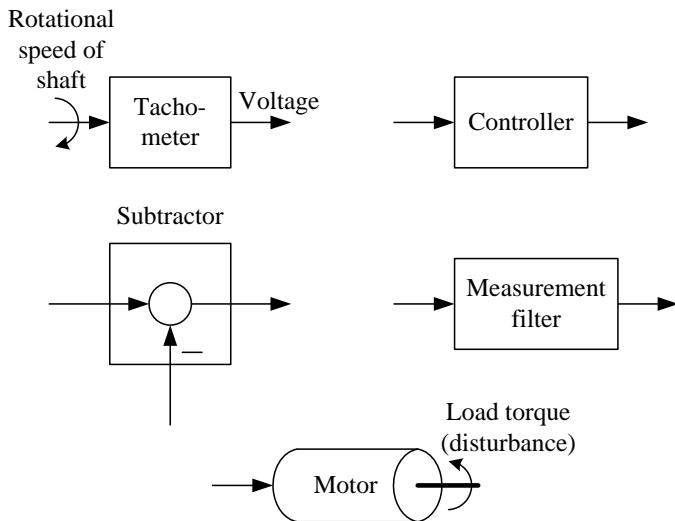


Figure 1.25: Components of a motor servo mechanism.

1. “Construct” a speed control system by connecting the components (draw a block diagram of the control system). Where is the control error in your block diagram?
2. How does the control system work? (Assume that the speed initially is equal to the speed reference (setpoint), and that the load torque is increased so that the motor speed is reduced.)

Problem 1.2 Automatic bartender

The weight control system shown in Figure 1.26 seems to be an automatic bartender.⁷

Explain how the control system works. (Explain the feedback control action.)

Do you know any other process of your daily life which uses the same principle of level control?

Problem 1.3 Evaporator

Figure 1.27 shows an evaporator where the product is created by evaporating the feed. (As an example, an evaporator is used to remove water from the half-finished fertilizer in a fertilizer factory.)

⁷From the book “Origins of Feedback Control” by O. Mayr.

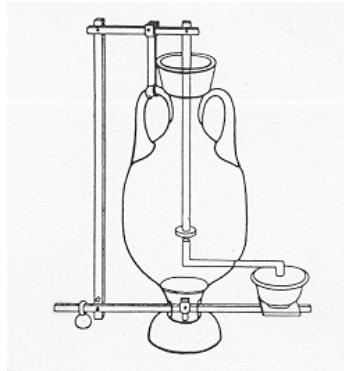


Figure 1.26: A weight control system from the Antics. An automatic bartender?

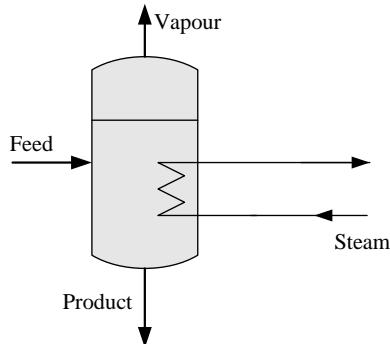


Figure 1.27: Evaporator.

Suggest a control structure by drawing a Process & Instrumentation Diagram of the evaporator according to the requirements listed below. (Process & Instrumentation Diagrams are covered in detail in Ch. 2, but I assume you have enough information from the examples in this chapter to draw such a diagram at this stage.)

- The feed flow is controlled to its setpoint. (Symbol of flow is F.)
- The liquid level is controlled to its setpoint. (Symbol of level is L.)
- The liquid temperature is controlled to its setpoint. (Symbol of temperature is T.)
- The vapour pressure is controlled to its setpoint. (Symbol of pressure is P.)
- You can use control valves as actuators. A symbol of a control valve is shown in Figure 1.28.

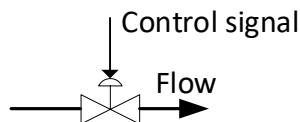


Figure 1.28: Symbol of a control valve.

Problem 1.4 *Control systems*

Below are mentioned three processes which are supposed to be controlled. The process output variable is indicated in parenthesis. For each of the processes:

- What is the control (manipulating) variable?
- What are the disturbances or loads or environmental variables (these are alternative names)?

Make your own assumptions. The processes are as follows:

1. Robot arm or manipulator driven by an electric motor (arm position)
2. Steam heated heat exchanger with some process fluid to be heated (temperature of fluid outlet)
3. Ship positioned with thrusters (denoted a dynamic positioning system) (ship position)

1.5 *Solutions to problems for Chapter 1*

Solution to Problem 1.1

- Figure 1.29 shows the speed control system. The control error, e , is the output of the

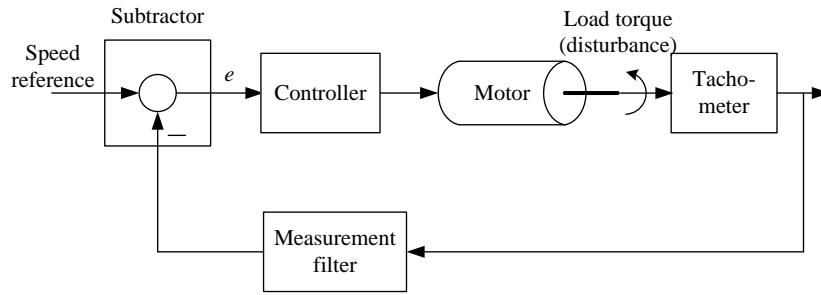


Figure 1.29: Speed control system.

subtractor.

- When the speed is reduced due to the increased load torque, the control error becomes different from zero, and positive. This non-zero, positive control error causes the controller to increase the control signal acting on the motor, so that the speed is increased. If the controller is properly chosen (it must have integral action, actually), the controller is able to adjust the control signal to exactly the new value that is needed to compensate for the load torque, and consequently the control error become zero – in steady state.

Solution to Problem 1.2

The purpose of the system is to fill just the right amount of liquid into the cup, i.e. level control. The system works as follows: As long as the level is low the inlet is open and the cup is being filled. The more liquid in the cup, the less opening in the inlet. Eventually, when the cup is full, the inlet is closed and hence, the desired level (the level setpoint) is reached.

Another system: Water toilet.

Solution to Problem 1.3

The control structure is shown in Figure 1.30.

Solution to Problem 1.4

- Robot arm: Control signal manipulates the motor. Disturbances: Torques due to the gravity and due to mechanical couplings to other links.

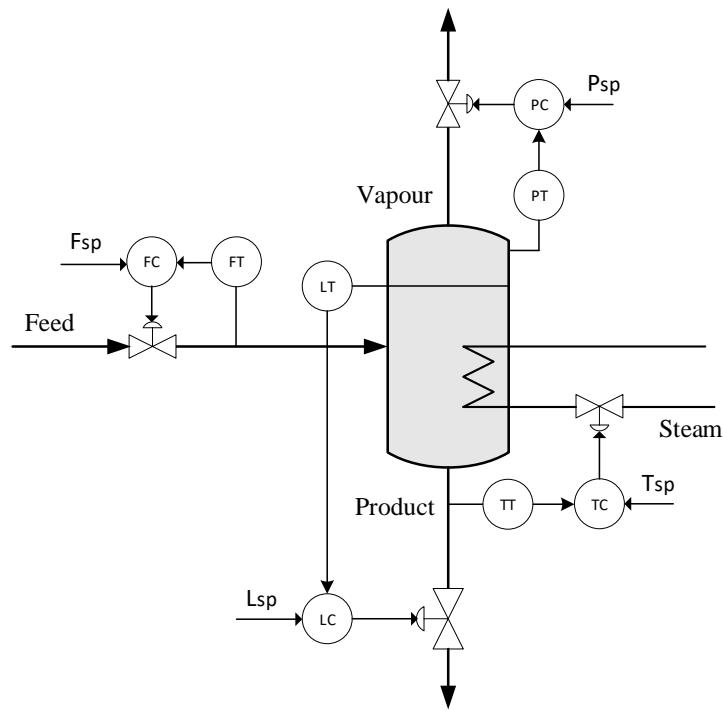


Figure 1.30: Control structure of an evaporator.

2. Heat exchanger: Control signal manipulates the valve. Disturbances: Temperature and pressure of inlet steam.
3. Ship: Control signal manipulates the propellers (thrusters). Disturbances: Wind, current, waves.

Chapter 2

Piping & Instrumentation Diagrams

In the industry, Process & Instrumentation Diagrams – P&I Ds are widely used as documentation of control systems. A P&I D shows the structure of the control system. It contains easily recognizable drawings of the processes to be controlled, e.g. tanks and heat exchangers, together with symbols for instrumentation equipment as sensors, controllers and actuators, e.g. valves and pumps. We have already seen examples of simple P&I Ds in the previous sections

This section gives a brief overview over codes and symbols used in the P&I D standard ISA-5.1¹.¹ There are other similar standards, both international, national and internal standards in factories.

2.1 Instrument codes

Instrument tags are used to give the instrument a unique name. The instrument code contains a letter code that expresses the function of the instrument (i.e. what it does), and a number code. For example, LC-102 is a Level Controller, while the LT-103 is a Level Transmitter (sensor). When we refer to the instrument code, we use a hyphen between the letter code and the numeric code, but when the instrument code is written in a P&I D, the hyphen is not included.

2.1.1 Letter codes

Table 2.1 shows the most commonly used letter codes used in Process & Instrumentation Diagrams.

Some examples:

- LC = Level Controller

¹ISA = International Society of Automation

Table 2.1: Common letter codes (identifiers) for instrument symbols in the ISA 5.1 standard

Letter code	1. letter	Subsequent modicator to 1. letter	Subsequent letter
A	Analysis		Alarm
B	Burner, Combustion		
C	User's choice		Control
D	User's choice	Differential	
E	Voltage		Sensor, Primary element
F	Flow rate	Ratio	
G	User's choice		Glass, Gauge
H	Hand		High
I	Current (electrical)		Indicate
J	Power		
L	Level		Low
P	Pressure		
Q	Quantity	Integrate, Totalize	
R	Radiation		Record
S	Speed, Frequency		Switch
T	Temperature		Transmit
V	Vibration		Valve
W	Weight, Force		
Y			Computation
Z	Position	Safety Instrumented System (Interlock)	

- LIC = Level Indicator Controller = level controller with level indicator
- LE = Level Element = transducer; the primary device used to detect the process variable to be measured, e.g. an ultrasound level sensor. Note: In P&I Ds, we will use the letter code LT, not LE, to represent the level sensor.
- TT = Temperature Transmitter = temperature sensor
- PDT = Pressure Differential Transmitter
- TY = Temperature computation = some formula related to temperature measurement or regulation, e.g. a feedforward controller based on temperature measurement. The formula must be indicated on an appropriate place in the P&I D.

It is the function that determines which letter symbol to be used. Example: If a differential pressure (dp) cell is used to measure fluid flow in a pipeline, the letter symbol FT – not PT – must be used.

In P&I Ds where emphasis is placed on documenting control structures, and with little emphasis on details such as whether the controller contains a display for showing the process value, you can use the simplest letter code, e.g. LC instead of LIC.

2.1.2 Numeric codes

Numeric codes are used to number the instruments. The ISA 5.1 standard does not define any specific standard for numbering instruments (such as controllers and sensors), but still recommends choosing between so-called parallel and serial numbering:

- **Parallel numbering:** New numbering is started for each new first letter, with 1 being the lowest number.
Examples: FIC-101. FT-102. LIC-101.
- **Serial numbering:** The numbering is continuous, regardless of letter.
Examples: FIC-101. FT-102. LIC-103. LIC-104.

Serial numbering is used in the P&I Ds in this book.

The first digit of the number code may represent e.g.

- Area or field
- Unit
- Plant (factory)

The number of digits is not defined by the standard. It must be chosen so that all elements can have their unique instrument code.

2.1.3 Localization

You can use various symbols to indicate the localization of the instrument, so the user knows where she/he can find it (this can be important information particularly in emergency situations). Three different localizations are shown in Figure 2.1.

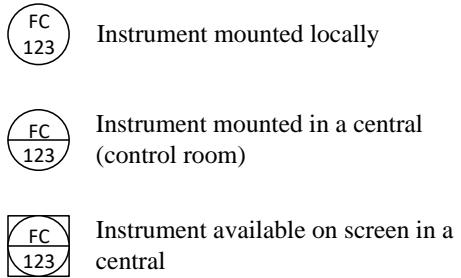


Figure 2.1: Main instrument symbols (FC123 is one example of instrument code)

If it is not important to show the location in a P&I D, you can use the simplest symbol, which is a circle with only the instrument code inside (the upper symbol in Figure 2.1).

2.2 Signals

Figure 2.2 shows various signal symbols.

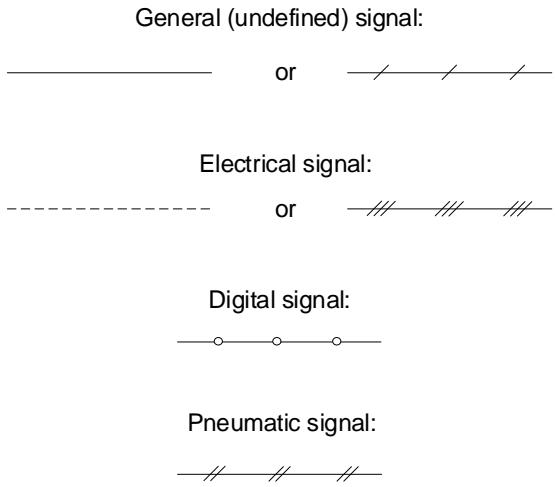


Figure 2.2: Signal symbols

2.3 Material flows

Figure 2.3 shows how material flows (process flows) can be drawn in a P&I D. Material flows should be drawn with lines that are clearly thicker than signal lines. If the plant is so

extensive that more than one flowchart is needed for all of the units, references to the adjacent diagrams are indicated, as shown in Figure 2.3.

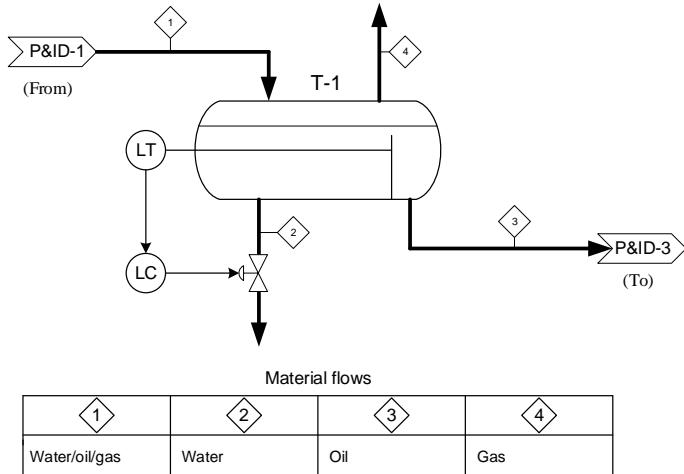


Figure 2.3: Drawing of material flows in a P&I D. (The example is a water/oil/gas separator. The vertical line inside the separator is a so-called weir that is used to separate the oil and water. The oil flows over the weir.)

In Figure 2.3, the material flows are identified with labels on the pipelines, and the flows are defined at the bottom of the diagram. However, it is not required in the ISA 5.1 standard to label material flows in P&I Ds. Instead, material flows can be labelled in so-called Process Flow Diagrams (PFDs), which are diagrams similar to technical flow charts, but which show fewer details of the instrumentation. ISA's documentation guidelines encourages to avoid giving the same information in different diagrams. However, this rule is not absolute, and control structures may appear both in P&I Ds and PFDs, and you may define material flows in P&I Ds if you find it appropriate.

Note: In Figure 2.3, I have drawn arrow heads on the signal lines pertaining to the level control loop. In the ISA 5.1 standard, arrow heads are generally not used on signal lines, only ordinary lines. However, in this book I will still draw arrow heads to make the direction of the signal flow completely clear. In comprehensive P&I Ds it may cause extra stress to understand the signal direction from the symbols, although the direction is obvious, but implicit. For example, the signal direction between a LT and a LC is obvious – the signal goes from the LT to the LC.

2.4 Process equipment

2.4.1 Codes of process equipment

P&I Ds contain drawings of process equipment such as tanks, heat exchangers, pumps, valves, etc. These can also be represented by letter and number codes, e.g. H-1 for Heat Exchanger number 1. The ISA 5.1 standard does not define process equipment letters, but some common letter codes are shown in Table . The numbering is usually serial.

Table 2.2: Letter codes of process equipment

Code	Equipment
C	Column
D	Drum
F	Furnace
H	Heat exchanger
K	Compressor
M	Motor
P	Pump
R	Reactor
T	Tank
V	Valve. Vessel

2.4.2 Valves

Some common valve symbols are shown in Figure 2.4.

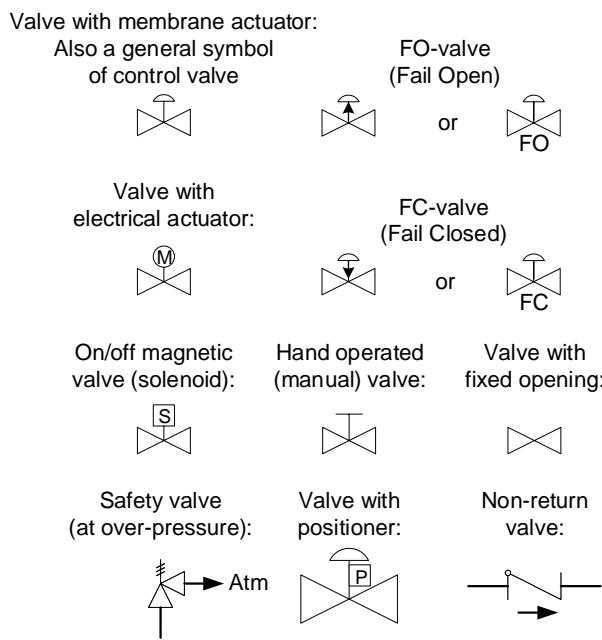


Figure 2.4: Valve symbols.

One comment:

- There are symbols for a Fail Open (FO) valve and a Fail Closed (FC) valve. The “fail” situation means that the power, e.g. air pressure, needed to operate the valve fails. So, a FC valve closes if the air pressure vanish.

2.4.3 Rotational flow components

Figure 2.5 shows various rotational flow components.

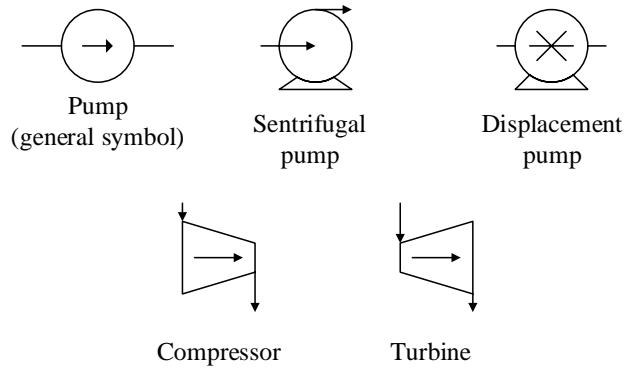


Figure 2.5: Symbols of pumps, compressors and turbines

2.4.4 Heat exchangers

Figure 2.6 shows symbols of heat exchangers.

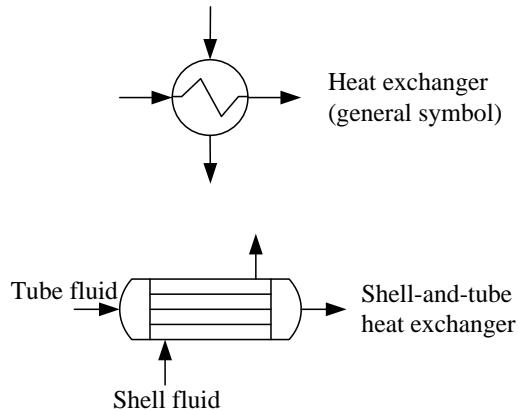


Figure 2.6: Symbols of heat exchangers

2.4.5 Vessels

Figure 2.7 shows symbols of various types of vessels.

2.4.6 Mathematical functions

Figure 2.8 shows two alternative ways of including mathematical functions in P&I Ds. In the example, the control signal u_{ff} is from a feedforward controller, to be added to the

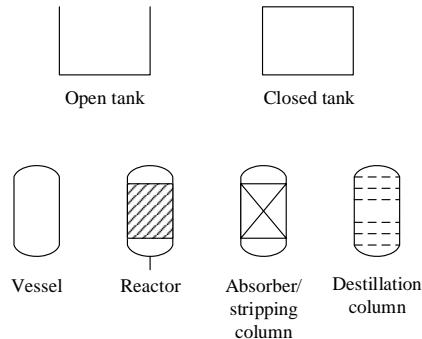


Figure 2.7: Symbols of various types of vessels

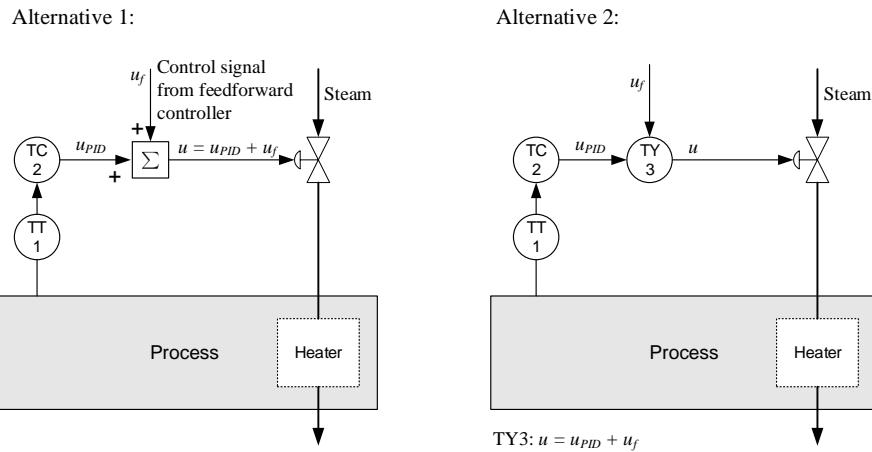


Figure 2.8: Two alternative ways of including mathematical functions in P&I Ds.

control signal u_{PID} generated by a feedback temperature controller (PID controller). The total control signal is $u = u_{PID} + u_{ff}$. Two alternatives are shown:

- **Alternative 1:** The mathematical function is specified directly with a suitable mathematical symbol.
- **Alternative 2:** A general letter symbol is used, here TY , where Y stands for “computation”, cf. Table 2.1. In this alternative, the mathematical function must be specified at a suitable location in the P&I D.

2.4.7 Logical functions

In addition to arithmetic functions as discussed above, one can enter logical functions into the diagram e.g. to express that a valve should be opened if the temperature in a tank is above a certain limit. Such logical functions are used to implement safety actions, i.e. interlocks, which are activated in critical situations.

A special diagram type denoted System and Control Diagram (SCD) has been developed in Norway for the oil and gas industry. SCDs defines a uniform documentation of both logical

control for safety, i.e. interlocks, and process control.

2.5 *Problems for Chapter 2*

Problem 2.1 *P&I D of a level control system*

Given a level control system for a water tank with inlet and outlet. The level control is based on manipulation of a pump in the inlet. The controller is accessible via a computer screen in a control room. Both the control signal and the measuring signal are electric. Draw a Piping & Instrumentation Diagram of the control system.

Problem 2.2 *P&I D of a temperature control system*

Draw a Piping & Instrumentation Diagram of a temperature control system of a process. You can select by yourself the process to be temperature controlled. It is assumed that the controller is accessible in the field. The instruments are numbered with parallel numbering with three digits. Both the control signal and the measurement signal are digital.

Problem 2.3 *P&I D of a separator*

Figure 2.9 shows an oil/water/gas separator. (The separation takes place by a sufficient retention time.) The vertical line inside the separator is a so-called weir that is used to separate the oil and water. The oil flows over the weir.

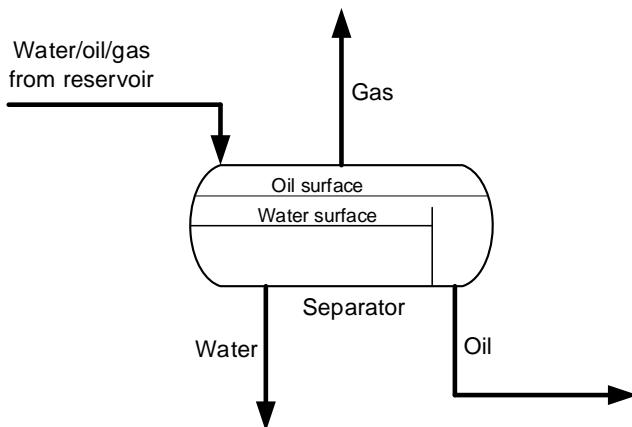


Figure 2.9: Oil/water/gas separator.

Draw a Process & Instrumentation Diagram for the separator according to the following requirements:

- The oil and water levels and the gas pressure in the separator are controlled to their setpoints.

- The setpoints can be shown explicitly with arrows.
- Signal lines may be drawn with arrows (although arrows are not normally used in P&I D standards).
- Control valves are used as actuators, and should be labeled.
- The separator should be labeled.
- Parallel numbering is used.

2.6 *Solutions to problems for Chapter 2*

Solution to Problem 2.1

See Figure 2.10.

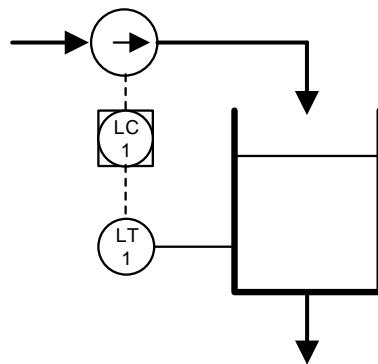


Figure 2.10: P&I Diagram of level control system.

Solution to Problem 2.2

See Figure 2.11.

Solution to Problem 2.3

The P&I D of the separator is shown in Figure 2.12.

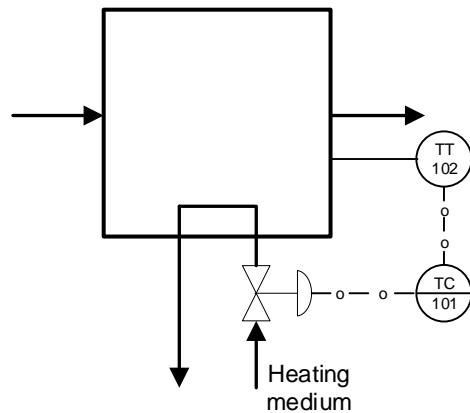


Figure 2.11: P&I Diagram of temperature control system.

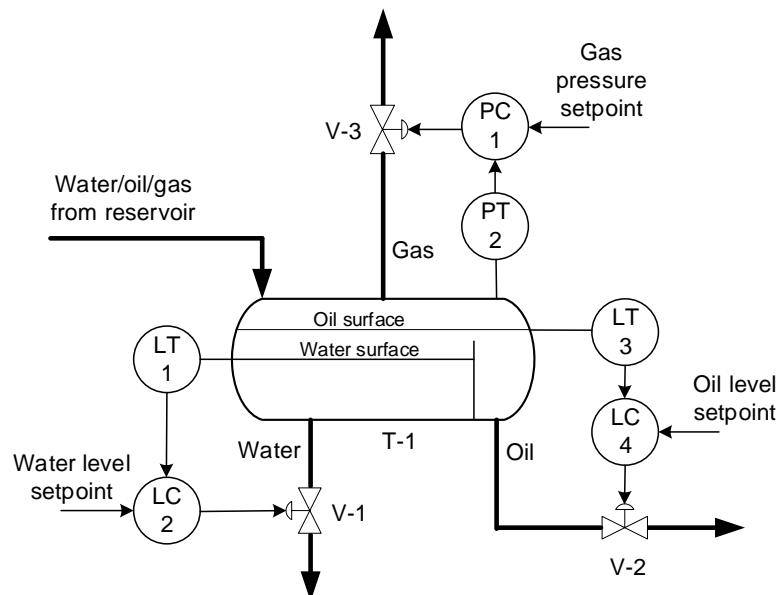


Figure 2.12: Control structure of an oil/water/gas separator.

Chapter 3

Components of control systems

To appear.

3.1 Introduction

To appear.

3.2 Controllers

To appear.

3.2.1 Introduction

3.2.2 Process controllers

3.2.3 Programmable logic controllers (PLCs)

3.2.4 Programmable automation controllers (PACs)

3.2.5 Plantwide control systems

3.2.6 Simple PC-based control systems

3.3 Actuators

To appear.

3.3.1 Introduction

3.3.2 Valves

3.3.3 Pumps

3.3.4 Motors

3.3.5 Heaters

3.3.6 Signal conditioning of control signals

3.3.6.1 Scaling

3.3.6.2 Digital-analog (DA) conversion

3.3.6.3 Pulse-width modulation

3.3.6.4 Current to voltage conversion

3.4 Sensors

To appear.

3.4.1 Introduction

3.4.2 Temperature sensors

3.4.3 Level sensors

3.4.4 Pressure sensors

3.4.5 Flow sensors

3.4.6 Position sensors

3.4.7 Speed sensors

3.4.8 Signal conditioning of measurement signals

3.4.8.1 Alternative forms of measurement signals

3.4.8.1.1 Current signals

3.4.8.1.2 Voltage signals

3.4.8.1.3 Current loop

3.4.8.1.4 Pneumatic signals

3.4.8.1.5 Digital signals and analog-digital (AD) conversion

3.4.8.2 Scaling

3.4.8.3 Accuracy of measurement signals

3.4.8.4 Adjustment of sensors

3.4.8.5 Calibration of sensors

3.4.8.6 Measurement filters

3.4.8.6.1 Introduction The measurement filter shown in Figure 12.1 is an important part of the feedback control loop. The purpose of the filter is to filter out, or attenuate, the more or less random measurement noise from the measurement signal, so that the measurement signal used by the controller is more smooth, in turn causing a smoother control signal.

Since the purpose of the filter is to filter out high-frequent measurement noise while letting assumeably low-frequent variations in the process variable (e.g. temperature) pass unfiltered, we must select a lowpass filter as a measurement filter. There are many types of signal filters. For example, a high-pass filter will not work here.

For illustration, Figure 3.1 shows the response of a lowpass filter output due to the following input signals:

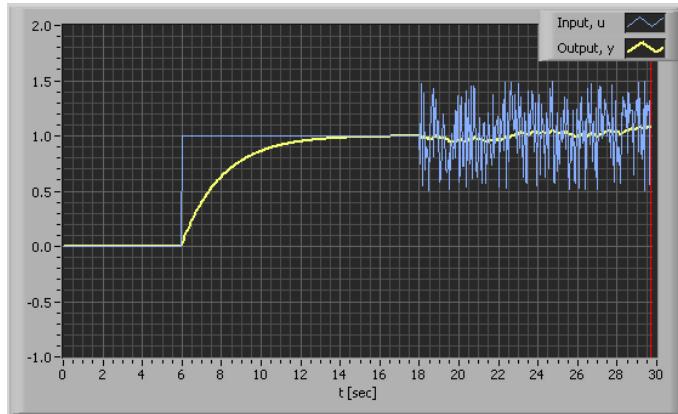


Figure 3.1: Response in a lowpass filter output due to a input step change at time 6 sec and random noise at the input from time 18 s.

- $0 < t < 6$ s: Zero input (no noise).
- $t = 6$ s: A step change of the input.
- $t \geq 18$ s: Random noise at the input

Comments to the responses shown in Figure 3.1:

- The filter attenuates the noise, as we want it to do.
- The filter has a sluggish response to the step change at the input. If this step represents the real changes of the process variable (although in practice such step changes would probably not appear), the filter actually have removed some information about the behaviour of the process variable, which may be unfortunate in an application.
- From the above two points, we can conclude that it is important to tune a filter for good noise attenuation while avoiding attenuation of variations in the process variables.

3.4.8.6.2 Continuous-time filter function There are various types of measurement filters in use. But my impression is that the most commonly used filter function is the time constant filter. It can be represented by the following differential equation:

$$T_f y'_{\text{mf}}(t) + y_{\text{mf}}(t) = y_{\text{m}}(t) \quad (3.1)$$

where: T_f [s] is the filter time constant. y_{m} is the unfiltered (raw) measurement signal, and y_{mf} is the filtered measurement signal.

Sometimes, the filter is represented by a transfer function corresponding to (3.1):

$$\frac{y_{\text{mf}}(s)}{y_{\text{m}}(s)} = H_f(s) = \frac{1}{T_f s + 1} \quad (3.2)$$

In (3.2), s is the Laplace variable. Transfer functions are described in Ch. 9.)

The tuning parameter of the filter is the filter time constant, T_f . How to set T_f ? That depends on how much noise smoothing you want and the dynamic properties of the process. If you do not have any other requirements, you can initially set it equal to one tenth of the time constant of the process to be controlled to avoid the filter adding too much sluggishness to the control loop. It turns out that a time constant of a few seconds is a typical value in industrial control loops (of e.g. temperature loops).¹

3.4.8.6.3 Analog RC circuit filter To appear

3.4.8.6.4 Discrete-time filter algorithm In computer-based automation systems the filter is available as a function block containing program code that implements the filter algorithm. We will derive a filter algorithm with (3.1) as the basis. Let us assume that the time step of the filter algorithm is T_s [s]. It is tradition to discretize a time constant filter using the Euler Backward method. Approximating the time derivative in (3.1) with Euler Backward approximation, gives

$$T_f \frac{y_{\text{mf},k} - y_{\text{mf},k-1}}{T_s} + y_{\text{mf},k} = y_{\text{m},k} \quad (3.3)$$

¹In one of the Fuji PID temperature controllers the preset value of the filter time constant is 5 sec.

We need a formula for the filter output at time index k . Therefore, we solve (3.3) with respect to $y_{\text{mf},k}$ to get the filter algorithm:

$$y_{\text{mf},k} = (1 - a)y_{\text{mf},k-1} + a y_{\text{m},k} \quad (3.4)$$

where a is a filter parameter

$$a = \frac{T_s}{T_f + T_s} \quad (3.5)$$

The filter algorithm (3.4) is sometimes denoted the exponentially weighted moving average (EWMA) filter.

It is important that T_f is considerably larger than T_s , or in other words, that T_s is considerably smaller than T_f . Otherwise, the filter algorithm (3.4) may behave quite differently from the continuous-time filter (3.1) from which it is derived. A rule of thumb for the upper limit of T_s is:

$$T_f \geq 10T_s \quad (3.6)$$

3.4.8.6.5 Moving averaging measurement filter ...

3.4.8.6.5.1 Batch MA filter algorithm

3.4.8.6.5.2 Recursive (online) MA filter algorithm

3.4.8.6.6 Comparing moving average filter with time constant filter

3.5 *Problems for Chapter 3*

To appear.

Problem 3.1 *To appear*

To appear.

3.6 *Solutions to problems for Chapter 3*

Solution to Problem 3.1

To appear.

Part II

MODELING and SIMULATION OF DYNAMIC SYSTEMS

Chapter 4

Mechanistic modeling

4.1 Introduction

This chapter describes basic principles of mathematical modeling of dynamic systems. A mathematical model is the set of equations which describes the behavior of the system. The chapter focuses on how to develop dynamic models, and you will see that the models are differential equations. The differential equations can be represented on various forms, for example state space models (Ch. 5), mathematical block diagrams (Ch. 7), or transfer functions (Ch. 9). One very important application of models is creating simulators (Ch. 6).

Unfortunately we can never make a completely precise model of a physical system. There are always phenomena which we will not be able to model. Thus, there will always be model errors or model uncertainties. But even if a model describes just a part of the reality it can be very useful for analysis and design – if it describes the dominating dynamic properties of the system. A saying is “All models are wrong, but some are useful.”

This chapter describes modeling based on physical principles. Such modeling has alternative names:

- Mechanistic modeling
- First principles modeling
- White-box modeling, as opposed to black-box modeling where the models are just mathematical expressions capable to represent the dynamic relation between the output and input of the system, like transfer functions.

4.2 What is a dynamic system?

Dynamic means “which has to do with the movement and change”. Dynamic systems are systems where the variables can vary or develop with time. We say that dynamic systems have dynamic responses. Figure 4.1

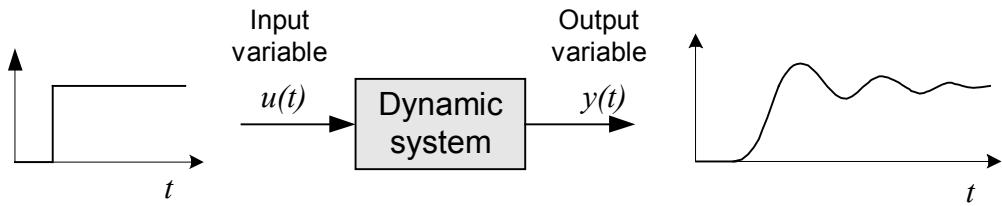


Figure 4.1: Dynamic systems are systems where the variables can vary or develop as functions of time.

gives an illustration. The figure shows a block diagram of a dynamic system. The input variable is here a step function, and the response in the output variable is dynamic since it changes with time. In general, dynamic systems may have more than one input variable and more than one output variable.

Here are some examples of dynamic systems:

- **A liquid tank.**

- Input (variable): Inflow.
- Output (variable): Level.

- **A motor.**

- Input: Motor control voltage.
- Output: Speed.

- **A heated water tank.**

- *Input*: Supplied heat.
- *Output*: Temperature in the water in the tank.

- **A robot manipulator.**

- *Input*: Control signal to motor.
- *Output*: Arm position.

- **A ship.**

- *Input*: Thruster force.
- *Output*: Ship position.

- **A signal filter:**

- *Input*: Filter input to be filtered (smoothed).
- *Output*: Filter output signal.

- **A control system for a physical process:**

- *Input*: Setpoint.
- *Output*: Process output variable.

4.3 A procedure for mathematical modeling

Below is described a procedure for developing dynamic mathematical models for physical systems:

1. **Define systems boundaries.** All physical systems works in interaction with other systems. Therefore it is necessary to define the boundaries of the system before we can begin developing a mathematical model for the system, but in most cases defining the boundaries is done quite naturally.
2. **Make simplifying assumptions.** One example is to assume that the temperature in a tank is the same everywhere in the tank, that is, there are homogeneous conditions in the tank.
3. **Use the Balance Law for the physical balances in the system.** See Figure 4.2.

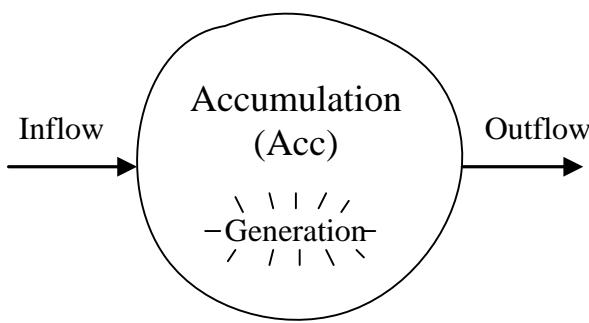


Figure 4.2: Illustration of the Balance Lawaw.

In the figure, “Accumulation” is a general term. Specifically, it can be accumulated mass, mole, energy, momentum, or electrical charge in a physical system. “Inflow” represents possibly several inflows. “Outflow” represents possibly several outflows. “Generation” can be e.g. material generated by chemical reactions, or generated energy in an exothermal reactor.

The accumulation in the system at time t is given by the following integral, where the term Acc is used for short:

$$\text{Acc}(t) = \text{Acc}(0) + \int_0^t \text{Acc}'(\theta) d\theta \quad (4.1)$$

where $t = 0$ is the initial time. The integrand, Acc' , is the rate of change, or time derivative, of the Acc, and is given by the following differential equation, which is often termed the the *Balance Law* of the system:

$$\text{Acc}' = \text{Inflow} - \text{Outflow} + \text{Generation} \quad (4.2)$$

Usually, only the (4.2) is said to be the *mathematical model* of the system, although (4.1) is also essential to calculate the Accumulation.

4. **Draw an overall block diagram showing inputs, outputs and parameters.** A block diagram makes the model appear more clearly. Figure 4.3 shows a general overall block diagram. In the figure, the single lines can represent a number of variables. The output variables are typically the accumulations, or accumulations

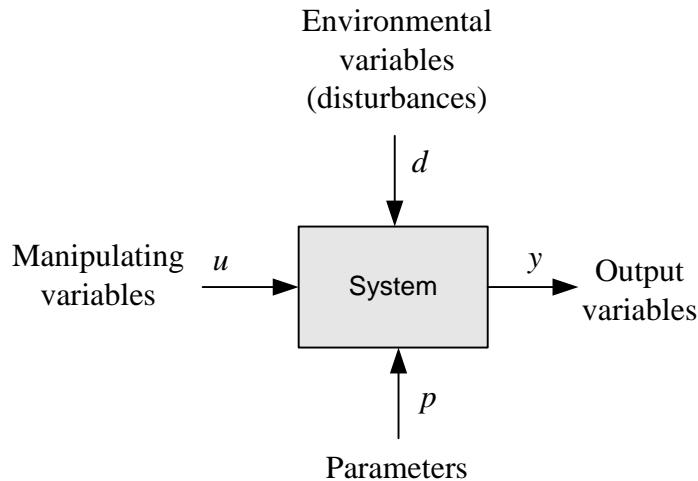


Figure 4.3: Overall block diagram.

multiplied by some constants. The input variables are of two kinds: (1) *Manipulating (adjustable) variables* which you can use to manipulate or control the system (like power from a heater), and (2) *environmental variables* which you can not manipulate (like environmental temperature). In the context of control systems, environmental variables are often denoted disturbance variables.

Mathematically, the input variables are *independent variables*, and the output variables are *dependent variables*.

The parameters of the model are quantities in the model which typically (but not necessarily) have constant values, like liquid density and spring constant. If a parameter have a varying value, it may alternatively be regarded as an environmental variable.

5. **Present the model on a proper form.** The choice of model form depends on the application of the model. The most common model forms are:

- Differential equation* with the highest order of the time derivative alone on the left side of the equation. This is the model form used in examples and problems in the present chapter.
- State space models* (Ch. 5.2), which are just the differential equations written with the time derivatives alone on the left side of the equation. (If the model is a differential equation of order two or higher, the state space model is an equivalent set of first order differential equation.)
- Mathematical block diagrams* (Ch. 7).
- Transfer functions* (Ch. 9) which applies only for linear differential equation models.

The following sections contain several examples of mathematical modeling. In those examples, items 1 and 2 above are applied more or less implicitly.

Comments on notation

For simplicity, I write the time-varying variables without the time as argument. For example, I write m instead of $m(t)$. However, in cases where time delay is involved, it is important to show the time argument explicitly, for example $u(t - \tau)$ where τ is the time delay. This is the case in Problem 4.2 at the end of this chapter, where there is a time delay due to a conveyor belt.

4.4 Mathematical modeling of material systems

In a system where the mass may vary, mass is the “accumulation” in the (4.2) which now becomes a *mass balance*:

$$m' = \sum_i F_i \quad (4.3)$$

where m [kg] is the mass, and F_i [kg/s] is mass inflow (no. i).

Example 4.1 Mass balance of a liquid tank

Figure 4.4 shows a liquid tank with inflow and outflow. Assume that the inflow can be manipulated with e.g. a pump. The outflow is due to some consumption which here is regarded as an environmental variable.

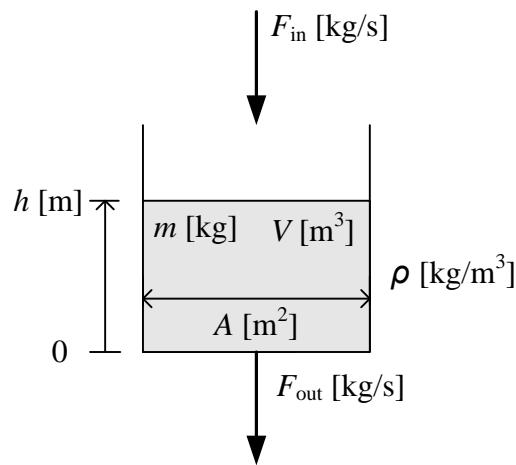


Figure 4.4: Example 4.1: Liquid tank.

The tank has straight, vertical walls. F_{in} and F_{out} are mass flows. h is liquid level. V is liquid volume. m is mass. A is cross sectional area. The density of the liquid, ρ , is assumed the same in the inlet, the outlet, and the tank.

The mass balance for the mass in the tank is:

$$m' = F_{\text{in}} - F_{\text{out}} \quad (4.4)$$

which is a differential equation for m . An additional condition for the differential equation is $m \geq 0$.

Now, m is the integral of m' :

$$m(t) = m(0) + \int_0^t m'(\theta) d\theta \quad (4.5)$$

with the condition $0 \leq h \leq m_{\max}$ where m_{\max} is the maximum liquid mass in the tank.

A model of the level

Maybe you are more interested in how level h varies than how m varies? The relation between h and m is given by

$$m = \rho V = \rho Ah \quad (4.6)$$

We insert this into the mass balance (4.4), which then becomes

$$m' = (\rho V)' = (\rho Ah)' = \rho Ah' = F_{\text{in}} - F_{\text{out}} \quad (4.7)$$

where parameters ρ and A have been moved outside the derivation (they are assumed constant).

By dividing by ρA we get the following differential equation of h :

$$h' = \frac{1}{\rho A} (F_{\text{in}} - F_{\text{out}}) \quad (4.8)$$

The level h is the integral of h' :

$$h(t) = h(0) + \int_0^t h'(\theta) d\theta \quad (4.9)$$

with the condition $0 \leq h \leq h_{\max}$.

Figure 4.5 shows an overall block diagram of the model (4.8). Note that F_{out} is an *input* variable despite it represents a physical *outflow* from the tank!

[End of Example 4.1]

4.5 Mathematical modeling of thermal systems

Mathematical modeling of thermal systems is based on the to set up energy balances. The term *energy* covers temperature-dependent energy, which we can call thermal energy, and kinetic and potential energy. In general we must assume that there is a transformation from

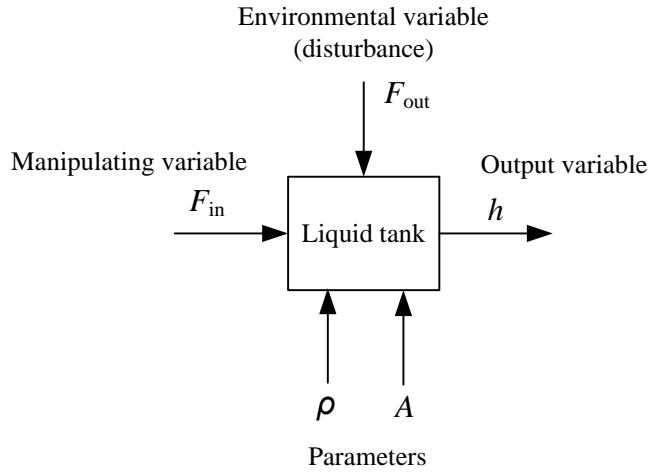


Figure 4.5: Example 4.1: Overall block diagram of the liquid tank.

one energy form to another within a given system. For example, kinetic energy can be transformed to thermal energy via friction. For many thermal systems we can assume that the energy consists of only thermal energy and we can neglect the transformation from kinetic and potential energy to thermal energy.

For thermal systems, the “accumulation” in the Balance Law (4.2) is thermal energy. Thus, the Balance Law becomes an *energy balance*:

$$E' = \sum_i Q_i \quad (4.10)$$

where E [J] is the thermal energy, and Q_i [J/s] is energy inflow no. i . The energy E is often assumed to be proportional to the temperature and the mass (or volume):

$$E = cmT = cpVT = CT \quad (4.11)$$

where T [K] is the temperature, c [J/(kg K)] is specific heat capacity, m [kg] is mass, V [m^3] volume, ρ [kg/ m^3] is density, C [J/K] is total heat capacity.

Example 4.2 Heated liquid tank

Figure 4.6 shows a liquid tank with continuous liquid inflow and outflow. There is heat transfer with the environment through the walls. The liquid receives power through a heating element. P is power from the heating element. T is temperature in the tank and in the outlet flow. T_i is the temperature in the inlet flow. F is mass flow. m is mass of liquid (constant). c is specific heat capacity. G is heat transfer coefficient.

We will now set up an energy balance for the liquid in the tank to find the differential equation which describes the temperature $T(t)$. We will then make the following assumptions:

- The temperature in the liquid in the tank is homogeneous (due to the stirring machine).

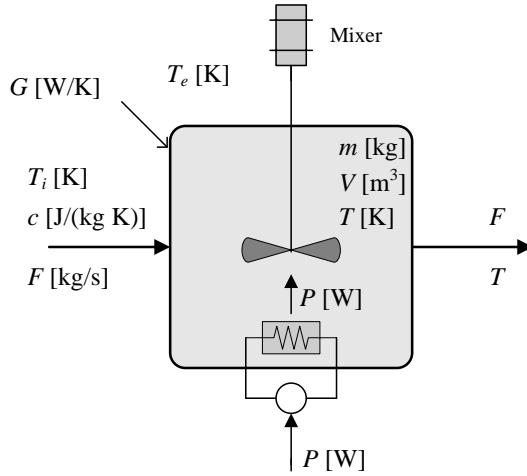


Figure 4.6: Example 4.2: Heated liquid tank.

- The inflow and outflow are equal, and the tank is filled by liquid.
- There is no storage of thermal energy in the heating element itself. This means that all of the supplied power to the heating element is supplied (immediately) to the liquid. (Thus, we do not write an energy balance for the heating element.)

The energy balance is based on the following energy transports (power):

1. Power from the heating element:

$$P = Q_1 \quad (4.12)$$

2. Power from the inflow:

$$cFT_i = Q_2 \quad (4.13)$$

3. Power removed via the outflow:

$$-cFT = Q_3 \quad (4.14)$$

4. Power via heat transfer from (or to) the environment:

$$G(T_e - T) = Q_4 \quad (4.15)$$

The energy balance is

$$E' = Q_1 + Q_2 + Q_3 + Q_4 \quad (4.16)$$

where the energy is given by

$$E = cmT$$

The energy balance can then be written as

$$(cmT)' = P + cFT_i - cFT + G(T_e - T) \quad (4.17)$$

If we assume that c and m are constant, we can move cm outside the derivative term. Furthermore, we can combine the terms on the right side. The result is

$$cmT' = P + cF(T_i - T) + G(T_e - T) \quad (4.18)$$

or:

$$T' = \frac{1}{cm} [P + cF(T_i - T) + G(T_e - T)] \quad (4.19)$$

The temperature T is the integral of T' :

$$T(t) = T(0) + \int_0^t T'(\theta) d\theta \quad (4.20)$$

Figure 4.7 shows an overall block diagram of the model (4.19).

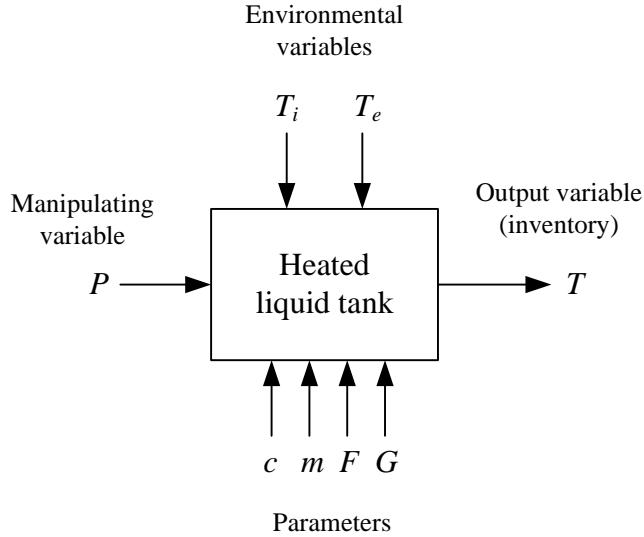


Figure 4.7: Example 4.2: Overall block diagram of heated tank.

[End of Example 4.2]

4.6 Mathematical modeling of kinetic systems

4.6.1 Systems with linear motion

For kinetic systems in the form of a body with linear motion (we will soon study rotational motion), the “accumulation” term in the Balance Law (4.2) is momentum. Thus, the Balance Law becomes a *momentum balance*, which is often denoted *force balance*:

$$I' = (mv)' = \sum_i F_i \quad (4.21)$$

where I [Ns] is the momentum (mass times speed), and F_i is force (no. i). I is

$$I = mv = mx' \quad (4.22)$$

where m [kg] is mass, v [m/s] is speed, and x [m] is position.

If m is constant, m can be moved outside the derivative term in (4.21), which then becomes

$$mv' = mx'' = ma = \sum_i F_i \quad (4.23)$$

where $v' = x'' = a$ is acceleration. (4.23) is the well-known Newton's second law (the sum of forces is equal to mass times acceleration).

Often the mass m is constant. Then (4.23) can be used for mathematical modeling. But if m is time-varying, (4.21) must be used. One example of a system with *varying* mass is a conveyor belt where the mass on the belt is varying.

Example 4.3 Block diagram of mass-spring-damper system

Figure 4.8 shows a mass-spring-damper-system.¹ y is position. F is applied force. D is

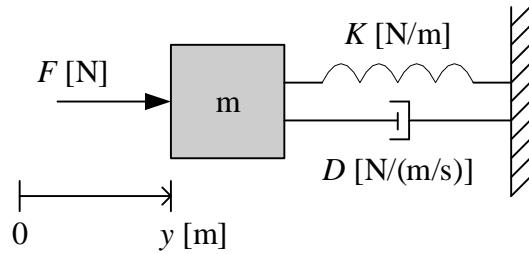


Figure 4.8: Mass-spring-damper.

damping constant. K is spring constant. It is assumed that the damping force F_d is proportional to the speed:

$$F_d = Dy' \quad (4.24)$$

and that the spring force F_s is proportional to the position of the mass:

$$F_s = Ky \quad (4.25)$$

The spring force is assumed to be zero when y is zero. Force balance (Newton's 2. Law) yields²

$$\begin{aligned} my'' &= F - F_d - F_s \\ &= F - D\dot{y} - Ky \end{aligned} \quad (4.26)$$

which is a second order differential equation, which we can write as

$$y'' = (F - D\dot{y} - Ky) / m \quad (4.27)$$

¹The mass-spring-damper system is not a typical system found in process control. It is chosen here because it is easy to develop a mathematical model using well known physical principles, here the Newton's second law. Examples of more relevance to process control are described in Chapter 4.

²Double-dot represents second order time-derivative: $\ddot{y}(t) \equiv d^2y(t)/dt^2$

Speed y' is the integral of acceleration y'' :

$$y'(t) = y'(0) + \int_0^t y''(\theta) d\theta \quad (4.28)$$

And position y is the integral of speed y' :

$$y(t) = y(0) + \int_0^t y'(\theta) d\theta \quad (4.29)$$

In other words, position is double-integral of the acceleration.

Figure 4.9 shows an overall block diagram for the model (4.26). The applied force F is the input variable, and the position y is the output variable.

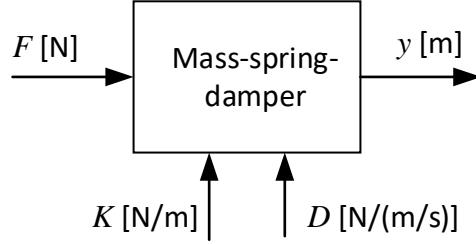


Figure 4.9: Overall block diagram of mass-spring-damper system.

[End of Example 4.3]

4.6.2 Systems with rotational motion

4.6.2.1 Momentum balance

Systems with rotational motion can be modelled in the same way as systems with linear motion (see above), but we must use *momentum balance*, which is often denoted *torque balance* for rotational systems:

$$S' = (J\omega)' = \sum_i T_i \quad (4.30)$$

Here, S [Nms] is momentum, J [kgm²] is inertia, ω [rad/s] is rotational speed, and T_i is torque (no. i). If J is constant, (4.30) can be written

$$J\omega' = Ja'' = \sum_i T_i \quad (4.31)$$

where $\omega' = a''$ is angular acceleration, and a [rad] is angular position.

(4.31) can be written as:

$$a'' = \left(\sum_i T_i \right) / J \quad (4.32)$$

Angular speed a' is the integral of angular acceleration a'' :

$$a'(t) = a'(0) + \int_0^t a''(\theta) d\theta \quad (4.33)$$

And angular position a is the integral of angular speed a' :

$$a(t) = a(0) + \int_0^t a'(\theta) d\theta \quad (4.34)$$

In other words, angular position is double-integral of the angular acceleration.

4.6.2.2 Relations between rotational and linear motion

In mathematical modeling of mechanical systems which consists of a combination of rotational and linear systems, the following relations are useful: Torque T is force F times arm L :

$$T = FL \quad (4.35)$$

Arc b is angle a (in radians) times radius r :

$$b = ar \quad (4.36)$$

4.6.2.3 Coupled mechanical systems

Mechanical systems often consist of coupled (sub)systems. Each system can have linear and/or rotational motion. Some examples: (1) A robot manipulator where the arms are coupled. (2) A traverse crane where a wagon moves a pending load. (3) A motor which moves a load with linear motion, as in a lathe machine.

A procedure for mathematical modeling of such coupled systems is as follows:

1. The force or torque balance is put up for each of the (sub)systems, and internal forces and torques acting between the systems are defined.
2. The final model is derived by eliminating the internal forces and torques.

This procedure is demonstrated in Example 4.4. An alternative way of modeling coupled systems is to use *Lagrange mechanics* where the model (the equations of motion) are derived from an expression which contains kinetic and potential energy for the whole system (this method is not described here).

Example 4.4 Modeling coupled rotational and linear motion systems

Figure 4.10 shows an electro-motor (which can be a current-controlled DC-motor) which moves a load with linear motion via a gear and a rod.

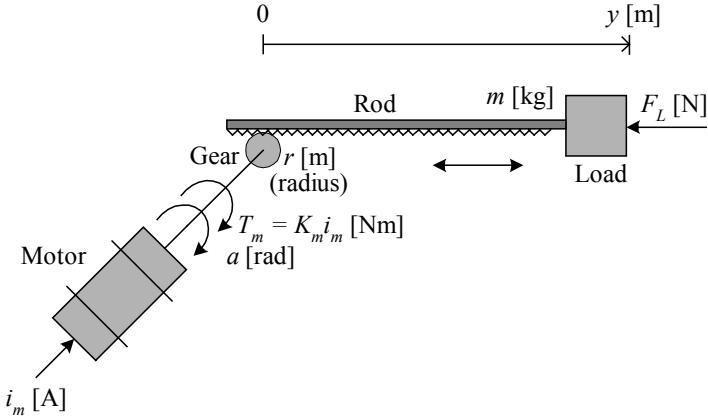


Figure 4.10: Example 4.4: Motor moving a linear load via a gear and a rod.

We set up a torque balance for the rotational part of the system and a force balance for the linear part, and then combines the derived equations. We shall finally have model which expresses the position y of the tool as a function of the applied motor current i_m . F_L is a load force acting on the load by the environment. (For simplicity the time argument t is excluded in the expressions below.)

1. Torque and force balance: The torque balance for the motor becomes

$$Ja'' = K_m i_m - T_1 \quad (4.37)$$

where T_1 is the torque which acts on the motor from the rod and the load via the gear. The force balance for rod and load becomes

$$my'' = F_1 - F_L \quad (4.38)$$

where F_1 is the force which acts on the rod and the load from the motor via the gear. The relation between T_1 and F_1 is given by

$$T_1 = F_1 r \quad (4.39)$$

The relation between y and θ is given by

$$y = ar \quad (4.40)$$

which yields

$$a'' = \frac{y''}{r} \quad (4.41)$$

By setting (4.41) and (4.39) into (4.37), (4.37) can be written

$$J \frac{y''}{r} = K_m i_m - F_1 r \quad (4.42)$$

2. Elimination of internal force: By eliminating the internal force F_1 between (4.38) and (4.42), we get

$$\left(m + \frac{J}{r^2} \right) y'' = \frac{K_m}{r} i_m - F_L \quad (4.43)$$

or:

$$y'' = \left(\frac{K_m}{r} i_m - F_L \right) / \left(m + \frac{J}{r^2} \right) \quad (4.44)$$

which is a mathematical model for the coupled system.

Speed y' is the integral of acceleration y'' :

$$y'(t) = y'(0) + \int_0^t y''(\theta) d\theta \quad (4.45)$$

And position y is the integral of speed y' :

$$y(t) = y(0) + \int_0^t y'(\theta) d\theta \quad (4.46)$$

Figure 4.11 shows an overall block diagram for the model (4.43). i_m and F_L are input variables, and y is the output variable.

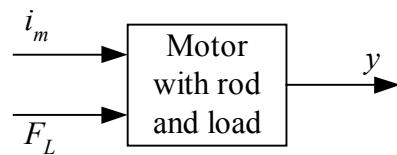


Figure 4.11: Overall block diagram of motor with rod and load.

[End of Example 4.4]

4.7 Mathematical modeling of electrical systems

This section gives a summary of some fundamental formulas for electrical systems which you will probably use in mathematical modeling of electrical systems.

4.7.1 Kirchhoff's law

4.7.1.1 Kirchhoff's Current Law

See the left part of Figure 4.12. The sum of currents into a junction in an electrical circuit is zero:

$$i_1 + i_2 + i_3 + \dots = 0 \quad (4.47)$$

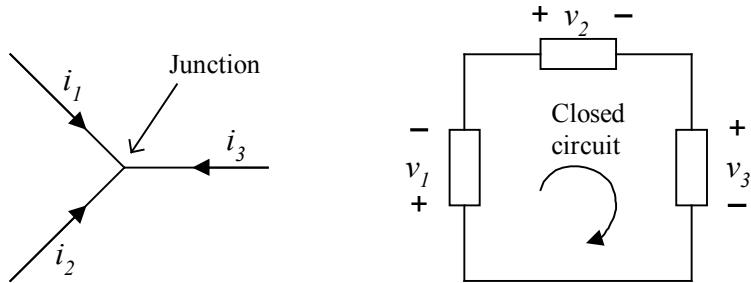


Figure 4.12: Kirchhoff's laws.

4.7.1.2 Kirchhoff's Voltage Law

: See the right part of Figure 4.12. The sum of voltage drops over the components on a closed electrical loop is equal to zero:

$$v_1 + v_2 + v_3 + \dots = 0 \quad (4.48)$$

4.7.2 Resulting resistance

Figure 4.13 shows series and parallel combination of resistors.

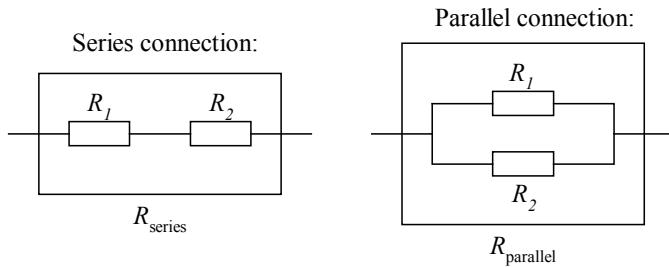


Figure 4.13: Series and parallel combination of resistors.

4.7.2.1 Resistors in series

Given two resistors R_1 and R_2 [Ω] in a series combination. The resulting resistance is

$$R_{\text{series}} = R_1 + R_2 \quad (4.49)$$

4.7.2.2 Resistors in parallel

Given two resistors R_1 and R_2 [Ω] in a parallel combination. The resulting resistance is

$$R_{\text{parallel}} = \frac{R_1 R_2}{R_1 + R_2} \quad (4.50)$$

4.7.3 Models of resistor, capacitor, and inductor

See Figure 4.14.

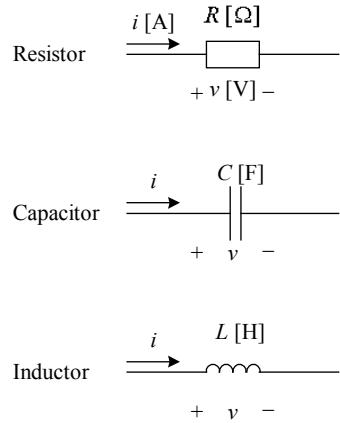


Figure 4.14: Resistor, capacitor and inductor.

Suppose that the current through a component is i [A] and that the corresponding voltage drop over the component v [V]. Current and voltage are then related as follows.

Resistor

$$v = Ri \quad (\text{Ohm's law}) \quad (4.51)$$

Capacitor

$$i = Cv' \quad (4.52)$$

Inductor

$$v = Li' \quad (4.53)$$

Example 4.5 Mathematical modeling of an RC-circuit

Figure 4.15 shows an RC-circuit (the circuit contains the resistor R and the capacitor C).

The RC-circuit is frequently used as an analog lowpass filter: Signals of *low* frequencies *passes* approximately unchanged through the filter, while signals of high frequencies are approximately filtered out (stopped).

We will now find a mathematical model relating v_{out} to v_{in} . First we apply the Kirchhoff's voltage law in the circuit which consists the input voltage terminals, the resistor, and the capacitor (we consider the voltage drops to be positive clockwise direction):

$$-v_{\text{in}} + v_R + v_{\text{out}} = 0 \quad (4.54)$$

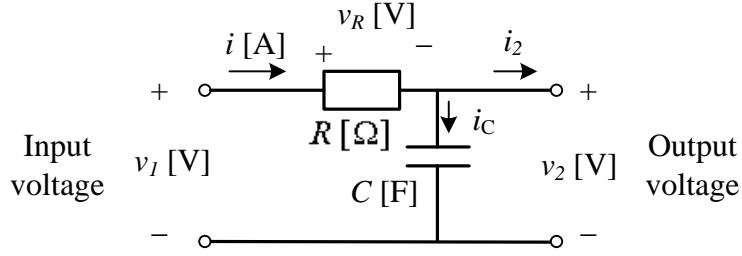


Figure 4.15: RC-circuit.

(v_{out} equals the voltage drop over the capacitor.) In (4.54) v_R is given by

$$v_R = Ri \quad (4.55)$$

We assume that there is no current going through the output terminals. (This is a common assumption, and not unrealistic, since it is typical that the output terminals are connected to a subsequent circuit which has approximately infinite input impedance, causing the current into it to be approximately zero. An operational amplifier is an example of such a load-circuit.) Thus, jf. (4.52),

$$i = i_C = Cv'_{out} \quad (4.56)$$

The final model is achieved by using i as given by (4.56) in (4.55) and then using v_R as given by (4.55) for v_R in (4.54). The model becomes

$$RCv'_{out} = v_{in} - v_{out} \quad (4.57)$$

or:

$$v_{out}' = (v_{in} - v_{out}) / (RC) \quad (4.58)$$

v_{out} is the integral of v'_{out} :

$$v_{out}(t) = v_{out}(0) + \int_0^t v_{out}(\theta)' d\theta \quad (4.59)$$

Figure 4.16 shows a block diagram for the model (4.57). v_{in} is the input variable, and v_{out} is the output variable.

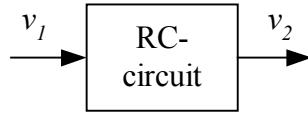


Figure 4.16: Overall block diagram of an RC-circuit.

[End of Example 4.5]

4.7.4 Power

4.7.4.1 Instantaneous power

When a current i flows through a resistor R , the instantaneous power delivered to the resistor is

$$P = ui \quad (4.60)$$

where $u = Ri$ is the voltage drop across the resistor.

4.7.4.2 Mean power

When an alternating (sinusoidal) current of amplitude I flows through a resistor R (for example a heating element), the mean or average value of the power delivered to the resistor is

$$\bar{P} = \frac{1}{2}UI = \frac{1}{2}RI^2 = \frac{1}{2}\frac{U^2}{R} \quad (4.61)$$

where U is the amplitude of the alternating voltage drop across the resistor. (4.61) is independent of the frequency.

4.8 Physical component based simulators

To appear.

4.8.1 OpenModelica

To appear.

4.8.2 Aspentech Hysys

To appear.

4.8.3 Simscape

To appear.

4.9 Problems for Chapter 4

Problem 4.1 Mass balance with volumetric flows

In Example 4.1 the inflow and outflow are mass flows. Now, assume that the flows are instead volumetric flows, $q_{in}[\text{m}^3/\text{s}]$ and $q_{out}[\text{m}^3/\text{s}]$, respectively. Derive the differential equation of h under this assumption.

Problem 4.2 Modeling of wood chips tank

Figure 4.17 shows a wood chips tank with a feed screw and conveyor belt (the belt has constant speed).³

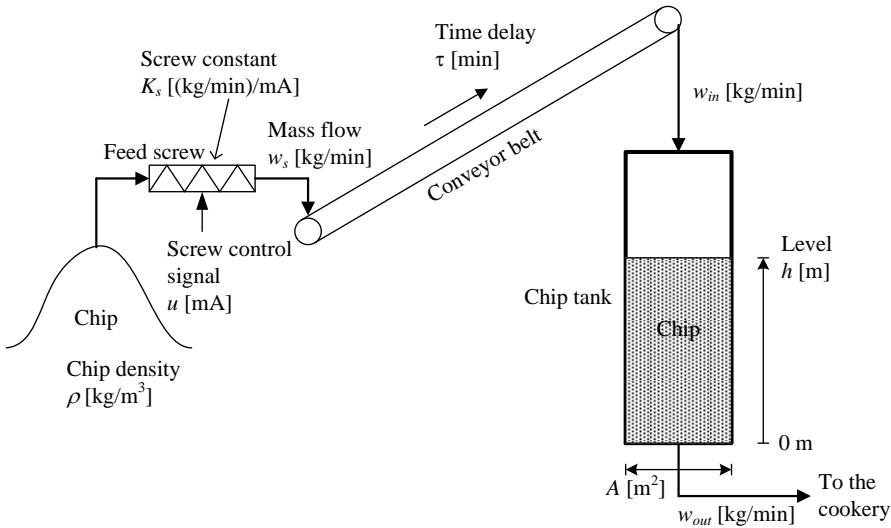


Figure 4.17: Wood chips tank.

There is an outflow of chip via an outlet at the bottom of the tank. The mass flow w_s from the feed screw to the belt is proportional to the screw control signal u :

$$w_s = K_s u \quad (4.62)$$

The mass flow w_{in} into the chip tank is equal to w_s but time delayed time τ :

$$w_{in}(t) = w_s(t - \tau) \quad (4.63)$$

1. Draw an overall input-output block diagram of the system. Define the input and output variables (it is assumed that the level is of particular interest).
2. Develop a mathematical model describing the behaviour of the chip level h .

Problem 4.3 Mole balance

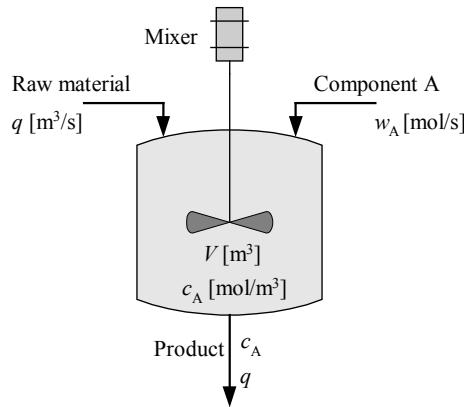


Figure 4.18: Problem 4.3: Blending tank.

Figure 4.18 shows a stirred blending tank where the material A is fed into a tank for blending with a raw material. The symbols in Figure 4.18 are as follows: V is the liquid volume in the tank. q is the volumetric inflow of the raw material. q is also the volumetric outflow. c_A is the mole density or concentration of material A in the tank. w_A is the mole flow of material A.

Assumptions:

- The contents of the tank has constant volume.⁴
 - The volumetric flow of material A is very small (negligible) compared to the volumetric flow of the raw material.
 - There are homogenous conditions (perfect stirring) in the tank.
 - The raw material does not contain A.
1. Develop a mathematical model which expresses how the concentration c_A varies.
 2. Draw an overall block diagram of the system.

Problem 4.4 *Modeling of blending tank*

Figure 4.19 shows a tank with cold water inflow and heated (blended) water outflow. The tank is constantly full, and the volumetric flow is thus equal to the sum of the inflows.

Assume homogeneous conditions in the tank. Develop a mathematical model of the water temperature T in the tank.

Problem 4.5 *Modeling of a ship*

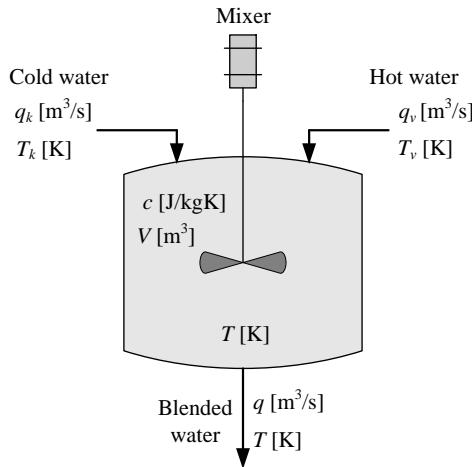


Figure 4.19: Tank with cold water inflow and heated (blended) water outflow.

Figure 4.20 shows a ship.

In this problem we concentrate on the so-called surge (forward) direction, i.e., the movements in the other directions are disregarded. The wind acts on the ship with the force F_w . The absolute value of the hydrodynamic force F_h (force from water acting on the ship) is proportional to the square of the difference between the ship speed u and the water current speed u_c .⁵ Assume that the proportionality constant is D (a positive number).

1. What is the mathematical relation between speed u and position y ?
2. Develop a mathematical model of the ship expressing the motion (the position y) in the surge direction.
Note: It is important to get the direction of the hydrodynamic force correct. Let us assume all speeds are positive in the positive surge direction (forwards). If the water current speed is larger than the ship speed, the hydrodynamic force acts on the ship in the forward direction. If the water current speed is smaller than the ship speed the hydrodynamic force acts on the ship in the backward direction. Your model must express this correctly.
3. Draw an input-output block diagram of the system. Assume that the ship position is the variable of particular interest.

Problem 4.6 *Modeling of a satellite*

Figure 4.21 shows a satellite with manoeuvering motors.

Develop a model of the angular motion of the satellite.

³Typically, there is such a wood chips tank in the beginning of the production line of a paper mass factory.

⁴This can be accomplished with for example a level control system.

⁵In the context of ship modeling, it is usual to use the symbol u for speed. In the control theory, however, u often represents the control signal, but control is not a topic in this problem.

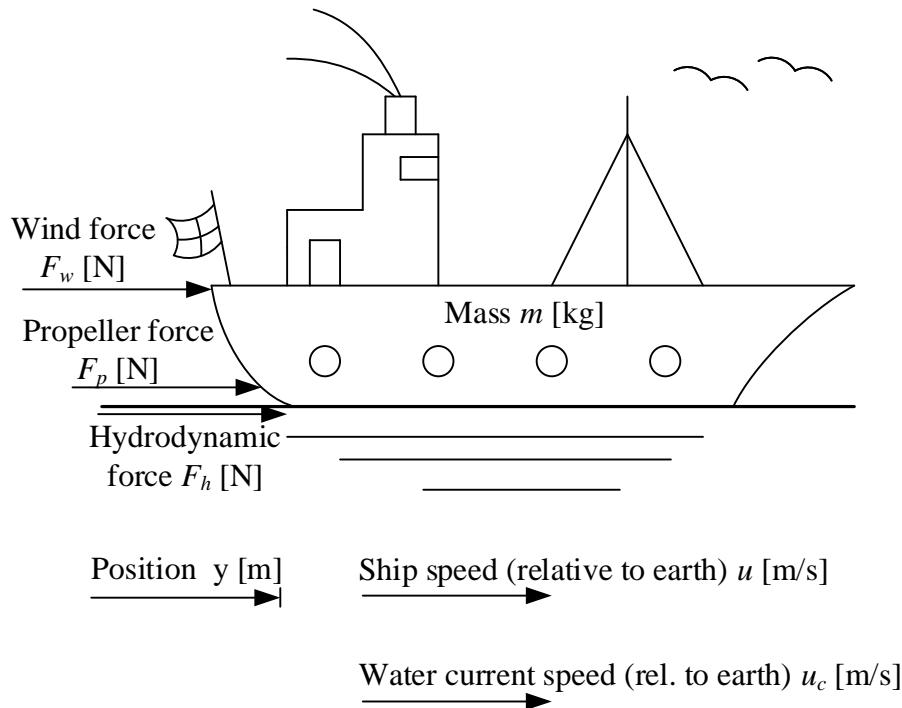


Figure 4.20: Ship.

Problem 4.7 Modeling of a pendulum

Figure 4.22 shows a cart with the pendulum. A motor (in the cart) acts on the cart with a force F .⁶

You can use the following variables and parameters in the model to be derived in this problem:

- I – the moment of inertia of the pendulum about its center of gravity. For the pendulum shown in Figure 1,

$$I = \frac{mL^2}{12} \quad (4.64)$$

- V and H – vertical and horizontal forces, respectively, in the pivot.
- d – a damping coefficient.

Derive a mathematical model of the motion of the system based on the following principles:

1. Force balance (Newton's Second Law) applied to the horizontal movement of the center of gravity of the pendulum.

⁶This force can be manipulated by the controller to stabilize the pendulum in a standing position or in a hanging position at a specified position of the cart, but this problem is not about control. The system can be well controlled with model-based control, for example optimal control based on state-variable feedback (cf. e.g. Lecture notes on Modes, Estimation and Control, TechTeach/F. Haugen).

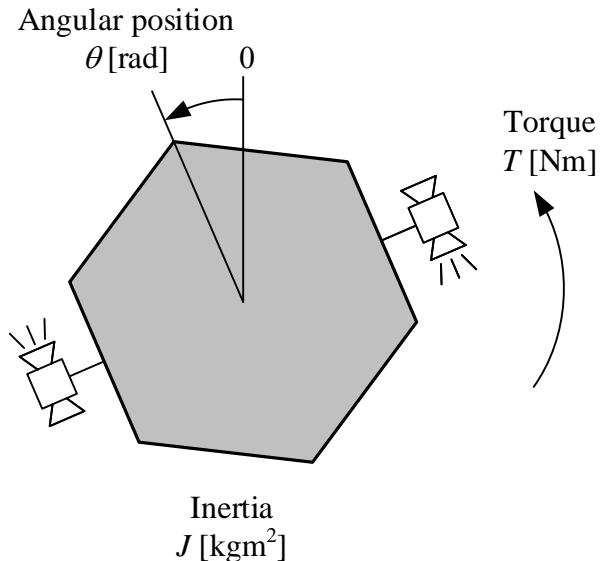


Figure 4.21: Satellite.

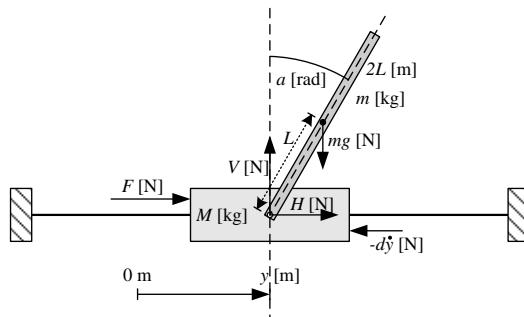


Figure 4.22: Pendulum.

2. Force balance applied to the vertical movement of the center of gravity of the pendulum.
3. Torque balance (the rotational version of the Newton's Second Law applied to the center of gravity of the pendulum).
4. Force balance applied to the cart.

(When using the model for developing a simulator or design of a stabilizing controller, it will probably be necessary to eliminate the internal forces V and H , but this elimination is not a part of this problem. Hence, it is ok that the resulting model in this problem contains V and H .)

Problem 4.8 Modeling of resistors

Figure 4.23 shows a combination of resistors.

What is the resulting resistance R_4 ?

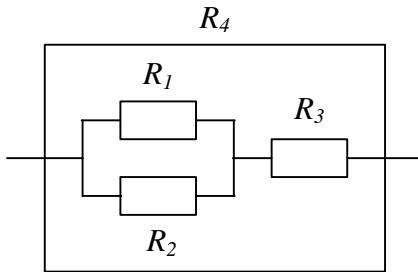


Figure 4.23: Combination of resistors.

Problem 4.9 *Calculation of resistance*

Given a lamp which receives $P = 100 \text{ W}$ mean (average) power when it is connected to the mains, which is an alternate voltage of amplitude $U = 220 \text{ V}$. Calculate the lamp restistance R .

Problem 4.10 *Modeling of electrical circuit (highpass filter)*

Figure 4.24 shows an (analog) highpass filter. (It attenuates low-frequent signals, while high-frequent signals pass through the filter.)

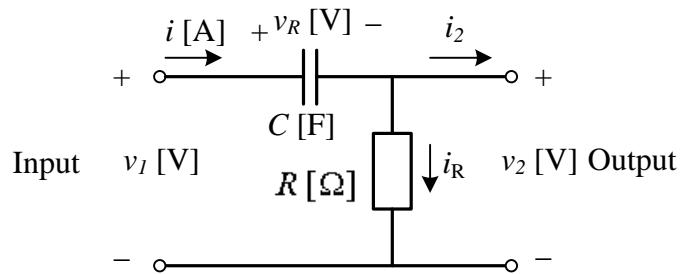


Figure 4.24: High-pass filter.

Find a mathematical model describing the behaviour of the output voltage v_2 .

4.10 Solutions to problems for Chapter 4

Solution to Problem 4.1

With volumetric flows, the mass balance (4.4) becomes:

$$m' = (\rho Ah)' = \rho Ah' = \rho q_{in} - \rho q_{out} \quad (4.65)$$

which gives

$$h' = \frac{1}{A} (q_{in} - q_{out}) \quad (4.66)$$

Solution to Problem 4.2

- Figure 4.25 shows the overall block diagram.

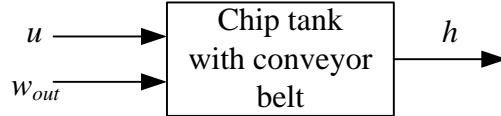


Figure 4.25: Overall block diagram of wood chips tank model.

- Since there is a time delay in the system (due to the transport delay of the conveyor belt) it is important to include the time argument in the equations. The mass balance if the wood chips contents of the tank is

$$\begin{aligned} [\rho Ah(t)]' &= \rho Ah(t)' = w_{in}(t) - w_{out}(t) \\ &= w_s(t - \tau) - w_{out}(t) \\ &= K_s u(t - \tau) - w_{out}(t) \end{aligned} \quad (4.67)$$

$h(t)$ is given by the integral of h' :

$$h(t) = h(0) + \int_0^t h'(\theta) d\theta \quad (4.68)$$

Solution to Problem 4.3

- The “accumulation” is the total mole number is Vc_A . The Balance Law (4.2) is in terms of a mole balance:

$$(Vc_A)' = w_A - c_A q \quad (4.69)$$

A state space model of c_A :

$$c'_A = \frac{1}{V} (w_A - c_A q) \quad (4.70)$$

$c_A(t)$ is given by the integral of c'_A :

$$c_A(t) = c_A(0) + \int_0^t c'_A(\theta) d\theta \quad (4.71)$$

- Figure 4.26 shows an overall block diagram of the model (4.70). w_A and q are input variables, and c_A is the output variable.

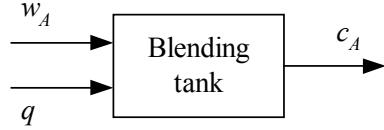


Figure 4.26: Problem 4.3: Overall block diagram for stirred blending tank.

Solution to Problem 4.4

Energy balance of the liquid in the tank:

$$(c\rho VT)' = c\rho VT' = c\rho q_k T_k + c\rho q_v T_v - c\rho q T \quad (4.72)$$

Cancelling ρ :

$$cVT' = cq_k T_k + cq_v T_v - cqT \quad (4.73)$$

Diving by cV gives a state space model of T :

$$T' = (cq_k T_k + cq_v T_v - cqT) / (cV) \quad (4.74)$$

Here, q is given by

$$q = q_k + q_v \quad (4.75)$$

$T(t)$ is given by the integral of T' :

$$T(t) = T(0) + \int_0^t T'(\theta) d\theta \quad (4.76)$$

Solution to Problem 4.5

1. The relation between position y and speed u is

$$y' = u \quad (4.77)$$

2. Force balance:

$$mu' = F_p + F_h + F_w \quad (4.78)$$

$$= F_p - D|u - u_c|(u - u_c) + F_w \quad (4.79)$$

(4.77) and (4.79) constitutes the model.

Alternatively, since

$$u' = y'' \quad (4.80)$$

the model can be expressed as

$$my'' = F_p - D|y' - u_c|(y' - u_c) + F_w \quad (4.81)$$

or:

$$y'' = (F_p - D|y' - u_c|(y' - u_c) + F_w) / m \quad (4.82)$$

Speed y' is the integral of acceleration y'' :

$$y'(t) = y'(0) + \int_0^t y''(\theta) d\theta \quad (4.83)$$

And position y is the integral of speed y' :

$$y(t) = y(0) + \int_0^t y'(\theta) d\theta \quad (4.84)$$

3. We can regard F_p , F_w and u_c as input variables, and y as the output variable.

Figure 4.27 shows the block diagram.



Figure 4.27: Overall block diagram of the ship model.

Solution to Problem 4.6

Torque balance:

$$J\theta'' = T \quad (4.85)$$

or:

$$\theta'' = T/J \quad (4.86)$$

Angular speed θ' is the integral of angular acceleration θ'' :

$$\theta'(t) = \theta'(0) + \int_0^t \theta''(\theta) d\theta \quad (4.87)$$

And angular position θ is the integral of angular speed θ' :

$$\theta(t) = \theta(0) + \int_0^t \theta'(\theta) d\theta \quad (4.88)$$

Solution to Problem 4.7

- Force balance (Newton's Second Law) applied to the horizontal movement of the center of gravity of the pendulum.

$$m(y + L \sin a)'' = H \quad (4.89)$$

(The differentiation of the additive term $(y + L \sin a)$ must be carried out in applications of this model, but it is not shown here.)

2. Force balance applied to the vertical movement of the center of gravity of the pendulum:

$$m(L \cos a)'' = V - mg \quad (4.90)$$

(The differentiation of the additive term $(L \cos a)$ is not shown here.)

3. Torque balance (the rotational version of the Newton's Second Law applied to the center of gravity of the pendulum):

$$Ia'' = V'L \sin a - HL \cos a \quad (4.91)$$

4. Force balance applied to the cart:

$$My'' = F - H - dy' \quad (4.92)$$

(From Eq. (4.89) – (4.92), the internal forces V and H can be eliminated, resulting in two differential equations not containing V and H .)

Solution to Problem 4.8

The circuit consists of two resistors in parallel in series with the third resistor. The resulting resistance is

$$R_4 = \frac{R_1 R_2}{R_1 + R_2} + R_3 \quad (4.93)$$

Solution to Problem 4.9

Mean power is

$$P = \frac{1}{2} \frac{U^2}{R} \quad (4.94)$$

which solved for R gives

$$R = \frac{1}{2} \frac{U^2}{P} = \frac{1}{2} \frac{220^2}{100} = 242 \Omega \quad (4.95)$$

Solution to Problem 4.10

There are many ways to find a mathematical model. Here is one: Kirchhoff's voltage law gives

$$-v_1 + v_C + v_2 = 0 \quad (4.96)$$

or

$$v_C = v_1 - v_2 \quad (4.97)$$

Kirchhoff's current law applied to the upper node gives

$$0 = i_C - i_R + \overbrace{i_2}^{=0} \quad (4.98)$$

$$= C v_C' - \frac{v_2}{R} \quad (4.99)$$

$$= C(v_1 - v_2)' - \frac{v_2}{R} \quad (4.100)$$

$$= C(v_1' - v_2') - \frac{v_2}{R} \quad (4.101)$$

Getting v_2' alone on the left side:

$$v_2' = v_1' - v_2 / (RC) \quad (4.102)$$

$v_2(t)$ is given by the integral of v_2' :

$$v_2(t) = v_2(0) + \int_0^t v_2(\theta)' d\theta \quad (4.103)$$

Chapter 5

State space models

5.1 Introduction

A *state space model* is just a structured form or representation of the differential equations for a system. Typically, the differential equations stem from mechanistic modeling of dynamic systems as explained in Ch. 4.

State space models are useful in a number of situations:

- **Linearization of non-linear models**
- **Calculation of time-responses** – both analytically and numerically
- **Using simulation tools:** Python, MATLAB, LabVIEW, Octave, and Scilab have simulation functions that assumes state space models.
- **Analysis of dynamic systems**, e.g. stability analysis
- **Analysis and design of advanced controllers and estimators:** Controllability and observability analysis; Design of LQ optimal controllers, Model-based predictive control; Design of state estimators (Kalman Filters).

5.2 The state space model

Most dynamic models can be represented with the following two equations:

- *The state space model*, which is a set of first order differential equations of the state

variables¹:

$$\begin{aligned} x_1' &= f_1(x, u, d, p) \\ &\vdots \quad \vdots \quad \vdots \\ x_n' &= f_n(x, u, d, p) \end{aligned} \tag{5.1}$$

- *The output model*, which is a set of algebraic equations defining the output variables:

$$\begin{aligned} y_1 &= g_1(x, u, d, p) \\ &\vdots \quad \vdots \quad \vdots \\ y_m &= g_m(x, u, d, p) \end{aligned} \tag{5.2}$$

The variables are (the indexes are dropped here, for simplicity):

- x is the state variable.
- y is the output variable.
- u is the input variable. In context of control, u represents the control variable (or signal).
- d is the disturbance, which also may be denoted the environmental variable, or the load variable.
- p is the parameter.

(5.1) and (5.2) can be written compactly on a vector form as:

$$x' = f(x, u, d, p) = f(\cdot) \tag{5.3}$$

$$y = g(x, u, d, p) = g(\cdot) \tag{5.4}$$

In (5.3) and (5.4), x, u, d, p, y are vectors. For example,

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \tag{5.5}$$

which also can be written as $x = [x_1, \dots, x_n]^T$ where super-index T means transpose.

In (5.3) and (5.4), f and g are vector functions:

$$f = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \tag{5.6}$$

¹Alternative symbols to x' are \dot{x} and $\frac{dx}{dt}$.

$$g = \begin{bmatrix} g_1 \\ \vdots \\ g_m \end{bmatrix} \quad (5.7)$$

The model (5.3) and (5.4) is often referred to as a *nonlinear* state space model because the vector functions $f(\cdot)$ and $g(\cdot)$ may contain nonlinear functions.

In many cases the mathematical modelling results in one or more first order differential equations. In such cases it is straightforward to write the model as a state space model; All you have to do is to ensure that the time derivatives appear alone on the left-hand side of the differential equations. However, when modeling kinetic systems, the model may consist of second order differential equations due to Newton's Second Law (since acceleration is the second order derivative of position). Example 5.1 demonstrates how to write a second order differential equation as a state space model.

Example 5.1 *Mass-spring-damper-model written as a state space model*

Figure 5.1 shows a mass-spring-damper-system. z is position. F is applied force. D is

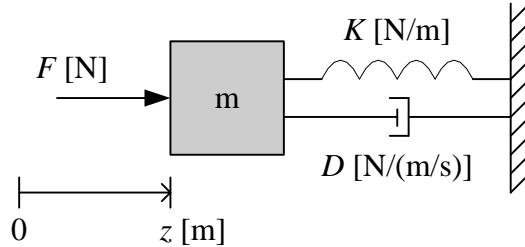


Figure 5.1: Mass-spring-damper.

damping constant. K is spring constant. It is assumed that the damping force F_d is proportional to the speed, and that the spring force F_s is proportional to the position of the mass. The spring force is assumed to be zero when z is zero. Force balance (Newton's Second Law) yields

$$\begin{aligned} mz'' &= F - F_d - F_s \\ &= F - Dz' - Kz \end{aligned} \quad (5.8)$$

which is a second order differential equation.

We define the following new variables: x_1 for position z , x_2 for speed z' and u for force F . Then the model (5.8) can be written as the following equivalent set of two first order differential equations:

$$x_1' = x_2 \quad (5.9)$$

$$m x_2' = -Dx_2 - Kx_1 + u \quad (5.10)$$

which can be written on the standard form (5.1) as follows:

$$x_1' = \underbrace{x_2}_{f_1} \quad (5.11)$$

$$x_2' = \underbrace{(-Dx_2 - K_f x_1 + u) / m}_{f_2} \quad (5.12)$$

Let us regard the position x_1 as the output variable y :

$$y = \underbrace{x_1}_g \quad (5.13)$$

The initial position, $x_1(0)$, and the initial speed, $x_2(0)$, define the initial state of the system.

(5.11) and (5.12) and (5.13) constitute a second order state space model which is equivalent to the original second order differential equation (5.8).

[End of Example 5.1]

5.3 The response of a state space model

5.3.1 Dynamic response

The output equation, (5.4), or (5.2), expresses that y is a function of x . So, x must exist. But x does not appear explicitly in the model (5.3) – (5.4)! So how is x is obtained? It is obtained by solving the differential equations, i.e. by integrating the differential equations. Let's assume that the state space model on the vector form (5.3). The response is:

$$x(t) = x(0) + \int_0^t x'(\theta) d\theta \quad (5.14)$$

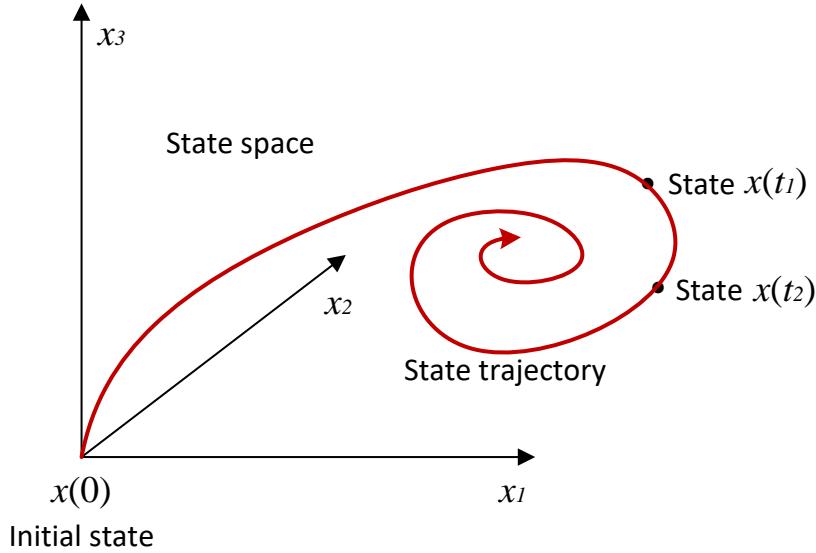
where x' is known from the differential equations:

$$x'(t) = f[x(t), u(t), d(t), p] \quad (5.15)$$

The response in the output variable, $y(t)$, is:

$$y(t) = g[x(t), u(t), d(t), p] \quad (5.16)$$

See Figure 5.2. The state vector $x(t) = [x_1(t), \dots, x_n(t)]^T$ defines or spans the *state space* of the system, with $x_1(t), \dots, x_n(t)$ as the coordinates of the space. $x(t)$ is the *state* of the system as a function of time. As time evolves, $x(t)$ creates a *state trajectory*. $x(t_1)$ is the state of the system at point of time t_1 , and $x(t_2)$ is the state of the system at point of time t_2 , etc.

Figure 5.2: The notion of *state*.

5.3.2 Static response

In some situations it is useful to calculate by hand the *static response* of a dynamic system, e.g. to check that the simulated response is correct. The static response is the steady-state value of the output variable of the model when the input variables have constant values. This response can be calculated directly from the model after the *time-derivatives have been set equal to zero*, since then the variables have constant values, their time-derivatives equal zero.

Example 5.2 Calculation of static response for mass-spring-damper

The mass-spring-damper system described in Example 4.3 has the following model:

$$my'' = -Dy' - Ky + F \quad (5.17)$$

Suppose the force F is constant of value F_s . The corresponding static response in the position y can be found by setting the time-derivatives equal to zero and solving with respect to y . The result is

$$y_s = \frac{F_s}{K} \quad (5.18)$$

[End of Example 5.2]

Can you calculate the static response in the process variable (the process output) for any differential equation model? No! Because there may be models for which there is no static response, as demonstrated in Example 5.3.

Example 5.3 Failed attempt to calculate static response

See Example 4.1 where (4.8) is the model of the liquid tank, repeated here for convenience:

$$h' = (q_i - q_o) / A \quad (5.19)$$

Assume that both q_i and q_o have constant values, Q_i and Q_o , respectively. Assuming there is a static response in h , called h_s . Its time derivative is $h'_s = 0$. Then, (5.19) gives

$$0 = (Q_i - Q_o) / A \neq 0$$

which make no sense unless $Q_i = Q_o$, and even in that case, no value of h_s is given. We know from physical insight that there is no static response in h (unless $Q_i = Q_o$); h will change all the time (unless $Q_i = Q_o$).

Only asymptotically stable systems have well-defined steady state responses which can be calculated from the static version of the mathematical model, as in Example 5.2.

Asymptotic stability is defined in Ch. 21.

[End of Example 5.3]

5.4 Linear state space models

5.4.1 Standard model form of linear state space models

Linear state space models are a special case of the general state space model (5.3)-(5.4). Many methods for analysis of differential equation models, as stability analysis, response calculation and model transformations, are based on linear state space models. Let us study a general second order linear state space model to see how linear state space models are defined. The model has two state-variables, x_1 and x_2 , and two input variables, u_1 and u_2 :

$$x_1' = a_{11}x_1 + a_{12}x_2 + b_{11}u_1 + b_{12}u_2 \quad (5.20)$$

$$x_2' = a_{21}x_1 + a_{22}x_2 + b_{21}u_1 + b_{22}u_2 \quad (5.21)$$

where the a and b coefficients are parameters (constants).

(5.20)-(5.21) can be written in matrix-vector form as follows:

$$\underbrace{\begin{bmatrix} x_1' \\ x_2' \end{bmatrix}}_x = \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_x + \underbrace{\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}}_B \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_u \quad (5.22)$$

or, more compact:

$$x' = Ax + Bu \quad (5.23)$$

where x is the state vector and u is the input vector. A is called the system-matrix, and is square in all cases.

Let us assume that the system has two output variables, which generally can be functions of both the state variables and the input variables. The output function can then be written

on the form

$$y_1 = c_{11}x_1 + c_{12}x_2 + d_{11}u_1 + d_{12}u_2 \quad (5.24)$$

$$y_2 = c_{21}x_1 + c_{22}x_2 + d_{21}u_1 + d_{22}u_2 \quad (5.25)$$

which can be written on matrix-vector form as follows:

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_y = \underbrace{\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}}_C \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_x + \underbrace{\begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}}_D \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_u \quad (5.26)$$

or, more compact:

$$y = Cx + Du \quad (5.27)$$

Example 5.4 Mass-spring-damper model as a state space model on matrix-vector form

The state space model (5.11), (5.12), (5.13) is linear. We get

$$\underbrace{\begin{bmatrix} x_1' \\ x_2' \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{K_f}{m} & -\frac{D}{m} \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_x + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_B u \quad (5.28)$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_C \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_x + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_D u \quad (5.29)$$

[End of Example 5.4]

5.4.2 Linearization of non-linear models

5.4.2.1 When do we have to linearize?

In many cases the mathematical model contains one or more non-linear differential equations. If the mathematical model is non-linear, there may be good reasons to *linearize* it, which means to develop a local linear model which approximates the original model about a given operating point. The reasons may be the following:

- We want to study the behavior of the system *about an operating point*, which is one specified state where the system can be. It is then the *deviations* from this operating point we study. Examples of such operating points are the level of 8.7 m in a tank, the temperature of 50 degrees Celcius in a heat exchanger, etc. It can be shown (and we will do it soon) that a model which describes the behavior of the deviations about the operating point, is approximately linear.

- We can use the large number of the methods which are available for analysis and design of linear systems, e.g. for stability analysis, frequency response, controller design and signal filter design. The number of methods for linear models are much larger than for non-linear models.

Note: If you have a non-linear model of a (physical) system, do *not use the linearized model for simulation* unless you have a good reason for using it. Instead, use the (original) non-linear model since it gives a more accurate representation of the system.

Figure 5.3 illustrates the relation between the original non-linear system and the local linear system (model).

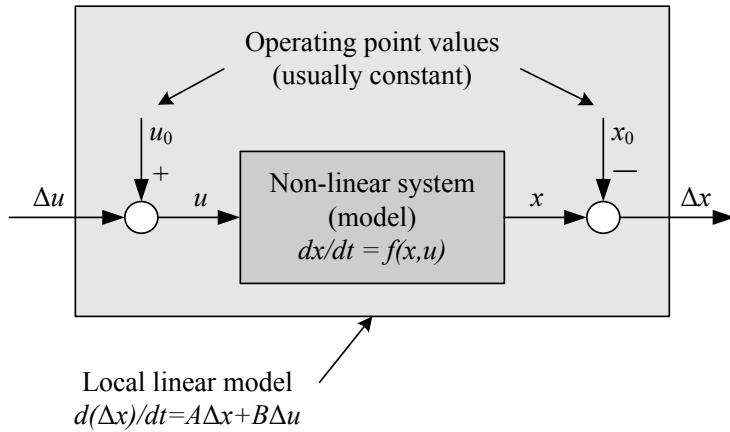


Figure 5.3: Illustration of the relation between the original non-linear system and the local linear system (model)

The input variable which excites the non-linear system is assumed to be given by

$$u = u_0 + \Delta u \quad (5.30)$$

where u_0 is the value in the operating point and Δu is the deviation from u_0 . Similarly,

$$x = x_0 + \Delta x \quad (5.31)$$

If you are going to experiment with the system to develop or adjust a linear model about the operating point, you must adjust Δu and observe the corresponding response in Δx (or in the output variable Δy).

5.4.2.2 Deriving the linearization formulas

We assume that the model is a non-linear state space model:

$$x' = f(x, u) \quad (5.32)$$

Suppose that the system is in an *operating point* defined by

$$x_0' = f(x_0, u_0) \quad (5.33)$$

If the input variable u is changed by Δu from the operating point value u_0 , the state-variable x is changed by Δx from x_0 . (5.33) can then be written

$$\frac{d(x_0 + \Delta x)}{dt} = f(x_0 + \Delta x, u_0 + \Delta u) \quad (5.34)$$

↓

$$x_0' + \Delta x' \approx f(x_0, u_0) + \left. \frac{\partial f}{\partial x} \right|_0 \Delta x + \left. \frac{\partial f}{\partial u} \right|_0 \Delta u \quad (5.35)$$

On the left side of (5.34) we have applied the summation rule of differentiation and on the right side we have used a first order Taylor series expansion of $f(\cdot)$. The expression $\left. \frac{\partial f}{\partial x} \right|_0$ means the partial time-derivative of f with respect to x , calculated in the operating point, that is, with x_0 and u_0 inserted into $\frac{\partial f}{\partial x}$. The same applies to $\left. \frac{\partial f}{\partial u} \right|_0$. Now we will exploit the fact that x_0' is equal $f(x_0, u_0)$, cf. (5.33). This implies that these two terms are cancelled against each other in (5.35). (5.35) then becomes

$$\Delta x' = \underbrace{\left. \frac{\partial f}{\partial x} \right|_0}_{A} \Delta x + \underbrace{\left. \frac{\partial f}{\partial u} \right|_0}_{B} \Delta u \quad (5.36)$$

$$= A\Delta x + B\Delta u \quad (5.37)$$

or, in more detail,

$$\begin{bmatrix} \Delta x'_1 \\ \Delta x'_2 \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}}_A \cdot \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}}_B \cdot \begin{bmatrix} \Delta u_1 \\ \Delta u_2 \\ \vdots \end{bmatrix} \quad (5.38)$$

which is the local linear model. A and B becomes Jacobi-matrices (which are partial derivative matrices) which generally are functions of the operating point. If the operating point is *constant*, A and B will be constant matrices, which can be calculated once and for all.

Similarly, linearization of the output equation

$$y = g(x, u) \quad (5.39)$$

gives

$$\Delta y = \underbrace{\left. \frac{\partial g}{\partial x} \right|_0}_{C} \Delta x + \underbrace{\left. \frac{\partial g}{\partial u} \right|_0}_{D} \Delta u = C\Delta x + D\Delta u \quad (5.40)$$

or

$$\begin{bmatrix} \Delta y_1 \\ \Delta y_2 \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}}_C \cdot \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{\partial g_1}{\partial u_1} & \frac{\partial g_1}{\partial u_2} & \cdots \\ \frac{\partial g_2}{\partial u_1} & \frac{\partial g_2}{\partial u_2} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}}_D \cdot \begin{bmatrix} \Delta u_1 \\ \Delta u_2 \\ \vdots \end{bmatrix} \quad (5.41)$$

If the operating point is a static *equilibrium* point, all variables have constant values and all time-derivatives are zero. Thus,

$$x_0' = 0 = f(x_0, u_0) \quad (5.42)$$

The values of the model variables in the static operating point can be found by solving the algebraic equation (5.42) with respect to the unknown variables.

Example 5.5 Linearization of a non-linear tank model

The nonlinear model

Figure 5.4 shows a liquid tank with inlet via a pump and outlet via a valve with fixed opening.

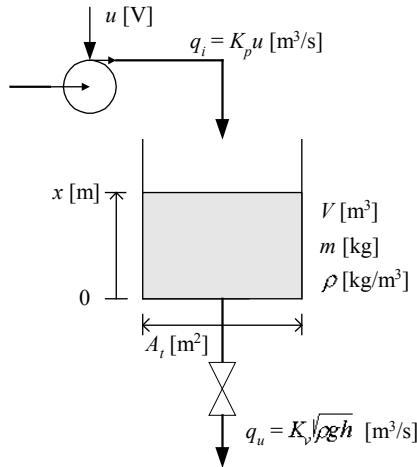


Figure 5.4: Liquid tank with non-linear mathematical model

The outflow is assumed to be proportional to the square root of the pressure drop over the valve, and this pressure is assumed to be equal to the hydrostatic pressure ρgh at the outlet. The mass balance becomes:

$$\begin{aligned} \rho A_t x' &= \rho q_i - \rho q_u \\ &= \rho K_p u - \rho K_v \sqrt{\rho g x} \end{aligned}$$

which can be written:

$$x' = \frac{K_p}{A_t} u - \frac{K_v}{A_t} \sqrt{\rho g x} \equiv f(x, u) \quad (5.43)$$

Static operating point

Before we linearize the original model, we will find the static operating point. Let us assume that the pump control signal is constant: $u = u_0$ (constant). We calculate the

corresponding static level, $x = x_0$ (constant), by setting $x' = 0$ in the dynamic model (5.43):

$$0 = \frac{K_p}{A_t} u_0 - \frac{K_v}{A_t} \sqrt{\rho g x_0} \equiv f(x_0, u_0) \quad (5.44)$$

Solving (5.44) with respect to x_0 gives:

$$x_0 = \frac{1}{\rho g} \left(\frac{K_p u_0}{K_v} \right)^2 \quad (5.45)$$

Linearization

Now that we have found the static operating point (u_0, x_0) , we can derive the local linear model by linearizing (5.43):

$$\begin{aligned} \Delta x' &= \left. \frac{\partial f}{\partial x} \right|_0 \Delta x + \left. \frac{\partial f}{\partial u} \right|_0 \Delta u \\ &= \underbrace{-\frac{K_v}{A_t} \frac{\rho g}{2\sqrt{\rho g x_0}}}_{A} \Delta x + \underbrace{\frac{K_p}{A_t}}_{B} \Delta u \\ &= A \Delta x + B \Delta u \end{aligned} \quad (5.46)$$

[End of Example 5.5]

5.5 Simulation tools for state space models (differential equation solvers)

To appear.

5.5.1 Introduction

5.5.2 Python

To appear.

5.5.3 Matlab

To appear.

5.5.4 LabVIEW

To appear.

5.6 Problems for Chapter 5

Problem 5.1 State space model of a system of tanks

Figure 5.5 shows two coupled liquid tanks. u_1 and u_2 are control signals.

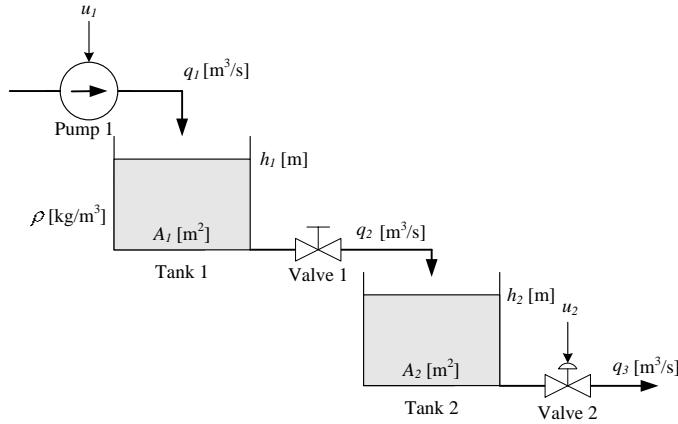


Figure 5.5: Two coupled liquid tanks

Material balance of the liquid in tank 1 gives:

$$\rho A_1 h_1' = \underbrace{\rho K_p u_1}_{q_1} - \underbrace{\rho K_{v_1} \sqrt{\frac{\rho g h_1}{G}}}_{q_2} \quad (5.47)$$

Material balance of the liquid in tank 2 gives:

$$\rho A_2 h_2' = \underbrace{\rho K_{v_1} \sqrt{\frac{\rho g h_1}{G}}}_{q_2} - \underbrace{\rho K_{v_2} u_2 \sqrt{\frac{\rho g h_2}{G}}}_{q_3} \quad (5.48)$$

Valve 1 has fixed opening. Valve 2 is a control valve with control signal u between 0 and 1. The square root functions stems from the common valve characteristic which expresses that the flow is proportional to the square root of the pressures drop across the valve. Here, the pressure drops are assumed to be equal to the hydrostatic pressures at the bottom the tanks. For example, for tank 1 the hydrostatic pressure is $\rho g h_1$. The parameter G is the relative density of the liquid.²

Assume that the input variables are u_1 and u_2 , and that the output variables are $y_1 = h_1$ and $y_2 = h_2$. Write the model (5.47) – (5.48) as a state space model. Is the state space model linear or nonlinear?

Problem 5.2 Calculation of static response of thermal system

Calculate the static response in the temperature T from the thermal model (4.19).

² $G = \rho/\rho_{\text{water}}$.

Problem 5.3 *State space model on matrix-vector form*

Write the following model as a state space model on matrix-vector form:

$$\begin{aligned} {x_1}' &= x_2 \\ 2{x_2}' &= 8u_2 - 6x_2 - 2x_1 + 4u_1 \\ y &= 5x_1 + 7u_1 + 6x_2 \end{aligned} \tag{5.49}$$

Problem 5.4 *Linearization*

Given the state space model (5.50) – (5.52) which are solutions to Problem (5.1). Linearize the model.

5.7 Solutions to problems for Chapter 5

Solution to Problem 5.1

Density ρ can be cancelled. (5.47) becomes

$$h_1' = \overbrace{\frac{1}{A_1} \left(K_p u_1 - K_{v_1} \sqrt{\frac{\rho g h_1}{G}} \right)}^{f_1(\cdot)} \quad (5.50)$$

(5.48) becomes

$$h_2' = \overbrace{\frac{1}{A_2} \left(K_{v_1} \sqrt{\frac{\rho g h_1}{G}} - K_{v_2} u_2 \sqrt{\frac{\rho g h_2}{G}} \right)}^{f_2(\cdot)} \quad (5.51)$$

The measurement equations become

$$y_1 = \overbrace{h_1}^{g_1(\cdot)} \quad (5.52)$$

$$y_2 = \overbrace{h_2}^{g_2(\cdot)} \quad (5.53)$$

The state space model is nonlinear due to the square root functions.

Solution to Problem 5.2

We start by setting

$$T' = 0 \quad (5.54)$$

in the model (4.19), which then becomes:

$$0 = \frac{1}{cm} [P + cF(T_i - T) + U_h(T_e - T)] \quad (5.55)$$

Solving for T gives the static response:

$$T_s = \frac{P + cFT_i + U_hT_e}{cF + U_h} \quad (5.56)$$

Solution to Problem 5.3

Firstly, we isolate the first order derivatives on the left side, and list the variables in the proper order to prepare for the matrix-vector form:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 - 3x_2 + 2u_1 + 4u_2 \\ y &= 5x_1 + 6x_2 + 7u_1 \end{aligned} \quad (5.57)$$

Finally,

$$\underbrace{\begin{bmatrix} x_1' \\ x_2' \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_x + \underbrace{\begin{bmatrix} 0 & 0 \\ 2 & 4 \end{bmatrix}}_B \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_u \quad (5.58)$$

and

$$y = \underbrace{\begin{bmatrix} 5 & 6 \end{bmatrix}}_C \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_x + \underbrace{\begin{bmatrix} 7 & 0 \end{bmatrix}}_D \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_u \quad (5.59)$$

or, compactly:

$$\dot{x}' = Ax + Bu \quad (5.60)$$

$$y = Cx + Du \quad (5.61)$$

Solution to Problem 5.4

Linearization of the differential equations:

$$\begin{bmatrix} \Delta h_1' \\ \Delta h_2' \end{bmatrix} = \left[\begin{bmatrix} \frac{\partial f_1}{\partial h_1} & \frac{\partial f_1}{\partial h_2} \\ \frac{\partial f_2}{\partial h_1} & \frac{\partial f_2}{\partial h_2} \end{bmatrix} \right]_0 \cdot \begin{bmatrix} \Delta h_1 \\ \Delta h_2 \end{bmatrix} \quad (5.62)$$

$$+ \left[\begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} \end{bmatrix} \right]_0 \cdot \begin{bmatrix} \Delta u_1 \\ \Delta u_2 \end{bmatrix} \quad (5.63)$$

$$= \underbrace{\begin{bmatrix} -\frac{K_{v_1}}{A_1} \sqrt{\frac{\rho g}{G}} \frac{1}{2\sqrt{h_1}} & 0 \\ \frac{K_{v_1}}{A_2} \sqrt{\frac{\rho g}{G}} \frac{1}{2\sqrt{h_1}} & -\frac{K_{v_2} u_2}{A_2} \sqrt{\frac{\rho g}{G}} \frac{1}{2\sqrt{h_2}} \end{bmatrix}}_A \cdot \begin{bmatrix} \Delta h_1 \\ \Delta h_2 \end{bmatrix} \quad (5.64)$$

$$+ \underbrace{\begin{bmatrix} \frac{K_p}{A_1} & 0 \\ 0 & -\frac{K_{v_2}}{A_2} \sqrt{\frac{\rho g h_2}{G}} \end{bmatrix}}_B \cdot \begin{bmatrix} \Delta u_1 \\ \Delta u_2 \end{bmatrix} \quad (5.65)$$

Linearization of the output equation:

$$\begin{bmatrix} \Delta y_1 \\ \Delta y_2 \end{bmatrix} = \left[\begin{bmatrix} \frac{\partial g_1}{\partial h_1} & \frac{\partial g_1}{\partial h_2} \\ \frac{\partial g_2}{\partial h_1} & \frac{\partial g_2}{\partial h_2} \end{bmatrix} \right]_0 \cdot \begin{bmatrix} \Delta h_1 \\ \Delta h_2 \end{bmatrix} \quad (5.66)$$

$$+ \left[\begin{bmatrix} \frac{\partial g_1}{\partial u_1} & \frac{\partial g_1}{\partial u_2} \\ \frac{\partial g_2}{\partial u_1} & \frac{\partial g_2}{\partial u_2} \end{bmatrix} \right]_0 \cdot \begin{bmatrix} \Delta u_1 \\ \Delta u_2 \end{bmatrix} \quad (5.67)$$

$$= \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_C \cdot \begin{bmatrix} \Delta h_1 \\ \Delta h_2 \end{bmatrix} \quad (5.68)$$

$$+ \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_D \cdot \begin{bmatrix} \Delta u_1 \\ \Delta u_2 \end{bmatrix} \quad (5.69)$$

Chapter 6

Simulation algorithms of state space models

6.1 Why simulate?

Why simulate? Some good reasons are:

- ***Education and training:*** A simulator can be a very good educational and training tool! Simulations can give students, operators and other people an experience of how systems behave, without running physical experiments. While running physical experiments is often the dream scenario, it may simply not be possible to run physical experiments – it can be too expensive or risky, and perhaps the physical system does not even exist!
- ***Testing:*** Different Scenarios – like “if-then” scenarios - may be tested without any risk to the environment. Just think about the benefits of being able to check whether an oil platform or aircraft management system will work without actually doing experiments with the platform or plane.
- ***Analyze:*** With simulations, you can observe how variables which you may not actually measure will develop in reality. This can give new deep insight into, and understanding of the system.
- ***Design:*** You can use a simulator to design – or redesign – a real system. For example, you can find out if a wastewater magazine in a so-called combined drainage system, where rain and sewage are mixed, is large enough to be able to collect the wastewater due to heavy rainfall.

Is simulation important and useful? You can think about it.

A modern term used for simulators is digital twin. Maybe a buzzword, but a very descriptive one.

6.2 Simulation algorithms

6.2.1 Introduction

A textual simulation algorithm is an algorithm that you can program in a textual programming tool as Python, MATLAB, Octave, JavaScript, C, etc. Although there are simulation tools (functions) – i.e. differential equation solvers – in Python and MATLAB etc., I will here show how you can develop a simulation algorithm completely from scratch. Your own implementation may run faster than with the simulation tools, and your implementation will become completely transparent, which may be useful for testing and documentation. (Personally, I usually develop simulation code from scratch in my projects.)

6.2.2 Simulation algorithm for state space models

6.2.2.1 Introduction

We will focus on developing a simulation algorithm ready for programming based on the state space model (5.3). Including the output model (5.4) in the simulator algorithm is straightforward since that model comprises only algebraic expressions.

Thus, we assume that the model of the system to be simulated has the form of (5.3), which is repeated here for convenience:

$$x' = f(x, u, d, p) \quad (6.1)$$

The simulation algorithm which we will develop, and eventually implement in a program in Python, will calculate the state, $x(t_k)$, at any discrete point of time, t_k . as illustrated in Figure 6.1. A simulation algorithm is based on some method of discretization of the given continuous-time model, (6.1).

Some comments about the symbols shown in Figure 6.1:

- T_s [s] is the time step, which defines the resolution along the simulation time axis.
- k is the time index, which is an integer, which counts the number of time steps corresponding to a given point of time, t_k . Example: Assume that $T_s = 0.1$ s. How many time steps are there from $t = 0$ to 5.0 s? Answer: 50. Then, we have the following alternative ways to represent the point of time $t = 5.0$ s:

$$t = 5.0 \Leftrightarrow t_{50} \Leftrightarrow k = 50$$

- Often, t_k means “now”, the present point of time. Then, t_{k-1} means the previous point of time, and t_{k+1} means the next (future) point of time.
- The following symbols are equivalent:

$$x(t_k) \Leftrightarrow x(k) \Leftrightarrow x_k$$

In the following, I will mainly use the symbol x_k . This makes the algorithm which we write on the “paper” look quite similar to the algorithm to appear in the program.

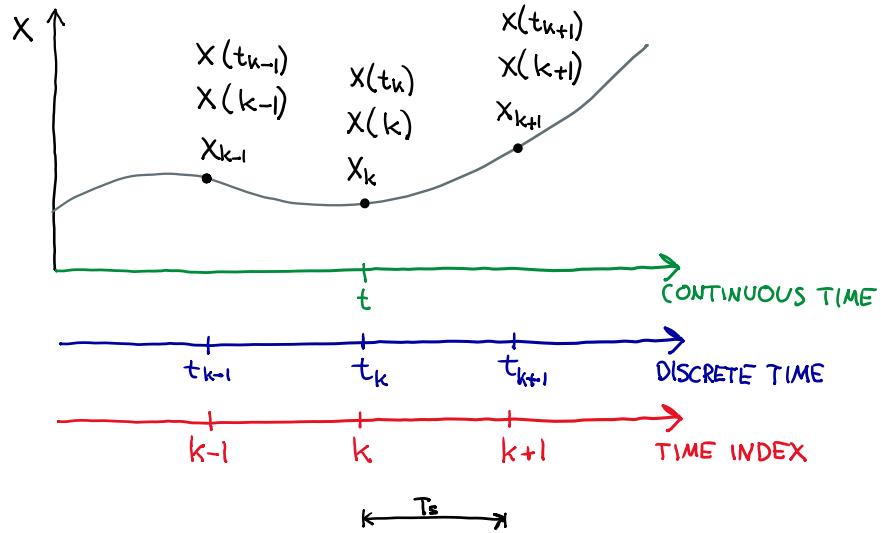


Figure 6.1: Discrete-time values of h , with three alternative symbols

6.2.2.2 The simulation algorithm

The following assumptions are made for the simulation algorithm based on (6.1):

1. The time-step of the algorithm is specified as T_s [s]. (How to select T_s will be discussed below.)
2. At the present time t_k , the following have known values:
 - (a) The state, x_k .
 - (b) The input, u_k .
 - (c) The disturbance, d_k .
 - (d) The parameter, p_k .

With these assumptions, we can predict x_{k+1} by integrating x'_k given by (6.1) between t_k and t_{k+1} :

$$x_{k+1} = x_k + \int_{t_k}^{t_{k+1}} x'_k dt \quad (6.2)$$

where x'_k is

$$x'_k = f(x_k, u_k, d_k, p_k) \quad (6.3)$$

The simulation program running on a computer must calculate (6.2) *numerically*. There are many ways to do that. It is my experience from a long life with models and simulations that the Euler Forward method of calculating (6.2) numerically is sufficient in, by far, the most cases, and I will therefore concentrate on that method here.

The Euler Forward regarded as an integration method

The Euler Forward method implies that the integrand of (6.2) is kept constant during the integration time interval. Thus,

$$x_{k+1} = x_k + T_s x'_k \quad (6.4)$$

where x'_k is given by (6.3). This corresponds to approximating the integral with rectangle integration, see Figure 6.2.

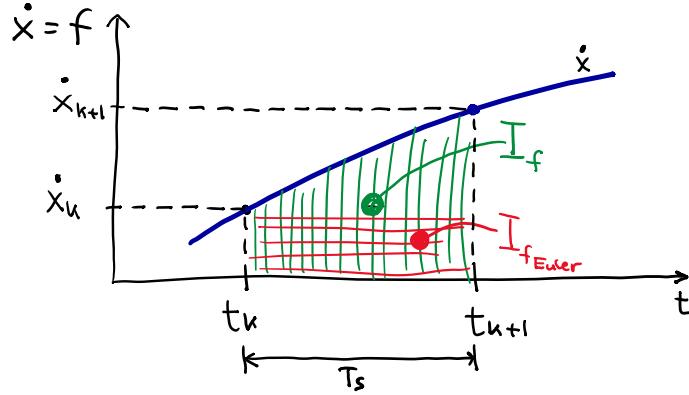


Figure 6.2: Euler Forward approximation, or rectangle approximation, of the integral of x' between t_k and t_{k+1} .

The Euler method regarded as a prediction method

Another graphical interpretation of the Euler Forward method is shown in Figure 6.3. x_{k+1} is predicted by assuming that the slope at time t_k is constant until t_{k+1} .

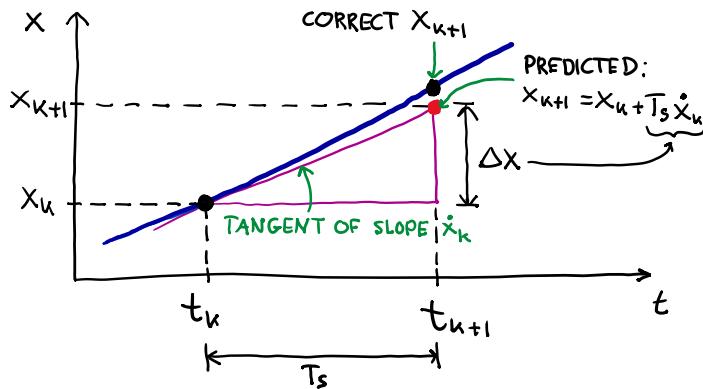


Figure 6.3: Graphical interpretation of the Euler Forward method: x_{k+1} is predicted by assuming that the slope at time t_k is constant until t_{k+1} .

Summary of the simulation algorithm:

Before the simulation loop:

- Initializations: Time settings, parameter values, preallocation of arrays for storing data.
- In particular, initialization of the state variable:

$$x_k = x_{\text{init}} \quad (6.5)$$

Inside the simulation loop:

- Assuming state variable x_k is known from initialization or from previous iteration of simulation loop:
 - State limitation: x_k must be limited between x_{\min} and x_{\max} to avoid unrealistic values (e.g. a negative liquid level):
- Assuming that the following quantities have known values at time t_k :
 - Set inputs:
 - * Control variable (or manipulating variable) u_k .
 - * Disturbance (or load or environmental variable) d_k .
 - Set parameter p_k .
- Apply x_k for storing in an array¹ for later plotting or analysis, signal processing, file saving, etc.
- Calculate the time derivative at time t_k :

$$x_k' = f(x_k, u_k, d_k, p_k) \quad (6.7)$$

- Calculate the state prediction for the next time step, t_{k+1} , i.e. the Euler integration step:

$$x_{k+1} = x_k + T_s x_k' \quad (6.8)$$

- Make a time index shift of x (to prepare for the next iteration of the algorithm):

$$x_k = x_{k+1} \quad (6.9)$$

After the simulation loop:

- Plotting, analysis, saving data to file, etc.

When presenting a simulation algorithm briefly, you may present it in terms of (6.5), (6.7), (6.8), and (6.9).

Figure 6.4 illustrates the simulation algorithm.

¹which should be preallocated to save computational time

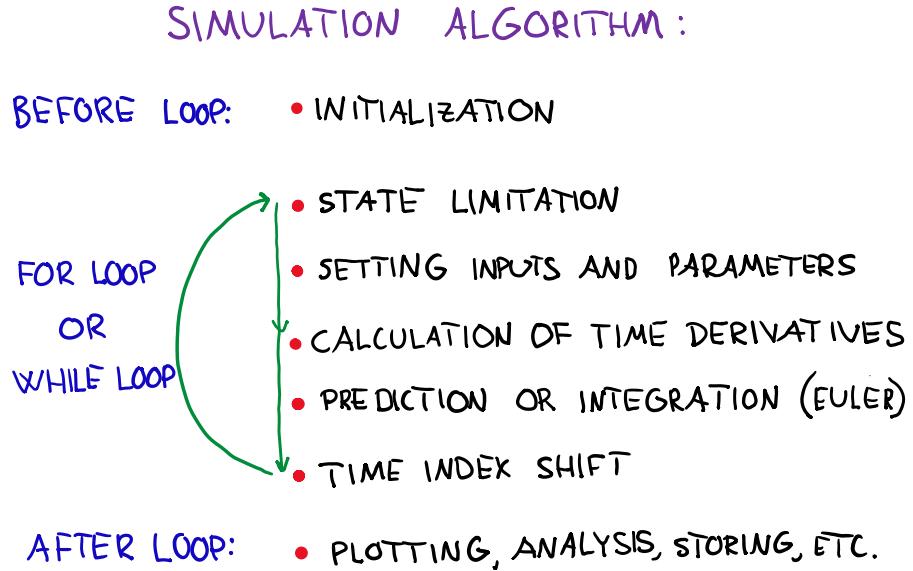


Figure 6.4: Simulation algorithm.

6.2.2.3 How to choose the simulation time step, T_s ?

As part of the discretization, we must select an appropriate value of the time step, T_s , which specifies the resolution along the simulation time axis. As illustrated in Figure 6.5, a “small” T_s provides greater (better) resolution along the simulation axis – which is a benefit. On the other hand, the computational demand on the computer is larger since it has to perform more calculations (more frequently), and also produce more simulated data.

If we are so unfortunate to choose T_s (too) large, the simulation algorithm itself may become unstable, so that the simulated responses “take off” and become completely different from the correct response. This is illustrated in Figure 6.6.

A general rule is to select T_s so that a smaller value does not make any appreciable change in the simulated response. You must expect some trial and error to find a suitable T_s . But with some insight into how fast or slow the simulated system is, I think you can estimate a reasonable value of T_s :

- Select T_s so that the simulation gives an accurate expression of the responses as they would actually have been in the original physical system (to be simulated).
- For a fast system, select a small T_s .
- For a slow system, you can choose a large T_s , but a small T_s will also work well, except that the computer will have a larger computational demand during the simulation, and more simulation data will be produced for plotting and/or file storage.

If you have no idea what to choose as T_s , try

$$T_s = 0.01 \text{ s} \quad (6.10)$$

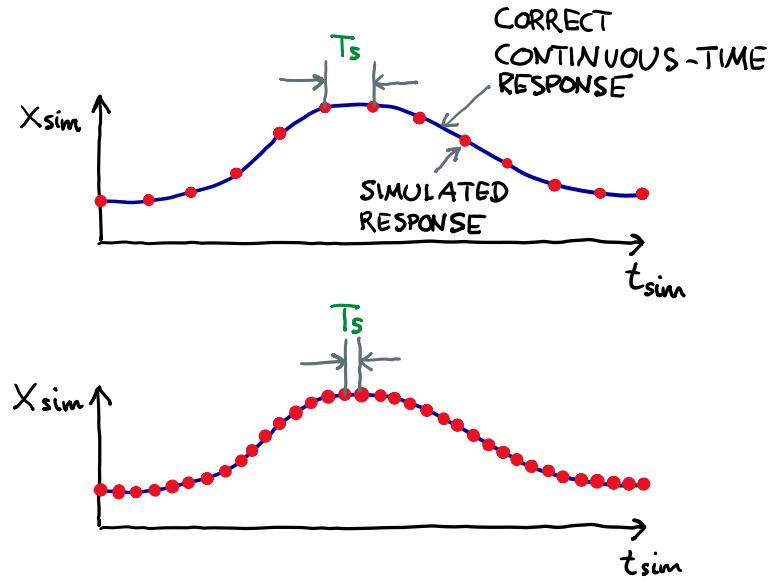


Figure 6.5: The importance of the time step T_s in simulations. Top: Relatively large T_s . Bottom: Relatively small T_s .

With (6.10), the simulator program calculates the response 100 times per second.

Simulation along real time or scaled real time?

If you want a “live” simulation – which can be very effective educationally, you need to run the simulation in real time so you can see that the simulation progresses as the simulated time runs. Also, if you want the simulator to run in parallel with the real system, maybe for monitoring purposes, you need to run the simulator in real time.

However:

- For systems that are very slow, real-time simulation will hardly be popular. Then it is better with scaled real time, so that the simulation runs eg. 100 times faster than real time.
- For systems that are very fast, real-time system is of little uses since there will be no time to experience the live simulation. In this case it may be suitable to run the simulation for example 100 times slower than real time.

Figure 6.7 illustrates the above (it is assumed 2 times faster and 2 times slower than real time).

How do you implement real-time, or possibly, scaled real-time simulation? Here is a procedure:

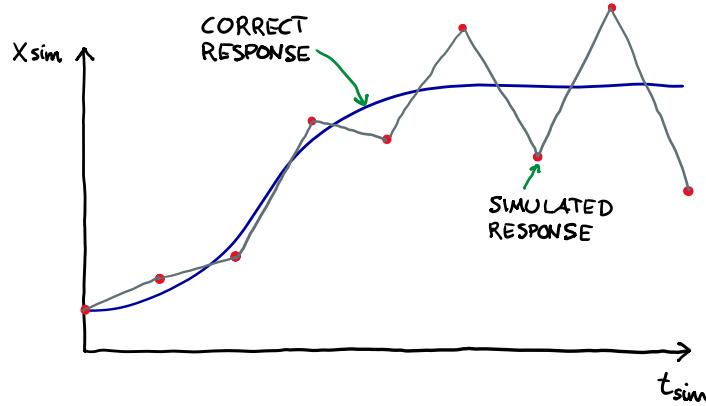


Figure 6.6: With a large T_s , the simulation algorithm may become unstable.

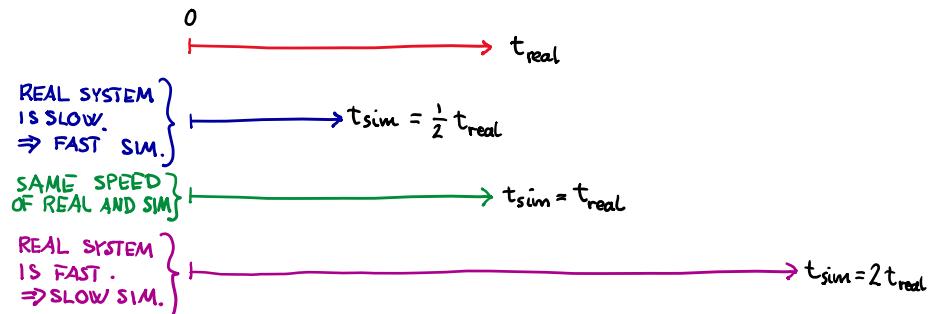


Figure 6.7: Real-time, or scaled real-time simulation

1. Select a proper value of the simulation time step, T_s . Note: The selection of T_s shall be independent of the speed of the simulator!
2. If you want the simulator to run k times faster than real time, then ensure to implement the real cycle time, or iteration time, of your simulation algorithm, $T_{\text{cycle,real}}$, according to

$$T_{\text{cycle,real}} = \frac{T_s}{k} \quad (6.11)$$

How to implement $T_{\text{cycle,real}}$, depends on the programming tool you are using, and I will not discuss this further here.

Example 6.1 Simulation algorithm of water tank

We will now develop a simulation algorithm of the water tank presented in Example 7.1. For convenience, the drawing of the tank is repeated in Figure 6.8, and the differential

equation for the level is reated in Eq. (6.12):

$$h' = \frac{1}{A} (F_{\text{in}} - F_{\text{out}}) = f() \quad (6.12)$$

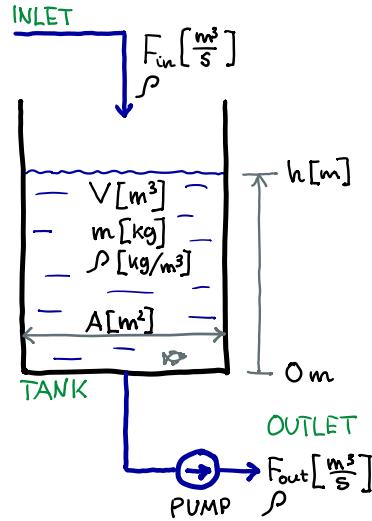


Figure 6.8: Water tank with inlet and outlet.

The simulation algorithm in terms of (6.5), (6.7), (6.8), and (6.9) becomes:

Before the simulation loop:

- Initialization (before the simulation loop):

$$h_k = h_{\text{init}} \quad (6.13)$$

Inside the simulation loop:

- Time derivative:

$$h_k' = \frac{1}{A} (F_{\text{in},k} - F_{\text{out},k}) \quad (6.14)$$

- State prediction; the Euler step:

$$h_{k+1} = h_k + T_s h_k' \quad (6.15)$$

- State limitation:

$$h \in [h_{\min}, h_{\max}] \quad (6.16)$$

- Time shift:

$$h_k = h_{k+1} \quad (6.17)$$

6.2.2.4 Implementation of the simulator in Python

Below is a Python program implementing the simulation algorithm based on the following specifications:

- Simulation time settings:

- Start time:

$$t_{\text{start}} = 0 \text{ s} \quad (6.18)$$

- Stop time:

$$t_{\text{stop}} = 5000 \text{ s} \quad (6.19)$$

- Time step (we can actually use a larger value without noticeable impact on the simulations):

$$T_s = 1 \text{ s}$$

- Simulation algorithm (6.13) – (6.17).

- Initial state:

$$h_{\text{init}} = 2 \text{ m}$$

- Model parameter: Cross-sectional area:

$$A = 2000 \text{ m}^2$$

- Disturbance (environmental variable): Inflow F_{in} to be varied as follows:

- $t_{\text{start}} \leq t < 1000 \text{ s}$: $2.0 \text{ m}^3/\text{s}$
 - $1000 \text{ s} \leq t < t_{\text{stop}}$: $3.0 \text{ m}^3/\text{s}$

- Input variable: Outflow F_{out} to be varied as follows:

- $t_{\text{start}} \leq t < 2000 \text{ s}$: $2.0 \text{ m}^3/\text{s}$
 - $2000 \text{ s} \leq t < t_{\text{stop}}$: $4.0 \text{ m}^3/\text{s}$

- Presentation of simulation results after the simulation: A figure with a (2,1)-subplot:

- h [m] is plotted in subplot no. (2,1,1).
 - F_{in} [m^3/s] and F_{out} [m^3/s] are plotted in subplot (2,1,2).

- Saving the figure with the subplots in a pdf file named plots_sim_water_tank.pdf.

Figure 6.9 shows a plot from the simulation.

Python program 11.3 implements the simulator.

http://techteach.no/control/python/prog_sim_water_tank.py

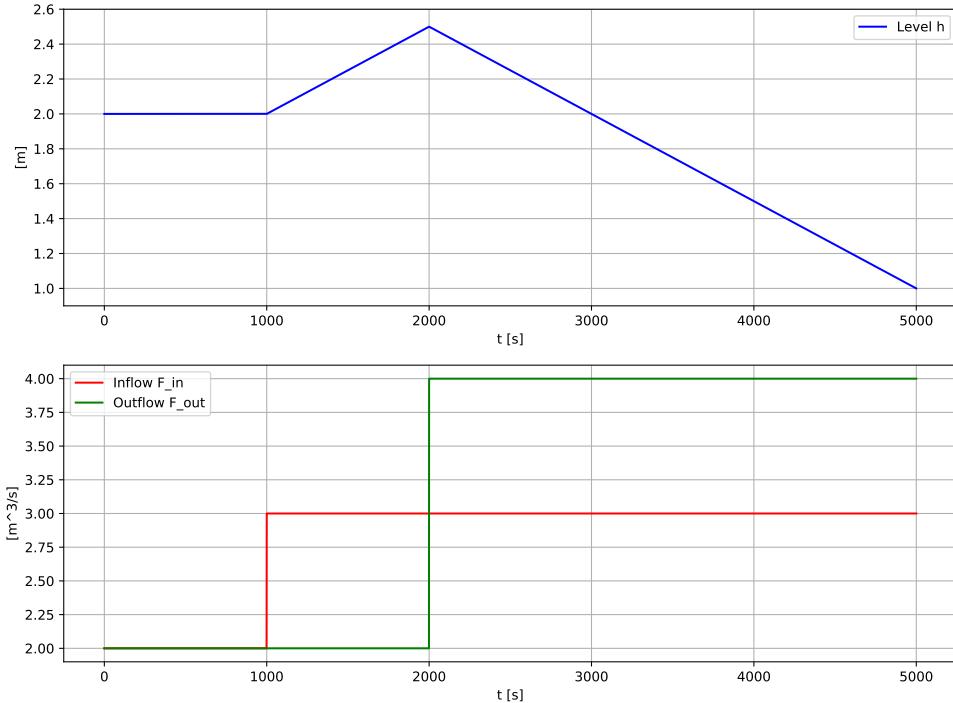


Figure 6.9: Plots from the simulation

Listing 6.1: prog_sim_water_tank.py

```
# %% Import of packages:
import matplotlib.pyplot as plt
import numpy as np

# %% Model parameters:
A = 2000 # [m2]
h_min = 0 # [m]
h_max = 4 # [m]

# %% Simulation time settings:
Ts = 1 # [s]
t_start = 0 # [s]
t_stop = 5000 # [s]
N_sim = int((t_stop - t_start)/Ts) + 1 # Number of time-steps

# %% Preallocation of arrays for plotting:
t_array = np.zeros(N_sim)
h_array = np.zeros(N_sim)
```

```

F_in_array = np.zeros(N_sim)
F_out_array = np.zeros(N_sim)

# %% Initialization:

h_k = h_init = 2.0 # [m]

# %% Simulation loop:

for k in range(0, N_sim):

    # State limitation:
    h_k = np.clip(h_k, h_min, h_max)

    t_k = k*Ts # Time

    # Selecting inputs:
    if (t_k >= t_start and t_k < 1000):
        F_in_k = 2.0
        F_out_k = 2.0
    elif (t_k >= 1000 and t_k < 2000):
        F_in_k = 3.0
        F_out_k = 2.0
    elif (t_k >= 2000 and t_k <= t_stop):
        F_in_k = 3.0
        F_out_k = 4.0

    # Time-derivative
    dh_dt_k = (1/A)*(F_in_k - F_out_k)

    # State updates using the Euler method:
    h_kp1 = h_k + dh_dt_k*Ts

    # Arrays for plotting:
    t_array[k] = t_k
    h_array[k] = h_k
    F_in_array[k] = F_in_k
    F_out_array[k] = F_out_k

    # Time shift:
    h_k = h_kp1

# %% Plotting:

plt.close('all') # Closes all figures before plotting
plt.figure(num=1, figsize=(12, 9))

plt.subplot(2, 1, 1)
plt.plot(t_array, h_array, 'b')
plt.grid()
plt.ylim(0.9, 2.6)
plt.xlabel('t [s]')
plt.ylabel('[m]')
plt.legend(labels=('Level h',))

```

```

plt.subplot(2, 1, 2)
plt.plot(t_array, F_in_array, 'r')
plt.plot(t_array, F_out_array, 'g')
plt.grid()
plt.ylim(1.9, 4.1)
plt.xlabel('t [s]')
plt.ylabel('[m^3/s]')
plt.legend(labels=('Inflow F_in', 'Outflow F_out'))

plt.show()

plt.savefig('plot_sim_water_tank.pdf')

```

6.2.2.5 Implementation of the simulator in LabVIEW

The simulation algorithm of the water tank may be implemented in any programming language. Let us take a look at LabVIEW², which is a commercial programming tool by National Instruments based primarily on graphical programming, but textual code in the C language may be included.³

LabVIEW programs consist of two parts:

- The *front panel* containing the user interface.
- The *block diagram* where the calculations, logics, etc, are coded using both graphical code and textual code.

Figure 6.10 shows the front panel and Figure 6.11 shows the block diagram of a simulator of the water tank programmed in LabVIEW. The program is available on:⁴

http://techtech.no/publications/mod_sim_con/labview/water_tank/vi/water_tank.vi

Comments to the LabVIEW program:

- Data elements on the front panel are either so-called controls or indicators. Controls can be adjusted by the user. Indicators only show values.
- The front panel elements corresponds to – i.e. have the same values as – the so-called terminals (terminals would be denoted variables in other programming languages) in the block diagram. For example, the F_in control on the front panel corresponds to the F_in terminal in the block diagram. And the tank indicator on the front panel corresponds to the h terminal to the right of the formula node in the block diagram.

²Video tutorials: <http://http://techtech.no/labview>

³I think this combination is very powerful. Using graphical programming when appropriate, and textual programming when appropriate, as in coding of mathematical formulas.

⁴Using it assumes you have installed LabVIEW 2020 (or later).

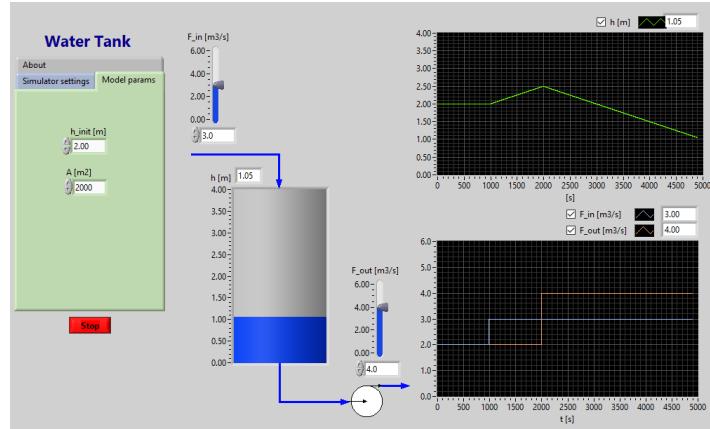


Figure 6.10: The front panel of a simulator of the water tank programmed in LabVIEW

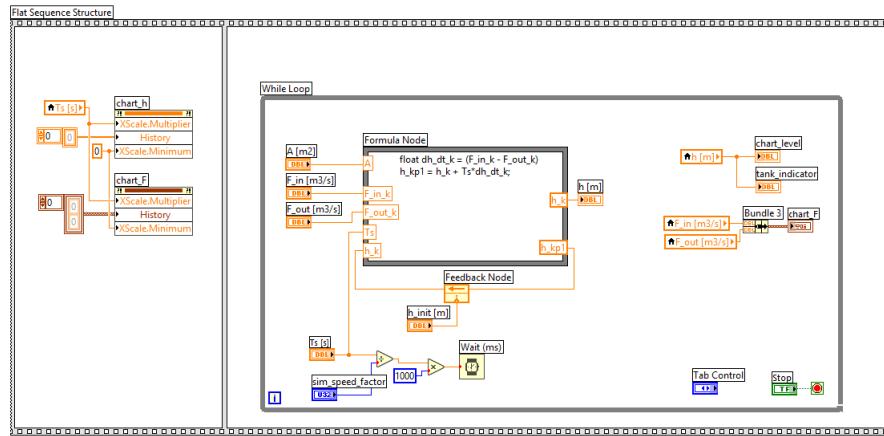


Figure 6.11: The block diagram of a simulator of the water tank programmed in LabVIEW

- The Flat Sequence Structure in the block diagram is like a movie strip. The code in the right frame is executed before the code in the subsequent frames. In the left frame is code to configure the charts. In the right frame is a while loop which is iterated until the condition for stopping the while loop is satisfied (the stopping is commented below).
- The while loop represents a frame. All code inside this frame is executed every time the while loop executes. The while loop stops when the user has clicked the Stop button on the front panel, thereby giving the Stop terminal at down-right in the while loop value of logical True.
- The formula node in the block diagram contains programming code in the C-language. I am sure you recognize the code as the simulation algorithm of the water tank! Index k , as in h_k , represents the time index of the simulation point of time. The float command is used to define the variable $dh_{dt,k}$. Such an explicit definition is needed if the code contains a variable which are neither an input nor an output from the formula node).
- The feedback node beneath the formula node is a memory element. It remember the

input value. The present output value is equal to the input value at the previous the feedback node was executed, i.e. the previous iteration of the while loop. The purpose of the feedback node here is to implement the time shift $h_k = h_kp1$ (“kp1” means “ $k+1$ ”).

- The Wait function in the block diagram makes the while loop execute with a cycle time equal to the number of milliseconds wired to the left side of the Wait function. The T_s terminal, which can be set to an appropriate value with a corresponding T_s control on the Simulator settings tab on the front panel (that tab is not shown in Figure 6.10). The T_s is actually set to 1 sec. With the Wait function you can make the LabVIEW program run with a simulation time which scaled real time (see comment below).
- The `sim_speed_factor` can be set to an appropriate value on the Simulator settings tab on the front panel (that tab is not shown in Figure 6.10) to determine how much faster than real time the simulator runs. I have set `sim_speed_factor` to 100 to make the simulator run 100 times faster than real time (the real tank is a slow process, so I want to speed up the simulation).
- The level h is plotted in the upper chart on the front panel, and inflow F_{in} and outflow F_{out} are plotted in the lower chart. These charts are updated with values from so-called local variables of h , F_{in} and F_{out} , respectively. The local variables are copies of the terminals of same names.

[End of Example 6.1]

6.2.2.6 Why predict?

Why calculate x_{k+1} when we actually need x_k ? After all, it is x_k we need. So why calculate the predicted value x_{k+1} ? Why not just subtract 1 from all of the time indexes in the algorithm above to get a formula for x_k instead of x_{k+1} ? Let us try. Then (6.7) becomes

$$x_{k-1}' = f(x_{k-1}, u_{k-1}, d_{k-1}, p_{k-1}) \quad (6.20)$$

and (6.7) becomes

$$x_k = x_{k-1} + T_s x_{k-1}' \quad (6.21)$$

While this may look ok as the basis of a simulation algorithm, a drawback is that the delayed values u_{k-1} and d_{k-1} and p_{k-1} are required. It is much more common to use the “prediction” algorithm (6.7) – (6.8).

6.2.2.7 Euler Forward vs. Euler Backward

The approximation (6.8), on which we have based the simulation algorithm, is the Euler Forward approximation of the time-derivative of x . As you may have guessed, or know already, there is also the Euler Backward method of calculating (approximating) the integral of (6.2):

$$x_k = x_{k-1} + T_s x_k' = x_{k-1} + T_s f(x_k, u_k, d_k, p_k) \quad (6.22)$$

Actually, if the function $f()$ is *linear* in x_k we will certainly be able to obtain a formula for x_k by solving (6.22) for x_k . This is actually the Euler Backward method. However, often process models are nonlinear, i.e. $f()$ is a nonlinear function of x_k , and then we may get problems. As an example, assume the following model:⁵

$$x' = -K_v \sqrt{x} \quad (6.23)$$

Then, (6.22) becomes:

$$x_k = x_{k-1} + T_s x_k' = x_{k-1} - T_s K_v \sqrt{x_k} \quad (6.24)$$

Do you see the problem? It is that x_k appears at both sides of the nonlinear equation (6.24), making it difficult (but not impossible) to calculate x_k , which we need in the simulation algorithm! With the Euler Forward method, the problem does not even appear – not for any function $f()$.

It can be shown that the Euler Forward method is somewhat less accurate than the Euler Backward method. But if we choose a small enough time step, T_s , the two methods give almost identical results.

Above, we have actually got a demonstration of why the Euler Backward method is also called the Euler Implicit method: (6.24) gives an implicit solution of x_k . And the Euler Forward method is also denoted the Euler Explicit method.

6.2.2.8 Simulation of second order differential equation models

In Section 6.2.2, the simulation algorithm assumed a first order differential equation. What if the model consists of a second order differential equation, like when you apply the Newton's Second Law to model the motion of a mechanical system?⁶ One common way to obtain a simulation algorithm for second order differential equation is to represent this differential equation by two first order differential equations, and applying the Euler method to each of them. This procedure is demonstrated in Example 6.2 where we develop a simulation algorithm for a mass-spring-damper system. Both the speed and the position of the system will be simulated.

Example 6.2 Simulator of a mass-spring-damper

Figure 6.12 shows a mass-spring-damper system. The damping force F_d is assumed proportional with the speed (i.e. so-called viscous damping is assumed):

$$F_d = -Dv \quad (6.25)$$

The spring force is assumed proportional with the position of the body relative to position $s = 0$:

$$F_s = -Ks \quad (6.26)$$

⁵A model of a water tank with a valve in the outlet may be modelled like this. The outflow through the valve is the cause of the square root function.

⁶Differential equation of higher order than two are very rare. You may meet them in e.g. signal processing and state estimator design, but such applications are not relevant here.

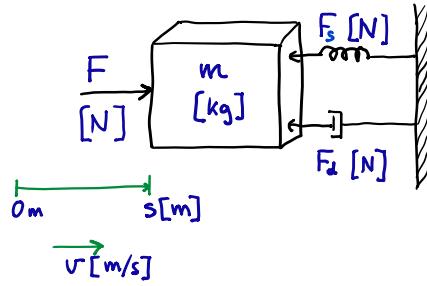


Figure 6.12: Mass-spring-damper system.

The Newton's 2nd law (force balance) is:

$$ms'' = mv' = F - F_d - F_s = F - Dv - Ks \quad (6.27)$$

The state space model is:

$$s' = v \quad (6.28)$$

$$v' = \frac{1}{m} [F - Dv - Ks] \quad (6.29)$$

The simulation algorithm is:

Before the simulation loop:

- Initialization:

$$s_k = s_{\text{init}} \quad (6.30)$$

$$v_k = v_{\text{init}} \quad (6.31)$$

Inside the simulation loop:

- State limitation:

$$s_k \in [s_{\min}, s_{\max}] \quad (6.32)$$

$$v_k \in [v_{\min}, v_{\max}] \quad (6.33)$$

- Time derivatives (the differential equations):

$$s_k' = v_k \quad (6.34)$$

$$v_k' = \frac{1}{m} [F_k - Dv_k - Ks_k] \quad (6.35)$$

- Euler integration:

$$s_{k+1} = s_k + T_s \cdot s_k' \quad (6.36)$$

$$v_{k+1} = v_k + T_s \cdot v_k' \quad (6.37)$$

- Time index shift:

$$s_k = s_{k+1} \quad (6.38)$$

$$v_k = v_{k+1} \quad (6.39)$$

Program 6.2 shows an implementation of the simulator, including various numerical settings (I do not show these values in the text). Figure 6.13 shows simulated responses due to a step change of the applied force F .

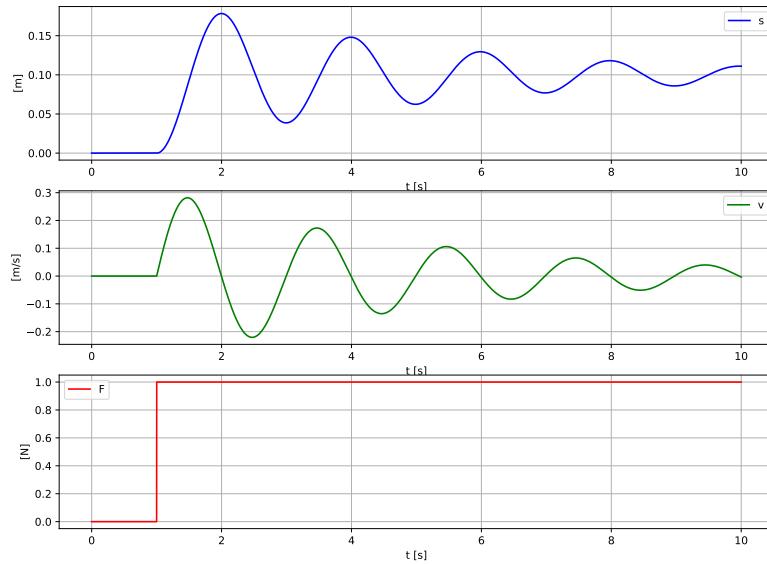


Figure 6.13: Simulation of a mass-spring-damper.

http://techteach.no/control/python/prog_sim_mfd.py

Listing 6.2: prog_sim_mfd.py

```
import matplotlib.pyplot as plt
import numpy as np

m = 1 # [kg]
K = 10 # [N/m]
D = 0.5 # [N/(m/s)]

# For state limitation, but not effective here.
s_min = -np.inf; s_max = np.inf,
v_min = -np.inf; v_max = np.inf

Ts = 0.001 # [s]
t_start = 0 # [s]
t_stop = 10 # [s]
N_sim = int((t_stop - t_start)/Ts) + 1

t_array = np.zeros(N_sim)
s_array = np.zeros(N_sim)
v_array = np.zeros(N_sim)
F_array = np.zeros(N_sim)
```

```

s_k = s_init = 0 # [m]
v_k = v_init = 0 # [m/s]

for k in range(0, N_sim):

    t_k = k*Ts

    if 0 <= t_k <= 1: F_k = 0 # [N]
    else: F_k = 1 # [N]

    s_k = np.clip(s_k, s_min, s_max)
    v_k = np.clip(v_k, v_min, v_max)

    ds_dt_k = v_k
    dv_dt_k = (1/m)*(F_k - D*v_k - K*s_k)

    s_kp1 = s_k + Ts*ds_dt_k
    v_kp1 = v_k + Ts*dv_dt_k

    t_array[k] = t_k
    s_array[k] = s_k
    v_array[k] = v_k
    F_array[k] = F_k

    s_k = s_kp1
    v_k = v_kp1

plt.close('all')
plt.figure(1, figsize=(12, 9))

plt.subplot(3, 1, 1)
plt.plot(t_array, s_array, 'b', label='s')
plt.xlabel('t [s]')
plt.ylabel('[m]')
plt.grid()
plt.legend()

plt.subplot(3, 1, 2)
plt.plot(t_array, v_array, 'g', label='v')
plt.grid()
plt.xlabel('t [s]')
plt.ylabel('[m/s]')
plt.legend()

plt.subplot(3, 1, 3)
plt.plot(t_array, F_array, 'r', label='F')
plt.grid()
plt.xlabel('t [s]')
plt.ylabel('[N]')
plt.legend()

# plt.savefig('plot_sim_mfd.pdf')
plt.show()

```

[Slutt på eksempel 6.2]

6.2.3 Simulation algorithm of time delays

Time delays – also denoted transportation time and time delay – appear in various systems:

- Transportation of material on conveyor belt or through pipelines.
- Apparent time delay due to imperfect mixing in tanks, e.g. a delayed response in the measured temperature of liquid in a tank after some change in the supplied power to the liquid.
- Delay of information through communication channels.

How to make a simulation algorithm of a time delay? Let us make a simulator of the transportation taking place on the conveyor belt shown in Figure 6.14. The belt runs with fixed speed.

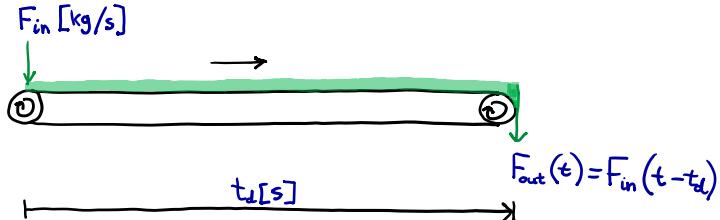


Figure 6.14: Conveyor belt

There is a time delay from inflow F_{in} to the belt to the outflow F_{out} from the belt.

Example: Suppose the time delay is 45 seconds. A change in the inflow at 12:30:00 will then give the same change in the outflow at 12:30:45. Mathematically, the relationship between F_{in} and F_{out} can be expressed as:

$$F_{out}(t) = F_{in}(t - t_d) \quad (6.40)$$

where $t_d = 45$ s is the time delay (transportation time).

Let us imagine that the conveyor belt in Figure 6.14 is represented by an *array*, where the input signal to the array is put into the first element of the array, and the time delayed signal is taken out from the last element of the array. At each discrete time in the simulation, the value of each element in the array is moved one step toward the end of the array. This is illustrated in Figure 6.15 where y_{in} is the input to the array and y_{out} is the output of the array. The mathematical relationship between y_{out} and y_{in} is

$$y_{out}(t) = y_{in}(t - t_d) \quad (6.41)$$

What should be the length of the array? If the time delay is t_d [s] and the simulation time step is T_s [s], the length of the array should be

$$N_f = \frac{t_d}{T_s} \quad (6.42)$$

or rounded upwards to the nearest integer if this fraction is not an integer (rounding upwards is safer than rounding downwards assuming it is safer to overstate the time delay than to underestimate it).

We can now state a simulation algorithm as follows. The algorithm is executed at each point of simulation time.

1. The time delayed signal, $y_{\text{out}}(t) = y_{\text{in}}(t - t_d)$, is the value of the last element of the array.
2. In principle, all the elements of the array are moved one step towards the end of the array. (The program example below shows a way to implement the move, but there are other ways.)
3. The value of the variable to be delayed, $y_{\text{in}}(t)$, is entered into the first element of the array.

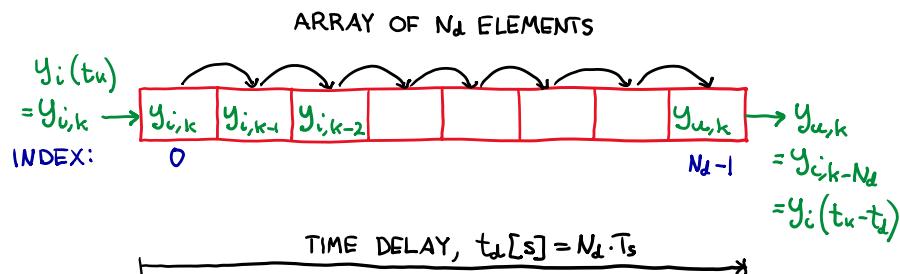


Figure 6.15: Realization of a time delay using an array. At each simulation point of time, all the elements of the array are moved one step toward the end of the array.

Example 6.3 A Python program for simulation of a time delay

Python program 6.3 implements the simulator of a time delay.

http://techteach.no/control/python/prog_sim_time_delay.py

Listing 6.3: prog_sim_time_delay.py

```
# %% Import of packages:
import matplotlib.pyplot as plt
import numpy as np
```

```

# %% Model parameters:

t_delay = 1.0 # [s]

# %% Simulation time settings:

Ts = 0.01 # [s]
t_start = 0 # [s]
t_stop = 5 # [s]
N_sim = int((t_stop - t_start)/Ts) + 1

# %% Preallocation of arrays for plotting:

t_array = np.zeros(N_sim)
y_in_array = np.zeros(N_sim)
y_out_array = np.zeros(N_sim)

# %% Initialization:

Y_out_init = 0.5

# %% Preallocation of array for time-delay:

Nd = int(round(t_delay/Ts)) + 1
delay_array = np.zeros(Nd) + Y_out_init

# %% Simulation loop:

for k in range(0, N_sim):

    t_k = k*Ts

    # Excitation:
    if (t_start <= t_k < 2.0):
        y_in_k = 0
    if (2.0 <= t_k):
        y_in_k = 1.0

    # Time delay:
    y_out_k = delay_array[-1]
    delay_array[1:] = delay_array[0:-1]
    delay_array[0] = y_in_k

    # Writing values to arrays for plotting:
    t_array[k] = t_k
    y_in_array[k] = y_in_k
    y_out_array[k] = y_out_k

# %% Plotting:

plt.close('all')
plt.figure(1)
plt.grid()

```

```

plt.plot(t_array, y_out_array, 'b')
plt.plot(t_array, y_in_array, 'r')
plt.xlabel('t [s]')
plt.legend(labels=('y_out', 'y_in'))
plt.show()

plt.savefig('plot_sim_time_delay.pdf')

```

The specifications of the simulator are as follows:

- Time step: $T_s = 0.01$ s
- Time delay: $t_d = 1$ s
- The input signal, y_{in} , is changed as a step at time $t_0 = 2$ s.
- The array representing the time delay is initially filled with values $Y_{out_init} = 0.5$. This causes the time delay value, y_{out} , to have value Y_{out_init} until the simulation time has become greater than the time delay.

Figure 6.16 shows the results of a simulation with the program.

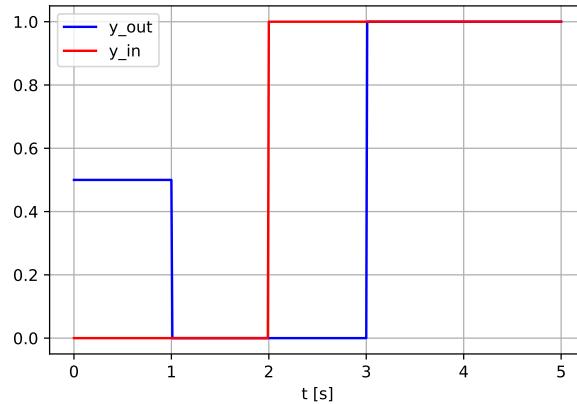


Figure 6.16: Simulation of a time delay

Comments to the above program:

- The code $Y_{out_init} = 0.5$ defines a constant to fill the array with an initial value.
- The code $Nd = \text{int}(\text{round}(td/Ts)) + 1$ calculates the number of elements of the array needed to represent the time delay.
- The code $\text{delay_array} = \text{np.zeros}(Nd) + y_{out_init}$ creates the array, and gives all the elements the initial value of y_{out_init} .
- The code beneath # Excitation: generates a step change of the input signal.

- The code beneath # Time delay: realizes the time delay according to the algorithm presented above:
 - *Step 1 of the algorithm:* The code $y_out_k = delay_array[-1]$ assigns the time delayed variable, y_out_k , the value of the last element of the array. Index -1 addresses the last element.
 - *Step 2 of the algorithm:* The code $delay_array[1:] = delay_array[0:-1]$ moves each of the elements one step towards the right. The access code $0:-1$ means “all elements but the last one”. The access code $1:$ means “all elements starting with element of index 1 (i.e. the element of index 0 is excluded)”.
 - *Step 3 of the algorithm:* The code $delay_array[0] = y_in_k$ assigns the element of index 0 (the leftmost element) of $delay_array$ the value of the input variable, y_in_k .
- The code beneath # Writing values to arrays for plotting: writes values to the arrays to be used for plotting (but the code for plotting is not shown here).

[End of Example 6.3]

6.3 *Problems for Chapter 6*

Problem 6.1 *Simulator of a heated water tank*

A mathematical model of a heated water tank with initial model parameter values is presented in Ch. 41.4. The model (41.8) includes a time delay of the supplied power P . However, in the problems below you can (for simplicity) disregard this time delay, i.e. not implement it in the simulator.

1. “Pseudo” simulation algorithm

Derive a “pseudo” simulation algorithm for the temperature (more or less ready to be implemented in a program).

2. Calculation of static operating point:

Calculate from the model the constant power, P_0 , needed to bring the temperature to a constant value of 25 deg C.

3. Programming and simulation:

Program a simulator in Python of the tank heater. The simulator must be implemented with “native” code in a For loop based on the Euler Forward discretization of the model (a built-in simulation function of Python should not be used). You can set the time-step to 1 s. The following variables should be plotted: T , T_{in} , and T_{env} in one subplot, and P in another subplot. Assume that the initial temperature is $T_{init} = 20$ deg C. Run a simulation with $P = P_0$ as calculated above. Is the static T the same as specified in Problem 2 above?

4. Stability of the simulator:

Demonstrate that the simulator becomes numerically inaccurate if you select a (too) large simulator time step. Also, run a simulation with such a large time step that the simulation becomes numerically unstable.

Problem 6.2 *Simulation of a time delay*

This problem is a continuation of Problem 6.1.

Modify program 6.5 (which is in the solution of Problem 6.1) as follows: Set the time step to 1 sec. Include a time delay of 60 sec in P .

Verify with a simulation that you have implemented the time delay correctly.

Problem 6.3 *Simulator of a ship*

A mathematical model of the surge or longitudinal motion of a ship is presented in Ch. 41.2. Program a simulator of the ship according to the following specifications:

- Time step is 1 s.
- Stop time is 1000 s.
- Position is initially 0 m, and speed is initially 0 m/s.
- The propeller force is initially 0 kN. It is changed as a step from 0 to 200 kN at time 200 s, and back to 0 kN at time 400 s.
- Water current can be assumed zero.
- Wind speed can be assumed zero.
- Position, speed, propeller force, and hydrodynamic force together with wind force are plotted in individual plots (four plots, altogether).

Hint: Write the model as a second order state space model with the following state variables: Position x_1 , and speed x_2 .

6.4 *Solutions to problems for Chapter 6*

Solution to Problem 6.1

1. “Pseudo” simulation algorithm

The model (41.10) without the time delay (i.e., $\tau = 0$) is:

$$c\rho VT'(t) = P(t) + c\rho F [T_{in}(t) - T(t)] + U [T_{env}(t) - T(t)] \quad (6.43)$$

We write this model as a state space model:

$$T' = \{P(t) + c\rho F [T_{in}(t) - T(t)] + U [T_{env}(t) - T(t)]\} / (c\rho V) \quad (6.44)$$

The model simulation algorithm:

- Before the simulation loop:
 - Initialization: $T_k = T_init$
- Inside the simulation loop:
 - Limitation of T_k between $T_min = 0$ deg C and $T_max = 100$ deg C (using e.g. the `numpy.clip` function)
 - Setting input signals P_k , and T_in_k , and T_env_k
 - Any use of T_k , e.g. storing in an array for later plotting
 - Time derivative: $dT_dt_k = (1/(c*\rho*V))*(P_k + c*\rho*F*(T_in_k - T_k) + U*(T_env_k - T_k))$
 - Prediction or integration (Euler forward): $T_kp1 = T_k + dt*dT_dt_k$
 - Time index shift: $T_k = T_kp1$
- After the simulation loop:
 - Plotting, analysis, saving simulation data to file, etc.

2. Calculation of static operating point:

Python program 6.4 implements a solution.

http://techteach.no/control/python/sim_heated_tank_1.py

Listing 6.4: sim_heated_tank_1.py

```
#%% Model params:
c = 4200 # [J/(kg*K)]
rho = 1000 # [kg/m3]
V = 0.2 # [m3]
```

```

U = 1000 # [W/K]
F = 0.25e-3 # [m3/s]
T_in = 20 # [deg C]
T_env = 20 # [deg C]

# %% Calculation of power giving specified static temp:

T_static = 25 # [deg C] Static temp

# From model after t' is set to zero (static value):
P0 = - (c*rho*F*(T_in-T_static) + U*(T_env-T_static)) # [W]
print('P0 [W] =', P0)

```

The result is:

P0 [W] = 10250.0

3. Programming and simulation:

Python program 6.5 implements a solution.

http://techteach.no/control/python/sim_heated_tank_2.py

Listing 6.5: sim_heated_tank_2.py

```

# %% Imports:

import numpy as np
import matplotlib.pyplot as plt

# %% Model params:

c = 4200 # [J/(kg*K)]
rho = 1000 # [kg/m3]
V = 0.2 # [m3]
U = 1000 # [W/K]
F = 0.25e-3 # [m3/s]
T_in = 20 # [deg C]
T_env = 20 # [deg C]

T_min = 0
T_max = 100

# %% Calculation of power giving specified static temp:

T_static = 25 # [deg C] Static temp

# From model after t' is set to zero (static value):
P0 = - (c*rho*F*(T_in-T_static) + U*(T_env-T_static)) # [W]

# %% Sim time settings:

dt = 1 # [s]

```

```

t_start = 0 # [s]
t_stop = 6000 # [s]
N_sim = int((t_stop - t_start)/dt) + 1

# %% Preallocation of arrays for storing:

t_array = np.zeros(N_sim)
T_array = np.zeros(N_sim)
T_in_array = np.zeros(N_sim)
T_env_array = np.zeros(N_sim)
P_array = np.zeros(N_sim)

# %% Sim loop:

T_k = T_init = 20 # [deg C] Initial temp

for k in range(0, N_sim):

    # State limitation:
    T_k = np.clip(T_k, T_min, T_max)

    t_k = k*dt

    P_k = P0
    T_in_k = T_in
    T_env_k = T_env

    t_array[k] = t_k
    T_array[k] = T_k
    T_in_array[k] = T_in_k
    T_env_array[k] = T_env_k
    P_array[k] = P_k

    # Time derivative:
    dT_dt_k = ((1/(c*rho*V))
                *(P_k
                  + (c*rho*F)*(T_in-T_k)
                  + U*(T_env-T_k)))
    T_kp1 = T_k + dt*dT_dt_k

    # Time index shift:
    T_k = T_kp1

# %% Plotting:

plt.close('all')
plt.figure(1)

plt.subplot(2, 1, 1)
plt.plot(t_array, T_array, 'r', label='T')
plt.plot(t_array, T_in_array, 'b', label='T_in')
plt.plot(t_array, T_env_array, 'g', label='T_env')
plt.legend()
plt.grid()

```

```

plt.xlabel('t [s]')
plt.ylabel('[deg C]')

plt.subplot(2, 1, 2)
plt.plot(t_array, P_array, 'm', label='P')
plt.legend()
plt.grid()
plt.xlabel('t [s]')
plt.ylabel('[W]')

plt.savefig('plot_sim_heated_tank_2.pdf')
plt.show()

```

Figure 6.17 shows the simulated responses.

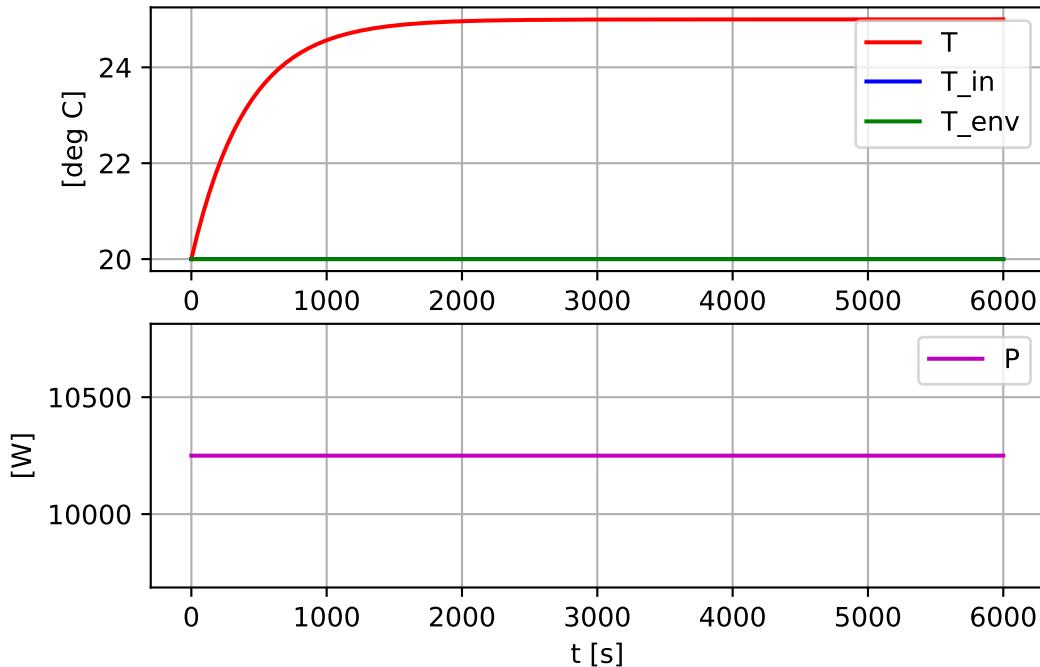


Figure 6.17: Problem 6.1: Simulated responses.

Comments:

- The static value of T can be read off from the plot in Figure 6.17, and can be found more precisely with the code $T_array[-1]$, and is 25.00, which is the same as the specified value, cf. Problem 2.

4. Stability of the simulator:

Python program 6.6 runs a simulation with time step

$$dt = 700 \text{ s}$$

http://techteach.no/control/python/sim_heated_tank_2.py

Listing 6.6: sim_heated_tank_3.py

```
#%% Imports:

import numpy as np
import matplotlib.pyplot as plt

#%% Model params:

c = 4200 # [J/(kg*K)]
rho = 1000 # [kg/m3]
V = 0.2 # [m3]
U = 1000 # [W/K]
F = 0.25e-3 # [m3/s]
T_in = 20 # [deg C]
T_env = 20 # [deg C]

T_min = 0
T_max = 100

#% Calculation of power giving specified static temp:

T_static = 25 # [deg C] Static temp

# From model after t' is set to zero (static value):
P0 = - (c*rho*F*(T_in-T_static) + U*(T_env-T_static)) # [W]

#% Sim time settings:

dt = 700 # [s]
t_start = 0 # [s]
t_stop = 10000 # [s]
N_sim = int((t_stop - t_start)/dt) + 1

#% Preallocation of arrays for storing:

t_array = np.zeros(N_sim)
T_array = np.zeros(N_sim)
T_in_array = np.zeros(N_sim)
T_env_array = np.zeros(N_sim)
P_array = np.zeros(N_sim)

#% Sim loop:

T_k = T_init = 20 # [deg C] Initial temp

for k in range(0, N_sim):
```

```

# State limitation:
T_k = np.clip(T_k, T_min, T_max)

t_k = k*dt

P_k = P0
T_in_k = T_in
T_env_k = T_env

t_array[k] = t_k
T_array[k] = T_k
T_in_array[k] = T_in_k
T_env_array[k] = T_env_k
P_array[k] = P_k

# Time derivative:
dT_dt_k = ((1/(c*rho*V))
            *(P_k
              + (c*rho*F)*(T_in-T_k)
              + U*(T_env-T_k)))
T_kp1 = T_k + dt*dT_dt_k

# Time index shift:
T_k = T_kp1

# %% Plotting:

plt.close('all')
plt.figure(1)

plt.subplot(2, 1, 1)
plt.plot(t_array, T_array, 'r', label='T')
plt.plot(t_array, T_in_array, 'b', label='T_in')
plt.plot(t_array, T_env_array, 'g', label='T_env')
plt.legend()
plt.grid()
plt.xlabel('t [s]')
plt.ylabel('[deg C]')

plt.subplot(2, 1, 2)
plt.plot(t_array, P_array, 'm', label='P')
plt.legend()
plt.grid()
plt.xlabel('t [s]')
plt.ylabel('[W]')

plt.savefig('plot_sim_heated_tank_3.pdf')
plt.show()

```

Figure 6.18 shows the simulated response. The simulation is (numerically) inaccurate.

Figure 6.19 shows the simulated response with time step $dt = 900$ s. The simulation is (numerically) unstable.

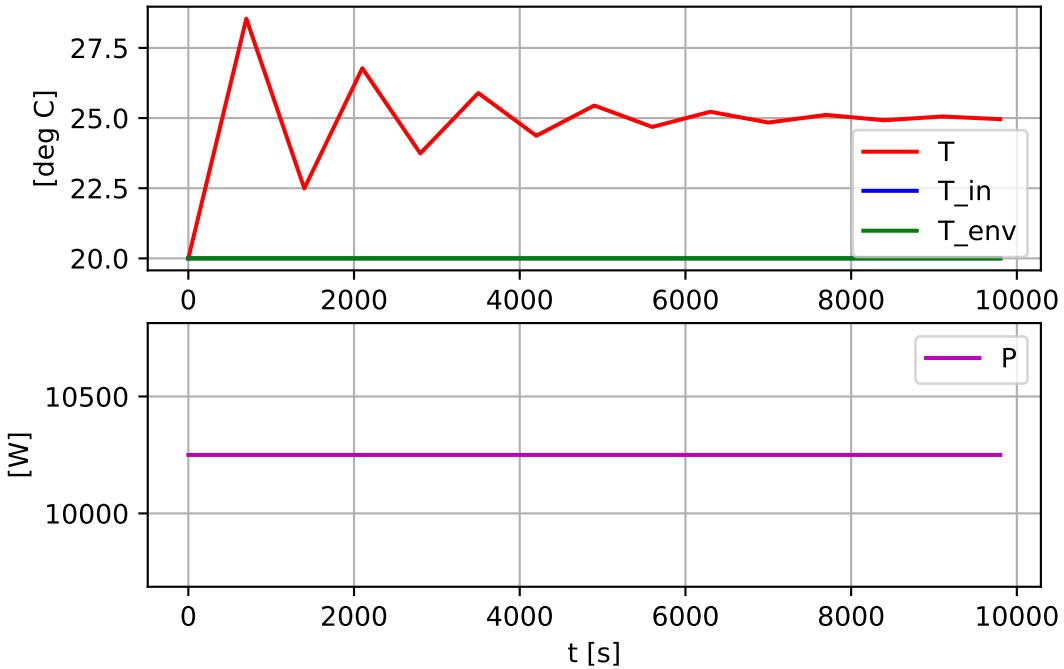


Figure 6.18: Problem 6.1: Simulated response with time step $dt = 700$ s.

Solution to Problem 6.2

Python program 6.7 implements a solution.

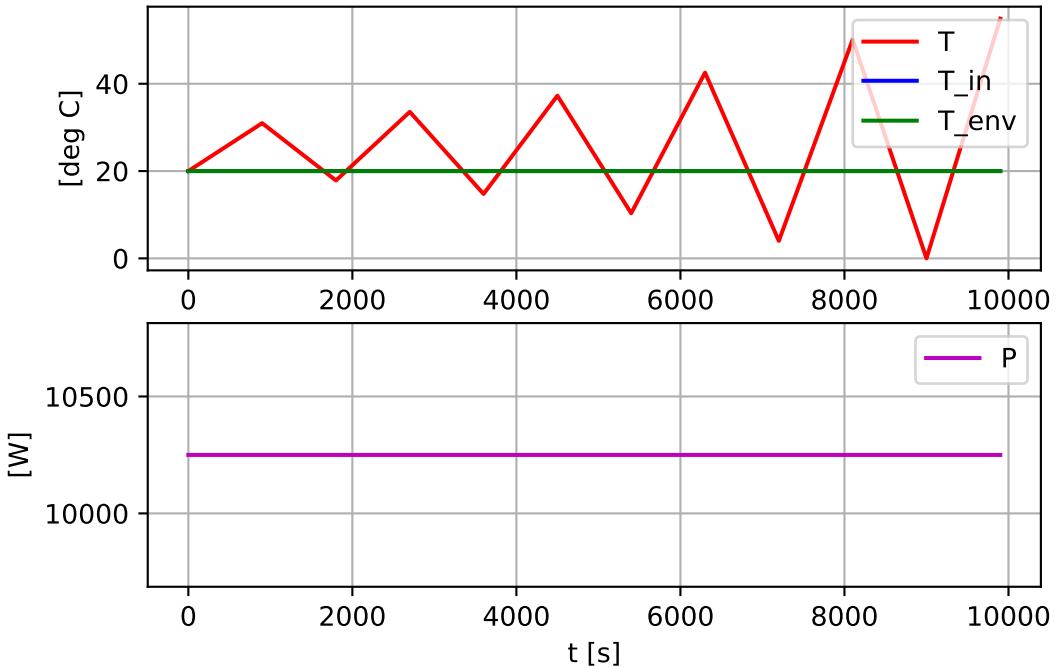
http://techteach.no/control/python/sim_heated_tank_time_delay.py

Listing 6.7: sim_heated_tank_time_delay.py

```
#%% Imports:
import numpy as np
import matplotlib.pyplot as plt

# Model params:
c = 4200 # [J/(kg*K)]
rho = 1000 # [kg/m^3]
V = 0.2 # [m^3]
U = 1000 # [W/K]
F = 0.25e-3 # [m^3/s]
T_in = 20 # [deg C]
T_env = 20 # [deg C]

T_min = 0
T_max = 100
```

Figure 6.19: Problem 6.1: Simulated response with time step $dt = 900$ s.

```

##% Sim time settings:

dt = 1 # [s]
t_start = 0 # [s]
t_stop = 10000 # [s]
N_sim = int((t_stop - t_start)/dt) + 1

##% Defining a step change in power P. A step is a
# convenient test signal to demonstrate the time delay
# of the system:

T_static = 25 # [deg C] Static temp

# From model after t' is set to zero (static value):
P0 = - (c*rho*F*(T_in-T_static) + U*(T_env-T_static)) # [W]

dP = 0.1*P0 # [W] Step amplitude of 10 percent
P1 = P0 + dP # [W] Power after step change
print('P0 [W] =', P0)
print('P1 [W] =', P1)

##% Array for transport delay of power:

t_delay = 60 # [s]
P_delayed_init = P0 # [W]
N_delay = int(round(t_delay/dt)) + 1

```

```

P_delay_array = np.zeros(N_delay) + P_delayed_init

#%% Preallocation of arrays for storing:

t_array = np.zeros(N_sim)
T_array = np.zeros(N_sim)
T_in_array = np.zeros(N_sim)
T_env_array = np.zeros(N_sim)
P_array = np.zeros(N_sim)

#%% Sim loop:

T_k = T_init = 25 # [deg C]
t0 = 5000 # [s] Time of step in P

for k in range(0, N_sim):

    # State limitation:
    T_k = np.clip(T_k, T_min, T_max)

    t_k = k*dt

    if (0 <= t_k <= t0):
        P_k = P0
        T_in_k = T_in
        T_env_k = T_env
    else:
        P_k = P1
        T_in_k = T_in
        T_env_k = T_env

    # Moving delay array elements one step:
    P_delayed_k = P_delay_array[-1]
    P_delay_array[1:] = P_delay_array[0:-1]
    P_delay_array[0] = P_k

    t_array[k] = t_k
    T_array[k] = T_k
    T_in_array[k] = T_in_k
    T_env_array[k] = T_env_k
    P_array[k] = P_k

    dT_dt_k = ((1/(c*rho*V))
                *(P_delayed_k
                  + (c*rho*F)*(T_in-T_k)
                  + U*(T_env-T_k)))

    T_kp1 = T_k + dt*dT_dt_k

    # Time index shift:
    T_k = T_kp1

# %% Plotting:

```

```

plt.close('all')
plt.figure(1)

# Defining time intervals for plotting:
# Start time for plotting:
t_plot_start = 4900 # [s]
# Stop time for plotting:
t_plot_end = 5200 # [s]
# Array index for start of plotting:
i_start = int(t_plot_start/dt) # [s]
# Array index for start of plotting
i_end = int(t_plot_end/dt) # [s]
# Index interval for plotting
i_interval = np.arange(i_start, i_end, dt)

plt.subplot(2, 1, 1)
plt.plot(t_array[i_interval], T_array[i_interval], 'b',
          label='T')
plt.legend()
plt.grid()
plt.xlabel('t [s]')
plt.ylabel('[C]')

plt.subplot(2, 1, 2)
plt.plot(t_array[i_interval], P_array[i_interval], 'm', label='P')
plt.legend()
plt.grid()
plt.xlabel('t [s]')
plt.ylabel('[W]')

plt.savefig('plot_sim_heated_tank_with_delay.pdf')
plt.show()
    
```

Figure 6.20 shows the simulated response. The time delay of 60 s is apparent in the plot.

Solution to Problem 6.3

With the following definitions:

- Position

$$y = x_1 \quad (6.45)$$

- Speed:

$$y' = x_2 \quad (6.46)$$

the original model, (41.3), can be represented with the following two first order differential equations:

$$x_1' = x_2 \quad (6.47)$$

$$x_2' = (F_p + F_h + F_w) / m \quad (6.48)$$

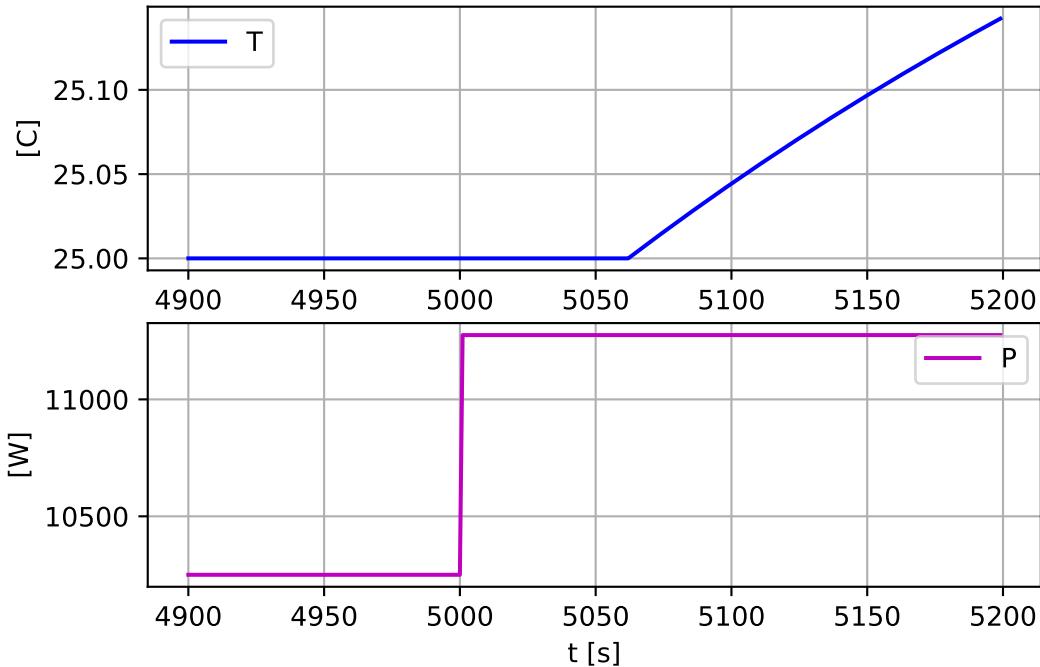


Figure 6.20: Problem 6.2: Simulated response.

where:

$$F_h = D_h (u_c - x_2) |u_c - x_2| \quad (6.49)$$

$$F_w = D_w (V_w - x_2) |V_w - x_2| \quad (6.50)$$

Python program 6.8 implements a simulator based on (6.47)-(6.48).

http://techteach.no/control/python/sim_ship.py

Listing 6.8: sim_ship.py

```
#%% Importing packages:
import matplotlib.pyplot as plt
import numpy as np

#%% Model parameters:
m = 71164*1000 # [kg]
Dh = 8.4*1000 # [N/(m/s)^2]
Dw = 0.177*1000 # [N/(m/s)^2]
```

```

%% State limits:

x1_min = -np.inf
x1_max = np.inf
x2_min = -np.inf
x2_max = np.inf

%% Simulation time settings:

dt = 1 # [s]
t_start = 0 # [s]
t_stop = 1000 # [s]
N_sim = int((t_stop - t_start)/dt) + 1

%% Preallocation of arrays for plotting:

t_array = np.zeros(N_sim)
x1_array = np.zeros(N_sim)
x2_array = np.zeros(N_sim)
Fp_array = np.zeros(N_sim)
Fh_array = np.zeros(N_sim)
Fw_array = np.zeros(N_sim)

%% Initialization:

x1_init = 0
x2_init = 0
x1_k = x1_init
x2_k = x2_init

%% Simulation loop:

for k in range(0, N_sim):

    # State limitation:
    x1_k = np.clip(x1_k, x1_min, x1_max)
    x2_k = np.clip(x2_k, x2_min, x2_max)

    # Time:
    t_k = k*dt

    # Excitations:
    if t_start <= t_k < 200:
        Fp_k = 0 # [N]
        uc_k = 0
        Vw_k = 0
    if 200 <= t_k:
        Fp_k = 200*1000
        uc_k = 0
        Vw_k = 0
    if 400 <= t_k:
        Fp_k = 0
        uc_k = 0
        Vw_k = 0

```

```

# Forces from water and wind:
Fh_k = Dh*(uc_k - x2_k)*np.abs(uc_k - x2_k)
Fw_k = Dw*(Vw_k - x2_k)*np.abs(Vw_k - x2_k)

# Time derivatives:
dx1_k = x2_k
dx2_k = (1/m)*(Fp_k + Fh_k + Fw_k)

# State prediction (Euler step):
x1_kp1 = x1_k + dx1_k*dt
x2_kp1 = x2_k + dx2_k*dt

# Arrays for plotting:
t_array[k] = t_k
x1_array[k] = x1_k
x2_array[k] = x2_k
Fp_array[k] = Fp_k
Fh_array[k] = Fh_k
Fw_array[k] = Fw_k

# Time shift:
x1_k = x1_kp1
x2_k = x2_kp1

#% Plotting:

plt.close('all') # Closes all figures before plotting
plt.figure(1)

plt.subplot(4, 1, 1)
plt.plot(t_array, x1_array, 'r', label='y')
plt.legend()
plt.grid()
plt.ylabel('[m]')

plt.subplot(4, 1, 2)
plt.plot(t_array, x2_array, 'b', label='dydt')
plt.legend()
plt.grid()
plt.ylabel('[m/s]')

plt.subplot(4, 1, 3)
plt.plot(t_array, Fp_array/1000, 'g', label='Fp')
plt.legend()
plt.grid()
plt.ylabel('[kN]')

plt.subplot(4, 1, 4)
plt.plot(t_array, Fh_array/1000, label='Fh')
plt.plot(t_array, Fw_array/1000, label='Fw')
plt.legend()
plt.grid()
plt.ylabel('[kN]')

```

```

plt.xlabel('t [s]')

%% Saving the plot figure as a pdf file:
plt.savefig('plot_simulation_ship.pdf')
plt.show()

```

Figure 6.21 shows the simulated responses.

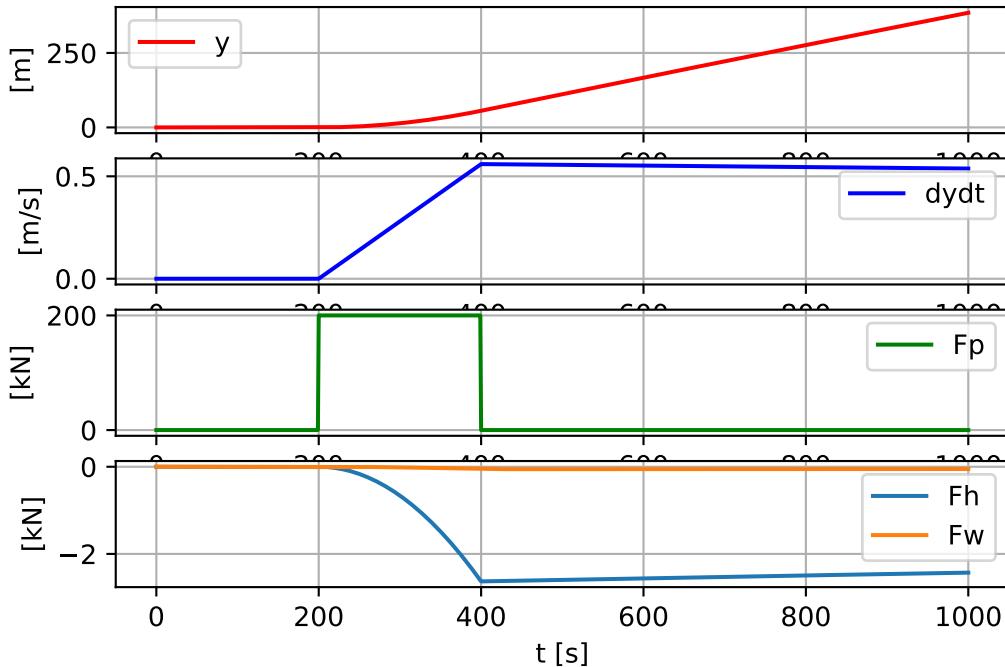


Figure 6.21: Problem 6.3: Simulated responses.

Comments to the simulation:

- The speed increases relatively fast between $t = 200$ and 400 s, which is due to the relatively large propeller force.
- The speed decreases relatively slowly after $t = 400$ s, which is due to the relatively small forces from water and wind.

Chapter 7

Block diagram models and simulators

7.1 Introduction

The differential equation models in Ch. 5 can be represented with *block diagrams* which display graphically the structure of the model. A block diagram may enhance your understanding of the model. Simulink, LabVIEW and OpenModelica are examples of simulation tools for block diagram models, cf. Section 7.3.

7.2 How to draw block diagrams

Assume that you have derived a state space model of some dynamic system. The differential equations are:

$$\begin{aligned} x_1' &= f_1(x, u, d, p) \\ &\vdots \quad \vdots \quad \vdots \\ x_n' &= f_n(x, u, d, p) \end{aligned} \tag{7.1}$$

The states are given by integrals:

$$\begin{aligned} x_1(t) &= x_1(0) + \int_0^t f_1(\cdot) d\tau \\ &\vdots \quad \vdots \quad \vdots \\ x_n(t) &= x_n(0) + \int_0^t f_n(\cdot) d\tau \end{aligned} \tag{7.2}$$

where $x_1(0) = x_{1,\text{init}}$, $x_n(0) = x_{n,\text{init}}$ are the initial states.

We can represent (7.1)-(7.2) with a (mathematical) block diagram. The block diagram

shows the state variables as solutions to their differential equations (the solutions are obtained with integrations), see Figure 7.1.

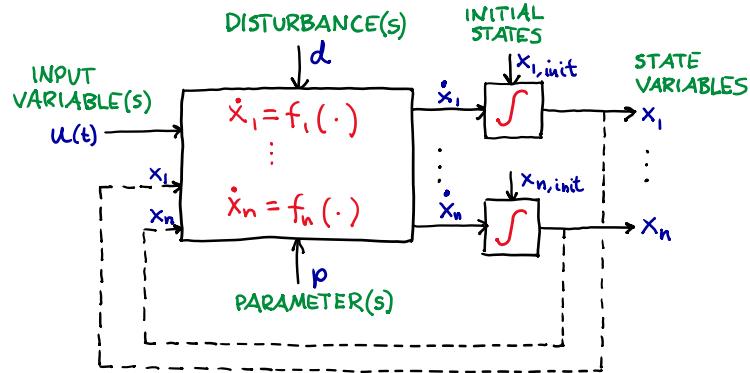


Figure 7.1: Mathematical block diagram of the model consisting of (5.1). In some models, namely pure integrator models, the dashed feedbacks do not exist.

In Figure 7.1, x_1, \dots, x_n , which are the integrator outputs, are fed back to the inputs of the block where f_1, \dots, f_n are calculated. However, in some models, namely so-called pure integrator models, those feedbacks do not exist. Therefore, I have drawn these feedbacks as dashed lines in Figure 7.1.

Example 7.1 Model with block diagram of water tank

Figure 7.2 shows a water tank with inlet and outlet.

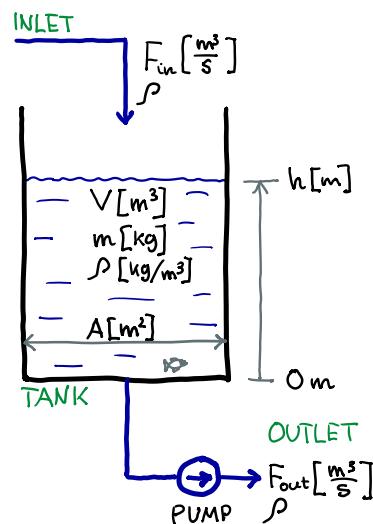


Figure 7.2: Water tank with inlet and outlet

In Figure 7.2:

- h [m] is the water level.
- F_{in} [m^3/s] is volumetric inflow. It can not be adjusted; its value is given, and it is here regarded as an environmental variable or a process disturbance.
- F_{out} [m^3/s] er volumetric outflow. It can be manipulated by the user or a computer.
- A [m^2] is the inner cross sectional area of the tank, which is also the water surface area.
- ρ (rho) [kg/m^3] is the water density.

We will develop a mathematical model that expresses how the level in the tank varies. The model will be based on a so-called material balance of the water in the tank:

The rate of change of mass of water [kg/s] = water inflow [kg/s] – water outflow [kg/s]

In mathematical terms, the mass balance is:

$$m' = \rho F_{\text{in}} - \rho F_{\text{out}} \quad (7.3)$$

Since we want the level h to appear explicitly in the mode, we express the mass m [kg] as a function of h and the area A as follows:

$$m = \rho Ah \quad (7.4)$$

By substituting m in (7.3) with (7.4), (7.3) becomes:

$$(\rho Ah)' = \rho F_{\text{in}} - \rho F_{\text{out}} \quad (7.5)$$

Here, we move the constants ρ and A outside the differentiation, then divide all terms by $A\rho$. The result is

$$h' = \frac{1}{A} (F_{\text{in}} - F_{\text{out}}) \quad (7.6)$$

which is the state equation of the model.

Assume that we are mainly interested in the value of the level, h . Then it is natural to define h as the output variable:

$$y = h \quad (7.7)$$

With general model terms:

- h is the state variable.
- h is also the process output variable.
- F_{in} is a process disturbance, or environmental variable, or process load.

- F_{out} is an input variable. Are you confused by thinking about the *outflow* F_{out} as an *input* variable? It *is* correct from a systems perspective.
- A is a model parameter.

Figure 7.3 shows a mathematical block diagram of the model of the water tank.

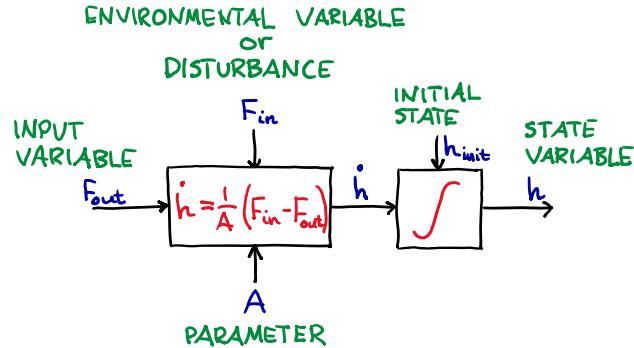


Figure 7.3: Mathematical block diagram of the model of the water tank.

[End of Example 7.1]

Using elementary mathematical blocks instead of textual math

In Figure 7.3, a *textual* mathematical expression is used to represent the mathematical expression making up the time derivative. Alternatively, you can use use elementary mathematical *blocks*. Figure 7.4 shows various blocks. There is no standard about how to design mathematical blocks, so you may actually invent blocks yourself.

Example 7.2 Model water tank using elementary blocks

Figure 7.5 shows a block diagram of the model (7.6) using some elementary mathematical blocks, here: summation and division, in addition to the integration block.

The block diagrams in Figures 7.3 and 7.5 have different forms, but are functionally equivalent. So, which form to select? That's up to you.¹

[End of Example 7.2]

¹I often prefer the block diagram form of Figure 7.3 since I find it easier and more flexible in the case of changes to represent mathematical formulas with text than with blocks.

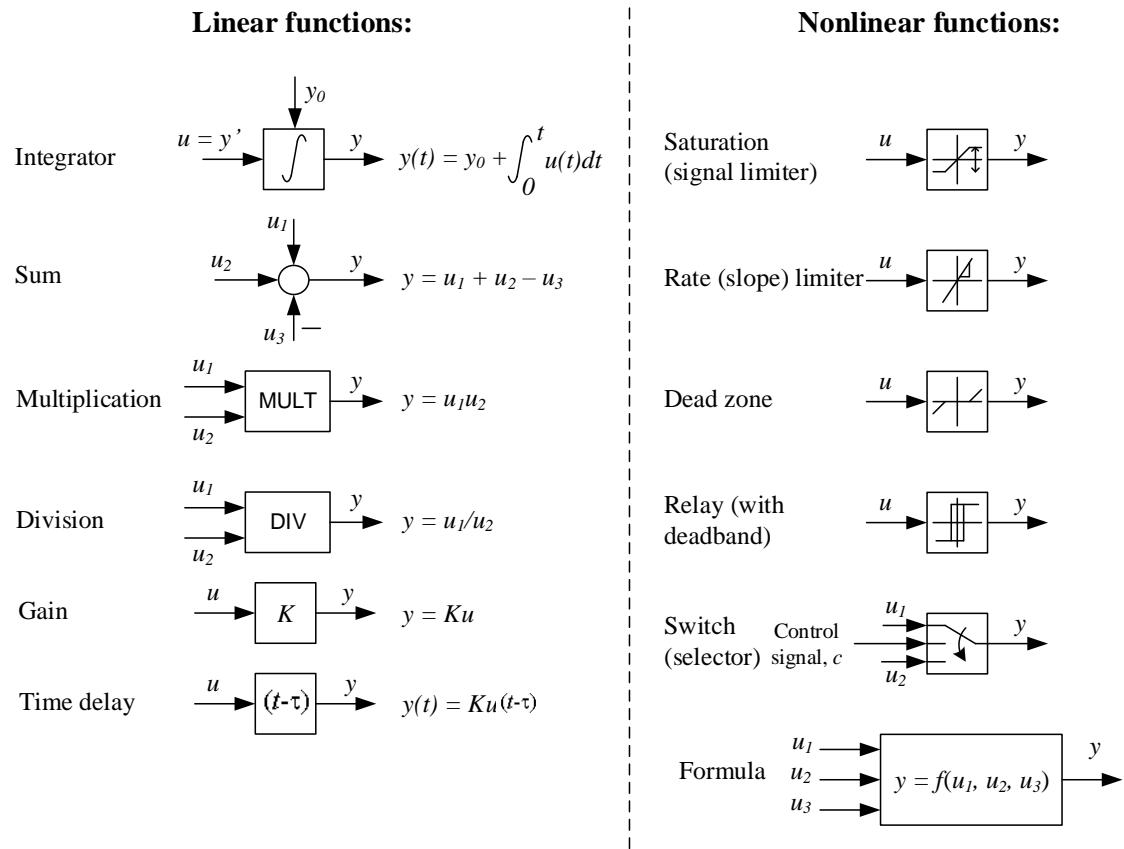


Figure 7.4: Elementary mathematical blocks.

7.3 Simulation tools for block diagrams

7.3.1 Introduction

You have seen how mathematical models can be illustrated with mathematical block diagrams, see for example Figure 7.1 which represents a general state space model and Figure 7.3 which represents the model of a water tank. In Section 6.2 we developed textual simulation algorithms from state space models.

As an alternative to textual simulation algorithms, there are block diagram-based simulation tools, as Simulink and LabVIEW with its Control and Simulation Toolkit, where the model can be represented by model blocks, and then you can run the simulation. In such tools, there are premade model blocks for:

- linear state space models, which is a set of linear first order differential equations, cf. Ch. 5,
- transfer functions, which is a model based on the Laplace transform of the underlying linear differential equations, cf. Ch. 9.

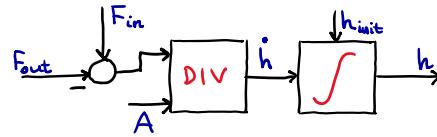


Figure 7.5: A block diagram of the model (7.6) using elementary mathematical blocks.

Block diagram simulation tools make it easy to combine models in

- series combination.
- parallel combination,
- feedback combination.

7.3.2 OpenModelica

7.3.3 Simulink

7.3.4 LabVIEW

Figure 7.6 shows an example of a feedback control system to be simulated in Simulink. The system contains blocks (models) of the PID controller, the process to be controlled, and the measurement filter – connected in a feedback structure. The models are transfer functions models (the s is the Laplace variable).

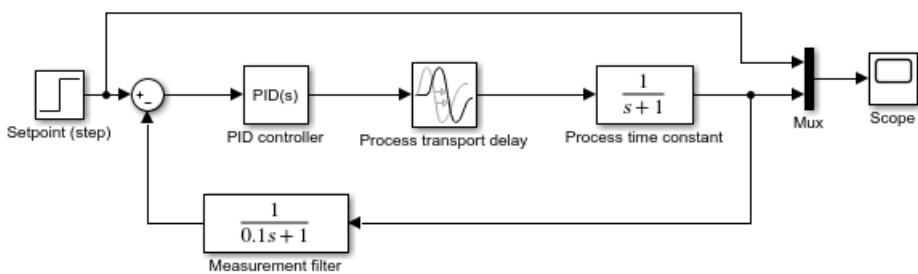


Figure 7.6: Feedback control loop modeled in Simulink

Block diagram simulation tools can be time-effective for you as a programmer, i.e. you may spend relatively little time on programming the simulation model comparing with developing and programming textual simulation algorithm from scratch. The effectiveness is particularly clear if the models are linear. However, these tools do not have any particular support for nonlinear models, i.e. you must program the model more or less from scratch, as you learnt in Section 6.2, and then the effectiveness is not clear. In most of the projects that I have been involved in, the models to be simulated have been nonlinear – requiring programming of the simulator algorithms more or less from scratch.

To summarize:

Block diagram-based simulation tools can be time-effective for you as developer (programmer), in particular when the models are linear. But, often the models to be simulated are nonlinear, and then you probably need to develop and program the simulator algorithm from scratch.

Furthermore, if you are able to develop and program the simulator algorithm from scratch, you can implement your simulator in any programming language!

7.4 *Problems for Chapter 7*

Problem 7.1 *Mathematical block diagram of an accumulation*

In Ch. 4, the general model of dynamic systems are presented as (4.1)-(4.2), which are repeated here for convenience:

$$\text{Acc}(t) = \text{Acc}(0) + \int_0^t \text{Acc}' d\theta \quad (7.8)$$

where:

$$\text{Acc}' = \text{Inflow} - \text{Outflow} + \text{Generation} \quad (7.9)$$

Draw a mathematical block diagram of (7.8)-(7.9).

Problem 7.2 *Mathematical block diagram of water tank*

Figure 7.7 shows a water tank with pump inflow and valve outflow.

A mathematical model of the level in a water tank is:

$$Ah' = F_{\text{in}} - F_{\text{out}} \quad (7.10)$$

where:

$$F_{\text{in}} = K_u u \quad (7.11)$$

$$F_{\text{out}} = K_v \sqrt{\rho gh} \quad (7.12)$$

Draw a mathematical block diagram of the model using one block with *textual* mathematical expressions for representing h' , and another block diagram of the model using elementary mathematical *blocks* for representing h' .

7.5 *Solutions to problems for Chapter 7*

Solution to Problem 7.1

See Figure 7.8.

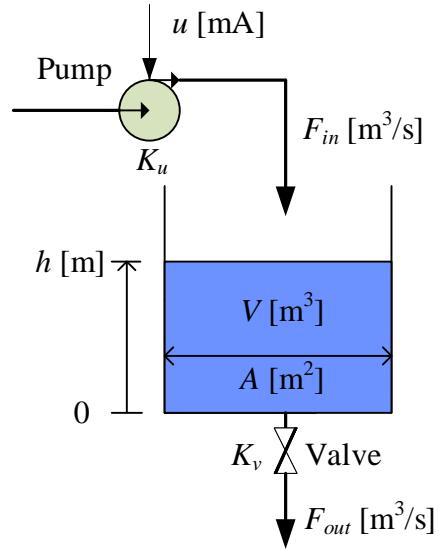


Figure 7.7: Water tank with pump inflow and valve outflow.

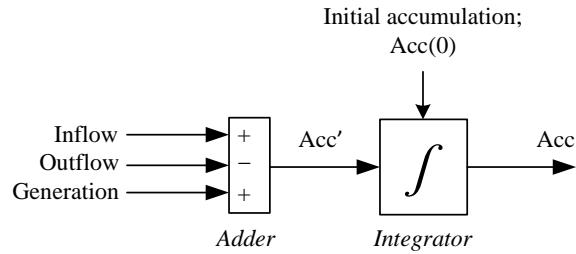


Figure 7.8: Mathematical block diagram of accumulation.

Solution to Problem 7.2

Figures 7.9 and 7.10 show the block diagrams.

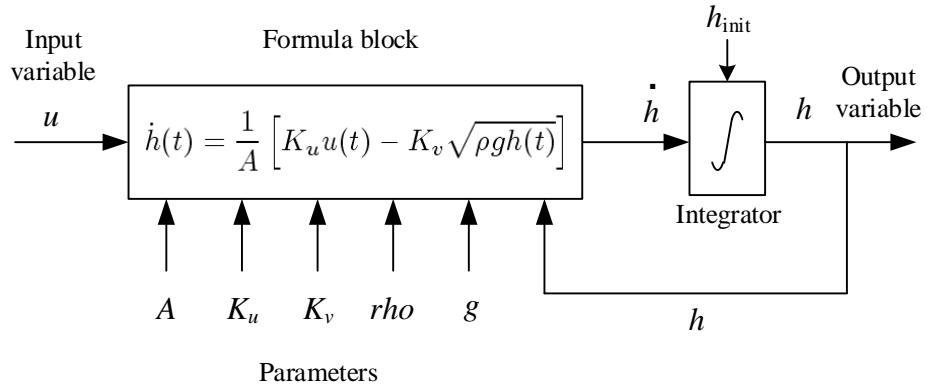


Figure 7.9: A mathematical block diagram of the model using one block with *textual* mathematical expressions for representing h' .

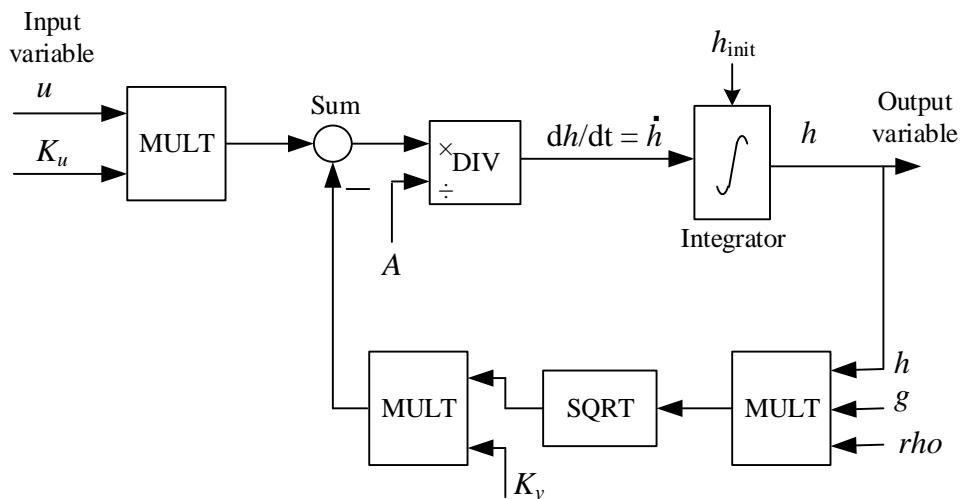


Figure 7.10: A mathematical block diagram of the model using elementary mathematical blocks for representing h' .

Chapter 8

The Laplace transform

8.1 Introduction

The Laplace transform is a mathematical tool which is useful in systems theory. It is the foundation of transfer functions which is a standard model form of dynamic systems.

Transfer functions are described in Chapter 9. Furthermore, with the Laplace transform you relatively easily calculate responses in dynamic systems by hand.¹

In this chapter, I present the Laplace transform at a minimum level. You can find much more information in a mathematics text-book.

8.2 Definition of the Laplace transform

Given a time-evaluated function $f(t)$ – that is, $f(t)$ is a function of time t . It can be a sinusoid, a ramp, an impulse, a step, a sum of such functions, or any other function of time. The Laplace transform of $f(t)$ can be denoted $F(s)$, and is given by the following integral:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt \quad (8.1)$$

Expressed with words, $f(t)$ is multiplied by the weight function e^{-st} , and the resulting product $e^{-st} f(t)$ is integrated from start time $t = 0$ to end time $t = \infty$. The Laplace transform does not care about any value that $f(t)$ might have at negative values of time t . In other words, you can think of $f(t)$ as being “switched on” at $t = 0$. (The so-called two-sided Laplace transform is defined also for negative time, but it is not relevant for our applications.)

¹It turns out that we rarely need to perform manual calculations of the responses. When we need to know the responses, it is in most situations more convenient to obtain them by simulating the system. With the Laplace transform you can calculate responses only for *linear systems*, that is, systems having a model which can be expressed as a linear differential equation.

s is the Laplace variable.² $F(s)$ is a function of s . The time t is not a variable in $F(s)$ – it disappeared through the time integration. $F(s)$ will look completely different from $f(t)$, cf. the following example.

Example 8.1 Laplace transform of a step

Given the function

$$f(t) = 1 \text{ (for } t \geq 0\text{)} \quad (8.2)$$

which is a step of amplitude 1 at time $t = 0$. Using (8.1), its Laplace transform becomes

$$F(s) = \int_0^\infty e^{-st} \cdot 1 \cdot dt = \left[-\frac{1}{s} e^{-st} \right]_{t=0}^{t=\infty} = \frac{1}{s} \quad (8.3)$$

[End of Example 8.1]

Calculating the time function $f(t)$ from its Laplace transform $F(s)$, or in other words: going from the Laplace domain to the time domain, is denoted *inverse Laplace transform*. This can be expressed as

$$f(t) = \mathcal{L}^{-1}\{F(s)\} \quad (8.4)$$

Figure 8.1 illustrates the time domain and the Laplace domain, and the Laplace transformation and the inverse Laplace transformation.

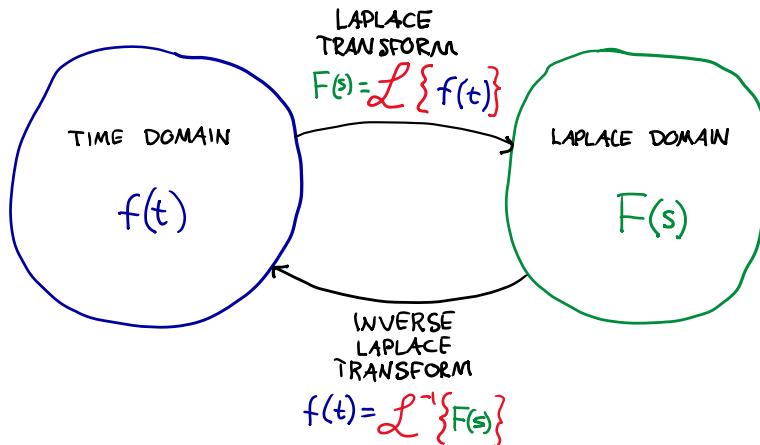


Figure 8.1: The time domain and the Laplace domain

The inverse Laplace transform is actually defined by a complicated complex integral.³ If you *really* want to calculate this integral, you should use the Residue Theorem in mathematics. However, I suggest you instead try the simplest method, namely to find $f(t)$ from the precalculated Laplace transform pairs, cf. Section 8.3, possibly combined with one or more of the Laplace transform properties, cf. Section 8.4.

²You may wonder what is the physical meaning of s . It can be interpreted as a complex frequency, but I think the best answer is that there is no meaningful physical meaning.

³ $f(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st} ds$

8.3 Laplace transform pairs

Laplace transform pairs:

$$f(t) \quad (\text{time function}) \iff F(s) \quad (\text{Laplace transform}) \quad (8.5)$$

$$k\delta(t) \quad (\text{impulse of strength or area } k) \iff k \quad (8.6)$$

$$k \quad (\text{step of amplitude } k) \iff \frac{k}{s} \quad (8.7)$$

$$kt \quad (\text{ramp of slope } k) \iff \frac{k}{s^2} \quad (8.8)$$

$$kt^n \quad (n^{\text{th}} \text{ order exponential of } t) \iff k \frac{n!}{s^{n+1}} \quad (8.9)$$

$$\frac{ke^{-t/T}}{T} \quad (\text{decaying 1. order exponential of } t) \iff \frac{k}{Ts + 1} \quad (8.10)$$

$$k(1 - e^{-t/T}) \quad (\text{raising 1. order exponential of } t) \iff \frac{k}{(Ts + 1)s} \quad (8.11)$$

$$k \left[1 + \frac{1}{T_2 - T_1} (T_1 e^{-t/T_1} - T_2 e^{-t/T_2}) \right] \iff \frac{k}{(T_1 s + 1)(T_2 s + 1)s} \quad (8.12)$$

8.4 Laplace transform properties

In calculations with the Laplace transform you will probably need one or more of the Laplace transform *properties* presented below.⁴ We will definitely use some of them for deriving transfer functions, cf. Chapter 9.

Linear combination:

$$k_1 f_1(t) + k_2 f_2(t) \iff k_1 F_1(s) + k_2 F_2(s) \quad (8.13)$$

Special case: Multiplication by a constant:

$$kf(t) \iff kF(s) \quad (8.14)$$

Time delay:

$$f(t - \tau) \iff F(s)e^{-\tau s} \quad (8.15)$$

⁴Additional properties could have been given here, too, but the ones presented are the most useful.

Time derivative:

$$\stackrel{(n)}{f(t)} \iff s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - \stackrel{(n-1)}{f}(0) \quad (8.16)$$

Special case: Time derivative with zero initial conditions:

$$s^n F(s) \iff \stackrel{(n)}{f(t)} \quad (8.17)$$

Special case: Time derivative with non-zero initial condition:

$$f'(t) \iff sF(s) - f_0 \quad (8.18)$$

Special case: First order time derivative with zero initial condition:

$$f'(t) \iff sF(s) \quad (8.19)$$

(So, differentiation corresponds to multiplication by s .)

Integration:

$$\int_0^t f(\tau) d\tau \iff \frac{1}{s} F(s) \quad (8.20)$$

(So, integration corresponds to division by s .)

Final Value Theorem:

$$\lim_{t \rightarrow \infty} f(t) \iff \lim_{s \rightarrow 0} sF(s) \quad (8.21)$$

Example 8.2 Calculation of time response (inverse Laplace transform)

Given the following differential equation:

$$y'(t) = -2y(t) \quad (8.22)$$

with initial value $y(0) = 4$. Calculate $y(t)$ using the Laplace transform.

To calculate $y(t)$ we start by taking the Laplace transform of both sides of the differential equation (8.22):

$$\mathcal{L}\{y'(t)\} = \mathcal{L}\{-2y(t)\} \quad (8.23)$$

Here, we apply the time derivative property (8.18) at the left side, and the linear combination property (8.14) to the right side, to get

$$sY(s) - 4 = -2Y(s) \quad (8.24)$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{4}{s+2} \quad (8.25)$$

To get the corresponding $y(t)$ from this $Y(s)$ we look for a proper Laplace transform pair. (27.9) fits. We have to write our $Y(s)$ on the form of (27.9). Dividing both the numerator and the denominator by 2 gives

$$Y(s) = \frac{4}{s+2} = \frac{2}{0.5s+1} = \frac{k}{Ts+1} \quad (8.26)$$

Hence, $k = 2$ and $T = 0.5$. Finally, according to (27.9) $y(t)$ becomes

$$\underline{\underline{y(t)}} = \frac{ke^{-t/T}}{T} = \frac{2e^{-t/0.5}}{0.5} = \underline{\underline{4e^{-2t}}} \quad (8.27)$$

[End of Example 8.2]

Example 8.3 Calculation of steady-state value using the Final Value Theorem

See Example 8.2. The steady-state value of y in (8.27) is

$$y_s = \lim_{t \rightarrow \infty} y(t) = \lim_{t \rightarrow \infty} 4e^{-2t} = 0 \quad (8.28)$$

Using the Final Value Theorem,

$$y_s = \lim_{s \rightarrow 0} sY(s) = \lim_{s \rightarrow 0} s \frac{4}{s+2} = 0 \quad (8.29)$$

So, the results are the same.

[End of Example 8.3]

8.5 *Problems for Chapter 8*

Problem 8.1 *Calculating Laplace transform from definition*

Calculate the Laplace transform, $F(s)$, of the following time function:

$$f(t) = e^{-t} \quad (8.30)$$

using the definition of the Laplace transform.

Can you find the same answer ($F(s)$) by using a proper Laplace transform pair?

Problem 8.2 *Calculating time response using the Laplace transform*

Given the following differential equation:

$$y'(t) = -2y(t) + u(t) \quad (8.31)$$

with initial value $y(0) = 4$. Assume that the input variable $u(t)$ is a step of amplitude 1 at time $t = 0$.

1. Calculate the time response in the output variable, $y(t)$, using the Laplace transform.
2. Calculate the steady-state value of $y(t)$ using the Final Value Theorem. Also calculate the steady-state value, y_s , from $y(t)$, and from (8.31) directly. Are all these (three) values of y_s the same?

8.6 *Solutions to problems for Chapter 8*

Solution to Problem 8.1

We set $f(t) = e^{-t}$ in the integral that defines the Laplace transform:

$$\begin{aligned}\underline{\underline{\mathcal{L}\{e^{-t}\}}} &= \int_0^\infty e^{-st} e^{-t} dt \\ &= \int_0^\infty e^{-(s+1)t} dt \\ &= \frac{1}{-(s+1)} \left[e^{-(s+1)t} \right]_{t=0}^{t=\infty} \\ &= \frac{1}{-(s+1)} [0 - 1] \\ &= \frac{1}{s+1}\end{aligned}$$

The proper Laplace transform pair is:

$$\frac{k}{Ts+1} \iff \frac{ke^{-t/T}}{T} = e^{-t} \quad (8.32)$$

Here, $T = 1$ and $k = 1$. Thus, $F(s)$ becomes

$$\underline{\underline{F(s) = \frac{1}{s+1} = \mathcal{L}\{e^{-t}\}}} \quad (8.33)$$

which is the same as found above using the definition of the Laplace transform.

Solution to Problem 8.2

- To calculate $y(t)$ we start by taking the Laplace transform of both sides of the given differential equation:

$$\mathcal{L}\{y'(t)\} = \mathcal{L}\{-2y(t) + u(t)\} \quad (8.34)$$

Here, we apply the time derivative property, cf. (8.18), at the left side, and the linear combination property, cf. (8.14), to the right side, to get

$$sY(s) - 4 = -2Y(s) + U(s) \quad (8.35)$$

Here,

$$U(s) = \frac{1}{s} \quad (8.36)$$

since the Laplace transform of a step of amplitude 1 is $\frac{1}{s}$, cf. the transform pair (8.7).

By now we have

$$sY(s) - 4 = -2Y(s) + \frac{1}{s} \quad (8.37)$$

Solving for $Y(s)$ gives

$$Y(s) = \underbrace{\frac{4}{s+2}}_{Y_1(s)} + \underbrace{\frac{1}{(s+2)s}}_{Y_2(s)} \quad (8.38)$$

To get the corresponding $y(t)$ from this $Y(s)$ we take the inverse Laplace transform of $Y_1(s)$ and $Y_2(s)$ to get $y_1(t)$ and $y_2(t)$ respectively, and then we calculate $y(t)$ as

$$y(t) = y_1(t) + y_2(t) \quad (8.39)$$

according to the linearity property of the Laplace transform. $y_1(t)$ and $y_2(t)$ are calculated below.

Calculation of $y_1(t)$:

We can use the transform pair (8.10), which is repeated here:

$$\frac{k}{Ts+1} \iff \frac{ke^{-t/T}}{T} \quad (8.40)$$

We have

$$Y_1(s) = \frac{4}{s+2} = \frac{2}{0.5s+1} \quad (8.41)$$

Hence, $k = 2$, and $T = 0.5$. Therefore,

$$y_1(t) = \frac{ke^{-t/T}}{T} = \frac{2e^{-t/0.5}}{0.5} = 4e^{-2t} \quad (8.42)$$

Calculation of $y_2(t)$:

We can use the transform pair (8.11), which is repeated here:

$$\frac{k}{(Ts+1)s} \iff k \left(1 - e^{-t/T}\right) \quad (8.43)$$

We have

$$Y_2(s) = \frac{1}{(s+2)s} = \frac{0.5}{(0.5s+1)s} \quad (8.44)$$

Hence, $k = 0.5$, and $T = 0.5$. Therefore,

$$y_2(t) = k \left(1 - e^{-t/T}\right) = 0.5 \left(1 - e^{-t/0.5}\right) = 0.5 \left(1 - e^{-2t}\right) \quad (8.45)$$

The final result becomes

$$\underline{\underline{y(t)}} = y_1(t) + y_2(t) \quad (8.46)$$

$$= 4e^{-2t} + 0.5 \left(1 - e^{-2t}\right) \quad (8.47)$$

$$= \underline{\underline{0.5 + 3.5e^{-2t}}} \quad (8.48)$$

2. Using the Final Value Theorem on (8.38):

$$\underline{\underline{y_s}} = \lim_{s \rightarrow 0} sY(s) = \lim_{s \rightarrow 0} s \left[\frac{4}{s+2} + \frac{1}{(s+2)s} \right] \quad (8.49)$$

$$= \lim_{s \rightarrow 0} s \frac{4}{s+2} + \lim_{s \rightarrow 0} s \frac{1}{(s+2)s} = 0 + \frac{1}{2} = \underline{\underline{0.5}} \quad (8.50)$$

From (8.48) we get

$$y_s = \lim_{t \rightarrow \infty} y(t) = 0.5 \quad (8.51)$$

And from the differential equation we get (because the time-derivative is zero in steady-state)

$$0 = -2y_s(t) + u_s(t) \quad (8.52)$$

which gives

$$y_s = \frac{u_s}{2} = \frac{1}{2} = 0.5 \quad (8.53)$$

So, the three results are the same.

Chapter 9

Transfer functions

9.1 Introduction

Transfer functions is a model form based on the Laplace transform (The Laplace transform is presented in Ch. 8.) Transfer functions are very useful in analysis and design of linear dynamic systems, in particular controller functions and signal filters. The main reasons why transfer functions are useful are as follows:

- **Compact model form:** If the original model is a higher order differential equation, or a set of first order differential equations, the relation between the input variable and the output variable can be described by one transfer function, which is a rational function of the Laplace variable s , without any time-derivatives.
- **Representation of standard models:** Transfer functions are often used to represent standard models of controllers and signal filters.
- **Simple to combine systems:** For example, the transfer function for a combined system which consists of two systems in a series combination, is just the product of the transfer functions of each system.
- **Simple to calculate time responses:** The calculations will be made using the Laplace transform, and the necessary mathematical operations are usually much simpler than solving differential equations. Calculation of time-responses for transfer function models is described in Chapter 9.5.
- **Simple to find the frequency response:** The frequency response is a function which expresses how sinusoid signals are transferred through a dynamic system. Frequency response is an important tool in analysis and design of signal filters and control systems. The frequency response can be found from the transfer function of the system. However, frequency response theory is not a part of this book (a reference is ?).

Before we start, I must say something about the mathematical notation: In the following sections, and in the remainder of the book, I use the *same symbol* (letter) for the time

function, say $y(t)$, as for the Laplace transform of $y(t)$, here $y(s)$ – although it is mathematically incorrect to do it. The reason is to simplify the presentation. Now, only one variable name (symbol) is needed for both the Laplace domain and the time domain. For example, assume that $y(t)$ is the time function of the level y in a water tank. Then I write $y(s)$, although I formally should have written $Y(s)$ or $y^*(s)$ or $\bar{y}(s)$ (or something else that is different from $y(s)$) for $\mathcal{L}\{y(t)\}$. It is my experience (from many years together with transfer functions) that this simplifying notation causes no problems.

9.2 Definition of the transfer function

The first step in deriving the transfer function of a system is taking the Laplace transform of the differential equation (which must be linear). Let us go on with an example, but the results are general. Given the following mathematical model having two input variables u_1 and u_2 and one output variable y . (I think you will understand from this example how to find the transfer function for systems with different number of inputs and outputs.)

$$y'(t) = ay(t) + b_1u_1(t) + b_2u_2(t) \quad (9.1)$$

a , b_1 and b_2 are model parameters (coefficients). Let the initial state (at time $t = 0$) be y_0 . We start by taking the Laplace transform of both sides of the differential equation:

$$\mathcal{L}\{y'(t)\} = \mathcal{L}\{ay(t) + b_1u_1(t) + b_2u_2(t)\} \quad (9.2)$$

By using the linearity property of the Laplace transform, cf. (8.13), the right side of (9.2) can be written as

$$\mathcal{L}\{ay(t)\} + \mathcal{L}\{b_1u_1(t)\} + \mathcal{L}\{b_2u_2(t)\} \quad (9.3)$$

$$= a\mathcal{L}\{y(t)\} + b_1\mathcal{L}\{u_1(t)\} + b_2\mathcal{L}\{u_2(t)\} \quad (9.4)$$

$$= ay(s) + b_1u_1(s) + b_2u_2(s) \quad (9.5)$$

The left side of (9.2) will be Laplace transformed using the differentiation rule, cf. (8.16), on $\mathcal{L}\{\dot{y}(t)\}$:

$$\mathcal{L}\{\dot{y}(t)\} = sy(s) - y_0 \quad (9.6)$$

Thus, we have found that the Laplace transformed (9.2) is

$$sy(s) - y_0 = ay(s) + b_1u_1(s) + b_2u_2(s) \quad (9.7)$$

Solving for the output variable $y(s)$ gives

$$y(s) = \underbrace{\frac{1}{s-a}y_0}_{H_1(s)} + \underbrace{\frac{b_1}{s-a}u_1(s)}_{H_2(s)} + \underbrace{\frac{b_2}{s-a}u_2(s)}_{H_3(s)} \quad (9.8)$$

In (9.8),

- y_1 is the contribution from input u_1 to the total response y ,

- y_2 is the contribution from input u_2 to the total response y ,
- y_{init} is the contribution from the initial state y_0 to the total response y .

Of course, these contributions to the total response are in the Laplace domain. The corresponding responses in the time domain are found by calculating the inverse Laplace transforms.

Now we have the following two *transfer functions* for our system:

- The transfer function from u_1 to y is

$$H_1(s) = \frac{b_1}{s - a} \quad (9.9)$$

- The transfer function from u_2 to y is

$$H_2(s) = \frac{b_2}{s - a} \quad (9.10)$$

Thus, *the transfer function from a given input variable to a given output variable is the s-valued function with which the Laplace transformed input variable is multiplied to get its contribution to the response in the output variable*. In other words: The transfer function expresses how the input variable is transferred through the system.

The transfer functions derived above can be illustrated with the block diagram shown in Figure 9.1

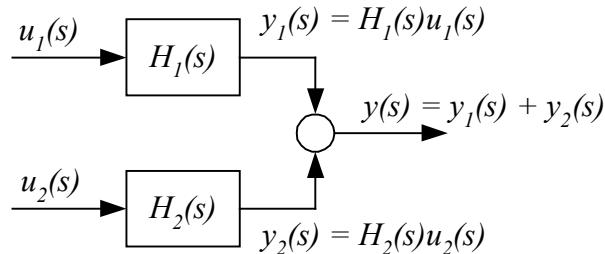


Figure 9.1: Block diagram representing the transfer functions $H_1(s)$ and $H_2(s)$ in (9.8).

One alternative way to express the definition of transfer function

From (9.8) we have

$$H_1(s) = \frac{b_1}{s - a} = \frac{y_1(s)}{u_1(s)} \quad (9.11)$$

So, we can define the transfer functions as the *ratio* between the Laplace transformed contribution to the total response in the output variable, here $y_1(s)$, and the Laplace transformed input variable, here $u_1(s)$. We may also say that the transfer functions is the ratio between the Laplace transformed total response in the output variable, here $y(s)$, and the Laplace transformed input variable, here $u_1(s)$, when all other inputs are set to zero and the initial state is zero.

9.3 Characteristics of transfer functions

A transfer function can be written on a factorized form – often called a *zero-pole form*:

$$H(s) = K \frac{(s - z_1)(s - z_2) \cdots (s - z_r)}{(s - p_1)(s - p_2) \cdots (s - p_n)} = \frac{b(s)}{a(s)} \quad (9.12)$$

Here z_1, \dots, z_r are the *zeros* and p_1, \dots, p_n are the *poles* of the transfer function. For example, the transfer function

$$H(s) = \frac{4s - 4}{s^2 + 5s + 6} = 4 \frac{s - 1}{(s + 3)(s + 2)} \quad (9.13)$$

have two poles, -3 and -2 , one zero, 1 , and the gain is 4 . (As shown in Ch. 21, the values of the poles determines the stability property of a system. The system is stable only if all the poles have negative real parts, in other words if all the poles lie in the left half part of the complex plane.)

The s -polynomial in the denominator of $H(s)$, which is $a(s)$ in (9.12), is denoted the *characteristic polynomial*. The poles are the roots of the characteristic polynomial, that is

$$a(s) = 0 \text{ for } s = s_1, s_2, \dots, s_n \text{ (the poles)} \quad (9.14)$$

The *order* of a transfer function is the order of the characteristic polynomial. For example, the transfer function (9.13) has order 2.

9.4 Combining transfer functions blocks in block diagrams

Transfer function blocks may be combined in a block diagram according to the rules shown in Figure 9.2.

One possible purpose of such a combination is to simplify the block diagram, or to calculate the resulting or overall transfer function. For example, the combined transfer function of two transfer functions connected in series is equal to the product of the individual transfer functions, jc. the Series connection rule in Figure 9.2.

9.5 How to calculate responses from transfer function models

It is quite easy to calculate responses in transfer function models manually (by hand). Assume given the following transfer function model:

$$y(s) = H(s)u(s) \quad (9.15)$$

To calculate the time-response $y(t)$ for a given input signal $u(t)$, we can do as follows:

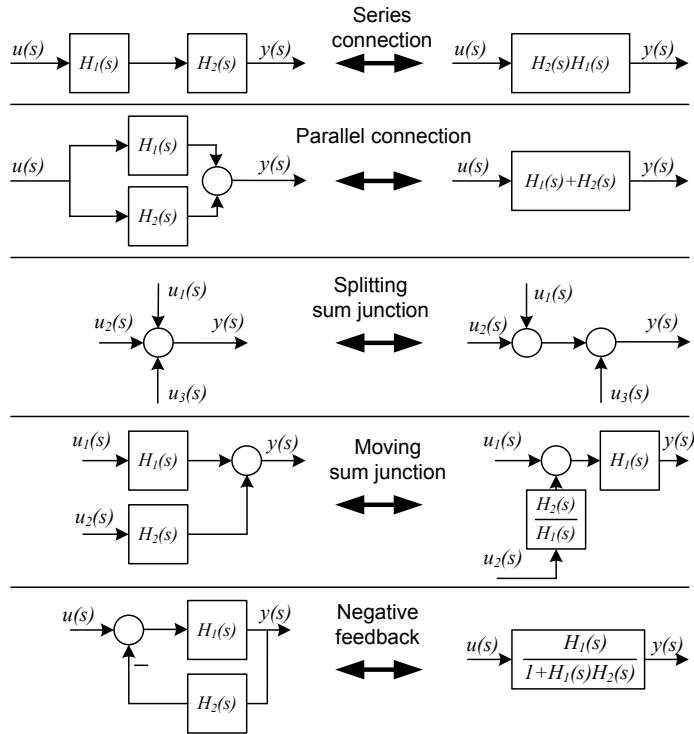


Figure 9.2: Rules for combining transfer function blocks.

- First, find $u(s)$ – the Laplace transform of the input signal. $u(s)$ can be found from precalculated Laplace transform pairs, cf. Section 8.3, possibly combined with one or more of the Laplace transform properties, cf. Section 8.4, where particularly the linearity property (8.13) is useful.
- The Laplace transform of the output signal, $y(s)$, is calculated from (9.15), that is,

$$y(s) = H(s)u(s) \quad (9.16)$$

where $u(s)$ is found as explained above.

Example 9.1 Calculation of time-response for transfer function model

Given the transfer function model

$$y(s) = \underbrace{\frac{3}{s+0.5}}_{H(s)} u(s) \quad (9.17)$$

Suppose that $u(t)$ is a step from 0 to 2 at $t = 0$. We shall find an expression for the time-response $y(t)$. The Laplace transform of $u(t)$ is, cf. (8.7),

$$u(s) = \frac{2}{s} \quad (9.18)$$

Inserting this into (9.17) gives

$$y(s) = \frac{3}{s+0.5} \cdot \frac{2}{s} = \frac{6}{(s+0.5)s} = \frac{12}{(2s+1)s} \quad (9.19)$$

(9.19) has the same form as the Laplace transform pair (8.11) which is repeated here:

$$\frac{k}{(Ts+1)s} \iff k [1 - e^{-t/T}] \quad (9.20)$$

Here $k = 12$ and $T = 2$. The time-response becomes

$$y(t) = 12 [1 - e^{-t/2}] \quad (9.21)$$

Figure 9.3 shows $y(t)$. The steady-state response is 12, which can be calculated from $y(t)$ by setting $t = \infty$.

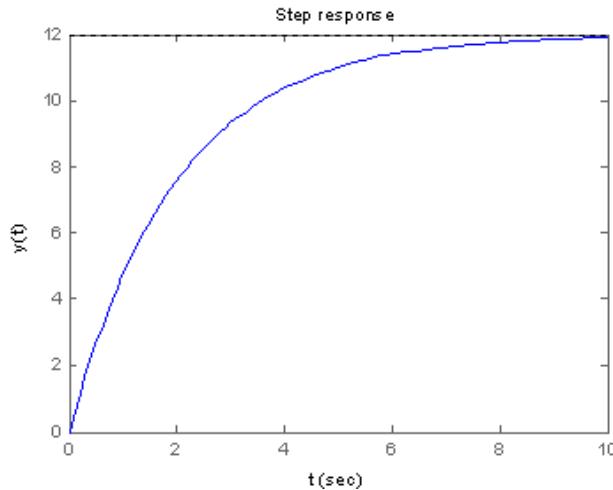


Figure 9.3: Example 9.1: The time-response $y(t)$ given by (9.70)

[End of Example 9.1]

9.6 Static transfer function and static response

Suppose that the input signal to a system is a step of amplitude u_s . The corresponding static time-response can be found from the Final Value Theorem:

$$y_s = \lim_{s \rightarrow 0} s \cdot y(s) = \lim_{s \rightarrow 0} s \cdot H(s) \frac{u_s}{s} = \underbrace{\lim_{s \rightarrow 0} H(s)}_{H_s} u_s \quad (9.22)$$

where H_s is the *static transfer function*. That is,

$$H_s = \lim_{s \rightarrow 0} H(s) \quad (9.23)$$

Thus, the static transfer function, H_s , is found by letting s approach zero in the transfer function, $H(s)$.

Once we know the static transfer function H_s the static (steady-state) response y_s due to a constant input of value u_s , is

$$y_s = H_s u_s \quad (9.24)$$

Example 9.2 *Static transfer function and static response*

See Example 9.1. The transfer function is

$$H(s) = \frac{3}{s + 0.5} \quad (9.25)$$

The corresponding static transfer function becomes

$$H_s = \lim_{s \rightarrow 0} H(s) = \lim_{s \rightarrow 0} \frac{3}{s + 0.5} = 6 \quad (9.26)$$

Assume that the input u has the constant value of $u_s = 2$. What is the corresponding static response y_s in the output? It can be calculated from the static transfer function as

$$y_s = H_s u_s = 6 \cdot 2 = 12 \quad (9.27)$$

which is the same result as found in Example 9.1.

[End of Example 9.2]

9.7 From transfer function to differential equation

Assume you have a transfer function from input u to output y :

$$H(s) = \frac{y(s)}{u(s)}$$

You can find an equivalent differential equation relating y and u as demonstrated in Example 9.3.

Example 9.3 *Converting a transfer function to an equivalent differential equation*

Given the transfer function

$$H(s) = \frac{y(s)}{u(s)} = \frac{2s}{5s + 1} \quad (9.28)$$

Here is how to find an equivalent differential equation:

- Cross-multiply to get:

$$(5s + 1)y(s) = 2su(s)$$

2. Dissolve the parenthesis to get:

$$5sy(s) + y(s) = 2su(s)$$

3. Apply the pertinent Laplace transform properties, cf. Section 8.4, to get:

$$5y' + y = 2u'$$

which is the differential equation.

[End of Example 9.3]

9.8 From transfer function to state space model

Some times we want to find a standard linear state space model which is equivalent to a given transfer function. We assume that the transfer function is:

$$H(s) = \frac{y(s)}{u(s)} = \frac{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} \quad (9.29)$$

It can be shown, e.g. using the block diagram manipulation rules presented in Figure 9.2, that the block diagram shown in Figure 9.4 has the transfer function (9.29) from u to y . Then, from the block diagram, you can write down a state space model. Hence, using the

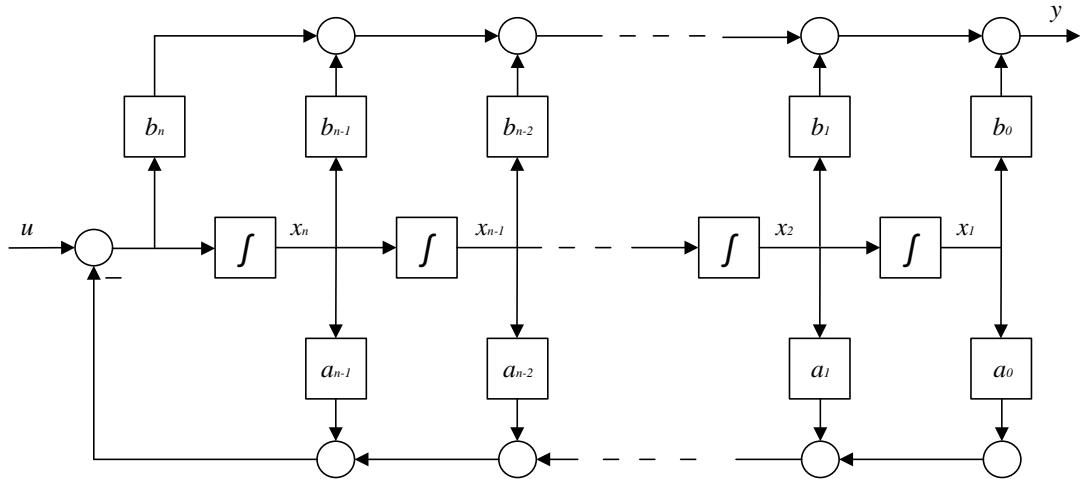


Figure 9.4: Canonical block diagram

block diagram, you can transform a given transfer function into an equivalent state space model.

The block diagram in Figure (9.4) is just one of an infinitely number of possible block diagram with (9.29) as transfer function. This block diagram has a special form denoted controller canonical form. (Canonical means “according to the rules”.).

Example 9.4 From transfer function to state space model

Given this transfer function:

$$H(s) = \frac{y(s)}{u(s)} = \frac{4s + 5}{s^2 + 2s + 3} = \frac{0s^2 + 4s + 5}{s^2 + 2s + 3} \quad (9.30)$$

Figure 9.5 shows a corresponding block diagram.

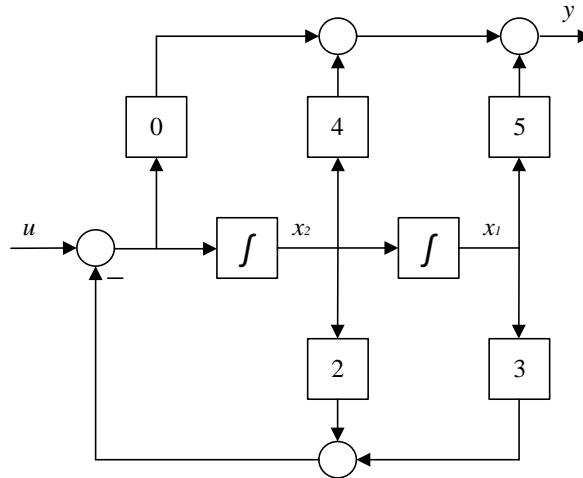


Figure 9.5: Block diagram corresponding to the transfer function (9.30).

From the block diagram, we find the following state space model:

$$x'_1 = x_2 \quad (9.31)$$

$$x'_2 = -3x_1 - 2x_2 + u \quad (9.32)$$

$$y = 5x_1 + 4x_2 \quad (9.33)$$

[End of Example 9.4]

9.9 From state space model to transfer function

Given a state space model:

$$x' = Ax + Bu \quad (9.34)$$

$$y = Cx + Du \quad (9.35)$$

We can derive the transfer function from u to y as follows: Take the Laplace transform of (9.34) – (9.35) to get (I is the identity matrix of equal dimension as of A)

$$sIx(s) - x_0 = Ax(s) + Bu(s) \quad (9.36)$$

$$y(s) = Cx(s) + Du(s) \quad (9.37)$$

We neglect the initial state x_0 , as we always can do when deriving transfer functions from differential equation models. Solving (9.36) for $x(s)$ gives

$$x(s) = (sI - A)^{-1}Bu(s) \quad (9.38)$$

Inserting this $x(s)$ into (9.37) gives

$$y(s) = [C(sI - A)^{-1}B + D] u(s) \quad (9.39)$$

from which we get the following transfer function from u to y :^{9.9}

$$H(s) = \frac{y(s)}{u(s)} = C(sI - A)^{-1}B + D \quad (9.40)$$

$$= C \frac{\text{adj}(sI - A)}{\det(sI - A)} B + D \quad (9.41)$$

$$= \frac{1}{\det(sI - A)} \cdot C \cdot \text{adj}(sI - A) \cdot B + D \quad (9.42)$$

One interesting observation: The *poles* of $H(s)$ are the roots of the denominator:

$$a(s) = \det(sI - A) = 0 \quad (9.43)$$

which is the characteristic equation. (9.43) defines the *eigenvalues* of A .¹ So, the poles of the transfer function are the same as the eigenvalues of A .

Example 9.5 Calculating the transfer function of a state space model

Given the following state space model:

$$\underbrace{\begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix}}_{x'} = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_B u \quad (9.44)$$

$$y = \underbrace{\begin{bmatrix} 3 & 1 \end{bmatrix}}_C x + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_D u \quad (9.45)$$

The transfer function from u to y using (9.42) is:

$$\begin{aligned} H(s) &= \frac{y(s)}{u(s)} = C(sI - A)^{-1}B + D \\ &= \frac{1}{\det(sI - A)} \cdot C \cdot \text{adj}(sI - A) \cdot B + D \\ &= \frac{1}{s^2 + 3s + 2} \cdot \begin{bmatrix} 3 & 1 \end{bmatrix} \cdot \text{adj}\left(sI - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\right) \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} + [0] \\ &= \frac{1}{s^2 + 3s + 2} \cdot \begin{bmatrix} 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} + [0] \\ &= \frac{s+3}{s^2 + 3s + 2} \end{aligned}$$

[End of Example 9.5]

¹In mathematics litterature it is more common to use the symbol λ instead of s for eigenvalues.

9.10 Problems for Chapter 9

Problem 9.1 From second order differential equation to transfer function

Given the following differential equation model:

$$y'' = -a_1 y' - a_0 y + b_0 u + c_0 d \quad (9.46)$$

where y is the output variable, u is the input variable, d is the interference variable, and a_1 , a_0 , b_0 and c_0 are parameters. Find the transfer function from u to y . What is the order of the transfer function?

Problem 9.2 Transfer function of a wood-chip tank

In Problem 4.2 the mathematical model of a wood-chip tank was derived. The model is

$$\rho A \dot{h}(t) = K_s u(t - \tau) - w_{\text{out}}(t) \quad (9.47)$$

Calculate the transfer function $H_1(s)$ from the screw control signal u to the level h and the transfer function $H_2(s)$ from the outflow w_{out} to the level h .

Problem 9.3 Transfer function of a mass-spring-damper-system

Figure 9.6 shows a mass-spring-damper-system.

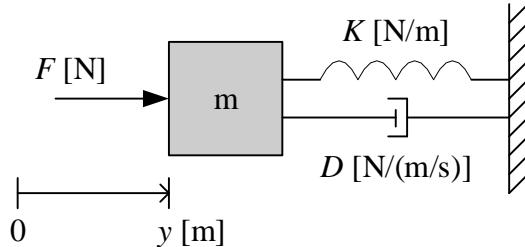


Figure 9.6: Mass-spring-damper-system

y is position. F is applied force. D is damping constant. K is spring constant. It is assumed that the damping force F_d is proportional to the speed, and that the spring force F_s is proportional to the position of the mass. The spring force is assumed to be zero when y is zero. Force balance (Newtons 2. Law) yields

$$my'' = F - Dy' - Ky \quad (9.48)$$

Calculate the transfer function from force F to position y .

Problem 9.4 *Characteristics of transfer function*

Given the following transfer function:

$$H(s) = \frac{s+3}{s^2 + 3s + 2} \quad (9.49)$$

1. What is the order?
2. What is the characteristic equation?
3. What is the characteristic polynomial?
4. What are the poles and the zeros?

Problem 9.5 *Transfer function block diagrams*

Given a thermal process with transfer function $H_p(s)$ from supplied power P to temperature T as follows:

$$T(s) = \underbrace{\frac{b_p}{s + a_p}}_{H_p(s)} P(s) \quad (9.50)$$

The transfer function from temperature T to temperature measurement T_m is as follows:

$$T_m(s) = \underbrace{\frac{b_m}{s + a_m}}_{H_m(s)} T(s) \quad (9.51)$$

a_p , b_p , a_m , and b_m are parameters.

1. Draw a transfer function block diagram of the system (process with sensor) with P as input variable and T_m as output variable.
2. What is the transfer function from P to T_m ? (Derive it from the block diagram.)

Problem 9.6 *Calculation of step response using transfer function*

Given the transfer function model

$$y(s) = \underbrace{\frac{5}{s}}_{H(s)} u(s) \quad (9.52)$$

Suppose that the input u is a step from 0 to 3 at $t = 0$. Calculate the response $y(t)$ due to this input.

Problem 9.7 *Static transfer function*

See Problem 9.3. It can be shown that the transfer function from force F to position y is

$$H(s) = \frac{y(s)}{F(s)} = \frac{1}{ms^2 + Ds + K} \quad (9.53)$$

Calculate the static transfer function H_s . From H_s calculate the static response y_s corresponding to a constant force, F_s .

Problem 9.8 *From transfer function to state space model*

The transfer function of a first order high-pass filter can be written as:

$$H(s) = \frac{y(s)}{u(s)} = \frac{\frac{s}{\omega_c}}{\frac{s}{\omega_c} + 1} = \frac{s}{s + \omega_c} \quad (9.54)$$

where ω_c [rad/s] is the crossover or corner frequency.

Find an equivalent state space model from a canonical block diagram corresponding to (9.54).

9.11 Solutions to problems for Chapter 9

Solution to Problem 9.1

Taking the Laplace transformation of the differential equation gives (here, any non-zero initial values of y and y' are neglected):

$$s^2y(s) = -a_1sy(s) - a_0y(s) + b_0u(s) + c_0d(s) \quad (9.55)$$

Solving for $y(s)$ gives:

$$y(s) = \frac{b_0}{s^2 + a_1s + a_0}u(s) + \frac{c_0}{s^2 + a_1s + a_0}d(s) \quad (9.56)$$

Hence, the transfer function from u to y is:

$$\frac{y(s)}{u(s)} = \frac{b_0}{s^2 + a_1s + a_0} \quad (9.57)$$

Solution to Problem 9.2

The Laplace transform of (9.47) is

$$\rho A [sh(s) - h_0] = K_s e^{-\tau s} u(s) - w_{out}(s) \quad (9.58)$$

Solving for output variable h gives

$$h(s) = \underbrace{\frac{1}{s}h_0}_{H_1(s)} + \underbrace{\frac{K_s}{\rho As}e^{-\tau s}u(s)}_{H_2(s)} + \underbrace{\left(-\frac{1}{\rho As}\right)w_{out}(s)}_{H_2(s)} \quad (9.59)$$

Thus, the transfer functions are

$$H_1(s) = \frac{K_s}{\rho As}e^{-\tau s} \quad (9.60)$$

and

$$H_2(s) = -\frac{1}{\rho As} \quad (9.61)$$

Solution to Problem 9.3

Laplace transform of (21.19) gives

$$m [s^2y(s) - sy'_0 - y_0] = F(s) - D [sy(s) - y_0] - Ky(s) \quad (9.62)$$

Setting initial values $y_0 = 0$ and $y'_0 = 0$, and then solving for $y(s)$ gives

$$y(s) = \underbrace{\frac{1}{ms^2 + Ds + K}}_{H(s)}F(s) \quad (9.63)$$

The transfer function is

$$H(s) = \frac{y(s)}{F(s)} = \frac{1}{ms^2 + Ds + K} \quad (9.64)$$

Solution to Problem 9.4

1. Order: 2.
2. $s^2 + 3s + 2 = 0$
3. $s^2 + 3s + 2$
4. We write the transfer function on pole-zero-form:

$$H(s) = \frac{s+3}{s^2+3s+2} = \frac{s+3}{(s+1)(s+2)} \quad (9.65)$$

We see that the poles are -1 and -2, and the zero is -3.

Solution to Problem 9.5

1. Figure 9.7 shows the block diagram.
2. According to the *series combination rule* the transfer function becomes

$$\underline{H(s) = \frac{T_m(s)}{P(s)} = H_m(s)H_p(s) = \frac{b_m}{s+a_m} \frac{b_p}{s+a_p}} \quad (9.66)$$

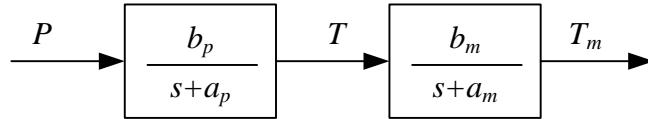


Figure 9.7: Block diagram of transfer functions in series.

Solution to Problem 9.6

The Laplace transform of $u(t)$ is (cf. the Laplace transform pair (8.7)):

$$u(s) = \frac{3}{s} \quad (9.67)$$

Inserting this into (9.52) gives

$$y(s) = \frac{5}{s} \cdot \frac{3}{s} = \frac{15}{s^2} \quad (9.68)$$

which has the same form as in the Laplace transform pair (8.8), which is repeated here:

$$\frac{k}{s^2} \iff kt \quad (9.69)$$

We have $k = 15$, so the response is

$$\underline{\underline{y(t) = 15t}} \quad (9.70)$$

Solution to Problem 9.7

Setting $s = 0$ in the transfer function gives

$$\underline{\underline{H_s}} = H(0) = \frac{1}{\underline{\underline{K}}} \quad (9.71)$$

The static response y_s corresponding to a constant force, F_s , is

$$\underline{\underline{y_s}} = H_s F_s = \frac{F_s}{\underline{\underline{K}}} \quad (9.72)$$

Solution to Problem 9.8

Figure 9.8 shows a canonical block diagram corresponding to (9.54). From the block

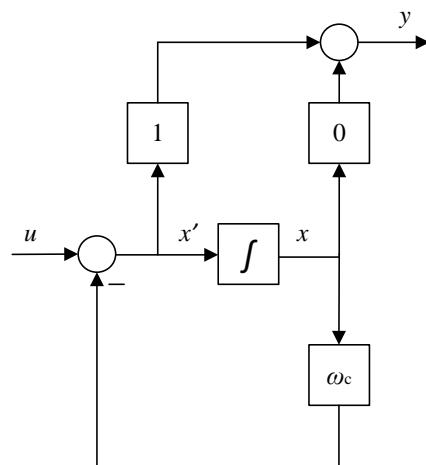


Figure 9.8: A canonical block diagram corresponding to (9.54).

diagram, we find the following state space model:

$$x' = -\omega_c x + u = Ax + Bu \quad (9.73)$$

$$y = -\omega_c x + u = Cx + Du \quad (9.74)$$

Chapter 10

Process dynamics

10.1 Introduction

In this chapter a number of standard dynamic models in the form of transfer functions will be defined. With such standard models you can characterize the dynamic properties of a physical system in terms of for example gain, time constant, and time delay. These terms are also useful for controller tuning, as in the Skogestad tuning method which is described in Section 16.8.

10.2 Integrators

An integrator is a system where *the output variable y is the time integral of the input variable u* , multiplied with some gain K :

$$y(t) = K \int_0^t u(\theta) d\theta \quad (10.1)$$

Taking the time derivative of both sides of (10.1) yields the following differential equation describing an integrator:

$$y'(t) = Ku(t) \quad (10.2)$$

Taking the Laplace transform using (8.19) gives

$$sy(s) = Ku(s) \quad (10.3)$$

which gives the following *transfer function* of an integrator:

$$H(s) = \frac{y(s)}{u(s)} = \frac{K}{s} \quad (10.4)$$

Let us now find the *step response* of the integrator. We assume that $u(t)$ is a step of amplitude U at $t = 0$. From (8.7) $u(s) = \frac{U}{s}$. Thus,

$$y(s) = H(s)u(s) = \frac{K}{s} \cdot \frac{U}{s} = \frac{KU}{s^2} \quad (10.5)$$

which, inverse Laplace transformed using (8.8), is

$$y(t) = KU t \quad (10.6)$$

Thus, the step response of an integrator is *a ramp* with rate KU . Figure 10.1 shows simulated response of an integrator.

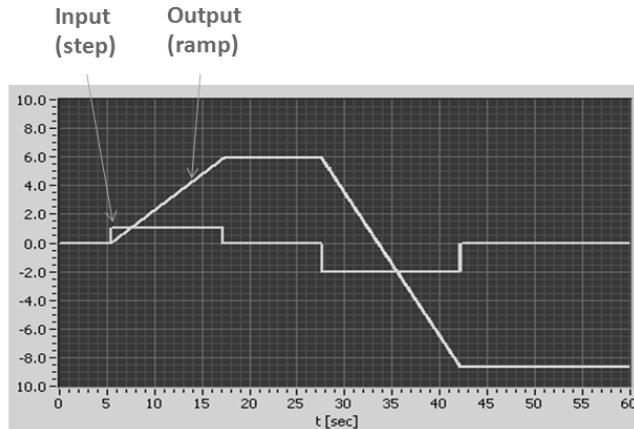


Figure 10.1: Simulated response of an integrator

Example 10.1 An integrator: A liquid tank

See Example 4.1 on page 77 which describes a liquid tank. Assume for simplicity that there is no outflow from the tank. The mathematical model of this system is then

$$h'(t) = \frac{1}{A} q_i(t) \quad (10.7)$$

Taking the Laplace transform of (10.7) gives

$$sh(s) - h_0 = \frac{1}{A} q_i(s) \quad (10.8)$$

which gives

$$h(s) = \frac{h_0}{s} + \underbrace{\frac{1}{As} q_i(s)}_{H(s)} \quad (10.9)$$

So, the transfer function is

$$H(s) = \frac{h(s)}{q_i(s)} = \frac{1}{A} \cdot \frac{1}{s} \quad (10.10)$$

The system is an integrator!

It is actually quite naturally that the liquid tank is an integrator, since the level is proportional to the integral of the inflow. This can be seen by integrating (10.7), which gives

$$h(t) = h(0) + \int_0^t \frac{1}{A} q_i(\theta) d\theta \quad (10.11)$$

Whenever you need a concrete example of an integrator, recall the tank!

[End of Example 10.1]

10.3 Time-constant systems

Many mathematical models, for material systems, thermal systems, mechanical systems and electrical systems, are first order differential equations, which can be written on the following general form:

$$y' = ay + bu \quad (10.12)$$

where y is the output variable, and u is the input variable. a and b are parameters. It has become a tradition within the fields of process technology and control to write (10.12) on a different form when the purpose is to analyse the dynamic properties of (10.12). That form is

$$Ty' = -y + Ku \quad (10.13)$$

where K is the *gain* and T is the *time constant*. The relation between the parameters of (10.12) and (10.13) is: $K = -b/a$ and $T = -1/a$. Parameters K and T give useful information about the dynamic properties of first order differential equations, as you will see soon.

Often the model is presented as a transfer function. Taking the Laplace transform of both sides of (10.13) gives

$$Tsy(s) = -y(s) + Ku(s)$$

Solving for the output variable gives

$$y(s) = \underbrace{\frac{K}{Ts+1}}_{H(s)} u(s) \quad (10.14)$$

where $H(s)$ is the transfer function corresponding to (10.13).

Here is an example showing how to read off K and T from the transfer function:

$$H(s) = \frac{y(s)}{u(s)} = \frac{3}{4s+2} = \frac{1.5}{2s+1} \quad (10.15)$$

The gain is $K = 1.5$, and the time constant is $T = 2$ (in a proper time unit, e.g. seconds).

Can you write the differential equation which corresponds to (10.15)? Cross-multiplying (10.15) by $(2s+1)$ gives

$$(2s+1)y(s) = 1.5u(s)$$

Taking the inverse Laplace transform gives the corresponding differential equation:

$$2y' + y = 1.5u$$

Analysis of dynamics from the step response

To analyse the dynamic properties of (10.13), or (10.14), it is common to study the *step response* of the system. We assume that the input signal $u(t)$ is changed as a step of amplitude U at time $t = 0$. From (8.7) $u(s) = \frac{U}{s}$. The Laplace transformed response becomes

$$y(s) = H(s)u(s) = \frac{K}{Ts+1} \cdot \frac{U}{s} \quad (10.16)$$

Taking the inverse Laplace transform using (8.11) gives

$$y(t) = KU(1 - e^{-\frac{t}{T}}) \quad (10.17)$$

Let us study the importance of the parameters K and T for the step response. It is assumed that the system is in an operating point where the input is u_0 and the output is y_0 before the step in the input u . Figure 10.2 shows a simulation of a first order system. The parameters used in the simulation are $U = 1$, $K = 2$, and $T = 5$.

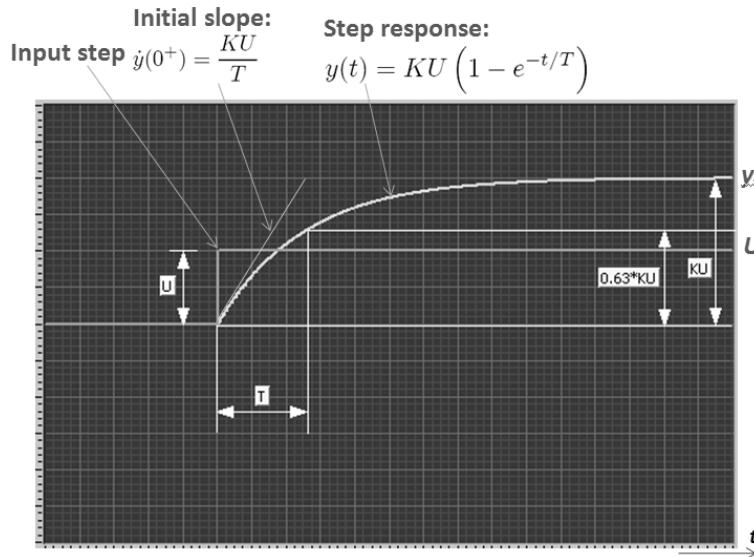


Figure 10.2: Step response of a time constant system

- **Importance of K :** The steady-state response due to the input step is

$$y_s = \lim_{t \rightarrow \infty} y(t) = KU \quad (10.18)$$

which is found from (10.17) with $t \rightarrow \infty$. Thus, the step is amplified with the gain K in steady-state. This is confirmed in Figure 10.2.

In Section 9.6 the static transfer function H_s was defined. What is the H_s of a time constant system? We get

$$H_s = \frac{y_s}{u_s} = \frac{KU}{U} = K \quad (10.19)$$

So, the H_s is equal to the gain parameter, K .

- **Importance of T :** Let us set $t = T$ in (10.17):

$$y(T) = KU(1 - e^{-\frac{T}{T}}) \quad (10.20)$$

$$= KU(1 - e^{-1}) \quad (10.21)$$

$$= 0.63 \cdot KU \quad (10.22)$$

Thus, at time $t = T$ the step response has increased to 63% of the total increase which is KU . This is confirmed in Figure 10.2. This suggests a practical way to read off the time constant from a step response curve.

Qualitatively, we can state the importance of the time constant as follows: *The less T , the faster the system.*

Does the steady-state response depend on the time constant? No, because the steady-state response is equal to $y_s = KU$ which is not dependent of T .

Gains and time constant for systems with several inputs

Above we assumed that the system (model) has one input signal, cf. (10.13). Let us assume that the system has two input signals, and that the model is

$$Ty' = -y + K_1 u_1 + K_2 u_2 \quad (10.23)$$

Taking the Laplace transform and gives

$$Tsy(s) = -y(s) + K_1 u_1(s) + K_2 u_2(s) \quad (10.24)$$

from which we find two transfer functions, $H_1(s)$ and $H_2(s)$:

$$y(s) = \underbrace{\frac{K_1}{Ts+1} u_1(s)}_{H_1(s)} + \underbrace{\frac{K_2}{Ts+1} u_2(s)}_{H_2(s)} \quad (10.25)$$

The system has one time constant: T , and two gains: K_1 and K_2 . You can choose whether to find these parameters from the differential equation (10.23) directly, or from the transfer functions in (10.25).

Example 10.2 First order system: Heated liquid tank

In Example 4.2 on page 79 we developed a mathematical model of heated liquid tank (a thermal system). The model is repeated here:

$$cmT' = P + cF(T_i - T) + U_h(T_e - T) \quad (10.26)$$

Let us for simplicity assume that the tank is well isolated so that

$$U_h \approx 0 \quad (10.27)$$

We will now calculate the transfer functions from P to T and from T_i to T . Taking the Laplace transform of (10.26) gives

$$cm[sT(s) - T_0] = P(s) + cF[T_i(s) - T(s)] \quad (10.28)$$

Since we are to find the transfer function, we may set the initial value to zero:

$$T_0 = 0 \quad (10.29)$$

From (10.28) we will find

$$T(s) = \underbrace{\frac{K_1}{T_1 s + 1}}_{H_1(s)} P(s) + \underbrace{\frac{K_2}{T_1 s + 1}}_{H_2(s)} T_i(s) \quad (10.30)$$

The gains and the time constant of each of the two transfer functions are

$$K_1 = \frac{1}{cF} \quad (10.31)$$

$$K_2 = 1 \quad (10.32)$$

$$T_1 = \frac{m}{F} = \frac{\text{Mass}}{\text{Flow}} \quad (10.33)$$

Comments:

- The time constant, which represents the “dynamics”, is the same for both transfer functions $H_1(s)$ and $H_2(s)$.
- In many applications the flow F may change. Assume that the flow is decreased. The dynamic properties of the system then change:
 - According to (10.31) the gain from P to T increases, and hence the T becomes more sensitive to P , giving higher value of T for a given change of P .
 - According to (10.33) the time constant increases, causing a more sluggish response in T to a change in P .

[End of Example 10.2]

10.4 Second order systems

10.4.1 Transfer function model

A standard second order transfer function model (with u as input variable and y as output variable) is

$$y(s) = \frac{K\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} u(s) \equiv \frac{K}{(\frac{s}{\omega_0})^2 + 2\zeta\frac{s}{\omega_0} + 1} u(s) \quad (10.34)$$

where K is the *gain*, ζ (zeta) [dimension 1] is the *relative damping factor*, and ω_0 [rad/s] is the *undamped resonance frequency*.

Example 10.3 *Second order system: Mass-spring-damper*

In Example 9.3 the transfer function from force F to position y of a mass-spring-damper system was found. The transfer function is repeated here:

$$H_{y,F}(s) = \frac{1}{ms^2 + Ds + K_f} \quad (10.35)$$

To find the standard parameters, we must transform the transfer function to one of the equivalent standard forms given by (10.34). Let us here choose the first one:

$$H_{y,F}(s) = \underbrace{\frac{1}{m}}_{\overbrace{s^2 + \frac{D}{m}s + \frac{K_f}{m}}^{s^2 + 2\zeta\omega_0s + \omega_0^2}} \underbrace{\frac{K\omega_0^2}{1}}_{\overbrace{\frac{D}{m}s + \frac{K_f}{m}}^{s^2 + 2\zeta\omega_0s + \omega_0^2}} \quad (10.36)$$

By equating the coefficients and using the following parameters values: $m = 20 \text{ kg}$, $D = 4 \text{ N/(m/s)}$ and $K_f = 2 \text{ N/m}$, we get

$$K = \frac{1}{K_f} = 0.5 \text{ [m/N]} \quad (10.37)$$

$$\omega_0 = \sqrt{\frac{K_f}{m}} = \sqrt{0.1} = 0.32 \text{ [rad/s]} \quad (10.38)$$

$$\zeta = \frac{D}{2\sqrt{mK_f}} = 0.32 \quad (10.39)$$

[End of Example 10.3]

10.4.2 Classification of second order systems

We will classify second order systems from the shape of the step response. We assume that the input variable $u(t)$ is a step of amplitude U , which Laplace transformed is $u(s) = U/s$. Then the Laplace transformed time-response becomes

$$y(s) = H(s)u(s) = \frac{K\omega_0^2}{s^2 + 2\zeta\omega_0s + \omega_0^2} \frac{U}{s} \quad (10.40)$$

The shape of the time-response $y(t)$, which is calculated as the inverse Laplace transform of $y(s)$, depends on the poles, cf. Section 9.5. The poles are the roots of the characteristic equation $a(s)$:

$$a(s) = s^2 + 2\zeta\omega_0s + \omega_0^2 = 0 \quad (10.41)$$

The poles p_1 and p_2 are the roots of $a(s)$ and becomes

$$p_1, p_2 = -\zeta\omega_0 \pm \sqrt{\zeta^2 - 1} \omega_0 \quad (10.42)$$

The value of ζ determines whether the poles are real or complex conjugate:

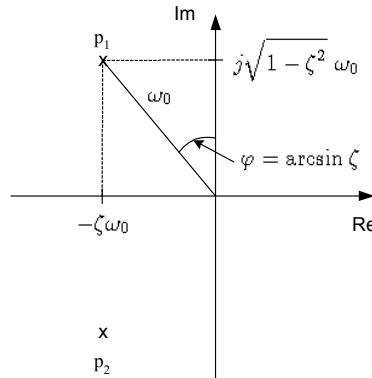


Figure 10.3: Pole placement for second order systems when the poles are complex conjugate. The poles are given by (10.43).

- If $\zeta \geq 1$, the poles are real and given by (10.42).
- If $0 \leq \zeta < 1$, the poles are complex conjugate:

$$p_1, p_2 = \underbrace{-\zeta \omega_0}_{\text{Re}} \pm j \underbrace{\sqrt{1 - \zeta^2} \omega_0}_{\text{Im}} \quad (10.43)$$

Figure 10.3 shows the pole placement when the poles are complex conjugate.

Figure 10.4 classifies second order systems by the value of ζ . (This is a common way to do the classification.) The step responses referenced in the figure can be calculated by taking the inverse Laplace transform of (10.40), but the detailed calculations are not shown here.

In the following are given simulated step responses and pole plots for representative examples of overdamped, underdamped, and undamped systems. The parameter values are shown on the front panels of the simulators in the respective figures.

In all cases the steady-state value of the step response is

$$y_s = KU \quad (10.44)$$

because the static transfer function is K .

10.4.2.1 Overdamped systems

Figure 10.5 shows the step response and the poles for an example of an overdamped system.

Comments:

- The step response has no overshoot.

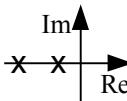
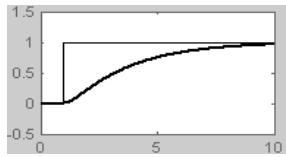
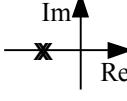
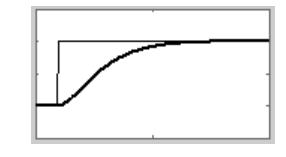
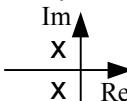
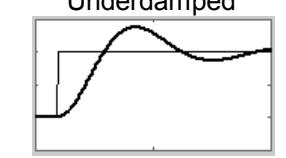
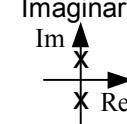
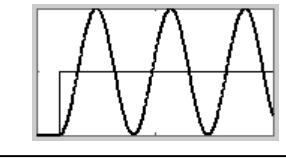
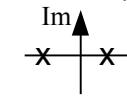
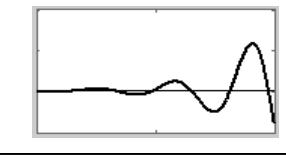
Value of ζ	Poles p_1 and p_2	Type of step response $y(t)$	Reference to $y(t)$ in Appendix
$\zeta > 1$	Real and distinct 	Overdamped 	(B.17)
$\zeta = 1$	Real and multiple 	Critically damped 	(B.18)
$0 < \zeta < 1$	Complex conj. 	Underdamped 	(B.23)
$\zeta = 0$	Imaginary 	Undamped 	(B.23)
$\zeta < 0$	Pos. real part 	Unstable 	(B.17) or (B.18) or (B.23)

Figure 10.4: Classification of second order systems by the value of ζ

- The poles p_1 and p_2 are real and distinct:

$$p_1, p_2 = -\zeta\omega_0 \pm \sqrt{\zeta^2 - 1} \omega_0 \quad (10.45)$$

The transfer function can therefore be written on the form

$$H(s) = \frac{Kp_1p_2}{(s - p_1)(s - p_2)} = \frac{K}{(T_1s + 1)(T_2s + 1)} \quad (10.46)$$

This implies that the second order system can be split into two first order subsystems having time constants T_1 and T_2 , respectively. The largest of these time constants can be denoted the *dominating time constant*.

10.4.2.2 Underdamped system

Figure 10.6 shows the step response and the poles for an example of an underdamped system.

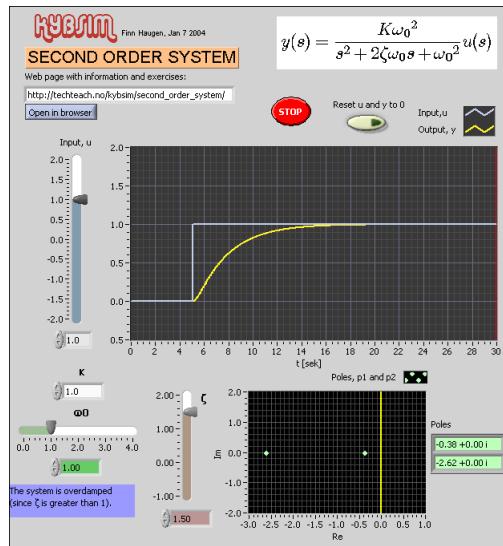


Figure 10.5: Step response and the poles for an example of an overdamped system

Comments:

- The poles are complex conjugate:

$$p_1, p_2 = -\zeta\omega_0 \pm j\sqrt{1-\zeta^2}\omega_0 \quad (10.47)$$

- The less ζ , the less damping in the step response. It can be shown that the less ζ , the more dominating imaginary part over the real part of the poles. This is a general property of poles: The larger imaginary part relative to the real part, the less damping in the time-response. Figure 10.7 shows the step-response for various values of ζ .
- The *overshoot factor* δ of the step response is defined as

$$\delta = \frac{y_{\max} - y_s}{y_s} \quad (10.48)$$

where y_s is the steady-state value of the step-response. It can be shown that δ is a function of the relative damping factor ζ , as follows:

$$\delta = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \quad (10.49)$$

δ is plotted as a function of ζ in Figure 10.8.

The inverse function of (10.49) is

$$\zeta = \frac{|\ln \delta|}{\sqrt{\pi^2 + (\ln \delta)^2}} \quad (10.50)$$

A few examples: Overshoot $\delta = 0.1$, that is, 10% overshoot, corresponds to $\zeta = 0.6$. Overshoot $\delta = 0$ (zero overshoot) corresponds to $\zeta = 1$ (critically damped system).

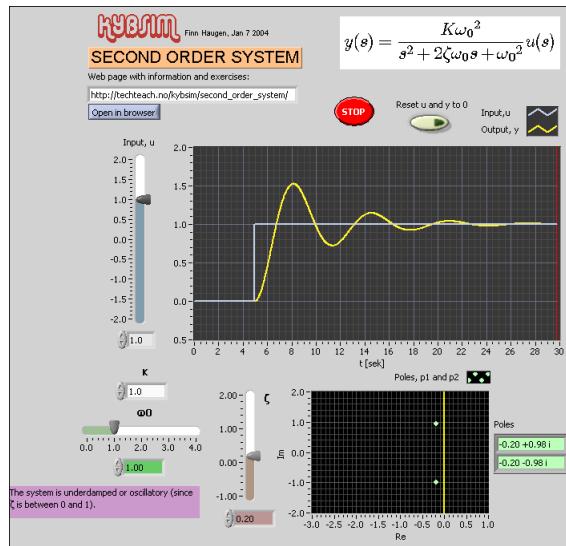


Figure 10.6: Step response and poles for an example of an underdamped system

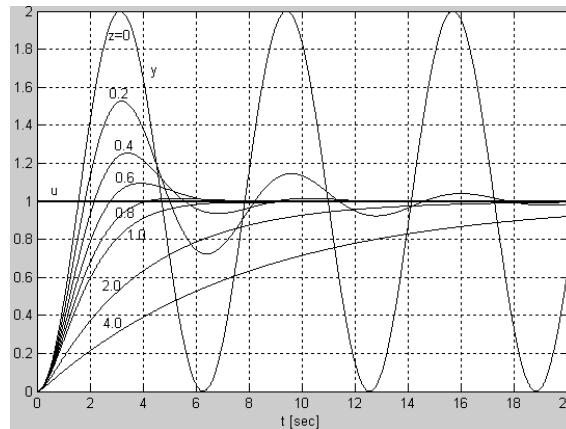


Figure 10.7: Step-response for various values of ζ for second order systems

- Simulations shows that the 63% response-time of the step response is approximately

$$T_r \approx \frac{1.5}{\omega_0} \quad (10.51)$$

ω_0 expresses in a way how quick the system is. ω_0 is the distance from origin to the poles, see Figure 10.3. This is a general property of poles: The longer distance from origin, the faster the dynamics of the system.

- It can be shown that the frequency in the oscillations are

$$\beta = \sqrt{1 - \zeta^2} \omega_0 \text{ [rad/s]} \quad (10.52)$$

10.4.2.3 Undamped system

Figure 10.9 shows the step response and the poles for an example of an undamped system.

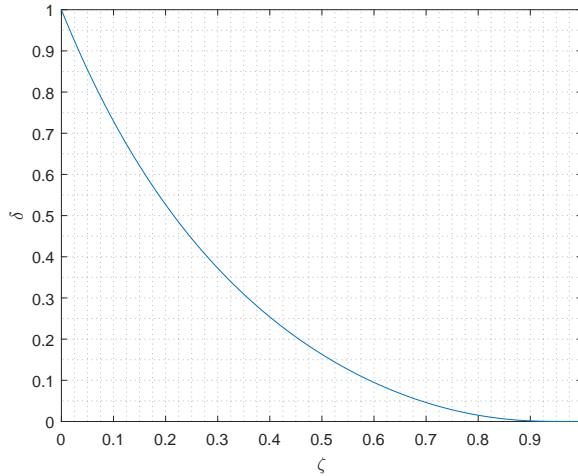


Figure 10.8: Overshoot factor δ plotted as a function of the relative damping factor ζ , cf. (10.49)

Comments:

- The step response is undamped, steady-state oscillations:

$$y(t) = KU(1 - \cos \omega_0 t) \quad (10.53)$$

The frequency of the oscillations in rad/s is ω_0 — therefore the name *undamped resonance frequency*.

- The poles are purely imaginary:

$$p_1, p_2 = \pm j\omega_0 \quad (10.54)$$

The real part is zero, which is an explanation of why the step response is undamped. In general, damping is due non-zero real part of poles.

Example 10.4 Control system

Figure 10.10 shows a control system for the angular position of an electro-motor.

Figure 10.11 shows a transfer function based block diagram for the control system.

We assume that the individual transfer functions with parameter values are as follows:

$$H_u(s) = \frac{K_u}{(Ts + 1)s} = \frac{1}{(s + 1)s} \quad (10.55)$$

$$H_d(s) = \frac{K_d}{(Ts + 1)s} = \frac{-1}{(s + 1)s} \quad (10.56)$$

$$H_c(s) = K_p \text{ (proportional controller)} \quad (10.57)$$

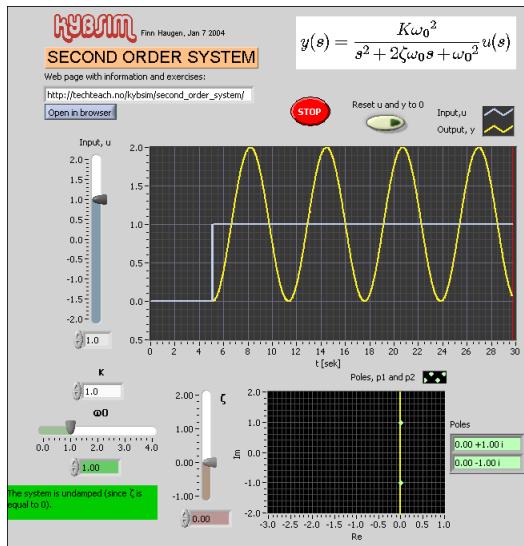


Figure 10.9: Step response and the poles for an example of an undamped system

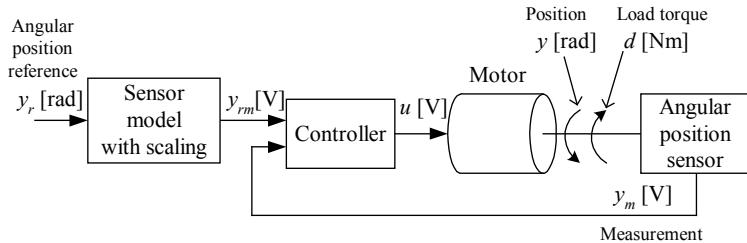


Figure 10.10: Feedback control system for the angular position of an electro-motor

$$H_s(s) = K_m = 1 \quad (10.58)$$

$$H_{sm}(s) = K_{sm} = 1 \quad (10.59)$$

(Some information about the background of $H_u(s)$: The transfer function from the manipulating or controlling variable u to the speed v is $K_1/(Ts + 1)$, and the transfer function from v to position y is $1/s$ which is an integrator. The process to be controlled is thus a “first order system in series with an integrator”.)

Let us calculate the controller gain K_c from specifications for the transfer function $H_{y_r,y}(s)$ from the position reference y_r to the position y (the specifications are presented soon).

First, we must find $H_{y_r,y}(s)$. From the block diagram in Figure 10.11, we can express y as follows (for simplicity, I drop the argument s here):

$$y = H_u H_c (H_{sm} y_r - H_s y)$$

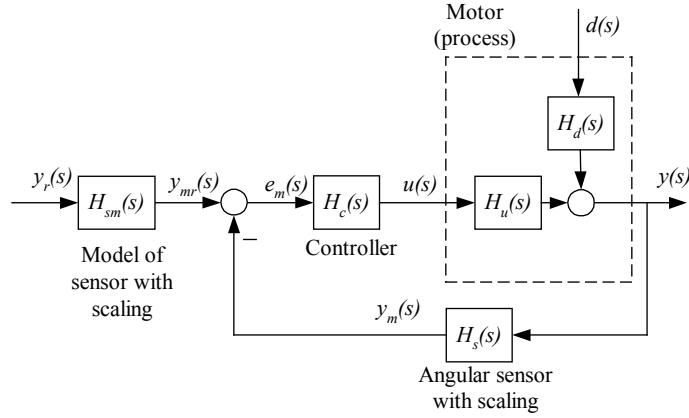


Figure 10.11: Transfer function based block diagram for the positional control system shown in Figure 10.10

som gir:

$$\frac{y(s)}{y_r(s)} = H_{y_r,y}(s) = \frac{H_u H_c H_{sm}}{1 + H_u H_c H_s} \quad (10.60)$$

$$= \frac{K_c \frac{1}{(s+1)s}}{1 + K_c \frac{1}{(s+1)s}} \quad (10.61)$$

$$= \frac{K_c}{s^2 + s + K_c} \quad (10.62)$$

$$= \frac{K \omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} \quad (10.63)$$

By equating coefficients between (10.62) and (10.63) we get:

$$K = 1 \quad (10.64)$$

$$\omega_0 = \sqrt{K_c} \quad (10.65)$$

$$\zeta = \frac{1}{2\sqrt{K_c}} \quad (10.66)$$

From (10.65) and (10.66) we can calculate K_c either from a specified ω_0 , which expresses the quickness of the system since the 63% response-time is $T_r \approx 1.5/\omega_0$, cf. (10.51), or from a specified ζ , which expresses the damping of the system. Good stability is the most important property of a control system, so we use (10.66) as the basis for calculation of K_p . Let us say that $\zeta = 0.6$ is a reasonable value. $\zeta = 0.6$ gives 10% overshoot ($\delta = 0.1$) in the step response, cf. (10.50). From (10.66) we get

$$K_p = \frac{1}{4\zeta^2} = \frac{1}{4 \cdot 0.6^2} = 0.69 \quad (10.67)$$

The response-time of the control system then becomes

$$T_r \approx \frac{1.5}{\omega_0} = \frac{1.5}{\sqrt{K_p}} = \frac{1.5}{\sqrt{0.69}} = 1.8 \text{ sec} \quad (10.68)$$

Figure 10.12

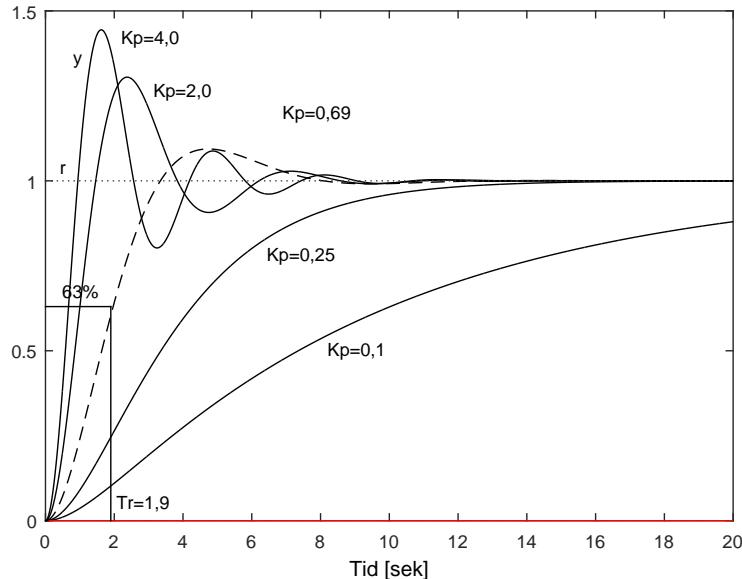


Figure 10.12: Simulated step responses in the angular position y (it is a step of amplitude 1 in the reference r) for several K_c -values

shows simulated step responses in the position y (it is a step of amplitude 1 in the reference r) for the calculated $K_c = 0.69$ and for several other K_c -values. We see that the step response is quicker and less damped the larger the K_c -value. (This is a typical consequence of increasing the controller gain in control systems.)

From Figure 10.12 we read off a response-time of $T_r = 1.9$ (for $K_c = 0.69$), so (10.68) is quite accurate.

[End of Example 10.4]

10.5 Time delays

In many systems there is a *time delay* or dead-time in the signal flow, for example with material transport on a conveyor belt, see Figure 10.13. In this application, the relation between the input variable F_{in} and the output variable F_{out} is

$$F_{\text{out}}(t) = F_{\text{in}}(t - \tau) \quad (10.69)$$

where τ is the time delay which is the transportation time on the belt. In other words: The outflow at time t is equal to the inflow τ time units ago.

What is the transfer function of a time delay? Taking the Laplace transform of (10.69)

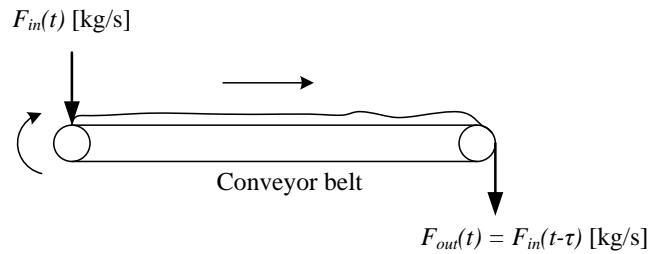


Figure 10.13: Time delay on a conveyor belt.

using (8.15):

$$F_{out}(s) = \underbrace{e^{-\tau s} F_{in}(s)}_{H(s)} \quad (10.70)$$

Thus, the transfer function of a time delay of τ [time unit] is

$$H(s) = e^{-\tau s} \quad (10.71)$$

Figure 10.14 shows a simulation of a time delay. The time delay is $\tau = 1$ sec.

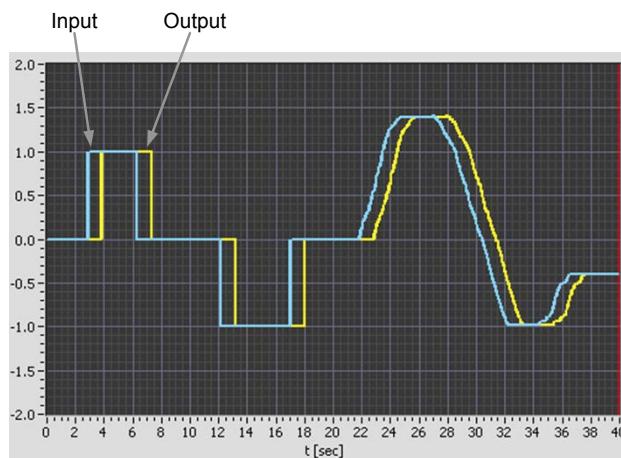


Figure 10.14: Output is equal to input, time delayed 1 sec.

10.5.1 Approximation of time delay by Padé approximation

10.6 Higher order systems

Systems having higher order of the denominator polynomial of the transfer function than one, are so-called higher order systems, or more specifically, second order systems, third order systems and so on. A serial connection of first order systems results in a higher order system. (But not all possible higher order systems can be constructed by serial connection of first order systems.) When transfer functions are connected in series, the resulting

transfer function is the product of the individual transfer functions, cf. Figure 9.2. As an example, Figure 10.15 shows a second order system consisting of “two time constants” connected in series.

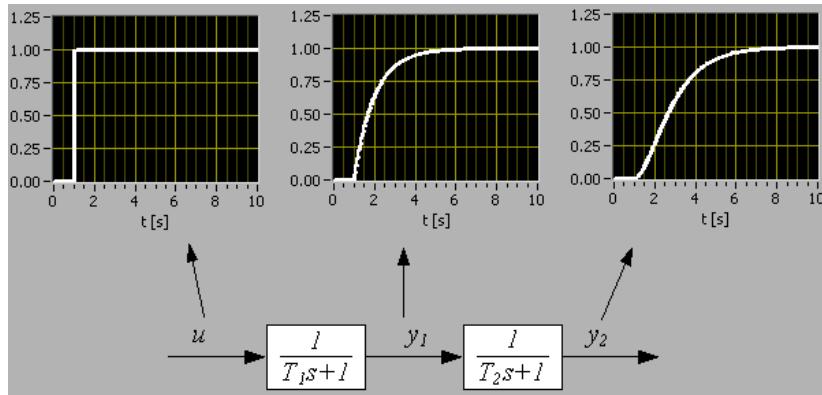


Figure 10.15: Step responses in a second order system

The combined transfer function becomes

$$H(s) = \frac{1}{(T_1 s + 1)(T_2 s + 1)} = \frac{y_2(s)}{u(s)} \quad (10.72)$$

The figure also shows the step responses in the system. It is assumed that $T_1 = 1$, $T_2 = 1$ and $K = 1$. Observe that each first order systems makes the response become more sluggish, as it has a smoothing effect.

Let us define the *response-time* T_r as the *time it takes for a step response to reach 63% of its steady-state value*. For time constant systems, the response-time is equal to the time constant:

$$T_r = T \quad (10.73)$$

For higher order systems (order larger than one) it turns out that the response-time can be roughly estimated as the sum of the time constants of the assumed serial subsystems that make up the higher order system:

$$T_r \approx \sum_i T_i \quad (10.74)$$

As an example, the response-time of the system shown in Figure 10.15 is

$$T_r \approx 1 + 1 = 2 \text{ s} \quad (10.75)$$

Does the simulation shown in Figure 10.15 confirm this?¹

¹Yes

10.7 Problems for Chapter 10

Problem 10.1 Dynamic response in tank

See Problem 9.2. The transfer function from w_{out} to h is

$$\frac{h(s)}{w_{out}(s)} = -\frac{1}{\rho As} = H_2(s) \quad (10.76)$$

1. Does this transfer function represent integrator dynamics?
2. Assume that $w_{out}(t)$ is a step from 0 to W at time $t = 0$. Calculate the response $h(t)$ that this excitation causes in the level h . You are required to base your calculations on the Laplace transform.

Problem 10.2 Tank as integrator

Figure 10.16 shows an isolated tank (having zero heat transfer through the walls).

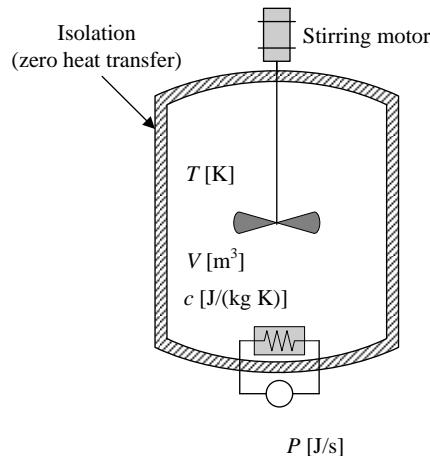


Figure 10.16: Isolated tank

Show that the tank dynamically is an integrator with the power P as input variable and the temperature T as output variable. (Hint: Study the transfer function from P to T .)

Problem 10.3 Time constant dynamics

Given a system with gain K , time constant T , and time delay T_d . The system is initially in steady (static) state, with the value of the output variable, y , equal to y_0 . At time t_0 , the system is affected by a step change from u_0 to u_1 at the input, u . Plot principally $u(t)$ and $y(t)$ manually in their respective plots, showing how the given information appears in the plots. The origin $(0, 0)$ shall appear in both plots.

Problem 10.4 Time constant dynamics - 2

Assume a system with gain -3 , time constant 10 s, and time delay 2 s. Sketch (manually) principally the step response of the system assuming the input step has amplitude 4 appearing at time 50 s. Before the step appears, the system output is at rest with value 15 .

Problem 10.5 Time constant dynamics from transfer function

Calculate the gain and the time constant of the transfer function

$$H(s) = \frac{y(s)}{u(s)} = \frac{2}{4s + 8} \quad (10.77)$$

and draw by hand roughly the step response of $y(t)$ due to a step of amplitude 6 in u from the following information:

- The steady-state value of the step response
- The time constant
- The initial slope of the step response, which is

$$S_0 = y'(0^+) = \frac{KU}{T} \quad (10.78)$$

Problem 10.6 Deriving transfer function from step response

Figure 10.17 shows the temperature response T of a thermal system due to a step of amplitude 1 kW in the supplied power P .

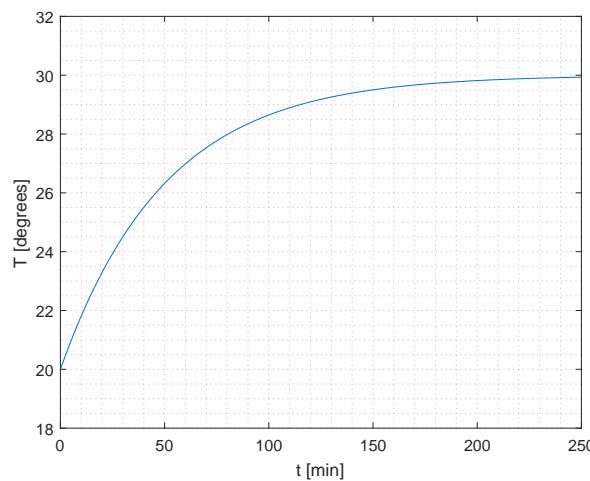


Figure 10.17: Problem 10.6: Temperature response due to step in supplied power

Find the transfer function from ΔP (power) to ΔT (temperature) where Δ indicates deviations from the steady-state values. Assume that the system is of first order (a time constant system).

Problem 10.7 Dynamics of an RC circuit

Figure 10.18 shows an RC-circuit (the circuit contains the resistor R and the capacitor C).

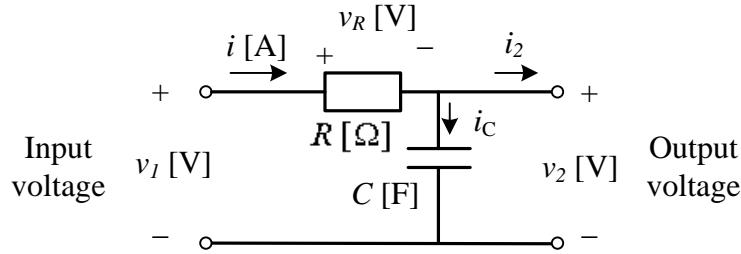


Figure 10.18: Problem 10.7: RC-circuit.

The RC-circuit is frequently used as an analog lowpass filter: Signals of *low* frequencies *passes* approximately unchanged through the filter, while signals of high frequencies are approximately filtered out (stopped). It can be shown that a mathematical model of the RC circuit is

$$RCv'_{\text{out}} = v_{\text{in}} - v_{\text{out}} \quad (10.79)$$

1. Calculate the transfer function $H(s)$ from v_{in} to v_{out} , and calculate the gain and the time constant of $H(s)$.
2. Assume that the RC circuit is used as a signal filter. Assume that the capacitance C [F] is fixed. How can you adjust the resistance R (increase or decrease) so that the filter performs stronger filtering or, in other words: is more sluggish.

Problem 10.8 Time delay of pipeline

For a pipeline of length 0.5 m and cross sectional area of 0.01 m^2 filled with liquid which flows with a volumetric flow $0.001 \text{ m}^3/\text{s}$, calculate the time delay (transport delay) from inlet to outlet of the pipe.

Problem 10.9 Response time of compound system

Assume that a system can be well described by 3 time constant systems in series, with the following time constants respectively: 0.5, 1, and 2 sec. What is the approximate response time of the system?

10.8 Solutions to problems for Chapter 10

Solution to Problem 10.1

1. Yes! Because the transfer function has the form of K_i/s .
2. The Laplace transform of the response is

$$h(s) = H_2(s)w_{\text{out}}(s) = -\frac{1}{\rho As}w_{\text{out}}(s) \quad (10.80)$$

Since $w_{\text{out}}(t)$ is a step of amplitude W at $t = 0$, $w_{\text{out}}(s)$ becomes, according to (8.7),

$$w_{\text{out}}(s) = \frac{W}{s} \quad (10.81)$$

With this $w_{\text{out}}(s)$, (10.80) becomes

$$h(s) = -\frac{1}{\rho As} \frac{W}{s} \quad (10.82)$$

According to (8.8),

$$h(t) = -\frac{W}{\rho A}t \quad (10.83)$$

That is, the response is a ramp with negative slope.

Comment: This $h(t)$ is only the *contribution* from the outflow to the level. To calculate the complete response in the level, the total model (9.47), where both u and w_{out} are independent or input variables, must be used.

Solution to Problem 10.2

Energy balance:

$$c\rho V T' = P \quad (10.84)$$

Laplace transformation:

$$c\rho V [sT(s) - T_0] = P(s) \quad (10.85)$$

which yields

$$T(s) = \frac{1}{s}T_0 + \underbrace{\frac{1}{c\rho Vs}P(s)}_{H(s)} \quad (10.86)$$

The transfer function is

$$H(s) = \frac{T(s)}{P(s)} = \frac{1}{c\rho Vs} = \frac{K}{s} \quad (10.87)$$

which is the transfer function of an integrator with gain $K = 1/c\rho V$.

Solution to Problem 10.3

See Figure 10.19.

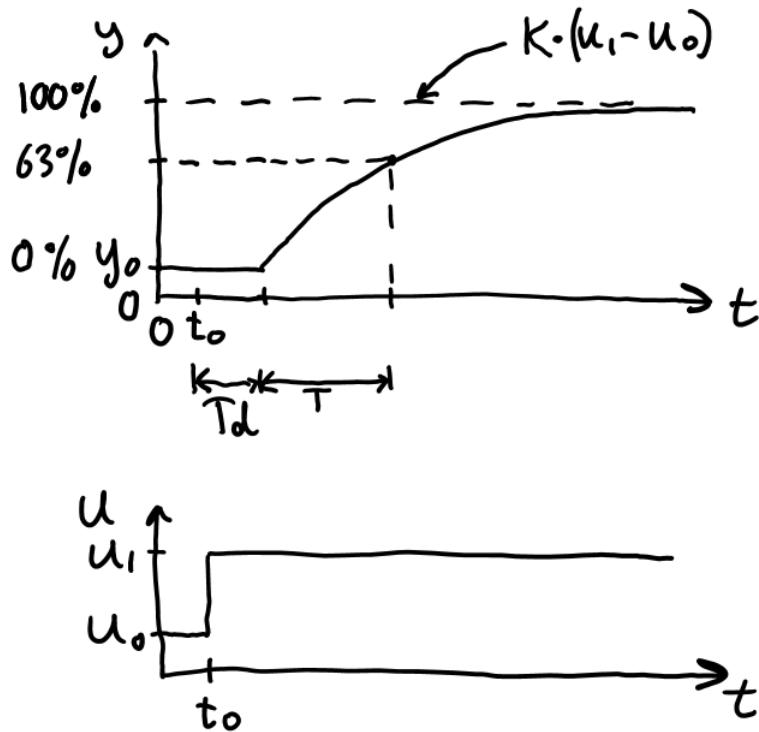


Figure 10.19: Problem 10.3: Time-constant dynamics.

Solution to Problem 10.4

See Figure 10.20.

Solution to Problem 10.5

We manipulate the transfer function so that the constant term of the denominator is 1:

$$H(s) = \frac{2}{4s+8} = \frac{2/8}{(4/8)s + 8/8} = \frac{0.25}{0.5s + 1} = \frac{K}{Ts + 1} \quad (10.88)$$

Hence,

$$K = 0.25$$

and

$$T = 0.5$$

We base the drawing of the step response on the following information:

- The steady-state value of the step response:

$$y_s = KU = 0.25 \cdot 6 = 1.5 \quad (10.89)$$

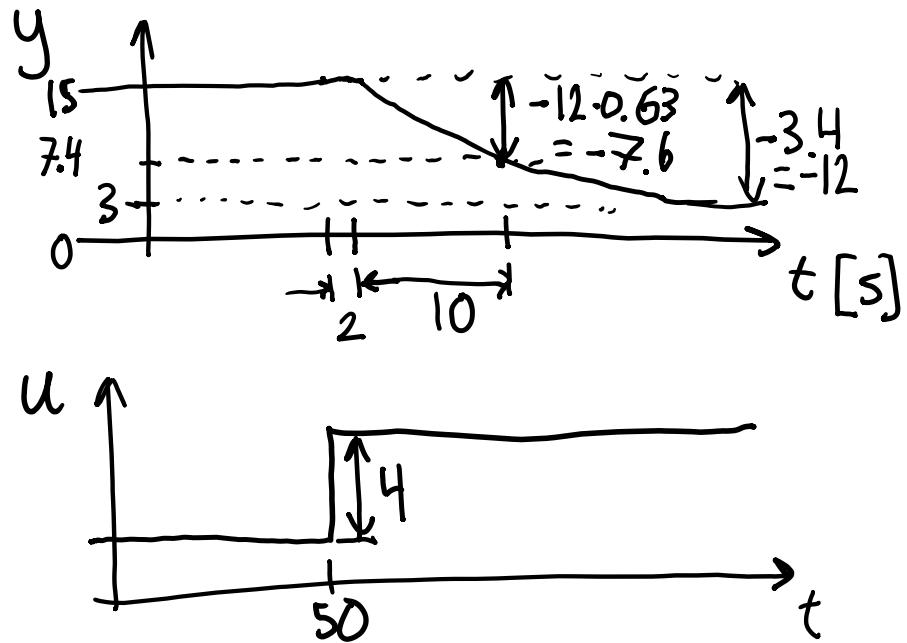


Figure 10.20: Problem 10.4. Time-constant dynamics.

- The time constant:

$$T = 0.5 \quad (10.90)$$

which is the time when the step response has reached value

$$0.63 \cdot y_s = 0.63 \cdot 1.5 = 0.95 \quad (10.91)$$

- The initial slope of the step response:

$$S_0 = y'(0^+) = \frac{KU}{T} = \frac{0.25 \cdot 6}{0.5} = 3 \quad (10.92)$$

Figure 10.21 shows the step response.

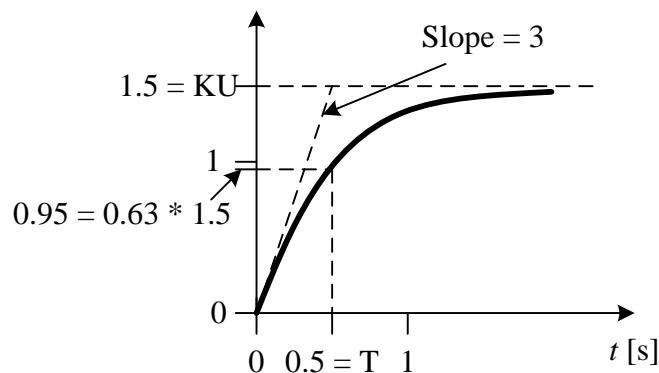


Figure 10.21: Problem 10.5: Step response.

Solution to Problem 10.6

From Figure 10.17 we see that the gain is

$$K = \frac{\Delta T}{\Delta P} = \frac{30 \text{ K} - 20 \text{ K}}{1 \text{ kW}} = 10 \frac{\text{K}}{\text{kW}} \quad (10.93)$$

and that the time constant (the 63% rise time) is

$$T_1 = 50 \text{ min} \quad (10.94)$$

The transfer function becomes

$$\frac{\Delta T(s)}{\Delta P(s)} = \frac{10}{50s + 1} \frac{\text{K}}{\text{kW}}$$

Solution to Problem 10.7

1. Laplace transformation of the differential equation (10.79) gives

$$RCsv_{\text{out}}(s) = v_{\text{in}}(s) - v_{\text{out}}(s) \quad (10.95)$$

Solving for $v_{\text{out}}(s)$ gives

$$v_{\text{out}}(s) = \frac{1}{RCs + 1} v_{\text{in}}(s) \quad (10.96)$$

The transfer function is

$$H(s) = \frac{1}{RCs + 1} = \frac{K}{Ts + 1} \quad (10.97)$$

The gain is

$$K = 1 \quad (10.98)$$

The time constant is

$$T = RC \quad (10.99)$$

2. The filtering is stronger if R is increased.

Solution to Problem 10.8

The time delay is

$$\tau = \frac{AL}{q} = \frac{0.01 \text{ m}^2 \cdot 0.5 \text{ m}}{0.001 \text{ m}^3/\text{s}} = 5 \text{ s} \quad (10.100)$$

Solution to Problem 10.9

The approximate response time is

$$T = 0.5 + 1 + 2 = 3.5 \text{ s} \quad (10.101)$$

Chapter 11

Adaptation of models to data

11.1 Introduction

Assume that you have a mathematical model of a system with parameters that characterizes the model. The model can be static or dynamic:

- **Static models** are models not containing time-derivatives. Thus, differential equations are not static models. One example of a static model is

$$y = a_1x_1 + a_2x_2 \quad (11.1)$$

where y is the output and x_j are inputs, and a_i are parameters to be estimated from known data of output and inputs.

- **Dynamic models** may be in the form of:

- Differential equations (continuous-time models), linear or nonlinear, possibly in the form of a state space model
- Difference equations (discrete-time models), linear or nonlinear, possibly in the form of a state space model
- Transfer functions:
 - * Laplace-transform based transfer functions (i.e. continuous-time transfer functions)
 - * Z-transform based transfer functions (i.e. discrete-time transfer functions)

Model adaptation is calculation of values of model parameters so that the model behaves in as good accordance as possible to given observations or measurements from the real system. “as good as possible” indicates that the model adaptation problem can be solved with optimization! Indeed, we will do that in Section 11.2.

But why do we want a model when we already have data? Good question! And here are two good answers:

- **Prediction or simulation:** We can use the model for prediction, ie to predict (predict) or simulate what may happen in the future, which we have no observations from yet.
- **Analysis:** We can use the model to analyze the data we already have, e.g. calculate increments for a trend that is in the data.

11.2 Model adaptation as an optimization problem

11.2.1 How to find the best model

We assume given a time series of observations or measurements of the system. The observations may consist of a time series of the input variable or signal and a time series of the output variable or signal, see Figure 11.1. In some cases we have no input excitation signal; only “data” or measurements, e.g. statistical data.

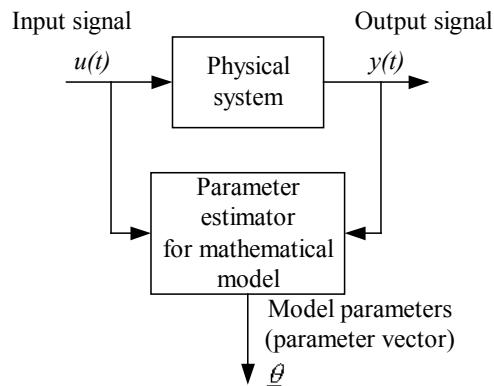


Figure 11.1: Estimation of parameters of a mathematical model from time-series of the observations which may consist of the input variable or signal (u in the figure) and the output variable or signal (y).

The formulation of the parameter estimation problem as an optimization problem is typically as follows:

- The objective function to be minimized is the sum of squared prediction errors – also denoted a least squares criterion:

$$\text{SSPE} = \sum_{k=1}^N e(k)^2 \quad (11.2)$$

where $e(i)$ is the prediction error which is the difference between the observations (measurements) and the model-based predicted or calculated observations:

$$e(i) = y_{\text{obs}}(i) - y_{\text{pred}}(i) \quad (11.3)$$

Figure 11.2 illustrates the prediction error. It is assumed that a linear model is to be adapted to the observations. y_{pred} is calculated in *simulations*, using the model.

Therefore, y_{pred} is a function of the parameters to be estimated, and SSPE is also a function of the parameters.

- The parameters to be estimated are used as optimization variables. All the parameters may be collected in a parameter vector:

$$P = [p(1), p(2), \dots, p(r)]^T \quad (11.4)$$

- In each iteration, the optimizer runs a simulation with parameter values that are adjusted based on previous iterations (simulations). The iterations stop when the parameter values that minimize the SSPE are found. *These “best” values are then used as the parameter estimates.*

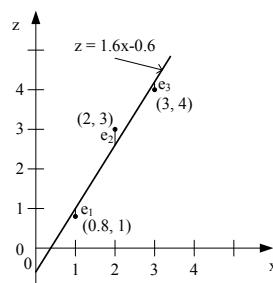


Figure 11.2: Illustration of the prediction error, $e(i)$. A linear model is assumed.

Mathematical formulation of the parameter estimation problem

Mathematically, the optimization problem can be stated as:

$$\min_{P=[p(1), p(2), \dots, p(r)]} \text{SSPE} \quad (11.5)$$

s.t. (subject to) the given mathematical model.

Figure 11.3 illustrates the principle of optimization-based parameter estimation.

Any nonlinear optimizer can be used to implement the parameter estimation, e.g. the slsqp optimizer in the Scipy package of Python or the fmincon optimizer in the Optimization toolbox of Matlab.

Selecting the best model among several model candidates

In some applications there are *several* candidates of models, for example, a first order differential equation and a second order differential equation that are both assumed to represent the real system. You can select the best model as the one that minimizes the SSPE index (11.2) *calculated from a validation data set*. The validation data set may be a part, e.g. a half of the original data set, while the data set used for parameter estimation is the other half of the original data set. This is illustrated in Figure 11.4.

Note that if you have only *one* (fixed) mathematical model that you want to adapt to the given data, there is no need to validate the model with any validation data set. In that

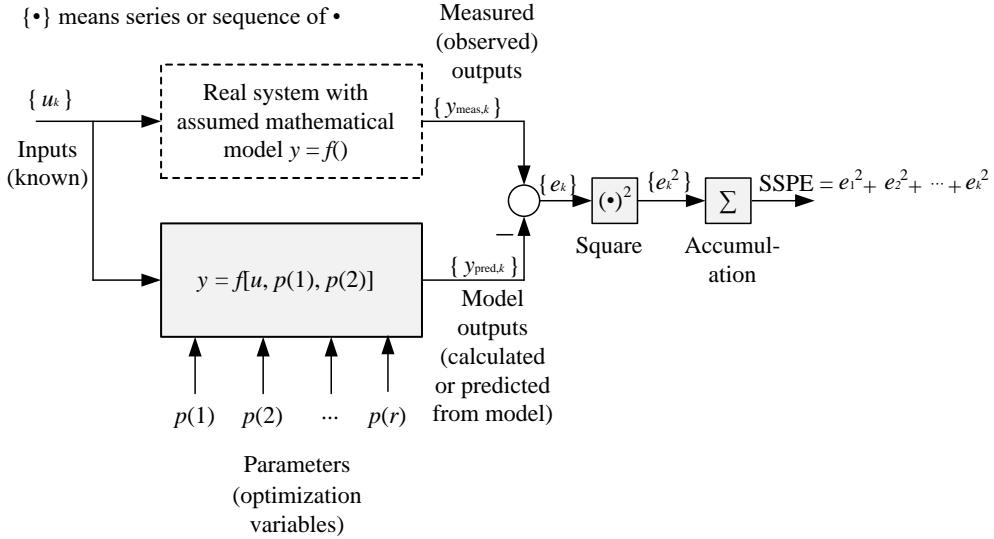


Figure 11.3: The principle of parameter estimation using optimization. The ultimate (estimated) parameter values are those that minimize SSPE. (SSPE = sum of squared prediction errors.)

case, you can use the whole data set for estimation. An example: If you assume that the following transfer function model:

$$H(s) = \frac{y(s)}{u(s)} = \frac{K}{Ts + 1} \quad (11.6)$$

is appropriate to represent the dynamics of a given system, then you can use the whole data for estimation of parameters K and T . In other words, model validation is not necessary. But if you have *two* model candidates, say model (11.6) *and* the following second order transfer function:

$$H_2(s) = \frac{y(s)}{u(s)} = \frac{K}{(T_1 s + 1)(T_2 s + 1)} \quad (11.7)$$

then you must use a validation data set to select among the two model candidates.

11.2.2 Good excitation is necessary!

Assume, as an example, that you want to estimate the time constant T of a first order transfer function. You can not estimate T , which is related to the dynamics of the system, if y and u have constant values all the time. Thus, it is necessary that the excitation signal, $u(t)$, has sufficiently rich variation to give the LS-method enough information about the dynamics of the system to produce an accurate estimate of T .

The up-down-up signal is probably a good excitation signal, see Figure 11.5. This signal is simple to generate manually during the experiment. This signal gives in many cases enough excitation for the estimator to calculate accurate parameter estimates, but the period of the signal shift must be large enough to give the system output a chance to approximately stabilize between the steps.

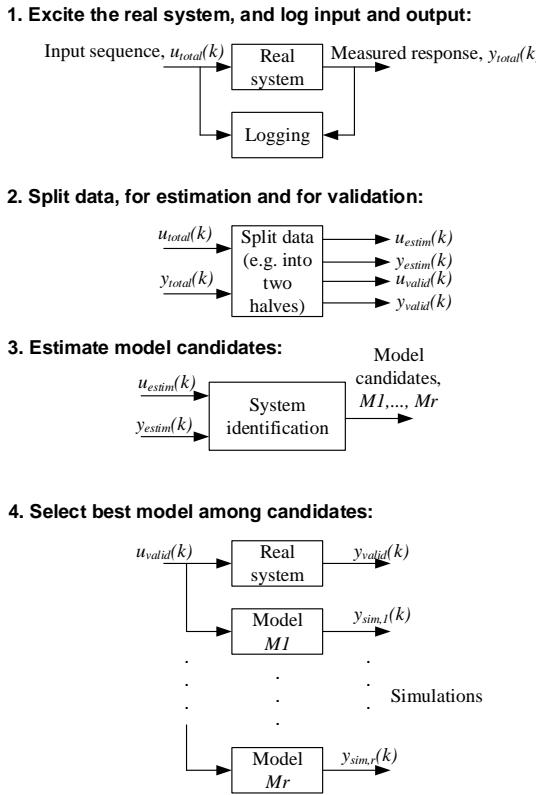


Figure 11.4: Procedure for selecting the best among several model candidates.

11.3 Adaptation of static models to data

11.3.1 Adaptation using grid optimization

11.3.1.1 Introduction

If the number of model parameters to be estimated is not so large, e.g. 5 or less, the straightforward grid method of optimization, cf. Chapter 42.2.3, can be used to find the estimates (the best values). As optimization criterion, we can use minimum of the squared sum of prediction error (sspe), cf. Section 11.5.

11.3.1.2 Model adaptation of static models using native grid optimization

The following examples demonstrates how to adapt a static model to given data using the grid method:

- Example 11.1: Adaptation of a model to data about greenhouse gas emission in Norway using the grid method implemented from scratch with nested for loops.
- Example 11.2: The same as Example 11.1, but using Python's `scipy.optimize.brute()`

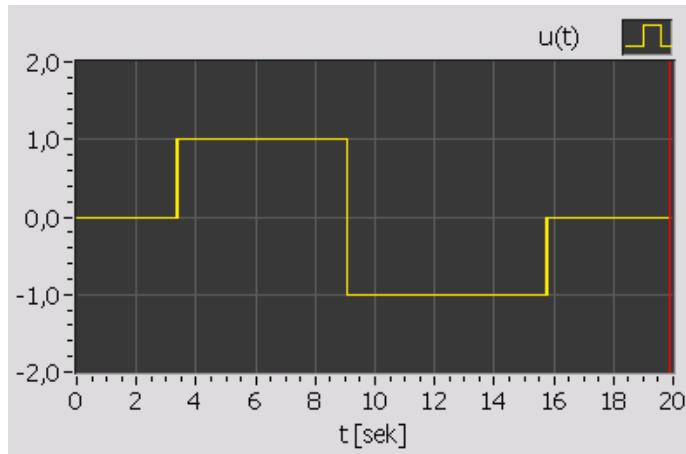


Figure 11.5: A good excitation signal: Up-down-up signal.

function which implements the grid method, cf. Section 11.3.1.1. The brute() function has an option to obtain an accurate optimal estimate with simplex optimization method started from the grid optimal solution. In the example, this option is activated.

As objective function (optimization criterion) we use the least squares deviation between observations and predictions. Often, the term regression is used on the least squares static model adaptation.

Example 11.1 *Adaptation of a static model to greenhouse gas emission data using the grid method*

Statistics Norway¹ has an overview of total greenhouse gas emissions in Norway since 1990. An extract of the overview, for the period 2000 – 2018, is shown in Table 11.1. The data are plotted in Figure 11.6.

We assume the following linear model (but a linear model is not necessarily a good model in all applications):

$$y = ax + b \quad (11.8)$$

As objective function we select the minimum SSPE, or least squares, function with a and b as optimization variables:

$$\min_{a,b} f(a, b) \quad (11.9)$$

where

$$f = \sum_{i=0}^{N-1=18} [e(i)]^2 \quad (11.10)$$

$N = 19$ is the number of observations. $e(i)$ is the prediction error:

$$e(i) = y_{\text{obs}}(i) - y_{\text{pred}}(i) \quad (11.11)$$

¹Statistisk sentralbyrå (SSB)

Table 11.1: Total greenhouse gas emissions in Norway over years 2000 – 2018.

Source: Statistics Norway:

<https://www.ssb.no/statbank/table/08940/tableViewLayout1/>

Index i	Time, x	Emission, y_{obs}	Index i	Time, x	Emission, y_{obs}
	[year]	[10^6 tons CO2-equiv.]		[year]	[10^6 tons CO2-equiv.]
0	2000	54.8	10	2010	55.5
1	2001	56.1	11	2011	54.6
2	2002	55.0	12	2012	54.1
3	2003	55.7	13	2013	54.0
4	2004	56.2	14	2014	54.1
5	2005	55.4	15	2015	54.4
6	2006	55.3	16	2016	53.6
7	2007	57.0	17	2017	52.7
8	2008	55.6	18	2018	52.9
9	2009	53.2	-	-	-

where $y_{\text{obs}}(i)$ is given in Table 11.1, and $y_{\text{pred}}(i)$ is predicted from the modell (11.8). Thus,

$$y_{\text{pred}} = ax + b \quad (11.12)$$

In detail, the objective function, (11.10), is:

$$f(a, b) = \sum_{i=0}^{N-1} [e(i)]^2 \quad (11.13)$$

$$= \sum_{i=0}^{N-1} [y_{\text{obs}}(i) - y_{\text{pred}}(i)]^2 \quad (11.14)$$

$$= \sum_{i=0}^{N-1} \{y_{\text{obs}}(i) - [ax(i) + b]\}^2 \quad (11.15)$$

where $y_{\text{obs}}(i)$ og $x(i)$ are given in Table 11.1.

Python program 11.1 solves this optimization problem using the grid method.

http://techteach.no/control/python/prog_grid_optim_greenhouse_gas.py

Listing 11.1: prog_grid_optim_greenhouse_gas.py

```
# %% Import of packages:

import numpy as np
import matplotlib.pyplot as plt
import time

# %% Definition of objective function:

def fun_obj(params):
    a = params[0]
```

```
b = params[1]
y_pred_array = a*x_array + b
e_array = y_obs_array - y_pred_array
sspe = sum(e_array*e_array)
return sspe

# %% Data:

x_array = np.arange(2000, 2019)
y_obs_array = np.array([54.8, 56.1, 55.0, 55.7, 56.2,
                      55.4, 55.3, 57.0, 55.6, 53.2,
                      55.5, 54.6, 54.1, 54.0, 54.1,
                      54.4, 53.6, 52.7, 52.9])

# %% Initialization:

N_resol_param = 1000

a_lb = -0.3
a_ub = 0
a_array = np.linspace(a_lb, a_ub, N_resol_param)

b_lb = 300
b_ub = 500
b_array = np.linspace(b_lb, b_ub, N_resol_param)

sspe_min = np.inf
a_opt = 0
b_opt = 0

# %% Starting the timer:

tic = time.time()

# %% Grid optimization:

for a in a_array:

    for b in b_array:

        # Calc of objective function:
        params = np.array([a, b])
        sspe = fun_obj(params)

        # Improvement of solution:
        if (sspe < sspe_min):
            sspe_min = sspe
            a_opt = a
            b_opt = b

# %% Stopping the timer:

toc = time.time()
t_elapsed = toc-tic
```

```
# %% Presentation of result:

print(f'a_opt = {a_opt:.3e}')
print(f'b_opt = {b_opt:.3e}')
print(f'sspe_min = {sspe_min:.3e}')
print(f'Elapsed time = {t_elapsed:.3e}')


# %% Plotting:

y_pred = a_opt*x_array + b_opt

plt.close('all')
plt.figure(num=1, figsize=(12, 9))

plt.plot(x_array, y_obs_array, 'ro')
plt.plot(x_array, y_pred, 'b-')
plt.xlim(2000, 2018)
plt.xlabel('x [year]')
plt.ylabel('[mill tons CO2-equiv]')
plt.grid()

plt.legend(labels=('y_obs', 'y_pred'),)

plt.savefig('prog_optim_grid_greenhouse_gas.pdf')
plt.show()
```

In the program, I have used the following value ranges of a and b :

- Value range of a : Since a is the slope of the linear function (11.12), we can estimate a value of a as the slope between two appropriate data point in Table 11.1. I select

$$a \approx \frac{52.9 - 56.1}{2018 - 2001} = -0.188 \quad (11.16)$$

Now, the value range of a may be set as

$$-0.3 \leq a \leq 0 \quad (11.17)$$

- Value range of b : Let us take e.g. the data set (2018, 52.9): By setting $y = 52.9$ and $x = 2018$ in (11.8) with $a = -0.188$, we get an estimate of b as

$$b = y - ax = 52.9 - (-0.188) \cdot 2018 = 432 \quad (11.18)$$

Now, the value range of b may be set as

$$300 \leq b \leq 500 \quad (11.19)$$

Since we have only two optimization variables (a and b), we can allow a fairly high resolution of the optimization variables. I set $N_{\text{resolution}} = 1000$ for both a and b , which gives 1000000 calculations, which run fast on a PC.

The result:

```
a_opt = -0.1553
b_opt = 366.7
sspe_min = 11.75
```

In Example 11.4, a og b are calculated with an exact implementation of the least squares method. The result, which can be regarded as the exact or true optimum, is $a = -1.517$ and $b = 359.6$, see Table 11.2 which shows the results with a number of optimization methods. The grid estimates shown above are in good accordance with the exact values.

Figure 11.6 shows the observations y_{obs} from Table 11.1 and the prediction y_{pred} calculated with the model (11.8) with the optimal parameter values, a_{opt} and b_{opt} , inserted.

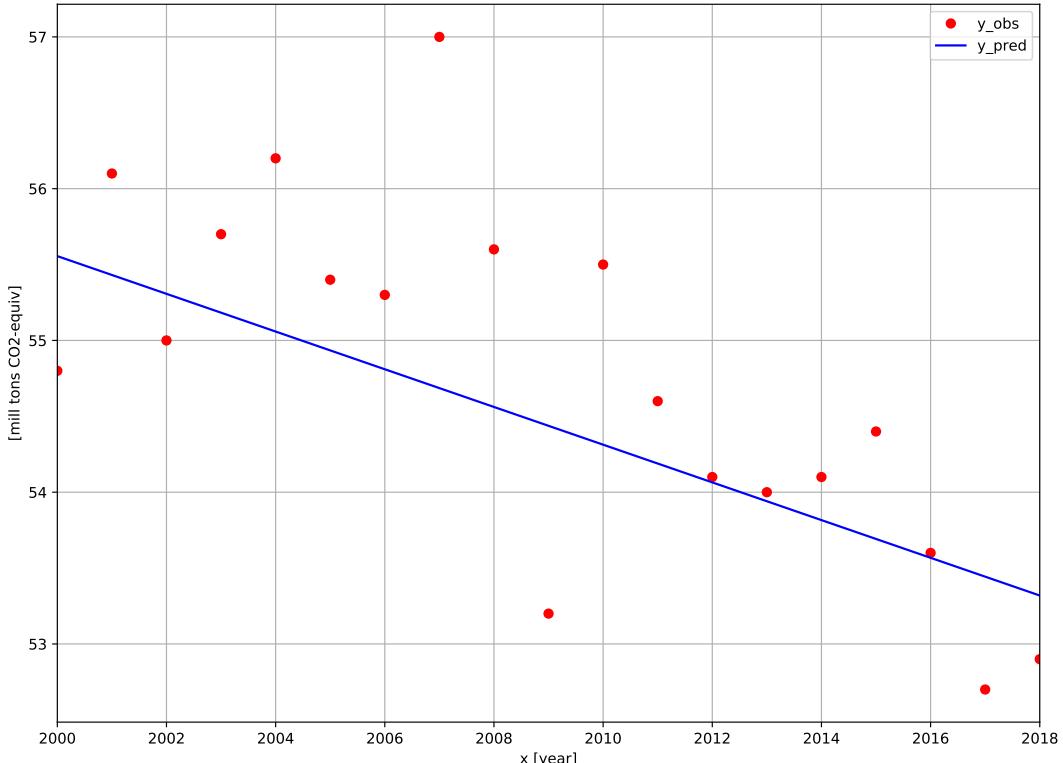


Figure 11.6: Plot of observations (y_{obs}) and prediction (y_{pred})

Models can be used, and abused. Let us calculate the year in which Norway does not emit greenhouse gases: We set $y = 0$ in (11.8):

$$0 = ax + b$$

which gives

$$x = -\frac{b}{a} = -\frac{366.7}{-0.155} = 2366$$

– a trustworthy result?

[End of Example 11.1]

11.3.1.3 Model adaptation of static models with Python's brute() function

As alternative to implementing the grid method from scratch, Python has a built-in optimization function based on the grid method (brute force method), namely `scipy.optimize.brute()`. You may be able to save some programming time using this feature compared to programming the grid method from scratch, but probably not much. And the execution time is almost the same.

However, `scipy.optimize.brute()` has one strong feature which is offered as an option: It can invoke an accurate iterative optimization method, namely a Nelder-Mead algorithm (also denoted downhill simplex algorithm), following after the grid optimization, to find an accurate optimal solution (i.e. a minimum). The starting point of this Nelder-Mead algorithm is the optimal solution that `brute()` function found from its built-in grid method. So, this option gives a final improvement of the optimal solution. Figure 11.7 illustrates this.

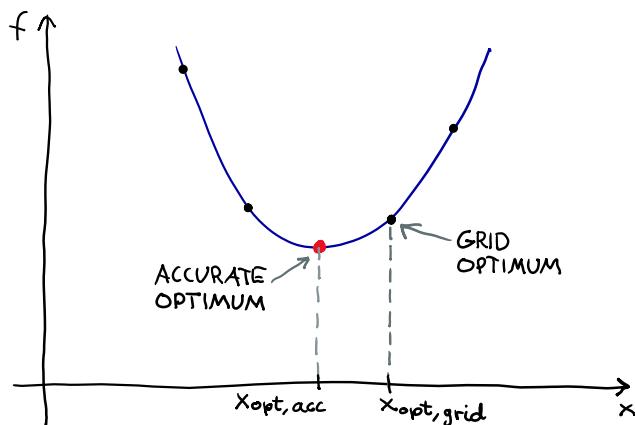


Figure 11.7: The starting point of the Nelder-Meads algorithm is the optimal solution that `brute()` function found from its built-in grid method.

Example 11.2 Adaptation of a static model to data using Python's brute() function

This example is based on the data and model used in Example 11.1.

In the present example I will use Python's `scipy.optimize.brute()` function which implements

the grid method, cf. Section 11.3.1.1. The standard use of brute() gives exactly the same result as a grid method implemented from scratch.

The brute() function has an option to obtain an accurate optimal estimate with the Nelder-Mead optimization algorithm started from the grid optimal solution. The options – without and with final optimization – can be selected as follows:

- With input argument finish=None to the brute() function, only grid optimization is calculated.
- With input argument finish=optimize.fmin to the brute() function, the Nelder-Mead algorithm is run with the grid solution as starting point.

I try both options in this example.

Program 11.2 implements the grid method using the brute() function.

http://techteach.no/control/python/prog_scipy_brute_estim_co2.py

Listing 11.2: prog_scipy_brute_estim_co2.py

```
# %% Import of packages:

import numpy as np
from scipy import optimize
import time

# %% Definition of objective function:

def fun_obj(params):
    a = params[0]
    b = params[1]
    y_pred_array = a*x_array + b
    e_array = y_obs_array - y_pred_array
    sspe = sum(e_array*e_array)
    return sspe

# %% Data:

x_array = np.arange(2000, 2019)
y_obs_array = np.array([54.8, 56.1, 55.0, 55.7, 56.2,
                      55.4, 55.3, 57.0, 55.6, 53.2,
                      55.5, 54.6, 54.1, 54.0, 54.1,
                      54.4, 53.6, 52.7, 52.9])

# %% Initialization:

N_resol_param = 100
```

```
a_lb = -0.3
a_ub = 0
a_step = (a_ub - a_lb)/(N_resol_param - 1)

b_lb = 300
b_ub = 500
b_step = (b_ub - b_lb)/(N_resol_param - 1)

params_ranges = (slice(a_lb, a_ub, a_step),
                  slice(b_lb, b_ub, b_step))

# %% Starting timer to calculate execution time:

tic = time.time()

# %% Solving the optim problem with optimize.brute():

# finish_setting = None
finish_setting = optimize.fmin # Nelder-Mead optim

result_optim = optimize.brute(fun_obj, params_ranges,
                               full_output=True,
                               finish=finish_setting)
params_optim = result_optim[0]
sspe_min = result_optim[1]

# %% Optimal parameter values:

a_opt = params_optim[0]
b_opt = params_optim[1]

# %% Stopping timer:

toc = time.time()
t_elapsed = toc-tic

# %% Presentation of result:

print('a_opt =', f'{a_opt:.3e}')
print('b_opt =', f'{b_opt:.3e}')
print('sspe_min =', f'{sspe_min:.3e}')
print('Elapsed time =', f'{t_elapsed:.3e}')
```

The result is shown in Table 11.2, which also shows other results for comparison. Also, the execution time as measured with the `time()` function of the `time` module is shown in the table.

Comments to the results shown in Table 11.2:

- The Python's `brute()` function with $N = 100$ and the option of accurate optimization with the Nelder-Mead algorithm activated, produces an accurate optimal solution,

Table 11.2: Results of parameter estimation of greenhouse gas emission model obtained with alternative optimization methods. The execution times, t , may vary substantially between computers, but their ratios should not differ so much.

Method	$N_{\text{resolution}}$	a	b	sspe	t [s]
Grid optim in native code (Example 11.1)	1000	-1.553	366.7	11.75	~ 12.8
Grid optim in native code (Example 11.1)	100	-0.1242	304.0	14.00	~ 0.12
Scipy's brute() without final optim (Ex. 11.2)	1000	-1.553	366.7	11.75	~ 13.2
Scipy's brute() without final optim (Ex. 11.2)	100	-0.1242	304.0	14.00	~ 0.14
Scipy's brute() with final optim (Ex. 11.2)	100	-0.1518	359.6	11.74	~ 0.14
Scipy's minimize(); Nelder-Mead solver (Ex. 11.3)	—	-0.1518	359.6	11.74	~ 0.005
LS (exact optimum) in native code (Ex. 11.4)	—	-0.1518	359.6	11.74	~ 0

and with almost 1/100 of execution time comparing with the brute() function with $N = 1000$ which gives comparable, however somewhat less accurate, results.

- The least squares (LS) method is extremely fast. A limitation of the LS method is that it works only on models which are linear in its parameters.

Comments to program 11.2:

- The function fun_sspe defines the function which calculates (returns) the optimization objective to be minimized.
- The list theta = [a, b] represents the parameters to be estimated. theta is the input argument to fun_sspe.
- fun_sspe is used by the brute() function which adjusts theta until a minimum of sspe is found.
- The brute() function returns the results of the optimization in a tuple named result_optim.
- The optimal solution, i.e. the optimal parameters, are the elements of the array theta_optim which is the element with index 0 in the tuple named result_optim.
- The minimum of the optimization objective is the element with index 1 in the tuple named result_optim.

[End of Example 11.2]

11.3.2 Adaptation of static models using nonlinear programming (NLP)

An nonlinear programming (NLP) solver can find the optimum (minimum) of an objective function which is a nonlinear function of the optimization variables.

Example 11.3 *Adaptation of a static model to CO₂ data using an NLP solver of Scipy*

Table 11.1 in Example 11.1 presents CO₂ emission data in Norway over the years 2000-2018. In that example, I used the grid method of optimization to estimate parameters a and b in this linear model:

$$y = ax + b \quad (11.20)$$

where: y are CO₂ emissions and x is year. How will the estimation go if I instead use an NLP solver? The Python program 11.3 estimates a and b using the `scipy.optimize.minimize()` function in the Scipy package of Python. There are several alternative solvers that the `minimize()` function can use. In program 11.3 I have selected the Nelder-Mead solver (also denoted the downhill simplex method) which finds the minimum after a kind of geometrical search for minimum based on subsequent calculations of objective function values.

http://techteach.no/control/python/prog_scipy_nelder_mead_adapt_static_model_co2.py

Listing 11.3: `prog_scipy_nelder_mead_adapt_static_model_co2.py`

```
# %% Import:
import numpy as np
import scipy.optimize

# %% Defining functions:

def fun_objective(params):
    a = params[0]
    b = params[1]
    co2_pred_array = a*year_array + b
    e_array = co2_obs_array - co2_pred_array
    sspe = sum(e_array*e_array)
    return sspe

# %% Data:
year_array = np.arange(2000, 2019)
co2_obs_array = np.array([54.8, 56.1, 55.0, 55.7, 56.2,
                        55.4, 55.3, 57.0, 55.6, 53.2,
                        55.5, 54.6, 54.1, 54.0, 54.1,
                        54.4, 53.6, 52.7, 52.9])

# %% Guessed values (initial values) of optim variables:
a_guess = -0.188
```

```

b_guess = 432
x_guess = np.array([a_guess, b_guess])

# %% Solving the optim problem:

res = scipy.optimize.minimize(fun_objective, x_guess,
                               method = 'nelder-mead',
                               options = {'ftol': 1e-9, 'disp': True})

# %% The results of the optimization:

(a_optim, b_optim) = res.x
sspe_optim = res.fun

# %% Displaying the optimal solution:

print('Optimal estimates:')
print('a_optim =', f'{a_optim:.3e}')
print('b_optim =', f'{b_optim:.3e}')
print(f'sspe_optim = {sspe_optim:.3e}')

```

As guessed or initial values for search, I selected the same values as (11.16) and (11.18) as derived in Example 11.1, namely:

$$a_{\text{guess}} = -0.188 \quad (11.21)$$

$$b_{\text{guess}} = 432 \quad (11.22)$$

The results of the estimation are:

$$a_{\text{est}} = -0.1518 \quad (11.23)$$

$$b_{\text{est}} = 359.6 \quad (11.24)$$

with

$$\text{SSPE} = 11.74 \quad (11.25)$$

These results are great as they are equal to the correct values as found with the Least Squares method – with the selected number of digits.

I have also tried the solver named “slsqp”(Sequential Least Squares Programming) in `minimize()`. It turned out that the resulting optimal values, which are the estimates, were quite sensitive to the selection of the guessed parameter values, i.e. the initial or starting point, used by the slsqp algorithm. For some guessed values, the results were accurate, while for other guessed values, the results were quite inaccurate.

[End of Example 11.3]

11.3.3 Adaptation of static models using standard least squares method

11.3.3.1 The standard regression model

Assume given the following model:

$$y = \varphi_1\theta_1 + \cdots + \varphi_n\theta_n \quad (11.26)$$

$$= \underbrace{\begin{bmatrix} \varphi_1 & \cdots & \varphi_n \end{bmatrix}}_{\phi} \underbrace{\begin{bmatrix} \theta_1 \\ \vdots \\ \theta_n \end{bmatrix}}_{\theta} \quad (11.27)$$

$$= \phi\theta \quad (11.28)$$

which is called the regression model. In (11.28):

- φ_i is the regression variable (with known value).
- ϕ is the regression vector (with known value).
- y is the observed variable (with known value).
- θ is an unknown parameter vector to be estimated with the LS-method.

Note that the regression model is *linear in the parameter vector θ* . If the model is nonlinear, you can not use the LS method. Instead, you can use some optimization method for nonlinear models, e.g. the grid method, cf. Ch. 11.3.1, or nonlinear programming, cf. Ch. 11.4.2.

Assume that we have m corresponding values of y and ϕ . Then we can write the following m equations according to the model:

$$y_1 = \varphi_{11}\theta_1 + \cdots + \varphi_{1n}\theta_n = \phi_1\theta$$

$$\vdots$$

$$y_i = \varphi_{i1}\theta_1 + \cdots + \varphi_{in}\theta_n = \phi_i\theta$$

$$\vdots$$

$$y_m = \varphi_{m1}\theta_1 + \cdots + \varphi_{mn}\theta_n = \phi_m\theta$$

These m “stacked” equations can more compactly be written as

$$\underbrace{\begin{bmatrix} \vdots \\ y_i \\ \vdots \end{bmatrix}}_Y = \underbrace{\begin{bmatrix} \vdots \\ \phi_i \\ \vdots \end{bmatrix}}_\phi \theta \quad (11.29)$$

or just:

$$Y = \Phi\theta \quad (11.30)$$

where Y and Φ consist of known data and θ is the unknown parameter of which we will calculate or estimate a value θ using the LS-method.²

11.3.3.2 The LS problem

We define the *equation-error vector* or *prediction-error vector*³, E , as the difference between the left side and the right side of (11.30):

$$E = \begin{bmatrix} \vdots \\ e_i \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ y_i - \varphi_i\theta \\ \vdots \end{bmatrix} = Y - \Phi\theta \quad (11.31)$$

Figure 11.8 illustrates the equation-errors or prediction errors for the case of the model

$$y = \phi\theta \quad (11.32)$$

to be fitted to two data points.

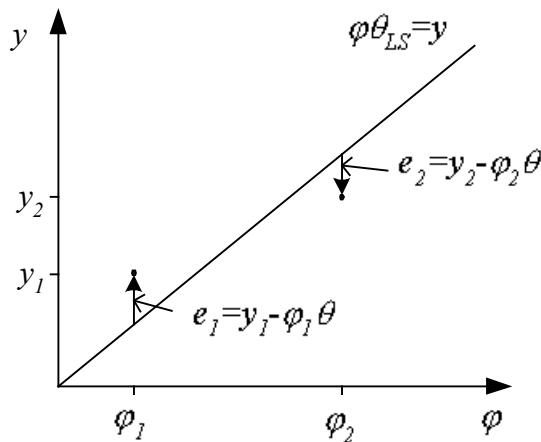


Figure 11.8: Equation-errors or prediction errors e_1 and e_2 for a simple case.

The problem is to calculate a value – an estimate – of the unknown parameter-vector θ so

²In mathematical literature, (11.30) is more often written on the form $b = Ax$. I have used symbols which are common in the field of system identification.

³The name prediction-error vector is because the term $\Phi\theta$ can be regarded as a prediction of the observed (known) “output” y .

that the following quadratic criterion function, $V(\theta)$, is minimized:

$$V(\theta) = e_1^2 + e_2^2 + \cdots + e_m^2 \quad (11.33)$$

$$= E^T E \quad (11.34)$$

$$= (Y - \Phi\theta)^T (Y - \Phi\theta) \quad (11.35)$$

$$= (Y^T - \theta^T \Phi^T) (Y - \Phi\theta) \quad (11.36)$$

$$= Y^T Y - Y^T \Phi\theta - \theta^T \Phi^T Y + \theta^T \Phi^T \Phi\theta \quad (11.37)$$

In other words, the problem is to estimate θ so that the sum of quadratic prediction errors is minimized.

11.3.3.3 The LS solution

Since $V(\theta)$ is a quadratic function of the unknown parameters θ , the minimum value of $V(\theta)$ can be calculated by setting the derivative of V with respect to θ equal to zero. This is illustrated in Figure 11.9.

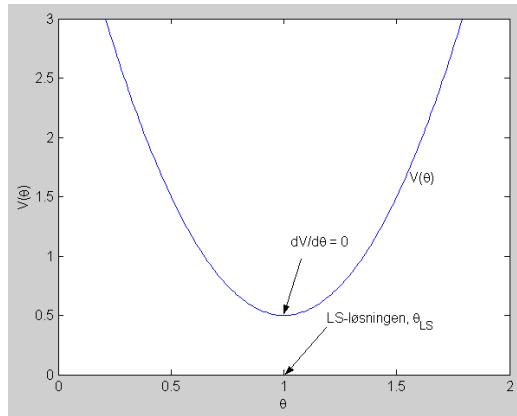


Figure 11.9: The LS solution θ_{est} corresponds to the minimum value of the quadratic function $V(\theta)$, and can be calculated by setting the derivative $V(\theta)/d\theta$ to zero.

Using the differentiation rule (44.5) on (11.37), and then setting the derivative equal to zero, gives

$$\frac{dV(\theta)}{d\theta} = 2\Phi^T \Phi\theta - 2\Phi^T Y \stackrel{!}{=} 0 \text{ (vector)} \quad (11.38)$$

or

$$\Phi^T \Phi\theta = \Phi^T Y \quad (11.39)$$

(11.39) is called the normal equation. The θ -solution of (11.39) can be found by pre-multiplying (11.39) with $(\Phi^T \Phi)^{-1}$. The result is

$$\theta_{\text{est}} = (\Phi^T \Phi)^{-1} \Phi^T Y \quad (11.40)$$

which is the *LS-solution* of (11.30)⁴. All right side terms in (11.30) are known. We may denote (11.40) as the “batch LS formula”.

⁴ $(\Phi^T \Phi)^{-1} \Phi^T$ is the so-called pseudo-inverse of Φ .

Note: To apply the LS-method, the model must be written on the regression model form (11.30), which consists of m (11.28) “stacked”.

Example 11.4 LS-estimation of parameters of a greenhouse gas emission model

In Example 11.1 we estimated the a linear model to the given data using the grid optimization method. The data are shown in Figure 11.6 in Example 11.1. The assumed model is

$$y = ax + b \quad (11.41)$$

Since this model is linear in its parameters a and b , we can estimate the parameters a and b using the LS method.

We start by writing the model on the standard regression form:

$$y = ax + b = \begin{bmatrix} x, & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \varphi\theta \quad (11.42)$$

The values of y and x are given in Table 11.1.

The estimate is given by (11.40), which is repeated here:

$$\theta_{\text{LS}} = (\Phi^T \Phi)^{-1} \Phi^T Y \quad (11.43)$$

In (11.43):

$$Y = \begin{bmatrix} y_0 \\ \vdots \\ y_{18} \end{bmatrix} = \begin{bmatrix} 54.8 \\ \vdots \\ 52.9 \end{bmatrix}$$

$$\Phi = \begin{bmatrix} \varphi_0 \\ \vdots \\ \varphi_{18} \end{bmatrix} = \begin{bmatrix} x_0 & 1 \\ \vdots & \vdots \\ x_{18} & 1 \end{bmatrix} = \begin{bmatrix} 2000 & 1 \\ \vdots & \vdots \\ 2018 & 1 \end{bmatrix}$$

Program 11.4 implements the above calculations.

The results are:

```
a_est = -1.518e-01
b_est = 3.596e+02
sspe = 1.174e+01
```

The results are also shown in Table 11.2 togheter with the results obtained with several other optimization methods.

Comparing with the grid method and the brute() function, one benefit of the LS method is that there is no need to define any range of the values of the parameters to be estimated. A drawback of the LS method is that the model has to be linear in the parameters, i.e. the LS method is selective regarding applicable models.

http://techteach.no/control/python/prog_ls_native_estim_co2.py

Listing 11.4: prog_ls_native_estim_co2.py

```
# %% Import of packages:  
  
import numpy as np  
  
# %% Data:  
  
x_obs_array = np.arange(2000, 2019)  
y_obs_array = np.array([54.8, 56.1, 55.0, 55.7, 56.2,  
                      55.4, 55.3, 57.0, 55.6, 53.2,  
                      55.5, 54.6, 54.1, 54.0, 54.1,  
                      54.4, 53.6, 52.7, 52.9])  
  
# %% Regression model:  
  
Phi_col_0 = np.array(x_obs_array)  
Phi_col_1 = np.array(np.ones(19))  
Phi = np.array([Phi_col_0, Phi_col_1]).T  
Y = np.array([y_obs_array]).T  
  
# %% Calculating LS estimate:  
  
theta = np.linalg.inv(Phi.T @ Phi) @ Phi.T @ Y  
a_est = theta[0, 0]  
b_est = theta[1, 0]  
  
# %% Calculating value of objective function:  
  
y_pred_array = a_est*x_obs_array + b_est  
pe = y_obs_array - y_pred_array # prediction error  
sspe = sum(pe*pe)  
  
# %% Presentation of result:  
  
print('a_est =', f'{a_est:.3e}')  
print('b_est =', f'{b_est:.3e}')  
print('sspe =', f'{sspe:.3e}')
```

Comments to Program 11.4:

- The LS estimate () is calculated with the code `theta = np.linalg.inv(Phi.T @ Phi) @ Phi.T @ y` which implements the batch LS formula () .
- The `np.linalg.inv()` function belongs to the `linalg` module in the `numpy` package.
- The `@` operator performs matrix multiplication of matrices represented as 2D-arrays in Python.

LS estimate with the `np.linalg.lstsq()` in Numpy

Of course, there is a built-in function in Python which implements least squares estimation, namely `lstsq()` in Numpy. Program 11.5 uses this function to estimate a and b in the CO₂ model. The results are as with our native LS code:

```
a_est = -1.518e-01
b_est = 3.596e+02
sspe = 1.174e+01
```

http://techteach.no/control/python/prog_lstsq_params_estim_co2.py

Listing 11.5: `prog_lstsq_params_estim_co2.py`

```
# %% Import of packages:

import numpy as np

# %% Data:

x_obs_array = np.arange(2000, 2019)
y_obs_array = np.array([54.8, 56.1, 55.0, 55.7, 56.2,
                      55.4, 55.3, 57.0, 55.6, 53.2,
                      55.5, 54.6, 54.1, 54.0, 54.1,
                      54.4, 53.6, 52.7, 52.9])

# %% Regression model:

phi_0 = np.array(x_obs_array)
phi_1 = np.array(np.ones(19))
Phi = np.array([phi_0, phi_1]).T
Y = np.array([y_obs_array]).T

# %% Results of LS estimate with lstsq() function:

ls_result = np.linalg.lstsq(Phi, Y, rcond=None)
theta = ls_result[0]
sspe_array = ls_result[1]

a_est = theta[0][0]
b_est = theta[1][0]
sspe = sspe_array[0]

# %% Presentation of results:

print('a_est =', f'{a_est:.3e}')
print('b_est =', f'{b_est:.3e}')
print('sspe =', f'{sspe:.3e}')
```

[End of Example 11.4]

11.3.3.4 Properties of the LS estimate

The variance of θ_{LS} is:

$$\text{Var}(\theta_{\text{est}}) \equiv \text{diag} \left\{ \text{Cov}(\theta_{\text{est}}) \equiv E \left[(\theta_{\text{est}} - \theta_0)(\theta_{\text{est}} - \theta_0)^T \right] = \sigma_e^2 (\Phi^T \Phi)^{-1} \right\} \quad (11.44)$$

where sub-index “e” represents prediction (or model) error. θ_0 are the “true” parameter values. σ_e^2 is the variance of the measurement noise (or model error). What do we mean with “variance of θ_{LS} ”? It is the variance of a large (ideally: infinite) number of θ_{LS} estimates where each θ_{LS} is calculated with some realization of the random model error e .

If you do not know the value of σ_e^2 , you can *estimate* it with:

$$\sigma_e^2 = \frac{V(\theta_{\text{est}})}{N - n} \quad (11.45)$$

where: N is the number of observations. n is the number of parameters to be estimated. $V(\theta_{\text{est}})$ is given by (11.37), i.e.:

$$V(\theta_{\text{est}}) = (Y_{\text{obs}} - \Phi \theta_{\text{est}})^T (Y_{\text{obs}} - \Phi \theta_{\text{est}}) \quad (11.46)$$

You can use (11.45) together with (11.46) to calculate confidence intervals of θ_{est} if you assume that θ_{est} is normally distributed. For example, as you may know from the field of statistics, the 95 % confidence interval of a random estimate θ_{est} given the known, or estimated σ_e given by (11.45), is

$$\theta_{\text{est}} - 2\sigma_e \leq \theta_{\text{est}} \leq \theta_{\text{est}} + 2\sigma_e \quad (11.47)$$

The 95 % confidence interval is the interval in which there is 95% probability that the true parameter values is, assuming that the statistical properties of the given data (time series) is representative for data from any other experiment.

Note that the confidence interval does not express any certainty of the calculated estimate itself. Actually, there is no uncertainty at all related to that estimate as the estimate is just a number calculated from the observations with given formulas.

11.3.3.5 Criterion for convergence of estimate towards the true value

The prediction-error vector is

$$E = Y - \Phi \theta_{\text{est}} \quad (11.48)$$

It can be shown that the LS-estimate θ_{LS} converges towards the true value θ_0 of the parameter vector as the number m of sets of observations goes to infinity, only if E is so-called white noise, which is defined in Ch. 33.3.1. White noise means that the elements of E are random numbers, and the vector E has zero mean value. The “opposite” of white noise is coloured noise. E becomes coloured if there are systematic equation errors. Such systematic equation errors can be reduced or eliminated by choosing a more accurate model structure.

A special example of coloured noise is a constant or bias, say b , having value different from zero. For example, a model like (11.49) includes the bias, b .

$$y = ax + b \quad (11.49)$$

11.4 Adaptation of dynamic models to data

11.4.1 Adaptation of dynamic models using grid optimization

11.4.1.1 Introduction

The grid method of optimization can be used for adaptation of dynamic models to given data. In Python, you can program the grid method from scratch as shown in Example 11.5, or using the built-in function `scipy.optimize.brute()` of the Scipy package, as shown in Example 11.6.

11.4.1.2 Adaptation of dynamic models using grid optimization

Example 11.5 DC motor model adaptation to simulated data with grid optimization

Figure 11.10 shows a DC-motor with tachometer which is presented in Ch. 41.6.

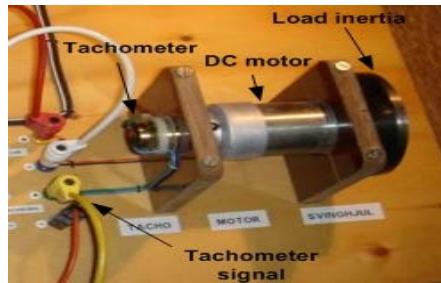


Figure 11.10: DC motor.

One reasonable mathematical model of the motor is the following time constant model:

$$TS' = K[u + L] - S \quad (11.50)$$

where: S [krpm = kilo revolutions per minute] is the rotational speed. u [V] is the control signal to the motor. L [V] is a voltage which represents the load torque acting on the motor. K [krpm/V] is the motor gain. T [s] is the motor time constant.

We will estimate K and T from a simulated experiment. We assume for simplicity that both L and the initial state S_{init} are known and have values zero, although we may estimate both L and S_{init} .

The objective function to be minimized, is:

$$\text{sspe} = \sum_{k=1}^N e_k^2 \quad (11.51)$$

where:

$$e_k = S_{\text{obs},k} - S_{\text{pred},k} \quad (11.52)$$

where $S_{\text{obs},k}$ is here the observed (but simulated) S , and $S_{\text{pred},k}$ is the predicted S .

We use grid optimization with the following intervals of K and T :

$$K \in [0.05, \dots, 0.55] \quad (11.53)$$

$$T \in [0.1, \dots, 1.1] \quad (11.54)$$

We assume that the true values of the model parameters are:

$$K_{\text{true}} = 0.15 \quad (11.55)$$

$$T_{\text{true}} = 0.30 \quad (11.56)$$

In the simulated experiment we vary the control signal, u , as an up-down-up-signal.

The Python program 11.6 implements the parameter estimation. Firstly, the program generates a simulated “experimental” data series, and then uses this data series as the basis of the parameter estimation.

http://techteach.no/control/python/prog_grid_motor_K_T_estim_sim.py

Listing 11.6: prog_grid_motor_K_T_estim_sim.py

```
# %% Import of packages:

import matplotlib.pyplot as plt
import numpy as np

# %% Definition of objective function:

def fun_calc_objfun(params, S_init, S_obs_array,
                     u_array, N, Ts):

    (K, T) = params
    S_pred_k = S_init
    pe_array = np.zeros(N)

    for k in range(0, N):

        # Simulation algorithm (Euler step):
        dS_sim_Ts_k = (1/T)*(-S_pred_k + K*u_array[k])
        S_pred_kp1 = S_pred_k + Ts*dS_sim_Ts_k
```

```
# Updating prediction error (pe):
pe_array[k] = S_obs_array[k] - S_pred_k

# Time shift:
S_pred_k = S_pred_kp1

sspe = sum(pe_array*pe_array)
return sspe

# %% Definition of simulation function:

def fun_sim(params, S_init, u_array, N, Ts):

    (K, T) = params
    S_sim_k = S_init
    S_sim_array = np.zeros(N)

    for k in range(0, N):
        # Simulation algorithm (Euler step):
        dS_sim_Ts_k = (1/T)*(-S_sim_k + K*u_array[k])
        S_sim_kp1 = S_sim_k + Ts*dS_sim_Ts_k
        S_sim_array[k] = S_sim_k
        S_sim_k = S_sim_kp1 # Time shift

    return S_sim_array

# %% Time settings:

t_start = 0 # [s]
t_stop = 7
Ts = 0.02
N = int(((t_stop - t_start)/Ts)) + 1
t_array = np.linspace(t_start, t_stop, N)

# %% Create input signal:

u_array = np.zeros(N)

for k in range(N):
    t_k = k*Ts
    if t_start <= t_k < 1: u_array[k] = 0
    elif 1 <= t_k < 3: u_array[k] = 1
    elif 3 <= t_k < 5: u_array[k] = -1
    else: u_array[k] = 0

# %% Creating simulated observation data:

K_true = 0.15
T_true = 0.30
params = (K_true, T_true)
S_init = 0
```

```
S_sim_array = fun_sim(params, S_init, u_array, N, Ts)
S_obs_array = S_sim_array

# %% Arrays of candidates of estimated param values:
N_params = 10

K_lb = 0.05 # Lower bound
K_ub = 0.55 # Upper bound

T_lb = 0.1
T_ub = 1.1

K_array = np.linspace(K_lb, K_ub, N_params)
T_array = np.linspace(T_lb, T_ub, N_params)

# %% For loop implementing estimation with grid optim:
sspe_optim = np.inf # Initialization of sspe

for K in K_array:

    for T in T_array:

        params = (K, T) # params with candidate K and T

        #Calculating objective function (sspe):
        sspe = fun_calc_objfun(params, S_init,
                               S_obs_array,
                               u_array, N, Ts)

        #Improving the previous optim solution:
        if sspe <= sspe_optim:
            sspe_optim = sspe
            K_est = K
            T_est = T

# %% Displaying the optimal solution = param estimates:
print('K_true =', f'{K_true:.4f}')
print('T_true =', f'{T_true:.4f}')
print('K_est =', f'{K_est:.4e}')
print('T_est =', f'{T_est:.4e}')
print('sspe_optim =', f'{sspe_optim:.7f}')

# %% Simulation with estimated parameters in the model:
S_init = 0

params = (K_est, T_est)
S_sim_adapted_model_array = fun_sim(params, S_init,
                                      u_array, N, Ts)

# %% Plotting:
```

```

plt.close("all")
plt.figure(1, figsize=(12, 9))

plt.subplot(2, 1, 1)
plt.plot(t_array, S_obs_array, 'b', label='S_obs')
plt.plot(t_array, S_sim_adapted_model_array, 'r',
         label='S_sim')
plt.grid()
plt.xlabel('t [s]')
plt.ylabel('[krpm]')
plt.legend()

plt.subplot(2, 1, 2)
plt.plot(t_array, u_array, 'g', label='u')
plt.grid()
plt.xlabel('t [s]')
plt.ylabel('[V]')
plt.legend()

plt.savefig('plot_grid_K_T_dcmotor_simdata.pdf')
plt.show()

```

Figure 11.11 shows a plot of the control signal u and the simulated speed S .

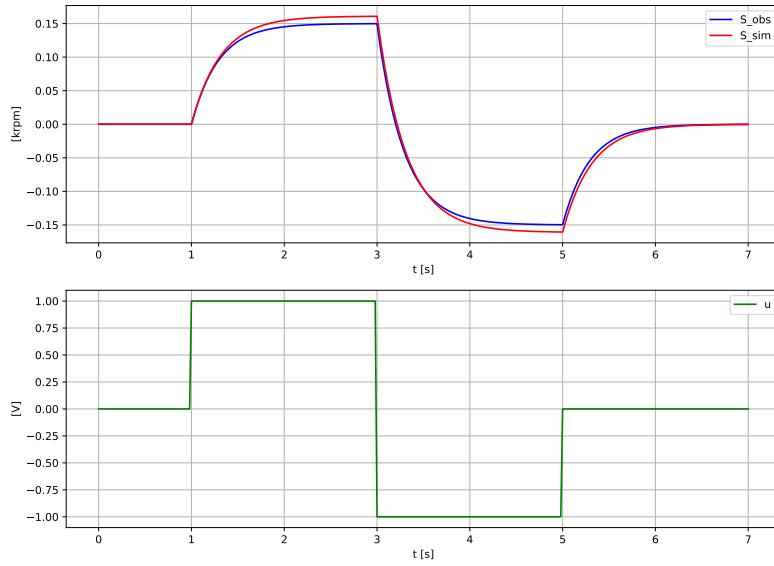


Figure 11.11: Plots of the control signal u and the simulated speed S used in the parameter estimation.

Table 11.3 shows the result of the parameter estimation with two different resolutions of the parameters K and T .

Table 11.3: Results of estimation of parameters K and T using grid optimization.

Parameter	$N_{\text{resolution}}$	K [krpm/V]	T [s]	sspe
True value	—	0.150	0.300	—
Estimated	10	0.161	0.322	0.0164
Estimated	100	0.151	0.302	0.000137

Above, the true values of K and T are not among the candidate values. If we had ensured that the true values were among the candidate values, the estimated values by the grid method would have been these true values.

[End of Example 11.5]

11.4.1.3 Model adaptation of dynamic models with Python's `brute()` function

In Example 11.5, the grid method was implemented from scratch. Alternatively, we can use the `scipy.optimize.brute()` function of the Python Scipy package which implements the grid method, cf. Section 11.3.1.3.

Example 11.6 DC motor model adaptation to simulated data with the `scipy.optimize.brute()` function

The parameter estimation problem is as in Example 11.5.

The intervals of K and T are the same as in Example 11.5, namely:

$$K \in [0.05, \dots, 0.55] \quad (11.57)$$

$$T \in [0.1, \dots, 1.1] \quad (11.58)$$

The Python program 11.7 below implements the parameter estimator. A resolution of 10 is selected for each of the parameters K and T . The `brute()` function is set to run a final Nelder-Mead optimization algorithm to obtain an accurate optimal solution (optimal parameter estimates). Example 11.2 explains how to set this option.

http://techteach.no/control/python/prog_scipy_brute_motor_K_T_estim_sim.py

Listing 11.7: `prog_scipy_brute_motor_K_T_estim_sim.py`

```
# %% Import of packages:

import numpy as np
from scipy import optimize

# %% Definition of functions:

def fun_objective(x):
```

```

K = x[0]
T = x[1]
S_pred_k = S_init
pe_array = np.zeros(N)

for k in range(0, N):

    # Simulation algorithm (Euler step):
    dS_sim_Ts_k = (1/T)*(-S_pred_k + K*u_array[k])
    S_pred_kp1 = S_pred_k + Ts*dS_sim_Ts_k

    # Updating prediction error (pe):
    pe_array[k] = S_obs_array[k] - S_pred_k

    # Time shift:
    S_pred_k = S_pred_kp1

sspe = sum(pe_array*pe_array)
return sspe

def fun_sim(modelparams, S_init, u_array, N, Ts):

    (K, T) = modelparams
    S_sim_k = S_init
    S_sim_array = np.zeros(N)

    for k in range(0, N):
        # Simulation algorithm (Euler step):
        dS_sim_Ts_k = (1/T)*(-S_sim_k + K*u_array[k])
        S_sim_kp1 = S_sim_k + Ts*dS_sim_Ts_k
        S_sim_array[k] = S_sim_k
        S_sim_k = S_sim_kp1 # Time shift

    return S_sim_array

# %% Time settings:

t_start = 0 # [s]
t_stop = 7
Ts = 0.02
N = int(((t_stop - t_start)/Ts)) + 1
t_array = np.linspace(t_start, t_stop, N)

# %% Create input signal:

u_array = np.zeros(N)

for k in range(N):
    t_k = k*Ts
    if t_start <= t_k < 1: u_array[k] = 0
    elif 1 <= t_k < 3: u_array[k] = 1
    elif 3 <= t_k < 5: u_array[k] = -1
    else: u_array[k] = 0

```

```
# %% Creating simulated observation data:  
  
K_true = 0.15  
T_true = 0.3  
params = (K_true, T_true)  
S_init = 0  
  
S_sim_array = fun_sim(params, S_init, u_array, N, Ts)  
S_obs_array = S_sim_array  
  
# %% Creating arrays of candidate parameter values:  
  
N_resolution_params = 10  
  
K_ub = 0.55 # Upper bound  
K_lb = 0.05 # Lower bound  
K_step = (K_ub - K_lb)/(N_resolution_params - 1)  
  
T_ub = 1.1  
T_lb = 0.1  
T_step = (T_ub - T_lb)/(N_resolution_params - 1)  
  
x_ranges = (slice(K_lb, K_ub, K_step),  
            slice(T_lb, T_ub, T_step))  
  
# %% Solving the optim problem with optimize.brute():  
  
# Options for the finish argument:  
finish_setting = optimize.fmin  
# finish_setting = None  
  
result_est = optimize.brute(fun_objective, x_ranges,  
                           full_output=True,  
                           finish=finish_setting)  
  
params_optim = result_est[0]  
sspe_optim = result_est[1]  
  
# %% The optimal parameters values:  
  
K_est = params_optim[0]  
T_est = params_optim[1]  
  
# %% Displaying the results:  
  
print('Optimal estimates: ')  
print('K_true = ', f'{K_true:.4f}')  
print('T_true = ', f'{T_true:.4f}')  
print(f'K_est = {K_est:.4f}')  
print(f'T_est = {T_est:.4f}')  
print(f'sspe_optim = {sspe_optim:.4f}')
```

The result of the parameter estimation is shown in Table 11.4. The `brute()` function was able to estimate the parameters accurately despite the relatively poor resolution of 10 for each of the parameters. The good result is due to the final optimization with the Nelder-Mead optimization algorithm (selected as an option in `brute()` function).

Table 11.4: Result of estimation of parameters K and T with the `brute()` function with a final Nelder-Mead optimization.

Parameter	$N_{\text{resolution}}$	K [krpm/V]	T [s]	sspe
True value	—	0.150	0.300	—
Estimated	10	0.150	0.300	0.0

[End of Example 11.6]

11.4.2 Adaptation of dynamic models using nonlinear programming (NLP)

The optimization problem of parameter estimation can be solved with nonlinear programming, cf. Chapter 42.2.8.

Example 11.7 DC motor model adaptation to simulated data with the Nelder-Mead optimizer in Scipy

The parameter estimation problem is as in Example 11.5: We will now use the Nelder-Mead NLP solver to estimate K and T .

Guessed (initial) values are selected as:

$$K_{\text{guess}} = 0.3 \quad (11.59)$$

$$T_{\text{guess}} = 0.6 \quad (11.60)$$

Python program 11.8 implements the parameter estimator.

http://techteach.no/control/python/prog_minimize_nelder_mead_estim_K_T_sim_motor.py

Listing 11.8: `prog_minimize_nelder_mead_estim_K_T_sim_motor.py`

```
# %% Import:

import numpy as np
import scipy.optimize

# %% Defining functions:

def fun_objective(x):
```

```

K = x[0]
T = x[1]
S_pred_k = S_init
pe_array = np.zeros(N)

for k in range(0, N):

    # Simulation algorithm (Euler step):
    dS_sim_Ts_k = (1/T)*(-S_pred_k + K*u_array[k])
    S_pred_kp1 = S_pred_k + Ts*dS_sim_Ts_k

    # Updating prediction error (pe):
    pe_array[k] = S_obs_array[k] - S_pred_k

    # Time shift:
    S_pred_k = S_pred_kp1

sspe = sum(pe_array*pe_array)
return sspe

# %% Definition of simulation function:

def fun_sim(modelparams, S_init, u_array, N, Ts):

    (K, T) = modelparams
    S_sim_k = S_init
    S_sim_array = np.zeros(N)

    for k in range(0, N):
        # Simulation algorithm (Euler step):
        dS_sim_Ts_k = (1/T)*(-S_sim_k + K*u_array[k])
        S_sim_kp1 = S_sim_k + Ts*dS_sim_Ts_k
        S_sim_array[k] = S_sim_k
        S_sim_k = S_sim_kp1 # Time shift

    return S_sim_array

# %% Time settings:

t_start = 0 # [s]
t_stop = 7
Ts = 0.02
N = int(((t_stop - t_start)/Ts)) + 1
t_array = np.linspace(t_start, t_stop, N)

# %% Create input signal:

u_array = np.zeros(N)

for k in range(N):
    t_k = k*Ts
    if t_start <= t_k < 1: u_array[k] = 0
    elif 1 <= t_k < 3: u_array[k] = 1

```

```

    elif 3 <= t_k < 5: u_array[k] = -1
    else: u_array[k] = 0

# %% Creating simulated observation data:

K_true = 0.15
T_true = 0.30
params = (K_true, T_true)
S_init = 0

S_sim_array = fun_sim(params, S_init, u_array, N, Ts)
S_obs_array = S_sim_array

# %% Guessed values (initial values) of optim variables:

K_guess = 0.3
T_guess = 0.6
x_guess = np.array([K_guess, T_guess])

# %% Solving the optim problem:

res = scipy.optimize.minimize(fun_objective, x_guess,
                               method = 'nelder-mead',
                               options = {'ftol': 1e-9, 'disp': True})

# %% The results of the optimization:
(K_optim, T_optim) = res.x
sspe_optim = res.fun

# %% Displaying the optimal solution:
print('Optimal estimates:')
print('K_true =', f'{K_true:.4f}')
print('T_true =', f'{T_true:.4f}')
print('K_optim =', f'{K_optim:.4f}')
print('T_optim =', f'{T_optim:.4f}')
print(f'sspe_optim = {sspe_optim:.4f}')

```

The result of the parameter estimation is shown in Table 11.5. The parameter estimates are virtually identical to the true values.

Table 11.5: Result of estimation of parameters K and T with the minimize() function with Nelder-Mead optimization solver.

Parameter	K [krpm/V]	T [s]	sspe
True value	0.150	0.300	—
Estimated	0.150	0.300	0.0

[End of Example 11.7]

11.4.3 Adaptation of dynamic models using the least squares method

Assume that the dynamic model which you want to adapt to data, can be written on the form of (11.28), which is repeated here for convenience:

$$y = \phi\theta \quad (11.61)$$

With m observations, the stacked model is:

$$\underbrace{\begin{bmatrix} \vdots \\ y_i \\ \vdots \end{bmatrix}}_Y = \underbrace{\begin{bmatrix} \vdots \\ \phi_i \\ \vdots \end{bmatrix}}_{\Phi} \theta \quad (11.62)$$

where $i \in [0, m - 1]$. Then, the parameters can be estimated with the ordinary least squares (LS) method. The parameter estimates are given by (11.40), which repeated here:

$$\theta_{\text{LS}} = (\Phi^T \Phi)^{-1} \Phi^T Y \quad (11.63)$$

Example 11.8 Estimation of parameters of a dynamic model using ordinary LS method

Given the following “time constant” model:

$$Tx' = Ku + d - x$$

where x is the measurement signal, u is the control signal, and d is the disturbance. K and T are model parameters. x and u have known values (at the sampling points of time). Assume that neither K , T , nor d are known. How can we estimate their values with the ordinary least squares method?

Let us use the Euler Forward (Euler Explicit) method of discretization:

$$x'_k = \frac{x_{k+1} - x_k}{T_s}$$

We write the model on the regression model form, which is

$$y_k = \varphi_k \theta \quad (11.64)$$

where θ is the parameter vector to be estimated.

Using the Euler Forward (Euler Explicit) approximation of the time-derivative, the model becomes:

$$T \frac{x_{k+1} - x_k}{T_s} = Ku_k + d - x_k \quad (11.65)$$

which can be written as:

$$x_k = Ku_k + T \frac{x_{k+1} - x_k}{T_s} + d = \left[u_k, -\frac{x_{k+1} - x_k}{T_s}, 1 \right] \begin{bmatrix} K \\ T \\ d \end{bmatrix} \quad (11.66)$$

which is on the form (11.64) with:

$$\begin{aligned} y_k &= x_k \\ \varphi_k &= \left[u_k, -\frac{x_{k+1} - x_k}{T_s}, 1 \right] \\ \theta &= \begin{bmatrix} K \\ T \\ d \end{bmatrix} \end{aligned} \tag{11.67}$$

The parameter vector estimate is calculated with the LS formula:

$$\theta_{\text{LS}} = (\Phi^T \Phi)^{-1} \Phi^T Y \tag{11.68}$$

[End of Example 11.8]

The model (11.30) is linear in the parameters. If the model is nonlinear in the parameters, you can not apply the ordinary least squares method. Instead, you can use e.g. the grid method, cf. Section 11.4.1, or nonlinear programming, cf. Section 11.4.2. Since the latter methods can be applied equally well to models being linear in its parameters (generally, linear models are just a special case of nonlinear models), I recommend that you consider those methods.

Still, one argument for using ordinary least squares method it that it is an extremely fast method as it uses a formula, and no iterative algorithm, to find the optimal solution, i.e. the parameter estimates.

11.5 Recursive (real-time) model adaptation

The parameter estimation described in this section is a *batch* estimation since it operates on the whole data series available. An alternative term to batch estimation is *full information estimation*. If you assume that the parameters may change continuously during the time interval of interest, the parameter estimates obtained with batch estimation may not be good estimates at recent (newest) times. In such cases, you may consider *online parameter estimation* where the estimates are updated continuously based on the most recent process measurement available.

Some alternative methods for online parameter estimation are:

- Moving horizon estimation (MHE) which is presented in Ch. 35.
- Kalman Filter which is presented in Ch. 34.
- Recursive least squares method (RLS), which is not covered by this book. RLS can be regarded as a special case of using a Kalman Filter for parameter estimation.

In MHE and Kalman Filter, the parameters are estimated as state variables. The original state vector is augmented with parameter states. Therefore, the Kalman Filter used for

parameter estimation is denoted *augmented* Kalman Filter, and the MHE may similarly be denoted augmented MHE.

Among those two alternatives, I generally recommend the Kalman Filter since it is easier to implement, executes faster, and the performance is typically comparable with that of a successful implementation of MHE.⁵

⁵My own experience is that MHE may actually fail to estimate parameters (except for simple models) while the Kalman Filter works well.

11.6 Problems for Chapter 11

Problem 11.1 Characterizing models as linear and nonlinear

Give an example of a model that is linear in the parameters, and an example of a model that is nonlinear in the parameters.

Problem 11.2 How to check if estimation has a chance to succeed

Assume that you have programmed some parameter estimation method. How can you check if your estimator is able to produce parameter estimates successfully?

Problem 11.3 Writing the regression model

Assume that the parameters a and b in the differential equation

$$h(k) + a\sqrt{h(k-1)} = h(k-1) + bu(k-1) \quad (11.69)$$

is to be estimated with the Least Squares (LS) method. Assume that the following samples of the variables h and u exist:

$$\{h(0), h(1), h(2), h(3), h(4)\} \quad (11.70)$$

$$\{u(0), u(1), u(2), u(3), u(4)\} \quad (11.71)$$

Write the total regression model

$$Y = \Phi\theta \quad (11.72)$$

which makes the basis for the LS estimation. However, you shall *not* calculate the estimate in this Problem. The regression model contains only the samples of h and u that are available. (You are to find the vector Y , matrix Φ , and the vector θ .)

Problem 11.4 Bias in LS estimate

This problem demonstrates that a LS-estimate converges towards an erroneous value – in other words the estimate is biased – if the noise or the model error has a mean value different from zero.⁶

The constant K in the model

$$y = K \quad (11.73)$$

is to be estimated. Assume that the real (true) value of K is K_0 . The observation y_k is

$$y_k = K_0 + e_k \quad (11.74)$$

where e_k is noise (or equation error or model error or prediction error). Calculate the LS estimate of K (you can assume that there are N observations). Assume that y_k in (11.73) is given by (11.74).

⁶In general, the estimate will be biased if the noise is *coloured* (non-white).

Problem 11.5 Which model order?

Figure 11.12 shows the criterion function V as a function of model order n in a fictitious problem about parameter estimation.

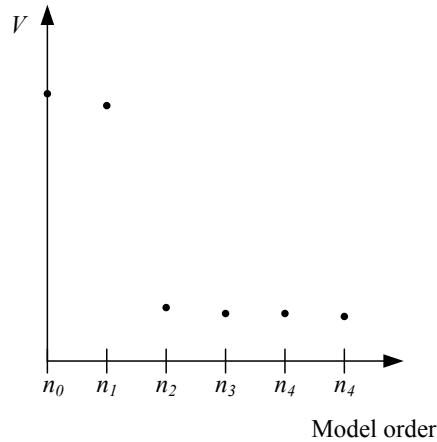


Figure 11.12: The criterion function V as a function of model order n

Which order should be selected?

Problem 11.6 Analysis of LS estimate

In Example 11.4, Program 11.4 uses the ordinary least squares method to estimate the parameters a and b in the model

$$y = ax + b$$

to given emission data (observations). Modify Program 11.4 to calculate the 95 % confidence intervals of a_{est} and b_{est} .

Problem 11.7 Minimum order at estimation of time delay

Assume that you know in advance that a given physical process has time delay of 2 sec. You are to estimate a discrete-time transfer function to from experimental input and output (measurement) sequences. The sampling time is 0.5 sec. You are not sure about the order of the transfer function, but what is the minimum order that it should have?

Problem 11.8 Mathematical modeling of a DC motor

Here is a mathematical model of an electric motor (a DC motor) with load torque:

$$J\omega' = \frac{K_T}{R_a} (v_a - K_e\omega) + T_L \quad (11.75)$$

Assume that the inertia J and the torque T_L will be estimated with the LS method. The motor is excited with the armature voltage v_a . The rotational speed ω is measured with a tachometer. K_T , R_a and K_e have known values.

Write the model on the standard form $y = \varphi\theta$. Use the center difference method to calculate ω' (see below).

The center difference method

Using y as the name of the variable or signal, the center difference method is

$$y(t_k)' \approx \frac{y(t_{k+1}) - y(t_{k-1})}{2T_s} \quad (11.76)$$

For a given time step T_s , the center difference method gives a somewhat more accurate approximation to y than the forward difference method and the backward difference method. Actually, the center difference method is the average of those two approximations.

Problem 11.9 *Adaption of linear dynamic models*

Assume that you want to adapt a linear dynamic model, e.g. a transfer function, to experimental data. If the model is actually nonlinear, a linear model will give a good representation of the system only near an operating point. How can you process the input and output data (the sequences) before using them in system identification to improve the accuracy of the estimated linear model, assuming the process is nonlinear?

Problem 11.10 *Is it possible to estimate parameters of a DC motor model?*

Given the following “time-constant” model:

$$Tx' = Ku + d - x$$

where x is the measurement signal, u is the control signal, and d is the disturbance. K and T are model parameters. x and u have known values (at the sampling points of time).

In the problems below: “Static” means “constant”. You should explain your answers, but no rigorous explanations are expected.

1. Assume that both K and d have known values. Is it possible to estimate T from static data?
2. Assume that T is known. Is it possible to estimate both K and d from static data?

Problem 11.11 *LS estimation of dynamic model*

Given the model:

$$x_{k+1} = ax_k + bu_k \quad (11.77)$$

Explain how you can estimate the assumed constant parameters a and b with the ordinary least squares method using data series (time series) of sampled values of x and u . In your answer, you are not expected to actually calculate the estimates, but the formula of the estimates should be given.

Problem 11.12 *Python program for estimation of DC motor load torque from simulated data*

In Example 11.5 we estimated the motor parameters K and T from simulated data.

Write a Python program, starting with the program available in Example 11.5, that also estimates an assumed constant load torque, L . You can assume that the true value is $L_{\text{true}} = -0.5$ V (an equivalent voltage which acts on the motor in the same way as a load torque).

Problem 11.13 *Python program for estimation of DC motor load torque from real data*

In Problem 11.12, K , T , and L of a DC motor model are to be estimated from simulated data. In the present problem, these parameters are estimated from real data.

Real data from an experiment with the motor are available at this link:

http://techteach.no/control/python/data_dc_motor.txt

The sampling time is 0.02 s (as you can also see in the data file). The file has the following columns of data:

- Column 0: Time in seconds.
- Column 1: The control signal u in voltage, adjusted manually more or less randomly.
- Column 2: The tachometer voltage S_{tacho} , which can be converted to a corresponding rotational speed in krpm (kilo revolutions per minute) with:

$$S_{\text{krpm}} = S_{\text{tacho}}/K_t \quad (11.78)$$

where:

$$K_t = 5.0 \text{ V/krpm} \quad (11.79)$$

You can load the data file to Python with e.g. the `loadtxt()` function of the Numpy package.

Based on the program given in the solution of Problem 11.12, create a program which estimates K , T , and L from u and S_{krpm} , and (after the estimation) simulates S_{krpm} with the estimated parameters, and plots the simulated S together with the real S_{krpm} . Does it look like the represents the real motor well?

Problem 11.14 *Python program for estimation of DC motor parameters with ordinary LS method*

To appear

Problem 11.15 *Python program for estimation parameters of an air heater with the `scipy.brute()` function*

Section 41.5 describes a laboratory air heater. A data file from an experiment on the air heater is available in Section 41.5.5.

Write a Python program which estimates the following variables using the data file: Gain K_h . Time constant θ_t . Time delay θ_d . Environmental temperature T_e . Initial value of the outlet temperature, T_{init} . In the call of the `brute()` function, you can select the option with final optimization, i.e. setting the `finish` argument to `optimize.fmin`.

Simulate the air heater with the control signal of the data file, and plot both the simulated temperature and the observed (measured) temperature in the same diagram. Does the adapted model represent the real air heater accurately?

Problem 11.16 *Comparing parameter estimation methods*

Range the following parameter estimation methods (approaches) in terms of program (algorithm) execution speed:

- Nonlinear programming (NLP)
- Ordinary Least Squares method
- Gridding (or brute force) method

Also, for each of these methods, give an important applicability limitation.

11.7 *Solutions to problems for Chapter 11*

Solution to Problem 11.1

Example of a model which is linear in the parameters (a and b):

$$x' = au + bx \quad (11.80)$$

A model which is nonlinear in the parameters (c and d):

$$x' = c(du - x) \quad (11.81)$$

However, in the latter example, the model can be written as a model being linear in the parameters e and c where

$$e = cd \quad (11.82)$$

The reformulated model is:

$$x' = eu - cx \quad (11.83)$$

Solution to Problem 11.2

1. Generate simulated data using a model with the known, true parameters.
2. Estimate the parameters from the simulated data.
3. Check if the parameter estimate becomes equal to the true parameter values.

Solution to Problem 11.3

Writing the model on standard regression form:

$$h(k) - h(k-1) = \begin{bmatrix} -\sqrt{h(k-1)} & u(k-1) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \quad (11.84)$$

The total model becomes

$$\underbrace{\begin{bmatrix} h(1) - h(0) \\ h(2) - h(1) \\ h(3) - h(2) \\ h(4) - h(3) \end{bmatrix}}_Y = \underbrace{\begin{bmatrix} -\sqrt{h(0)} & u(0) \\ -\sqrt{h(1)} & u(1) \\ -\sqrt{h(2)} & u(2) \\ -\sqrt{h(3)} & u(3) \end{bmatrix}}_{\Phi} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\theta} \quad (11.85)$$

Solution to Problem 11.4

K is given by

$$y_k = K \quad (11.86)$$

$$= 1 \cdot K \quad (11.87)$$

$$= \varphi \theta \quad (11.88)$$

The LS estimate is

$$\theta_{\text{LS}} = K_{\text{LS}} \quad (11.89)$$

$$= (\Phi^T \Phi)^{-1} \Phi^T y \quad (11.90)$$

$$= \left(\begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} K_0 + e_1 \\ K_0 + e_2 \\ \vdots \\ K_0 + e_N \end{bmatrix} \quad (11.91)$$

$$= N^{-1} \cdot \left[N \cdot K_0 + \sum_{k=1}^N e_k \right] \quad (11.92)$$

$$= K_0 + \frac{1}{N} \sum_{k=1}^N e_k \quad (11.93)$$

$$= K_0 + m_e \quad (11.94)$$

where m_e is the mean value of the noise. From (11.94) you can see that K_{LS} does not converge towards K_0 if the mean value m_e is different from zero.

Solution to Problem 11.5

According to the Parsimony Principle, you should select the smallest order with relatively small value of the criterion function. So, you should select order n_2 .

Solution to Problem 11.6

See the following Program:

http://techteach.no/control/python/prog_ls_optim_greenhouse_gas_estimate_analysis.py

The results are shown in the box below.

```
a_opt = -0.152
b_opt = 359.622
f_min = V = 11.741
std_a = 0.035
std_b = 69.930
confint_a = [-0.221 -0.152 -0.082]
confint_b = [219.761 359.622 499.482]
```

So, the 95 % confidence intervals are:

$$a_{\text{est}}: [-0.221, -0.082] \quad (11.95)$$

$$b_{\text{est}}: [219.761, 499.482] \quad (11.96)$$

Solution to Problem 11.7

The order should be large enough to include the time delay. One time step of 0.5 sec corresponds to a time delay of 0.5 s, and one such time delay is represented by the factor z^{-1} in the transfer function. The model should therefore include $2/0.5 = 4$ such factors. Hence, the minimum order of the transfer function is 4.

Solution to Problem 11.8

The model written on the standard LS form:

$$\underbrace{\frac{K_T}{R_a} [v_a(t_k) - K_e \omega(t_k)]}_y = \underbrace{[\omega(t_k)' \quad -1]}_\varphi \underbrace{\begin{bmatrix} J \\ T_L \end{bmatrix}}_\theta \quad (11.97)$$

$\omega(t_k)'$ is calculated with the center difference method:

$$\omega(t_k)' \approx \frac{\omega(t_{k+1}) - \omega(t_{k-1})}{2T_s} \quad (11.98)$$

where T_s is the time step.

Solution to Problem 11.9

You can remove the mean values of both the input and the output signals (time series or sequences), and use the deviations as new input and output. In more detail: Assume that $\{u(k)\}$ is the original input signal and $\{y(k)\}$ is the output signal, and that m_u and m_y are the respective mean values. The deviation signals are then

$$\{du(k)\} = \{u(k) - m_u\} \quad (11.99)$$

and

$$\{dy(k)\} = \{y(k) - m_y\} \quad (11.100)$$

The signals $\{du(k)\}$ and $\{dy(k)\}$ are used as input and output signals.

Solution to Problem 11.10

Given the following “time-constant” model:

$$Tx' = Ku + d - x$$

where x is the measurement signal, u is the control signal, and d is the disturbance. K and T are model parameters. x and u have known values (at the sampling points of time).

In the problems below: “Static” means “constant”. You should explain your answers, but no rigorous explanations are expected.

1. T can not be estimated from static data because dx/dt is zero, effectively removing T from the model, and therefore, whatever static values of x and u , we can not calculate an estimate of T .
2. The static model becomes:

$$0 = Ku_s + d - x_s$$

giving:

$$x_s = Ku_s + d$$

Since x_s and u_s are known, the above equation is one quation with two unknowns, K and d , so there is no unique solution for K and d . Hence, we can not expect any estimate of K and d to be equal to the true value (i.e. the estimates are not consistent).

Solution to Problem 11.11

The given model must be written on the regression model form, which is

$$y_k = \varphi_k \theta \quad (11.101)$$

where θ is the parameter vector to be estimated.

The model written on the regression model form:

$$\underbrace{y_k}_{x_{k+1}} = \underbrace{[\varphi_k]}_{[x_k, u_k]} \underbrace{\theta}_{\begin{bmatrix} a \\ b \end{bmatrix}}$$

The parameter vector estimate is calculated with the LS formula given in the formula list:

$$\theta_{LS} = (\Phi^T \Phi)^{-1} \Phi^T Y$$

where:

$$\Phi = \begin{bmatrix} \vdots \\ \varphi_i \\ \vdots \end{bmatrix}$$

$$Y = \begin{bmatrix} \vdots \\ y_i \\ \vdots \end{bmatrix}$$

For simplicity, the lower and upper indexes of φ_i and y_i are omitted in the vectors (arrays) above.

Solution to Problem 11.12

The Python program named 'prog_grid_motor_K_T_L_estim_sim.py' which is available via the link below implements the model adaptation. Firstly, the program generates a simulated data series, and then uses this data series as the basis of the parameter estimation.

http://techteach.no/control/python/prog_grid_motor_K_T_L_estim_sim.py

Figure 11.13 shows a plot of the control signal u and the observed (simulated) S .

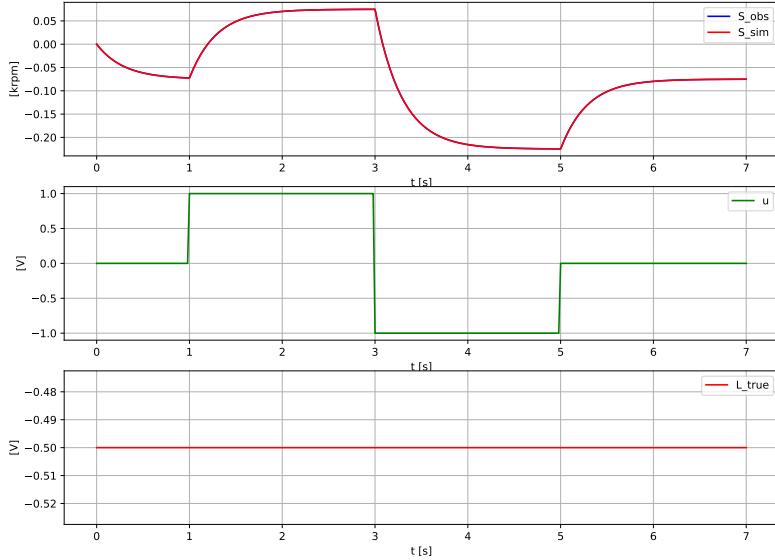


Figure 11.13: Control signal u and observed (simulated) S which is the basis of the model adaptation.

Table 11.6 shows the result of the parameter estimation. The estimated values are equal to the true values, and sspe is zero – i.e. perfect model adaptation.

Table 11.6: The result of the parameter estimation of K , T , and L .

Parameter	Estimated	True
K [krpm/V]	0.15	0.15
T [s]	0.30	0.30
L [V]	-0.5	-0.5
sspe	0	

Solution to Problem 11.13

The Python program named 'prog_grid_motor_K_T_L_estim_real.py' which is available via the link below implements the parameter estimation. Firstly, the program generates a simulated data series, and then uses this data series as the basis of the parameter estimation.

http://techteach.no/control/python/prog_grid_motor_K_T_L_estim_real.py

Figure 11.14 shows a plot of the control signal u and the simulated S together with the real S . It seems that the model represents the motor well.

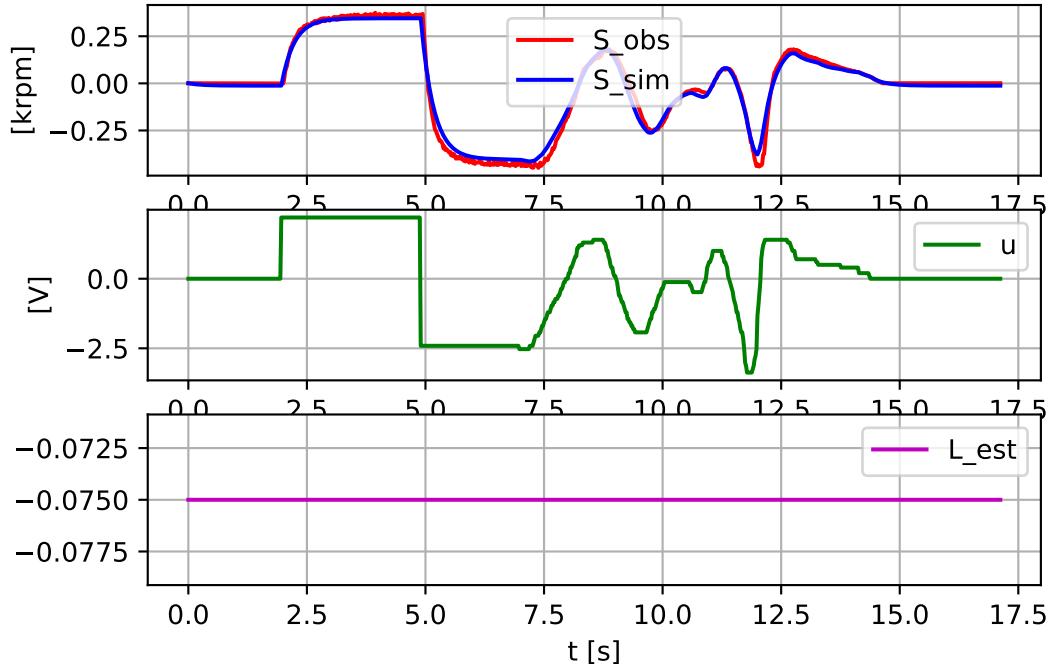


Figure 11.14: Plots of control signal u , and simulated S and real S .

The results of the parameter estimation are:

- $K_{\text{est}} = 0.165$ [krpm/V]
- $T_{\text{est}} = 0.280$ s.
- $L_{\text{est}} = -0.0750$ V (which indicates a breaking torque, to be expected).

Solution to Problem 11.14

To appear.

Solution to Problem 11.15

The Python program named 'prog_estim_brute_airheater_real.py' which is available via the link below implements the model adaptation.

http://techteach.no/control/python/prog_estim_brute_airheater_real.py

The result of the estimation is shown in the box below.

```
sspe_optim = 32.5108
Kh_optim = 3.9730
theta_t_optim = 27.8173
theta_d_optim = 2.7852
T_env_optim = 22.0161
T_init_optim = 23.1379
```

Figure 11.15 shows a plot of the control signal u and the simulated T together with the observed (real) T . It seems that the model represents the air heater well.

Solution to Problem 11.16

Range in terms of program (algorithm) execution speed:

1. Ordinary Least Squares method.

Applicability limitation: The model must be linear in the parameters.

2. Nonlinear programming (NLP).

Applicability limitation: There is a chance that the parameter estimates only satisfy a local minimum of the optimization criterion, and hence are not the “globally” optimal estimates.

3. Gridding (or brute force) method.

Applicability limitation: Due to limited resolution in the parameter estimate candidates, the precisely optimal parameter estimates are not found.

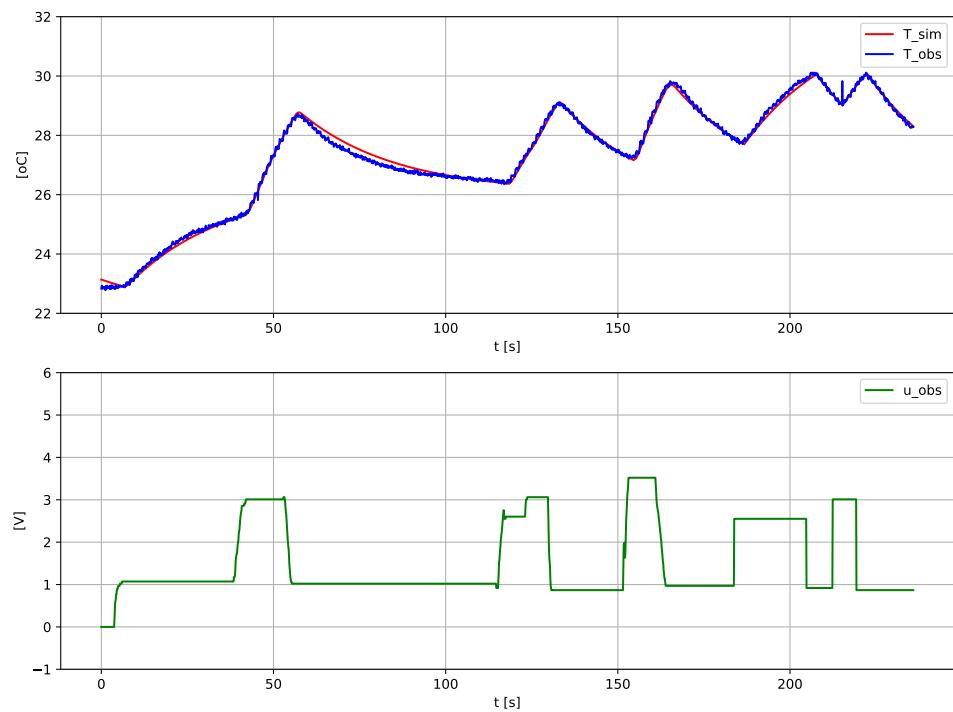


Figure 11.15: Plots of control signal u , and simulated T and observed (real) T .

Part III

BASIC CONTROL METHODS

Chapter 12

PID controller

12.1 Introduction

The principle of feedback control was introduced in Section 1.2, but without going into details about the controller function. The PID – proportional-integral-derivative – controller function has become a standard for technical and industrial applications. It is available in all commercial automation products. You will now get to know the PID controller in detail.

12.2 Computer-based PID control loop

In automation equipment, the PID controller is implemented in a computer (microprocessor) in some form, e.g. process controller or PLC (programmable logic controller) or PAC (programmable automation controller) or DCS (distributed control system). The PID controller is therefore inherently in a form of a programmed, discrete-time algorithm which calculates the control signal at discrete point of times. Figure 12.1 shows a computer-based PID control system. In this chapter, we focus on the PID controller block, but also on the filter which is used to attenuate measurement noise, n , affecting the measurement, y_m .

In Figure 12.1, I have introduced a simplifying notation of discrete-time signals. For example,

$$y_{\text{mf},k} = y_{\text{mf}}(t_k)$$

which is the filtered measured process value at discrete time t_k .

The discrete-time PID control loop operates as follows:

- The analog process measurement, y_m , is sampled and converted into a digital value in the ADC with a sampling time of T_s [s], which may be 0.1 s in automation equipment in the process industry, and considerably smaller in automation equipment for small motors.

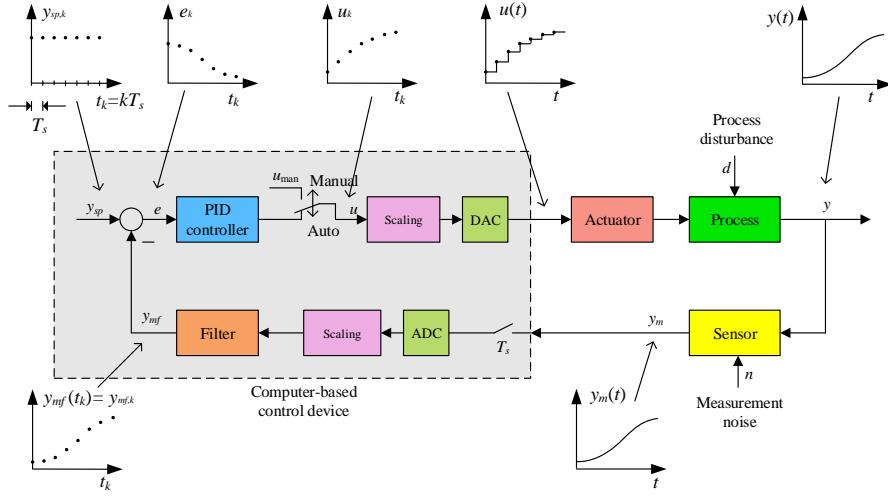


Figure 12.1: A computer-based PID control system. (DAC = Digital-Analog Converter. ADC = Analog-Digital Converter.)

- The controller (the program) calculates the control error as the difference between the discrete-time setpoint value, r_k , and the discrete-time filtered process measurement, $y_{mf,k}$:

$$e_k = r_k - y_{mf,k} \quad (12.1)$$

- The controller calculates a discrete-time value, u_k , according to the PID algorithm, which we will meet later in this chapter.
- Typically, u_k is converted into a milliampere value, which becomes a physical milliampere value in the DAC (digital-analog converter). A holding element in the DAC ensures that this value is kept constant in the current time step, which typically has the same duration as the sampling time of the ADC. Therefore actuator actually receives a step-wise control signal. However, often the time step, T_s , is very small compared to the dynamics of the actuator, and therefore the actuator feels like receiving a smooth control signal.

12.3 Continuous-time PID controller

We will in the subsequent section develop a discrete-time PID controller – or a PID algorithm – ready for programming. This is the contents of the PID controller block in Figure 12.1. In the present section, I present the continuous-time, or analog, PID controller since it is the basis of the discrete-time PID controller.

The continuous-time PID controller is:

$$\begin{aligned} u &= u_{\text{man}} + \underbrace{K_c e}_{u_p} + \underbrace{\frac{K_c}{T_i} \int_0^t e d\tau}_{u_i} + \underbrace{K_c T_d e'}_{u_d} \\ &= u_{\text{man}} + u_p + u_i + u_d \end{aligned} \quad (12.2)$$

where e is the control error:

$$e = r - y_{\text{mf}} \quad (12.3)$$

In (12.2), u_{man} is the nominal value of the control variable. It is the control signal available for adjustment by the operator while the controller is in manual mode. While the controller is in automatic mode, u_{man} can usually not be adjusted.

In (12.2), u_p is the P term. u_i is the I term, with time $t = 0$ being the latest time when the controller was set to automatic mode. u_d is the D term.

The PID controller (12.2) is denoted the parallel PID controller since the terms appear in parallel in a mathematical block diagram of the controller, see Figure 12.2. We may denote

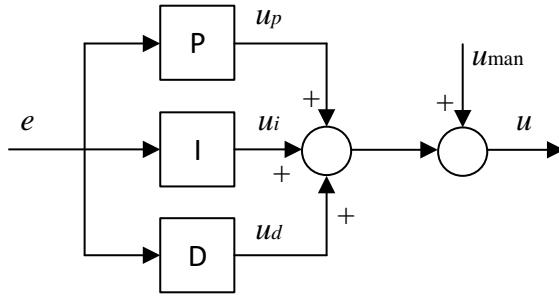


Figure 12.2: The PID controller (12.2) is denoted the parallel PID controller

it the additive PID controller. It is sometimes denoted the academic controller.¹ There is also a serial form, which is discussed in Section 16.8.4.2.

The controller parameters in (12.2) are commented in detail below.

Proportional gain

K_c is the controller gain. An alternative symbol of this gain is K_p (p for proportional).

K_c is in unit of [unit of u /unit of y_{mf}]. Example: If u is in unit of W and y_{mf} is in unit of °C, K_c is in unit of W/°C, or strictly speaking, unit of W/K.

Several commercial controllers uses the *proportional band* P_B , as a parameter instead of the proportional gain, K_c . The proportional band is given by

$$P_B = \frac{100\%}{K_c} \quad (12.4)$$

where K_c is the gain, which here is assumed to be dimensionless. (It will be dimensionless if the control error e and the control variable u have the same unit, typically percent). It is typical that P_B has a value in the range of $10\% \leq P_B \leq 500\%$, which corresponds to K_c being in the range of $0.2 \leq K_c \leq 10$. It is important to note that P_B is *inversely proportional* to K_c . Thus, a small P_B corresponds to a large K_c , and vice versa.

¹Maybe that name was coined to indicate that the form can not be used in practice. However, it is indeed useful in practice.

What does *proportional band* actually mean? One explanation is that P_B is the change of the control error interval Δe (or the size of the measurement signal interval) which gives a change of the control signal interval Δu equal to 100% (i.e. a full range change of u): From the P-term $u = K_c e$ we see that $\Delta e = \Delta u / K_c = 100\% / K_c = P_B$.

Integral time

T_i [s] (or som other time unit, e.g. minutes) is the integral time.

In some controllers the value of $1/T_i$ is used instead of the value of T_i . The unit of $1/T_i$ is repeats per minute. For example, 5 repeats per minute means that $T_i = 1/5 = 0.2$ min. The background of the term repeats per minute is as follows: Assume that the control error e is constant, say E . The P-term has value $u_p = K_c E$. During a time interval of 1 minute the I-term equals $\frac{K_c}{T_i} \int_0^1 E d\tau = K_c E \cdot 1[\text{min}] / T_i = u_p \cdot 1/T_i$. Thus, the I-term has repeated the P-term $1/T_i$ times.

Derivative time

T_d [s] is the derivative time.

Alternative parameterization

Note that some controllers uses the following alternative parameterization:

- Integral gain:

$$K_i = \frac{K_p}{T_i}$$

- Derivative gain:

$$K_d = K_p T_d$$

Special cases of the PID controller

A P controller (proportional controller) is achieved by setting $T_i = \infty$ (or to a very large value) and $T_d = 0$. In some commercial controllers, you can set T_i to 0 (zero), which is a code expressing that the integral term is deactivated.

A PI controller (proportional-integral controller) is obtained by setting $T_d = 0$.

12.4 Transfer function of the PID controller

12.4.1 Ideal PID controller

The ideal continuous-time PID controller function in the time domain is:

$$u(t) = u_{\text{man}} + K_c e(t) + \frac{K_c}{T_i} \int_0^t e d\tau + K_c T_d e'(t) \quad (12.5)$$

Taking the Laplace transform of (12.5), while omitting the term u_{man} :

$$u(s) = K_c e(s) + \frac{K_c}{T_i} \cdot \frac{1}{s} \cdot e(s) + K_c T_d \cdot s \cdot e(s) = \left(K_c + \frac{K_c}{T_i s} + K_c T_d s \right) e(s) \quad (12.6)$$

The transfer function is

$$C(s) = \frac{u(s)}{e(s)} = K_c + \frac{K_c}{T_i s} + K_c T_d s \quad (12.7)$$

12.4.2 Practical PID controller with filter in the derivative term

In a practical PID controller, the derivative term includes a time constant filter to attenuate high frequent measurement noise. The transfer function of such a PID controller is:

$$C(s) = \frac{u(s)}{e(s)} = K_c + \frac{K_c}{T_i s} + K_c T_d s \cdot \frac{1}{T_f s + 1} \quad (12.8)$$

where T_f is the filter time constant. T_f can be set as a fraction of T_d :

$$T_f = a T_d \quad (12.9)$$

A typical value of a is

$$a = 0.1 \quad (12.10)$$

12.5 Discrete-time PID algorithm

Using the continuous-time PID controller, (12.2), as the basis, we will now develop a discrete-time PID controller – or a PID algorithm – ready for programming. The PID algorithm has the same basic form as (12.2), i.e.

$$u_k = u_{\text{man},k} + u_{p,k} + u_{i,k} + u_{d,k} \quad (12.11)$$

where each of the terms are discretized version of their corresponding continuous-time terms.

In the following, the discrete-time control error is calculated as

$$e_k = r_k - y_{\text{mf},k} \quad (12.12)$$

The manual term

u_{man} is a constant in both the continuous-time PID controller and the discrete-time PID controller.

The P term

It is straightforward to discretize the P term:

$$u_{p,k} = K_c e_k \quad (12.13)$$

The I term

The continuous-time I term is

$$u_i(t) = \frac{K_c}{T_i} \int_0^t e(\tau) d\tau \quad (12.14)$$

It has become a tradition to discretize this integral with the Euler Backward method, which gives

$$\begin{aligned} u_{i,k} &= \frac{K_c}{T_i} \int_0^{t_k} e d\tau \\ &= \frac{K_c}{T_i} (T_s e_1 + \dots + T_s e_{k-1} + T_s e_k) \end{aligned} \quad (12.15)$$

$$\begin{aligned} &= \frac{K_c T_s}{T_i} (e_1 + \dots + e_{k-1} + e_k) \\ &= \underbrace{\frac{K_c T_s}{T_i} (e_1 + \dots + e_{k-1})}_{u_{i,k-1}} + \frac{K_c T_s}{T_i} e_k \\ &= u_{i,k-1} + \frac{K_c T_s}{T_i} e_k \end{aligned} \quad (12.16)$$

which is a recursive version of the I term algorithm, which I repeated here for easier reference:

$$u_{i,k} = u_{i,k-1} + \frac{K_c T_s}{T_i} e_k \quad (12.17)$$

It is much more practical to implement (12.17), which is a recursive or online algorithm, than the batch algorithm (12.15), although they are equivalent. In (12.17), the only term that we have to store is $u_{i,k-1}$, while in (12.15) we have to store all older control errors, from $t = 0$ (i.e. the last time when the controller was set to automatic mode), whenever back in time that is (years maybe).

The D term

The continuous-time D term is, cf. (12.2),

$$u_d = K_c T_d e' \quad (12.18)$$

It has become a tradition to discretize e' with the Euler Backward method, which gives the following discrete-time D term:

$$u_{d,k} = K_c T_d \frac{e_k - e_{k-1}}{T_s} \quad (12.19)$$

The PID algorithm summarized

- Control error:

$$e_k = r_k - y_{mf,k} \quad (12.20)$$

- Manual control signal:

$$u_{\text{man},k} = \text{constant}$$

- P term:

$$u_{p,k} = K_c e_k \quad (12.21)$$

- I term:

$$u_{i,k} = u_{i,k-1} + \frac{K_c T_s}{T_i} e_k \quad (12.22)$$

- D term:

$$u_{d,k} = K_c T_d \frac{e_k - e_{k-1}}{T_s} \quad (12.23)$$

- Total control signal:

$$u_k = u_{\text{man}} + u_{p,k} + u_{i,k} + u_{d,k} \quad (12.24)$$

(12.20) – (12.24) is actually an idealized PID algorithm. Several practical modifications should be considered:

- Lowpass filtering of the D term
- Integral anti windup
- Reducing P kick and D kick
- Bumpless transfer between manual and automatic modes

These modifications are described in Section 12.7.

12.6 How does the PID controller work?

The PID algorithm is (12.20) – (12.24). How does it work (in automatic mode)?

The manual term, u_{man} , is constant, and hence “passive” when the controller is in automatic mode. Its contribution in automatic mode is to provide kind of a reasonable initial value at the moment of switching from manual to automatic mode.

The P term, u_p , contributes with a term in the total control signal, u , which is proportional to the control error, e . It brings some speed to the control. However, assuming u_{man} is not “perfect” to give zero control error, i.e. $e = 0$, the P term by itself can not ensure $e = 0$ either. This is because, with $e = 0$, $u_p = 0$, which mean no contribution from the P term. In other words, P controller can not ensure zero error in steady state.

The I term is the most important part of the PID controller because *it ensures zero steady state control error*, i.e. $e_s = 0$. How? Look at the I term, (12.22). As long as e is different from zero, u_i will change. In other words, e is an “improvement term”. Or, e drives the control. This change keeps on until e has become zero, and then u_i , and u , are

kept constant, until some disturbance or setpoint change causes e again to become nonzero, but then the improvement starts again.

The D term

- Assume that, for some reason, the control error, e , is increasing, i.e. the measurement is moving away from (lower than) the setpoint. The difference $e_k - e_{k-1}$ in 12.23) is then positive, making $u_{d,k}$ positive. So, the D term contributes positively to the total control signal, u_k , and we can expect *faster control* with the D term (PID control) comparing with no D term (only PI control).
- Now, assume that e is decreasing, i.e. the measurement is getting closer to the setpoint. The difference in 12.23) is now negative, making $u_{d,k}$ negative. So, the D term contributes negatively to the total control signal, u_k , and we can expect *a breaking or stabilizing control* with the D term (PID control) comparing with no D term (only PI control).

So, the D term may “step on the gas” when appropriate, and “step on the break” when appropriate.

One implication of the above is that the D term may stabilize a control system which otherwise can not be stabilized with a P or a PI controller. This is the case with dynamic positioning of ships. With a PI controller the control system is deemed to be unstable, while it is stable with a properly tuned PID controller.

There is one serious practical problem with the D-term: It amplifies the random measurement noise, causing large variations in the control signal. These variations will be reduced with a lowpass filter acting on the process measurement, cf. Section 3.4.8.6.

Conclusion

From the above it can be concluded that the controller should definitely have an I-term, i.e. the controller should be either a PID controller or a PI controller, to ensure zero steady-state control error. The D-term should be omitted if the control signal is too noisy even with a measurement lowpass filter. However there are processes where the D-term is essential for obtaining a stable control system (as in position control of ships and other “free-body” mechanical systems). In fact, the PI controller is by far the most used feedback controller in the industry. I have heard that more than 90% of PID controllers runs as PI controllers.

Example 12.1 Temperature control of a liquid tank

Figure 12.3 shows a simulation of a temperature control system of a liquid tank. Observations:

- The steady state control error is non-zero with the P controller.
- The steady state control error is zero with the PI controller.

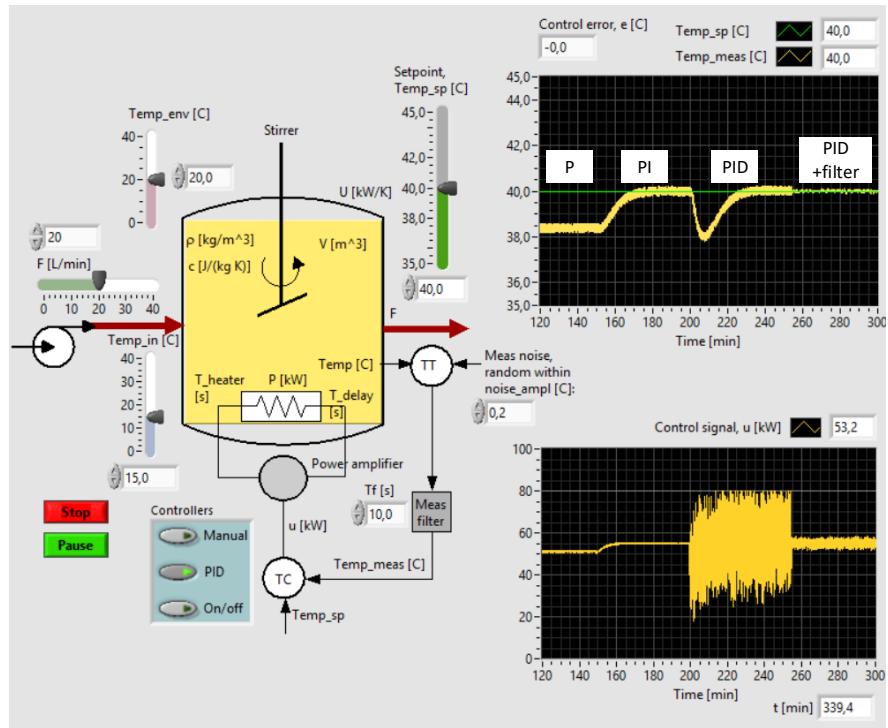


Figure 12.3: Temperature control with P, PI, PID controller without meas. filter, and PID with filter

- The control signal is very noisy when the PID controller is used. (The large transient response in the temperature around $t = 210$ s is due to the D term suddenly being activated, causing a jump in the control signal and hence in the temperature.)
- The measurement filter attenuates the noise, thereby reducing the noise in the control signal.

[End of Example 12.1]

12.7 Practical aspects of the PID controller

This Section describes several practical aspects of PID controller. Some of these modifications are necessary to have a controller that actually works in practice.

12.7.1 Reverse or direct controller action?

12.7.1.1 What is meant by reverse action and direct action?

A PID controller shall have either reverse action mode or direct action mode. It is crucial that you select correctly between these two options. If set incorrectly, *the control system*

becomes unstable, totally useless, and perhaps even dangerous, since the controller will adjust the control signal in the wrong direction.

By definition:

- Reverse action mode = Positive K_c
- Direct action mode = Negative K_c

What is behind the terms reverse action and direct action? Assume for simplicity a P controller:

$$u = u_{\text{man}} + K_c(r - y) \quad (12.25)$$

Assume there is a change, Δy , in y . The corresponding change in the control signal is then (u_{man} and r are assumed constant):

$$\Delta u = -K_c \Delta y \quad (12.26)$$

Assume a positive Δy (i.e. the process variable increases). Then:

- If K_c is positive, Δu becomes negative, i.e. Δu and Δy has the opposite sign – hence the controller reacts “reversely”.
- If K_c is negative, Δu becomes positive, i.e. Δu and Δy has the same sign – hence the controller reacts “directly”.

Note that for industrial controllers it is common that the user always enters a positive value for the controller gain parameter (this is actually the absolute value of the controller gain). To set the sign of the resulting gain, the user must in addition select either reverse action or direct action using a dedicated parameter. This is illustrated in Figure 12.4.

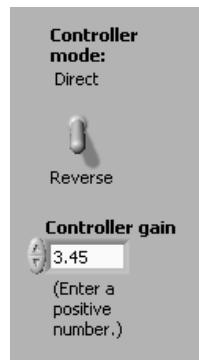


Figure 12.4: Setting the controller mode (reverse, direct) and the (absolute value of) the controller gain

12.7.1.2 How to select between reverse action and direct action modes?

Definition of process gain

Let us start by defining the term process gain, K_{process} . Figure 12.5 shows two processes to be controlled, Process 1 and Process 2. For both, the control signal is denoted u , and the process variable, or actually its measurement, is denoted y . The process blocks include the actuator and the sensor, in addition to the “process” itself (tank, motor, reactor). The figure shows the responses due to a *positive* step change, Δu , in the control signal. The sign of the change of y , denoted Δy , decides the sign of K_{process} :

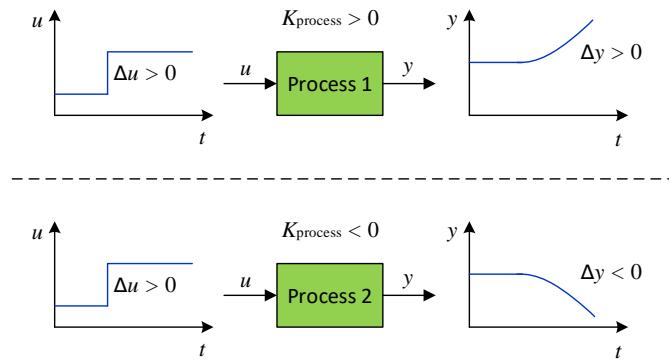


Figure 12.5: Step response test to determine the signal of the process gain, K_{process} , for two processes

- Assume $\Delta u > 0$. If $\Delta y > 0$, then $K_{\text{process}} > 0$, i.e. positive process gain.
- Assume $\Delta u > 0$. If $\Delta y < 0$, then $K_{\text{process}} < 0$, i.e. negative process gain.

So, from a step response test on the process, we can conclude about the sign of K_{process} .

The step response test above does not have to be a real or simulated test. Probably it is sufficient that you imagine (simulate in your head) a step response test. Alternatively, you can conclude about the sign of K_{process} directly from a process model, e.g. a transfer function. Some examples:

- Positive K_{process} , namely 5:

$$H_p(s) = \frac{y(s)}{u(s)} = \frac{5}{2s + 1}$$

- Negative K_{process} , namely -5:

$$H_p(s) = \frac{-5}{2s + 1}$$

- Positive K_{process} , in the form of the integrator gain, 10:

$$H_p(s) = \frac{10}{s}$$

- Negative K_{process} , in the form of the integrator gain, -10 :

$$H_p(s) = \frac{-10}{s}$$

Rule for selection between reverse action and direct action

To ensure that the control loop is stable, it is necessary that the signs of the controller gain, K_c , and the process gain, K_{process} , are the same. Thus, we can use the sign of K_{process} to select between reverse and direct actiod mode:

- For $K_{\text{process}} > 0$, select $K_c > 0$, i.e. reverse action mode.
- For $K_{\text{process}} < 0$, select $K_c < 0$, i.e. direct action mode.

Example 12.2 Reverse or Direct action in the level controller?

Figure 12.6 shows a level control system for a liquid tank where the control variable controls the outflow of the tank, and it is assumed that increasing control signal gives increasing outflow.

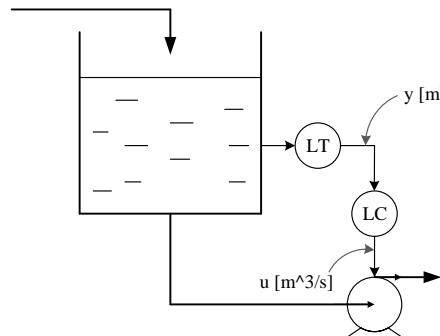


Figure 12.6: Tank with level manipulated by a pump in the outlet

Shall the level controller, LC, have reverse or direct action mode? To answer the question, let's find the signal of K_{process} . Assume a positive step change in the control signal u , i.e. in the outflow through the pump. Does this change cause a positive or a negative change in the level y ? Negative! Therefore, $K_{\text{process}} < 0$, and we conclude that shall have *direct action*.

What if the pump was in the inlet instead?²

What if the pump in the outlet is replaced with a valve where the valve opening, and hence the flow, is reduced if the control signal is increased? (This could be a Fail Open valve, see Figure 2.4.)³

[End of Example 12.2]

²Then the controller shall have reverse action.

³Then the controller shall have reverse action.

12.7.2 Lowpass filter in the D-term

The D-term is very sensitive to random measurement noise. The noise may be amplified through the D term to cause unfortunate variations in the control signal out from the controller. These variations may have too large amplitude, even if there is a measurement lowpass filter in the feedback path, cf. Figure 12.1. To reduce the problem of the noisy D-term, a *lowpass filter* can be applied to the D term. This can be done by lowpass filtering the control error, e , before it is differentiated. This is a common feature of commercial controllers, but there are controllers that do not have it. Typically, the filter is a time constant filter.

Let us use the symbol e_f for the filtered error. The modified D term is:

$$u_d = K_p T_d e'_f \quad (12.27)$$

e_f is the output of the following lowpass filter:

$$e_f(s) = \frac{1}{T_f s + 1} e(s) \quad (12.28)$$

where T_f is the filter time constant which usually is expressed as a fraction of the derivative time T_d :

$$T_f = a T_d \quad (12.29)$$

a is a constant which typically is chosen between 0.05 and 0.2. If no special requirements exist, we can set $a = 0.1$.

A discrete time filtering algorithm corresponding to (12.28) can be derived as shown in Section 3.4.8.6.

12.7.3 Reducing P-kick and D-kick caused by setpoint changes

Abrupt changes of the setpoint y_{sp} , for example step changes, creates an abrupt change of the control error, which in turn may cause unfortunate kicks in the control variable. The problem is related to the P term and, to an even higher degree, to the D term of the PID controller. These kicks are denoted proportional kick or P-kick and derivative kick or D-kick, respectively. Such kicks may cause mechanical actuators to move abruptly, resulting in excessive wear.

Why does these kicks come?

P-term kick

The P-term is:

$$u_p = K_p (w_p r - y_{mf}) \quad (12.30)$$

In the standard PID controller, $w_p = 1$. Assume that r changes as a step. Then there is also a step in u_p , causing a proportional kick, a P-kick, in the total control signal (in which u_p is one additive term).

D-term kick

The D-term is:

$$u_d = K_p T_d (w_d r - y)' \quad (12.31)$$

In the standard PID controller the weight w_d is 1. Assume that y_{sp} changes as a step. Since the time-derivative of a step is an impulse, the step causes an impulse in u_d – a D-kick – in the total control signal, u .

How to solve the problems about P-kick and D-kick?

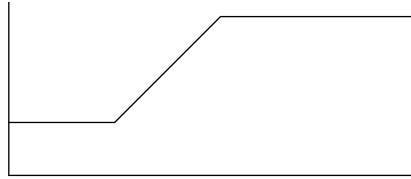


Figure 12.7: The change of the setpoint from one value to another may follow a ramp instead of a step to avoid kicks in the control signal.

Two solutions are:

- **Smooth setpoint changes:** Perhaps the best solution to this problem is to allow only smooth setpoint changes. Commercial controllers have functions to implement rampwise changes between two setpoint values. This is denoted ramping the setpoint, see Figure 12.7.
- **Reduce setpoint weights:** Another solution is to reduce the setpoint weights w_p and w_d . Regarding w_p , it is not so common to reduce it, but if it is, $w_p = 0.3$ is suggested [Åström & Hägglund \(1995\)](#). Regarding w_d , it is actually quite common to set $w_d = 0$, causing the setpoint to be removed completely from the derivative term. In many commercial controllers $w_d = 0$ is a fixed factory setting.

Example 12.3 Avoiding controller kicks during setpoint step change

Figure 12.8 shows responses in simulated PID control system with and without reduction of the setpoint weight in the P-term and the D-term, and with setpoint ramping. The reduced weights are $w_p = 0.3$ and $w_d = 0$. The PID parameters are $K_c = 3.6$, $T_i = 2.0$ s, $T_d = 0.5$ s. The simulations demonstrates that the control signal is smoother with reduced weights and with setpoint ramping.

[End of Example 12.3]

12.7.4 Integrator anti wind-up

The problem

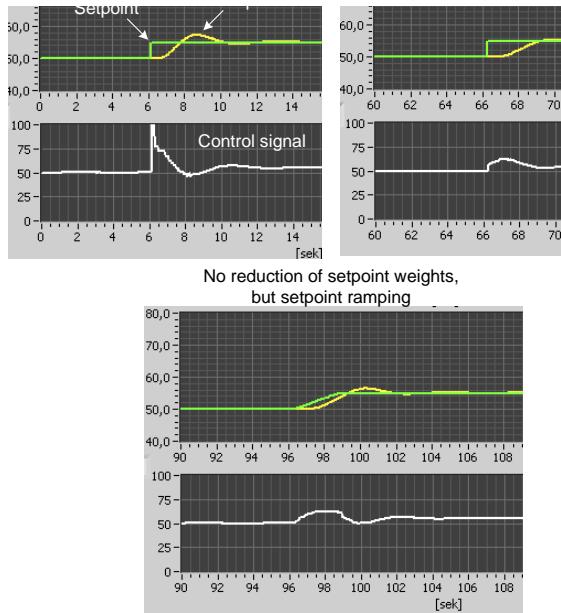


Figure 12.8: Example 12.3: Responses in PID control system with and without reduction of setpoint weight and with setpoint ramping

Wind-up is a problem related to the integral term of a PID controller – the I-term may get a large and continuously increasing value – *it winds up* – if large disturbances or large setpoints or sensor failure is causing the actuator to reach its saturation limits, i.e. a maximum limit and a minimum limit. Examples of such limits are: A power amplifier (for a heater or a motor) not deliver an infinitely amount of power; A valve can not have an infinitely large opening and can not be more closed than closed(!). Under normal process operation the control variable should not reach the saturation limits, but at extreme conditions, it can reach the limits.

We will study a concrete example: Figure 12.9 shows the front panel of a simulator for a temperature control system for a liquid tank with continuously mass flow. The responses shown in the figure are commented below. The disturbance is here the inlet temperature T_{in} , which is changed as a step from 40 °C to 10 °C at approx. 210 min and back to 40 °C at approx. 300 min. The temperature setpoint T_{SP} is 70 °C (constant). The controller has been tuned with the Ziegler-Nichols closed loop method. The maximum value of the control variable is 100 % and the minimum value is 0 %.

The PID controller used in the simulation shown in Figure 12.9 has no limitation on its integral term – hence it has no anti wind-up (later we will activate anti wind-up). In the simulation, T_{in} is reduced to 10 °C, causing the actuator (heating element) to go into saturation (100 %) trying to compensate for the (cold) disturbance. It can be shown that the control variable u should have a value of 122.5 % (which corresponds to more heat power than what is available) to be able to compensate for $T_{in} = 10$ °C. But because of the limitation of the heater capacity, the controller is not able to compensate fully for the disturbance, and the control error remains non-zero.

The non-zero error makes the integral term, u_i , increase, or: *wind up*. At $t = 300$ min, the

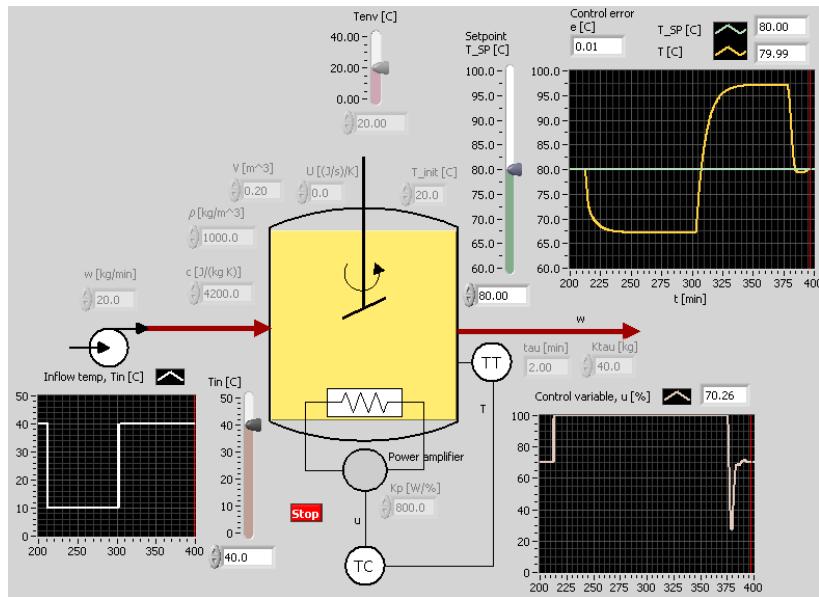


Figure 12.9: Temperature control without anti wind-up.

disturbance was set back to the normal value of 40 °C. It was observed (but not shown in Figure 12.9), that the integral term at that time reached appox. 2200 %! Due to the very large I-term and the disturbance being back at normal (“hot”) value, the total power delivered to the tanks is too large, causing the temperature to become too large. The control error has become negative, causing the I-term to decrease, which is good. But, it takes a long time until the I-term is again within a normal range, and in that time the temperature is about 17 degrees above the setpoint. Hence, there is a long-lasting large control error, which is a problem.

The solution

A practical PID controller must be able to cope with the possibility of integrator wind-up, that is, it must have some *integral anti wind-up* mechanism. Fortunately, you can assume that anti wind-up is implemented in commercial controllers.

The principle of an anti wind-up mechanism is simple: Since the problem is that the integral term increases continuously during actuator saturation, the solution is to halt the integration of the control error when the control signal reaches either its maximum or its minimum limit. Figure 12.10 illustrates the solution. The I term is represented with the tank, and anti windup is realized with a weir so that the integration of the liquid (the error) is halted at a certain maximum level.

How to realize anti windup in a discrete-time PID algorithm? Recall the I term of the PID algorithm:

$$u_{i,k} = u_{i,k-1} + \frac{K_p T_s}{T_i} e_k \quad (12.32)$$

Anti windup can be realized by stopping the integration (or accumulation) of u_i when the following condition is met, namely that a (preliminary) calculation of u is greater than the maximum of u , typically 100%, or less than the minimum of u , typically 0%. In (12.32), the

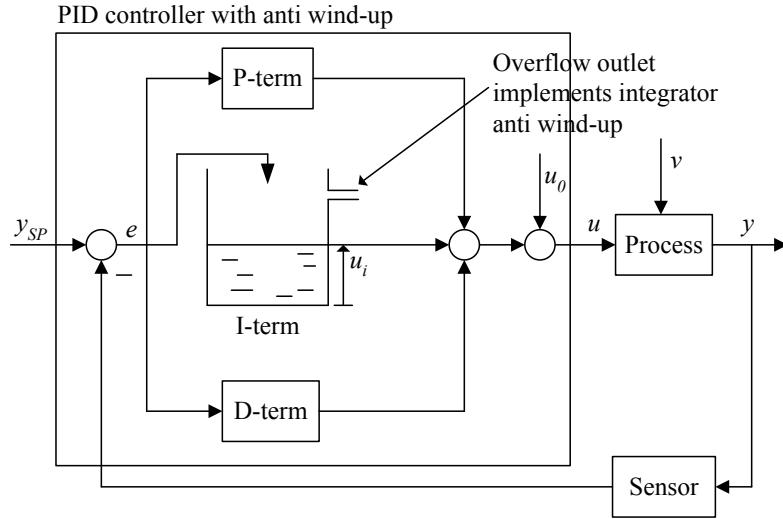


Figure 12.10: An analogy of integrator anti windup: A tank with weir to stop the integration at a certain level.

integration is stopped by forcing the term at the right to zero, so that the I term becomes

$$u_{i,k} = u_{i,k-1} + 0 \cdot e_k \quad (12.33)$$

Back to the temperature control system. Figure 12.11 shows the responses *with* integrator anti windup, which means that the updating of the I-term is stopped when the control signal is at its maximum (here 100 %) or minimum value (here 0 %). The simulations

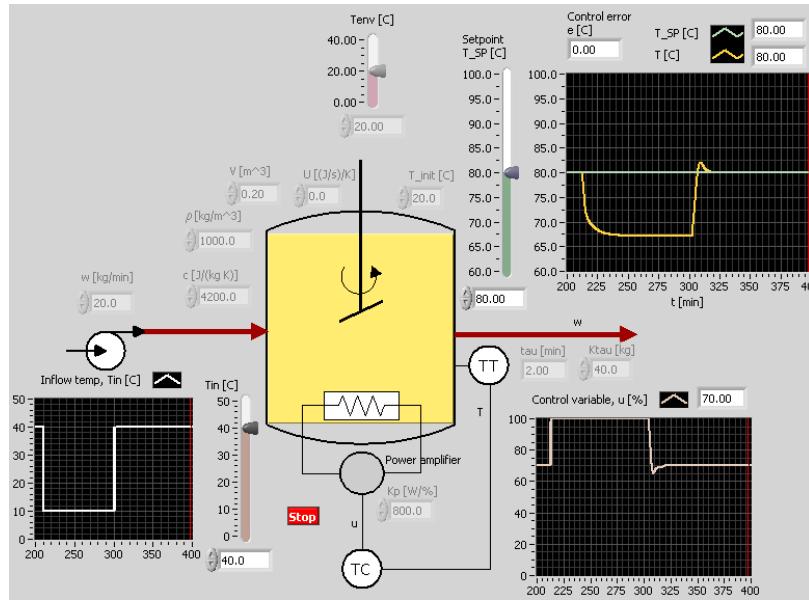


Figure 12.11: Temperature control with anti wind-up.

clearly show that it is beneficial to use integrator anti windup: The temperature returns much sooner to the setpoint after the disturbance has changed back to its normal value.

This is because the I-term stayed at a reasonable value (in the simulation it was observed that the I-term had the value of a few %) despite the control error was large due to the limit of the heater.

12.7.5 Bumpless transfer between manual/auto mode

It is important that the control signal does not jump (too much) when the controller is switched from automatic to manual mode, or from manual to automatic mode. In other words, the transfer between the automatic and manual modes should be *bumpless*. Bumpless transfer is implemented in commercial controllers.

We can implement bumpless transfer in a PID algorithm as follows:

- **Bumpless transfer from manual to automatic mode:** Set $u_{i,k-1} = 0$ to empty the I term just before switching between the modes.
- **Bumpless transfer from automatic to manual mode:** Set $u_{man} = u$, and halt the I term by setting $u_{i,k} = u_{i,k-1}$ during manual mode.

12.8 Problems for Chapter 12

Problem 12.1 Reverse or direct action?

Figure 12.12 shows a pressure control system. Assume that increasing the control signal to the valve increases the valve opening. Will you set the controller to have reverse action or direct action?

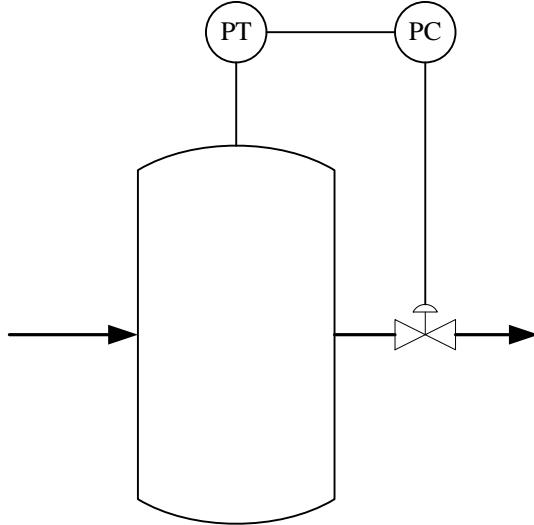


Figure 12.12: Pressure control system

Problem 12.2 Filter time constant

Assume that a time constant filter with time constant T_f is to be used on a tachometer⁴ reading on an electrical motor which itself can be modelled as a time constant system with time constant of 0.2 sec. Why is it not wise to set T_f equal to say 2 sec, which is a typical value of T_f in certain control loops in the process industry? Estimate a reasonable value of T_f .

Problem 12.3 Measurement scaling

Given a temperature sensor which represents temperatures, T , in the range [15 – 55 °C] with a measurement signal (current), M , in the range [4 – 20 mA], with a linear relation between these ranges. Find the scaling function with M as input and T as output on the following form:

$$T = aM + b \quad (12.34)$$

⁴A tachometer is a rotational speed sensor which gives a voltage being proportional to the speed.

Problem 12.4 Step response of PI controller

Suppose you want to verify that a PI controller works correctly according to the mathematical PI controller function. This can be done with a step response test, where a step is applied to the setpoint r input to the controller while the measurement input y_m is kept constant. Assume that

$$y_m(t) = A \quad (12.35)$$

and that the setpoint is increased from A to

$$r(t) = A + E \quad (12.36)$$

This implies that the control error e is increased as a step from zero to

$$e(t) = r(t) - y_m(t) = (A + E) - A = E \quad (12.37)$$

By comparing the observed (experimental) step response in the controller output u with the theoretical output, you can (hopefully) confirm that the mathematical operation of the controller is correct.

The PI controller function is

$$u = K_p e + \frac{K_p}{T_i} \int_0^t e dt \quad (12.38)$$

1. Calculate the step response in u (as a function of time) assuming that the control error is a step of amplitude E , and plot $u(t)$.
2. Figure 12.13 shows the step response in u for a given PI controller. The step amplitude of the control error was

$$E = 1 \quad (12.39)$$

Calculate K_c and T_i from the step response.

Problem 12.5 Gain and PB

What is the value of the controller gain K_c corresponding to proportional band $PB = 250\%$?

Problem 12.6 Temperature response with various controllers

Figure 12.14 shows an air heater. A fan with fixed speed blows air through the pipe. The fan opening can be varied manually. The air is heated by an electrical heater. The control signal u is the voltage signal which controls (adjusts) the power supplied to the heater. The temperature is measured with a thermistor which is a temperature-dependent resistance. In the experiments described below the controller is implemented in a PC with I/O-device (Input/Output-device). (In general, a control system should contain a measurement filter, but in this particular system a filter was not used.)

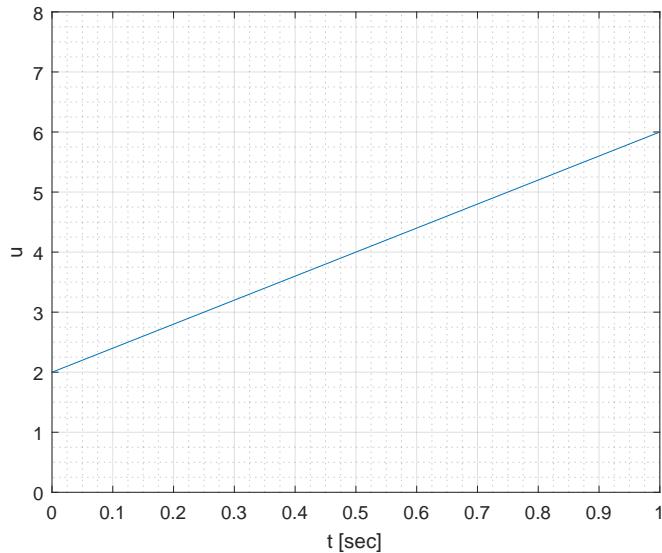


Figure 12.13: Step response in PI controller output

1. Draw a block diagram of the control system, including a measurement filter.
2. Figure 12.15 shows the response in the temperature y after a step in the temperature reference (setpoint) and after a step in the air inflow due to an increase of the fan opening. The air inflow or – equivalently the fan opening – can be regarded as a disturbance. In this experiment the control signal is constant, hence there is no feedback (no measurement-based or error-driven) control. Explain why there is no response in the temperature due to the reference change. And explain why there *is* a response after the disturbance (fan opening) change.
3. Assume that the temperature is controlled with a PID-controller (with proper parameter settings). Draw the principal temperature response after a reference step and after a disturbance step. You can “add” your curves to 12.15.)

Problem 12.7 *Filter time constant in D-term*

Given a PID controller with a lowpass filter acting on the derivative term. Assume that the derivative time T_d is 2.0 sec. Suggest a proper value of the filter time constant T_f .

Problem 12.8 *Derivative kick*

A derivative term – or D-term – with the possibility of setpoint weight reduction is

$$u_d = K_p T_d (w_d r - y)' \quad (12.40)$$

(It is here, for simplicity, assumed that there is no lowpass filter acting on the D-term.) Assume that there is no reduced weight of the setpoint, i.e.

$$w_d = 1 \quad (12.41)$$

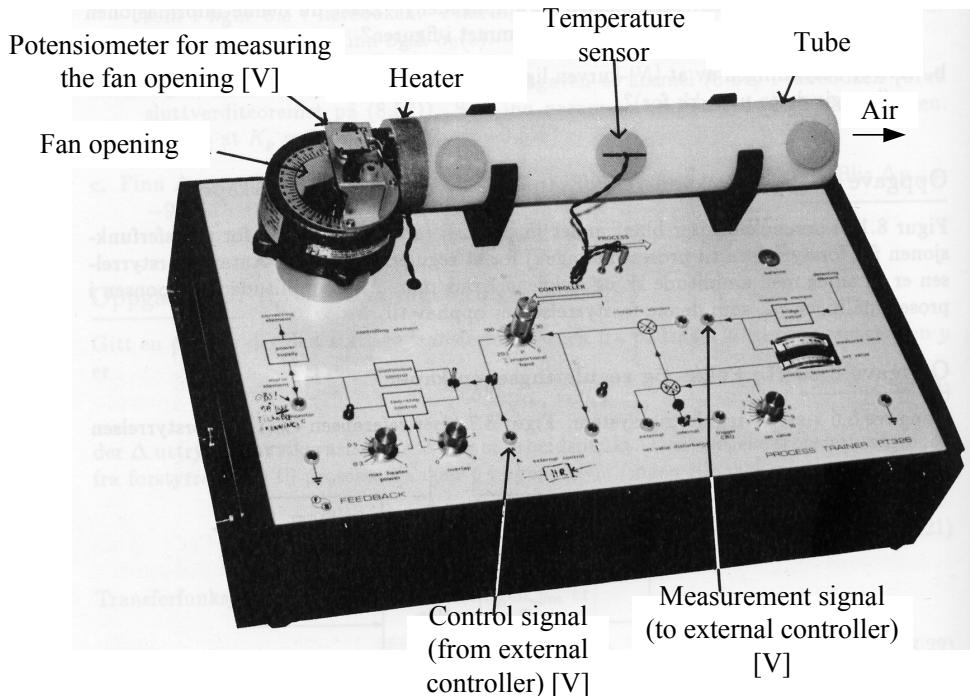


Figure 12.14: Process trainer

Assume that the process measurement y is constant, and that the setpoint r is changed as a step at time $t = 0$. Describe (qualitatively) the corresponding response in the control signal u_d due to this step. What kind of signal is this response?

Problem 12.9 *Anti windup*

One application where it is particularly important with anti windup is limiting control. Figure 12.16 shows a gas tank with one inlet and two outlets. The purpose of the normal control loop is to keep the gas pressure at the normal setpoint SP_1 , say 2 bar. The purpose of the limiting control loop is to limit the pressure to the higher setpoint SP_2 , say 4 bar. What is the control error of controller PC_2 under normal conditions? Why is it particularly important that this controller has anti windup?

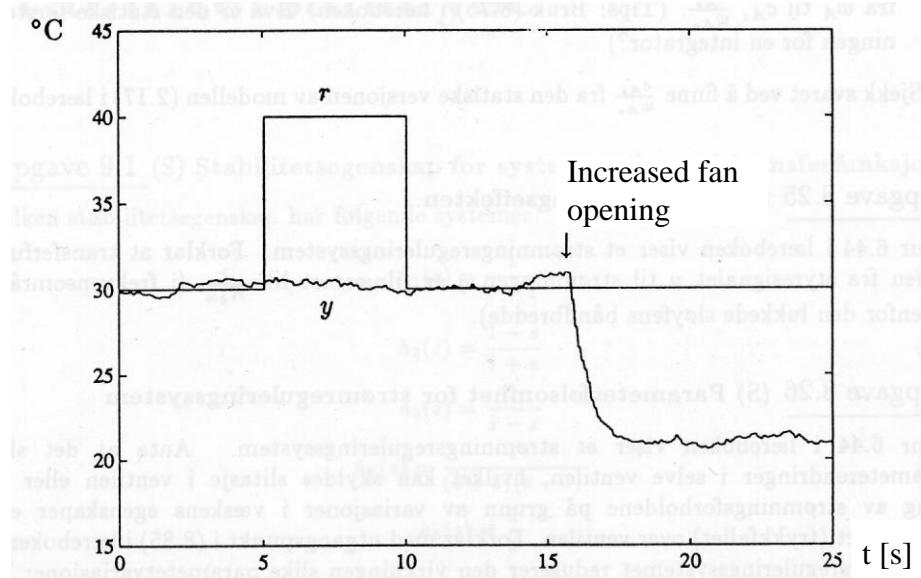


Figure 12.15: Temperature response with constant control signal

12.9 *Solutions to problems for Chapter 12*

Solution to Problem 12.1

Assume that for some reason the pressure is larger than the pressure reference. The controller must react to this by ensuring that the valve opening is increased, which means that the valve control signal is increased. Therefore, the controller must have direct action.

Solution to Problem 12.2

With $T_f = 2$ sec the filter will be much more sluggish than the motor. Quick motor speed changes will then be filtered or smoothed out (depending on how quick the real speed actually varies).

A good estimate of the filter time constant is one tenth of the process time constant:

$$\underline{\underline{T_f}} = \frac{0.2\text{s}}{10} = \underline{\underline{0.02\text{ s}}} \quad (12.42)$$

Solution to Problem 12.3

The slope a can be calculated from

$$\underline{\underline{a}} = \frac{T_{\max} - T_{\min}}{M_{\max} - M_{\min}} = \frac{55 - 15}{20 - 4} = \frac{40}{16} = \underline{\underline{2.5 \frac{\text{°C}}{\text{mA}}}} \quad (12.43)$$

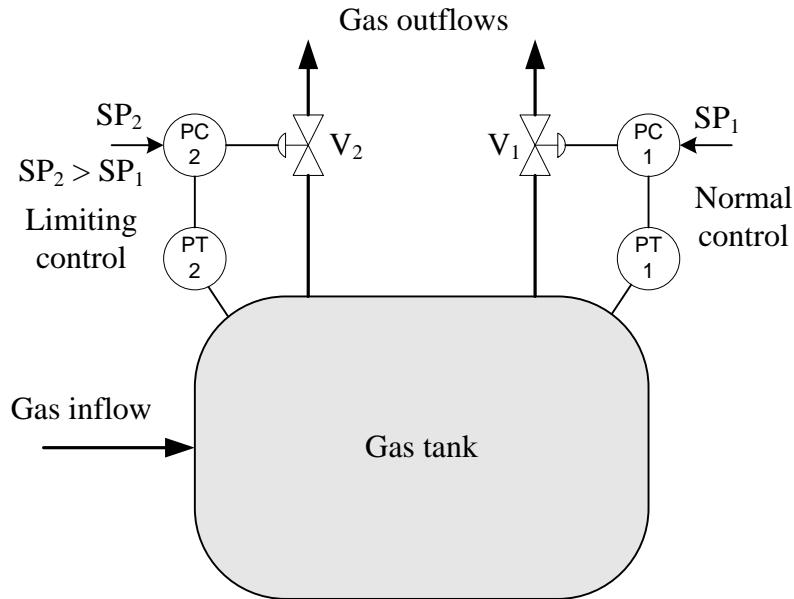


Figure 12.16: Limiting control

and

$$\underline{b} = T_{\min} - aM_{\min} = 15 \text{ }^{\circ}\text{C} - 2.5 \frac{\text{ }^{\circ}\text{C}}{\text{mA}} \cdot 4 \text{ mA} = \underline{5 \text{ }^{\circ}\text{C}} \quad (12.44)$$

Solution to Problem 12.4

1. Setting $e = E$ in the controller function gives

$$\underline{\underline{u(t)}} = K_c E + \frac{K_c}{T_i} \int_0^t E dt = K_c E + \frac{K_c}{T_i} Et \quad (12.45)$$

which is “constant plus ramp”. Figure 12.17 shows this step response.

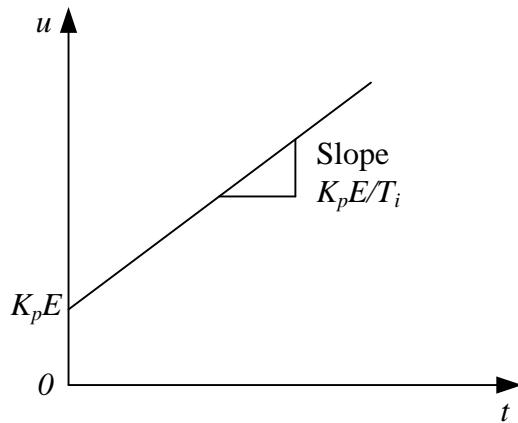


Figure 12.17: Step response of PI controller.

2. From Figure 12.13 we see that

$$u(0_+) = \underline{\underline{2}} = K_c E = K_c \cdot 1 = \underline{\underline{K_c}} \quad (12.46)$$

And we see that the slope is

$$\text{Slope} = 4 = \frac{K_c}{T_i} E = \frac{2}{T_i} \cdot 1 \quad (12.47)$$

which gives

$$\underline{\underline{T_i = 0.5 \text{ sec}}} \quad (12.48)$$

Solution to Problem 12.5

$$\underline{\underline{K_c}} = \frac{100}{\text{PB}} = \frac{100}{250} = \underline{\underline{0.4}} \quad (12.49)$$

Solution to Problem 12.6

1. The block diagram is shown in Figure 12.18.

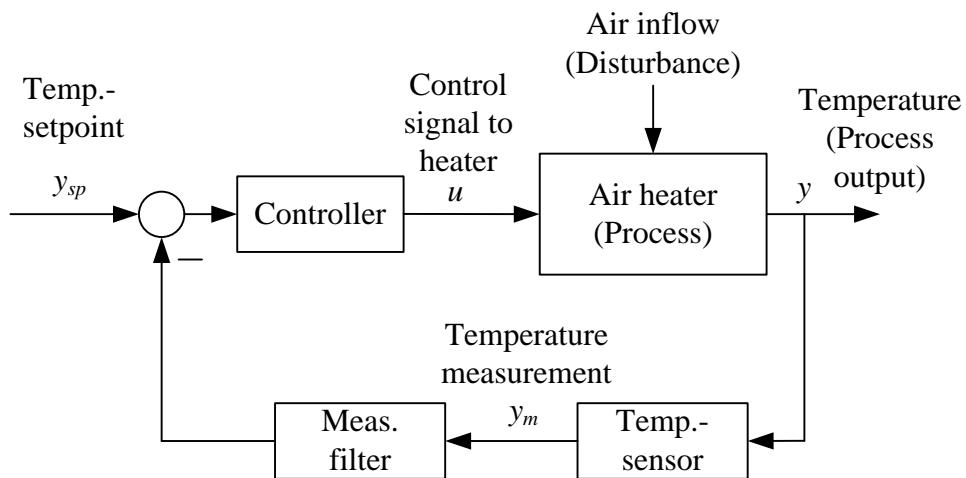


Figure 12.18: Block diagram of temperature control system

2. Since u_0 is not influenced by the temperature reference (setpoint), the control signal remains constant. Therefore, the reference does not influence the actual temperature. However, an increase of the fan opening will influence (reduce) the temperature because more cold air is blown into the pipe.
3. The PID-controller gives zero control error (in average), see Figure 12.19.

Solution to Problem 12.7

$$\underline{\underline{T_f}} = 0.1 T_d = 0.1 \cdot 2 = \underline{\underline{0.2 \text{ s}}} \quad (12.50)$$

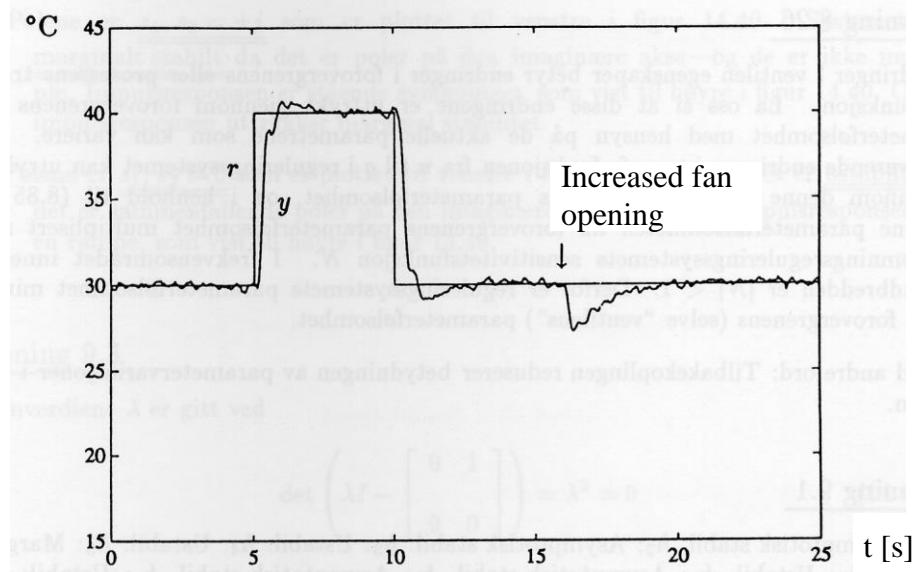


Figure 12.19: Temperature response with PID controller

Solution to Problem 12.8

The control signal is

$$u_d = K_c T_d \frac{d(\text{Setpoint step} - \text{constant})}{dt} \quad (12.51)$$

The time-derivative of a setpoint step is an *impulse*, which is a signal of infinite amplitude and with infinite duration, see Figure 12.20. So, the stepwise change of the setpoint causes an impulse-like change of the control signal. That is the *derivative kick*.

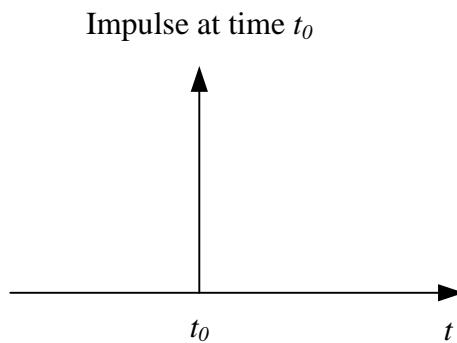


Figure 12.20: Impulse.

Solution to Problem 12.9

Under normal conditions the control error of PC₂ is

$$e_2 = SP_1 - SP_2 = 4 - 2 \text{ bar} = 2 \text{ bar} \quad (12.52)$$

This sustained non-zero control error would have caused the integral term of controller PC₂ to wind up – or, actually, to wind down to a very large negative value, making the controller virtually inactive if the pressure rises and comes close to SP₂, which makes the controller useless for limiting control.

Chapter 13

On-off controller

13.1 The ideal On-off controller

The ideal On-off controller is probably the simplest controller there is. On-off controllers may be an alternative to PID controllers, especially in temperature control. For example, room temperature is commonly controlled with a thermostat, which is an On-off controller.

The On-off controller function is:

$$\begin{aligned} \text{if } e \geq 0 : & \quad u = u_{\max} \\ \text{if } e < 0 : & \quad u = u_{\min} \end{aligned} \tag{13.1}$$

where

$$e = r - y \tag{13.2}$$

is the control error. r is the setpoint (reference), and y is the process measurement. u_{\max} is the maximum control signal, typically 100%. u_{\min} is the minimum control signal, typically 0%.

Figure 13.1 illustrates the On-off controller.

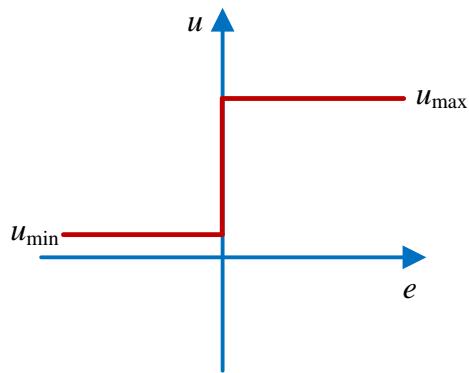


Figure 13.1: On-off controller.

The On-off controller works as follows: When the e is positive, i.e. when y is below r , u is u_{\max} , causing y to increase. Eventually y becomes larger than r , so that e becomes negative, which sets u to u_{\min} , causing y to decrease. Eventually y becomes less than r , so that e becomes positive, and then the scenario repeats. Hence, the On-off controller causes the control system to oscillate.

The period of the oscillations is given by the dynamic properties of the process. The amplitude of the oscillations in u and in y are also given by the dynamic properties of the process. We can reduce the amplitudes by reducing the difference between u_{\max} and u_{\min} , but with the danger of limiting the control signal so that it will stay at either u_{\max} or u_{\min} in case of large process disturbances.

The benefits of the On-off controller are:

- It is a very “quick” controller. It compensates quickly for process disturbances – quicker than a PID controller.
- It is tuned very easily. In principle, the only tuning is selecting the values of u_{\max} and u_{\min} .

The main drawback about the On-off controller is the unavoidable oscillations.

Example 13.1 Temperature control with On-off controller

Figure 13.2 shows a simulation of a temperature control system. $u_{\max} = 80$ kW, and $u_{\min} = 0$ kW. The oscillatory behaviour is clear. Note that the mean value of the control error is different from zero (the mean temperature is slightly below its setpoint), which is typical for On-off control.

[End of Example 13.1]

On-off controller for processes with negative process gain

Actually, the On-off controller (13.1) applies only to processes which have a so-called positive process gain, i.e. the process measurement increases when the control signal increases (like in a “heating” thermal process where an increase of the control signal to the heater makes the process temperature increase). If the process has a negative gain, i.e. the process measurement decreases when the control signal increases (like in a “cooling” thermal process where an increase of the control signal to the cooler makes the process temperature decrease), the On-off controller is instead:

$$\begin{aligned} \text{if } e \geq 0 : & \quad u = u_{\min} \\ \text{if } e < 0 : & \quad u = u_{\max} \end{aligned} \tag{13.3}$$

Figure 13.3 illustrates the On-off controller for processes with negative process gain.

A practical problem with the ideal On-off controller

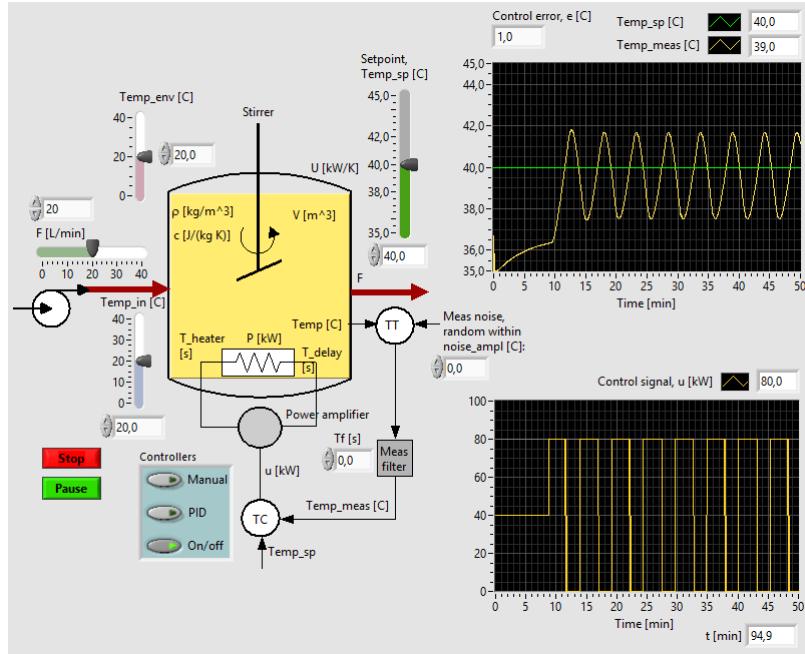


Figure 13.2: Simulation of a temperature control system with an On-off controller. (http://techteach.no/simview/temp_control_pid_onoff)

In the following, we assume that the On-off controller is (13.1).

There is a practical problem with the ideal On-off controller. If the process measurement, y , is noisy, also the controller error, e , is noisy. A noisy e may make the control signal, u , switch abruptly between u_{\min} and u_{\max} . This very unfortunate, especially if the actuator is mechanical.

Example 13.2 Temperature control with On-off controller with noisy measurement

Figure 13.4 shows a simulation of the same temperature control system as in Example 13.1, but where uniformly distributed random noise between ± 1.0 °C has been added to the temperature measurement. We can see that the control signal varies abruptly when the error is around zero.

[End of Example 13.2]

13.2 The practical On-off controller

In Section 13.1 we saw that measurement noise caused the ideal On-off controller to change abruptly between u_{\min} and u_{\max} when the control error varied around zero. Such abrupt changes are unfortunate, and should be avoided, particularly if the actuator is mechanical, like a mechanical relay, or a pump, or a valve, or a motor.

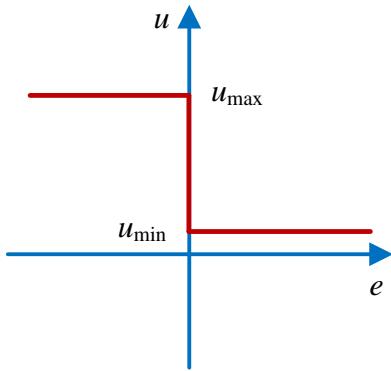


Figure 13.3: On-off controller for processes with negative process gain.

How can we avoid such abrupt changes? We can include a deadband in the On-off controller! The deadband should be larger than the maximum amplitude of the measurement noise. This solution is shown in Figure 13.5.

A drawback about including the deadband is that the amplitude of the oscillations in y will be somewhat larger than without deadband. Furthermore, the period of the oscillations will increase, but this is hardly a drawback.

Example 13.3 On-off controller with deadband

Figure 13.4 in Example 13.2 shows a simulation of a temperature control system where uniformly distributed random noise between ± 1.0 °C was added to the temperature measurement, causing u to change abruptly when e is around zero. Now we include a deadband with $D_e = 2$ °C in the On-off controller. Figure 13.6 shows the results of a simulation. As expected, the abrupt changes in u are now eliminated. The period of the oscillations is somewhat larger than without deadband, cf. Figure 13.4.

[End of Example 13.3]

13.3 Programming the On-off controller

To appear.

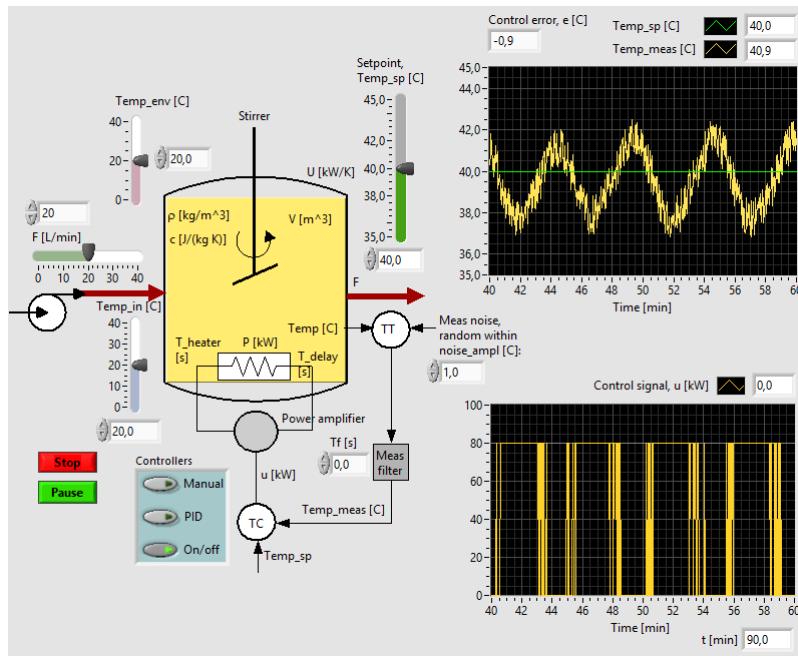


Figure 13.4: Simulation of a temperature control system where uniformly distributed random noise between ± 1.0 °C has been added to the temperature measurement. (http://techteach.no/simview/temp_control_pid_onoff)

13.4 Problems for Chapter 13

Problem 13.1 To appear

To appear

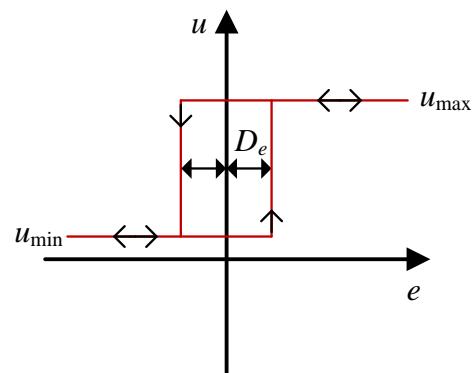


Figure 13.5: On-off controller with deadband.

13.5 *Solutions to problems for Chapter 13*

Solution to Problem 13.1

To appear

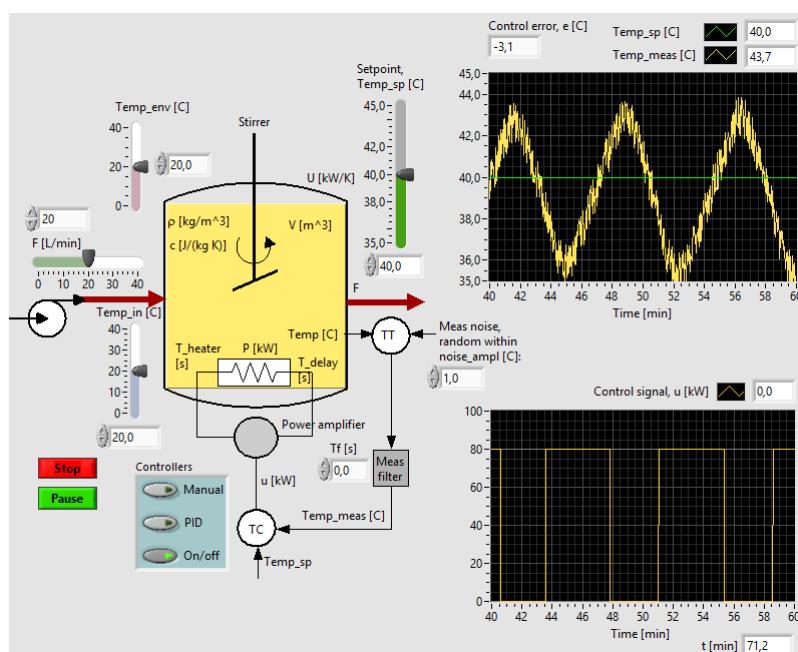


Figure 13.6: Temperature control with On-off controller with deadband

Chapter 14

Transfer functions of feedback control systems

14.1 Introduction

Transfer functions of feedback control systems can be useful for:

- simulation
- analytical calculation of responses
- frequency response analysis

14.2 Definition of setpoint tracking and disturbance compensation

Figure 14.1 shows a principal block diagram of a control system.

There are two input signals to the control system, namely the setpoint y_{sp} and the disturbance d . The value of the control error e is our primary concern (it should be small, preferably zero). Therefore we can say that e is the (main) output variable of the control system. The value of e expresses the *performance* of the control system: The less e , the higher performance. e is influenced by r and d . Let us therefore define the following two properties of control systems:

- The **setpoint tracking property** of the control system concerns the relation between y_{sp} and e .
- The **disturbance compensation property** of the control system concerns the relation between d and e .

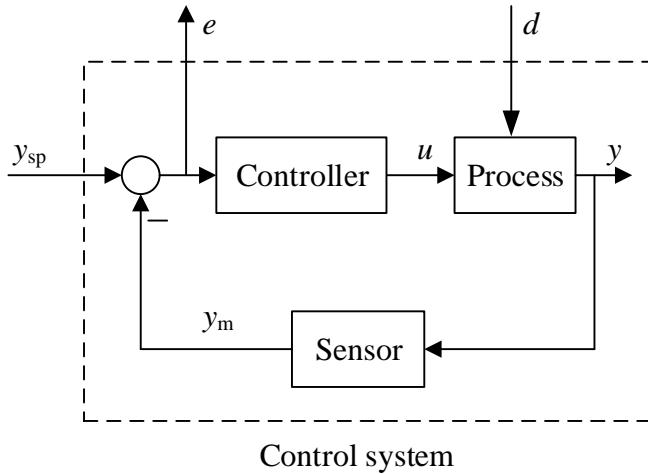


Figure 14.1: Principal block diagram of a control system.

Totally, the setpoint tracking and disturbance compensation properties determine the performance of the control system.

14.3 The Sensitivity transfer function

We assume that the control system has a transfer function-based block diagram as shown in Figure 14.2.

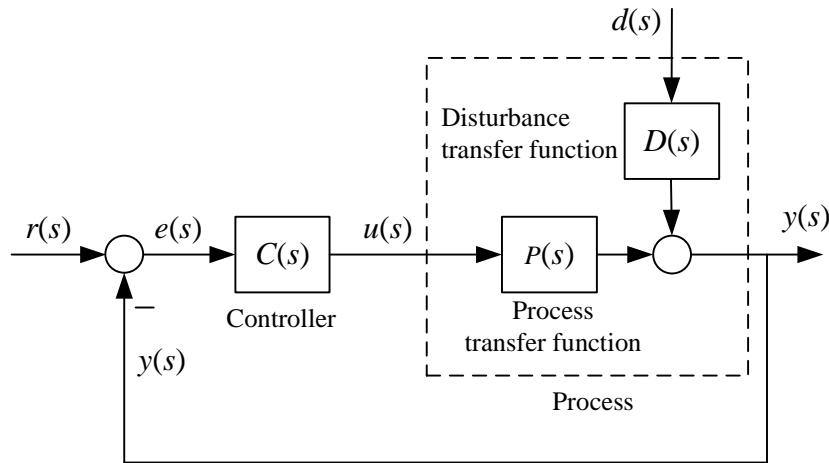


Figure 14.2: Transfer function-based block diagram of a control system.

We regard the reference r and the disturbance d as input variables and the control error e as the output variable of the system. Thus, we will derive the transfer function from r to e and the transfer function from d to e . From the block diagram in Figure 14.2 we can write

the following expressions for $e(s)$ (for simplicity, dropping the argument s in the following):

$$e = r - y \quad (14.1)$$

$$= r - (Dd + Pu) \quad (14.2)$$

$$= r - (Dd + PCe) \quad (14.3)$$

Solving (14.3) for e gives:

$$e = \left(\frac{1}{1+CP} \right) r + \left(-\frac{1}{1+CP} D \right) d \quad (14.4)$$

Let us define the *loop transfer function* as the product of the transfer functions in the loop, i.e.:

$$L \stackrel{\text{def}}{=} CP \quad (14.5)$$

and the *sensitivity transfer function*:

$$S \stackrel{\text{def}}{=} \frac{1}{1+L} \quad (14.6)$$

With these two definitions, (14.4) can be written:

$$e = \underbrace{S \cdot r}_{e_r} + \underbrace{(-SD) \cdot d}_{e_d} \quad (14.7)$$

where:

$$e_r = S \cdot r \quad (14.8)$$

and

$$e_d = (-SD) \cdot d \quad (14.9)$$

From (14.7) we can calculate the control error for any reference and any disturbance signal (assuming we know their Laplace transforms).

In the following we discuss e_r and e_d .

The response in the error due to the setpoint

The response in the control error due to the *reference (setpoint)* is

$$e_r(s) = S(s)r(s) = \frac{1}{1+L(s)}r(s) \quad (14.10)$$

which gives a quantitative expression of the *tracking property* of the control system.

The *static* tracking property is given by static error when r is constant. This error can be

calculated as follows:¹

$$e_{r,s} = \lim_{t \rightarrow \infty} e_r(t) \quad (14.11)$$

$$= \lim_{s \rightarrow 0} s \cdot e_r(s) \quad (14.12)$$

$$= \lim_{s \rightarrow 0} s \cdot S(s)r(s) \quad (14.13)$$

$$= \lim_{s \rightarrow 0} s \cdot S(s) \frac{r_s}{s} \quad (14.14)$$

$$= S(0)r_s \quad (14.15)$$

From (14.15) we see that *the setpoint tracking property of the control system are good if the sensitivity transfer function S has small (absolute) value – ideally zero.*

The response in the error due to the disturbance

The response in the control error due to the disturbance is

$$e_d(s) = -S(s)D(s)d(s) \quad (14.16)$$

which expresses the *disturbance compensation property* of the control system.

The *static* compensation property is given by

$$e_{d_s} = \lim_{t \rightarrow \infty} e_d(t) = \lim_{s \rightarrow 0} s \cdot e_d(s) \quad (14.17)$$

$$= \lim_{s \rightarrow 0} s \cdot [-S(s)D(s)d(s)] \quad (14.18)$$

$$= \lim_{s \rightarrow 0} s \cdot \left[-S(s)D(s) \frac{d_s}{s} \right] \quad (14.19)$$

$$= -S(0)D(0)d_s \quad (14.20)$$

From (14.20) we see that the *disturbance compensation property is good if the sensitivity transfer function S has a small (absolute) value (close to zero).*

14.4 The Tracking transfer function

The *tracking transfer function T(s)* is the transfer function from the setpoint y_{sp} to the process output measurement y_m :

$$y_m(s) = T(s)r(s) \quad (14.21)$$

From the block diagram in Figure 14.2, or by setting $e_r(s) \equiv r(s) - y_m(s)$ for $e_r(s)$ in (14.10), we can find the tracking transfer function $T(s)$ as the transfer function from y_{sp} to y_m :

$$\frac{y_m(s)}{r(s)} = T(s) = \frac{C(s)P(s)}{1 + C(s)P(s)} = \frac{L(s)}{1 + L(s)} = 1 - S(s) \quad (14.22)$$

¹Here the Final Value Theorem of the Laplace transform is used.

The *static* tracking property is given by the static tracking transfer function $T(0)$:

$$y_{m_s} = \lim_{t \rightarrow \infty} y(t) \quad (14.23)$$

$$= \lim_{s \rightarrow 0} s \cdot y_{m_s}(s) \quad (14.24)$$

$$= \lim_{s \rightarrow 0} s \cdot T(s)r(s) \quad (14.25)$$

$$= \lim_{s \rightarrow 0} s \cdot T(s) \frac{r_s}{s} \quad (14.26)$$

$$= T(0)r_s \quad (14.27)$$

The *tracking property is good if the tracking transfer function T has (absolute) value equal to or close to 1* (since then y will be equal to or close to r).

14.5 Analytical calculation of responses with transfer functions

To appear

14.6 Problems for Chapter 14

Problem 14.1 Transfer function of feedback control system

Given a process with the following transfer function from control signal to measurement signal:

$$P(s) = \frac{K}{s} \quad (14.28)$$

1. Characterize this process in terms of its dynamics (just a single term is expected in your answer).
2. Calculate the gain K_c of a P controller for the process so that the control system gets a time constant of T_c [s].
3. What is the pole of the control system?
4. Assuming K is negative, does the controller have reverse action or direct action?

Problem 14.2 Block diagram of feedback control system with noise and filter

Figure 14.1 shows a principal block diagram of a control system. The setpoint y_{sp} and the disturbance v are two input signals to the control system. There is actually a third input that is present in practical control systems, namely measurement noise n , which is typically a random signal. Assume that n is added to the measurement signal which is the output of the sensor. Most control systems contain a lowpass filter which attenuates the measurement noise so that the resulting measurement signal entering the controller becomes smoother.

Draw a block diagram of feedback control system where measurement noise and filter are included.

Problem 14.3 Finding the loop transfer function

Assume given a control system as shown in Figure 14.2. The transfer functions are as follows:

$$P(s) = \frac{K_u}{T_u s + 1} e^{-\tau s} \quad (14.29)$$

$$D(s) = \frac{K_d}{T_d s + 1} e^{-\tau s} \quad (14.30)$$

$$C(s) = K_c \frac{T_i s + 1}{T_i s} \text{ (PI controller)} \quad (14.31)$$

Find the loop transfer function $L(s)$, the sensitivity transfer function $S(s)$, and the tracking transfer function $T(s)$.

14.7 *Solutions to problems for Chapter 14*

Solution to Problem 14.1

1. Process dynamics: Integrator.
2. Tuning the P controller: Controller transfer function is

$$H_c(s) = K_c \quad (14.32)$$

Loop transfer function:

$$L(s) = C(s)P(s) = K_c \frac{K}{s} \quad (14.33)$$

Tracking transfer function:

$$T(s) = \frac{L(s)}{1 + L(s)} = \frac{K_c K}{s + K_c K} = \frac{1}{\frac{1}{K_c K} s + 1} = \frac{1}{T_c s + 1} \quad (14.34)$$

The closed loop time constant is

$$T_c = \frac{1}{K_c K} \quad (14.35)$$

Solving for K_c gives

$$K_c = \frac{1}{T_c K} \quad (14.36)$$

3. The pole of control system is the root of the characteristic equation which is:

$$s + K_c K = 0 \quad (14.37)$$

Hence, pole becomes

$$p = -K_c K = -\frac{1}{T_c} \quad (14.38)$$

4. Assuming K is negative, K_c becomes negative (since T_c must be positive to have a stable control system). A negative K_c means that the controller has direct action.

Solution to Problem 14.2

Figure 14.3 shows a block diagram where the measurement noise n and a (lowpass) filter is included.

Solution to Problem 14.3

Loop transfer function:

$$L(s) = C(s)P(s) = K_c \frac{T_i s + 1}{T_i s} \frac{K_u}{T_u s + 1} e^{-\tau s} \quad (14.39)$$

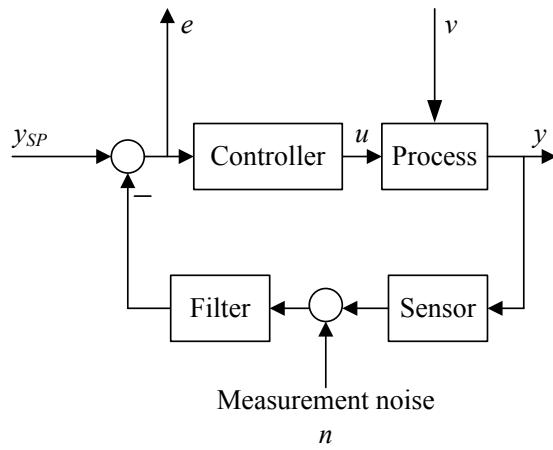


Figure 14.3: Block diagram of feedback control system where measurement noise and filter are included.

Sensitivity transfer function:

$$S(s) = \frac{1}{1 + L(s)} \quad (14.40)$$

$$= \frac{1}{1 + K_c \frac{T_i s + 1}{T_i s} \frac{K_u}{T_u s + 1} e^{-\tau s}} \quad (14.41)$$

$$= \frac{T_i s (T_u s + 1)}{T_i s (T_u s + 1) + K_c K_u (T_i s + 1) e^{-\tau s}} \quad (14.42)$$

Tracking transfer function:

$$T(s) = \frac{L(s)}{1 + L(s)} \quad (14.43)$$

$$= \frac{K_c \frac{T_i s + 1}{T_i s} \frac{K_u}{T_u s + 1} e^{-\tau s}}{1 + K_c \frac{T_i s + 1}{T_i s} \frac{K_u}{T_u s + 1} e^{-\tau s}} \quad (14.44)$$

$$= \frac{K_c K_u (T_i s + 1) e^{-\tau s}}{T_i s (T_u s + 1) + K_c K_u (T_i s + 1) e^{-\tau s}} \quad (14.45)$$

Chapter 15

Simulation of control systems

To appear.

15.1 Introduction

To appear.

15.2 Simulation with block-diagram simulation tools (OpenModelica)

To appear.

15.3 Simulation with programmed simulation algorithms (Python)

To appear.

15.4 Simulation with transfer functions (Python Control package)

To appear.

Chapter 16

Tuning of PID controllers

16.1 Introduction

PID controller tuning is to find proper values of the controller parameters K_c , T_i and T_d so that the specifications of the control system are met. Generally, the specifications comprise:

- Stability
- Speed, either
 - Fastness. In most cases, we want the control system to act as fast as possible.
 - Or: Slowness. Some control systems should act slowly, as in averaging level control, cf. Section 18.4.

Below are some comments to these specifications:

Stability

Ziegler and Nichols [Ziegler & Nichols \(1942\)](#) defined “ok stability” as “one quarter decay ratio”, see Figure 16.1. The figure shows the response due to a step change in the setpoint. The ratio between amplitudes A_2 and A_1 is one quarter:

$$\frac{A_2}{A_1} = \frac{1}{4} \quad (16.1)$$

The one quarter decay ratio is the stability you can expect for a control system where you have used one of the Ziegler-Nichols tuning methods, however, there is no guarantee that the ratio is actually one quarter. Most control practitioners are not so happy with such a stability – it should be better (smaller ratio; i.e. better damping). This chapter presents several methods that typically gives better stability.

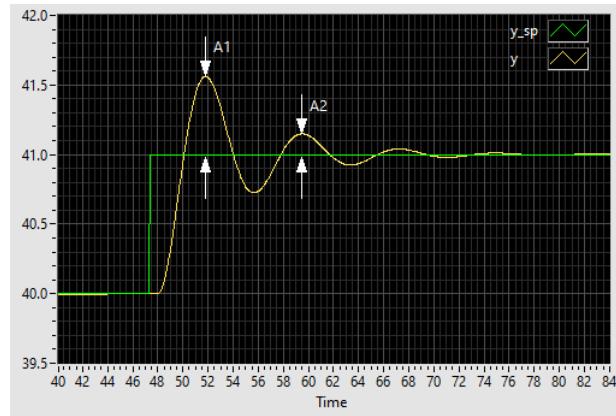


Figure 16.1: The definition of “one quarter decay ratio”

Fastness

Typically, there is a compromise between stability and fastness. This is illustrated in Figure 16.2. It shows the response in the process output variable due to a step change of the setpoint. If you want very good stability in the control system, the system will not be so fast. If you want very fast control, the system will have poor stability.

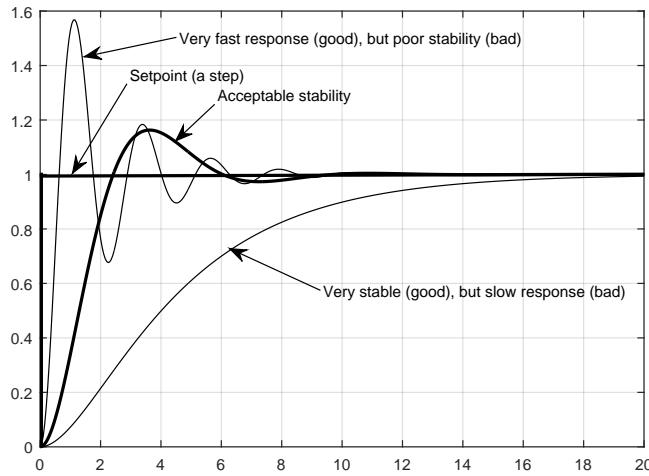


Figure 16.2: The typical compromise between good stability as fastness of a control systems

Since stability is the most important specification, the controller tuning aims at giving satisfactory stability, and the obtained fastness must be accepted.

Slowness

Averaging level control, cf. Section 18.4, is one of relatively few cases where we want sufficiently slow control action. In averaging control, the level control of a magazine or reservoir is made slow by purpose. The slow level control will attenuate variations in the inflow to the magazine, so the outflow, which is manipulated by the level controller, varies less than the inflow – therefore the term averaging control. Stability is “guaranteed” in such a control system.

The controller tuning methods

This chapter presents several controller tuning methods. Most of them are experimental methods. But that does not mean that these methods can be applied only to a physical system on which you perform experiments; You may use the methods on a *simulator* of the control system!

Some times I am asked which of the controller tuning methods is the best, or which do I recommend trying first. Based on my work is quite a few practical projects, I think the Skogestad method, cf. Section 16.8, is the best method overall. It can be used both for fast control and for slow control, and the controller can be retuned directly from the tuning formulas if the process dynamics, for example the process gain or the process time delay, varies. And as an experimental method, it requires in principle only one relatively short experiment, while other methods typically requires several experiments.

16.2 The Ziegler-Nichols closed loop method

Introduction

Ziegler and Nichols published in 1942 a paper [Ziegler & Nichols \(1942\)](#) where they described two methods for tuning the parameters of P-, PI- and PID controllers. These two methods go under various names, e.g. the Ziegler-Nichols closed loop method (which is described in this section) and the Ziegler-Nichols open loop method, cf. Section 16.9. One may say that the Ziegler-Nichols closed loop method remains as the most known PID tuning method, despite decades of research and development. The Ziegler-Nichols closed loop method is also often used as a reference method when new methods are tested.

There is no guarantee that a given control system tuned with one of the Ziegler-Nichols methods actually gets a stability as “one quarter decay ratio”, but the stability is probably not very different.

The method

The Ziegler and Nichols closed loop method is based on experiments executed on an established control loop (a real system or a simulated system), see Figure 16.3.

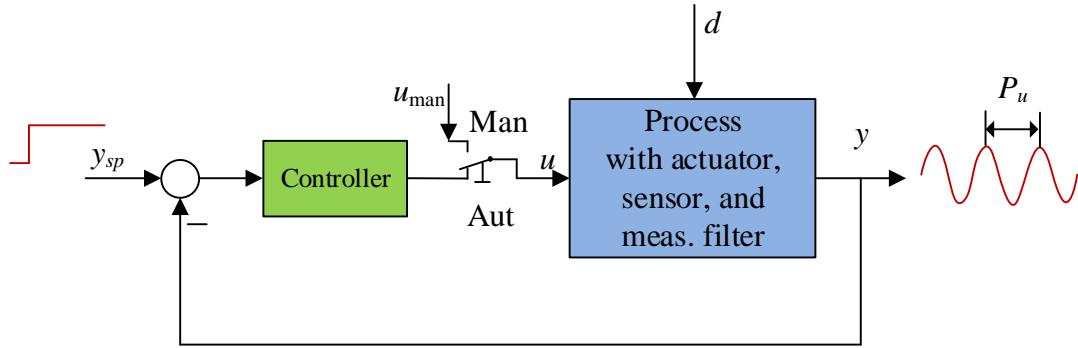


Figure 16.3: The Ziegler-Nichols closed loop method is applied to an established control system.

The tuning procedure is as follows:

1. Bring the process to (or as close to as possible) the specified *operating point* of the control system to ensure that the controller during the tuning is “feeling” representative process dynamic¹ and to minimize the chance that variables during the tuning reach limits. You can bring the process to the operating point by manually adjusting the control variable, with the controller in manual mode, until the process variable is approximately equal to the setpoint.
2. Turn the PID controller into a *P controller* with gain $K_c = 0$ (set $T_i = \infty$ and $T_d = 0$). Close the control loop by setting the controller in automatic mode.
3. Increase K_c until there are *sustained oscillations* in the signals in the control system, e.g. in the process measurement, after an excitation of the system. (The sustained oscillations corresponds to the system being on the stability limit.) This K_c value is denoted the *ultimate (or critical) gain*, $K_{c,u}$.

The excitation can be a step in the setpoint. This step must be small, for example 5% of the maximum setpoint range, so that the process is not driven too far away from the operating point where the dynamic properties of the process may be different. On the other hand, the step must not be too small, or it may be difficult to observe the oscillations due to the inevitable measurement noise.

It is important that $K_{c,u}$ is found without the actuator being driven into any saturation limit (maximum or minimum value) during the oscillations. If such limits are reached, you will find that there will be sustained oscillations for any (large) value of K_c , e.g. 1000000, and the resulting K_c -value (as calculated from the Ziegler-Nichols formulas, cf. Table 16.1) is useless (the control system will probably be unstable).

¹This may be important for nonlinear processes.

One way to say this is that $K_{c,u}$ must be the smallest K_c value that drives the control loop into sustained oscillations.

4. Measure the *ultimate (or critical) period* P_u of the sustained oscillations.
5. Calculate the controller parameter values according to Table 16.1, and use these parameter values in the controller.

The lowpass filter time constant T_f (cf. Section 3.4.8.6) can be set to

$$T_f = 0.1T_d \quad (16.2)$$

(if no other specification exists).

If the stability of the control loop is poor, try to improve the stability by decreasing K_c .

Table 16.1: The Ziegler-Nichols controller settings.

	K_c	T_i	T_d
P controller	$0.5K_{c,u}$	∞	0
PI controller	$0.45K_{c,u}$	$\frac{P_u}{12}$	0
PID controller	$0.6K_{c,u}$	$\frac{P_u}{2}$	$\frac{P_u}{8} = \frac{T_i}{4}$

Example 16.1 PI controller tuning with the Ziegler-Nichols closed loop method

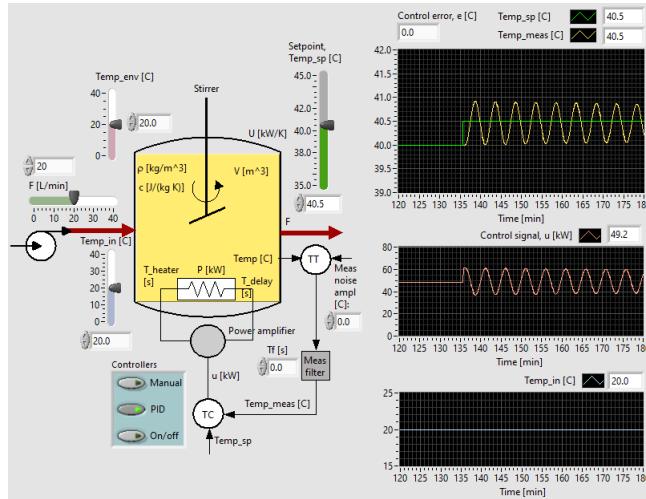


Figure 16.4: The Ziegler-Nichols tuning experiment

Figure 16.4 shows a temperature control system with simulated responses of the Ziegler-Nichols tuning experiment. The ultimate gain is

$$K_{c,u} = 26.5 \quad (16.3)$$

The ultimate periode is read off as

$$P_u = 5 \text{ min} = 300 \text{ s} \quad (16.4)$$

This gives the following Ziegler-Nichols PI settings:

$$K_c = 0.45K_{c,u} = 0.45 \cdot 26.5 = 11.9 \quad (16.5)$$

and

$$T_i = \frac{P_u}{1.2} = \frac{300 \text{ s}}{1.2} = 250 \text{ s} \quad (16.6)$$

Figure 16.5 shows the simulated responses with the above PI settings. The excitations are step changes in setpoint and disturbance (inlet temperature). The stability can be characterized with an amplitude damping ratio of approximately 0.30 which is somewhat larger (worse) than 0.25 which is the ratio that Ziegler and Nichols prescribed.

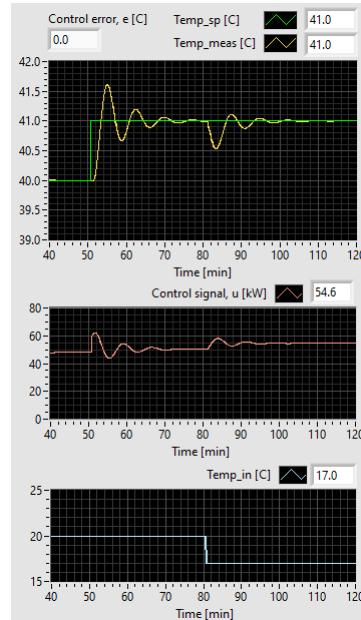


Figure 16.5: Simulated temperature control system with PI settings according to the Ziegler-Nichols method.

[End of Example 16.1]

Some comments to the Ziegler-Nichols closed loop method

1. *You do not know in advance the amplitude of the sustained oscillations.* The amplitude depends partly of the initial value of the process measurement. By using the Åström-Hägglund's tuning method described in Section 16.5 instead of the Ziegler-Nichols closed loop method, you have full control over the amplitude, which is beneficial, of course.
2. *For sluggish processes it may be time consuming to find the ultimate gain in physical experiments.* The Åström-Hägglund's method reduces this problem since the oscillations come automatically.

3. If the operating point varies and if the process dynamic properties depends on the operating point, you should consider using some kind of *adaptive control or gain scheduling*, where the PID parameter are adjusted as functions of the operating point. If the controller parameters shall have fixed value, they should be tuned in the worst case as stability is regarded. This ensures proper stability if the operation point varies. The worst operating point is the operation point where the process gain has its greatest value and/or the time delay has its greatest value.
4. *The responses in the control system may become unsatisfactory* with the Ziegler-Nichols method. 1/4 decay ratio may be too much, that is, the damping in the loop is too small. A simple re-tuning in this case is to reduce the K_c somewhat, for example by 20%.

In the beginning

The Ziegler and Nichols methods have definitely proven to be useful, but they actually met some resistance in the beginning. In [Blickley \(1990\)](#) Ziegler reports from a meeting in the American Society of Mechanical Engineers (ASME): “*The questions at the end were pretty bitter because they (the ‘old-timers’) could not stomach this ultimate sensitivity². The questions got worse and worse, and I was answering them. Finally a little guy in the back of the room got up. He was from Goodyear. Since he was on the committee he had received an advance copy of the paper. He stuttered some, and stammered out for all to hear: ‘We had one process in our plant, a very bad one, and so I tried this method and it just worked perfectly.’ That broke up the meeting.*”

16.3 Relaxed Ziegler-Nichols PI settings

It may happen that the stability of the control system becomes somewhat poor with the Ziegler-Nichols method. In [Haugen & Lie \(2013\)](#), alternative formulas for tuning PI controllers based on the Ziegler-Nichols oscillations are derived. The method will provide better stability than tuning with the original Ziegler-Nichols method. We can denote the resulting settings the Relaxed Ziegler-Nichols settings.

The PI settings are derived from a combination of the Ziegler-Nichols method and the Skogestad method. The PI setting formulas are:

$$K_c = \frac{2}{\pi(1+k_r)} K_{c,u} \quad (16.7)$$

$$T_i = \frac{k_r + 1}{2} P_u \quad (16.8)$$

where k_r is a tuning parameter. With $k_r = 1$, the PI settings are equal to the Skogestad PI settings for a process assumed to be “integrator with time delay”, cf. Section 16.8. Since the purpose of the Relaxed Ziegler-Nichols settings is relaxed tuning, I suggest even more relaxation than with $k_r = 1$, namely

$$k_r = 1.5 \quad (16.9)$$

²which implies that the control system is on the stability limit and oscillates

Using $k_r = 1.5$ in (16.7) and (16.8) gives

$$K_c = 0.25K_{c,u} \quad (16.10)$$

$$T_i = 1.25P_u \quad (16.11)$$

which are the Relaxed Ziegler-Nichols PI settings. They are presented in Table 16.2 together with the original Ziegler-Nichols PI settings.

Table 16.2: Relaxed Ziegler-Nichols PI settings and the original Ziegler-Nichols settings

PI settings	K_c	T_i
Relaxed Ziegler-Nichols	$0.25K_{c,u}$	$1.25P_u$
Original Ziegler-Nichols	$0.45K_{c,u}$	$\frac{P_u}{1.2}$

Example 16.2 Relaxed Ziegler-Nichols PI tuning

In Example 16.1 we found the following parameters in the Ziegler-Nichols experiment:

$$K_{c,u} = 26.5$$

$$P_u = 5 \text{ min} = 300 \text{ s}$$

This gives the following Relaxed Ziegler-Nichols PI settings:

$$K_c = 0.25K_{c,u} = 0.25 \cdot 26.5 = 6.63$$

and

$$T_i = 1.25P_u = 1.25 \cdot 300 = 375 \text{ s}$$

Figure 16.6 shows simulations with these PI settings. The stability is clearly improved comparing with the stability with the Ziegler-Nichols method, cf. Figure 16.5.

[End of Example 16.2]

16.4 Repeated Ziegler-Nichols tuning

Here is another method to cope with poor stability of a control loop – assuming PI control.³ The poor stability may be the result of a Ziegler-Nichols tuning applied to a process which has relatively small time delay, e.g. less than 10% of the process time constant. However, the method may be used also on other control systems showing poor stability.

Assume that the original PI settings are $K_{c,0}$ and $T_{i,0}$, and that the control system shows poor stability with these settings. Assume that you can observe poorly damped oscillations

³This method has not a solid scientific background. It stems from an idea I got while working on tuning a PI controller for biogas production on a pilot plant around year 2012. The method worked out very well on both a simulator of the plant and the real plant.

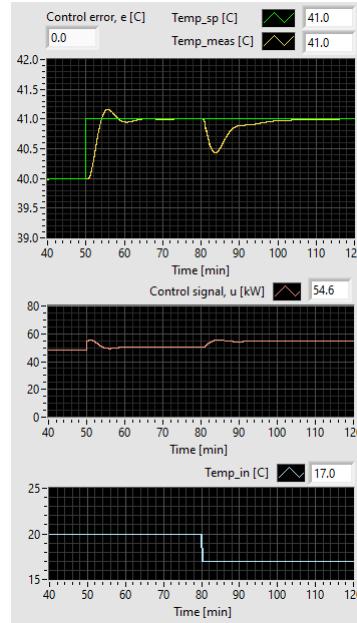


Figure 16.6: Simulation of the temperature control system with PI settings according to the Relaxed Ziegler-Nichols method.

with period $P_{u,0}$. Then, improved PI setting can be obtained by applying Ziegler-Nichols PI tuning *pretending* that these oscillations are true Ziegler-Nichols oscillations:

$$K_c = 0.45K_{c,0} \quad (16.12)$$

$$T_i = \frac{P_{u,0}}{1.2} \quad (16.13)$$

Note that the original value of $T_{i,0}$ is not used in these formulas (although I mentioned $T_{i,0}$ above).

Example 16.3 Repeated Ziegler-Nichols method

In Example 16.1 we found the following PI settings for the temperature control system with the Ziegler-Nichols method:

$$K_c = 11.9 \quad (16.14)$$

$$T_i = 250 \text{ s} \quad (16.15)$$

Figure 16.5 shows simulations with these settings. Assume that we are not happy with the stability. From Figure 16.5 we read off

$$P_{u,0} = 8 \text{ min} = 480 \text{ s}$$

The Repeated Ziegler-Nichols method gives:

$$K_c = 0.45K_{c,0} = 0.45 \cdot 11.9 = 5.36 \quad (16.16)$$

$$T_i = \frac{P_{u,0}}{1.2} = \frac{480 \text{ s}}{1.2} = 400 \quad (16.17)$$

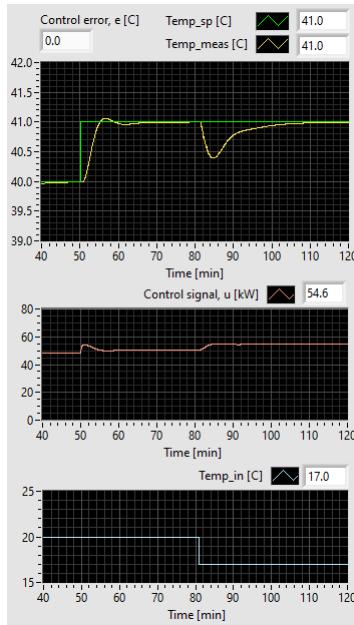


Figure 16.7: Simulation of the temperature control system with PI settings according to the Repeated Ziegler-Nichols method

Figure 16.7 shows simulations with these PI settings. The stability is clearly improved comparing with the stability with the Ziegler-Nichols method, cf. Figure 16.5.

[End of Example 16.3]

16.5 The Åstrøm-Hägglund Relay tuning method

If the process has slow dynamics, the trial-and-error procedure to find the critical gain $K_{c,u}$ in the Ziegler-Nichols method may take a long time. Åstrøm and Hägglund [Åstrøm & Hägglund \(1995\)](#) found a clever solution that may reduce the time considerably as the trial-and-error procedure is avoided. The PID controller to be tuned is replaced (in the tuning session) by an On-off controller! As we saw in Ch. 13, an On-off gives sustained oscillations in the control loop, and the sustained oscillations come automatically. Figure 16.8 illustrates this solution. An On-off controller behaves like a relay. Hence, the method is often referred to as the Åstrøm and Hägglund Relay tuning method.

Now, the important questions are:

1. Is the period of the oscillation the same with the On-off controller as with a P controller with ultimate gain for the given control loop?
2. Can we estimate the ultimate gain, $K_{c,u}$, from oscillations with the On-off controller?

Answer to the first question:

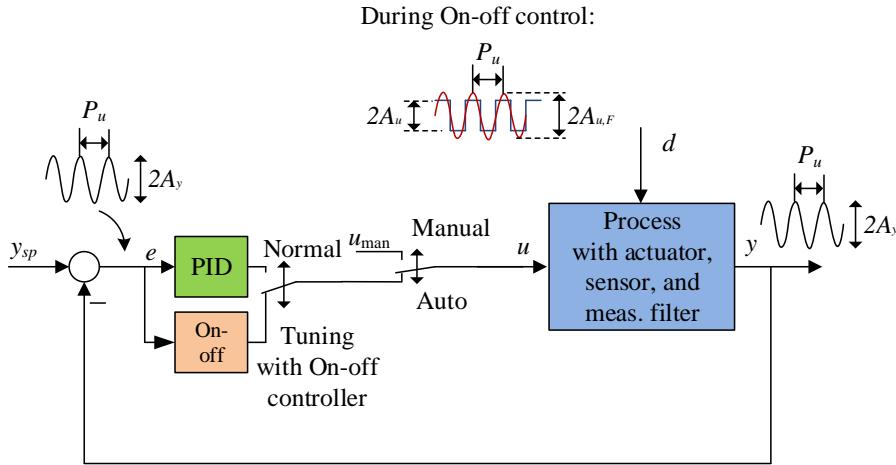


Figure 16.8: With an On-off in the place of the PID controller oscillations come automatically.

It turns out that the period is very similar. Hence, the oscillations with the On-off controllers can be used as P_u in the Ziegler-Nichols formulas.

Answer to the second question:

In general, the gain of a system can be estimated from the ration

$$K = \frac{A_{\text{out}}}{A_{\text{in}}} \quad (16.18)$$

where A_{in} is the amplitude of an assumed sinusoid at the input, and A_{out} is the amplitude of the corresponding sinusoid at the output. In Figure 16.8, the amplitude at the input of the On-off controller is the same the amplitude of the process output. Hence,

$$A_{\text{in}} = A_y \quad (16.19)$$

which can be estimated from the observed response (with the On-off controller).

What is A_{out} ? With On-off control, the control signal, u , is a square wave (created automatically as a result of the On-off controller function). Åström and Hägglund approximated this square wave with the basic sinusoid of the Fourier series approximation to the square wave. According to the Fourier theory, the amplitude of the sinusoid is

$$A_{u,F} = \frac{4A_u}{\pi} \quad (16.20)$$

where A_u is the amplitude of the square wave, cf. Figure 16.8. Often the range of the control signal is [0%, 100%], giving $A_u = 50\%$. Now, A_{out} in (16.18) is

$$A_{\text{out}} = A_{u,F} = \frac{4A_u}{\pi}$$

As a result of the above, the ultimate gain to be used in the Ziegler-Nichols tuning method can be estimated as

$$K_{c,u} = \frac{\frac{4A_u}{\pi}}{A_y} = \frac{4A_u}{\pi A_y} = 1.27 \frac{A_u}{A_y} \quad (16.21)$$

Example 16.4 PI controller tuning with the On-off tuning method

See Example 16.1. We will now tune the temperature controller TC as PI controller using On-off tuning. Figure 16.9 shows the responses with On-off controller with

$$A_u = 40 \text{ kW} \quad (16.22)$$

From the simulation, we read off

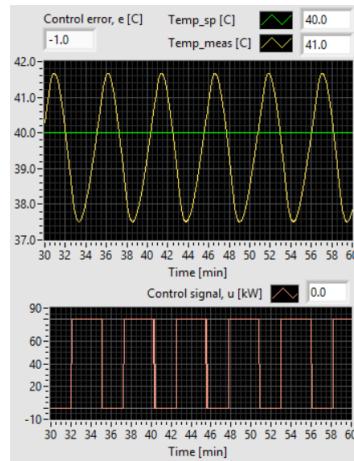


Figure 16.9: Simulated responses with On-off controller.

$$A_y = \frac{41.7^\circ\text{C} - 37.5^\circ\text{C}}{2} = 2.1^\circ\text{C} \quad (16.23)$$

giving

$$K_{c,u} = \frac{4A_u}{\pi A_y} = \frac{4 \cdot 40 \text{ kW}}{\pi \cdot 2.1^\circ\text{C}} = 24.3 \text{ kW}/^\circ\text{C}$$

and

$$P_u = 5.2 \text{ min} = 312 \text{ s}$$

which are quite similar to the values found with the Ziegler-Nichols method in Example 16.1, namely $K_{c,u} = 26.5 \text{ kW}/^\circ\text{C}$ and $P_u = 300 \text{ s}$, respectively. (I do not show the final PI tuning nor simulations here.)

[End of Example 16.4]

16.6 Auto-tuning

Auto-tuning is automatic tuning of controller parameters in a well-defined experiment. It is common that commercial controllers offers auto-tuning. The operator starts the auto-tuning via some button or menu choice on the controller. The controller then executes automatically a pre-planned experiment either on the uncontrolled process (i.e. the open loop system), or on the control loop (i.e. the closed loop system) – depending on the

auto-tuning method implemented. After this experiment, the controller is automatically set into normal operation with the tuned parameters.

One example of an auto-tuner is the so-called relay tuner which uses an automatic implementation of the On-off PID tuning method presented in Section 16.5. Such an auto-tuner is found in e.g. PID controllers by ABB and by Fuji. When the auto-tuning phase is started, an On-off (or relay) controller is used as the controller in the control loop. The On-off controller sustained oscillations in control loop come automatically. From the amplitude and the period of these oscillations, proper PID controller parameters are calculated by an algorithm in the controller (the algorithm does not necessarily use the Ziegler-Nichols formulas). Just a couple of periods are needed for the autotuner to have enough information to accomplish the tuning. The autotuner activates the tuned PID controller automatically after the tuning has finished.

16.7 The Good Gain method

The Good Gain method aims at obtaining acceptable stability as explained above. It is a simple method which has proven to give good results on laboratory processes and on simulators. The method is based on experiments on a real or simulated control system, see Figure 16.10.

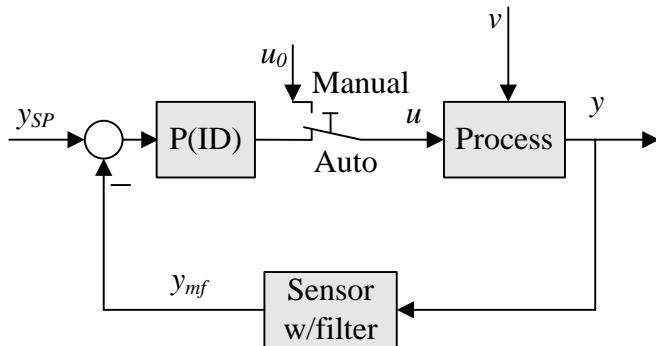


Figure 16.10: The Good Gain method for PID tuning is applied to the established control system

The procedure described below assumes a PI controller, which is the most commonly used controller function. However, a comment about how to include the D-term, so that the controller becomes a PID controller, is also given.

1. Bring the process to or close to the normal or specified operation point by adjusting the nominal control signal u_0 (with the controller in manual mode).
2. Ensure that the controller is a P controller with $K_c = 0$ (set $T_i = \infty$ and $T_d = 0$). Increase K_c until the control loop gets good (satisfactory) stability as seen in the response in the measurement signal after e.g. a step in the setpoint or in the disturbance (exciting with a step in the disturbance may be impossible on a real

system, but it is possible in a simulator). If you do not want to start with $K_c = 0$, you can try $K_c = 1$ (which is a good initial guess in many cases) and then increase or decrease the K_c value until you observe some overshoot and a barely observable undershoot (or vice versa if you apply a setpoint step change the opposite way, i.e. a negative step change), see Figure 16.11. This kind of response is assumed to represent good stability of the control system. This gain value is denoted $K_{c,GG}$.

It is important that *the control signal is not driven to any saturation limit* (maximum or minimum value) during the experiment. If such limits are reached the K_c value may not be a good one – probably too large to provide good stability when the control system is in normal operation. So, you should apply a relatively small step change of the setpoint (e.g. 5% of the setpoint range), but not so small that the response drowns in noise.

3. Set the integral time T_i equal to

$$T_i = 1.5T_{ou} \quad (16.24)$$

where T_{ou} is the time between the overshoot and the undershoot of the step response (a step in the setpoint) with the P controller, see Figure 16.11.⁴ Note that for most systems (those which does not contain a pure integrator) there will be offset from setpoint because the controller during the tuning is just a P controller.

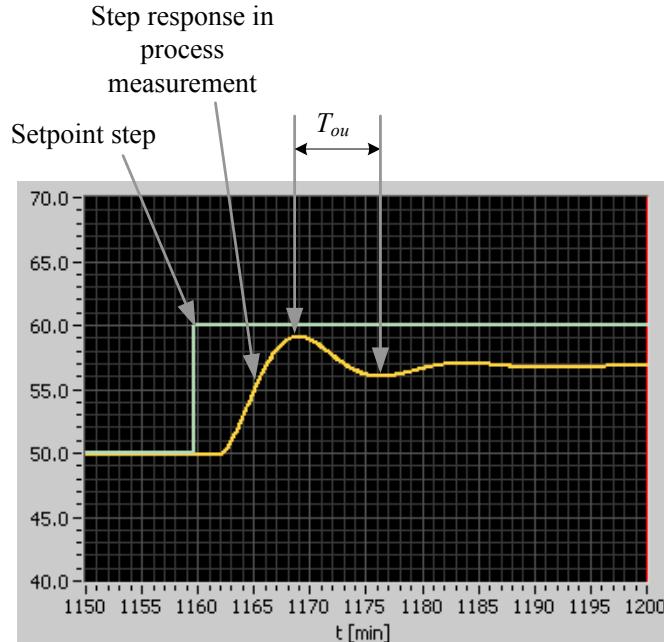


Figure 16.11: The Good Gain method: Reading off the time between the overshoot and the undershoot of the step response with P controller.

4. Because of the introduction of the I-term, the loop with the PI controller in action will probably have somewhat reduced stability than with the P controller only. To

⁴Alternatively, you may apply a negative setpoint step, giving a similar response but downwards. In this case T_{ou} is time between the undershoot and the overshoot.

compensate for this, the K_c can be reduced somewhat, e.g. to 80% of the original value. Hence,

$$K_c = 0.8K_{c,GG} \quad (16.25)$$

5. If you want to include the D-term, so that the controller becomes a PID controller⁵, you can try setting T_d as follows:

$$T_d = \frac{T_i}{4} \quad (16.26)$$

which is the T_d-T_i relation that was used by Ziegler and Nichols ([Ziegler & Nichols 1942](#)).

6. You should check the stability of the control system with the above controller settings by applying a step change of the setpoint. If the stability is poor, try reducing the controller gain somewhat, possibly in combination with increasing the integral time.

Example 16.5 PI tuning with the Good Gain method

Figure 16.12 shows the Good Gain tuning experiment on the simulated temperature system shown in Figure 16.4.

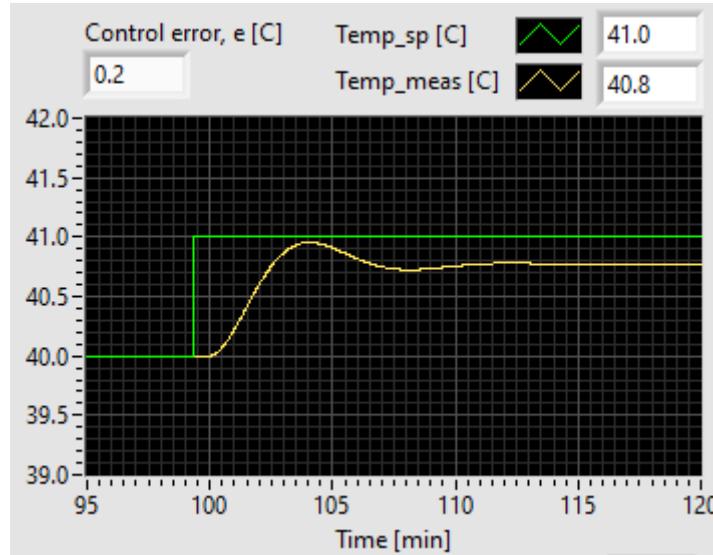


Figure 16.12: The Good Gain tuning experiment on the simulated temperature system shown in Figure 16.4

From the response we read off:

$$K_{c,GG} = 8.0 \quad (16.27)$$

and

$$T_{ou} = 4 \text{ min} = 240 \text{ s} \quad (16.28)$$

⁵But remember the drawbacks about the D-term, namely that it amplifies the measurement noise, causing a more noisy controller signal than with a PI controller.

which give the following Good Gain PI settings

$$K_c = 0.8K_{c,GG} = 0.8 \cdot 8.0 = 6.4 \quad (16.29)$$

$$T_i = 1.5T_{ou} = 1.5 \cdot 240 \text{ s} = 360 \text{ s} \quad (16.30)$$

Figure 16.13 shows simulations with these PI settings. The stability is clearly improved comparing with the stability with the Ziegler-Nichols method, cf. Figure 16.5.

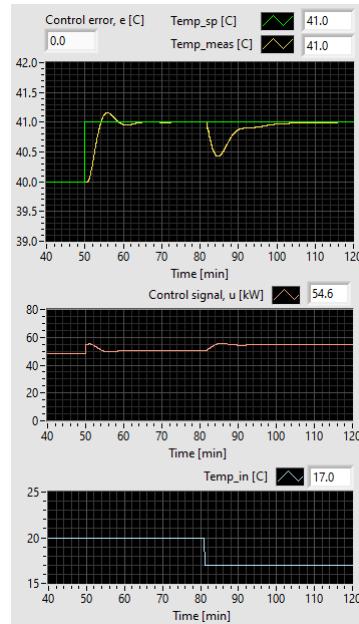


Figure 16.13: Simulation of the temperature control system with PI settings according to the Good Gain method.

[End of Example 16.5]

16.8 The Skogestad controller tuning method

16.8.1 Background of the Skogestad method

The Skogestad controller tuning method [Skogestad \(2003\)](#)⁶ is a model-based tuning method where the controller parameters are expressed as functions of the process model parameters. It is assumed that the control system has a block diagram as shown in Figure 16.14.

Comments to this block diagram:

- The transfer function $H_p(s)$ is a *combined transfer function of the process, the sensor, and the measurement lowpass filter*. For simplicity we may denote this transfer function the “process transfer function”, although it is a combined transfer function.

⁶I named this method after the originator, Prof. Sigurd Skogestad at NTNU, Norway.

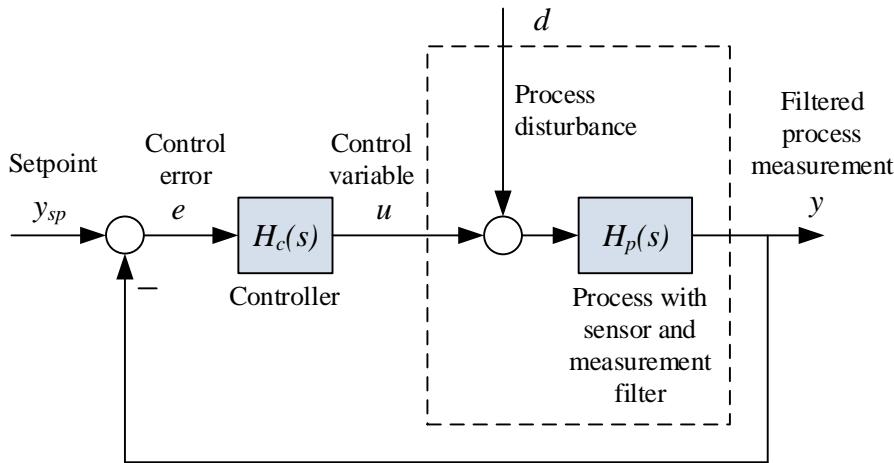


Figure 16.14: Block diagram of the control system in the Skogestad controller tuning method

- The process transfer function may stem from a single step-response experiment with the process, as explained in subsequent sections.
- The block diagram shows a disturbance acting on the process. Information about this disturbance is not used in the tuning, but if you are going to test the tuning on a simulator to see how the control system compensates for a process disturbance, you should add a disturbance at the point indicated in the block diagram, which is at the process input. It turns out that in most processes the dominating disturbance influences the process dynamically at the “same” point as the control variable. Such a disturbance is called an *input disturbance*. Here are a few examples:
 - Liquid tank: The control variable controls the inflow. The outflow is a disturbance.
 - Motor: The control variable controls the motor torque. The load torque is a disturbance.
 - Thermal process: The control variable controls the power supply via a heating element. The power loss via heat transfer through the walls and heat outflow through the outlet are disturbances.

The design principle of Skogestad method is as follows. The control system *tracking transfer function* $T(s)$, which is the transfer function from the setpoint to the (filtered) process measurement, is *specified* as a first order transfer function with time delay:

$$T(s) = \frac{y(s)}{r(s)} = \frac{1}{T_c s + 1} e^{-\tau s} \quad (16.31)$$

where T_c is the time constant of the control system which *the user must specify*, and τ is the process time delay which is *given* by the process model (the method can however be used for processes without time delay, too). Figure 16.15 shows as an illustration the response in y after a step in the setpoint y_{sp} for (16.31).

From the block diagram shown in Figure 16.14 the tracking transfer function is, cf. the

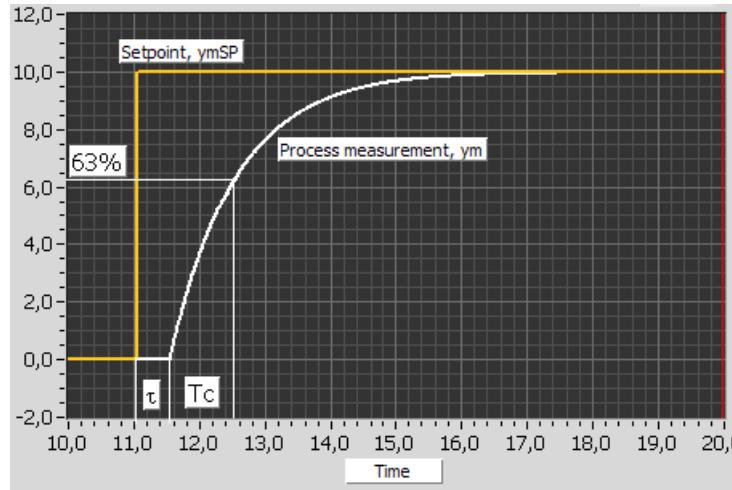


Figure 16.15: Step response (setpoint step) of the specified tracking transfer function (16.31) in Skogestad controller tuning method

Feedback rule in Figure 9.2,

$$T(s) = \frac{H_c(s)H_p(s)}{1 + H_c(s)H_p(s)} \quad (16.32)$$

Here, the only unknown is the controller transfer function, $H_c(s)$. Solving for $H_c(s)$, gives

$$H_c(s) = \frac{T(s)}{[1 - T(s)] H_p(s)}$$

By setting (16.32) equal to (16.31) and making some proper simplifying approximations to the time delay term, the controller, $H_c(s)$, becomes a PID controller or a PI controller for the assumed process transfer function, $H_p(s)$.

16.8.2 Controller tuning for “integrator with time delay” processes

16.8.2.1 Mathematical model and dynamics

The process model of an integrator with time delay is

$$y'(t) = K_i u(t - \tau) \quad (16.33)$$

The corresponding transfer function is

$$\frac{y(s)}{u(s)} = H_p(s) = \frac{K_i}{s} e^{-\tau s} \quad (16.34)$$

Figure 16.16 illustrates the dynamics of “integrator with time delay” processes with a step response, which is in the form of a delayed ramp. Note that the step is in the *control signal* (the step is applied with the controller – if present – in manual mode).

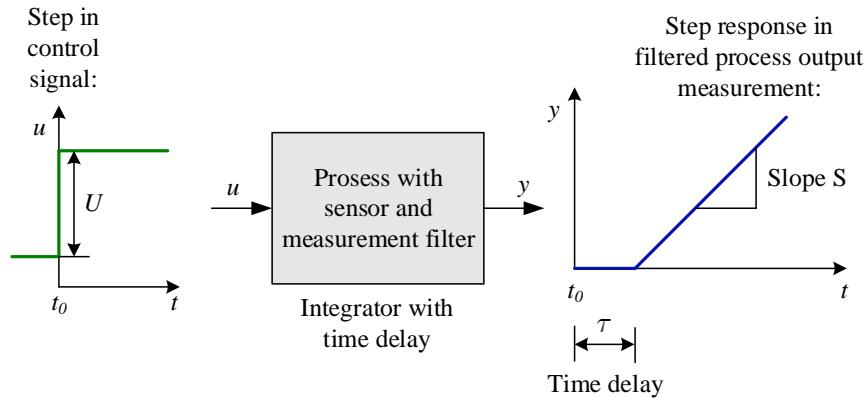


Figure 16.16: Step response of an “integrator with time delay” process.

16.8.2.2 Controller settings

The Skogestad controller settings are:

$$K_c = \frac{1}{K_i(T_c + \tau)} \quad (16.35)$$

$$T_i = 2(T_c + \tau) \quad (16.36)$$

$$T_d = 0 \quad (16.37)$$

Thus, the controller is a PI controller.

In the original Skogestad formulas, the factor 2 in (16.36) is 4. This gives good setpoint tracking. But the disturbance compensation may become unnecessarily sluggish. To obtain faster disturbance compensation, factor 2 can be used Haugen & Lie (2013).⁷ The drawback of using 2 instead of 4 is that there will be somewhat more overshoot in the setpoint step response, and that the stability of the control loop will be somewhat reduced. Also, the robustness against changes of process parameters (e.g. increase of process gain and increase of process time delay) will be somewhat reduced.

Skogestad’s suggestion for selecting T_c

If you do not know how to specify T_c in (16.35) and (16.36), you may use Skogestad’s advice:

$$T_c = \tau \quad (16.38)$$

Consequently, (16.35) and (16.36) becomes

⁷So, we may denote the above settings as modified Skogestad settings, but I prefer to say just Skogestad settings.

The golden PI settings:

$$K_c = \frac{1}{2K_i\tau} \quad (16.39)$$

$$T_i = 4\tau \quad (16.40)$$

The PI settings (16.39)-(16.40) are the most useful controller formulas I know. I will refer to them many times throughout the book. They can be used even if the process dynamics is actually not “integrator with time delay”, as explained in Section 16.8.2.3.

Example 16.6 Skogestad tuning of the level PI controller of a wood chip tank

A level control system of a wood chip tank with conveyor belt is presented in Ch. 1.2.6. A mathematical model of the tank with conveyor belt based on mass balance of the wood chip is described in Ch. 41.1. The model is:

$$\rho Ay'(t) = K_s u(t - \tau) - w_{\text{out}}(t) \quad (16.41)$$

where ρ [kg/m³] is chip density, A [m²] is cross-sectional area, K_s [(kg/min)/%] is feed screw gain, τ [min] is time delay of the conveyor belt. The level y is measured with a level sensor, and the measurement is

$$y_m = K_m y \quad (16.42)$$

where K_m [%/m] is the sensor gain.

We will solve the following problems:

1. Calculate the transfer function $H(s)$ from control variable u to process measurement y_m .
2. Calculate the PI(D) parameters for the process using Skogestad method.

Solution:

1. The Laplace transform of (16.41) is

$$\rho A [sy(s) - y_0] = K_s e^{-\tau s} u(s) - w_{\text{out}}(s) \quad (16.43)$$

Solving for output variable h gives

$$y(s) = \frac{1}{s} h_0 + \underbrace{\frac{K_s}{\rho As} e^{-\tau s} u(s)}_{H_1(s)} + \left(-\frac{1}{\rho As} \right) w_{\text{out}}(s) \quad (16.44)$$

Thus, the transfer function from u to y is

$$H_1(s) = \frac{y(s)}{u(s)} = \frac{K_s}{\rho As} e^{-\tau s} \quad (16.45)$$

The Laplace transform of (16.42) is

$$H_2(s) = \frac{y_m(s)}{y(s)} = K_m \quad (16.46)$$

Combining these two transfer functions gives

$$H(s) = \frac{y_m(s)}{u(s)} = \frac{y_m(s)}{y(s)} \cdot \frac{y(s)}{u(s)} = \frac{K_s K_m}{\rho A s} e^{-\tau s} = \frac{K_i}{s} e^{-\tau s} \quad (16.47)$$

where

$$K_i = \frac{K_s K_m}{\rho A} \quad (16.48)$$

$H_i(s)$ is “integrator with time delay”.

2. We set

$$T_c = \tau \quad (16.49)$$

According to Skogestad tuning formulas for “integrator with time delay” processes:

$$K_c = \frac{1}{K_i(T_c + \tau)} = \frac{1}{2K_i\tau} = \frac{1}{2\frac{K_m K_s}{\rho A}\tau} = \frac{\rho A}{2K_m K_s \tau} \quad (16.50)$$

$$= \frac{145 [\text{kg}/\text{m}^3] \cdot 13.4 [\text{m}^2]}{2 \cdot 6.67 [\%/\text{m}] \cdot 33.36 [(\text{kg}/\text{min})/\%] \cdot 4.17 [\text{min}]} \quad (16.51)$$

$$= 1.05 [\% \text{ control} / \% \text{ meas}] \quad (16.52)$$

$$T_i = 4\tau = 4 \cdot 250 \text{ s} = 1000 \text{ s} \quad (16.53)$$

Figure 16.17 shows simulated responses due to a step change of the level setpoint and a step change of the outflow (disturbance). The responses indicated that the stability of the control loop is satisfactory.

[End of Example 16.6]

16.8.2.3 PI tuning for pretended “integrator with time delay” processes

The PI settings (16.39) – (16.40) may be used even if the process dynamics is not actually “integrator with time delay”. In such cases, we have to pretend that the process dynamics is “integrator with time delay”. I have experienced in several practical applications that this is a useful simple approach to PI controller tuning.

Figure 16.18 shows the actual step response (blue curve) of a process and the step response (red curve) of a pretended “integrator with time delay” process. From the pretended response, we read off the pretended time delay, τ , and the slope, S , of the steepest tangent:

- τ is the time measured from the point of time that the step change was made to the point of time where the tangent crosses the base line of the response.
- S is the slope of the steepest tangent. If the process response is measured in say %, and the time unit is say seconds, S has the unit of %/s.

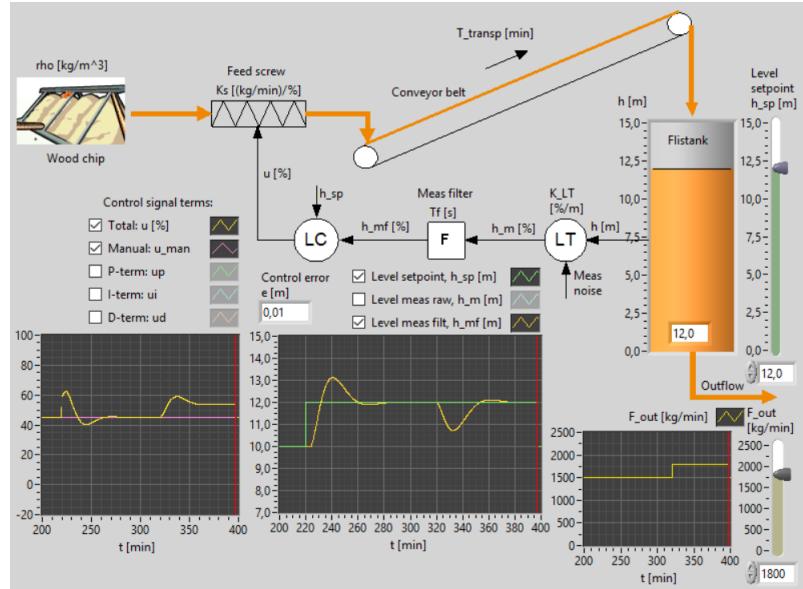


Figure 16.17: Level control of wood chip tank: Simulated responses due to a step change of the level setpoint and a step change of the outflow (disturbance).

For an integrator, the slope of the step response is

$$S = K_i U \quad (16.54)$$

where K_i is the integrator gain and U is the amplitude of the step applied to the control signal. That the slope is $K_i U$, can be seen directly from (16.33) by setting u equal to U and neglecting the time delay. From (16.54) we can calculate K_i as

$$K_i = \frac{S}{U} \quad (16.55)$$

Hence, K_i can be interpreted as the normalized slope of the step response of the integrator.

Now that have found the values of τ and K_i , we can tune a PI controller for the process using the golden PI tuning formulas, (16.39) – (16.40).

If uncertain: Overestimate!

If you are uncertain when reading off the values of S and τ from the step response, cf. Figure 16.18, you should *overestimate* their values, i.e. give them (slightly) too large values rather than (slightly) too small because it is *safe regarding stability* of the control system. This is because relatively large values of K_i and τ in (16.39) – (16.40) gives smaller values of K_c and T_i , which improves the stability of the control loop.

You may wonder

The blue curve in Figure 16.18 indicates “time constant with time delay” process dynamics, or actually some higher order dynamics. Skogestad has actually developed PI tuning formulas also for such process dynamics. However, it is more challenging, and definitely more time-consuming, to adapt such models than to adapt the “integrator with time delay”

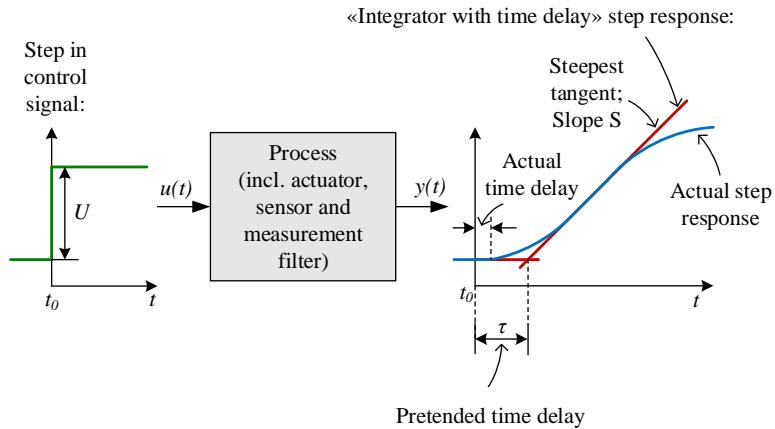


Figure 16.18: Actual step response of a process and the step response of a pretended “integrator with time delay” process

process model to the step response. As an example, I once did a PI controller tuning of a biogas flow controller on a real biogas reactor, which are a very slow processes, with a time constant of days. If I should have waited for the biogas response to have settled to adapt a model, I would wait for days. Instead, I pretended that the reactor was a “integrator with time delay” process, and I read off the time delay and the slope after a few some hours (when I saw that the response in the biogas measurement was about to flatten). So, I saved a lot of time.

Relating to PI for “time constant with time delay” processes

In Section 16.8.3 the Skogestad PI tuning formulas for “time constant with time delay” processes are presented. From those formulas, we can find that the PI settings for “time constant with time delay” processes are identical with the PI settings for “integrator with time delay” processes if

$$\tau \leq \frac{T}{4} \quad (16.56)$$

Example 16.7 Skogestad PI tuning of a temperature controller

Let us tune a PI controller for the heated liquid tank shown in Figure 16.4. Figure 16.19 shows the step response in the temperature, T , due to a step change in the control signal, u , to the heater.

The step amplitude is

$$U = 50 \text{ kW} - 45 \text{ kW} = 5 \text{ kW}$$

I have drawn the steepest tangent to the temperature curve. From Figure 16.19 we can read off the following values (approximately):

$$t_0 = 400.5 \text{ min}$$

$$t_1 = 401.7 \text{ min}$$

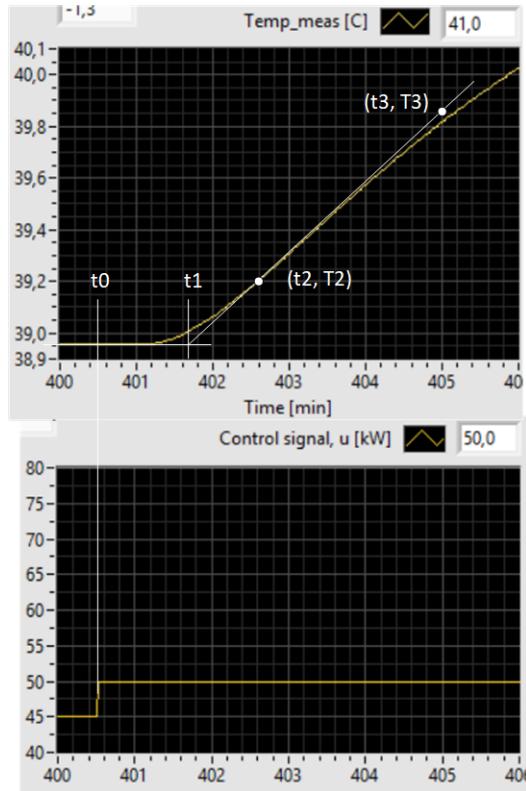


Figure 16.19: Step response in the temperature, T , due to a step change in the control signal, u , to the heater

$$t_2 = 402.6 \text{ min}$$

$$t_3 = 405.0 \text{ min}$$

$$T_2 = 39.20 \text{ }^{\circ}\text{C}$$

$$T_3 = 39.85 \text{ }^{\circ}\text{C}$$

This gives:

$$S = \frac{T_3 - T_2}{t_3 - t_2}$$

$$K_i = \frac{S}{U}$$

$$\tau = t_1 - t_0$$

The PI settings become:

$$K_c = \frac{1}{2K_i\tau} = 7.69 \text{ kW}/{}^{\circ}\text{C} \quad (16.57)$$

$$T_i = 4\tau = 4.8 \text{ min} = 288 \text{ s} \quad (16.58)$$

Figure 16.20 shows the response in the temperature with these PI settings. The setpoint is changed as a step from 40 to 41 °C at $t = 50$ min, and the inlet temperature, T_{in} , which is a

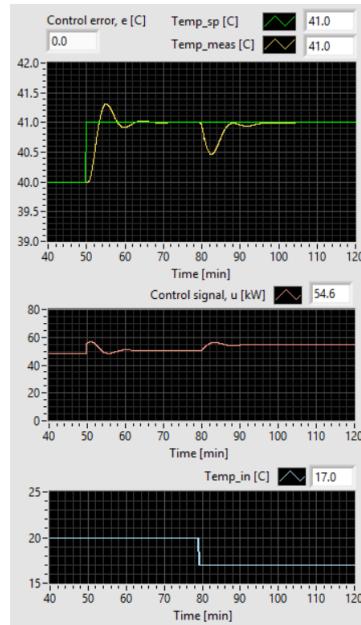


Figure 16.20: Skogestad PI settings: Temperature response due to a setpoint step

disturbance, is changed from 20 to 17°C at $t = 80$ min. The stability of the control system seems to be ok.

[End of Example 16.7]

16.8.2.4 Tuning for integrator *without* time delay

What if the process $H_p(s)$ is *without time delay*? Then you can not specify T_c according to (16.38) since that would give $T_c = 0$ (zero response time of the control system). You must specify T_c to some reasonable value larger than zero. If you do not know what could be a reasonable value, you can simulate the control system for various values of T_c . If the control signal (controller output signal) is changing too quickly, or often reaches the maximum and minimum values for reasonable changes of the setpoint or the disturbance, the controller is too aggressive, and you can try increasing T_c . If you don't want to simulate, then just try setting $T_c = T/2$ where T is the dominating (largest) time constant of the process (assuming the process is a time constant system, of course).

For the double integrator (without time delay) I have seen in simulations that the actual response-time (or 63% rise-time) of the closed-loop system may be about twice the specified time constant T_c . Consequently, you can set T_c to about half of the response-time you actually want to obtain.

16.8.3 Controller tuning for “time constant with time delay” processes

16.8.3.1 Mathematical model and dynamics

The process model of a time constant with time delay is

$$Ty'(t) = Ku(t - \tau) - y \quad (16.59)$$

The corresponding transfer function is

$$\frac{y(s)}{u(s)} = H_p(s) = \frac{K}{Ts + 1} e^{-\tau s} \quad (16.60)$$

Figure 16.21 illustrates the dynamics of “time constant with time delay” processes with a step response, which is in the form of a delayed exponential curve.

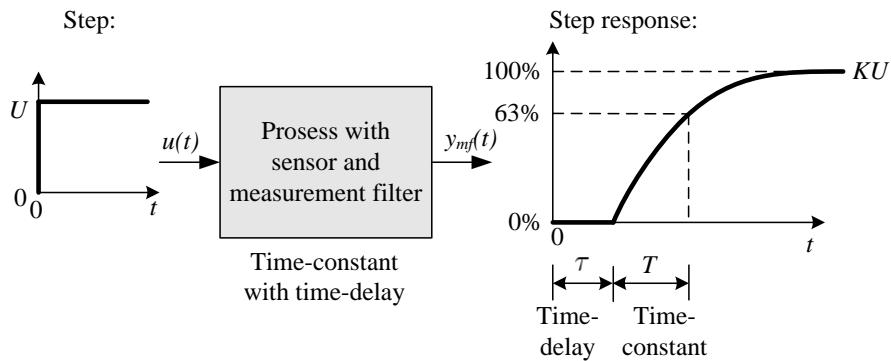


Figure 16.21: How the transfer function parameters K , T , and τ appear in the step response of a Time-constant with time delay process

16.8.3.2 Controller settings

The Skogestad controller settings are

$$K_c = \frac{T}{K(T_c + \tau)} \quad (16.61)$$

$$T_i = \min [T, 2(T_c + \tau)] \quad (16.62)$$

$$T_d = 0 \quad (16.63)$$

Thus, the controller is a PI controller.

In the original Skogestad formulas, the factor 2 in (16.62) is 4. I suggest using 2 instead of 4 since this gives faster disturbance compensation.

Skogestad's suggestion for selecting T_C

If you do not know how to specify T_c in (16.61) and (16.62), you may use Skogestad's advice:

$$T_c = \tau \quad (16.64)$$

With this choice, the PI settings are

$$K_c = \frac{T}{2K\tau} \quad (16.65)$$

$$T_i = \min(T, 4\tau) \quad (16.66)$$

Note that if $\tau > T/4$, the PI settings are identical to the settings for “integrator with time delay” processes, cf. Section 16.8.2.

Example 16.8 *Control of first order system with time delay*

Let us try Skogestad method for tuning a PI controller for the (combined) process transfer function

$$H_{\text{psf}}(s) = \frac{K}{Ts + 1} e^{-\tau s} \quad (16.67)$$

(time constant with time delay) where

$$K = 1; T = 1 \text{ s}; \tau = 0.5 \text{ s} \quad (16.68)$$

We use (16.38):

$$T_c = \tau = 0.5 \text{ s} \quad (16.69)$$

The controller parameters are as follows:

$$K_c = \frac{T}{K(T_c + \tau)} = \frac{1}{1 \cdot (0.5 + 0.5)} = 1 \quad (16.70)$$

$$T_i = \min[T, c(T_c + \tau)] \quad (16.71)$$

$$= \min[1, 2(0.5 + 0.5)] \quad (16.72)$$

$$= \min[1, 2] \quad (16.73)$$

$$= 1 \text{ s} \quad (16.74)$$

$$T_d = 0 \quad (16.75)$$

Figure 16.22 shows control system responses with the above PID settings. At time 5 sec the setpoint is changed as a step, and at time 15 sec the disturbance is changed as a step. The responses, and in particular the stability of the control systems, seem ok.

[End of Example 16.8]

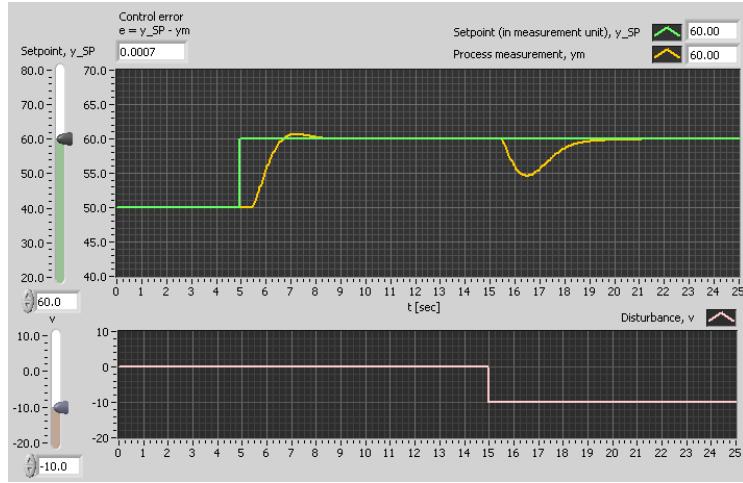


Figure 16.22: Example 16.8: Simulated responses in the control system with Skogestad controller tuning

16.8.4 Controller tuning for “double integrator” processes

16.8.4.1 Mathematical model

Typically, there is no or just a negligible time delay in processes which have models like double integrators (motion systems). Although Skogestad has developed PID settings for double integrators with time delay processes, I here neglect any time delay.

The differential equation model of a double integrator is:

$$y'' = K_{ii}u \quad (16.76)$$

The corresponding transfer function is

$$\frac{y(s)}{u(s)} = H_p(s) = \frac{K_{ii}}{s^2} \quad (16.77)$$

Example 16.9 Model of a double integrator

An example of a double integrator is a body (ship, robot, satellite) actuated by a force, but not by any speed-dependent damping forces (i.e. no so-called viscous damping). One example is a ship. Chapter 41.2 presents a model of a ship that also includes hydrodynamic forces and wind forces. Now, assume that the ship speed is zero relative to water and wind. (If the water and wind speeds are zero, this means that the ship is at rest.) Then, there are no damping forces acting on the ship. The model of the ship is given by Newton’s 2nd law:

$$my'' = u \quad (16.78)$$

or:

$$y'' = \frac{1}{m}u = K_{ii}u \quad (16.79)$$

where: m [kg] is ship mass, y [m] is position, u [N] is propeller force.

(16.79) is a double integrator.

The still state will be the most critical operating point in position control because there is no natural damping on the movement by the environment. If you are going to tune a position controller for the ship in a dynamic positioning system (DP system), you should tune the controller in this operating point since the stability will be imporved in any other operating point (where the ship moves).

[End of Example 16.9]

16.8.4.2 Controller settings

It can be shown using theoretical stability analysis that the controller for a double integrator must have derivative action, otherwise the control loop is guaranteed to be unstable. Thus, controller should be selected as a PID controller. The control loop will be unstable with a PI controller.

Skogestad has specified the following PID settings of a so-called serial PID controller:

$$K_c = \frac{1}{4K_{ii} T_c^2} \quad (16.80)$$

$$T_i = 4T_c \quad (16.81)$$

$$T_d = 4T_c \quad (16.82)$$

A serial PID controller comprises a PI controller in series with a PD controller, which means that the transfer function of the PID controller is equal to the product of the transfer function of the PI term and the transfer function of the PD term:

$$H_{\text{PID, serial}}(s) = H_{\text{PI}}(s) \cdot H_{\text{PD}}(s) \quad (16.83)$$

where

$$H_{\text{PI}}(s) = K_c \frac{T_i s + 1}{T_i s} \quad (16.84)$$

and

$$H_{\text{PD}}(s) = T_d s + 1 \quad (16.85)$$

Skogestad PID settings for a parallel PID controller

In this book, the parallel PID controller is assumed. In the parallel PID controller, the P, I, and D terms are added. In terms of transfer functions:

$$H_{\text{PID, parallel}}(s) = H_{\text{P}}(s) + H_{\text{I}}(s) + H_{\text{D}}(s) \quad (16.86)$$

where:

$$H_{\text{P}}(s) = K_c \quad (16.87)$$

$$H_{\text{I}}(s) = \frac{K_c}{T_i s} \quad (16.88)$$

$$H_D(s) = K_c T_d s \quad (16.89)$$

From the above two PID transfer function models, we can derive the following transformation formulas:

$$K_{c,\text{par}} = K_{c,\text{ser}} \left(1 + \frac{T_{d,\text{ser}}}{T_{i,\text{ser}}} \right) \quad (16.90)$$

$$T_{i,\text{par}} = T_{i,\text{ser}} \left(1 + \frac{T_{d,\text{ser}}}{T_{i,\text{ser}}} \right) \quad (16.91)$$

$$T_{d,\text{par}} = T_{d,\text{ser}} \frac{1}{1 + \frac{T_{d,\text{ser}}}{T_{i,\text{ser}}}} \quad (16.92)$$

So, to find the Skogestad settings for a parallel PID controller, we can do as follows:

1. Calculate the settings for a serial PID controller, (16.80)-(16.82).
2. Transform those serial settings to settings for a parallel PID controller using the transformation formulas (16.90)-(16.92).

It turns out that the actual time constant of the control system with the parallel PID controller becomes about 2 times the specified time constant, T_c . To compensate for this inaccuracy, I suggest we simply use $T_c/2$ instead of T_c in (16.80)-(16.82). Following the two-step procedure above, we get the following (improved) PID setting for a parallel PID controller:

Parallel PID controller settings for a double integrator:

$$K_c = \frac{2}{K_{ii} T_c^2} \quad (16.93)$$

$$T_i = 4T_c \quad (16.94)$$

$$T_d = T_c \quad (16.95)$$

Example 16.10 *PID tuning of a dynamic positioning system*

Ch. 41.2 presents a mathematical model of a ship. Assume

$$m = 71164 \text{ tonn} \quad (16.96)$$

Assume the following specification:

$$T_c = 2.0 \text{ min} \quad (16.97)$$

According to the formulas of PID settings⁸ er følgende PID-innstilling benyttet:

$$K_c = 9884 \quad (16.98)$$

$$T_i = 8.0 \text{ min} \quad (16.99)$$

$$T_d = 2.0 \text{ min} \quad (16.100)$$

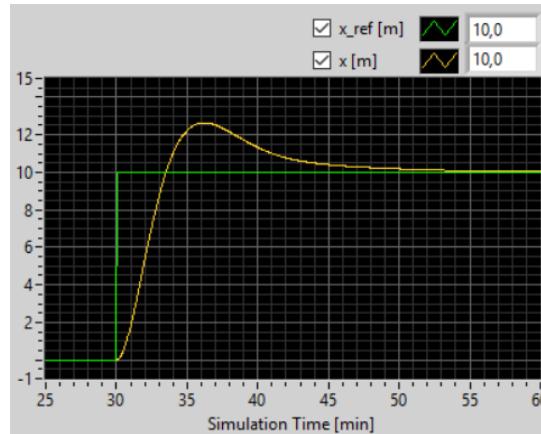


Figure 16.23: Step response in the position in a simulated dynamic positioning system.

Figure 16.23 shows the step response in position in a simulated dynamic positioning system. The observed time constant (63% rise time) is approx. 2.2 min, which is quite close to the specified value of 2.0 min.

[End of Example 16.10]

16.9 The Ziegler-Nichols open loop method

In [Ziegler & Nichols \(1942\)](#), Ziegler and Nichols presented their closed loop method. In that article, they presented a second PID tuning method which is denoted the Ziegler-Nichols open loop method, or the Ziegler-Nichols process reaction curve method. In this method, the PID settings are calculated from the integrator gain, K_i , and the time delay, τ , as in the Skogestad method assuming “integrator with time delay” process dynamics, cf. Section [16.8.2.3](#). The Ziegler og Nichols PI settings for a PI controller is (they derived also P settings and PID settings) are:

$$K_c = \frac{0.9}{K_i\tau} \quad (16.101)$$

$$T_i = 3.3\tau \quad (16.102)$$

Comparison with the Skogestad PI formulas, [\(16.39\)](#) – [\(16.40\)](#): In the Ziegler and Nichols PI settings, the gain is $0.9/0.5 = 1.8$ times larger, and T_i is $4/3.3 = 1.2$ times smaller than in the (modified) Skogestad PI settings. In general, both increasing K_c and reducing T_i reduce the stability of a control loop. We can therefore expect that Ziegler and Nichols tuning gives reduced stability comparing with Skogestad tuning.

⁸SimView simulator: *Dynamic Positioning*. <http://techteach.no>.

16.10 PID tuning when process dynamics varies

16.10.1 Introduction

A well tuned PID controller has parameters which are adapted to the dynamic properties to the process, so that the control system becomes fast and stable. If the process dynamic properties varies without re-tuning the controller, the control system

- gets *reduced stability*, or
- becomes *more sluggish*.

Problems with variable process dynamics can be solved in the following alternative ways:

- **The controller is tuned in the most critical operation point**, so that when the process operates in a different operation point, the stability of the control system is just better — at least the stability is not reduced. However, if the stability is too good the tracking quickness is reduced, giving more sluggish control.
- **The controller parameters are varied in the “opposite” direction of the variations of the process dynamics**, so that the performance of the control system is maintained, independent of the operation point. Some ways to vary the controller parameters are:
 - *Model-based PID parameter adjustment*, cf. Section 16.10.2.
 - *PID controller with gain scheduling*, cf. Section 16.10.3.
 - *Model-based adaptive controller*, cf. Section 16.10.4.

Commercial control equipment is available with options for gain scheduling and/or adaptive control.

16.10.2 PID parameter adjustment based on the Skogestad PID tuning method

Assume that you have tuned a PID or a PI controller for some process that has a model as assumed in the Skogestad PI(D) controller tuning method, cf. Section 16.8. Assume that some of the parameters of the process model changes. How should the controller parameters be adjusted? The answer is directly given by the Skogestad tuning formulas because those formulas contains the process parameters!

Example 16.11 *Adjustment of PI controller parameters for integrator with time delay process*

Assume that the process transfer function is

$$H_{psf}(s) = \frac{K_i}{s} e^{-\tau s} \quad (16.103)$$

(integrator with time delay). The Skogestad PI settings for this process are (16.39) – (16.40), which I repeat here:

$$K_c = \frac{1}{2K_i\tau} \quad (16.104)$$

$$T_i = 4\tau \quad (16.105)$$

As an example, assume that the process gain K_i is increased to, say, twice its original value. How should the PI parameters be adjusted to maintain good behaviour of the control system? From (16.104) we see that K_c should be halved, and from (16.105) we see that T_i should remain unchanged.

As another example, assume that the process time delay τ is increased to, say, twice its original value. From (16.104) we see that K_c should be halved, and from (16.105) we see that T_i should get a doubled value. One concrete example of such a process change is the wood-chip tank. If the speed of the conveyor belt is halved, the time delay (transport delay) is doubled. And now you know how to quickly adjust the PI controller parameters if such a change of the conveyor belt speed should occur.⁹

[End of Example 16.11]

You may use the Skogestad tuning formulas as the basis for adjusting the PID parameters even if you used some other method than Skogestad method for the initial tuning. In Ch. 17.1 the Skogestad tuning formulas are used to retune the PI controller of the level control of the wood chip tank when parameter changes of the process (tank with conveyor belt) occur.

16.10.3 Gain scheduling of PID parameters

Figure 16.24 shows the structure of a control system for a process which may have varying dynamic properties, for example a varying gain.

The *Gain scheduling variable GS* is some measured process variable which at every instant of time expresses or represents the dynamic properties of the process. As you will see in Example 16.12, GS may be the mass flow through a liquid tank.

Assume that proper values of the PID parameters K_c , T_i and T_d are found (using e.g. the Good Gain method) for a set of values of the GS variable. These PID parameter values can be stored in a parameter table – the gain schedule – as shown in Table 16.3. From this table proper PID parameters are given as functions of the gain scheduling variable, GS .

There are several ways to express the PID parameters as functions of the GS variable:

⁹What may happen if you do not adjust the controller parameters? The control system may get poor stability, or it may even become unstable.

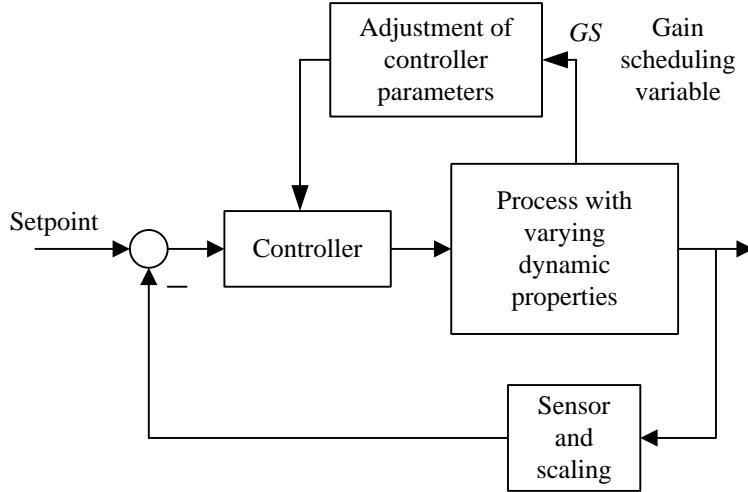


Figure 16.24: Control system for a process having varying dynamic properties. The GS variable expresses or represents the dynamic properties of the process.

Table 16.3: Gain schedule or parameter table of PID controller parameters

GS	K_c	T_i	T_d
GS_1	K_{c1}	T_{i1}	T_{d1}
GS_2	K_{c2}	T_{i2}	T_{d2}
GS_3	K_{c3}	T_{i3}	T_{d3}

- **Piecewise constant:** An interval is defined around each GS value in the parameter table. The controller parameters are kept constant as long as the GS value is within the interval. This is a simple solution, but is seems nonetheless to be the most common solution in commercial controllers.

When the GS variable changes from one interval to another, the controller parameters are changed abruptly, see Figure 16.25 which illustrates this for K_c , but the situation is the same for T_i and T_d . In Figure 16.25 it is assumed that GS values toward the left are critical with respect to the stability of the control system. In other words: It is assumed that it is safe to keep K_c constant and equal to the K_c value in the left part of the interval.

Using this solution there will be a disturbance in the form of a step in the control variable when the GS variable shifts from one interval to another, but this disturbance is probably of negligible practical importance for the process output variable. Noise in the GS variable may cause frequent changes of the PID parameters. This can be prevented by using a hysteresis, as shown in Figure 16.25.

- **Piecewise linear**, which means that a linear function is found relating the controller parameter (output variable) and the GS variable (input variable) between two adjacent sets of data in the table. The linear function is of the form

$$K_c = a \cdot GS + b \quad (16.106)$$

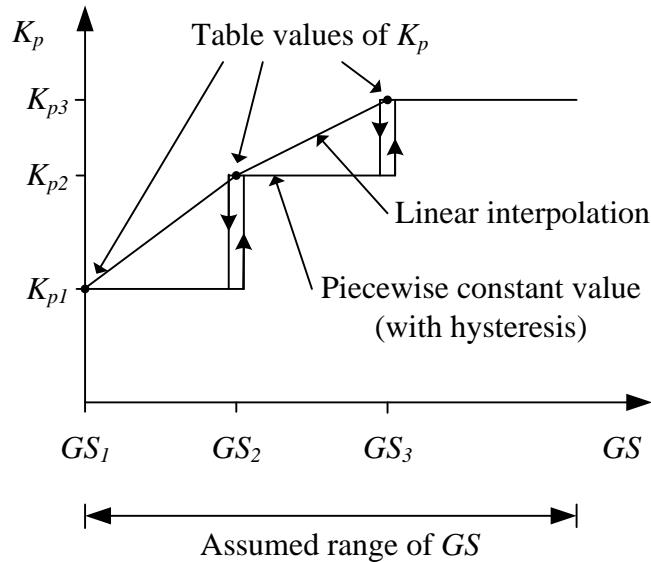


Figure 16.25: Two different ways to interpolate in a PID parameter table: Using piecewise constant values and linear interpolation.

where a and b are found from the two corresponding data sets:

$$K_{c1} = a \cdot GS_1 + b \quad (16.107)$$

$$K_{c2} = a \cdot GS_2 + b \quad (16.108)$$

(Similar equations applies to the T_i parameter and the T_d parameter.) (16.107) and (16.108) constitute a set of two equations with two unknown variables, a and b (the solution is left to you).¹⁰

- **Other interpolations** may be used, too, for example a polynomial function fitted exactly to the data or fitted using the least squares method.

Example 16.12 PID temperature control with gain scheduling during variable mass flow

Figure 16.27 shows the front panel of a simulator for a temperature control system for a liquid tank with variable mass flow, w , through the tank. The control variable u controls the power to heating element. The temperature T is measured by a sensor which is placed some distance away from the heating element. There is a time delay from the control variable to measurement due to imperfect blending in the tank.

The process dynamics

We will initially, both in simulations and from analytical expressions, that the dynamic properties of the process *varies with the mass flow w* . The response in the temperature T after a step change in the control signal (which is proportional to the supplied power) is simulated for a large mass flow and a small mass flow. (Feedback temperature control is not active, thus open loop responses are shown.) The responses are shown in Figure 16.26.

¹⁰Note that both MATLAB/SIMULINK and LabVIEW have functions that implement linear interpolation between tabular data. Therefore gain scheduling can be easily implemented in these environments.

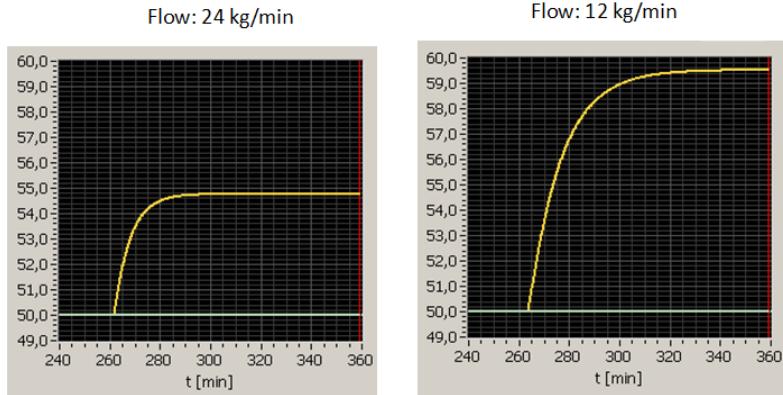


Figure 16.26: Responses in temperature T after a step in u of amplitude 10% at large mass flow and small mass flow

The simulations show that the following happens when the mass flow w is reduced (from 24 to 12 kg/min): *The gain process K is larger.* It can be shown that in addition, the time constant T_t is larger, and the time delay τ is larger. (These terms assumes that system is a first order system with time delay. The simulator is based on such a model. The model is described below.)

Let us see if the way the process dynamics seems to depend on the mass flow w as seen from the simulations, can be confirmed from a mathematical process model.¹¹ Assuming perfect stirring in the tank to have homogeneous conditions in the tank, we can set up the following energy balance for the liquid in the tank:

$$c\rho V T'_1(t) = K_P u(t) + cw [T_{in}(t) - T_t(t)] \quad (16.109)$$

where T_1 [K] is the liquid temperature in the tank, T_{in} [K] is the inlet temperature, c [J/(kg K)] is the specific heat capacity, V [m^3] is the liquid volume, ρ [kg/m^3] is the density, w [kg/s] is the mass flow (same out as in), K_p [W/%] is the gain of the power amplifier, u [%] is the control variable, $c\rho V T_1$ is (the temperature dependent) energy in the tank. It is assumed that the tank is isolated, that is, there is no heat transfer through the walls to the environment. To make the model a little more realistic, we will include a time delay τ [s] to represent inhomogeneous conditions in the tank. Let us for simplicity assume that the time delay is inversely proportional to the mass flow. Thus, the temperature T at the sensor is

$$T(t) = T_1 \left(t - \frac{K_\tau}{w} \right) = T_1 (t - \tau) \quad (16.110)$$

where τ is the time delay and K_τ is a constant. It can be shown that the transfer function from control signal u to process variable T is

$$T(s) = \underbrace{\frac{K}{T_t s + 1}}_{H_u(s)} e^{-\tau s} u(s) \quad (16.111)$$

where

$$\text{Gain } K = \frac{K_p}{cw} \quad (16.112)$$

¹¹Well, it would be strange if not. After all, we will be analyzing the same model as used in the simulator.

$$\text{Time-constant } T_t = \frac{\rho V}{w} \quad (16.113)$$

$$\text{Time delay } \tau = \frac{K_\tau}{w} \quad (16.114)$$

This confirms the observations in the simulations shown in Figure 16.26: Reduced mass flow w implies *larger process gain*, and larger time constant and larger time delay.

Heat exchangers and blending tanks in a process line where the production rate or mass flow varies, have similar dynamic properties as the tank in this example.

Control without gain scheduling (with fixed parameters)

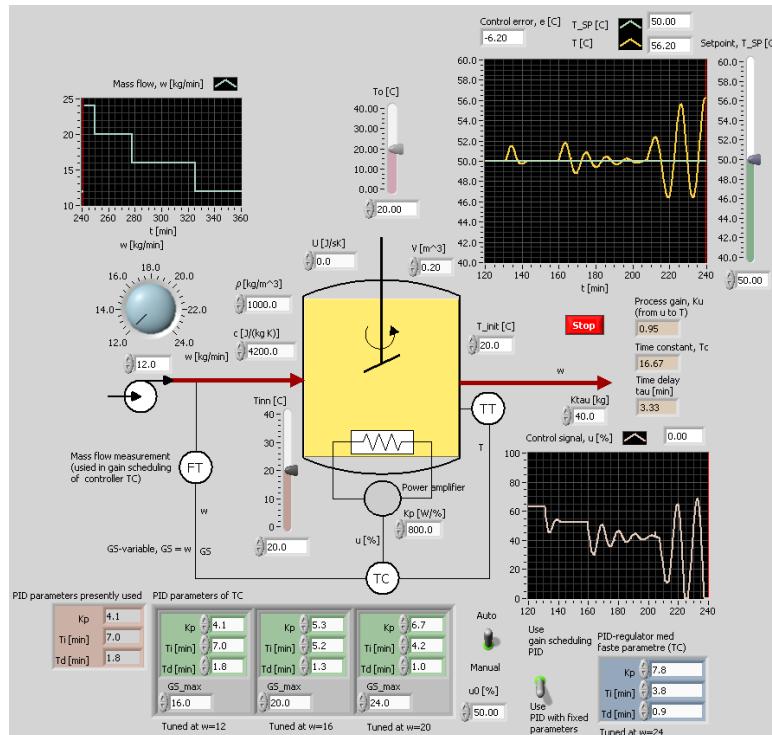


Figure 16.27: Example 16.12: Simulation of temperature control system with PID controller with fixed parameters tuned at maximum mass flow, which is $w = 24\text{kg}/\text{min}$.

Let us look at temperature control of the tank. The mass flow w varies. In which operating point should the controller be tuned if we want to be sure that the stability of the control system is not reduced when w varies? In general the stability of a control loop is reduced if the gain increases and/or if the time delay of the loop increases. (16.112) and (16.114) show how the gain and time delay depends on the mass flow w . According to (16.112) and (16.114) the PID controller should be tuned at minimal w . If we do the opposite, that is, tune the controller at the maximum w , the control system may actually become unstable if w decreases.

Let us see if a simulation confirms the above analysis. Figure 16.27 shows a temperature control system. The PID controller is in the example tuned at the maximum w value, which

here is assumed 24 kg/min.¹² The PID parameters are

$$K_c = 7.8; T_i = 3.8 \text{ min}; T_d = 0.9 \text{ min} \quad (16.115)$$

Figure 16.27 shows what happens at a stepwise reduction of w : The stability becomes worse, and the control system becomes *unstable* at the minimal w value, which is 12 kg/min.

Instead of using the PID parameters tuned at maximum w value, we can tune the PID controller at minimum w value, which is 12 kg/min. The parameters are then

$$K_c = 4.1; T_i = 7.0 \text{ min}; T_d = 1.8 \text{ min} \quad (16.116)$$

The control system will now be stable for all w values, but the system behaves sluggish at large w values. (Responses for this case is however not shown here.)

Control with gain scheduling

Let us see if gain scheduling maintains the stability for varying mass flow w . The PID parameters will be adjusted as a function of a measurement of w since the process dynamics varies with w . Thus, w is the gain scheduling variable, GS :

$$GS = w \quad (16.117)$$

A gain schedule consisting of three PID parameter value sets will be used. The PID controller are tuned at the following GS or w values: 12, 16 and 20 kg/min. These three PID parameter sets are shown down to the left in Figure 16.27. The PID parameters are held piecewise constant in the GS intervals. In each interval, the PID parameters are held fixed for an increasing $GS = w$ value, cf. Figure 16.25.¹³ Figure 16.28 shows the response in the temperature for decreasing values of w .

The simulation shows that the *stability of the control system is maintained even if w decreases*.

Finally, assume that you have decided not to use gain scheduling, but instead a PID controller with fixed parameter settings. What is the most critical operating point, at which the controller should be tuned? Is it at maximum flow or at minimum flow?¹⁴

[End of Example 16.12]

16.10.4 Adaptive controller

In an adaptive control system, see Figure 16.29,

a mathematical model of the process to be controlled is continuously estimated from samples of the control signal (u) and the process measurement (y_m). The model is typically

¹²Actually, the controller was tuned with the Ziegler-Nichols Ultimate Gain method. This method is however not described in this book. The Good Gain method could have been used instead.

¹³The simulator uses the inbuilt gain schedule in LabVIEW's PID Control Toolkit.

¹⁴The answer is minimum flow, because at minimum flow the process gain is at maximum, and also the time delay (transport delay) is at maximum.

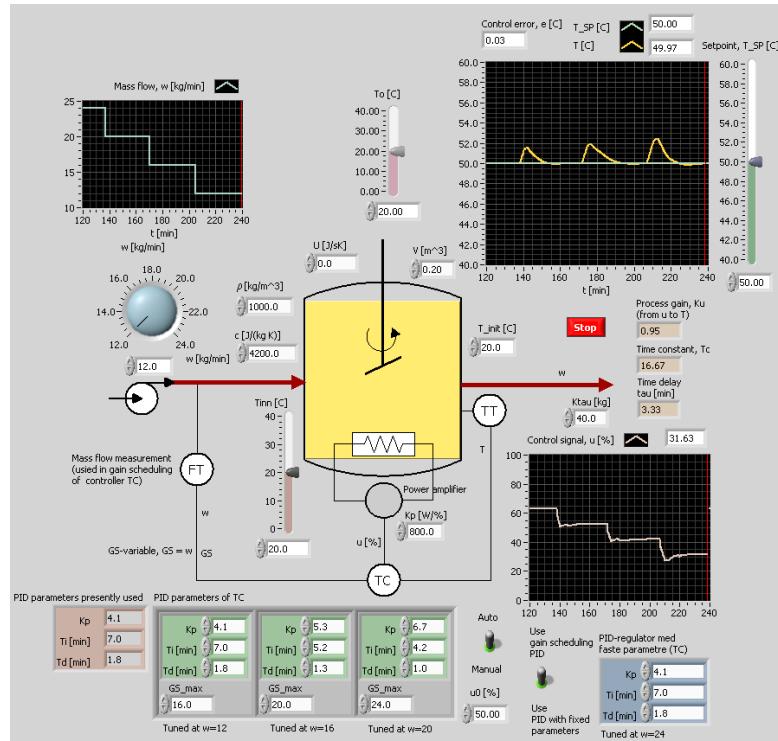


Figure 16.28: Example 16.12: Simulation of temperature control system with a gain schedule based PID controller

a transfer function model. Typically, the structure of the model is fixed. The model parameters are estimated continuously using e.g. the least squares method. From the estimated process model the parameters of a PID controller (or of some other control function) are continuously calculated so that the control system achieves specified performance in form of for example stability margins, poles, bandwidth, or minimum variance of the process output variable?. Adaptive controllers are commercially available, for example the ECA60 controller (ABB).

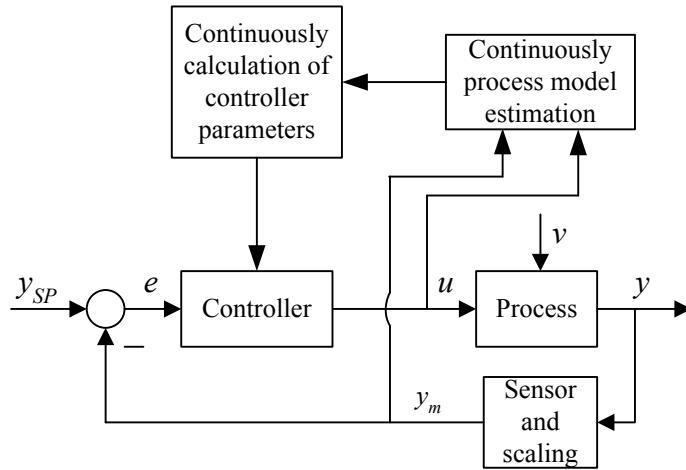


Figure 16.29: Adaptive control system

16.11 Problems for Chapter 16

Problem 16.1 Good gain tuning

Figure 16.30 shows the response in the temperature of a simulated temperature control system with P controller with the following “good gain” value:

$$K_c = 4.0 \quad (16.118)$$

Tune a PI controller for this process using the Good Gain method.

What can you do with the controller tuning if turns out that the stability of the control system is too bad with this value of K_c ?

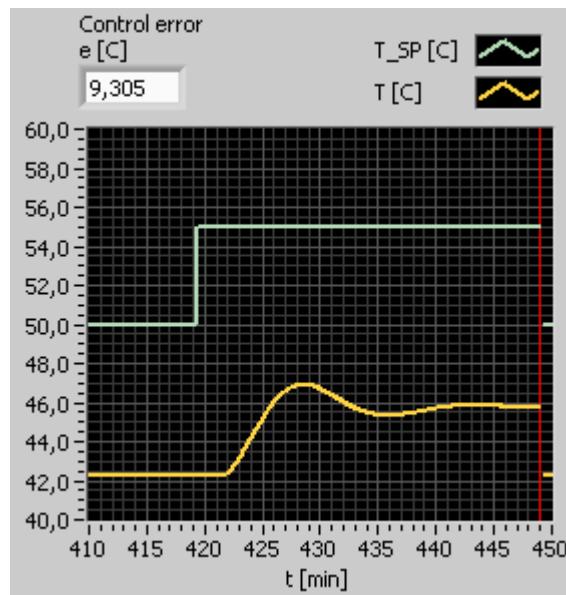


Figure 16.30: Temperature response

Problem 16.2 Tuning of controller for time constant with time delay process

Given a process where the relation between the control signal and the process measurement can be well represented as “time constant with time delay” with gain $K = 0.5$, time-constant $T = 5$ s, and time delay $\tau = 1$ s.

1. Tune a PI controller for this process using Skogestad method assuming “time constant with time delay” process dynamics.
2. Tune a PI controller for this process using Skogestad method assuming “integrator with time delay” process dynamics. Compare with the result of Problem 1.

Problem 16.3 Skogestad PI tuning from process step response

Figure 16.31 shows the response in the process measurement y_m due to a step of amplitude $U = 2$ in the control signal u at time $t = 0$ s. Calculate settings for a PI controller for this process.

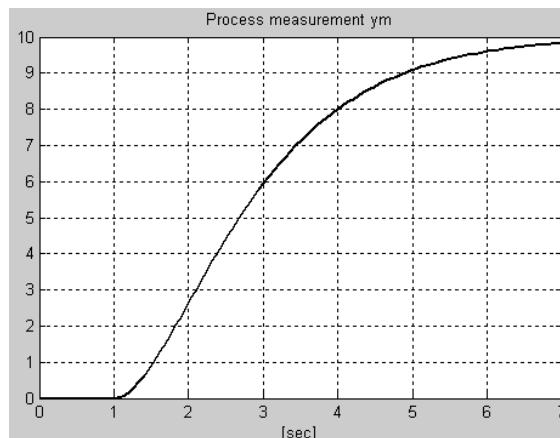


Figure 16.31: Process measurement step response for Skogestad tuning.

Problem 16.4 PI tuning for averaging level control

Given a level control system with PI controller for averaging level control for a water tank with cross-sectional area $A = 50 \text{ m}^2$. The largest sudden change that the inflow to the tank can have is 100 L/s . It is specified that this largest change must give a maximum level change of 20 cm . Tune the PI controller.

Problem 16.5 Controller tuning when process dynamics varies – 1

Assume that the process to be controlled has varying process dynamics, which may cause stability problems or sluggish control. Both of the solutions A and B below are possible. Which is the best one with respect to control performance, and which is the simplest one?

- A: The controller is tuned at the most critical operating point, and the controller parameters are then kept constant.
- B: The controller parameters are adjusted continually so that they fit to the dynamic properties of the process at any operating point.

Problem 16.6 *Controller tuning when process dynamics varies – 2*

Assume that you in a given control system for a “time constant with time delay” process have found proper PI parameters in one specific operating point. Assume that the process gain increases.

1. How would this process gain increase influence the stability of the control system?
2. Derive formulas for the new controller parameters. You can indicate the initial values of the controller parameters and the process parameters (before the change) with index 0, and new values (after the changes) with index 1.

Problem 16.7 *In which operating point to tune the controller?*

Figure 16.32 shows a chemical reactor and a PID parameter table which is the basis of a PID controller with gain scheduling. Assume that gain scheduling is not to be used, but fixed PID settings instead. Should the controller be tuned at high temperature or at low temperature, given that it is crucial that the stability of the control system is satisfactory at any temperature?

Problem 16.8 *Interpolation in a gain scheduling table*

Table 16.4 shows parts of a gain scheduling based PID controller.

Table 16.4: PID Gain Schedule

GS	K_c	T_i	T_d
\vdots	\vdots	\vdots	\vdots
20 %	0.4	5.2	1.3
30 %	0.5	4.5	1.6
\vdots	\vdots	\vdots	\vdots

Find K_c as a function of the gain scheduling variable GS between the operating points shown in the table. The function should be based on linear interpolation.

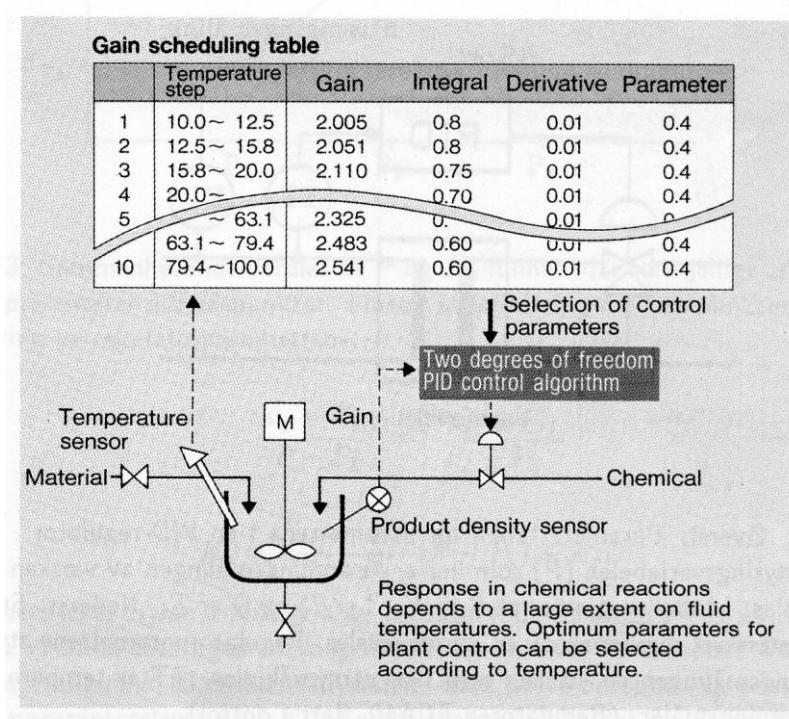


Figure 16.32: Chemical reactor and PID parameter table

Problem 16.9 Compensation for process nonlinearity

Figure 16.33 shows a process with a PID control system where the actuator is represented with a nonlinear relation between the control signal u and the internal process variable z :

$$z = f(u) \quad (16.119)$$

For example, the actuator can be a control valve with some nonlinear relation between the control signal (u) and the flow (z).

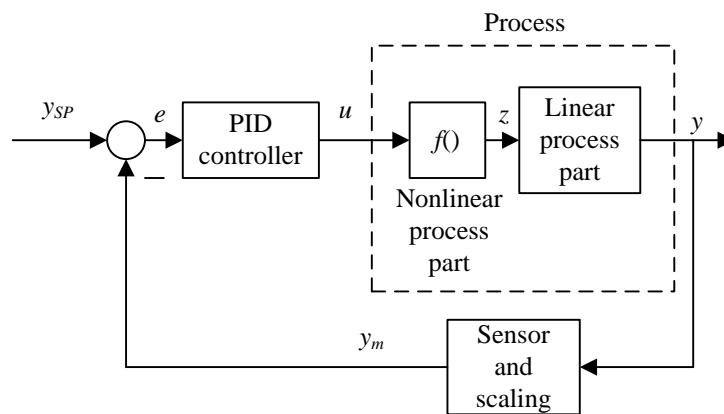


Figure 16.33: Control valve.

In most cases it is beneficial if the PID controller sees a *linear* process – not a nonlinear process, because this makes the controller tuning easier, and the dynamic properties of the

control system may be independent of the operating point. This can be achieved by including the *inverse* of the nonlinear function in the controller:

$$u = f^{-1}(z) \quad (16.120)$$

The z -value that the PID controller demands can be denoted z_{PID} . See Figure 16.34.

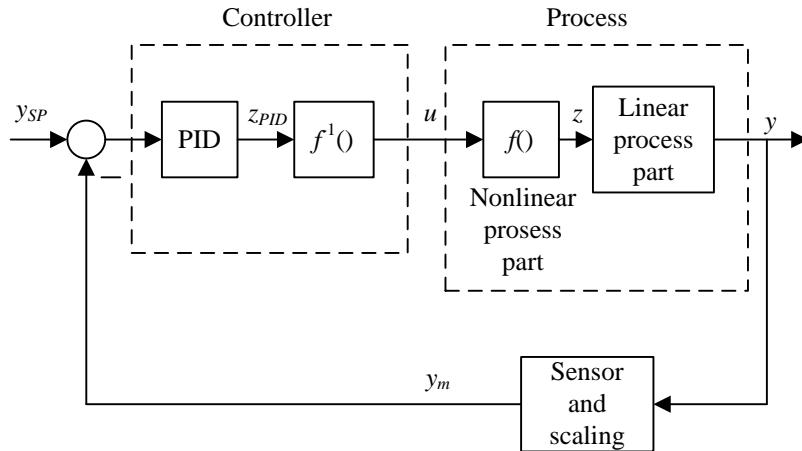


Figure 16.34: Nonlinear compensation.

Assume that the nonlinear function $z = f(u)$ can be represented with n tabular data points (which can stem from a data sheet or from experiments) as shown in Table 16.5.

Table 16.5: Data points

z	u
z_1	u_1
z_2	u_2
\vdots	\vdots
z_n	u_n

Explain how you can implement the inverse function using *table-lookup*. Table-lookup functions implements linear interpolation between the data points in the table.¹⁵

¹⁵Table lookup functions are available in computer tools as MATLAB and LabVIEW.

16.12 Solutions to problems for Chapter 16

Solution to Problem 16.1

We set the controller gain to

$$K_c = 0.8 \cdot 4.0 = 3.2 \text{ min} \quad (16.121)$$

In Figure 16.35 we read off

$$T_{ou} = 8.0 \text{ min} \quad (16.122)$$

which gives the integral time

$$T_i = 1.5T_{ou} = 12.0 \text{ min} \quad (16.123)$$

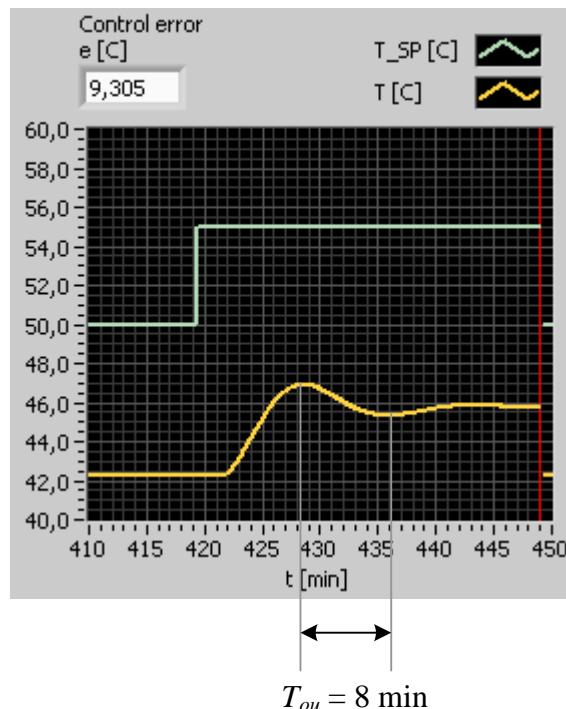


Figure 16.35: Problem 16.1: Good Gain response.

If it turns out that the stability of the control system is too bad with this value of K_c , you can try reducing the gain and/or increasing the integral time.

Solution to Problem 16.2

- With the Skogestad PI tuning formulas for “time constant with time delay”, (16.65)-(16.66), and using the Skogestad hand-rule

$$T_C = \tau = 1 \text{ s} \quad (16.124)$$

the PI settings become:

$$K_c = \frac{T}{2K\tau} = \frac{5}{2 \cdot 0.5 \cdot 1} = 5 \quad (16.125)$$

$$T_i = \min [T, 4\tau] = \min [5, 4 \cdot 1] = \min [5, 4] = 4 \text{ s} \quad (16.126)$$

Comment: Above, $\min[x, y]$ means the minimum of the two numbers x and y .

2. The Skogestad PI tuning formulas for “integrator with time delay”, (16.39)-(16.40), with the Skogestad hand-rule

$$T_C = \tau \quad (16.127)$$

are:

$$K_c = \frac{1}{2K_i\tau} \quad (16.128)$$

$$T_i = 4\tau \quad (16.129)$$

Here, K_i is the normalized steepest slope of the process step response. For a “time constant” system, the steepest slope appears just after the point of time of the input step. E.g. from the analytical step response (10.17), we can find:

$$K_i = \frac{K}{T} \quad (16.130)$$

With this K_i , (16.128) and (16.129) become:

$$K_c = \frac{1}{2K_i\tau} = \frac{T}{2K\tau} = \frac{5}{2 \cdot 0.5 \cdot 1} = 5 \quad (16.131)$$

$$T_i = 4\tau = 4 \text{ s} \quad (16.132)$$

which are the same PI settings as in Problem 1 above. This is in accordance with a fact presented in Ch. 16.8.3.2: If $\tau > T/4$, the PI settings for “time constant with time delay” processes and for “integrator with time delay” processes are identical.

Solution to Problem 16.3

Figure 16.36 shows the steepest tangent drawn on the process step response. From the two points labelled P2 and P3, the slope of the tangent can be calculated as:

$$S = \frac{(10.0 - 0.0)}{(4.0 - 1.3)} = 3.7 \quad (16.133)$$

The normalized slope is:

$$K_i = \frac{S}{U} = \frac{3.7}{2} = 1.85 \quad (16.134)$$

From the two points labelled P1 and P2, the (approximate) time delay is:

$$\tau = 1.3 - 0.0 = 1.3 \text{ s} \quad (16.135)$$

The PI settings become:

$$K_c = \frac{1}{2K_i\tau} = \frac{1}{2 \cdot 1.85 \cdot 1.3} = 0.21 \quad (16.136)$$

$$T_i = 4\tau = 4 \cdot 1.3 = 5.2 \text{ s} \quad (16.137)$$

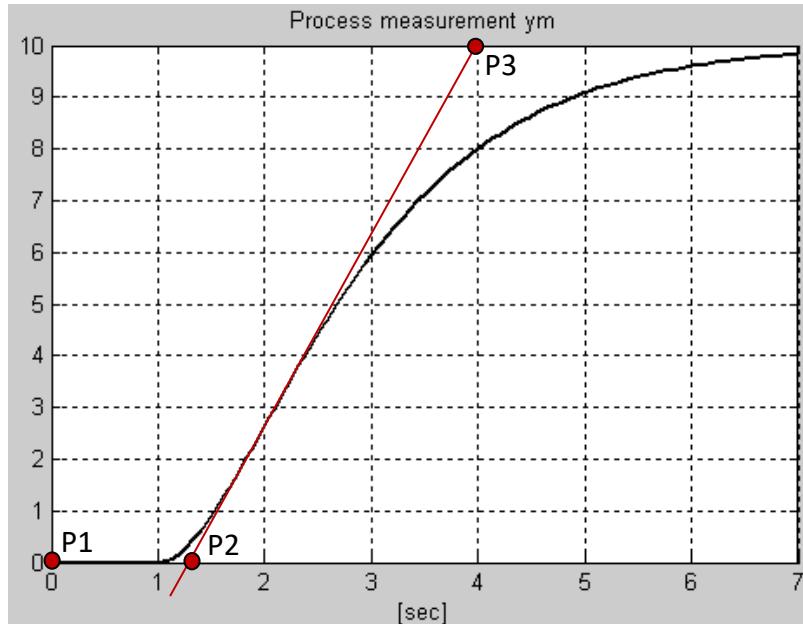


Figure 16.36: Problem 16.3: Steepest tangent drawn on the process step response.

Solution to Problem 16.4

Given:

$$A = 50 \text{ m}^2 \quad (16.138)$$

$$\Delta F_{\text{in}} = 100 \text{ L/s} = 0.1 \text{ m}^3/\text{s} \quad (16.139)$$

$$\Delta h = 20 \text{ cm} = 0.2 \text{ m} \quad (16.140)$$

This gives:

$$T_c = A \frac{\Delta h}{\Delta F_{\text{in}}} = 50 \cdot \frac{0.2}{0.1} = 100 \text{ s} \quad (16.141)$$

$$K_c = -\frac{A}{T_c} = -\frac{\Delta F_{\text{in}}}{\Delta h} = -\frac{0.1}{0.2} = -0.5 \text{ (m}^3/\text{s})/\text{m} \quad (16.142)$$

$$T_i = 2T_c = 200 \text{ s} \quad (16.143)$$

Solution to Problem 16.5

For best control performance: B. Simplest: A.

Solution to Problem 16.6

1. The stability would be decreased.¹⁶
2. Let us indicate initial values (before the change) with index 0, and new values (after the changes) with index 1. According to Skogestad PID tuning rule for “time constant with time delay” processes:

$$K_{c_0} = \frac{T_0}{K_0(T_C + \tau_0)} \quad (16.144)$$

$$T_{i_0} = \min[T_0, c(T_C + \tau_0)] \quad (16.145)$$

and, of course,

$$K_{c_1} = \frac{T_1}{K_1(T_C + \tau_1)} \quad (16.146)$$

$$T_{i_1} = \min[T_1, c(T_C + \tau_1)] \quad (16.147)$$

If the process gain K is the only parameter that has changed, we get from (16.144) and (16.146)

$$\frac{K_{c_0}}{K_{c_1}} = \frac{\frac{T_0}{K_0(T_C + \tau_0)}}{\frac{T_1}{K_1(T_C + \tau_1)}} = \frac{\frac{T_0}{K_0(T_C + \tau_0)}}{\frac{T_0}{K_1(T_C + \tau_1)}} = \frac{K_1}{K_0} \quad (16.148)$$

from which we get the following formula for the new controller gain:

$$K_{c_1} = K_{c_0} \frac{K_0}{K_1} \quad (16.149)$$

For the integral time there will be no change because the process gain is not in the formula of T_i :

$$T_{i_1} = T_{i_0} \quad (16.150)$$

Solution to Problem 16.7

From the table shown in Figure 16.32 we see that Gain = K_c has (should have) less value and Integral = T_i has (should have) larger value the lower the temperature. This indicates that minimum temperature is “worst case”. Therefore, a PID controller with fixed settings should be tuned at mimum temperature.

Solution to Problem 16.8

$$K_c = \frac{0.5 - 0.4}{30\% - 20\%}(GS - 20\%) + 0.4 = \underline{0.01\%^{-1} \cdot GS + 0.2} \quad (16.151)$$

¹⁶For processes that are unstable, the controller gain must actually be “large” for the control system to be stable, but such processes are relatively rare (one example is exothermal reactors).

Solution to Problem 16.9

The table-lookup is used straightforward: The input to the table-lookup function is z_{PID} which is the z -value that the PID controller demands, and the output from the table-lookup function is u which is used as control signal to the nonlinear process part (e.g. valve).

Chapter 17

Control loop stability

17.1 Heuristic stability analysis

It is important to be aware that there may be stability problems in a control loop. It is a basic requirement to a control loop that it is stable. Simply stated this means that the response in any signal in control loop converges towards a finite value after a limited change (with respect to the amplitude) of the setpoint or the disturbance or any other input signal to the loop.

There is always a possibility that a feedback control system which is originally stable, *may become unstable* due to parameter changes in the loop. Instability implies that signals in the loop starts to increase in amplitude until some saturation limit is reached (for example, a valve have a limited opening).

As a start of the discussion of how a feedback control system can become unstable, let us repeat the block diagram of a feedback control system from the beginning of this book, see Figure 17.1.

Instability in a feedback system can be explained in two ways:

- **Too large gain in the loop:** The loop gain is the product of the gains in each of the subsystems (controller, process, sensor) in the loop. If the signal in the loop is amplified too much through the loop, it “comes back” amplified. Then, this signal is again amplified through the loop, and eventually the amplitude just increases. In other words, the control loop is unstable.
- **Too large time delay in the loop.** The effect or response due to the controller action is fed back to the controller with too much time delay so that the information in the response is out-of-date and does not reflect the state of the process good enough. Based on this poor information, the controller may generate too much or too little control action. This repeats through the loop, causing larger and larger deviation from the setpoint. In other words, the control loop is unstable.
- **Too large time constant in the loop.** Note: The impact of a large time constant

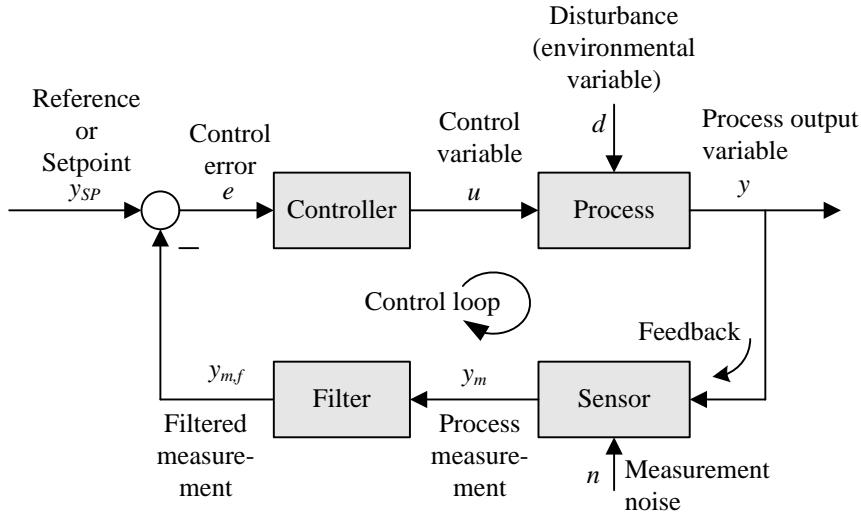


Figure 17.1: Block diagram of a feedback control system.

on the loop stability presented here assumes that the controller has integral action, as in a PI or PID controller. The effect of a too large time constant is similar to that of the a too large time delay: The response due to a change in the controller output come back to the controller with a large lag, which resembles a time delay. Furthermore, the signal is smoothed due to the filtering effect of a time constant. Both these effects cause the controller to get a relatively “false” information about the process, and it may generate a too large control signal which gives an over-compensation, eventually causing oscillations to occur.

Example 17.1 Instability in the wood-chip tank level control system

We study here the level control system of the wood chip tank described in Ch. 1.2.6. The controller is a PI controller with Skogestad settings as found in Example 16.6:

$$K_c = 1.05 \text{ \%}/\text{m} \quad (17.1)$$

$$T_i = 1000 \text{ s} \quad (17.2)$$

Too large gain

Figure 17.2 shows the response in the level when the screw gain of the feed screw, K_s , was increased by a factor of approximately three, from $33.36 \text{ (kg/min)}/\%$ to $100 \text{ (kg/min)}/\%$, at time 40 min. (This parameter change may be cause by the installation of a new large screw with three times larger capacity.) This process gain increase causes a similar increase in the control loop gain. The control system becomes unstable. (The amplitude of the oscillations are limited due to the limits of the control variable. The maximum value is 100%, and the minimum value is 0% are reached during the simulations.)

Can we get the stability back by retuning the controller? Recall the Skogestad PI settings, (16.39) – (16.40), which I repeat here:

$$K_c = \frac{1}{2K_i\tau}; T_i = 4\tau \quad (17.3)$$

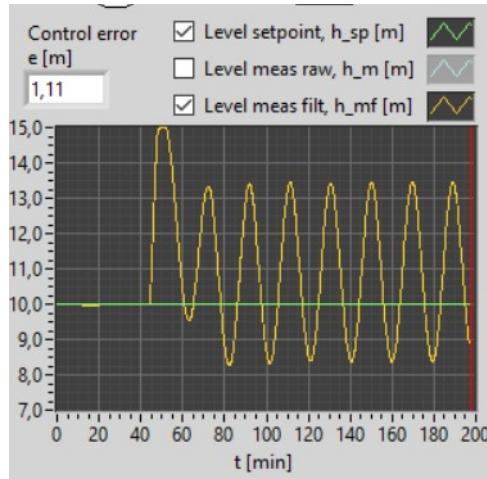


Figure 17.2: Large process gain: The response in the level when the screw gain, was increased by a factor of approximately three, from 33,36 (kg/min)/% to 100 (kg/min)/%, at time 40 min. The control system has become unstable.

In our case, K_i has been increased by a factor of 3. From the PI tuning formulas above, we can conclude that K_c should be reduced by a factor of 3, from 1.05 to 0.35, while T_i should not be changed. Figure 17.3 shows a simulation when these new PI settings were applied at time approx. 100 min. We see that the control system is again stable.

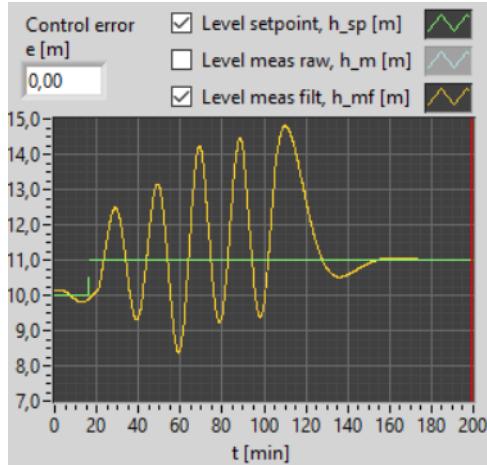


Figure 17.3: Large process gain: The level control system is again stable when the controller was retuned at time approx. 100 min.

Too large time delay

Figure 17.4 shows the setpoint step response in the level when the time delay of the conveyor belt has been increased from the nominal value of 250 s (= 4.17 min) to 500 s (= 8.33 min). (The controller has the orginal settings of $K_c = 1.05$ and $T_i = 1000$ s.) This process time delay increase causes a similar increase in the control loop time delay. The control system has become close to unstable due to the increase in the time delay.

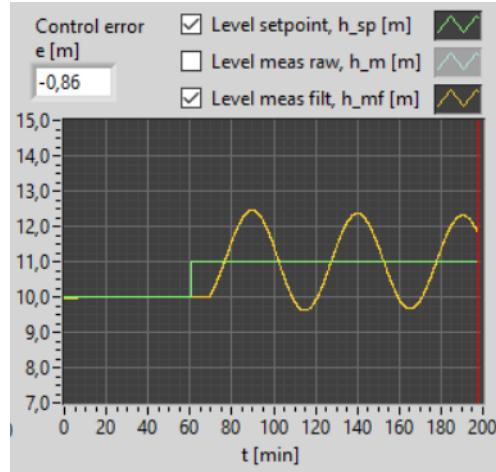


Figure 17.4: Large process time delay: The level control system has become close to unstable when the time delay of the conveyor belt is increased from the nominal value of 250 s to 500 s.

Can we get the stability back by retuning the controller? The Skogestad PI settings:

$$K_c = \frac{1}{2K_i\tau}; T_i = 4\tau \quad (17.4)$$

tell us that if the time delay, τ , is increased by a factor of 2, as in our example, K_c should be reduced by a factor of 2, from 1.05 to 0.525, and T_i should be increased by a factor of 2, from 1000 s to 2000 s. Figure 17.5 shows a simulation when these new PI settings applied at time approx. 480 min. We see that the control system is again stable.

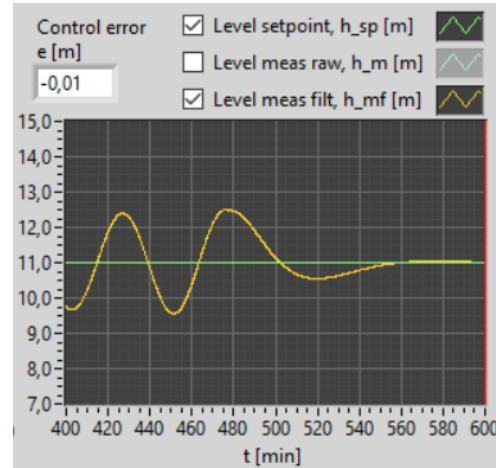


Figure 17.5: Large process time delay: The level control system is again stable when the PI controller was retuned at time approx. 480 min.

Too large time constant

In the level control system of the wood chip tank, the only time constant system is the measurement filter. We will study the effect on control system stability of increasing the

filter time constant, T_f . We increase it from 20 s to 520 s (this particular increase is to make the calculations somewhat easier; see below). Figure 17.6 shows the setpoint step response in the level with this increase in time constant. (The controller has the orginal settings of $K_c = 1.05$ and $T_i = 1000$ s.) The yellow curve in Figure 17.6 is the level measurement signal after the filter, i.e. the filtered measurement, while the blue curve in Figure 17.6 is the level measurement signal before the filter, i.e. the raw measurement. (We see that the filtered measurement has a big lag compared to the unfiltered measurement. The lag is due to the sluggish filter.) The increase in time constant has made the control loop unstable.

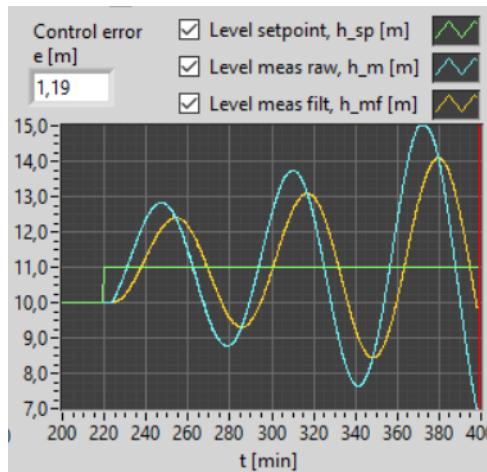


Figure 17.6: Large process time constant: The level control system has become close to unstable when the time constant of the meausurement filter is increased from the nominal value of 20 s to 520 s.

Can we get the stability back by retuning the controller? The Skogestad PI settings are:

$$K_c = \frac{1}{2K_i\tau}; T_i = 4\tau \quad (17.5)$$

Here we have a challenge, since there is no time constant in (17.5). However, we can try *approximating* the time constant with a time delay. We have increased the time constant, T_f , by 500 s. The time delay, τ , is 250 s. Thus, the increase of T_f is regarded as an increase in τ from 250 by 500 to 750 s which is an increase by factor 3. According to (17.5), a 3 times larger τ implies a reduction of K_c by factor 3 and an increase of T_i by factor 3. Therefore, the new settings are: $K_c = 1.05/3 = 0.35$ and $T_i = 3 \cdot 1000 = 3000$ min. Figure 17.7 shows a simulation when these new PI settings applied at time approx. 130 min. We see that the control system is again stable.

I emphasize the particular trick used above: Pretending that a time constant is a time delay. Such an approximation can be useful when retuning controllers after an increase in the filter time constant in the control loop. This approximation is safe (or robust or conservative) because an increase in the time delay has a larger (worse) impact on dynamics and stability of a control loop than an increase in the time constant has.

[End of Example 17.1]

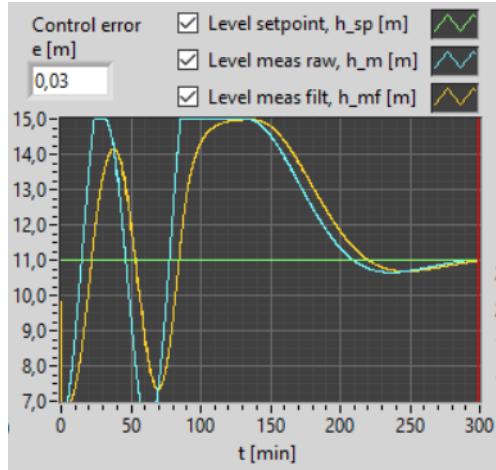


Figure 17.7: Large process time delay: The level control system is again stable when the PI controller was retuned at time approx. 480 min.

17.2 Experimental gain margin (GM) and phase margin (PM)

In Section 17.1 we saw that a feedback control system gets worse stability

- if the loop gain in the loop becomes larger, and/or
- if the time delay in the loop becomes larger.

A stability margin indicates how much change in the parameters in a (feedback) control system that can be tolerated before the control system comes to the stability limit, i.e. becomes marginally stable. When the control system is marginally stable, the system variables show undamped oscillations.

In the feedback control theory, there are two common stability margins:

- **Gain margin GM**, which expresses how large increase of the loop gain that feedback loop can tolerate before it becomes marginally stable.
- **Phase margin PM**, which (indirectly) expresses how large increase of the loop time delay that the feedback loop can tolerate before it becomes marginally stable.

GM and PM can be calculated from a linear mathematical model of the control system, cf. Ch. 23.3. GM and PM can also be found experimentally on a real or simulated control system (the simulator model may be nonlinear), as explained in the following.

Figure 17.8 shows a feedback control system where two special blocks for the purpose of stability analysis have been included:

- One block with a multiplicative gain ΔK

- One block with an additional time delay increase $\Delta\tau$.

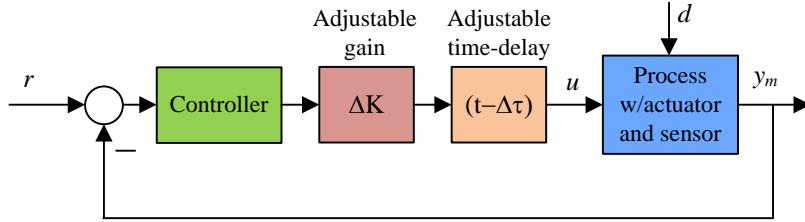


Figure 17.8: A feedback control system with blocks of multiplicative gain increase ΔK and additional time delay increase $\Delta\tau$.

You can find GM and PM as explained below.

- **GM:** Let $\Delta\tau$ be 0, i.e. no time delay increase. Start with $\Delta K = 1$, i.e. no gain increase to start with. By trial-and-error, find the value of ΔK , here denoted ΔK_u (u for ultimate), that makes the control system become marginally stable (oscillatory). This gives

$$GM = \Delta K_u \quad (17.6)$$

- **PM:** Let ΔK be 1, i.e. no gain increase. Start with $\Delta\tau = 0$, i.e. no time delay increase to start with. By trial-and-error, find the value of $\Delta\tau$, here denoted $\Delta\tau_u$ (u for ultimate), that makes the control system become marginally stable (oscillatory). It can be shown that

$$PM = \frac{\Delta\tau_u}{P_u} \cdot 360^\circ \quad (17.7)$$

where P_u [s] is the period of the oscillations. (17.7) is derived at the end of this section, after Example 17.2. Comment: PM measured in degrees originates from traditional frequency response theory of feedback systems.

Acceptable intervals for GM and PM are given in Seborg et al. (2004) as

$$1.7 \leq GM \leq 4.0 \quad (17.8)$$

$$30^\circ \leq PM \leq 45^\circ \quad (17.9)$$

Since too poor stability *must* be avoided, while too good stability *may* be accepted, the lower limits of GM and PM are critical, while the upper limits for GM and PM are recommended, but not critical. In other words: GM should not be less than 1.7 and PM not less than 30° , while there is no disaster if GM is greater than 4.0 or if PM is greater than 45° .

Practical problems of including the extra blocks?

Figure 17.8 contains a gain block and a time delay block where we adjust ΔK and $\Delta\tau$ to find marginal stability. If, for some reason, it is difficult to include two such blocks, you can use the following alternative method to find ΔK and $\Delta\tau$ (that procedure is demonstrated in Example 17.2):

- Finding ΔK_u : We adjust the *controller gain* until the feedback loop becomes marginally stable. Then, ΔK_u is equal to the multiplicative increase of the controller gain.
- Finding $\Delta\tau_u$: It is here assumed here that there is a time delay somewhere in the feedback loop, e.g. in the process, that can be adjusted. Then, $\Delta\tau_u$ is equal to the additive increase of the time delay that makes the feedback loop become marginally stable. Note: If there is no time delay in the loop at all and you can not include a time delay in the loop, either, there will not be any time delay to adjust. In such a case you must simply drop finding PM.¹

Example 17.2 *Stability margins of the level control system of the wood-chip tank*

Ch. 1.2.6 presents a level control system of a wood-chip tank. Assume that the controller is a PI controller with these settings:

$$K_c = 1.35 \quad (17.10)$$

$$T_i = 900 \text{ s} \quad (17.11)$$

Let us find the GM and PM for the control system from a simulation.² To find GM, we adjust the regulator gain, K_c , in a simulation. To find PM, we adjust the conveyor time delay (transport delay), τ , in a simulation. Initially, $\tau = 250 \text{ s} = 4.17 \text{ min}$.

- **GM:** Simulations show that the control system becomes marginally stable with $K_c = 2.63$, see Figure 17.9. The gain margin becomes

$$\text{GM} = \Delta K_u = \frac{2.63}{1.35} = 1.95$$

which is an acceptable value since it is within the limits in (17.8).

- **PM:** Before the time delay is adjusted in a simulation, K_c is reset to 1.35. A simulation show that with $\tau = 6.73 \text{ min}$, the control system becomes marginally stable, see Figure 17.10. The period of the oscillations is measured as $P_u = 37.5 \text{ min}$. The phase margin becomes

$$\text{PM} = \frac{\Delta\tau_u}{P_u} \cdot 360^\circ \quad (17.12)$$

$$= \frac{(6.73 \text{ min} - 4.17 \text{ min})}{37.5 \text{ min}} \cdot 360^\circ \quad (17.13)$$

$$= 24.6^\circ \quad (17.14)$$

which is an unacceptably low value, cf. (17.9). Actually, the PI controller was tuned with the Ziegler-Nichols turning method, which is known to give the control system a relatively poor stability. So, the low phase margin in this example is not surprising.

¹With a simulator, we should always be able to include a time delay, but it can be difficult to do it in a real feedback control system.

²A SimView simulator of the system: Level control of wood chip tank, <http://techteach.no>.

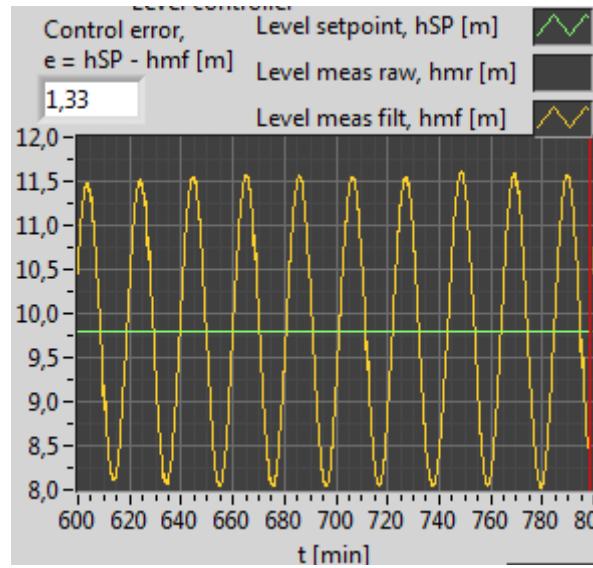


Figure 17.9: The control system is marginally stable with $K_c = 2.63$.

[End of Example 17.2]

Derivation of (17.7):

I will now derive (17.7) from the theory of stability analysis of linear feedback systems based on frequency response (this theory is presented in Ch. 23). See Figure 17.8. Assume that the feedback system is stable with $\Delta\tau = 0$ (and $\Delta K = 1$), and that $\Delta\tau$ is then increased to the value $\Delta\tau_u$ that makes the system become marginally stable. In general, when the time delay in the loop is increased, the amplitude gain characteristic is unchanged, and in particular the gain crossover frequency, ω_c [rad/s], is unchanged, while the phase characteristic is reduced by $\omega\Delta\tau$ [rad] where ω [rad/s] is frequency. In frequency response theory, the phase margin, PM, is defined as the phase reduction (given as a positive value) which causes the phase of the loop to become -180° at the gain crossover frequency, ω_c . Since this phase reduction stems from the time delay increase only, the phase margin is

$$PM = \omega_c \Delta\tau_u \text{ [rad]} \quad (17.15)$$

or

$$PM = \omega_c \Delta\tau_u \cdot \frac{180}{\pi} \text{ [deg]} \quad (17.16)$$

When the system is marginally stable, its response in the time domain is oscillatory, and the frequency of the oscillations is equal to ω_c (because the purely imaginary poles, $\pm j\omega_c$, are among the poles of the system). ω_c is related to the period, P_u [s], of the oscillations as follows:

$$\omega_c = \frac{2\pi}{P_u} \quad (17.17)$$

Finally, combining (17.17) and (17.16) gives (17.7), which is what I wanted to derive.

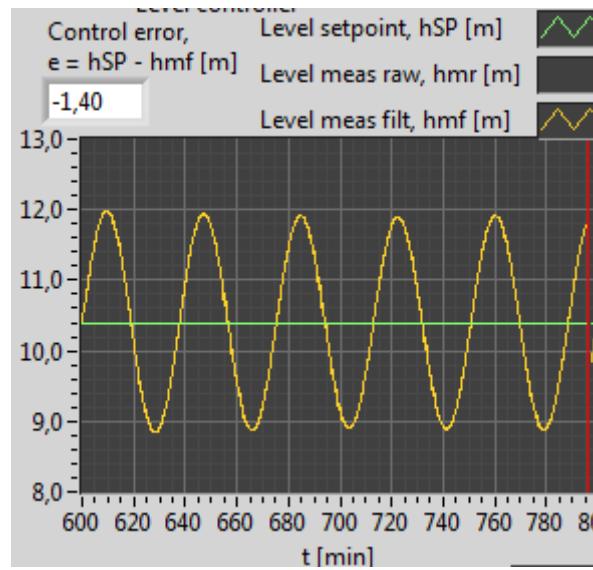


Figure 17.10: The control system is marginally stable with $\tau = 6.63$ min.

17.3 Problems for Chapter 17

Problem 17.1 PB and stability

How will the stability of the control loop change if the proportional band PB is reduced?

Problem 17.2 Control system stability at process changes

Figure 17.11 shows a temperature control system with two different positions of the temperature sensor. Assume that the temperature controller is tuned so that the stability of the control system is satisfactory with the sensor in position 1.

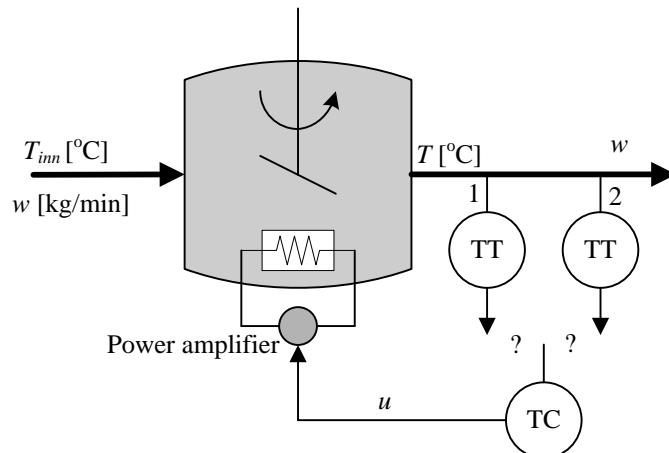


Figure 17.11: Temperature control system

- What will happen to the stability if the sensor is moved to position 2?

2. Assume that the sensor is in position 1. How will the stability change if the liquid flow decreases?
3. How will the stability change if the sensor gain is increased (the gain is the ratio of the sensitivity of the measurement signal in volts or amperes to the temperature)?
4. How will the stability change if the heater is substituted by a heater delivering more power per unit of the control signal?

Problem 17.3 *Explain the unstable response!*

A temperature control system is described in Problem 12.6. Figure 17.12 shows the temperature response after a relatively large reduction of the fan opening. Explain the unstable response!

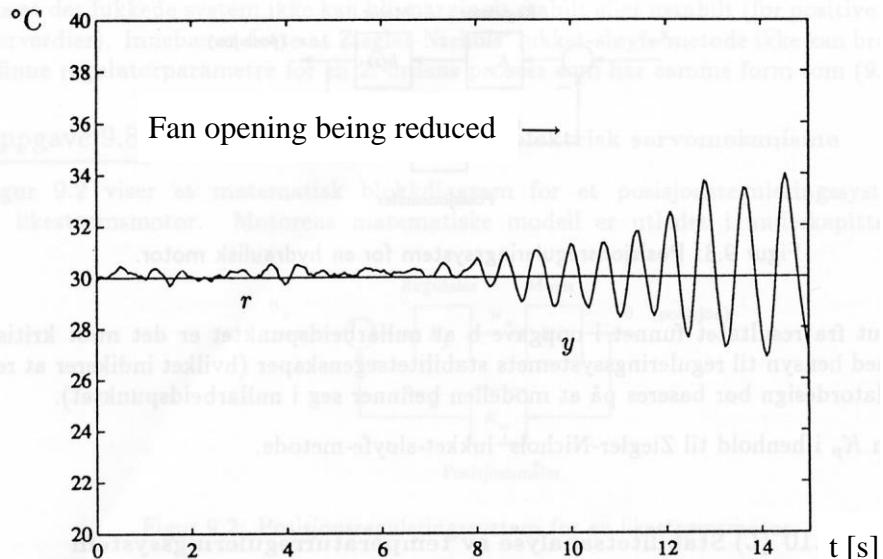


Figure 17.12: Unstable temperature control system

Problem 17.4 *Stability margins*

Given a control system that is initially stable when $\Delta K = 1$ and $\Delta\tau = 0$. The controller gain is then $K_c = 5.0$. It is stated that the control system has $GM = 2.5$.

1. Which value of K_c will make the control system become marginally stable?
2. In an experiment on the control system with $K_c = 5.0$, it has been found that a dead time increase of $\Delta\tau = 10$ s causes the control system to oscillate with a period of 100 s. What is the phase margin PM? Is this an ok value?

17.4 *Solutions to problems for Chapter 17*

Solution to Problem 17.1

Reduced PB value means increased controller gain, and therefore increased loop gain. This implies that the stability is reduced if the PB value is decreased.

Solution to Problem 17.2

1. The transport time from tank to sensor increases, thus increasing the time delay in the control loop, causing reduced stability. If the time delay gets too large, the control system becomes unstable.
2. With reduces flow, the time delay is increased, causing reduced stability.
3. The stability is reduced since the loop gain is increased.
4. The stability is reduced since the loop gain is increased.

Solution to Problem 17.3

The response is oscillatory because the control system is unstable. The instability is due to:
(1) The reduced air flow which causes *increased transport delay (time delay) in the process* from heating element to temperature sensor. (2) *Increased process gain* because the reduced air flow makes the temperature more sensitive to the supplied heat (adjusted by the control signal), and increased process gain implies increased control loop gain.

Solution to Problem 17.4

1.

$$\underline{\underline{K_c}} = 5 \cdot 2.5 = \underline{\underline{12.5}}$$

2.

$$\underline{\underline{\text{PM}}} = \frac{\Delta\tau_u}{P_u} \cdot 360^\circ = \frac{10 \text{ s}}{100 \text{ s}} \cdot 360^\circ = \underline{\underline{36^\circ}}$$

which is an ok value, cf. (17.9).

Chapter 18

Control structures based on the PID control loop

This chapter describes a number of control methods and control structures based on the PID controller that are quite common in industry.

Also, sequential control is described. Sequential control is used to implement batch process control and automated mechanical operations. In a sequential control system PID control loops typically play a minor role, because the overall control task is not about continuous control. For example, PID control may be used to make the temperature of a batch reactor track a ramped temperature profile during the heating step of the control sequence.

18.1 Cascade control

18.1.1 The principle of cascade control

From earlier chapters we know that a control loop compensates for disturbances. If the controller has integral action the steady-state control error is zero despite a change in the disturbance. What more can we wish? We may want the compensation to happen faster, so that the control error goes to zero faster. This can be achieved by *cascade control*, see Figure 18.1.

In a cascade control system there is one or more control loops inside the *primary loop*, and the controllers are in cascade.

There is usually one, but there may be two and even three internal loops inside the primary loop. The (first) loop inside the primary loop is called the *secondary loop*, and the controller in this loop is called the *secondary controller* (or slave controller). The outer loop is called the *primary loop*, and the controller in this loop is called the *primary controller* (or master-controller). The control signal calculated by the primary controller is used as the setpoint of the secondary controller.

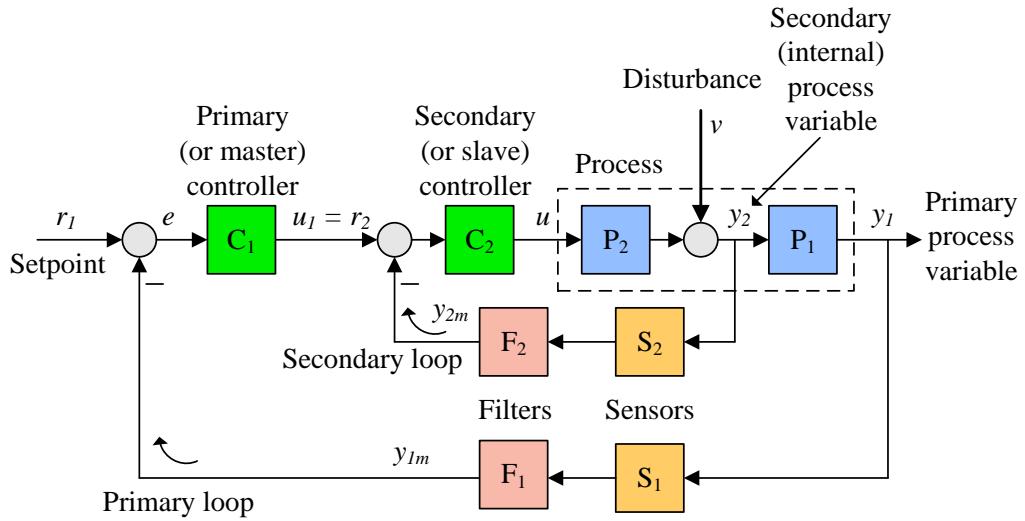


Figure 18.1: Cascade control system.

18.1.2 Benefits of cascade control

To repeat: One important reason for using cascade control is that it *gives faster (better) disturbance compensation*. In most applications the purpose of the secondary loop is to compensate quickly for the disturbance so that the response of the disturbance in the primary output variable is small. For this to happen the secondary loop must register the disturbance. This is done with the sensor S_2 in Figure 18.1.

One additional benefit of cascade control is that *the internal process variable (y_2) becomes more directly controllable, making it easier to tune the primary controller to obtain satisfactorily good overall stability and fastness of the whole control system*. In many applications process part 2 (P_2 in Figure 18.1) is the actuator (valve, pump, motor), or is closely related to the actuator. The secondary control loop can be regarded as a *new actuator* which can be more directly controlled or manipulated by the primary controller. See Figure 18.2. The internal process variable y_2 is controlled or manipulated directly by the primary controller: The output of the primary controller, u_1 , demands a value of y_2 which is then a setpoint r_2 of y_2 , and the secondary control loop makes y_2 track r_2 .

Examples:

- The internal process variable (y_2) is valve flow. The secondary loop is then a flow control loop.
- The internal process variable (y_2) is pressure after (downstream) the valve. The secondary loop is then a pressure control loop.
- The internal process variable (y_2) is valve position (valve opening). The secondary loop is then a position control loop.

The generally improved control with cascade control can be explained by the increased

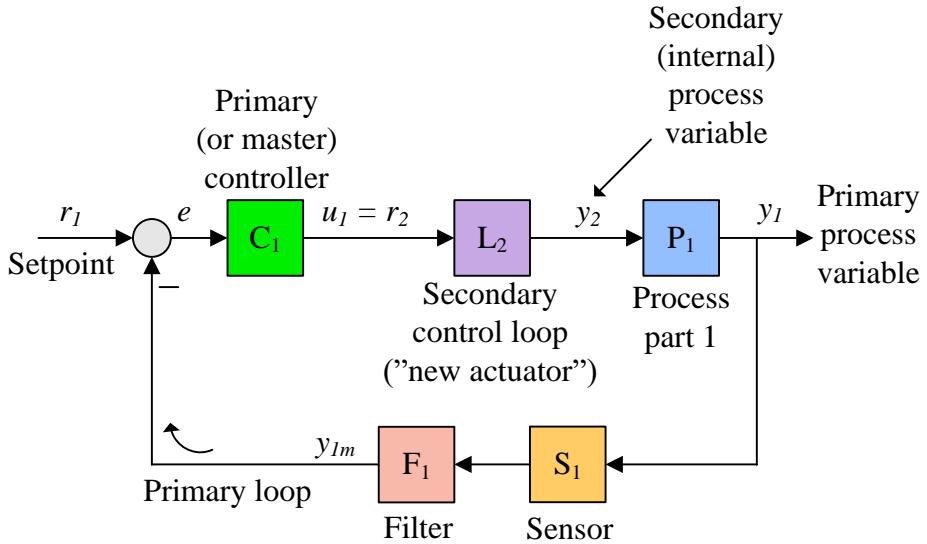


Figure 18.2: The secondary control loop can be regarded as a *new actuator*.

information about the process – there is at least one more measurement. It is a general principle that the more information you have about the process to be controlled, the better it can be controlled. Note however, that there is still only one control variable to the process, but it is based on two or more measurements.

As explained above cascade control can give substantial disturbance compensation improvement. Cascade control can also give improved tracking of a varying setpoint, but only if the secondary loop has faster dynamics than the process part P_2 itself, cf. Figure 18.1, so that the primary controller “sees” a faster process. But, if there is a time delay in P_2 , the secondary loop will not be faster than P_2 (this is demonstrated in Example 18.1), and then faster setpoint tracking can not be expected.

Are there any drawbacks of cascade control? Since cascade control requires at least two sensors a cascade control system is somewhat more expensive than a single loop control system. Except for cheap control equipment, commercial control equipment are typically prepared for cascade control, so no extra control hardware or software is required.

18.1.3 Controller selection and controller tuning

The primary controller is typically a PID controller or a PI controller. The secondary controller is typically a P controller or a PI controller. The derivative action is usually not needed to speed up the secondary loop since process part 2 anyway has faster dynamics than process part 1, so the secondary loop becomes fast enough. And in general the noise sensitive derivative term is a drawback.

In the secondary controller the P- and the D-term should not have reduced setpoint weights, cf. Section 12.7.3. Why?¹

¹Because attenuating or removing the time-varying setpoint (which is equal to the control signal produced

How do you *tune* the controllers of a cascade control? You can follow this procedure:

- First the secondary controller is tuned, with the primary controller in manual mode.
- Then the primary controller is tuned, the secondary controller in automatic mode.

So, you start with the inner loop, and then move outwards.

The tuning of the controllers (finding good PID settings) can be made using any of the tuning methods described in Section 16.

18.1.4 Cascade control and state feedback

Cascade control is quite similar to *state feedback control*, which is a control principle where the controller generates the control signal as a function of measurements (or estimates) of *all the state variables* of the process to be controlled, see Figure 18.3.

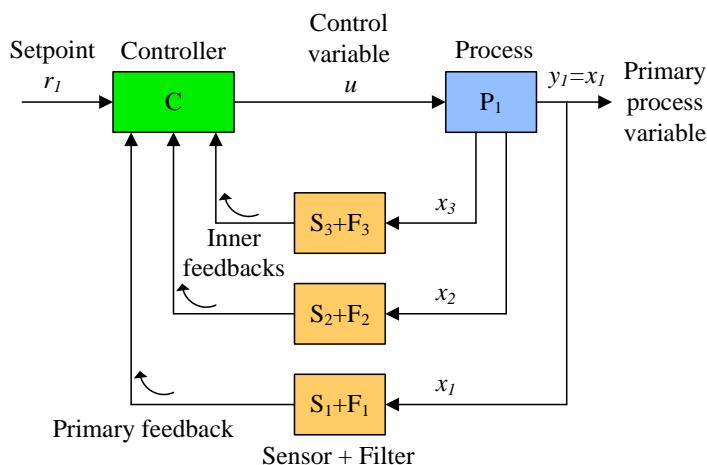


Figure 18.3: State feedback control.

It can be shown that with state feedback, you can design a control system with specified performance regarding stability and fastness for virtually any stable or unstable process. Hence, state feedback is a very powerful control principle. To design a state feedback controller you need the mathematical model of the process. There are several control methods – linear and nonlinear – that are based on state feedback control. The most basic method is linear state feedback where the control variable is a linear combination of the state variables.

Example 18.1 Temperature control using cascade control

by the primary controller) of the secondary loop will reduce the ability of the secondary loop to track these setpoint changes, causing slower tracking of the total control system.

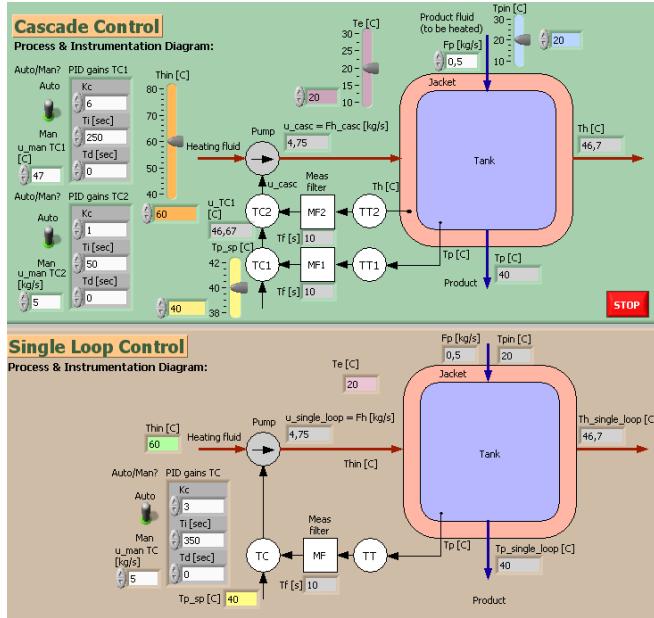


Figure 18.4: Example 18.1: Cascade and single-loop control systems of the product temperature of a tank

This example is about temperature control system of a liquid tank with continuous product flow heated with a heating liquid which flows continuously through a jacket around the tank. The tank can represent a reactor. In the simulator that is used in this example both the product and the heating liquid are water. A cascade control system and single-loop control system are simulated simultaneously (i.e. in parallel) so that the differences between these two control structures are easily demonstrated. The two control systems are as follows:

- **A cascade control system** with temperature control of both the tank and the jacket.
- **A single-loop control system** with temperature control of the tank only.

Figure 18.4 shows the P&IDs (Process & Instrumentation Diagrams) of the two control systems.

Both control systems are excited by the same temperature setpoint and the same disturbances. The controller parameter settings are shown in Figure 18.4.

Figure 18.5 shows block diagrams of the cascade and single-loop control systems.

The mathematical model on which the simulator is based contains the following energy balances (assuming homogenous conditions in both the tank and the jacket):

$$\text{Tank: } c_p d_p V_p T'_p = [c_p F_p (T_{p_{in}} - T_p) + U_{hp} (T_h - T_p)] \quad (18.1)$$

$$\text{Jacket: } c_h d_h V_h T'_h = [c_h F_h (T_{h_{in}} - T_h) - U_{hp} (T_h - T_p) + U_{eh} (T_e - T_h)] \quad (18.2)$$

where T_p is product liquid temperature, T_h is heating liquid temperature, c_p and c_h are specific heat capacities, F_p and F_h are flows, $T_{p_{in}}$ and $T_{h_{in}}$ are inlet temperatures, V_p and

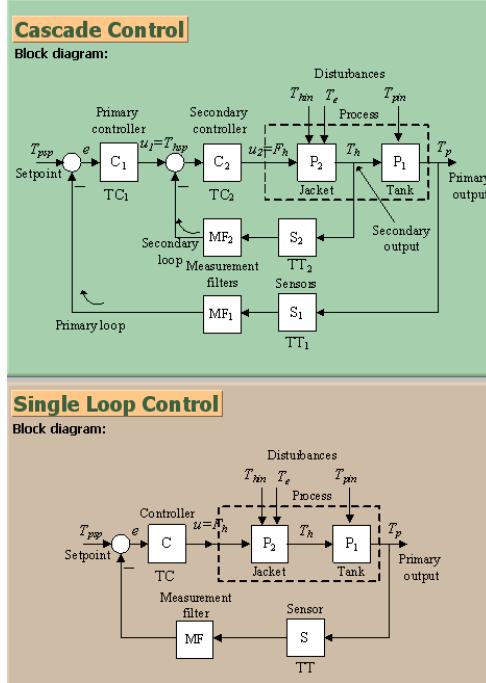


Figure 18.5: Example 18.1: Block diagrams of the cascade and single-loop control systems

V_h are volumes of tank and jacket respectively, T_e is environmental temperature, d_p and d_h are liquid densities, U_{hp} and U_{eh} are heat transfer coefficients between jacket and tank, and between jacket and environment (air) respectively. (All parameter values are available from the front panel of the simulator.) To account for unmodelled dynamics, a pure time delay is added to the measured temperatures T_p and T_h . The measurements of T_p and T_h are fed to measurement lowpass filters with time constants T_f .

Figure 18.6 shows simulated responses due to the following excitations (applied at different points of time):

- **Disturbance change:** The inflow temperature $T_{h_{in}}$ was changed as a step from 60 to 55 °C.
- **Disturbance change:** The environmental (ambient) temperature T_e was changed as a step from 20 to 25 °C.
- **Disturbance change:** The product inlet temperature $T_{p_{in}}$ was changed as a step from 20 to 25 °C.
- **Setpoint change:** The setpoint r_T of the product temperature was changed as a step from 40 to 40.5 °C.

From the simulations we observe the following:

1. The disturbance compensations regarding changes in the jacket inflow temperature $T_{h_{in}}$ and in the environmental (ambient) temperature T_e are substantially better with

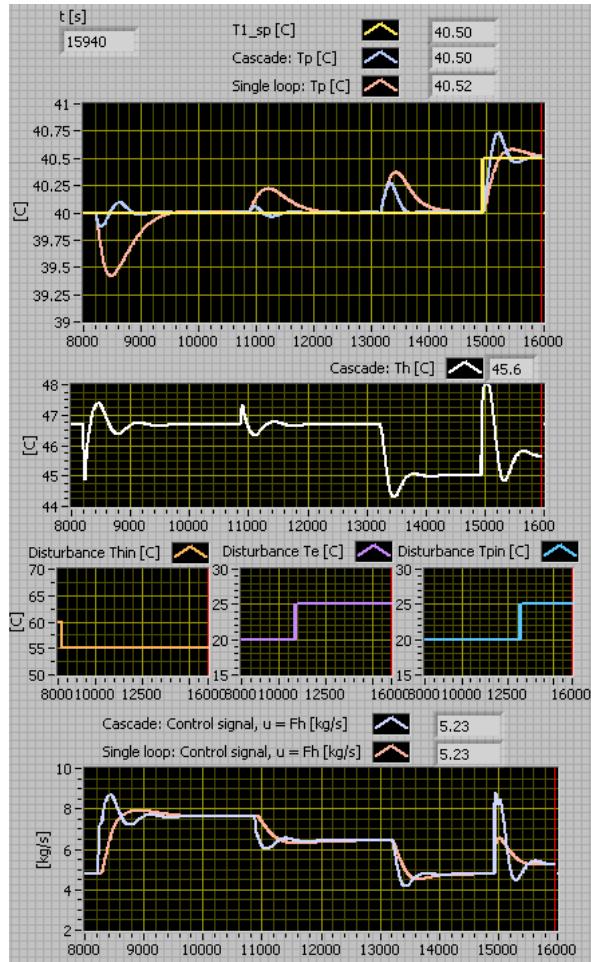


Figure 18.6: Example 18.1: Simulated responses of the cascade and single-loop control systems. The excitations are described in the text.

the cascade control system (see the responses in the upper plot in Figure 18.6). This is because the secondary process variable T_h (the jacket temperature) is directly influenced by these disturbances, and hence the secondary control loop can compensate for their changes.

2. The disturbance compensation regarding the change in the product inflow temperature T_{pin} is *not* substantially better with the cascade control system. This is because the secondary process variable T_h (the jacket temperature) is not directly influenced by this disturbance, and hence the secondary control loop can not compensate immediately for its change.
3. The setpoint tracking is somewhat better – but *not* substantially better – with the cascade control system.
4. The control signal in the cascade control system works more aggressively than in the single-loop control system, which is due to the relatively quick secondary loop.

[End of Example 18.1]

Cascade control is frequently used in the industry. A few examples are described in the following.

Example 18.2 Cascade control of the level in wood-chip tank

Level control of a wood-chip tank has been a frequent example in this book. In the real level control system² cascade control is used, although not described in the previous examples. The primary loop performs level control. The secondary loop is a control loop for the mass flow on the conveyor belt, see Figure 18.7. The mass flow is measured by a flow sensor (which actually is based on a weight measurement of the belt with chip between two rollers). The purpose of the secondary loop is to give a quick compensation for disturbances, d , in the chip flow due to variations in the chip density since the production is switched between spruce, pine and eucalyptus. (In Figure 18.7 the disturbance d actually has a negative value, hence, it represents a reduction of the inflow to the belt.)

In addition to this compensation the secondary flow control loop is a “new feed screw” with flow that is more directly controllable by the level controller.

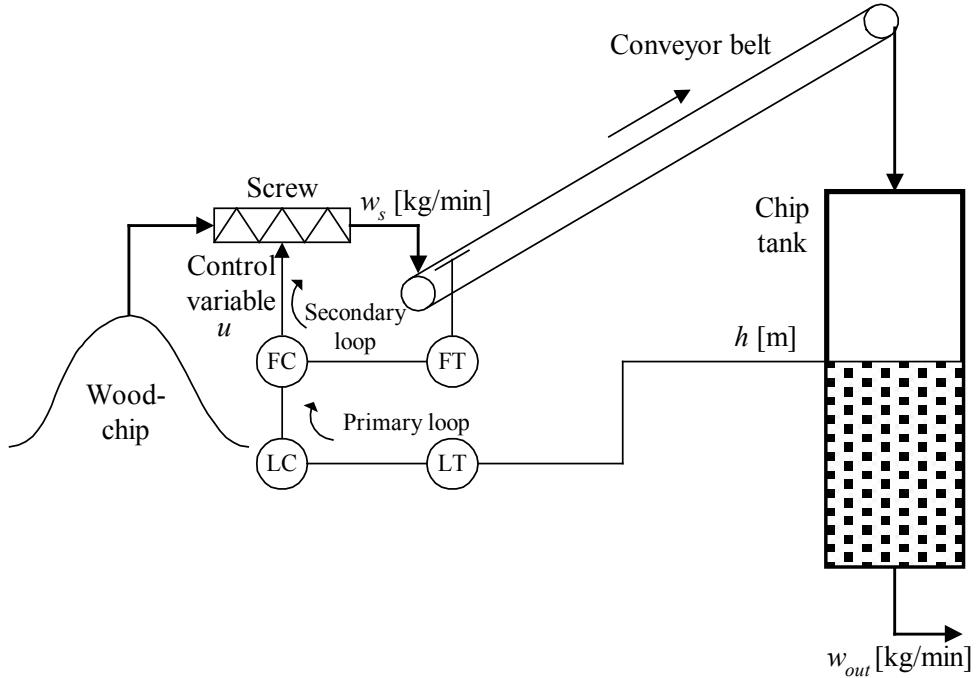


Figure 18.7: Example 18.2: Level control system of a wood-chip tank where the primary loop performs level control, and the secondary loop performs mass flow control. (FT = Flow Transmitter. FC = Flow Controller. LT = Level Transmitter. LC = Level Controller.)

[End of Example 18.2]

Example 18.3 Cascade control of a heat exchanger

²at Södra Cell Tofte in Norway

Figure 18.8 shows a temperature control system of a heat exchanger. The control variable controls the opening of the hot water valve. The primary loop controls the product temperature. The secondary loop controls the heat flow to compensate for flow variations (disturbances). The valve with flow control system can be regarded a new valve with an approximate proportional relation between the control variable and the heat flow.

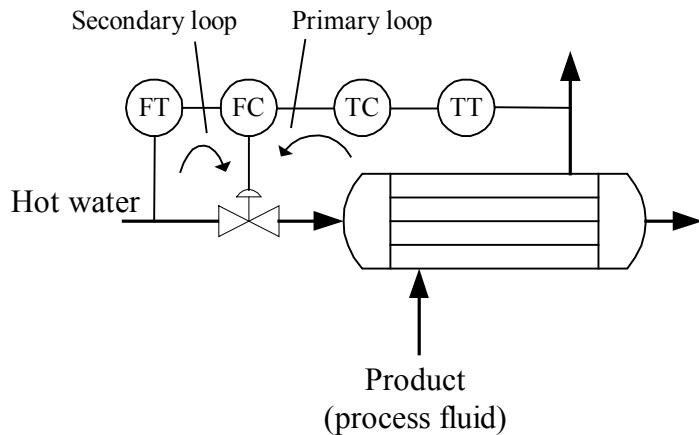


Figure 18.8: Example 18.3: Cascade control of the product temperature of a heat exchanger. (TC = Temperature Controller. TT = Temperature Transmitter. FC = Flow Controller. FT = Flow Transmitter.)

[End of Example 18.3]

There are many other examples of cascade control, e.g.:

- **DC-motor:**

- Primary loop: Speed control based on measurement of the rotational speed using a tachometer as speed sensor.
- Secondary loop: Control of armature current which compensates for nonlinearities of the motor, so that the speed controller can manipulate the current more directly. This may improve the speed control of the motor.

- **Hydraulic motor:**

- Primary loop: Positional control of the cylinder
- Secondary loop: Control of the servo valve position (the servo valve controls the direction of oil flow into the cylinder), so that the primary position controller can manipulate the servo valve position more directly. This may improve the position control of the cylinder.

- **Control valve:**

- The primary loop: Flow control of the liquid or the gas through the valve.
- Secondary loop: Positional control of the valve stem, so that the flow controller can manipulate the valve position more directly. This may improve the flow control. Such an internal positional control system is called *positioner*.

18.2 Ratio control

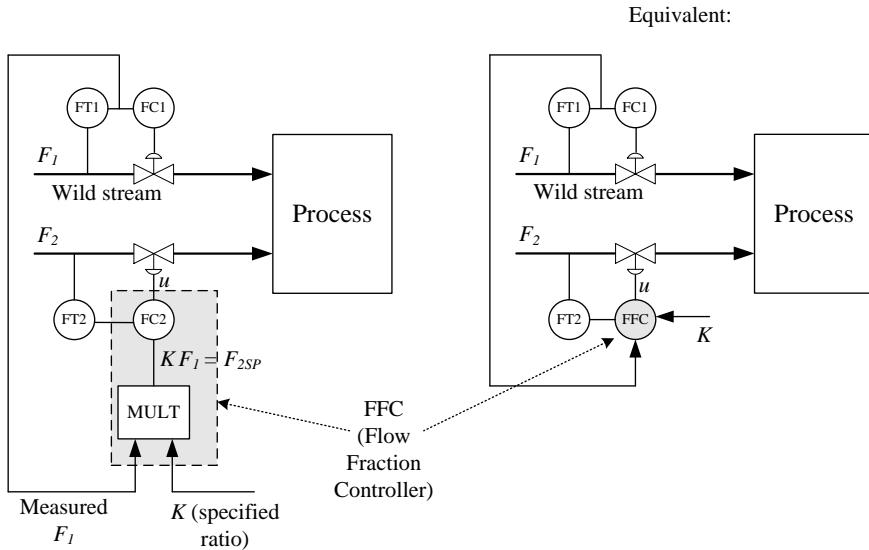


Figure 18.9: Ratio control

The purpose of *ratio control* is to control a mass flow, say F_2 , so that the ratio between this flow and another flow, say F_1 , is

$$F_2 = KF_1 \quad (18.3)$$

where K is a specified ratio which may have been calculated as an optimal ratio from a process model. One example is the calculation of the ratio between oil inflow and air inflow to a burner to obtain optimal operating condition for the burner. Another example is the nitric acid factory where ammonia and air must be fed to the reactor in a given ratio.

Figure 18.9 shows the most common structure of ratio control. As shown with the left diagram in the figure, the setpoint of the flow F_2 is calculated as K times the measured value of F_1 , which is denoted the “wild stream” (this name is used, even if this stream is controlled, as shown in Figure 18.9). The diagram to the right in the figure shows a compact but equivalent representation of ratio control with the symbol FFC (Flow Fraction Controller).

18.3 Split-range control

In *split-range control* one controller controls two actuators in different ranges of the control signal span, which here is assumed to be 0 – 100%. See Figure 18.10.

Figure 18.11 shows an example of split-range temperature control of a thermal process. Two valves are controlled – one for cooling and one for heating, as in a reactor.

The temperature controller controls the cold water valve for control signals in the range 0 – 50 %, and it controls the hot water valve for control signals in the range 50 – 100 %, cf. Figure 18.10.

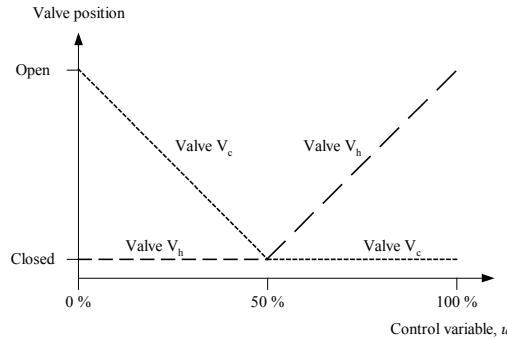


Figure 18.10: Split-range control of two valves

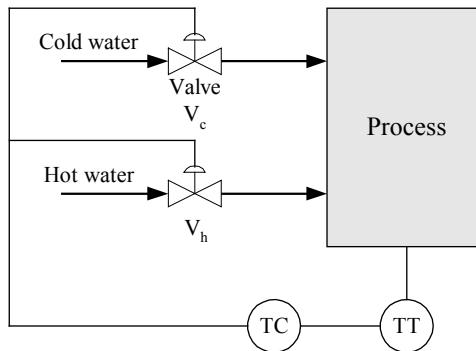


Figure 18.11: Split-range temperature control using two control valves

In Figure 18.10 it is indicated that one of the valves are open while the other is active. However in certain applications one valve can still be open while the other is active, see Figure 18.12.

One application is pressure control of a process: When the pressure drop compensation is small (as when the process load is small), valve V_1 is *active* and valve V_2 is *closed*. And when the pressure drop compensation is large (as when the process load is large), valve V_1 is *open* and valve V_2 is *still active*.

18.4 Averaging level control

We got to know averaging level control already in Ch. 1.2.7. Now, I will show how to tune the level controller. Figure 18.13 shows a buffer tank with a level control system aiming at averaging (or equalizing, or attenuating) inflow variations so that the outflow becomes smoother than the inflow.

Averaging level control is an important part of several process systems. Some important practical examples are:

- The equalization or buffer magazine at the inlet of a waste treatment plant, see Figure 18.14.

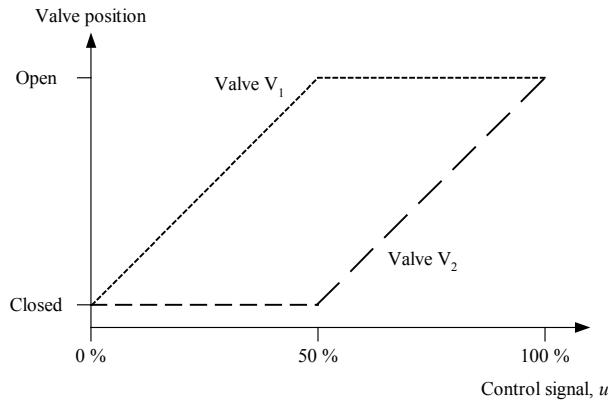


Figure 18.12: In split-range control one valve can be active while another valve is open simultaneously.

- Oil/water separators in the oil industry

Both the equalization magazine and the separator can here be regarded as liquid buffer tanks. In both examples, the level should be compliant to flow variations so that variations in the inflow are damped through the tank, making the outflow considerably smoother than the inflow. Smoother outflow is advantageous for the subsequent processes, e.g. for the biological treatment processes and the oil production. The level controller must be tuned for compliant (or soft, or sluggish) level control so that the volume of the tank can absorb the inlet variations.

How to tune the level controller of the buffer tank? The Ziegler-Nichols controller tuning methods - both their ultimate gain method (or closed loop method), and the process reaction method (or open loop method) are designed for giving as fast control as possible. These methods are therefore useless for tuning of the level controller of a buffer tank since we want sluggish control of such tanks. However, the Skogestad method is suitable here since the speed of the controller can be tuned with the closed loop time constant, T_c , as tuning parameter. Skogestad tuning of the level controller is explained below.

Dynamically, buffer tanks are integrators (or accumulators) since the process output variable, which is the liquid level, is proportional to the integral of (or accumulation of) the control signal, which is here assumed to be the inflow pump control signal. Let us recall the Skogestad formulas for integrating processes with zero time delay (in liquid buffer tanks, we can assume a negligible or zero time delay between the pump flow control signal and the level):

$$K_c = \frac{1}{K_i T_c} \quad (18.4)$$

$$T_i = 2T_c \quad (18.5)$$

Here:

- K_i is the process integrator gain, or the normalized step slope of the process step

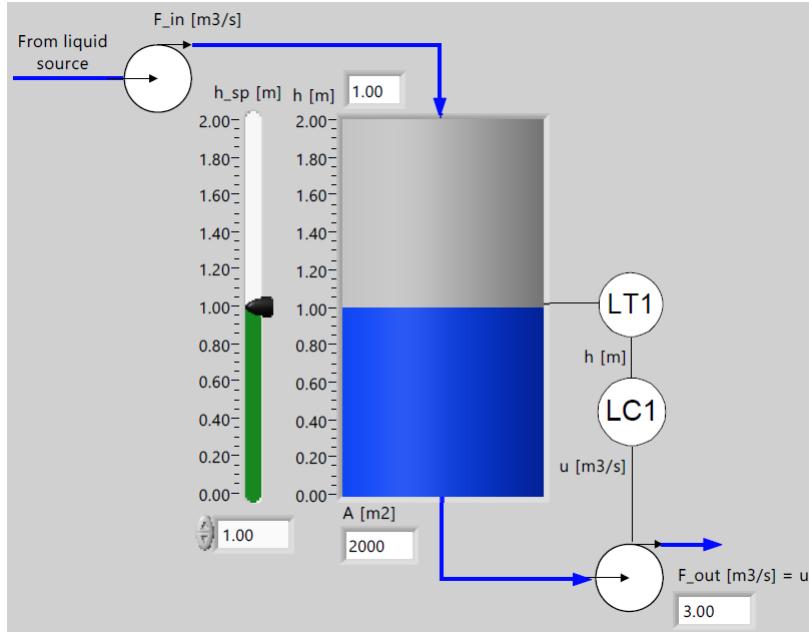


Figure 18.13: Averaging level control of a buffer tank.

response (step in the control signal). As explained below,

$$K_i = -\frac{1}{A} \quad (18.6)$$

where A [m^2] is the surface area of the liquid in the tank.

- T_c [s] is the time constant of the level control system, i.e. of the closed loop system from level set point, h_{sp} , to level measurement, h . A way of tuning T_c is suggested below.

Using (18.6), the PI controller settings of the level controller of a buffer tank is:

$$K_c = \frac{1}{K_i T_c} = -\frac{A}{T_c} \quad (18.7)$$

$$T_i = 2T_c \quad (18.8)$$

Note: The negative sign in K_c means that the controller has *direct action*.

Derivation of (18.6), $K_i = -1/A$

Let us study an example in the form of a simulator of a level controlled buffer tank. Figure 18.15 shows the front panel of the simulator. The controller is in manual mode. The upper plot in Figure 18.15 shows the ramp-shaped response in the level, h [m], due to a step in the pump control signal, u . (The values of the model parameters are representative of wastewater treatment plants for cities of similar size as Oslo.)

Assume that the pump control signal is changed as a step of amplitude U . Figure 18.15 shows such an experiment (applied there with $U = 0.3 \text{ m}^3/\text{s}$). The ramp-shaped level

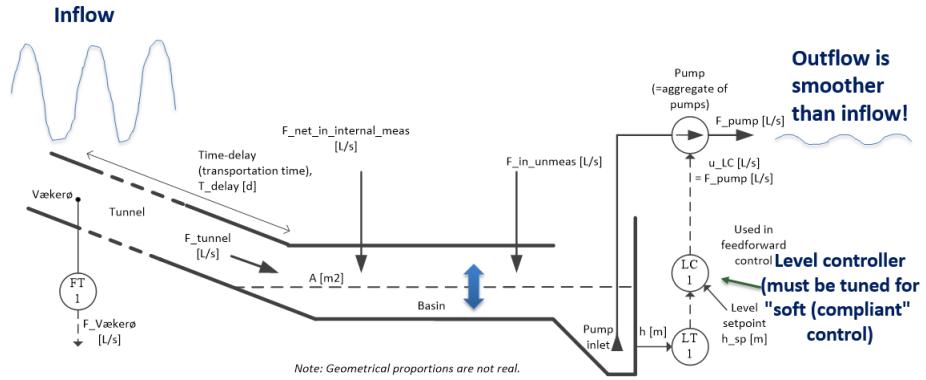


Figure 18.14: Level control of equalization magazine upstreams the VEAS water resource recovery facility (wrrf) or resource recovery facility (wrrf), at Slemmestad, south of Oslo, Norway:

response has slope S [m/s] equal to

$$S = -\frac{U}{A} \quad (18.9)$$

and the normalized slope becomes

$$K_i = \frac{S}{U} = \frac{-U/A}{U} = -\frac{1}{A} \quad (18.10)$$

How to specify a reasonable value of T_c ?

How to specify a reasonable value of T_c [s] needed in the PI settings of K_c and T_i in (18.7) and (18.8), respectively? An interpretation of this time constant is how fast, in the form of 63% rise time, the level approaches to the level set point when the set point is changed as a step. This is illustrated in Figure 18.16 which shows an ideal, principal time constant response of a control system (numbers will, of course, be unique to each pertinent control system).

However, in level control systems of buffer tanks the setpoint is typically constant, so specifying the 63% rise time of the setpoint step response is not particularly meaningful for such control systems. Instead, let's make a more meaningful specification of T_c , related to the response of an inflow change. We start by assuming that the level controller is a P (proportional) controller. It can be shown, from a mathematical model of the level control system, that

$$\Delta h_{\max} = \frac{T_c}{A} \Delta F_{in} \quad (18.11)$$

Here, Δh_{\max} is the maximum steady-state change in level due to an assumed maximum inflow step change of amplitude ΔF_{in} [m^3/s], see the upper plot of Figure 18.17, which shows a simulated response with P-controller (yellow curve) due to an inflow step change at $t = 2000$ s (shown in the bottom plot of the figure). The response with a PI-controller is also shown (blue curve).

With a PI controller that has the same T_c as with the P controller, the following applies, as

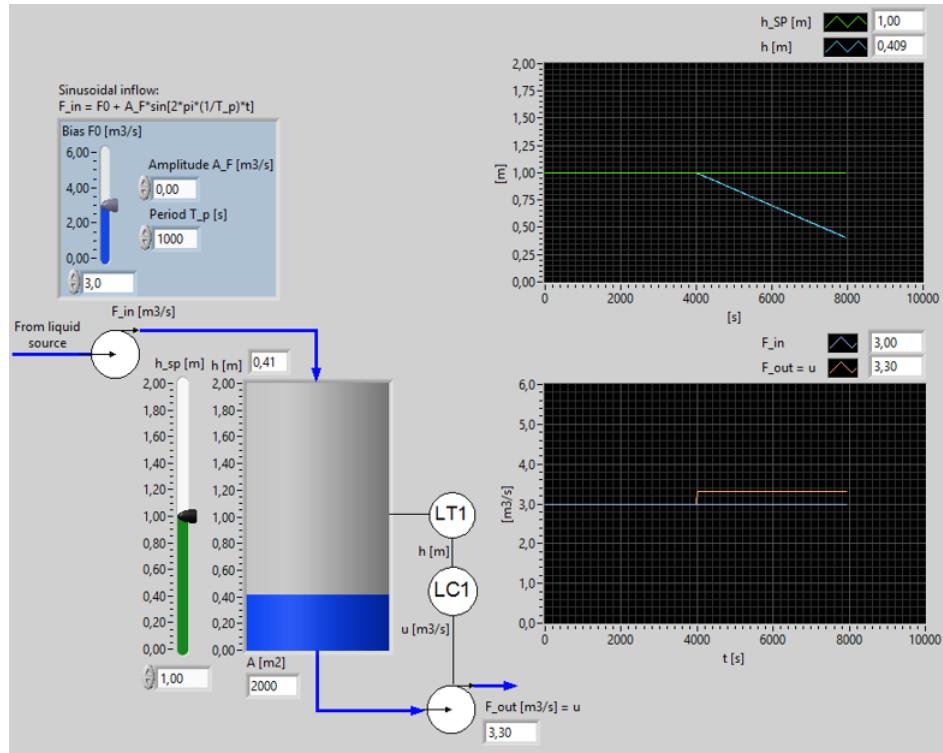


Figure 18.15: The upper plot shows the ramp-shaped response in the level due to a step in the pump control signal shown in the bottom plot. (Controller LC is in manual mode.)

illustrated in Figure 18.17:

$$\Delta h_{\max} \leq \frac{T_c}{A} \Delta F_{in} \quad (18.12)$$

Solving this inequality for T_c gives

$$T_c \geq \frac{A \Delta h_{\max}}{\Delta F_{in}} \quad (18.13)$$

We may use this upper limit of T_c in (18.13) to specify T_c :

$$T_c = \frac{A \Delta h_{\max}}{\Delta F_{in}} \quad (18.14)$$

Summing up the PI tuning formulas

- From the allowed Δh_{\max} due to an inflow step change of amplitude ΔF_{in} , calculate T_c as

$$T_c = \frac{A \Delta h_{\max}}{\Delta F_{in}} \quad (18.15)$$

- Then, T_c is used to tune the PI controller as follows:

$$K_c = -\frac{A}{T_c} \quad (18.16)$$

$$T_i = 2T_c \quad (18.17)$$

Note: The negative sign in K_c means that the controller has direct action.

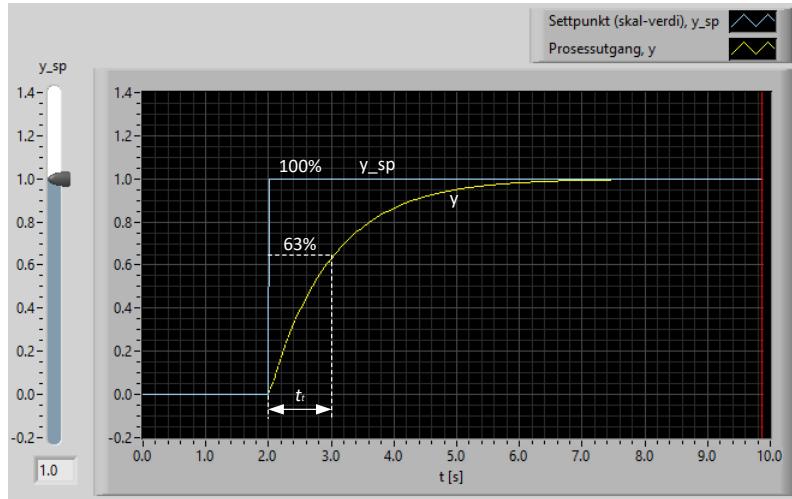


Figure 18.16: Ideal time constant response of a control system

Example 18.4 Tuning of the PI controller for averaging level control

Assumptions:

$$A = 2000 \text{ m}^2$$

$$\Delta F_{in} = 1.0 \text{ m}^3/\text{s}$$

$$\Delta h_{\max} = 0.5 \text{ m}$$

From (18.15):

$$T_c = \frac{A \Delta h_{\max}}{\Delta F_{in}} = \frac{2000 \text{ m}^2 \cdot 0.5 \text{ m}}{1.0 \text{ m}^3/\text{s}} = 1000 \text{ s} \quad (18.18)$$

Then, from (18.16) and (18.17):

$$K_c = -\frac{A}{T_c} = -\frac{2000 \text{ m}^2}{1000 \text{ s}} = -2 \text{ m}^2/\text{s} = -2 (\text{m}^3/\text{s})/\text{s} \quad (18.19)$$

$$T_i = 2T_c = 2 \cdot 1000 \text{ s} = 2000 \text{ s} \quad (18.20)$$

Simulation with step change in inflow

Figure 18.17 shows a simulation with step change in the inflow of $\Delta F_{in} = 1.0 \text{ m}^3/\text{s}$. The blue curves pertain to PI control (the orange is P control). We see that the level, as expected, is reduced somewhat less 0.5 due to the inflow step change. Thus, the specification of the control system is satisfied.

Simulation with sinusoidal change in inflow

Figure 18.18 shows another simulation with the controller in automatic mode. The inflow is now sinusoidal, with amplitude of $1 \text{ m}^3/\text{s}$ and period of 1200 s (this intake variation may be washwater from the treatment processes subsequent to the inlet magazine in a wastewater treatment plant). The bottom plot of Figure 18.18 shows that the level controlled buffer tank (magazine) attenuates the flow variations. Even greater attenuation can be achieved

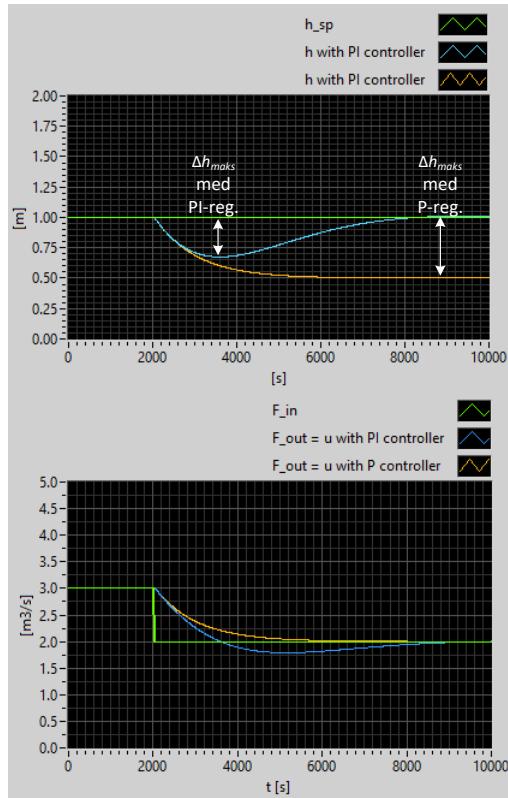


Figure 18.17: Simulations with P controller and with PI controller due to a step change in F_{in}

by increasing the control system time constant T_c , but then with the disadvantage that the level may deviate too far from the setpoint in the case of a large inflow change.

[End of Example 18.4]

18.5 Plantwide control

In the process industry products are created after materials being processed in a number of vessels in series, which are typically unit processes as evaporators, blending or heated tanks, buffer tanks, reactors, distillation columns, separators, absorbers, etc. Typical basic control requirements of such a production line are as follows:

- **The product flow must be controlled (to follow a setpoint).** This is typically done by implementing flow control of one key component or feed.
- **The product quality must be controlled** – more or less directly. Ideally, the product quality is analyzed or measured in real time, and this measurement is used to adjust feeds, supplied heat etc. so that the product quality is controlled to follow its setpoint using feedback. In practice, however, such real time or online measurements are typically not implemented since they rely on time consuming laboratory

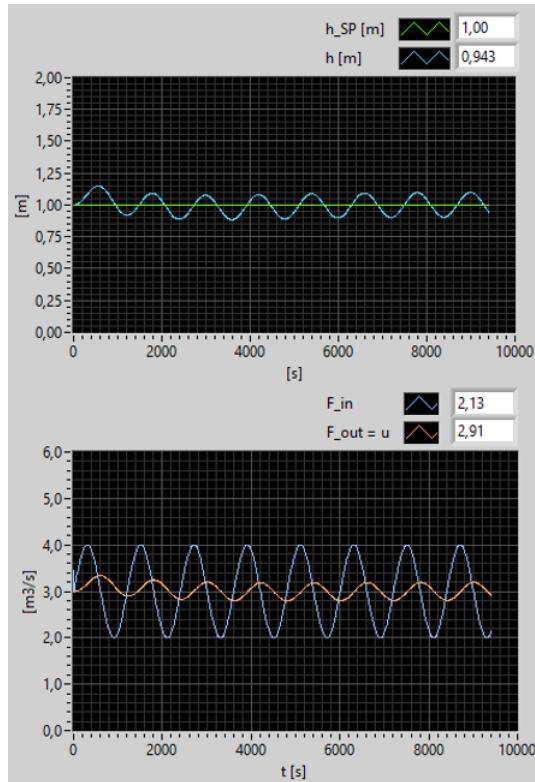


Figure 18.18: Simulation with the PI level controller in automatic mode. The controller is tuned with the Skogestad method. F_{in} has a sinusoidal variation.

analysis. Instead, the product quality is controlled indirectly by controlling the flow of the feeds needed to produce the final product, typically using ratio control.

- **The mass balance of each process vessel (tank, reactor, etc.) must be maintained** – otherwise it may run full or empty. The mass balance is maintained with level control systems for liquids, and with pressure control systems for gases. In vessels containing gas there may still not be any pressure control. It depends on the process if it necessary to control the process.

It is crucial to select the correct reverse or direct action for each of the controllers (otherwise the control loop becomes unstable), cf. Section 12.7.1. The general rules are as follows³:

- Level/gas controllers that are *upstream* relative to the flow control loop (i.e. are placed before the loop) implementing product flow control of the plant (see above) shall be set into *reverse action mode*.
- Level/gas controllers that are *downstream* relative to the flow control loop (i.e. are placed after the loop) implementing product flow control of the plant (see above) shall be set into *direct action mode*.
- **The temperature in some of the process lines or vessels must be controlled.**

³It is assumed that increased control signal to the actuator increases flow through the actuator.

Figure 18.19 shows a basic, principal example where the above control requirements are implemented.

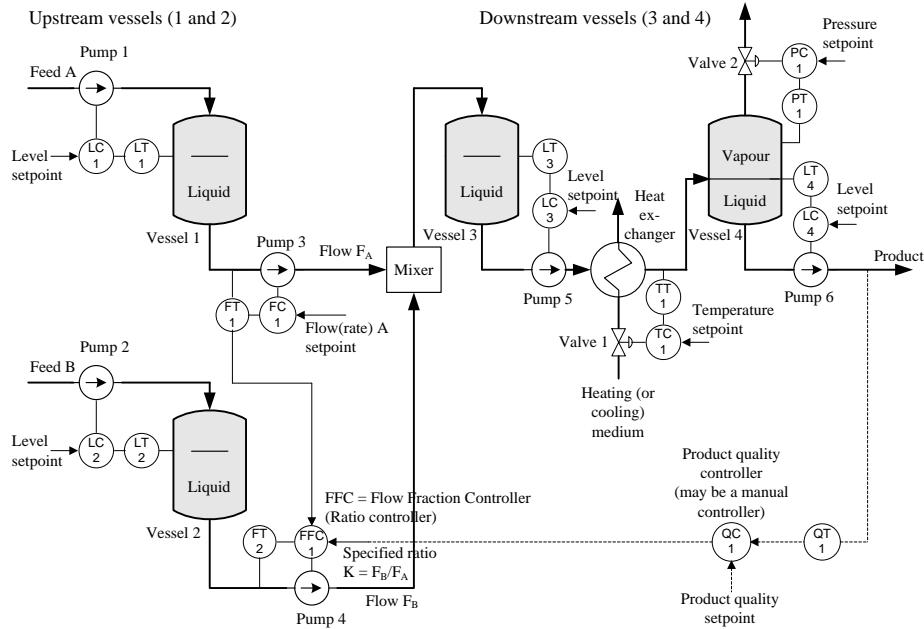


Figure 18.19: Process plant with control loops

Comments to the control system shown in Figure 18.19:

- Two flows, F_A and F_B , are mixed. In general, the mix of two feeds can be fed to a subsequent reactor. It is assumed that F_A contains the key component of the product, and that F_A defines the production rate. Ratio control is used to control F_B so that the ratio between the flows is as specified:

$$K = \frac{F_B}{F_A} \quad (18.21)$$

- The mass balance of upstream vessels 1 and 2 (upstream relative to the point of production flow control in the process line) is controlled by manipulating *inflows* while the mass balance of downstream vessels 3 and 4 tank are controlled by manipulating the *outflows*.
- In vessel 4 both the liquid level and the vapour pressure are controlled.
- The mass balances of liquids are maintained using level control. The mass balance of vapour is maintained using pressure control.
- The product quality may be controlled by adjusting the ratio K between the flows into the mixer. Ideally, this quality controller is operating in automatic mode based on online (continuous) quality measurement of the product. However, in many cases

(as in the nitric acid production plant described in Example 18.5) the product quality is measured off-line using laboratory analysis which gives quality measures only a few times per day. In these cases, the quality controller is actually in the form of an operator adjusting the ratio K .

- The temperature of the inflow to vessel 4 is controlled.
- It is crucial to select correctly between Reverse action mode and Direct action mode in the controllers, cf. Section 12.7.1). Assume that for each of the pumps and the valves that increased control signal increase pump speeds and increase valve openings. Furthermore, assume that heating is needed by the heat exchanger, the control valve manipulates the amount of heating medium. Then, the controller mode of each of the controllers are as follows:
 - LC1: Reverse
 - LC2: Reverse
 - LC3: Direct
 - LC4: Direct
 - PC1: Reverse
 - TC1: Reverse
 - All flow controllers: Reverse

Is this correct?⁴

A specific, industrial example is given in the following.

Example 18.5 Control structures of a nitric acid plant

The example is about production of nitric acid, which is used in e.g. the production of fertilizers.⁵ The description of the process is based on information given by Yara company in Porsgrunn, Norway. However, I have simplified the process somewhat because too many details may obscure the basic principles. (One simplification is that the bleacher, which is used to clean the raw, brown-coloured nitric acid which is the output from the absorber, is not included. Another simplification is that the condenser between compressor K2 and absorber A1 is not included.) The control structure is shown in the figure is based on information given by Yara.

Figure 18.20 shows a Process & Instrumentation Diagram (P & ID) of the (simplified) nitric acid plant.

Description of the nitric acid plant

(Control structures are described below.)

The feeds (input flows) to the production are

⁴No! PC1: Direct.

⁵There are many Nitric Acid production plants around the world. One of them is at the Yara industrial plant area in Porsgrunn, Norway.

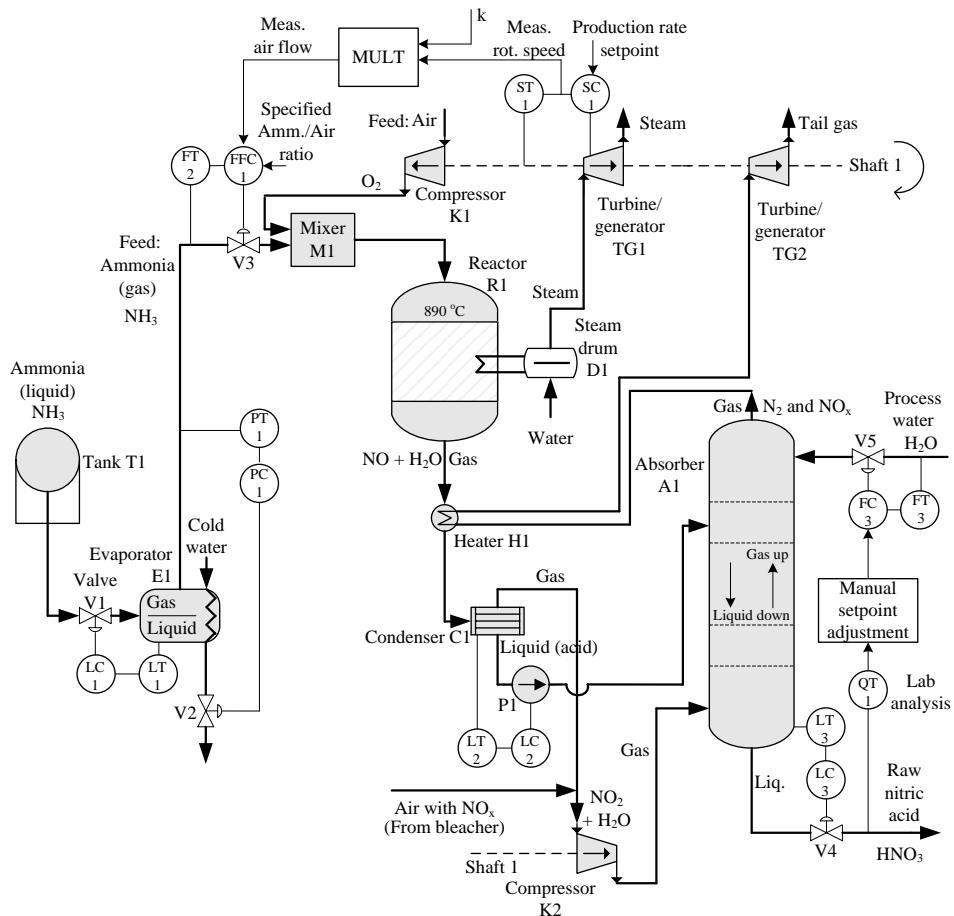


Figure 18.20: Example 18.5: Process and Instrumentation Diagram (P & ID) of simplified nitric acid plant.

- Ammonia
- Air
- Process water

The product (output flow) is

- Nitric acid

Below are descriptions of the components and flows related to these components:

- **Evaporator E1:** Ammonia liquid (NH_3) is evaporated using water (cold!) as heat supply.
- **Compressor K1:** Primary air is compressed.
- **Mixer M1:** Ammonia gas and air are mixed.

- **Reactor R1** (exothermal): Ammonia and oxygen (in the primary air) reacts with a platinum-rhodium catalyst to produce nitrogen monoxide (NO) and water (H_2O) which are passed to condenser C1.
- **Condenser C1**: The water and some of the nitrogen dioxide (gases) are condensed into acid liquid.
- **Condenser C1**: The gaseous output is mixed with secondary air from the bleacher (the bleacher is omitted in this simplified plant), causing oxydation. The product of the oxydation is nitrogen dioxide (NO_2) and water (H_2O).
- **Compressor K2**: Nitrogen dioxide (NO_2) and water (H_2O) is compressed in compressor K2.
- **Condenser C1**: The acid liquid outflow is pumped by pump P1 to a proper tray in the absorber.
- **Absorber A1**: Process water is added at the top. This water and the acid liquid (from condenser C1) flow downwards to meet gaseous nitrogen dioxide (from condenser C2) with a chemical reaction taking place. The product of this reaction is raw nitric acid (HNO_3) liquid collected at the bottom of the absorber, to be passed to bleacher (the bleacher is omitted here), and nitrous gases exiting at the top.
- **Heater H1**: The nitrous gases are heated by the hot product out of reactor R1, before being passed to, and driving, turbine/generator TG2 which produces electrical power that is exported.
- **Steam drum D1**: The reaction taking place in reactor R1 is exothermal. Some of the produced heat is used to heat water, and the resulting high-pressure steam drives turbine/generator TG1 which produces electrical power that is exported.
- **Shaft 1**: The compressors and the turbines/generators have one common shaft. The turbines transfers chemical energy to motional (kinetic) energy. The generators produce electrical power that is exported. The rotational speed of the common shaft determines the production rate of the plant.

Description of the control structure of the plant

- **Production rate** is controlled with rotational speed control of the common turbine/generator shaft. The mass flow of the air, which is a feed to the process, is proportional to the speed (indicated with the MULT block with a gain k in the upper part of the diagram). The stoichiometric balances are maintained with a ratio of 10 % between the mass flow of ammonia to mass flow of air. Therefore, the ammonia gas flow (feed) is controlled with ratio control based on the rotational speed.

Also the *process water flow* into the absorber is a feed to the process. This flow is controlled with flow control with the flow setpoint adjusted manually according to a laboratory analysis of the nitric acid outflow (from the absorber) accomplished twice a day. The QT (Quality Transmitter) represents this analysis.

- **Mass balances** of the various vessels are maintained as follows:

- *Evaporator E1*: Ammonia liquid level is controlled using inlet valve V1 as actuator. Since this level control loop is upstream relative to the production flow control loop, the level controller LC1 shall be set in reverse action mode.
 - *Evaporator E1*: Ammonia gas pressure is controlled using water valve V2 as actuator.
 - *Reactor R1*: The contents of the reactor is gaseous only. There is however no control of gas pressure.
 - *Condenser C1*: Acid liquid level is controlled using pump P1 as actuator. Since this level control loop is downstream relative to the production flow control loop, the level controller LC2 shall be set into direct action mode.
 - *Condenser C1*: Gas pressure is not controlled.
 - *Absorber A1*: Raw nitric acid liquid level is controlled using valve V4 as actuator. Since this level control loop is downstream relative to the production flow control loop, the level controller LC3 shall be set into direct action mode.
 - *Absorber A1*: Nitrous gas pressure at the “top” is not controlled.
- **Temperature control:** None.

[End of Example 18.5]

18.6 *Problems for Chapter 18*

Problem 18.1 *Cascade control for level control*

When pumps are used as actuators in level control systems, they are typically flow-controlled. What can be the purpose of the flow control of the pumps? In such level control systems, which of the general terms primary loop and secondary loop apply to the flow control loop and the level control loop? What may be a tertiary control loop in such a control system?

Problem 18.2 *Purpose of the ammonia flow control loop?*

In the neutralization section of a fertilizer production plant, intermediate mother liquor flows into and out of a tank. In the tank the pH value of the liquid is controlled by adjusting the inflow of ammonia gas to the tank. The ammonia flow is flow controlled using a control valve.

Draw an instrumentation diagram of this process section. (You can use Q (for quality) as symbol for pH.) What can be the purpose of the ammonia flow control loop?

Problem 18.3 *Three control loops*

Figure 18.21 shows a control valve being used to manipulate the flow of a heating medium (liquid) into a heat exchanger where the temperature is to be controlled. The output of the

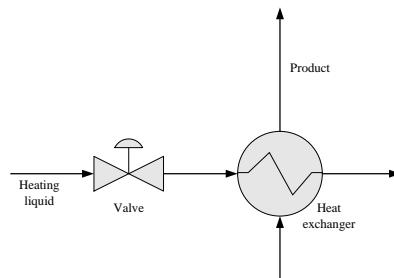


Figure 18.21: Heat exchanger.

temperature controller is flow command signal (flow setpoint) to the valve, and the output of the flow controller is a valve stem position command (position setpoint) to the stem moving mechanism.

Draw an instrumentation diagram of the total control system. You can use symbol G for position of the valve stem. (Hint: There are three control loops.)

Problem 18.4 *Which controllers?*

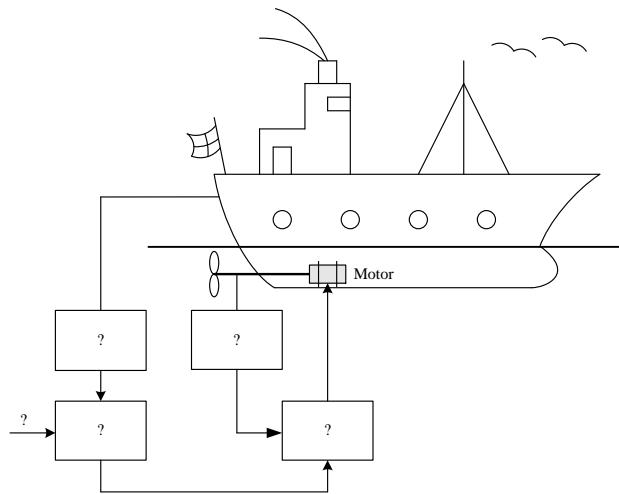


Figure 18.22: Ship

Figure 18.22 shows a ship. The position of the ship is controlled. Assume that it is beneficial for the positional control system that the rotational speed of the propeller is controlled.

Based on the given information, substitute the question marks with proper functions (text). What are the purposes of the control loops?

Problem 18.5 *Control structure for level and flow control*

Figure 18.23 shows a tank with two inlet flows. The liquid level of the tank is to be

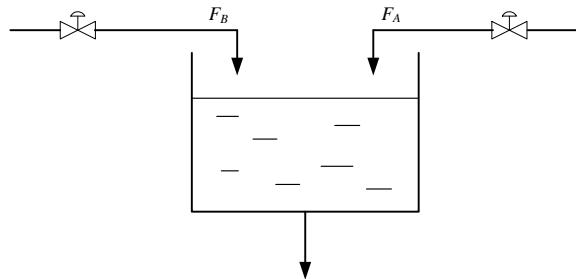


Figure 18.23: Tank

controlled by manipulating (controlling) flow A. It is assumed that flow A is much larger than flow B. The ratio between flow B and A is specified as

$$\frac{F_B}{F_A} = k \quad (18.22)$$

where k is a given ratio. Assume that it is necessary to have local flow control loops around each valve.

Draw a Process & Instrumentation diagram of a control system for this process.

Problem 18.6 *pH control*

Figure 18.24 shows a liquid tank where the pH value of the liquid is to be controlled with split-range control where acid flow and base flow are adjusted. Both the acid flow and the

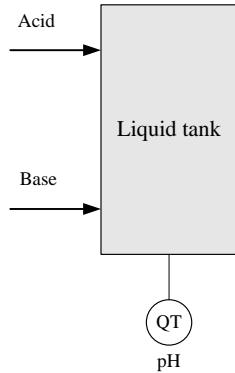


Figure 18.24: Liquid tank

base flow are controlled with (local) flow control loops. Draw an instrumentation diagram of the tank with control system.

Problem 18.7 *P control of tank level*

To appear.

Problem 18.8 *Control structure of process line*

Figure 18.25 shows an uncontrolled process line consisting of a serial connection of liquid tanks. Draw a Piping & Instrumentation Diagram of the control structure of the process

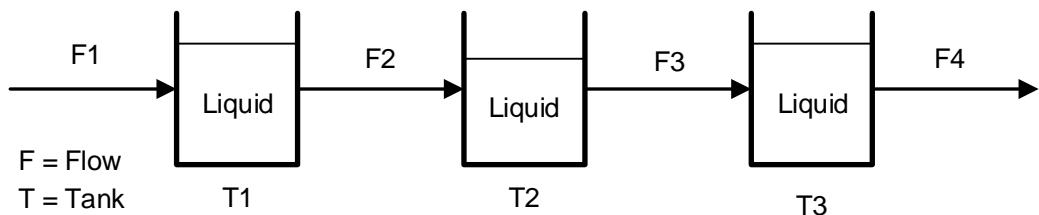


Figure 18.25: Process line.

line based on the following specifications: The production rate (-flow) is controlled by controlling F3 to its setpoint. The mass balance of the liquid in each tank should be maintained. Pumps are used as actuators.

Problem 18.9 *Control structure of process line*

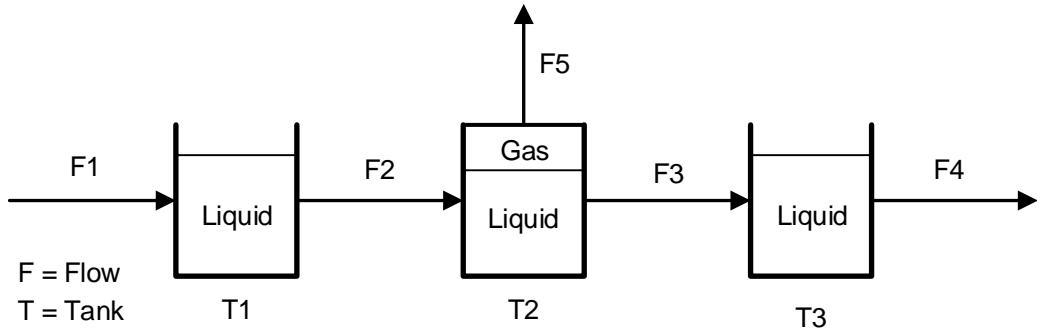


Figure 18.26: Process line.

Figure 18.26 shows an uncontrolled process line consisting of a serial connection of liquid tanks. Draw a Piping & Instrumentation Diagram of the control structure of the process line based on the following specifications: The production rate (-flow) is controlled by controlling F1 to its setpoint. The mass balance of the liquid in each tank should be maintained using pumps as actuators. The gas pressure is controlled to a setpoint using a control valve as actuator.

Problem 18.10 Control structure of level control system

Figure 18.27 shows a wood-chip tank, which is in the beginning of the pulp & paper production line. Spruce, pine and eucalyptus are used as feeds into the tank, via a conveyor

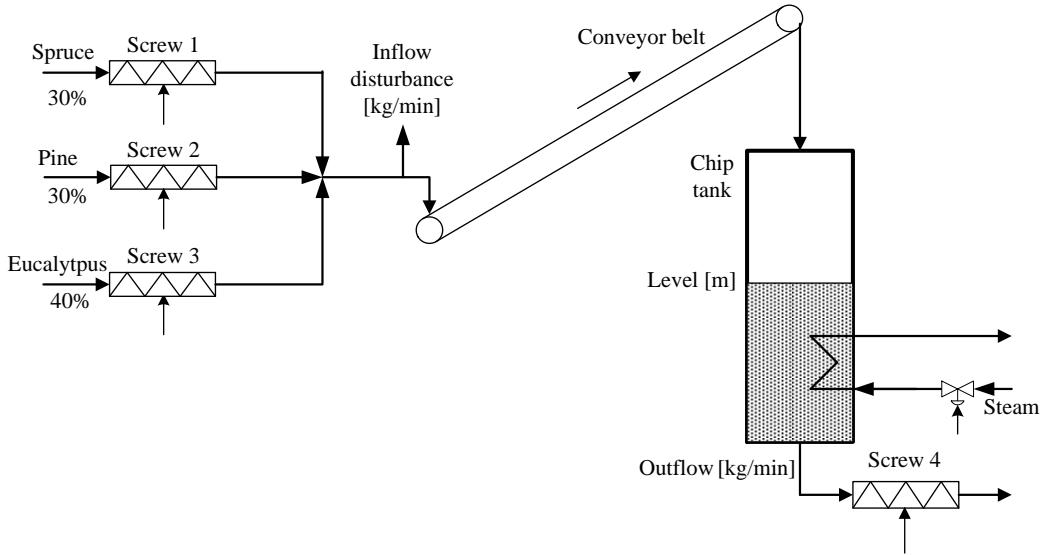


Figure 18.27: Wood-chip tank

belt. The percentages of each of these flows are indicated in Figure 18.27. There is a flow disturbance before the belt which is due to sieving the chip flow to remove large parts of chip.

Draw a P&I (Process & Instrumentation) diagram of a control system for this part of the production line according to the following specifications:

- The production rate is controlled to a setpoint with flow control of Screw 4.
- The level of chip in the tank is controlled to a setpoint by manipulating the total inflow to the conveyor belt.
- The total chip flow into the belt is splitted into percentage flows shown in Figure 18.27. The splitting can be represented with a block with total control signal (100%) as input and three flow value outputs (30%, 30% and 40% respectively). The flows out of the three inflow screws are flow controlled.
- A flow control loop is used to compensate for the flow disturbance due to the sieving. This flow loop is based on the measurement of the flow with a flow sensor at the beginning of the belt.
- The temperature of the chip in the tank is controlled to a setpoint using the steam valve.

Problem 18.11 *Control structure of distillation column*

Figure 18.28 shows an incomplete P&I (Process & Instrumentation) diagram of a controlled distillation column.

If you need it, here is some basic information about distillations columns: A distillation column contains a number of trays from where liquid can pour downwards (to the next tray) and vapour can rise upwards (to the next tray). The purpose of the distillation column is to separate the “light” component and the “heavy” component by exploiting their different boiling points of temperature. Heat is supplied to the boiler at the bottom of the column. Vapour leaving the column is condensed in the condenser. The liquid leaving the condenser is accumulated or stored in the accumulator. Part of the liquid leaving the accumulator is directed back to the column, and the rest – the distillation product – is directed to e.g. a storage tank. Ideally, the concentration of “heavy” component in the top product is zero, and the concentration of the “light” component in the bottoms product is zero. In principle this can be achieved by one quality control loop for the top product and one quality control loop for the bottoms product, but due to the dynamic properties of distillation columns such “two-point” control is difficult to realize. Therefore, there is typically either quality control of the top product or quality control of the bottoms product.

Make the diagram shown in Figure 18.28 complete by entering letter codes in the instrumentation symbols according to these specifications: The quality of the distillate product is controlled, and there is mass balance control of various parts of the column. (The heating medium supplied to the boiler is manually controlled, so it is not adjusted by an automatic controller.)

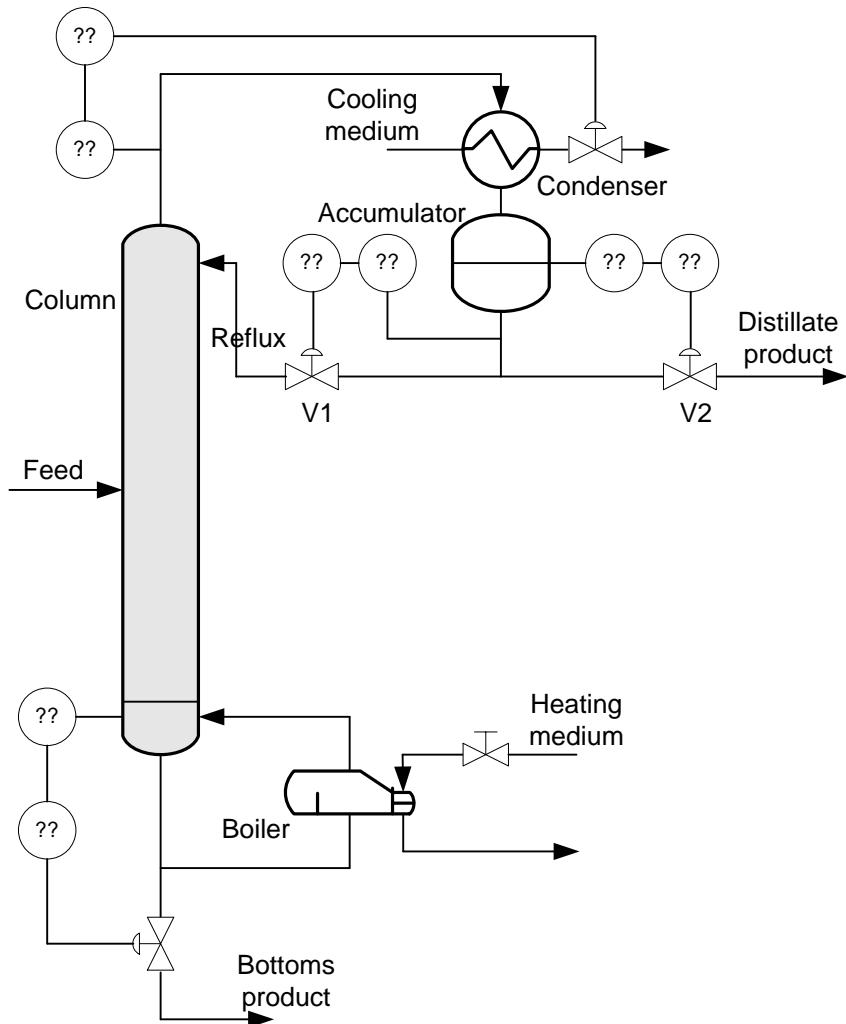


Figure 18.28: Distillation column

18.7 *Solutions to problems for Chapter 18*

Solution to Problem 18.1

The purpose of the flow control is to compensate for disturbances, e.g. pressure variations, which affect the flow, and for nonlinear relation between the pump control signal and the actual flow, so that the actual flow value becomes equal to the flow value requested by the level controller (flow setpoint).

- Primary control loop: Level control loop.
- Secondary loop: Flow control loop.
- Tertiary control loop: Rotational speed of the pump motor.

Solution to Problem 18.2

See Figure 18.29.

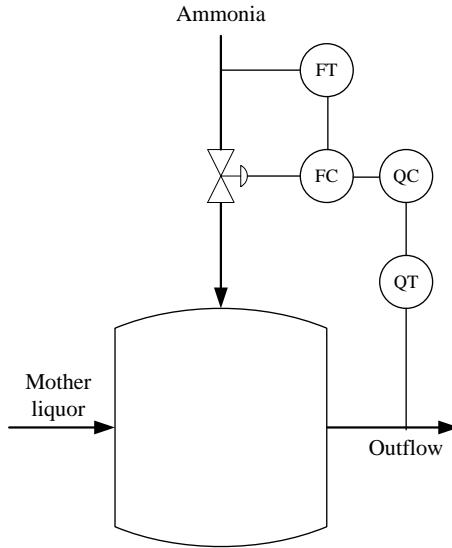


Figure 18.29: pH control system

The purpose of the ammonia flow control loop can be to obtain an ammonia flow that tracks the flow value (flow setpoint) that the pH controller demands. The flow control will compensate for flow variations caused by e.g. pressure variations in the ammonia gas supply.

Solution to Problem 18.3

Figure 18.30 shows the instrumentation diagram.

Solution to Problem 18.4

Figure 18.31 shows the ship with control system.

The purpose of the position control loop is to make the ship track the position reference.

The purpose of the propeller speed control loop is to make the propeller speed track the speed command or reference generated by the positional controller. The speed control loop will compensate for disturbances acting on the propeller so that the propeller speed is more smooth. In addition, this speed control loop may make the tuning of the position controller easier because the position controller will control the speed more directly.

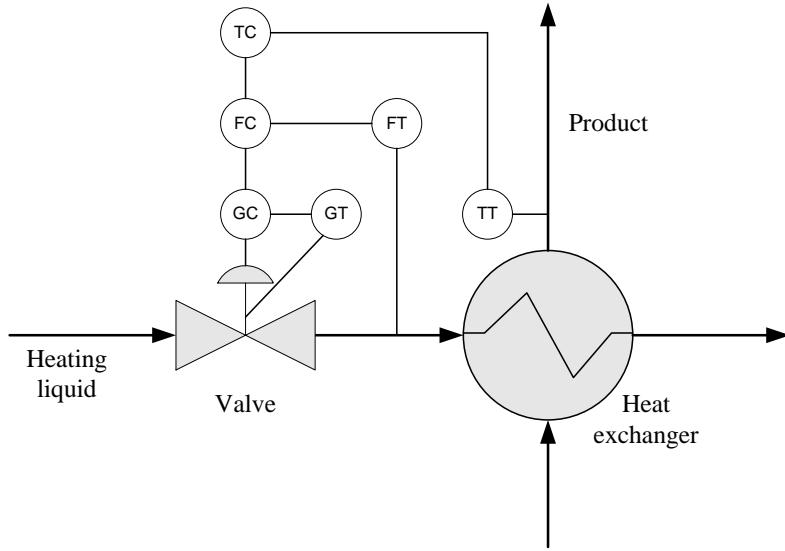


Figure 18.30: Cascade control of heat exchanger

Solution to Problem 18.5

See Figure 18.32.

Solution to Problem 18.6

Figure 18.33 shows the pH control system.

Solution to Problem 18.7

- Substituting F_{out} in the process model by F_{out} from the controller gives the following model of the control system:

$$\rho A \dot{h} = F_{in} - F_{out} = F_{in} - K_c(r_h - h) \quad (18.23)$$

Assuming static conditions the model is

$$0 = F_{in_s} - K_c(r_{h,s} - h_s) \quad (18.24)$$

which gives

$$\underline{\underline{h_s = r_h + \frac{1}{-K_c} F_{in_s}}} \quad (18.25)$$

- From (18.25) we see that the difference between the level and level setpoint is

$$\Delta h_s = e_s = r_{h,s} - h_s = \frac{1}{K_c} F_{in_s} \quad (18.26)$$

Solving for K_c :

$$\underline{\underline{K_c = \frac{1}{\Delta h_s} F_{in_s} = \frac{1}{e_s} F_{in_s}}} \quad (18.27)$$

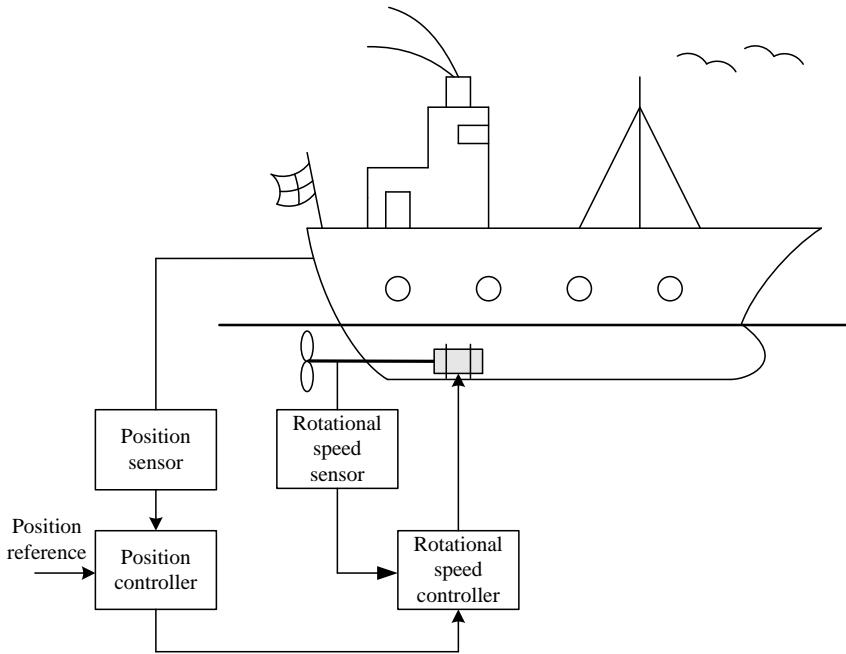


Figure 18.31: Cascade control of ship

Δh_s is reduced if K_p is increased.

3. The drawback about using a P controller is that the static control error $e_s = \Delta h_s$ becomes different from zero.

Solution to Problem 18.8

See Figure 18.34.

Solution to Problem 18.9

See Figure 18.35.

Solution to Problem 18.10

See Figure 18.36.

Solution to Problem 18.11

See Figure 18.37.

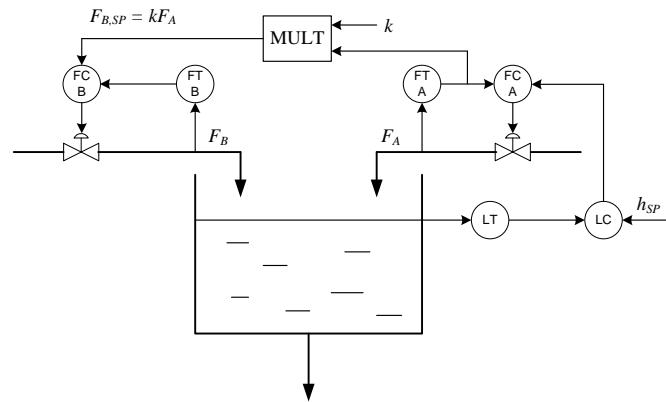


Figure 18.32: Ratio control and cascade control

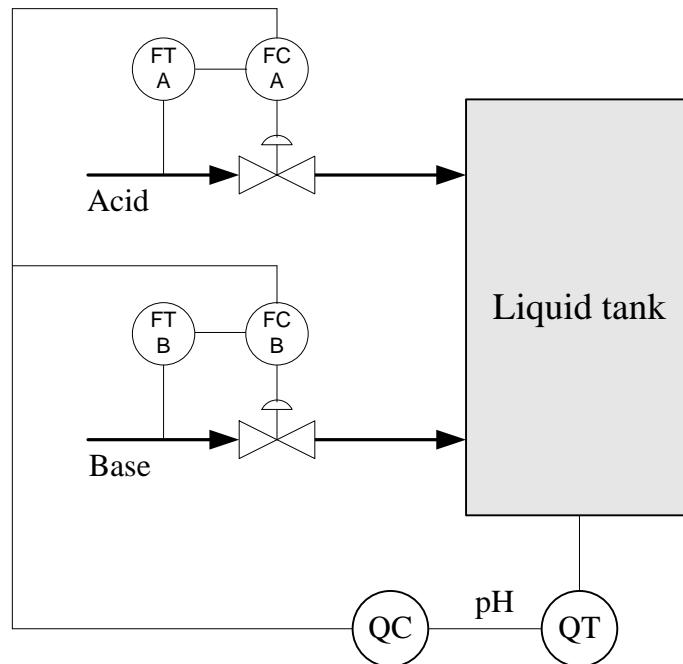


Figure 18.33: Split-range control system

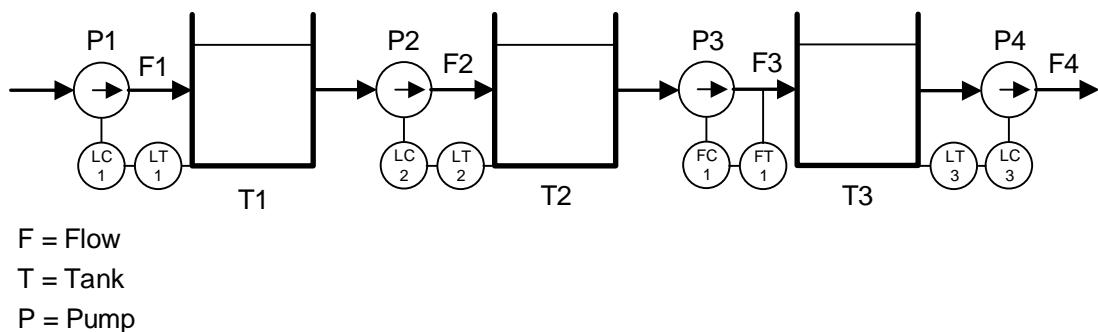


Figure 18.34: Process line with control system.

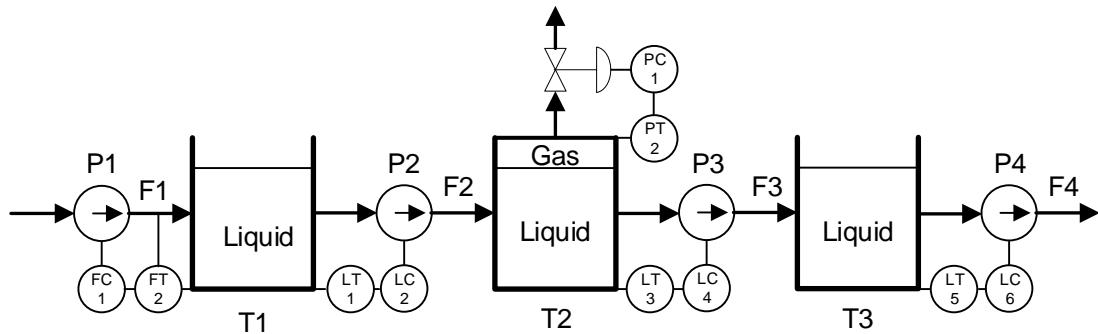


Figure 18.35: Process line with control system.

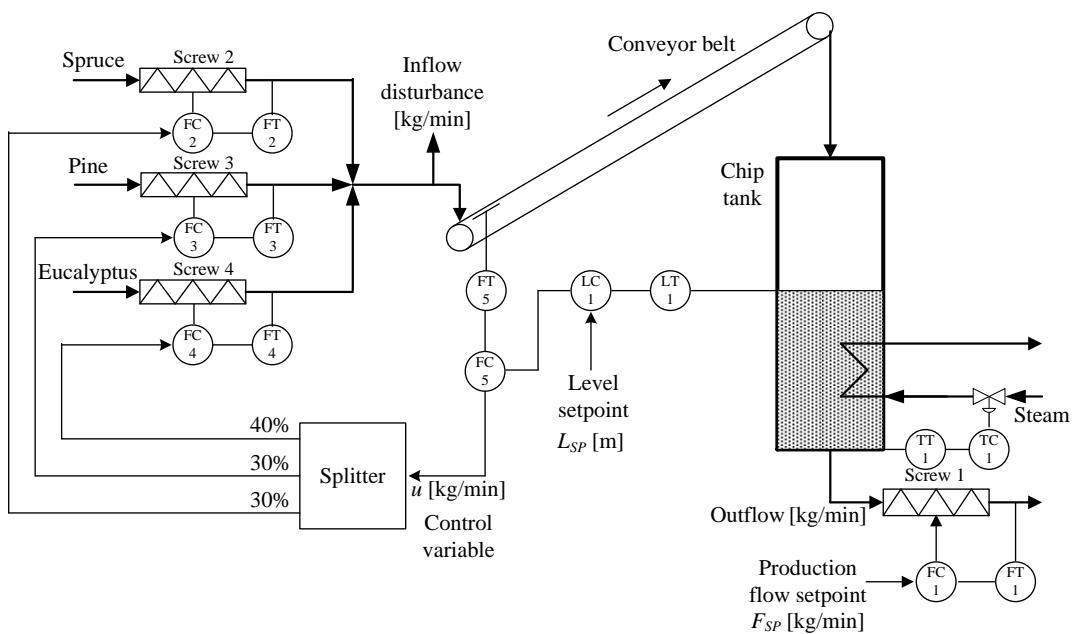


Figure 18.36: Level control system of wood-chip tank

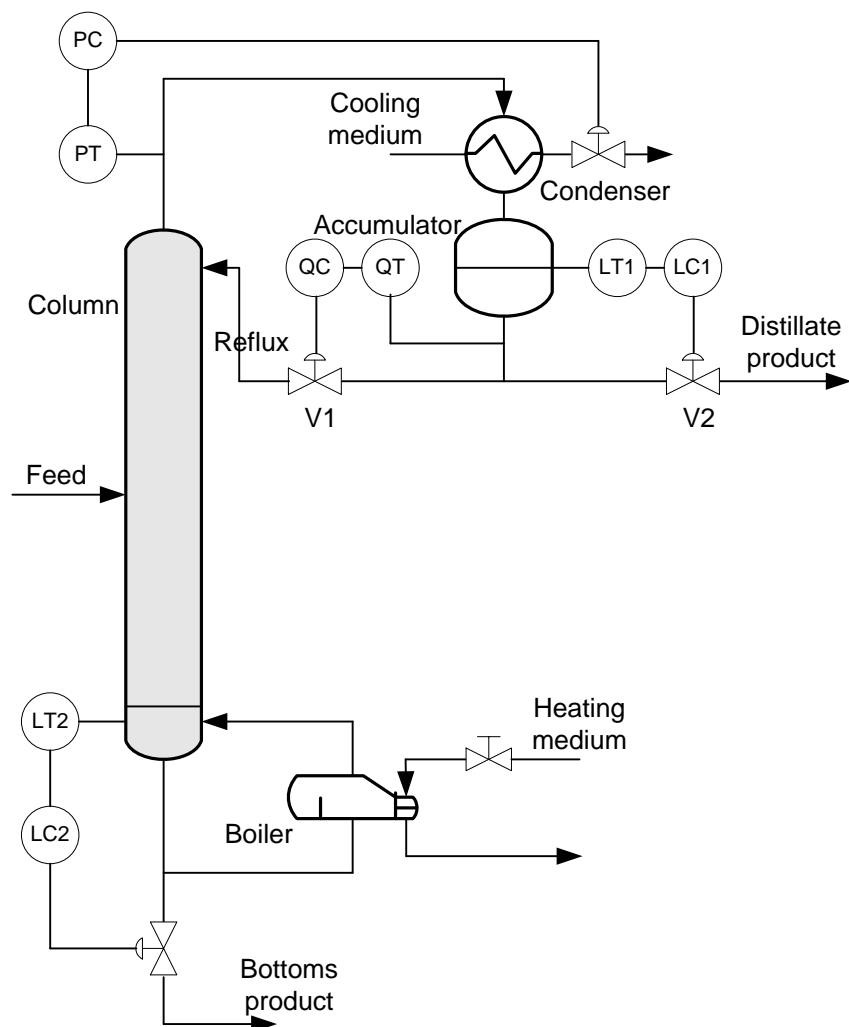


Figure 18.37: Control system of distillation column

Chapter 19

Feedforward control

19.1 Introduction

We know from Chapters 1.2 and 12 that feedback control – or error-driven control – can bring the process output variable to or close to the setpoint. Feedback control is in most cases a sufficiently good control method. But improvements can be made, if required. A problem with feedback is that there is no adjustment of the control variable before the control error is different from zero, since the control variable is adjusted as a function of the control error. This problem does not exist in *feedforward control*.

Feedforward control generates a contribution to the control signal using knowledge about the following three components:

- The setpoint, which is always known.
- The process disturbance(s), assumed measured so that its value is known.
- A mathematical process model (linear or non-linear) which conveys process knowledge in mathematical terms.

Using the above information, the feedforward controller generated (ideally) a perfect control signal making the process output variable, y , equal to its setpoint, r .

However, a perfect feedforward is not possible to realize since there are always some process disturbances which are not measured, and because the process model is never perfect. In other words, perfect feedforward can not be realized fully because of model errors, causing the control error to become different from zero. But by including feedback control, this error may be reduced. In practice, feedforward control is used together with feedback control. Figure 19.1 shows the structure of a control system with both feedforward and feedback control.

In Figure 19.1, the feedforward from disturbance can be interpreted as a technical coupling from the disturbance to the process output variable which is supposed to counteract, or

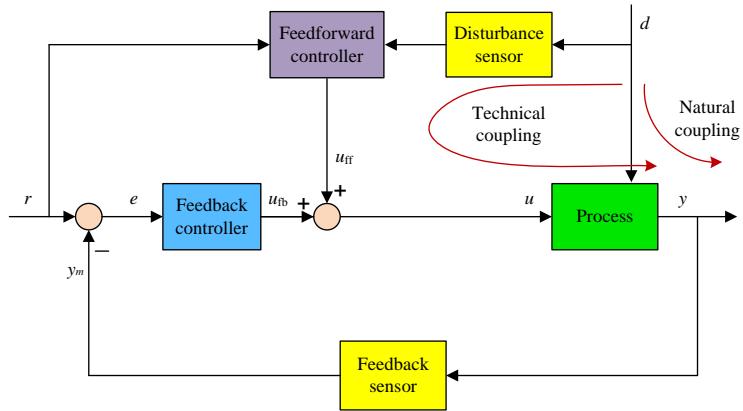


Figure 19.1: Control system with both feedforward and feedback control

cancel out, the natural coupling so that the net effect that the disturbance has on the process output y is zero, which is what we want.

The feedforward control signal is added to the feedback control signal to make the total control signal:

$$u = u_{fb} + u_{ff} \quad (19.1)$$

The feedforward controller, which generated u_{ff} in Figure 19.1, can be developed in several ways:

- From a differential equations process model, cf. Section 19.2.
- From a transfer functions process model. This is not described in this book, but the procedure is as for differential equation models, assuming you have found the differential equation(s) that corresponds to the given transfer functions.
- From experimental data, cf. Section 19.3. This method is model-free, and should be regarded as an approximative method, which still may give substantial improvement of control system performance.

Note that using feedforward together with feedback does not influence the stability of the feedback loop because the feedforward does not introduce new dynamics in the loop.

19.2 Designing feedforward control from differential equation models

The feedforward controller can be derived with the following two-step procedure:

1. Substitute the process output variable, y , by its desired value, the setpoint (reference), r .
2. Solve the process model for the control signal, u . Denote this u by u_{ff} , the feedforward control signal.

One practical issue: Typically, the feedforward controller comprise the time-derivative of r . The time-derivative is very sensitive to abrupt changes of r . Two solutions to this problem are:

- r is lowpass filtered before the differentiation. If the setpoint appears in the feedforward controller with its first order time derivative, a time constant filter will probably be ok, see Figure 19.2 where a time constant filter is assumed. If the setpoint appears in the feedforward controller with its second order time derivative, a second order filter should be used. A simple second order filter may be constructed with two time constant filters in series.
- Only smooth changes of r are allowed, e.g. a sigmoid or sinusoidal change between setpoint values.

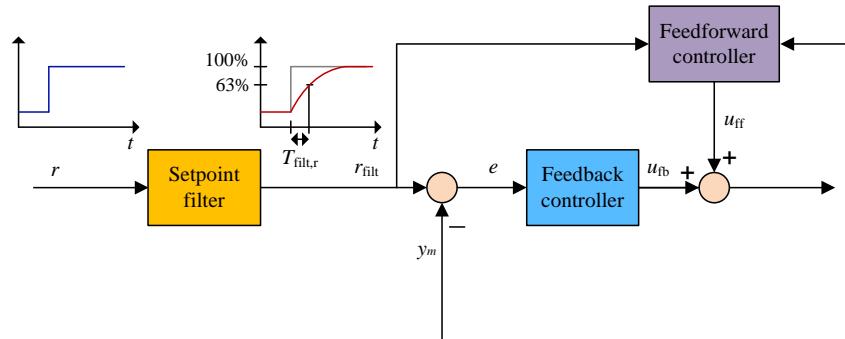


Figure 19.2: A setpoint filter generates a smooth setpoint to be used by the feedforward controller (and the feedback controller).

Example 19.1 Feedforward level control of a liquid tank

Figure 19.3 shows a liquid tank with inflow and outflow. The level h is to be controlled using feedback with PID controller in combination with feedforward control. (The responses are explained later in this example.)

From a mass balance of the liquid in the tank we get the following process model:

$$\rho Ah' = F_{in} - F_{out} \quad (19.2)$$

where h [m] is liquid level, F_{in} [kg/s] is mass inflow, F_{out} [kg/s] is mass outflow, A [m^2] is cross sectional area of the tank, ρ [kg/ m^3] is the liquid density. F_{in} is assumed to be equal in value to the applied control signal u . By using $F_{in} = u$ the model becomes:

$$\rho Ah' = u - F_{out} \quad (19.3)$$

Now, let us derive the feedforward control function from the process model (19.3), following the two-step procedure presented above:

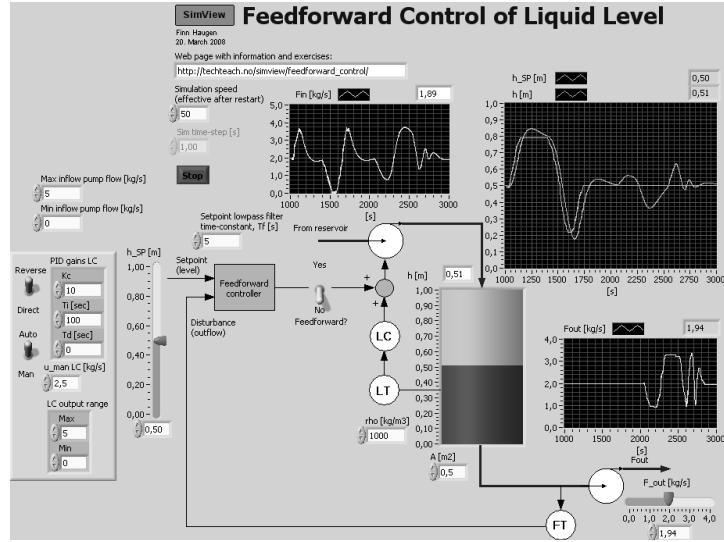


Figure 19.3: Example 19.1: Liquid tank where the level h is controlled with feedback control. Here, feedforward control is not applied.

Firstly, we substitute the level h by its setpoint r_h :

$$\rho A r'_h = u - F_{\text{out}} \quad (19.4)$$

Secondly, we solve (19.4) for the control variable u to get the feedforward control variable u_{ff} :

$$u_{\text{ff}} = \rho A r'_h + F_{\text{out}} \quad (19.5)$$

which is the ideal feedforward controller. $u_{\text{ff},\text{sp}} = \rho A r'_h$ represents feedforward from setpoint, and $u_{\text{ff},\text{d}} = F_{\text{out}}$ represents feedforward from disturbance. We see that calculation of feedforward control signal u_{ff} requires measurement or knowledge of the following four quantities: ρ , A , F_{out} , and r'_{sp} . (Figure 19.3 indicates that flow F_{out} is measured.)

Does the feedforward controller (19.5) make sense?

- The term

$$u_{\text{f,d}} = F_{\text{out}} \quad (19.6)$$

tells that the inflow should be equal to the outflow at any instant of time to cancel the impact of the outflow in the level. Makes sense?¹

- The term

$$u_{\text{f,sp}} = \rho A r'_h \quad (19.7)$$

tells that if the setpoint is changing, the inflow should be increased, or decreased – depending on the sign of r'_h , equal to the specified rate of change of the mass in the tank, which is $\rho A r'_h$.

¹Yes.

In (19.5), r should be lowpass filtered before it is time-differentiated. A first order lowpass filter in the form of a time constant filter may be used:

$$r_{\text{filt}}(s) = \frac{1}{T_{\text{filt},r}s + 1} r(s) \quad (19.8)$$

where $T_{\text{filt},r}$ is the filter time constant which can be tuned by trial-and-error.

In the following, two cases are simulated: Level control *without* and *with feedforward control*. In both cases there is feedback control with PI controller with parameters $K_c = 10$ and $T_i = 100$ s. In the first part of the simulation which is the first 1000 seconds, the level setpoint r is varied while the outflow (disturbance) is constant, while in the last part of the simulation which is the last 1000 seconds, the level setpoint is kept constant while the outflow F_{out} is varied.

- **Without feedforward control** (only feedback control): Figure 19.3 shows the responses in the level h as the setpoint is varied and as the disturbance (outflow) is varies. The level deviates clearly from the setpoint in both situations.
- **With feedforward control** (in addition to feedback control): Figure 19.4 shows the responses in the level with almost the same variations of the setpoint and the disturbance as when only feedback control is used. We see that the level now deviates very little from the setpoint. The control performance is substantially improved.

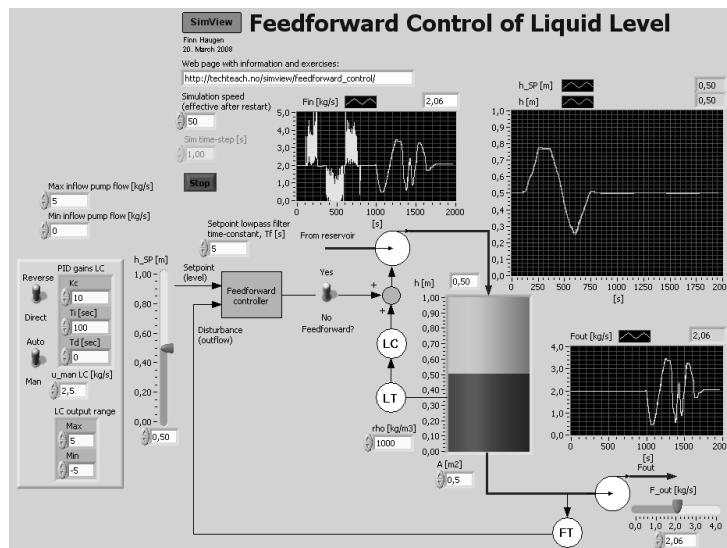


Figure 19.4: Example 19.1: Liquid tank where the level h is controlled with feedback control. Here, feedforward control is applied.

Note: Without feedforward control the control signal range of the PID controller is $[0, 5]$ (in unit of m^3/s). With feedforward the output signal range of the PID controller was set to $[-5, +5]$ so that the contribution, u_{PID} , from the PID controller can be negative. If u_{PID} can not become negative, the total control signal, which is

$$u = u_{\text{PID}} + u_{\text{ff}} \quad (19.9)$$

may not get small enough value to give proper control when the outflow is small. (This was confirmed by a simulation, but the responses are not shown here.)

[End of Example 19.1]

19.3 Designing feedforward control from experimental data

Feedforward control can be designed from experimental data as follows:

- Decide a proper set of N different values of the disturbance d on which the feedforward control will be based, for example $N = 6$ different values of the disturbance.
- For each of these N distinct disturbance values, find (experimentally or by simulation) the value of the control signal u which corresponds to zero steady state control error. This can (should) be done with PI or PID feedback control. (Typically feedback control is used together with feedforward control, so no extra effort is needed to run the feedback control here.)
- The set of N corresponding values of d and u can be represented by a table, cf. Table 19.1, or in a coordinate system, cf. Figure 19.5.
- For any given (measured) value of the disturbance, the feedforward control signal u_f is calculated using interpolation, for example linear interpolation as shown in Figure 19.5. In practice, this linear interpolation can be implemented using a table lookup function.²

Table 19.1: N corresponding values of u and d

u	d
u_1	d_1
u_2	d_2
u_3	d_3
u_4	d_4
u_5	d_5
u_6	d_6

Note: This feedforward design method is based on *steady state* data. Therefore, the feedforward control will not be ideal or perfect. However, it is easy to implement, and it may give substantial better control compared to only feedback control.

Example 19.2 Temperature control with feedforward from flow

²Both MATLAB/SIMULINK and LabVIEW have functions that implement linear interpolation between tabular data.

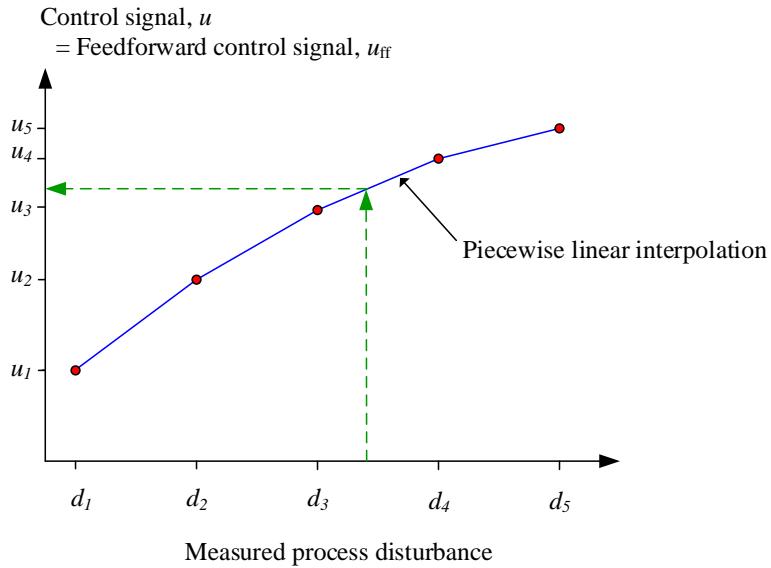


Figure 19.5: Calculation of feedforward control signal from known disturbance value

Figure 41.10 shows a lab process consisting of a heated air tube where the air temperature will be controlled. The control signal adjusts the power to the heater. The air temperature is measured by the primary Pt100 element. The feedback PID control is based on this temperature measurement. (The control system is implemented with a LabVIEW program running on a laptop PC.)

Variations of the air flow act as disturbances to the process. The feedback controller tries to compensate for such variations using the temperature measurement. Can we obtain improved control by also basing the control signal on measured air flow, which is here available as the fan speed indication? First, ordinary PID control without feedforward is tried. The fan speed was changed from minimum to maximum, and then back again. The temperature setpoint was 40 %. Figure 19.6 shows the fan speed and the response in the temperature (which is represented in % with the range [0–100%] corresponding to [20–70°C]).

The maximum control error was 1.0 %.

Will there be any improvement by using feedforward control from the fan speed (air flow)? A number of corresponding values of fan speed and control signal was found experimentally. The feedforward control signal, u_f , was calculated by linear interpolation with **Interpolate 1D Array** function in LabVIEW, and was added to the PID control signal to make up the total control signal: $u = u_{PID} + u_f$. Figure 19.7 shows the fan speed and the response in the temperature. Also, the set of 6 corresponding values of control signal and fan speed, on which the feedforward control signal is based, is shown.

In this case the maximum control error was 0.27, which is a large improvement compared to using no feedforward control!

[End of Example 19.2]

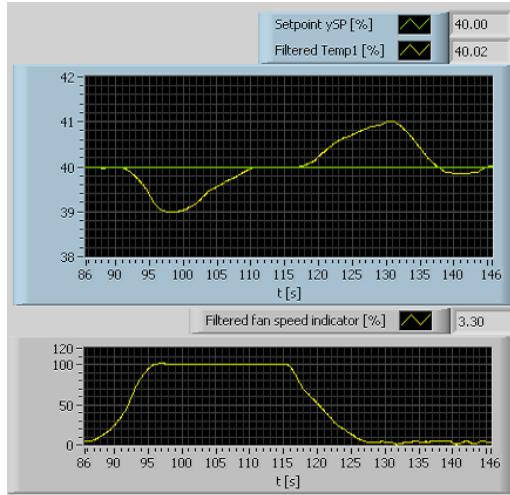


Figure 19.6: Example 19.2: The response in the temperature after a change in the fan speed. Only feedback control (no feedforward) is used.

19.4 Problems for Chapter 19

Problem 19.1 Deriving a feedforward controller

Given the following model of a process to be controlled:

$$Ty' = -y + K_1 u + K_2 d$$

where y is the output variable, u is the control variable, d is the process disturbance variable, and T , K_1 and K_2 are parameters. The setpoint of y is r .

1. Derive a feedforward controller for this process.
2. What quantities must be known for the feedforward controller to be realizable?
3. In general, why is it typically necessary in practice to combine feedforward control with feedback control?

Problem 19.2 Feedforward controller for ship

A mathematical model of a ship is presented in Ch. 41.2. The model is shown below, for convenience:

In mathematical terms:

$$my'' = F_p + D_h (u_c - y') |u_c - y'| + D_w (V_w - y') |V_w - y'| \quad (19.10)$$

F_p is the control variable. Assume that the positional reference is y_r [m].

Design a feedforward controller for the ship. What information is needed to implement the feedforward controller? Is it realistic to get this information?

Figure 1.24 shows a simulation with the feedforward controller, and feedback control with a PID controller.

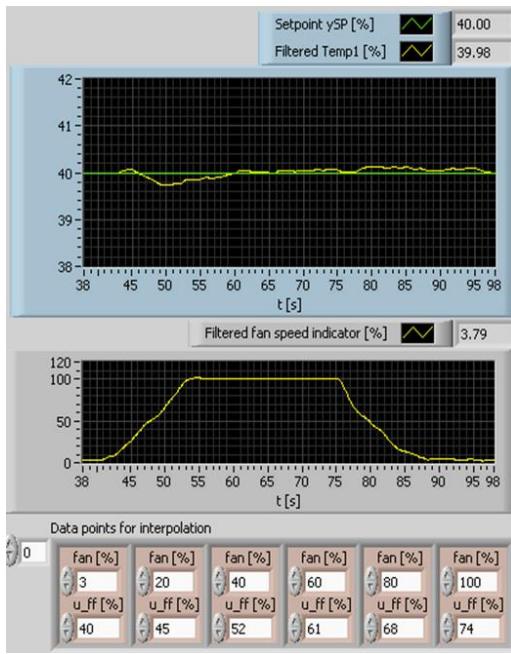


Figure 19.7: Example 19.2: The response in the temperature after a change in the fan speed. Feedforward from fan speed (air flow) is used together with feedback control. u_{ff} is the feedforward control signal, u_f .

Problem 19.3 Feedforward controller for heated tank

Figure 19.8 shows a heated liquid tank where the temperature T shall be controlled using feedback with PID controller in combination with feedforward control.

We assume the following process model, which is based on energy balance:

$$c\rho VT'(t) = \underbrace{K_h u(t)}_P + cw [T_{in}(t) - T(t)] + U [T_e(t) - T(t)] \quad (19.11)$$

where T [K] is the temperature of the liquid in the tank, T_{in} [K] is the inlet temperature, T_e [K] is environmental temperature, c [J/(kg K)] is specific heat capacity, w [kg/s] is mass flow (same in as out), V [m^3] is the liquid volume, ρ [kg/m^3] is the liquid density, U [(J/s)/K] is the total heat transfer coefficient, $P = K_h u$ [J/min] is supplied power via heating element where K_h is a parameter (gain) and u [%] is the control signal applied to the heating element. $c\rho VT$ is the (temperature dependent) energy of the liquid in the tank. We can consider T_{in} and T_e as disturbances, but the derivation of the feedforward function F_f is not dependent of such a classification.

Derive the feedforward function from the process model (19.11). The temperature setpoint is r_T [K]. Which parameters and variables must have known values to implement the feedforward control?

Problem 19.4 Interpolation in feedforward controller

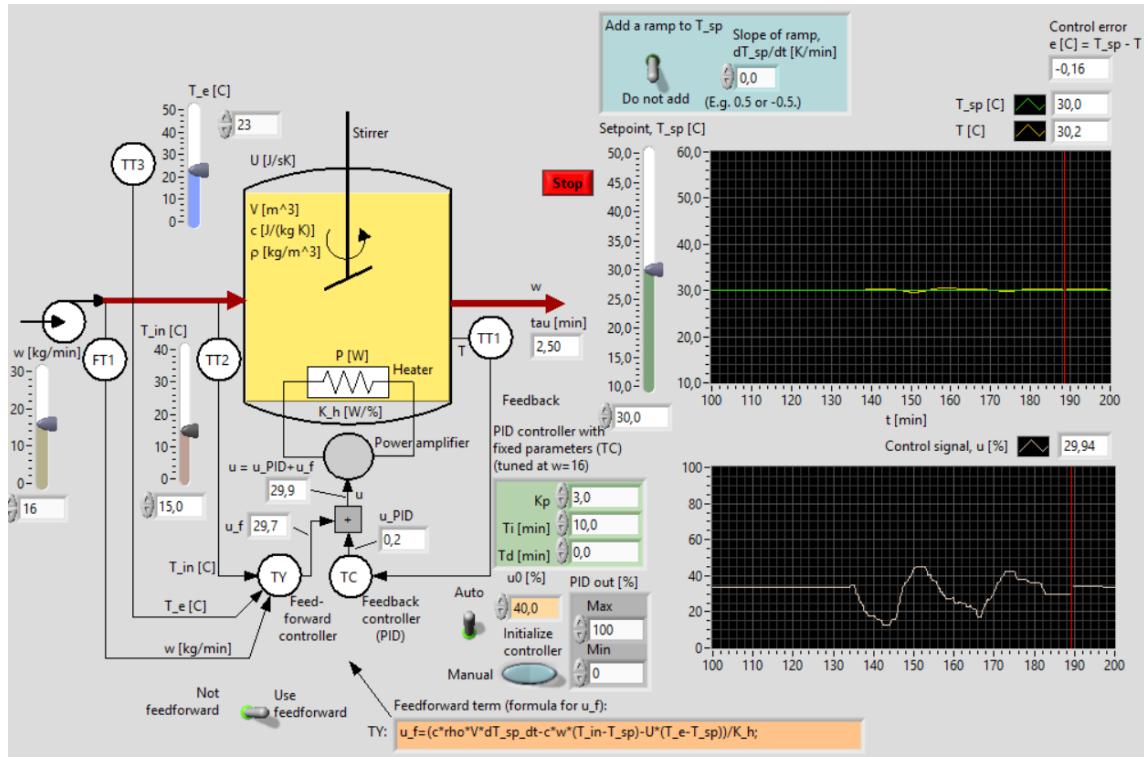


Figure 19.8: Liquid tank with temperature control system

Assume that you will implement the feedforward controller as a set of linear functions (lines) between the data points (fan, u_{ff}). The linear functions are on the form

$$u_{ff} = a \cdot \text{fan} + b \quad (19.12)$$

Let us select the points (40,52) and (60,61). Calculate a and b of the line between these two points.

19.5 *Solutions to problems for Chapter 19*

Solution to Problem 19.1

1. Solving the model for the control signal, u , and substituting y with y_{sp} gives the feedforward controller:

$$u_{ff} = [Tr' + r - K_2 d] / K_1 \quad (19.13)$$

2. All quantities on the right side of (19.13) must be known.
3. The feedforward controller is based on a mathematical model of the process, and on the measurements (or otherwise known values) of the disturbances which appears in the model. In practice, there will always be errors in the model (the model is not perfect) and inaccuracies in the measurements. Therefore, the practical feedforward controller will not calculate the perfect control signal, causing the control error to become different from zero. The feedback controller implements error-driven control, and will therefore reduce this error. In particular, assuming that the feedback controller has integral action, the feedback controller will ensure that the steady state control error is zero (despite model and measurement errors).

Solution to Problem 19.2

Firstly, we substitute y by its reference r . Secondly, we solve for the control variable F_p which we denote the feedforward control variable $F_{p,ff}$. The result is:

$$\underline{F_{p,ff} = mr'' - D_h (u_c - r') |u_c - r'| - D_w (V_w - r') |V_w - r'|} \quad (19.14)$$

Requirements of implementing the feedforward controller: y_r must be known – no problem. Parameters m , D_h and D_w must be known. We can assume they are known approximately by the ship designer. Water current u_c may be measured. However, in real DP applications (Dynamic Positioning) u_c is estimated using a state estimator algorithm named Kalman Filter. Wind speed V_w must be known. In real DP applications it is measured with wind sensors. (The sensor for measuring wind direction and speed is placed on the top of the ship.)

Solution to Problem 19.3

We substitute the temperature T by the temperature setpoint r_T (the time argument t is omitted for simplicity, but it should not be omitted if the model contain time delay terms):

$$c\rho V r'_T = K_h u + c w (T_{in} - r_T) + U (T_e - r_T) \quad (19.15)$$

We solve (19.15) for the control variable u to get the feedforward control variable u_f :

$$\underline{\underline{u_f}} = \frac{1}{K_h} [c\rho V r'_T - cw(T_{in} - r_T) - U(T_e - r_T)] \quad (19.16)$$

$$= \underbrace{\frac{1}{K_h} [c\rho V r'_T + cwr_T + Ur_T]}_{\underline{\underline{u_{fSP}}}} + \underbrace{\frac{1}{K_h} [-cwT_{in} - UT_e]}_{\underline{\underline{u_{fd}}}} \quad (19.17)$$

Implementation of feedforward control signal u_f requires measurement or knowledge of the following five quantities: c , ρ , V , h , w , K_h and T_{in} , in addition to the setpoint time-derivative, r'_T .

Solution to Problem 19.4

With the two data points given, we have

$$52 = a \cdot 40 + b \quad (19.18)$$

and

$$61 = a \cdot 60 + b \quad (19.19)$$

From these two equations we get (the mathematics to solve these two equations is not shown here)

$$\underline{\underline{a = 0.45}} \quad (19.20)$$

$$\underline{\underline{b = 34}} \quad (19.21)$$

Chapter 20

Sequential control

Sequential control is used to implement e.g. chemical batch process control and automated mechanical operations.

A sequential control procedure can be represented graphically by

- Sequential function chart (SFC), or
- State diagram.

SFCs have the following elements:

- **Steps** with a number of associated actions to be executed when the step is active. A step is either active or passive. One particular step is defined as the initial step. It can be symbolized with e.g. a double-lined box, while ordinary steps are represented with single-lined boxes.
- **Actions** are control actions made by the control device (typically a PLC or a PC), e.g. opening a valve, setting a PID controller into automatic mode, starting a motor, lighting a lamp on, etc. The action box indicates the changes of control actions from previous steps (control actions which are not changed, does not have to be listed in the action box).
- **Transitions** from one active step to another taking place when the transition condition is satisfied. Transition conditions are in the form of logical expressions – simple or complicated – having value either TRUE or FALSE. Here is one example: T Step1.Step2: Level > 0.9 m.

A transition may take place *unconditionally*, that is, it takes place automatically after the actions of the presently active state have been accomplished. The transition condition of an unconditional transition has permanent value TRUE, and it may be expressed for example as T.Step2.Step3: TRUE.

Figure 20.1 shows how these basic elements (step, action, transition) appear in a Sequential

function chart. The `:=` symbol is the assignment operator. The `==` symbol is the equality check operator.

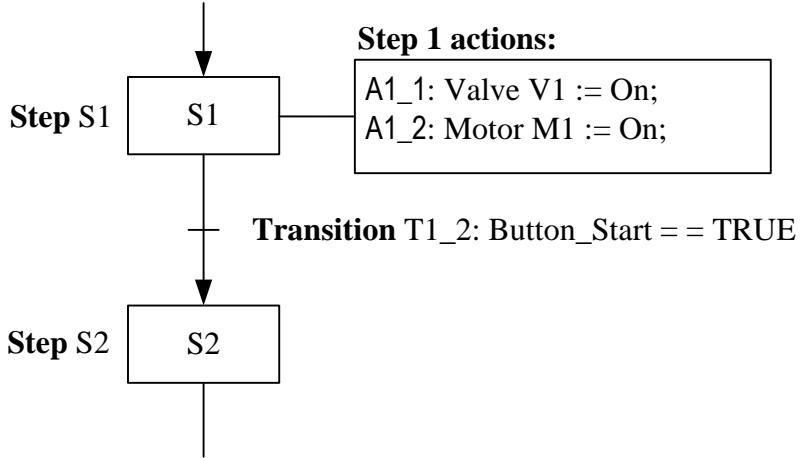


Figure 20.1: Elements of a Sequential Function Chart (SFC): Step, action, and transition

Sequential function charts may also contain *branches* defining simultaneous or alternative parts of the chart to be executed.

State diagrams can be drawn similar to Sequential function charts. In State diagram the term *state* is used instead of *step*. The control system represented with a state diagram is denoted a *state machine*.¹

SFC is standardized in the IEC 61131-3 standard about PLC programming², and is available as a programming tool in most PLC systems, e.g. Mitsubishi PLCs, Simatic PLCs, etc. State diagrams are supported in e.g. the Statchart Module of LabVIEW, the Stateflow Toolbox of Matlab/Simulink, and the Graph7 programming tool of Simatic PLCs.

Example 20.1 Sequential control of a batch process

Figure 20.2 shows a simple batch process which is to be controlled by sequential control.³

The tank is filled with water up to a certain level. The water is then heated using temperature control (with a PID controller) for a specific time defined in timer, and stirred⁴. Finally the heated water is discharged from the tank. Then the batch can be restarted, or ended.

The *control signals* are

¹The state machine is a very useful concept in programming, independently of the programming tool you are using. You can structure a programming task that contains several alternative paths dependent on certain conditions with a state diagram. After the structure has been made, the implementation with programming code is more or less straightforward.

²PLC = Programmable Logic Controller

³You can find a simulator of this system at <http://techteach.no/simview>.

⁴The motor is assumed to ensure homogenous thermal conditions in the water in the tank.

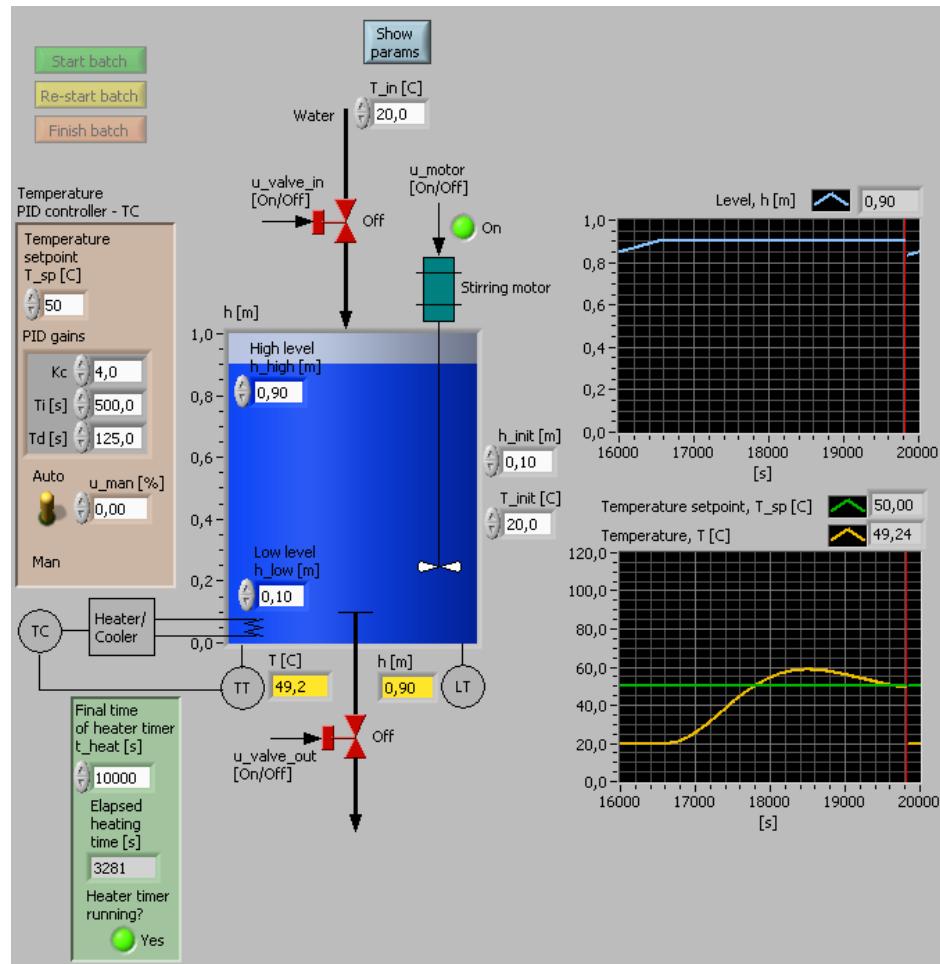


Figure 20.2: A batch process to be controlled by sequential control

- u_{valve_in} (boolean, i.e. having value TRUE or FALSE, or ON or OFF)
- u_{valve_out} (boolean)
- u_{motor} (boolean)
- u_{heat} (continuous having any value between 0% and 100%)

The *measurements* are

- Temperature T of the water in the tank
- Level h of the water in the tank

Figure 20.3 shows a Sequential Function Chart defining the control function.

[End of Example 20.1]

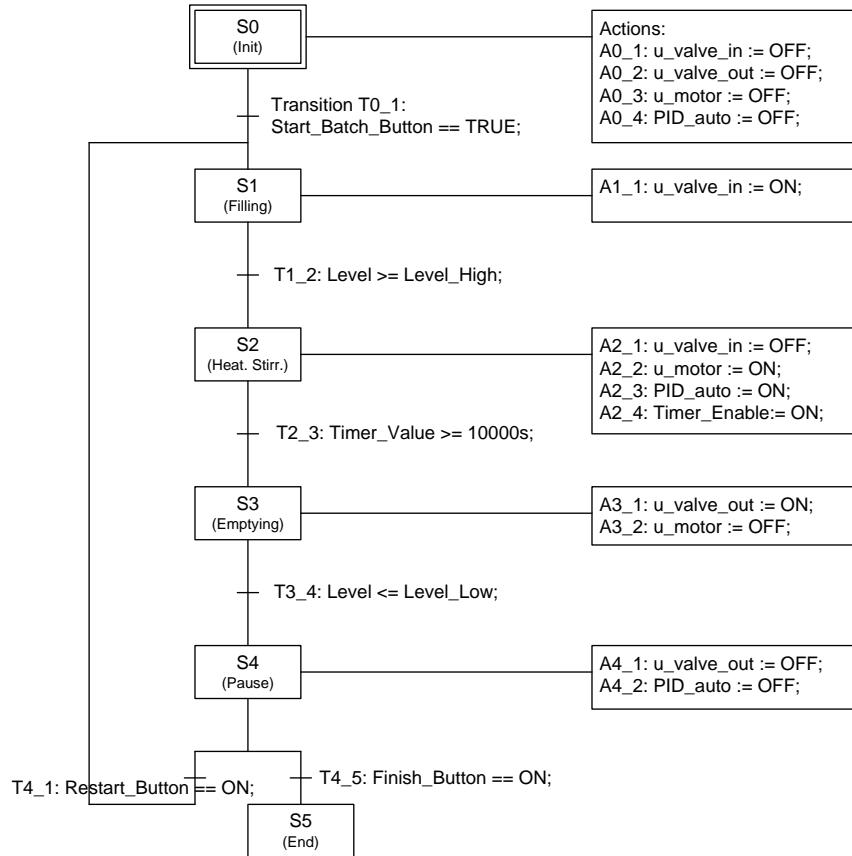


Figure 20.3: Sequential Function Chart (SFC) defining the control function

20.1 *Problems for Chapter 20*

Problem 20.1 *Sequential control of drilling machine*

Figure 20.4 shows a simple drilling machine.

The machine operates as follows: The drilling operation is started with the Start button which sets the control signal Control_start to value On. Just after the button has been pressed, it pops up automatically and Control_start is automatically set back to Off (this reset is not a part of the control task in this problem). When the drilling operation has been started, the clamps are activated by setting the control signal Control_clamp to On, the drill starts rotating with Control_drill set to On, and the cart is moved downwards with Control_cart set to Down until the measured drill position Meas_p becomes p_low. Then, the cart is automatically moved upwards with Control_cart set to Up. When the Meas_p has become p_high, the cart is stopped with Control_cart set to Steady, the clamp is released with Control_clamp set to Off, and the drill is stopped with Control_drill set to Off. Then the drill is idle, waiting until the Start button is again pressed.

Placing the workpiece in the correct position is not a part of this control task.

Draw a Sequential Function Chart (SFC) with steps, actions and transitions solving the control task given above.

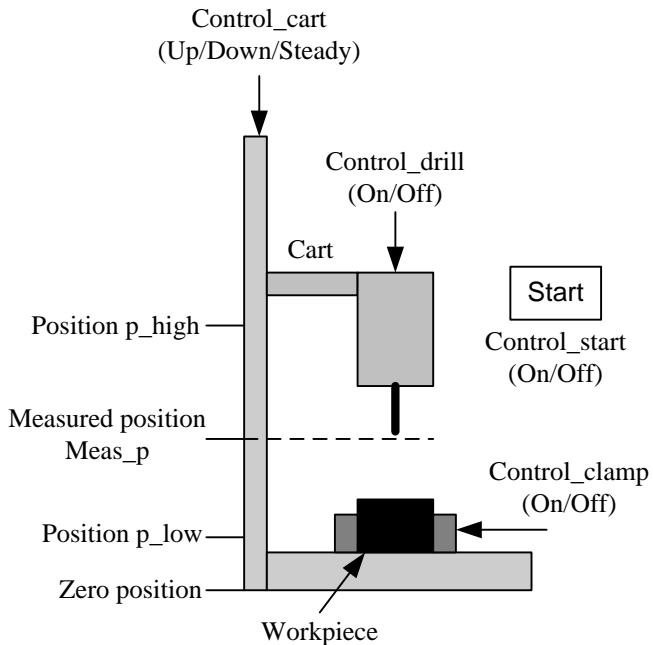


Figure 20.4: Drilling machine.

20.2 *Solutions to problems for Chapter 20*

Solution to Problem 20.1

Figure 20.5 shows the Sequential Function Chart (SFC) solving the given control task.

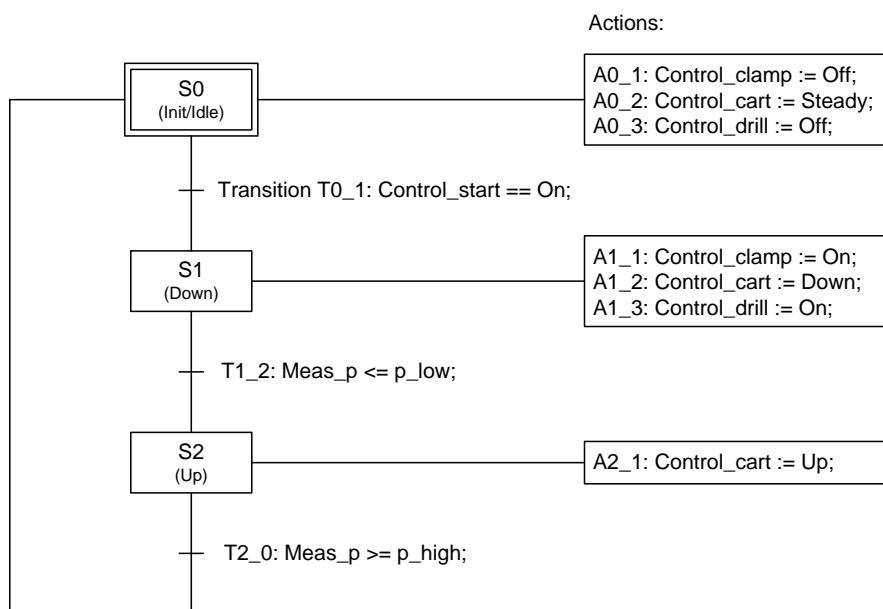


Figure 20.5: Sequential Function Chart

Part IV

**ANALYSIS OF
CONTINUOUS-TIME
FEEDBACK SYSTEMS**

Chapter 21

Stability analysis based on poles

21.1 Introduction

In some situations we want to determine if a dynamic system is stable or unstable. Particularly in the control theory, stability analysis is important, since feedback control systems may become unstable if the controller parameters have been given erroneous values. In this chapter, different stability properties will be defined in terms of the placement of the pole or eigenvalues of the system in the complex plane.

Traditionally Routh's stability criterion has been taught as a tool for stability analysis. Routh's criterion is a method for determining the correspondence between stability properties and values of system (model) parameters, *without* calculating the actual poles or eigenvalues. But, since it is my clear impression and experience that Routh's stability criterion does not play an important role in practical stability analysis of dynamic systems, so I have decided not to include the method in this book.

There exists a special graphical method — the Nyquist's stability criterion — for stability analysis of feedback systems (as control systems) based on the frequency response. This method is presented in Chapter 23.3.

21.2 Stability properties and impulse response

This section defines the different stability properties that a dynamic system can have in terms of impulse response of the system. Then, the corresponding transfer function pole locations in the complex plane are derived. In Section 21.5, and subsequent sections, these results are applied to feedback (control) systems, which are a special case of dynamic systems. The different stability properties can be defined in several ways. I have chosen to use the *impulse response* of the system as the basis for definition of the stability properties. The impulse response is the time-response in output variable of the system due to an impulse on the input. Using the impulse response makes it relatively simple to relate stability properties to the *poles* of the system (this is because the impulse response and the

poles are closely connected), as you will see soon.

Some words about the impulse signal: It is a time-signal which in principle has infinitely short duration and infinite amplitude, but so that the integral of the signal – the integral is equal to the area under the time-function of the signal – is finite. This area is also called the *strength* of the impulse. An impulse of strength one is called a unit impulse, $\delta(t)$. The square pulse in Figure 21.1 approaches an impulse function as Δ goes to zero.

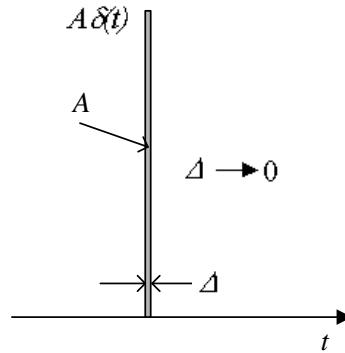


Figure 21.1: The square pulse approaches an impulse function as Δ goes to zero.

Here are the stability definitions: A dynamic system has one of the following stability properties:

- **Asymptotically stable system:** The stationary impulse response, $h(t)$, is zero:

$$\lim_{t \rightarrow \infty} h(t) = 0 \quad (21.1)$$

- **Marginally stable system:** The stationary impulse response is different from zero, but limited:

$$0 < \lim_{t \rightarrow \infty} h(t) < \infty \quad (21.2)$$

- **Unstable system:** The stationary impulse response is unlimited:

$$\lim_{t \rightarrow \infty} h(t) = \infty \quad (21.3)$$

A system is *stable* if it is either asymptotically stable or marginally stable. Figure 21.2 depicts the different stability properties.

In the definition above it is assumed that the impulse in the input u starts from zero and that the impulse response in the output y has zero initial value. In practice these initial values may be different from zero if the system initially is some other operating point than the “zero” operating point.

One problem with the ideal impulse function is it can not be generated fully in practice, but in practice there is hardly ever a need to perform impulse response experiments to determine stability properties. It is more useful as a conceptual definition of stability, cf. the next section.

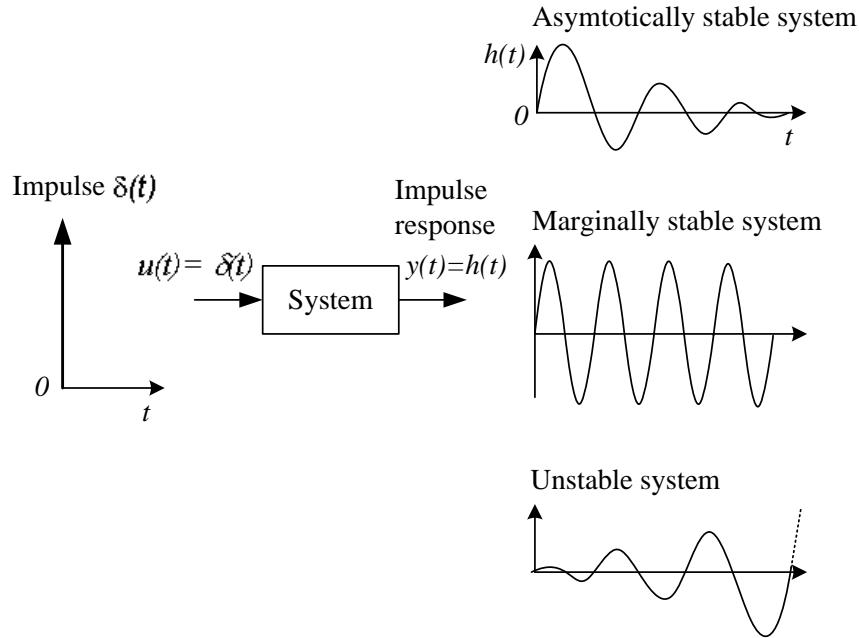


Figure 21.2: Different stability properties

21.3 Stability properties and poles

In many cases it would be quite impractical if the only way to determine the stability property of a system was to do experiments or to run simulations, to obtain the impulse response (or the step response). Fortunately, we can conclude about the stationary part of the impulse response, and hence conclude about the stability property, by just analyzing the mathematical model of the system. This is because the stationary impulse response is a function of the *poles* of the system. The connection between the impulse response and the poles is derived in the following.

Let us assume that the mathematical model of the system is a transfer function, $H(s)$, from input signal u to output signal y :

$$y(s) = H(s)u(s) \quad (21.4)$$

The input u will be a unit impulse, that is, $u(t) = \delta(t)$. It can be shown that the Laplace transform of $\delta(t)$ is 1. Let us denote the impulse response $h(t)$. The Laplace transform of $h(t)$ is

$$h(s) = H(s)\mathcal{L}\{\delta(t)\} = H(s) \cdot 1 = H(s) \quad (21.5)$$

Thus, *the Laplace transform of the impulse response equals the transfer function of the system*. We need the impulse response time-function $h(t)$ since it is the basis for the definitions of the different stability properties.

With results from Laplace transform theory, $h(t)$ can be calculated using the following

formula:

$$h(t) = \sum_i \lim_{s \rightarrow p_i} \frac{1}{(m-1)!} \left\{ \frac{d^{m-1}}{ds^{m-1}} \left[(s-p_i)^m \underbrace{H(s)}_{h(s)} e^{st} \right] \right\} \quad (21.6)$$

$$= \sum_i \lim_{s \rightarrow p_i} \left\{ (s-p_i) \underbrace{H(s)}_{h(s)} e^{st} \right\} \text{ if } m = 1 \quad (21.7)$$

$\{p_i\}$ is the set of poles in $H(s)$, and hence the roots of the denominator of $H(s)$. m is the multiplicity of the poles (so-called simple poles have $m = 1$). The denominator polynomial of $H(s)$ is called the *characteristic polynomial*, $a(s)$. The poles are the roots of $a(s)$. Consequently, the poles are the solutions of the *characteristic equation*

$$a(s) = 0 \quad (21.8)$$

An example: Transfer function

$$H(s) = \frac{1}{(s+2)(s+3)} \quad (21.9)$$

has the poles $p_1 = -2$ and $p_2 = -3$, and the characteristic polynomial is

$$a(s) = (s+2)(s+3) = s^2 + 5s + 6 \quad (21.10)$$

We will now use (21.6) and (21.7) to connect the different stability properties to the pole placement in the complex plane. Let us first assume that the poles of $H(s)$ are simple. Then $m = 1$, and $h(t)$ is given by (21.7). A pole is generally a complex number:

$$p_i = a_i + jb_i \quad (21.11)$$

where a_i is the real part and b_i is the imaginary part of the pole. (21.7) implies that $h(t)$ is equal to the sum of the i partial responses of the total impulse response:

$$h(t) = \sum_i h_i(t) \quad (21.12)$$

where

$$h_i(t) = k_i e^{p_i t} = k_i e^{(a_i + jb_i)t} = k_i e^{a_i t} e^{jb_i t} \quad (21.13)$$

Here k_i is some constant. The term $e^{jb_i t}$ is a complex number on the unity circle and therefore it has absolute value equal to 1.¹ Thus, it is the term $e^{a_i t}$ which determines the steady-state ($t \rightarrow \infty$) absolute value of the partial response $h_i(t)$ according to the following analysis:

- Suppose that the real part, a_i , of the pole is strictly negative, that is, $a_i < 0$, which means that the pole lies in the left half plane. This implies $e^{a_i t} \rightarrow 0$, and therefore $h_i(t) \rightarrow 0$ as $t \rightarrow \infty$.

¹If a pole has a imaginary part b differen from zero, there must be a complex conjugate pole with imaginary part $-b$. However, this fact does not influate the conclusions of the analysis we are performing.

- Suppose that the real part, a_i , of the pole is zero, that is $a_i = 0$, which means that the pole lies on the imaginary axis. This implies $e^{a_i t} = 1$, and therefore $h_i(t)$ goes towards a constant value different from zero as $t \rightarrow \infty$.
- Suppose that the real part, a_i , of the pole is strictly positive, that is, $a_i > 0$, which means that the pole lies in the right half plane. This implies $e^{a_i t} \rightarrow \infty$, and therefore $h_i(t) \rightarrow \infty$ as $t \rightarrow \infty$.

From the above analysis we can conclude as follows for transfer functions having pole multiplicity one: (1) If each of the poles lies in the left half plane, the system is *asymptotically stable*, because then each of the partial impulse response terms, $h_i(t)$, goes towards 0 as $t \rightarrow \infty$. (2) If one pole lies on the imaginary axis while the rest of the poles lies on the left half plane, the system is *marginally stable*, because then one of the $h_i(t)$ -terms goes towards a constant value different from 0 as $t \rightarrow \infty$. (3) If at least one of the poles lies in the right half plane, the system is *unstable*, because then at least one term $h_i(t)$ goes to ∞ as $t \rightarrow \infty$.

Multiple poles: It would have been nice to conclude about stability and pole placement now, but we have to look closer at the case of *multiple* poles of $H(s)$. The impulse response $h(t)$ is given by (21.6). Suppose that that the multiplicity of the pole p_i is $m = 2$. The corresponding partial impulse response becomes

$$h_i(t) = \lim_{s \rightarrow p_i} \left\{ \frac{d}{ds} \left[(s - p_i)^2 H(s) e^{st} \right] \right\} \quad (21.14)$$

Here, the term $\frac{d}{ds}(e^{st})$ is equal to te^{st} , which means that $h_i(t)$ will contain terms as $te^{p_i t}$. By performing the same analysis as for simple poles, we will find the following: (1) $h_i(t) \rightarrow 0$ for a pole with negative real part (since $te^{p_i t}$ goes towards zero because $e^{p_i t}$ decreases faster than t increases). (2) $h_i(t) \rightarrow \infty$ for a pole on the imaginary axis ($te^{p_i t}$ equals t). (3) $h_i(t) \rightarrow \infty$ for a pole having positive real part. We will get the same results if the multiplicity m is greater than two.

Now we can conclude by stating the following correspondence between stability and pole placement:

- **Asymptotically stable system:** Each of the poles of the transfer function lies strictly in the left half plane (has strictly negative real part).
- **Marginally stable system:** One or more poles lies on the imaginary axis (have real part equal to zero), and all these poles are distinct. Besides, no poles lie in the right half plane.
- **Unstable system:** At least one pole lies in the right half plane (has real part greater than zero). Or: There are multiple poles on the imaginary axis.

Figure 21.3 gives a illustration of the relation between stability property and pole placement.

Example 21.1 *Stability property of some simple dynamic systems*

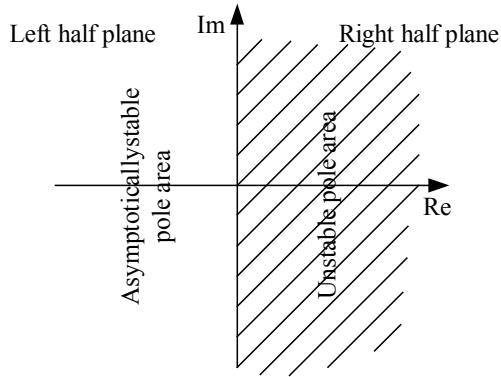


Figure 21.3: The relation between stability property and pole placement

The first order transfer function

$$H_1(s) = \frac{1}{s+1} \quad (21.15)$$

has pole $p = -1$ which lies in the left half plane. Thus, the system is asymptotically stable.

The transfer function

$$H_2(s) = \frac{1}{s} \quad (21.16)$$

(which is the transfer function of an integrator) has pole $p = 0$, which lies on the imaginary axis and has multiplicity one. So, the system is marginally stable.

The transfer function

$$H_3(s) = \frac{1}{s^2} \quad (21.17)$$

have poles $p = 0$, which is on the imaginary axis with multiplicity two. The system is therefore unstable.

The transfer function

$$H_4(s) = \frac{1}{s-1} \quad (21.18)$$

has pole $p = +1$, which lies in the right half plane. The system is therefore unstable.

[End of Example 21.1]

Example 21.2 Stability property of a mass-spring-damper

Figure 21.4 shows a mass-spring-damper-system.

y is position. F is applied force. D is damping constant. K is spring constant. Newton's 2. Law gives the following mathematical model:

$$my''(t) = F(t) - Dy'(t) - Ky(t) \quad (21.19)$$

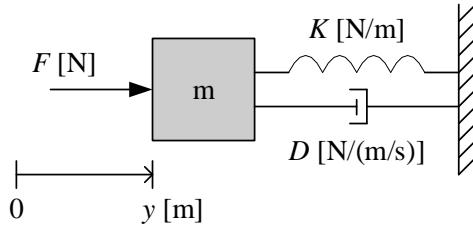


Figure 21.4: Mass-spring-damper

The transfer function from the force F to position y is

$$H(s) = \frac{y(s)}{F(s)} = \frac{1}{ms^2 + Ds + K} \quad (21.20)$$

Assume that $m = 20$ kg, $D = 4$ N/(m/s), and $K = 2$ N/m. What is the stability property of the system? The characteristic polynomial becomes

$$a(s) = ms^2 + Ds + K = 20s^2 + 4s + 2 \quad (21.21)$$

which has roots

$$p_1, p_2 = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 20 \cdot 2}}{2 \cdot 20} = -0.1 \pm j0.3 \quad (21.22)$$

which are the poles of $H(s)$. Both these poles have strictly negative real parts (-0.1). The system is therefore asymptotically stable. Figure 21.5 shows the poles (marked as crosses) in the complex plane. Figure 21.6 shows the impulse response.

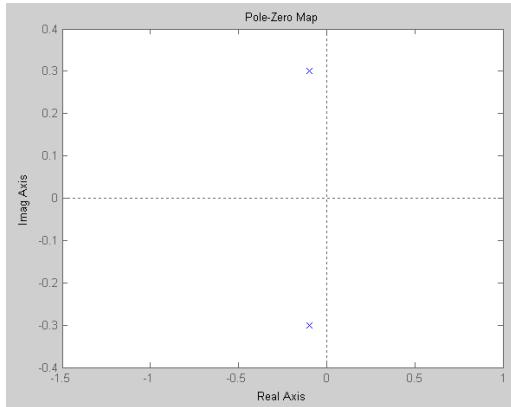


Figure 21.5: The poles of a mass-spring-damper plotted in the complex plane. The poles are $p_{1,2} = -0.1, 1 \pm j0.3$.

Assume now that the damper is removed so that $D = 0$. Then the characteristic polynomial is

$$a(s) = ms^2 + K \quad (21.23)$$

and the poles are

$$p_1, p_2 = \pm j\sqrt{\frac{K}{m}} = \pm j0.32 \quad (21.24)$$

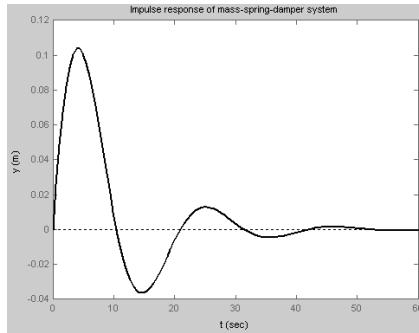


Figure 21.6: The impulse response for a mass-spring-damper with $m = 20 \text{ kg}$, $D = 4 \text{ N}/(\text{m/s})$ and $K_f = 2 \text{ N/m}$. The system is asymptotically stable.

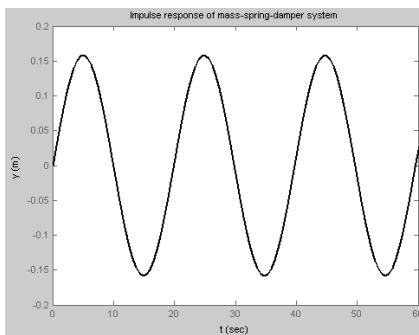


Figure 21.7: The impulse response of the mass-spring-damper with $m = 20 \text{ kg}$, $D = 0 \text{ N}/(\text{m/s})$ og $K_f = 2 \text{ N/m}$. The system is marginally stable.

which lies on the imaginary axis, and they have multiplicity one. The system is then marginally stable. Figure 21.7 shows the impulse response.

[End of Example 21.2]

21.4 Stability analysis of state space models

In Section 21.3, the different stability properties of a transfer function were related to the poles of the transfer function. If the model originally is given as a state space model,

$$x' = Ax + Bu \quad (21.25)$$

$$y = Cx + Du \quad (21.26)$$

we can determine the stability properties of that system by finding the corresponding transfer function from u to y , and calculating the poles of the transfer function. Then, these poles determines the stability of the state space system!

What is this transfer function, then? In Section 9.9, we found that it is

$$H(s) = \frac{y(s)}{u(s)} = C(sI - A)^{-1}B + D \equiv C \frac{\text{adj}(sI - A)}{\det(sI - A)} B + D \quad (21.27)$$

The poles of $H(s)$ are the roots of the denominator:

$$a(s) = \det(sI - A) = 0 \quad (21.28)$$

Thus, we can conclude about the stability property of the state space model from the roots of (21.28).

But (21.28) also defines the *eigenvalues* of the system matrix A of the state space model.² Therefore, the poles of the transfer function are the same as the eigenvalues of A .

Consequently, we can *conclude about the stability of the state space model from the eigenvalues* – you can just substitute “pole” by “eigenvalue” in the criteria for asymptotic stability, marginal stability and instability in Section 21.3.

Note: In some state space models factors of type $(s - p_i)$ in the denominator can be cancelled against factors $(s - z_i)$ in the numerator of the transfer function. Such pole/zero-cancellations implies that some of the poles (and zeros) “disappears” from the transfer function. Consequently, the set of poles will then be just a subset of the set of eigenvalues. Thus, there may exist eigenvalues which are not poles, so that stability analysis based on eigenvalues placement (in the complex plane) may give a different result than stability analysis based on pole placement.

21.5 Stability analysis of feedback systems

Figure 21.8 shows a general transfer function-based block diagram of a feedback control system. The stability of the feedback system can be determined from the poles of the

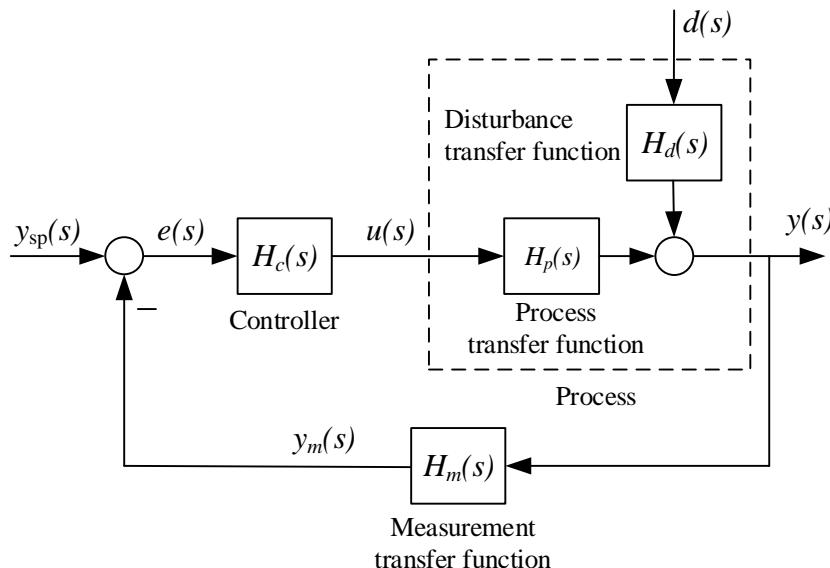


Figure 21.8: Transfer function-based block diagram of a feedback control system.

transfer function from setpoint r to process measurement y_m . (Actually, we can determine

²In mathematics literature it is more common to use the symbol λ instead of s for eigenvalues.

the stability from any input to the control system to any output of the control system – the conclusion about stability will be the same.)

It is convenient to make the block diagram in Figure 21.8 more compact, see Figure 21.9 where $L(s)$ is the *loop transfer function* of the feedback control system. $L(s)$ is the product

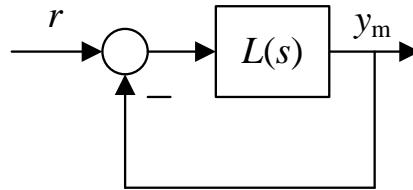


Figure 21.9: Compact block diagram of a control system with setpoint r as input variable and process measurement y_m as output variable.

of the series-connected transfer functions in the loop:

$$L(s) = H_c(s)H_p(s)H_m(s) \quad (21.29)$$

From the block diagram in Figure 21.9, we have:

$$y_m(s) = L(s) [r(s) - y_m(s)]$$

which gives:

$$y_m(s) = \underbrace{\frac{L(s)}{1 + L(s)} r(s)}_{T(s)} = T(s)r(s) \quad (21.30)$$

where:

$$T(s) = \frac{L(s)}{1 + L(s)} \quad (21.31)$$

is the tracking transfer function of the control system.

The stability of the feedback control system is determined by the poles of $T(s)$.

These poles are the roots of the characteristic polynomial of $T(s)$. Let us elaborate $T(s)$ further to get an expression of the characteristic polynomial of $T(s)$: We write $L(s)$ in terms of its numerator and denominator polynomials:

$$L(s) = \frac{n_L(s)}{d_L(s)} \quad (21.32)$$

We can then write $T(s)$ as:

$$T(s) = \frac{L(s)}{1 + L(s)} = \frac{\frac{n_L(s)}{d_L(s)}}{1 + \frac{n_L(s)}{d_L(s)}} = \frac{n_L(s)}{d_L(s) + n_L(s)} \quad (21.33)$$

So, the characteristic polynomial of the tracking transfer function is:

$$a(s) = d_L(s) + n_L(s) \quad (21.34)$$

Thus, the stability of the feedback control system is determined by the roots of characteristic polynomial (21.34).

$K_c = 1$ (asympt. stable)	$K_c = 2$ (marg. stable)	$K_c = 4$ (unstable)
$p_1 = -1.75$	$p_1 = -2$	$p_1 = -2.31$
$p_2 = -0.12 + j0.74$	$p_2 = j$	$p_2 = 0.16 + j1.31$
$p_3 = -0.12 - j0.74$	$p_3 = -j$	$p_3 = 0.16 - j1.31$

Table 21.1: Poles of the tracking transfer function for various values of K_c .

Example 21.3 *Stability analysis of a feedback control system*

Assume the following specific transfer functions in Figure 21.8:

$$H_m(s) = 1 \quad (21.35)$$

$$H_c(s) = K_c \quad (\text{proportional controller}) \quad (21.36)$$

$$H_p(s) = \frac{1}{(s+1)^2 s} \quad (21.37)$$

$$H_d(s) = \frac{-1}{(s+1)^2 s} \quad (\text{but not used in the stability analysis}) \quad (21.38)$$

The loop transfer function is:

$$\begin{aligned} L(s) &= H_c(s)H_p(s)H_m(s) \\ &= \frac{K_c = d_L(s)}{(s+1)^2 s = n_L(s)} \end{aligned}$$

We get:

$$T(s) = \frac{L(s)}{1 + L(s)} = \frac{n_L(s)}{d_L(s) + n_L(s)} = \frac{K_c}{s^3 + 2s^2 + s + K_c} \quad (21.39)$$

Thus, the characteristic polynomial of $T(s)$ is:

$$a(s) = s^3 + 2s^2 + s + K_c \quad (21.40)$$

Table 21.1 shows the poles for three different values of K_c .³

Figure 21.10 shows the step response in y for the three K_c -values (it is a unit step in r).

[End of Example 21.3]

³The poles can be calculated using the pole() or the pzmap() function function of Python Control Package.

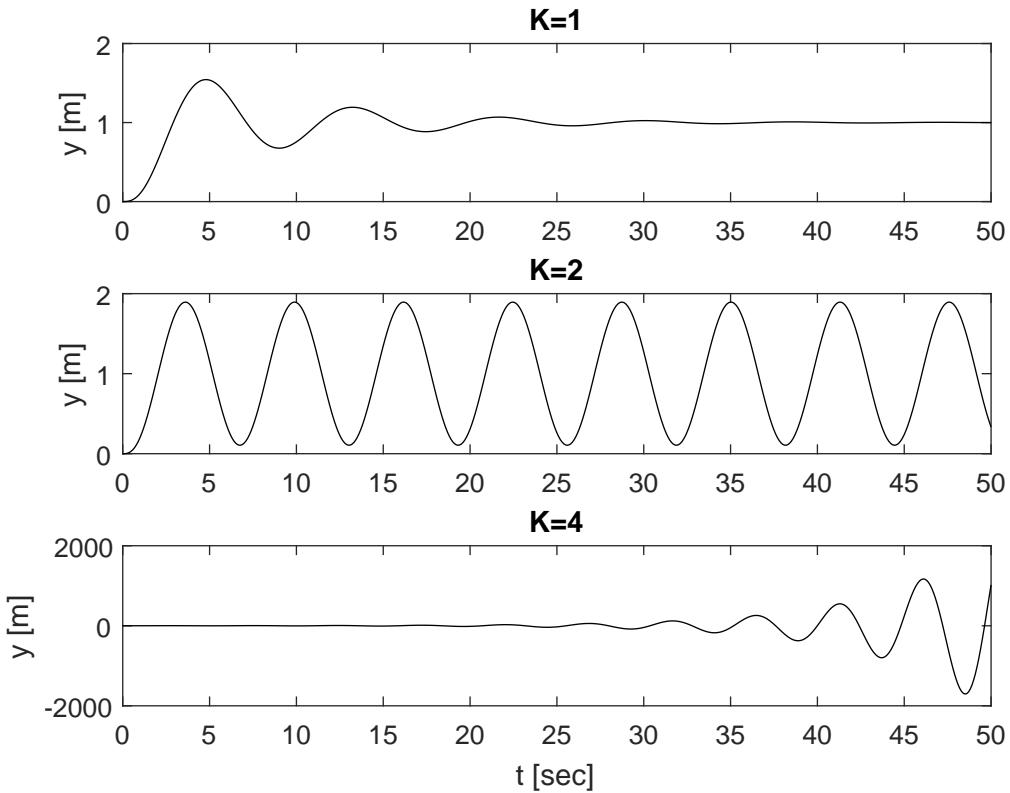


Figure 21.10: Example 21.3: Step response in the process output variable y for three different values of K_c .

21.6 Problems for Chapter 21

Problem 21.1 Stability property of a transfer function

Determine the stability property of the transfer function (21.41) by calculating its pole and also by studying its impulse response, $h(t)$ (make a rough sketch of it).

$$H(s) = \frac{y(s)}{u(s)} = \frac{1}{s+1} \quad (21.41)$$

To calculate $h(t)$, you can use the following Laplace transform:

$$\frac{k}{Ts+1} \iff \frac{ke^{-t/T}}{T} \quad (21.42)$$

Problem 21.2 Stability properties of several transfer functions

Determine the stability property of the following transfer functions:

$$H_1(s) = \frac{1}{s+1} \quad (21.43)$$

$$H_2(s) = \frac{1-s}{1+s} \quad (21.44)$$

$$H_3(s) = \frac{1}{1-s} \quad (21.45)$$

$$H_4(s) = \frac{1}{(s+1)(s-1)} \quad (21.46)$$

$$H_5(s) = \frac{1}{s} \quad (21.47)$$

$$H_6(s) = \frac{1}{s^3} \quad (21.48)$$

$$H_7(s) = \frac{e^{-s}}{s+1} \quad (21.49)$$

$$H_8(s) = -\frac{1}{s+1} \quad (21.50)$$

$$H_9(s) = \frac{1}{s^2 + s + 1} \quad (21.51)$$

$$H_{10}(s) = \frac{1}{s^2 + 1} \quad (21.52)$$

$$H_{11}(s) = \frac{1}{(s+1)s} \quad (21.53)$$

Problem 21.3 Stability property of state space model

Determine the stability property of the following state space model:

$$\begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad (21.54)$$

Problem 21.4 Control system stability with P controller

Figure 21.11 shows a feedback control system.

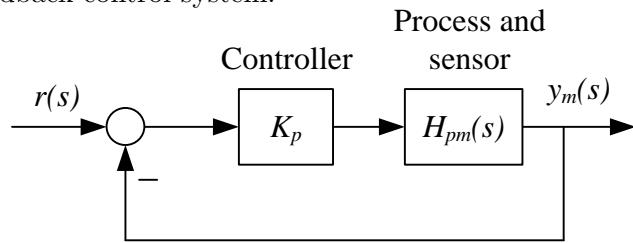


Figure 21.11: Feedback control system.

The transfer function of the process and sensor is

$$H_{pm}(s) = \frac{1}{s} \quad (21.55)$$

1. What is the stability property of $H_{pm}(s)$, i.e. of the process itself – also called the open-loop system?
2. For which values of the controller gain K_c is the control system asymptotically stable?
3. Has this problem demonstrated that it is possible to obtain an asymptotically stable feedback system even though the process itself is asymptotically stable?

21.7 *Solutions to problems for Chapter 21*

Solution to Problem 21.1

The pole of $H(s)$ is the root of

$$(s + 1) = 0 \quad (21.56)$$

The pole is

$$p = -1$$

which is in the left half plane. Therefore, the system is asymptotically stable.

The Laplace transform of the impulse response is

$$h(t) = y(s) = H(s) \underbrace{u(s)}_{=1} = H(s) = \frac{1}{s + 1} \quad (21.57)$$

Using (8.10) with $k = 1$ and $T = 1$ we get

$$h(t) = \frac{ke^{-t/T}}{T} = e^{-t} \quad (21.58)$$

Figure 21.12 shows $h(t)$.

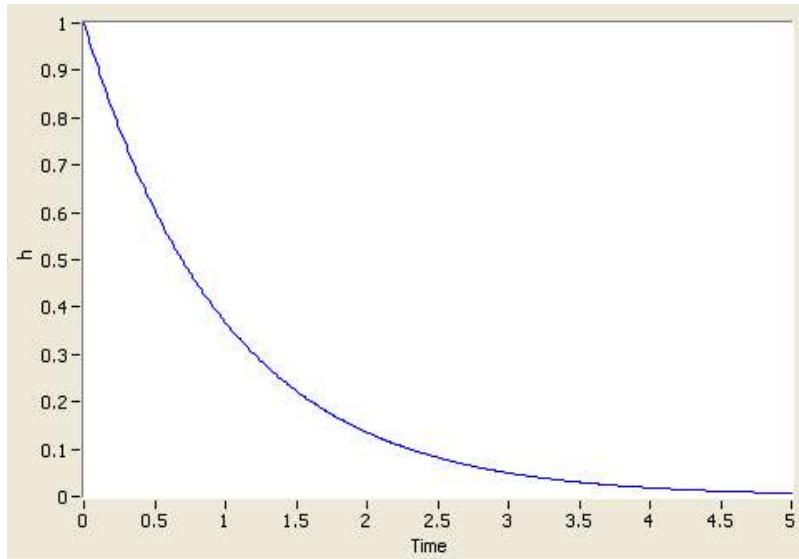


Figure 21.12: Impulse response.

Since $h(t)$ goes to zero as time goes to infinity, the system is asymptotically stable.

Solution to Problem 21.2

The transfer function

$$H_1(s) = \frac{1}{s + 1} \quad (21.59)$$

is asymptotically stable since the pole $p = -1$ is in the left half plane.

The transfer function

$$H_2(s) = \frac{1-s}{1+s} \quad (21.60)$$

is asymptotically stable since the pole $p = -1$ is in the left half plane. The value of the zero, which is 1, does not determine the stability.

The transfer function

$$H_3(s) = \frac{1}{1-s} \quad (21.61)$$

is unstable since the pole $p = 1$ is in the right half plane.

The transfer function

$$H_4(s) = \frac{1}{(s+1)(s-1)} \quad (21.62)$$

is unstable since one of the poles, $p_1 = 1$, is in the right half plane.

The transfer function

$$H_5(s) = \frac{1}{s} \quad (21.63)$$

is marginally stable since the pole $p = 0$ is in origin, which is on the imaginary axis, and this pole is single (there are no multiple poles).

The transfer function

$$H_6(s) = \frac{1}{s^3} \quad (21.64)$$

is unstable since there are multiple poles, $p_{1,2,3} = 0$, on the imaginary axis.

The transfer function

$$H_7(s) = \frac{e^{-s}}{s+1} \quad (21.65)$$

is asymptotically stable since the pole $p = -1$ is in the left half plane.

The transfer function

$$H_8(s) = -\frac{1}{s+1} \quad (21.66)$$

is asymptotically stable since the pole $p = -1$ is in the left half plane.

The transfer function

$$H_9(s) = \frac{1}{s^2 + s + 1} \quad (21.67)$$

is asymptotically stable since both the poles

$$p_{1,2} = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot 1}}{4 \cdot 1} = \frac{-1 \pm j\sqrt{3}}{4} \quad (21.68)$$

are in the left half plane.

The transfer function

$$H_{10}(s) = \frac{1}{s^2 + 1} \quad (21.69)$$

is marginally stable since the poles

$$p_{1,2} = \pm j \quad (21.70)$$

are on the imaginary axis and they are single (not multiple).

The transfer function

$$H_{11}(s) = \frac{1}{(s+1)s} \quad (21.71)$$

has poles

$$p_{1,2} = 0, -1 \quad (21.72)$$

One pole is on the imaginary axis, and the other is in the left half plane. The system is marginally stable.

Solution to Problem 21.3

The stability property is determined by the system eigenvalues, which are the roots of the characteristic equation:

$$\det(sI - A) = \det \left(s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \right) \quad (21.73)$$

$$= \det \left(\begin{bmatrix} s & -1 \\ 0 & s+2 \end{bmatrix} \right) \quad (21.74)$$

$$= s(s+2) + 0 \quad (21.75)$$

The roots are

$$s_{1,2} = 0, -2 \quad (21.76)$$

One pole is in the origin, and the other pole is in the left half plane. Thererfore, the system is marginally stable.

Solution to Problem 21.4

1. $H_{pm}(s)$ has pole equal to 0. Consequently the system is marginally stable.

2. The transfer function of the controller is

$$H_c(s) = K_c \quad (21.77)$$

The loop transfer function is

$$L(s) = H_c(s)H_{pm}(s) = \frac{K_c}{s} = \frac{n_0(s)}{d_0(s)} \quad (21.78)$$

The characteristic polynomial is

$$c(s) = d_0(s) + n_0(s) = s + K_c \quad (21.79)$$

The pole of the control system is the root of (21.79):

$$p = -K_c \quad (21.80)$$

The control system is asymptotically stable with $\text{Re}(p) < 0$, i.e.

$$\underline{\underline{K_c > 0}} \quad (21.81)$$

3. Sure!

Chapter 22

Frequency response

22.1 Introduction

The *frequency response* of a system is a frequency dependent function which expresses how a sinusoidal signal of a given frequency on the system input is transferred through the system. Time-varying signals – at least periodical signals – which excite systems, as the reference (setpoint) signal or a disturbance in a control system or measurement signals which are inputs signals to signal filters, can be regarded as consisting of a sum of *frequency components*. Each frequency component is a sinusoidal signal having a certain amplitude and a certain frequency. (The Fourier series expansion or the Fourier transform can be used to express these frequency components quantitatively.) The frequency response expresses how each of these frequency components is transferred through the system. Some components may be amplified, others may be attenuated, and there will be some phase lag through the system.

The frequency response is an important tool for analysis and design of signal filters (as lowpass filters and highpass filters), and for analysis, and to some extent, design, of control systems. Both signal filtering and control systems applications are described (briefly) later in this chapter.

The definition of the frequency response – which will be given in the next section – *applies only to linear models*, but this linear model may very well be the local linear model about some operating point of a non-linear model.

The frequency response can be found experimentally or from a transfer function model. It can be presented graphically or as a mathematical function.

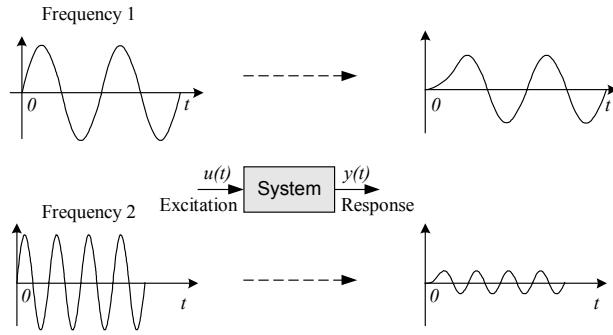


Figure 22.1: Sinusoidal signals in the input and the resulting responses on the output for two different frequencies

22.2 How to calculate frequency response from sinusoidal input and output

We can find the frequency response of a system by exciting the system with a sinusoidal signal of amplitude U and frequency ω [rad/s] and observing the response in the output variable of the system.¹ Mathematically, we set the input signal to

$$u(t) = U \sin \omega t \quad (22.1)$$

See Figure 22.1. This input signal will give a transient response (which will die, eventually) and a *steady-state* response, $y_s(t)$, in the output variable:

$$y_s(t) = Y \sin(\omega t + \phi) \quad (22.2)$$

$$= \underbrace{YA}_{Y} \sin(\omega t + \phi) \quad (22.3)$$

Here A is the (*amplitude*)*gain*, and ϕ (phi) is *the phase lag* in radians. The frequency of $y_s(t)$ will be the same as in $u(t)$. Figure 22.2 shows in detail $u(t)$ and $y(t)$ for a simulated system.

The system which is simulated is

$$y(s) = \frac{1}{s+1} u(s) \quad (22.4)$$

(a first order system with gain 1 and time constant 1). The input signal $u(t)$ has frequency $\omega = 3$ rad/s and amplitude $U = 1$.

A is the ratio between the amplitudes of the output signal and the input signal (in steady-state):

$$A = \frac{Y}{U} \quad (22.5)$$

For the signals shown in Figure 22.2,

$$A = \frac{Y}{U} = \frac{0.32}{1} = 0.32 \quad (22.6)$$

¹The correspondance between a given frequency ω in rad/s and the same same frequency f in Hz is $\omega = 2\pi f$.

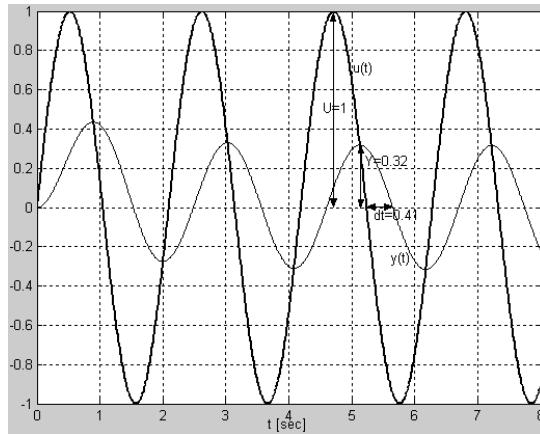


Figure 22.2: The input signal $u(t)$ and the resulting (sinusoidal) response $y(t)$ for a simulated system. $u(t)$ has frequency $\omega = 3 \text{ rad/s}$ and amplitude $U = 1$. The system is given by (22.4).

ϕ can be calculated by first measuring the time-lag Δt between $u(t)$ and $y_s(t)$ and then calculating ϕ as follows:

$$\phi = -\omega \Delta t \quad [\text{rad}] \quad (22.7)$$

In Figure 22.2 we find $\Delta t = 0.41 \text{ sec}$, which gives

$$\phi = -\omega \Delta t = -3 \cdot 0.41 = -1.23 \text{ rad} \quad (22.8)$$

The gain A and the phase-lag ϕ are functions of the frequency. We can use the following terminology: $A(\omega)$ is the *gain function*, and $\phi(\omega)$ is the *phase shift function* (or more simply: phase function). We say that $A(\omega)$ and $\phi(\omega)$ expresses the *frequency response* of the system.

22.3 Bode diagram

It is common to present $A(\omega)$ and $\phi(\omega)$ graphically in a *Bode diagram*, which consists of two subdiagrams, one for $A(\omega)$ and one for $\phi(\omega)$, where the phase values are usually plotted in degrees (not radians). Figure 22.3 shows a Bode diagram of the frequency response of the system given by (22.4). The curves may stem from a number of A -values and ϕ -values found in experiments (or simulations) with an sinusoidal input signal of various frequencies. The curves may also stem from the transfer function of the system, as described in Section 22.4. The frequency axes usually show the 10-logarithm of the frequency in rad/s or in Hz.

Actually, the system (22.4) is used to generate $u(t)$ and $y(t)$ shown in Figure 22.2. We have earlier in this chapter calculated $A(3) = 0.32 = -10.2 \text{ dB}$ (the dB-unit is described below) and phase lag $\phi(3) = -1.23 \text{ rad} = -72 \text{ degrees}$. This gain value and phase lag value are indicated in the Bode diagram in Figure 22.3.

The $A(\omega)$ -axis is usually drawn with decibel (dB) as unit. The decibel value of a number x is calculated as

$$x \text{ [dB]} = 20 \log_{10} x \quad (22.9)$$

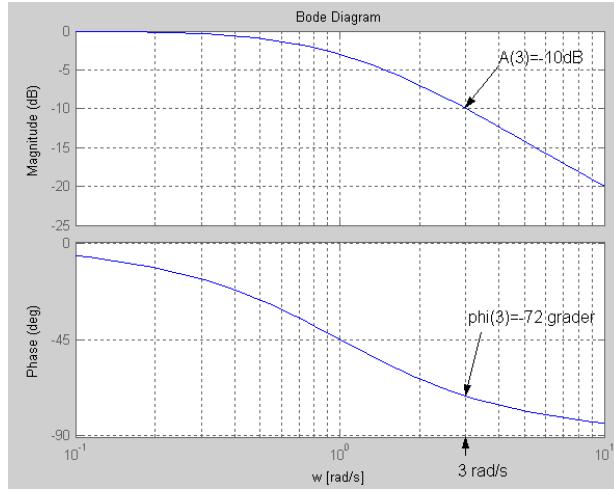


Figure 22.3: The frequency response of the system given by (22.4) presented in a Bode diagram

Table 22.1 shows some examples of dB-values.

22.4 How to calculate frequency response from transfer functions

In Section 22.2 we saw how to find the frequency response from experiments on the system. No model was assumed. However, if we know a transfer function model of the system, we can *calculate the frequency response from the transfer function*, as explained below.

Suppose that system has the transfer function $H(s)$ from input u to output y , that is,

$$y(s) = H(s)u(s) \quad (22.10)$$

By setting

$$s = j\omega \quad (22.11)$$

(j is the imaginary unit) into $H(s)$, we get the complex quantity $H(j\omega)$, which is the *frequency response* (function). The *gain function* is

$$A(\omega) = |H(j\omega)| \quad (22.12)$$

and the *phase shift function* is the angle or argument of $H(j\omega)$:

$$\phi(\omega) = \arg H(j\omega) \quad (22.13)$$

(The formulas (22.12) and (22.13) can be derived using the Laplace transform.)

Example 22.1 Frequency response calculated from a transfer function

Table 22.1: Some dB-values

0	=	-∞dB
0.01	=	-40dB
0.1	=	-20dB
0.2	=	-14dB
0.25	=	-12dB
0.5	=	-6dB
$\frac{1}{\sqrt{2}}$	=	-3dB
1	=	0dB
$\sqrt{2}$	=	3dB
2	=	6dB
$\sqrt{10}$	=	10dB
4	=	12dB
5	=	14dB
10	=	20dB
100	=	40dB

We will find the frequency response for the transfer function

$$H(s) = \frac{K}{Ts + 1} \quad (22.14)$$

The frequency response becomes

$$H(j\omega) = H(s)|_{s=j\omega} = \frac{K}{Tj\omega + 1} = \underbrace{\frac{K}{1}}_{\text{Re}} + j\underbrace{\frac{K}{T\omega}}_{\text{Im}} \quad (22.15)$$

which we write on polar form:

$$H(j\omega) = \frac{K}{\sqrt{1^2 + (T\omega)^2} e^{j \arctan(\frac{T\omega}{1})}} \quad (22.16)$$

$$= \frac{1}{\sqrt{1 + (T\omega)^2}} e^{j[-\arctan(T\omega)]} \quad (22.17)$$

$$= |H(j\omega)| e^{j \arg H(j\omega)} \quad (22.18)$$

Thus, the gain function is

$$|H(j\omega)| = \frac{K}{\sqrt{1 + (T\omega)^2}} \quad (22.19)$$

and the phase function is

$$\arg H(j\omega) = -\arctan(T\omega) \quad [\text{rad}] \quad (22.20)$$

Figure 22.4 shows the curves of $|H(j\omega)|$ and $\arg H(j\omega)$ drawn in a Bode diagram. The numerical values along the axes assume $K = 1$ and $T = 1$. (The asymptotes indicated in the figure are not explained in this document.)

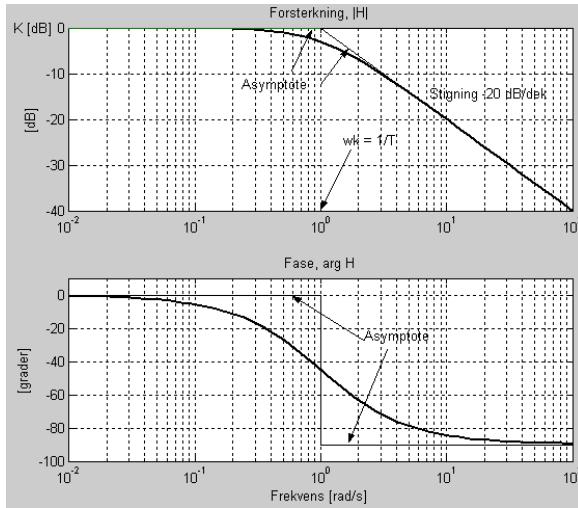


Figure 22.4: Bode diagram for the frequency response of the first ordens system (22.14). The asymptotes are not explained in this document.

To illustrate the use of (22.19) and (22.20), let us calculate the gain and phase lag values for the frequency $\omega = 3$ rad/s. We assume that $K = 1$ and $T = 1$. (22.19) gives

$$|H(j3)| = \frac{1}{\sqrt{1+3^2}} = \frac{1}{\sqrt{10}} = 0.316 = -20 \log_{10} \left(\frac{1}{\sqrt{10}} \right) = -10.0 \text{ dB} \quad (22.21)$$

(22.20) gives

$$\arg H(j3) = -\arctan(3) = -1.25 \text{ rad} = -71.6 \text{ degrees} \quad (22.22)$$

[End of Example 22.1]

The next example shows how the frequency response can be found of a transfer function which consists of several factors in the numerator and/or the denominator.

Example 22.2 Frequency response of a (more complicated) transfer function

Given the transfer function

$$H(s) = K \frac{T_1 s + 1}{(T_2 s + 1) s} e^{-\tau s} \quad (22.23)$$

(The term $e^{-\tau s}$ represents a time delay of τ sec.) We set $s = j\omega$ in $H(s)$ and then sets the individual factors on *polar form*. Finally, we combine these factors so that we end up with a

polar form of $H(j\omega)$:

$$H(j\omega) = K \frac{T_1 j\omega + 1}{(T_2 j\omega + 1) j\omega} e^{-\tau j\omega} \quad (22.24)$$

$$= K \frac{\sqrt{1^2 + (T_1\omega)^2} e^{j \arctan(\frac{T_1\omega}{1})}}{\left[\sqrt{1^2 + (T_2\omega)^2} e^{j \arctan(\frac{T_2\omega}{1})} \right] \left[\sqrt{0^2 + \omega^2} e^{j \frac{\pi}{2}} \right]} e^{-\tau j\omega} \quad (22.25)$$

$$= \underbrace{\frac{K \sqrt{1 + (T_1\omega)^2}}{\sqrt{1 + (T_2\omega)^2} \omega}}_{|H(j\omega)|} e^{j \underbrace{\arctan(T_1\omega) - \arctan(T_2\omega) - \frac{\pi}{2} - \tau\omega}_{\arg H(j\omega)}} \quad (22.26)$$

So, the amplitude gain function is

$$A(\omega) = |H(j\omega)| = \frac{K \sqrt{1 + (T_1\omega)^2}}{\sqrt{1 + (T_2\omega)^2} \omega} \quad (22.27)$$

and the phase shift function is

$$\phi(\omega) = \arg H(j\omega) = \arctan(T_1\omega) - \arctan(T_2\omega) - \frac{\pi}{2} - \tau\omega \quad (22.28)$$

[End of Example 22.2]

22.5 Application of frequency response: Signal filters

22.5.1 Introduction

A *signal filter* – or just *filter* – is used to attenuate (ideally: remove) a certain frequency interval of frequency components from a signal. These frequency components are typically noise. For example, a lowpass filter is used to attenuate high-frequent components (low-frequent components passes).

Knowledge about filtering functions is crucial in signal processing, but it is useful also in control engineering because control systems can be regarded as filters in the sense that the controlled process variable can follow only a certain range or interval of frequency components in the reference (setpoint) signal, and it will be only a certain frequency range of process disturbances that the control system can compensate for effectively. Furthermore, knowledge about filters can be useful in the analysis and design of physical processes. For example, a stirred tank in a process line can act as a lowpass filter since it attenuates low-frequent components in the inflow to the tank.

In this section we will particularly study *lowpass filters*, which is the most commonly used filtering function, but we will also take a look at *highpass filters*, *bandpass filters* and *bandstop filters*.

Figure 22.5 shows the gain function for ideal filtering functions and for practical filters (the phase lag functions are not shown).

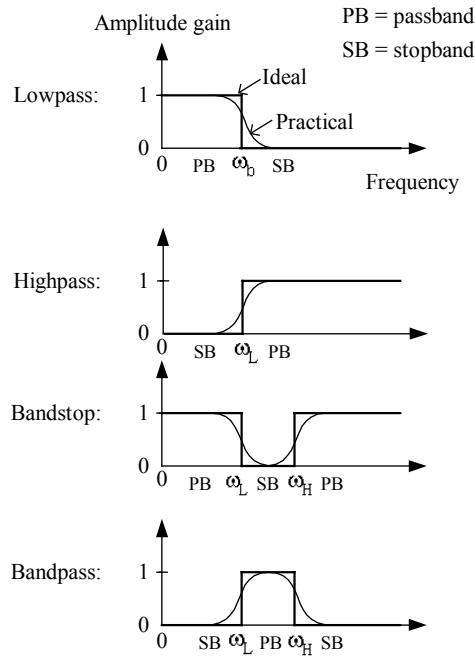


Figure 22.5: The gain functions for ideal filters and for practical filters of various types

The *passband* is the frequency interval where the gain function has value 1, ideally (thus, frequency components in this frequency interval passes through the filter, unchanged). The *stopband* is the frequency interval where the gain function has value 0, ideally (thus, frequency components in this frequency interval are stopped through the filter).²

It can be shown that transfer functions for ideal filtering functions will have infinitely large order. Therefore, ideal filters can not be realized, neither with analog electronics nor with a filtering algorithm in a computer program.

22.5.2 First order lowpass filters

The most commonly used signal filter is the first order lowpass filter. As an example, it is the standard measurement filter in a feedback control system.

The transfer function of a first order lowpass filter with input variable u and output variable y is usually written as

$$H(s) = \frac{1}{\frac{s}{\omega_b} + 1} \quad (22.29)$$

²It is a pity that lowpass filters were not called highstop filters instead since the main purpose of a lowpass filter is to stop high-frequency components. Similarly, highpass filters should have been called lowstop filters, but it is too late now...

where ω_b [rad/s] is the *bandwidth* of the filter. This is a first order transfer function with gain $K = 1$ and time constant $T = 1/\omega_b$. The frequency response is

$$H(j\omega) = \frac{1}{\frac{j\omega}{\omega_b} + 1} \quad (22.30)$$

$$\begin{aligned} &= \frac{1}{\sqrt{\left(\frac{\omega}{\omega_b}\right)^2 + 1}} e^{j \arctan \frac{\omega}{\omega_b}} \\ &= \frac{1}{\sqrt{\left(\frac{\omega}{\omega_b}\right)^2 + 1}} e^{j(-\arctan \frac{\omega}{\omega_b})} \end{aligned} \quad (22.31)$$

The gain function is

$$|H(j\omega)| = \frac{1}{\sqrt{\left(\frac{\omega}{\omega_b}\right)^2 + 1}} \quad (22.32)$$

and the phase lag function is

$$\arg H(j\omega) = -\arctan \frac{\omega}{\omega_b} \quad (22.33)$$

Figure 22.4 shows exact and asymptotic curves of $|H(j\omega)|$ and $\arg H(j\omega)$ drawn in a Bode diagram. In the figure, $K = 1$ and $\omega_b = \omega_c$.

The bandwidth defines the upper limit of the passband. It is common to say that the bandwidth is the frequency where the filter gain is $1/\sqrt{2} = 0.71 \approx -3$ dB (above the bandwidth the gain is less than $1/\sqrt{2}$). This bandwidth is therefore referred to as the “ -3 dB-bandwidth”. Now, what is the -3 dB-bandwidth of a first order lowpass filter? It is the ω -solution of the equation

$$|H(j\omega)| = \frac{1}{\sqrt{\left(\frac{\omega}{\omega_b}\right)^2 + 1}} = \frac{1}{\sqrt{2}} \quad (22.34)$$

The solution is $\omega = \omega_b$. Therefore, ω_b [rad/s] given in (22.29) is the -3 dB-bandwidth in rad/s. In Hertz the bandwidth is

$$f_b = \frac{\omega_b}{2\pi} \quad (22.35)$$

Figure 22.6 shows the front panel of a simulator of a first order filter where the input signal consists of a sum of two sinusoids or frequency components of frequency less than and greater than, respectively, the bandwidth. The simulation shows that the low frequent component (0.5 Hz) passes almost unchanged (it is in the passband of the filter), while the high-frequent component (8 Hz) is attenuated (it is in the stopband).

Example 22.3 The RC-circuit as a lowpass filter

Figure 22.7 shows an RC-circuit (the circuit contains the resistor R and the capacitor C). The RC-circuit is frequently used as an analogue lowpass filter: Signals of *low* frequencies

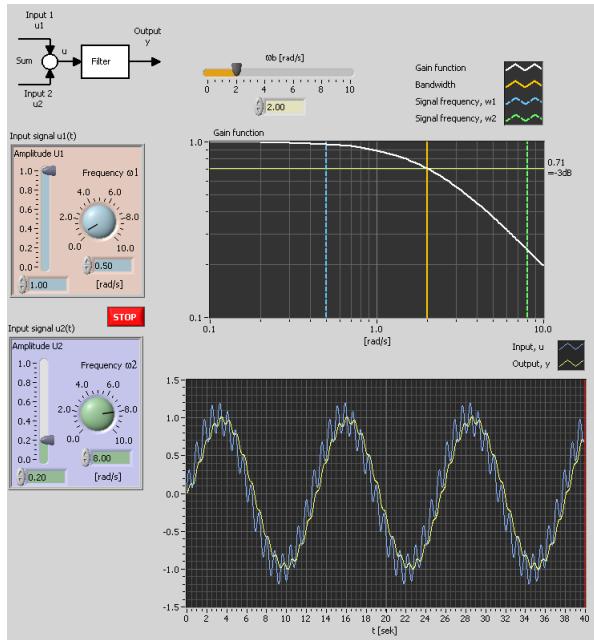


Figure 22.6: Simulator for a first order lowpass filter where the input signal consists of a sum of two frequency components

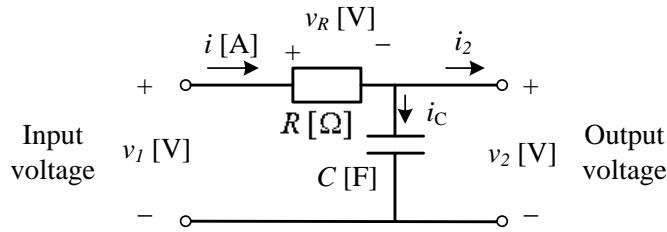


Figure 22.7: RC-circuit

passes approximately unchanged through the filter, while signals of high frequencies are approximately filtered out (stopped). v_1 is the signal source or input voltage to be filtered, while v_2 is the resulting filtered output voltage.

We will now find a mathematical model relating v_2 to v_1 . First we apply the Kirchhoff's voltage law in the circuit which consists the input voltage terminals, the resistor, and the capacitor (we consider the voltage drops to be positive clockwise direction):

$$-v_1 + v_R + v_2 = 0 \quad (22.36)$$

(v_2 equals the voltage drop over the capacitor.) In (22.36) v_R is given by

$$v_R = Ri \quad (22.37)$$

We assume that there is no current going through the output terminals. (This is a common assumption, and not unrealistic, since it is typical that the output terminals are connected to a subsequent circuit which has approximately infinite input impedance, causing the current into it to be approximately zero. An operational amplifier is an example of such a

load-circuit.) Therefore,

$$i = i_C = Cv'_2 \quad (22.38)$$

The final model is achieved by using i as given by (22.38) in (22.37) and then using v_R as given by (22.37) for v_R in (22.36). The model becomes

$$RCv'_2 = v_1 - v_2 \quad (22.39)$$

The transfer function from the input voltage v_1 to the output voltage v_2 becomes

$$H_{v_2, v_1}(s) = \frac{1}{RCs + 1} = \frac{1}{\frac{s}{\omega_b} + 1} \quad (22.40)$$

Thus, the RC-circuit is a first order lowpass filter with bandwidth

$$\omega_b = \frac{1}{RC} \text{ rad/s} \quad (22.41)$$

If for example $R = 1 \text{ k}\Omega$ and $C = 10 \mu\text{F}$, the bandwidth is $\omega_b = 1/RC = 100 \text{ rad/s}$. (22.41) can be used to design the RC-circuit (calculate the R- and C-values).

[End of Example 22.3]

22.6 Problems for Chapter 22

Problem 22.1 Frequency response measures from sinusoids

Figure 22.8 shows the input signal and the corresponding output signal of a system.

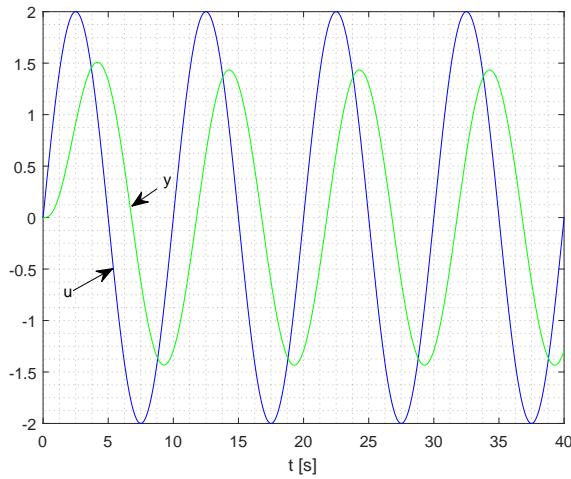


Figure 22.8: Input signal u and output signal y

1. What is the frequency of the signal in Hz and in rad/s?
2. Calculate the amplitude gain and the phase lag at the frequency found in Problem 1 above. What is the amplitude gain in dB?

Problem 22.2 Steady-state response decided from Bode diagram

Figure 22.9 shows a Bode diagram of a system.

Assume that the input signal u is a sinusoid of amplitude $U = 0.8$ and frequency $\omega = 1.0$ rad/s. Write the corresponding steady-state output response $y_s(t)$.

Problem 22.3 Frequency response from transfer function

Calculate the frequency response functions $A(\omega)$ and $\phi(\omega)$ of the transfer function

$$H(s) = \frac{K}{(1 + T_1 s)(1 + T_2 s)} e^{-\tau s} \quad (22.42)$$

Problem 22.4 Design of RC filter from bandwidth

Assume that it is specified that a given RC filter shall have bandwidth 100 Hz. Find proper values of the resistance R and the capacitance C . (Tip: C can be selected between 10^{-4} and 10^{-6} F because this gives a practical size of the capacitor. Unless you insist on some other value, you can use 10^{-5} F.)

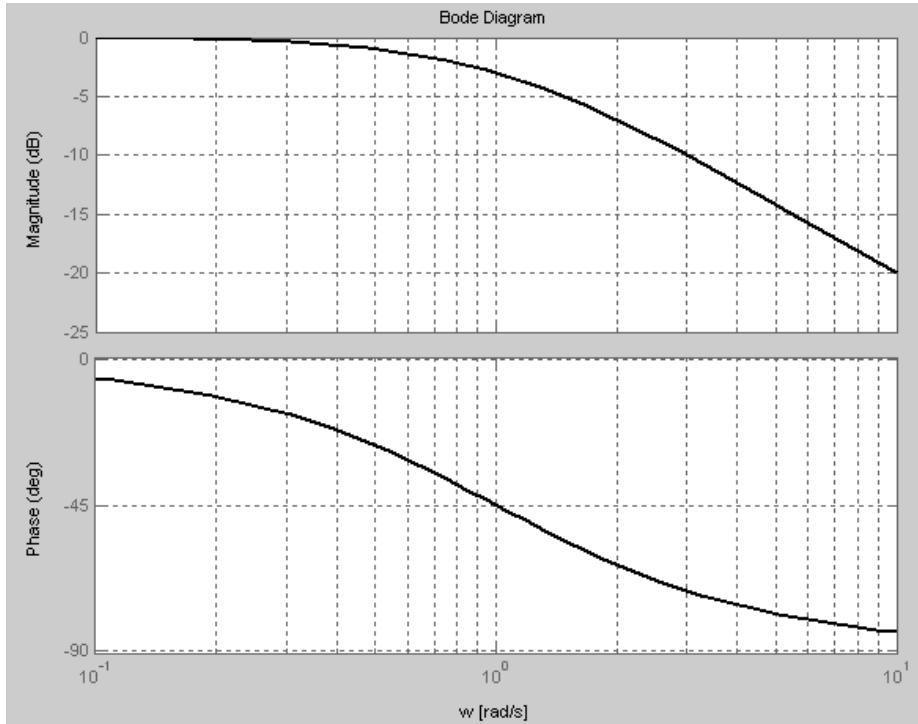


Figure 22.9: Bode diagram

22.7 *Solutions to problems for Chapter 22*

Solution to Problem 22.1

- Figure 22.10 shows the signals with amplitudes and time-lag indicated.

From the figure we see that the period of the input signal is

$$T_p = 10 \text{ sec} \quad (22.43)$$

which corresponds to the frequency

$$\underline{\underline{f_1}} = \frac{1}{T_p} = \frac{1}{10} = \underline{\underline{0.1 \text{ Hz}}} \quad (22.44)$$

or, alternatively,

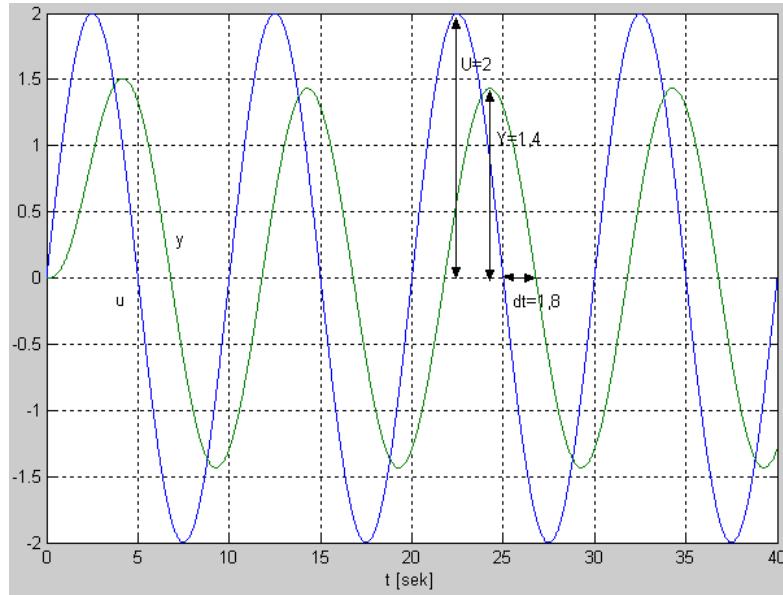
$$\underline{\underline{\omega_1}} = 2\pi f_1 = 2 \cdot \pi \cdot 0.1 = \underline{\underline{0.63 \text{ rad/s}}} \quad (22.45)$$

- The amplitude gain (at frequency f_1) is

$$\underline{\underline{A}} = \frac{Y}{U} = \frac{1.4}{2} = \underline{\underline{0.7}} = 20 \cdot \log_{10}(0.7) \text{ dB} = \underline{\underline{-3.1 \text{ dB}}} \quad (22.46)$$

The phase lag ϕ can be calculated by firstly measuring the time lag Δt between input $u(t)$ and output $y(t)$ and then calculating ϕ with

$$\phi = -\omega \Delta t \text{ [rad]} \quad (22.47)$$


 Figure 22.10: Input signal u and output signal y

From Figure 22.10 you can find

$$\underline{\underline{\Delta t}} = 1.8 \text{ s} \quad (22.48)$$

Hence,

$$\underline{\underline{\phi}} = -\omega \Delta t = -0.63 \cdot 1.8 = \underline{\underline{-1.13 \text{ rad}}} = -1.13 \cdot \frac{180}{\pi} = \underline{\underline{-65 \text{ deg}}} \quad (22.49)$$

Alternatively, we can calculate ϕ from the following ratio (360 degrees corresponds to one period):

$$\underline{\underline{\phi}} = -\frac{\Delta t}{T_p} \cdot 360 \text{ deg} = -\frac{1.8 \text{ s}}{10 \text{ s}} \cdot 360 \text{ deg} = \underline{\underline{-65 \text{ deg}}} \quad (22.50)$$

Solution to Problem 22.2

The steady-state response is

$$y_s(t) = UA \sin(\omega t + \phi) \quad (22.51)$$

The amplitude of the input signal is $U = 0.8$. The amplitude gain A is read off from the upper curve in the Bode diagram at frequency $\omega = 1.0 \text{ rad/s}$:

$$A = -3 \text{ dB} \quad (22.52)$$

which is

$$A = 10^{-3/20} = 0.71 \quad (22.53)$$

The phase lag ϕ at frequency $\omega = 1.0 \text{ rad/s}$ is read off from the lower curve in the Bode diagram:

$$\phi = -45 \text{ deg} = -45 \cdot \frac{\pi}{180} \text{ rad} = -0.79 \text{ rad} \quad (22.54)$$

Hence,

$$\underline{\underline{y_s(t) = 0.8 \cdot 0.71 \cdot \sin(1.0 \cdot t - 0.79) = 0.57 \sin(t - 0.79)}} \quad (22.55)$$

Solution to Problem 22.3

We set $s = j\omega$ in $H(s)$, and then turn the factors into *polar forms*, which we then combine to finally get a polar form of $H(j\omega)$:

$$\underline{\underline{H(j\omega)}} = \frac{K}{(1 + T_1 j\omega)(1 + T_2 j\omega)} e^{-j\omega\tau} \quad (22.56)$$

$$= \frac{K}{\left[\sqrt{1^2 + (T_1\omega)^2} e^{j \arctan(\frac{T_1\omega}{1})} \right] \left[\sqrt{1^2 + (T_2\omega)^2} e^{j \arctan(\frac{T_2\omega}{1})} \right]} e^{-j\omega\tau} \quad (22.57)$$

$$= \underbrace{\frac{K}{\sqrt{1 + (T_1\omega)^2} \sqrt{1 + (T_2\omega)^2}}}_{|H(j\omega)|} e^{\underbrace{j[-\arctan(T_1\omega) - \arctan(T_2\omega) - \omega\tau]}_{\arg H(j\omega)}} \quad (22.58)$$

So, we have

$$\underline{\underline{A(\omega)}} = |H(j\omega)| = \frac{K}{\sqrt{1 + (T_1\omega)^2} \sqrt{1 + (T_2\omega)^2}} \quad (22.59)$$

and

$$\underline{\underline{\phi(\omega)}} = \arg H(j\omega) = -\arctan(T_1\omega) - \arctan(T_2\omega) \quad (22.60)$$

Solution to Problem 22.4

The bandwidth is

$$f_b \text{ [Hz]} = \frac{\omega_b \text{ [rad/s]}}{2\pi} = \frac{1}{2\pi} \cdot \frac{1}{RC} \quad (22.61)$$

With

$$C = 10^{-5} \text{ F} \quad (22.62)$$

we get

$$\underline{\underline{R}} = \frac{1}{2\pi f_b C} = \frac{1}{2\pi \cdot 100 \text{ Hz} \cdot 10^{-5} \text{ F}} = \underline{\underline{159 \Omega}} \quad (22.63)$$

Chapter 23

Frequency response analysis of feedback control systems

23.1 Introduction

With frequency response – using Bode plots – we can analyze dynamic properties of feedback control systems. These properties refers to

- dynamic setpoint tracking, and
- dynamic disturbance compensation.

By definition, in frequency response analysis all signals in the system are assumed to be sinusoids. This seems to limit the usefulness of such analysis because in real systems signals are rarely sinusoids. Still, the frequency response analysis provides useful insight about the dynamic properties of a control system because varying signals can be decomposed into certain frequency components.

Frequency response analysis assumes a *linear* model of the control system. However, practical control systems are nonlinear due to phenomena as saturation, hysteresis, stiction, nonlinear signal scaling etc. Such nonlinearities can influence largely the dynamic behaviour of the control system. To perform “linear” analysis of a non-linear model, this model must be linearized about some operating point. Thus, the results of the analysis will be valid at or close to the operation point where the linearization was made. This fact limits the usefulness of a theoretical analysis of a given nonlinear control system using linear systems methods, but the results may still be useful, particularly if the system most of the time operates close to the chosen or specified operating point.

Although a “linear” analysis of a given nonlinear control system may have limited practical value, you will get much general understanding about the behaviour of control systems through analysis of examples of linear control systems.

Note: Once you have a mathematical model of a given control system, you should definitely

run simulations as a part of the analysis. This applies for both linear and nonlinear control systems. Simulations will give you information about dynamics responses, whether the static control error is zero or not, the impact of process disturbances, the effects of measurement noise, effects of parameter variations, etc.

23.2 Frequency response analysis of setpoint tracking and disturbance compensation

23.2.1 Introduction

Frequency response analysis of control systems expresses the tracking and compensation properties under the assumption that the setpoint and the disturbance are *sinusoidal signals* or *frequency components* in a compound signal. We assume that the control system has a transfer function-based block diagram as shown in Figure 23.1.

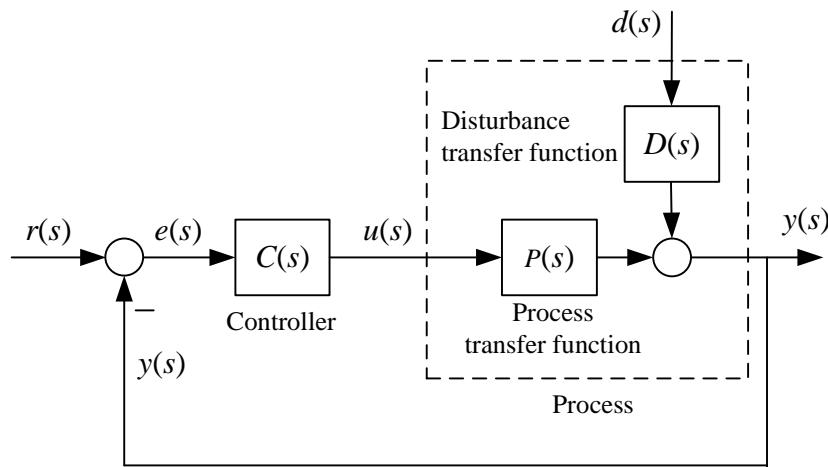


Figure 23.1: Transfer function-based block diagram of a control system.

The Laplace transformed control error is given by (14.7), which is repeated here:

$$e(s) = \underbrace{[S(s)r(s)]}_{e_r(s)} + \underbrace{[-S(s)D(s)d(s)]}_{e_d(s)} \quad (23.1)$$

where $S(s)$ is the sensitivity transfer function:

$$S(s) = \frac{1}{1 + L(s)} \quad (23.2)$$

where $L(s)$ is the loop transfer function. In the following we will study both $S(s)$ and the tracking ratio $T(s)$:

$$T(s) = \frac{L(s)}{1 + L(s)} = \frac{y(s)}{r(s)} \quad (23.3)$$

23.2.2 Frequency response analysis of setpoint tracking

From (23.1) we see we that the response in the control error due to the setpoint is

$$e_{\text{sp}}(s) = S(s)r(s) \quad (23.4)$$

By plotting the *frequency response* $S(j\omega)$ we can easily calculate how large the error is for a given frequency component in the setpoint: Assume that the setpoint is a sinusoid of amplitude R and frequency ω . According to frequency response theory, cf. Section 22.4, the steady-state response in the control error is

$$e_r(t) = R |S(j\omega)| \sin [\omega t + \arg S(j\omega)] \quad (23.5)$$

Thus, the error is small and consequently the tracking property is good if $|S(j\omega)| \ll 1$, while the error is large and the tracking property poor if $|S(j\omega)| \approx 1$.

The tracking property can be indicated by the tracking transfer function $T(s)$, too. The response in the process output measurement due to the setpoint is

$$y_m(s) = T(s)r(s) \quad (23.6)$$

Assume that the setpoint is a sinusoid of amplitude Y_{sp} and frequency ω . According to frequency response theory, cf. Chapter 22, and in particular, Section 22.4, the steady-state response in the process output due to the setpoint is:

$$y(t) = R |T(j\omega)| \sin [\omega t + \arg T(j\omega)] \quad (23.7)$$

Thus, $|T(j\omega)| \approx 1$ indicates that the control system has good tracking property, while $|T(j\omega)| \ll 1$ indicates poor tracking property.

Since both $S(s)$ and $T(s)$ are functions of the loop transfer function $L(s)$, cf. (23.2) and (23.3), there is a relation between $L(s)$ and the tracking property of the control system. Using (23.2) and (23.3) we can conclude as follows:

$$\text{Good setpoint tracking: } |S(j\omega)| \ll 1, |T(j\omega)| \approx 1, |L(j\omega)| \gg 1 \quad (23.8)$$

$$\text{Poor setpoint tracking: } |S(j\omega)| \approx 1, |T(j\omega)| \ll 1, |L(j\omega)| \ll 1 \quad (23.9)$$

Figure 23.2 shows typical Bode plots of $|S(j\omega)|$, $|T(j\omega)|$ and $|L(j\omega)|$. Usually we are interested in the amplitude gains, not the phase lags. Therefore plots of $\arg S(j\omega)$, $\arg T(j\omega)$ and $\arg L(j\omega)$ are not shown nor discussed here.

The *bandwidth* of a control system is the frequency which divides the frequency range of good tracking and the frequency range of poor tracking. From (23.8) and (23.9) and Figure 23.2 we can list the following three candidates for a definition of the bandwidth:

- ω_t , which is the frequency where the amplitude gain of the tracking transfer function has value $1/\sqrt{2} \approx 0.71 = -3$ dB. This definition is in accordance with the usual bandwidth definition of lowpass filters. The ω_t bandwidth is also called the -3 dB bandwidth $\omega_{-3\text{dB}}$.

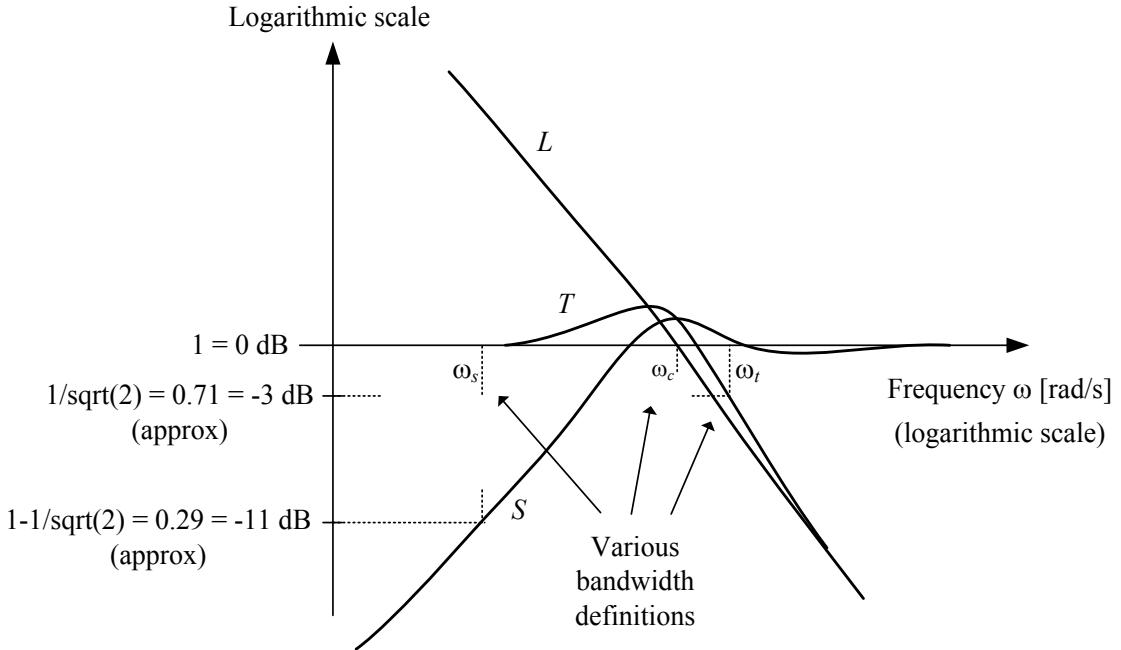


Figure 23.2: Typical Bode plots of $|S(j\omega)|$, $|T(j\omega)|$ and $|L(j\omega)|$.

- ω_c , which is the frequency where the amplitude gain of the loop transfer function has value $1 = -0$ dB. ω_c is called the *crossover frequency* of L .
- ω_s , which is the frequency where the amplitude gain of the sensitivity transfer function has value $1 - 1/\sqrt{2} \approx 1 - 0.71 \approx 0.29 \approx -11$ dB. This definition is derived from the -3 dB bandwidth of the tracking transfer function: Good tracking corresponds to tracking gain between $1/\sqrt{2}$ and 1. Now recall that the sensitivity transfer function is the transfer function from setpoint to control error, cf. (23.4). Expressed in terms of the control error, we can say that good tracking corresponds to sensitivity gain $|S|$ less than $1 - 1/\sqrt{2} \approx -11$ dB ≈ 0.29 . The frequency where $|S|$ is -11 dB is denoted the *sensitivity bandwidth*, ω_s .

Of the three bandwidth candidates defined above, the sensitivity bandwidth ω_s is the one that is most closely related to the control error. Therefore, we may say that ω_s is the most convenient bandwidth definition as far as the tracking property of a control system is concerned. In addition, ω_s is a convenient bandwidth related to the compensation property of a control system (we will look into this in more detail below). However, the crossover frequency ω_c and the -3 dB bandwidth are the commonly used bandwidth definitions.

As indicated in Figure 23.2, the numerical values of the various bandwidth definitions will be different (and this is demonstrated in Example 23.1).

If you need a (possibly rough) estimate of the *response time* T_r of a control system, which is time it takes for a step response to reach 63% of its steady-state value, you can use

$$T_r \approx \frac{k}{\omega_t} \text{ [s]} \quad (23.10)$$

where ω_t is the -3 dB bandwidth in rad/s.¹ k can be set to some value between 1.5 and 2.0 , say 2.0 if you want to be conservative.

Example 23.1 Frequency response analysis of setpoint tracking

See the block diagram in Figure 14.2. Assume the following transfer functions:

PID controller:

$$C(s) = K_c \left(1 + \frac{1}{T_i s} + \frac{T_d s}{T_f s + 1} \right) \quad (23.11)$$

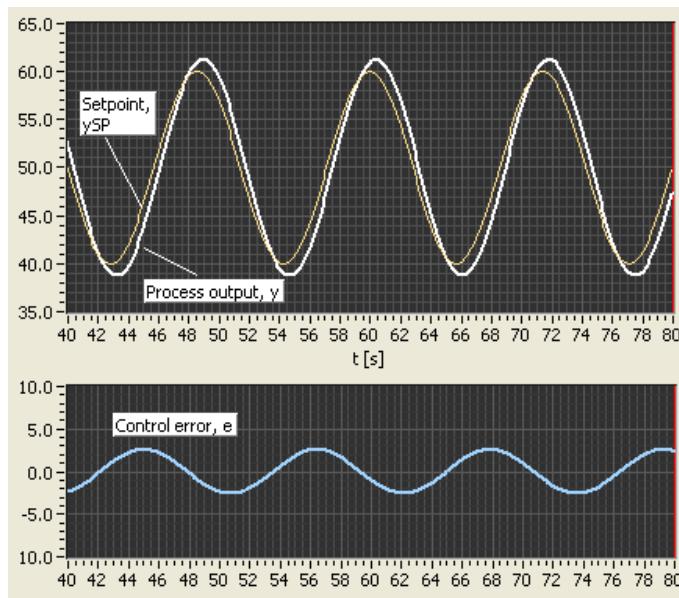


Figure 23.3: Example 23.1: Simulated responses of the control system. The setpoint r is sinusoid of frequency $\omega_1 = 0.55$ rad/s.

Process transfer functions (second order with time delay):

$$P(s) = \frac{K_u}{(T_1 s + 1)(T_2 s + 1)} e^{-\tau s} \quad (23.12)$$

$$D(s) = \frac{K_v}{(T_1 s + 1)(T_2 s + 1)} e^{-\tau s} \quad (23.13)$$

Measurement transfer function:

$$H_m(s) = K_m \quad (23.14)$$

The parameter values are $K_c = 4.3$, $T_i = 1.40$, $T_d = 0.35$, $T_f = 0.1 T_d = 0.035$, $K_u = 1$, $K_d = 1$, $T_1 = 2$, $T_2 = 0.5$, $\tau = 0.4$, $K_m = 1$. (The PID parameter values are calculated using the Ziegler-Nichols closed loop method.) The operation point is at setpoint value 50% and disturbance $d = 10\%$ (constant).

¹How can you find the exact value of the response time? Answer: Simulate!

Figure 23.3 shows simulated responses in the process output measurement y_m and in the control error $e = r - y_m$ when the setpoint r is a sinusoid of amplitude 10 % (about a bias of 50 %) and frequency $\omega_1 = 0.55$ rad/s. The frequency of the sinusoidal is chosen equal to the sensitivity bandwidth ω_s . The amplitude of the control error should be $0.29 \cdot 10\% = 2.9\%$, and this is actually in accordance with the simulation, see Figure 23.3.

Figure 23.4 shows Bode plots of $|S(j\omega)|$, $|T(j\omega)|$ and $|L(j\omega)|$.

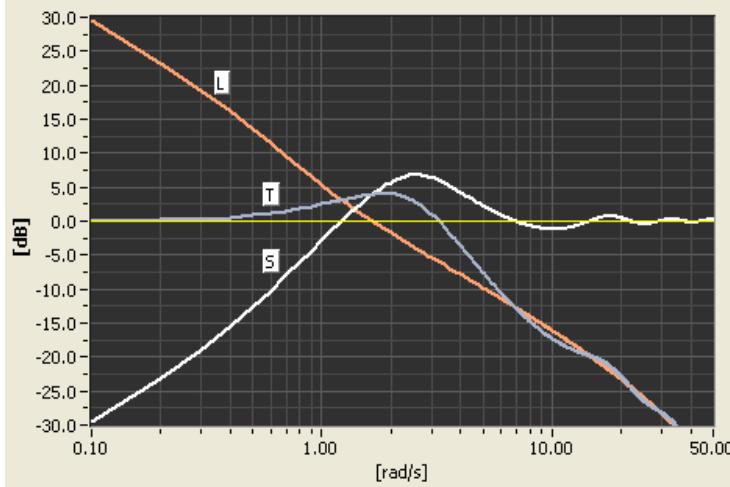


Figure 23.4: Example 23.1: Bode plots of $|L(j\omega)|$, $|T(j\omega)|$ and $|S(j\omega)|$.

Let us compare the various bandwidth definitions. From Figure 23.4 we find

- -3 dB bandwidth: $\omega_t = 3.8$ rad/s
- Crossover frequency: $\omega_c = 1.7$ rad/s
- Sensitivity bandwidth: $\omega_s = 0.55$ rad/s

These values are actually quite different. (As commented in the text above this example, it can be argued that the ω_s bandwidth gives the most expressive measure of the control system dynamics.)

Finally, let us read off the response time T_r . Figure 23.5 shows the response in y_m due to a step in y_{sp} . From the simulation we read off $T_r \approx 1.1$ s. The estimate (23.10) with $k = 2$ gives $T_r \approx 2/\omega_t = 2/3.8 = 0.53$ s, which is about half the value of the real (simulated) value.

[End of Example 23.1]

23.2.3 Frequency response analysis of disturbance compensation

(23.1) gives the response in the control error due to the disturbance, d . It is repeated here:

$$e_d(s) = -S(s)D(s)d(s) \quad (23.15)$$

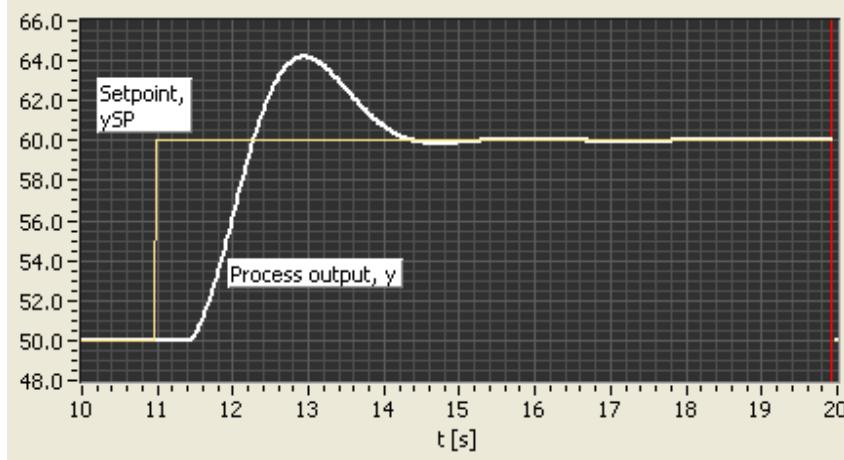


Figure 23.5: Example 23.1: Step response in process output measurement y_m after a step in setpoint r .

Thus, the sensitivity transfer function $S(s)$ is a factor in the transfer function from d til e for the control system. However, $S(s)$ has an additional meaning related to the compensation of a disturbance, namely it expresses the degree of the reduction of the control error due to using closed loop control, as explained below.

With feedback (i.e. closed loop system) the response in the control error due to the disturbance is

$$e_d(s) = -S(s)D(s)d(s) \quad (23.16)$$

Without feedback (open loop) this response is

$$e_d(s) = -D(s)d(s) \quad (23.17)$$

The ratio between these responses is

$$\frac{e_d(s)_{\text{with feedback}}}{e_d(s)_{\text{without feedback}}} = \frac{-S(s)D(s)d(s)}{-D(s)d(s)} = S(s) \quad (23.18)$$

Assuming that the disturbance is sinusoidal with frequency ω rad/s, (23.18) with $s = j\omega$, that is $S(j\omega)$, expresses the ratio between sinusoidal responses.

Again, effective control, which here means effective disturbance compensation, corresponds to a small value of $|S|$ (value zero or close to zero), while ineffective control corresponds to $|S|$ close to or greater than 1. We can define the *bandwidth* of the control system with respect to its compensation property. Here are two alternate bandwidth definitions:

- The bandwidth ω_s – the sensitivity bandwidth – is the upper limit of the frequency range of effective compensation. One possible definition is

$$|S(j\omega_s)| \approx 0.29 \approx -11 \text{ dB} \quad (23.19)$$

which means that the amplitude of the error *with* feedback control is less than 29% of amplitude *without* feedback control. The number 0.29 is chosen to have the same bandwidth definition regarding disturbance compensation as regarding setpoint tracking, cf. page 460.

- The bandwidth ω_c is the crossover frequency of the loop transfer functions ω_c , that is,

$$|L(j\omega_c)| = 0 \text{ dB} \approx 1 \quad (23.20)$$

Note: The feedback does not reduce the control error due to a sinusoidal disturbance if its frequency is above the bandwidth. But still the disturbance may be well attenuated through the (control) system. This attenuation is due to the typical inherent lowpass filtering characteristic of physical systems (processes). Imagine a liquid tank, which attenuates high-frequent temperature variations existing in the inflow fluid temperature or in the environmental temperature. This inherent lowpass filtering is *self regulation*.

Example 23.2 Frequency response analysis of disturbance compensation

This example is based on the control system described in Example 23.1.

Figure 23.6 shows simulated responses in the process output y due to a sinusoidal disturbance v of amplitude 10% (with bias 10%) and frequency $\omega_1 = 0.55\text{rad/s}$.

This frequency is for illustration purpose chosen equal to the sensitivity bandwidth of the control system, cf. Figure 23.4. The setpoint y_{sp} is 50%. The control error can be read off as the difference between y_{sp} and y_m . In the first 40 seconds of the simulation the PID controller is in manual mode, so the control loop is open. In the following 40 seconds the PID controller is in automatic mode, so the control loop is closed. We clearly see that the feedback control is effective to compensate for the disturbance at this frequency (0.55 rad/s). The amplitude of the control error is 6.6 without feedback and 1.9 with feedback. Thus, the ratio between the closed loop error and the open loop error is $1.9/6.6 = 0.29$, which is in accordance with the amplitude of the sensitivity transfer function at this frequency, cf. Figure 23.4.

Figure 23.7 shows the same kind of simulation, but with disturbance frequency $\omega_1 = 1.7\text{ rad/s}$, which is higher than the sensitivity bandwidth, which is 0.55 rad/s.

From the simulations we see that closed loop control at this relatively high frequency, 1.7 rad/s, does not compensate for the disturbance — actually the open loop works better. This is in accordance with the fact that $|S(j\omega)|$ is greater than 1 at $\omega = 1.7\text{ rad/s}$, cf. the Bode plot in Figure 23.4.

Finally, let us compare the simulated responses shown in Figure 23.7 and in Figure 23.3. The amplitude of the control error is less in Figure 23.7, despite the fact that the closed loop or feedback control is not efficient (at frequency 1.7 rad/s). The relatively small amplitude of the control error is due to the self regulation of the process, which means that the disturbance is attenuated through the process, whether the process is controlled or not.

[End of Example 23.2]

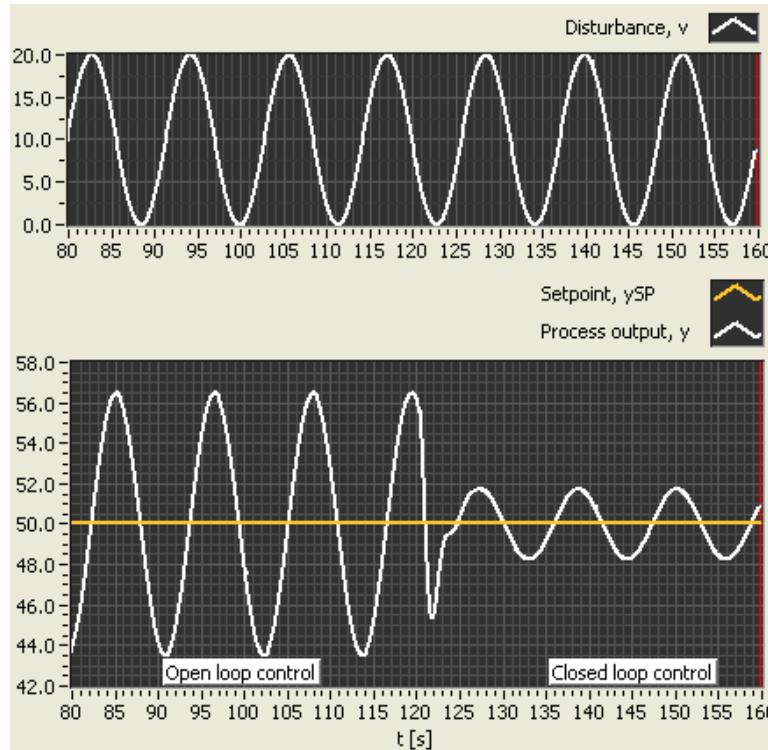


Figure 23.6: Example 23.2: Simulated responses of the control system. The disturbance v is sinusoidal with frequency $\omega_1 = 0.55 \text{ rad/s}$. The PID-controller is in manual mode (i.e. open loop control) the first 40 seconds, and in automatic mode (closed loop control) thereafter.

In Example 23.2 I did not choose the disturbance frequency, 1.7 rad/s , by random. 1.7 rad/s is actually the loop transfer function crossover frequency of the control system. Thus, the example demonstrates that the crossover frequency may give a poor measure of the performance of the control system. The sensitivity bandwidth is a better measure of the performance.

23.3 Stability analysis of feedback systems

23.3.1 Introduction

In Chapter 21 we analyzed the stability property of a feedback control systems in terms of poles of the tracking transfer function. I will now show how to analyze the stability of a control system from the frequency response of the loop transfer function², $L(j\omega)$.

There is an algebraic stability analysis method named Routh's stability criterion which is based on the coefficients of the characteristic polynomial of the control system. I have decided to not present this method since I think it has quite limited practical importance, and the mathematical operations become quite complicated except for simple models. A

²the product of all the transfer functions in the control loop, cf. (14.5)

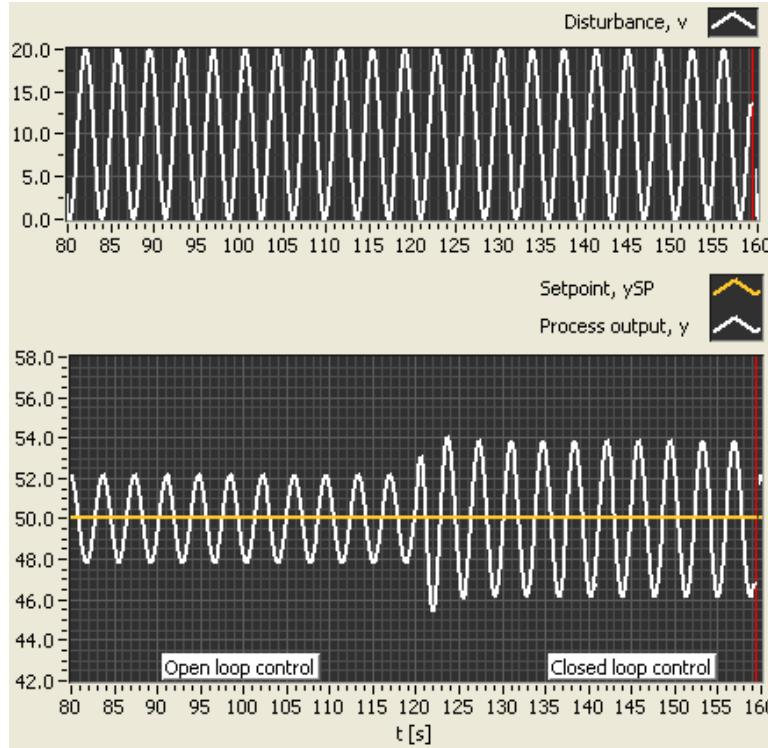


Figure 23.7: Example 23.2: Simulated responses of the control system. The disturbance v is sinusoidal with frequency $\omega_1 = 1.7 \text{ rad/s}$. The PID-controller is in manual mode (i.e. open loop control) the first 40 seconds, and in automatic mode (closed loop control) thereafter.

reference to Routh's stability criterion is e.g. [Seborg et al. \(2004\)](#).

23.3.2 Nyquist's stability criterion

Let us start with a quick review from Section 21.5: The stability of a feedback control system is determined by the placement of the roots of the characteristic polynomial, $a(s)$, in the complex plane. $a(s)$ is:

$$a(s) = d_L(s) + n_L(s) \quad (23.21)$$

where $d_L(s)$ is the denominator polynomial, and $n_L(s)$ is the numerator polynomial of the loop transfer function:

$$L(s) = \frac{d_L(s)}{n_L(s)} \quad (23.22)$$

To continue with deriving the Nyquist's stability criterion, we start with a rewriting. The roots of (23.21) are the same as the roots of:

$$\frac{d_L(s) + n_L(s)}{d_L(s)} = 1 + \frac{n_L(s)}{d_L(s)} = 1 + L(s) = 0 \quad (23.23)$$

Therefore, we can denote also (23.23) as the characteristic equation of the feedback control system. (23.23) is the equation from which the Nyquist's stability criterion will be derived. In the derivation we will use the so-called Argument variation principle:

Argument variation principle: Given a function $f(s)$ where s is a complex number.

Then $f(s)$ is a complex number, too. As with all complex numbers, $f(s)$ has an angle or argument. If s follows a closed contour Γ (gamma) in the complex s -plane which encircles a number of poles and a number of zeros of $f(s)$, see Figure 23.8,

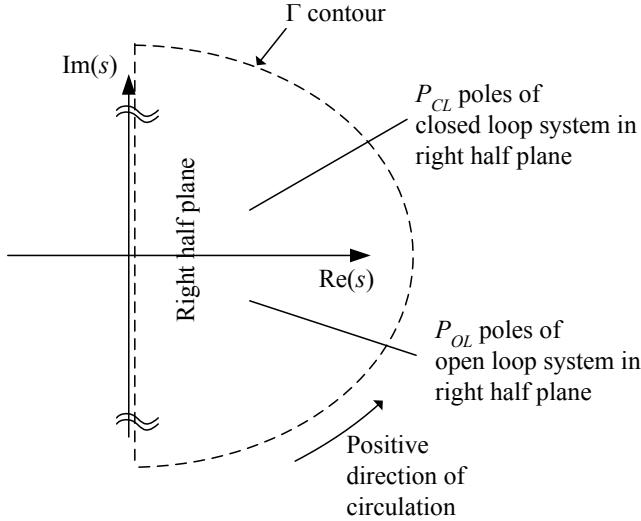


Figure 23.8: s shall follow the Γ contour once in positive direction (counter clockwise).

then the following applies:

$$\arg_{\Gamma} f(s) = 360^\circ \cdot (\text{number of zeros minus number of poles of } f(s) \text{ inside } \Gamma) \quad (23.24)$$

where $\arg_{\Gamma} f(s)$ means the change of the angle of $f(s)$ when s has followed Γ once in positive direction of circulation (i.e. clockwise).

For our purpose, we let the function $f(s)$ in the Argument variation principle be

$$f(s) = 1 + L(s) \quad (23.25)$$

The Γ contour must encircle the entire right half s -plane, so that we are certain that all poles and zeros of $1 + L(s)$ are encircled. From the Argument Variation Principle we have:

$$\arg_{\Gamma} [1 + L(s)] = \arg_{\Gamma} \frac{d_L(s) + n_L(s)}{d_L(s)} \quad (23.26)$$

$$= 360^\circ \cdot (\text{number of roots of } (d_L + n_L) \text{ in RHP} \\ \text{minus number roots of } d_L \text{ in RHP}) \quad (23.27)$$

$$= 360^\circ \cdot (\text{number poles of closed loop system in RHP} \\ \text{minus number poles of open system in RHP}) \\ = 360^\circ \cdot (P_{CL} - P_{OL}) \quad (23.28)$$

where RHP means right half plane. By “open system” we mean the (imaginary) system having transfer function $L(s) = n_L(s)/d_L(s)$, i.e., the original feedback system with the feedback broken. The poles of the open system are the roots of $d_L(s) = 0$.

Finally, we can formulate the Nyquist's stability criterion. But before we do that, we should remind ourselves what we are after, namely to be able to determine the number poles P_{CL} of the closed loop system in RHP. It those poles which determines whether the closed loop system (the control system) is asymptotically stable or not. *If $P_{CL} = 0$ the closed loop system is asymptotically stable.*

Nyquist's stability criterion: Let P_{OL} be the number of poles of the open system in the right half plane, and let $\arg_\Gamma L(s)$ be the angular change of the vector $L(s)$ as s have followed the Γ contour once in positive direction of circulation. Then, the number poles P_{CL} of the closed loop system in the right half plane, is

$$P_{CL} = \frac{\arg_\Gamma L(s)}{360^\circ} + P_{OL} \quad (23.29)$$

If $P_{CL} = 0$, the closed loop system is asymptotically stable.

Let us take a closer look at the terms on the right side of (23.29): P_{OL} are the roots of $d_L(s)$, and there should not be any problem calculating these roots. To determine the angular change of the vector $1 + L(s)$. Figure 23.9 shows how the vector (or complex number) $1 + L(s)$ appears in a *Nyquist diagram* for a typical plot of $L(s)$. A Nyquist diagram is simply a Cartesian diagram of the complex plane in which L is plotted. $1 + L(s)$ is the vector from the point $(-1, 0j)$, which is denoted the *critical point*, to the Nyquist curve of $L(s)$.

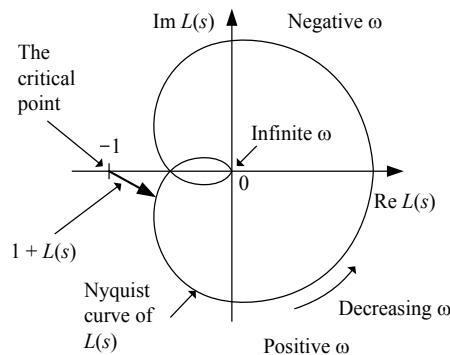


Figure 23.9: Typical Nyquist curve of $L(s)$. The vector $1 + L(s)$ is drawn.

More about the Nyquist curve of $L(j\omega)$ Let us take a more detailed look at the Nyquist curve of L as s follows the Γ contour in the s -plane, see Figure 23.8. In practice, the denominator polynomial of $L(s)$ has higher order than the numerator polynomial. This implies that $L(s)$ is mapped to the origin of the Nyquist diagram when $|s| = \infty$. Thus, the whole semicircular part of the Γ contour is mapped to the origin.

The imaginary axis constitutes the rest of the Γ contour. How is the mapping of $L(s)$ as s runs along the imaginary axis? On the imaginary axis $s = j\omega$, which implies that $L(s) = L(j\omega)$, which is the frequency response of $L(s)$. A consequence of this is that we can

in principle determine the stability property of a feedback system by just looking at the frequency response of the open system, $L(j\omega)$.

ω has negative values when $s = j\omega$ is on the negative imaginary axis. For $\omega < 0$ the frequency response has a mathematical meaning. From general properties of complex functions,

$$|L(-j\omega)| = |L(j\omega)| \quad (23.30)$$

and

$$\angle L(-j\omega) = -\angle L(j\omega) \quad (23.31)$$

Therefore the Nyquist curve of $L(s)$ for $\omega < 0$ will be identical to the Nyquist curve of $\omega > 0$, but mirrored about the real axis. Thus, we only need to know how $L(j\omega)$ is mapped for $\omega \geq 0$. The rest of the Nyquist curve then comes by itself! Actually we need not draw more of the Nyquist curve (for $\omega > 0$) than what is sufficient for determining if the critical point is encircled or not.

We must do some extra considerations if some of the poles in $L(s)$, which are the poles of the open loop system, lie in the origin. This corresponds to pure integrators in control loop, which is a common situation in feedback control systems because the controller usually has integral action, as in a PI or PID controller. If $L(s)$ contains integrators, the Γ contour must go *outside* the origo. But to the left or to the right? We choose to the right, see Figure 23.10.

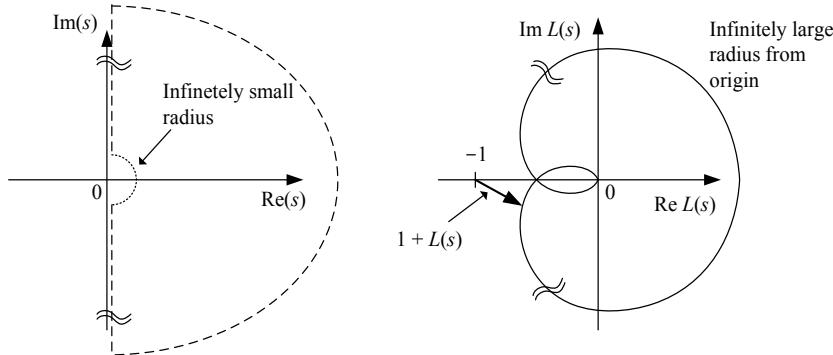


Figure 23.10: Left diagram: If $L(s)$ has a pole in origin, the Γ contour must pass the origin along an arbitrarily small semicircle to the right. Right diagram: A typical Nyquist curve of L .

(We have thereby decided that the origin belongs to the left half plane. This implies that P_{OL} does not count these poles.) The radius of the semicircle around origin is arbitrarily small. The Nyquist curve then becomes as shown in the diagram to the right in the same figure. The arbitrarily small semicircle in the s -plane is mapped to an infinitely large semicircle in the L -plane. This is because as $s \rightarrow 0$, the loop transfer function is approximately

$$L(s) \approx \frac{K}{s}$$

(if we assume one pole in the origin). On the small semicircle,

$$s = re^{j\theta} \quad (23.32)$$

which gives

$$L(s) \approx \frac{K}{r} e^{-j\theta} \quad (23.33)$$

When $r \rightarrow 0$ and when simultaneously θ goes from $+90^\circ$ via 0° to -90° , the Nyquist curve becomes an infinitely large semicircle, as shown.

The Nyquist's stability criterion for non-rational transfer functions The Nyquist's stability criterion gives information about the poles of feedback systems. So far it has been assumed that the loop transfer function $L(s)$ is a rational transfer function. What if $L(s)$ is irrational? Here is one example:

$$L(s) = \frac{1}{s} e^{-\tau s} \quad (23.34)$$

where $e^{-\tau s}$ represents time delay. In such cases the tracking ratio $T(s)$ will also be irrational, and the definition of poles does not apply to such irrational transfer functions. Actually, the Nyquist's stability criterion can be used as a graphical method for determining the stability property on basis of the frequency response $L(j\omega)$.

Nyquist's special stability criterion In most cases the open system is stable, that is, $P_{OL} = 0$. (23.29) then becomes

$$P_{CL} = \frac{\arg_\Gamma[L(s)]}{360^\circ} \quad (23.35)$$

This implies that the feedback system is asymptotically stable if the Nyquist curve does not encircle the critical point. This is the *Nyquist's special stability criterion* or the *Nyquist's stability criterion for open stable systems*.

The Nyquist's special stability criterion can also be formulated as follows: *The feedback system is asymptotically stable if the Nyquist curve of L has the critical point on its left side for increasing ω .*

Another way to formulate Nyquist's special stability criterion involves the *amplitude crossover frequency* ω_c and the *phase crossover frequency* ω_{180} . ω_c is the frequency at which the $L(j\omega)$ curve crosses the unit circle, while ω_{180} is the frequency at which the $L(j\omega)$ curve crosses the negative real axis. In other words:

$$|L(j\omega_c)| = 1 \quad (23.36)$$

and

$$\arg L(j\omega_{180}) = -180^\circ \quad (23.37)$$

See Figure 23.11. Note: The Nyquist diagram contains no explicit frequency axis.

We can now determine the stability properties from the relation between these two crossover frequencies:

- Asymptotically stable closed loop system: $\omega_c < \omega_{180}$
- Marginally stable closed loop system: $\omega_c = \omega_{180}$
- Unstable closed loop system: $\omega_c > \omega_{180}$

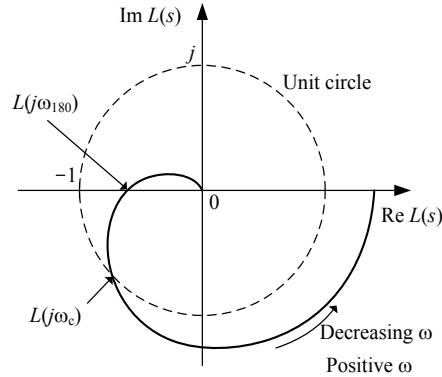


Figure 23.11: Definition of amplitude crossover frequency ω_c and phase crossover frequency ω_{180} .

The frequency of the sustained oscillations There are sustained oscillations in a marginally stable system. *The frequency of these oscillations is $\omega_c = \omega_{180}$.* This can be explained as follows: In a marginally stable system, $L(\pm j\omega_{180}) = L(\pm j\omega_c) = -1$. Therefore, $d_L(\pm j\omega_{180}) + n_L(\pm j\omega_{180}) = 0$, which is the characteristic equation of the closed loop system with $\pm j\omega_{180}$ inserted for s . Therefore, the system has $\pm j\omega_{180}$ among its poles. The system usually have additional poles, but they lie in the left half plane. The poles $\pm j\omega_{180}$ leads to sustained sinusoidal oscillations. Thus, ω_{180} (or ω_c) is the frequency of the sustained oscillations in a marginally stable system.

23.3.3 Stability margins

23.3.3.1 Stability margins in terms of gain margin and phase margin

An asymptotically stable feedback system may become marginally stable if the loop transfer function changes. The *gain margin* GM and the *phase margin* PM [radians or degrees] are *stability margins* which in their own ways expresses how large parameter changes can be tolerated before an asymptotically stable system becomes marginally stable. Figure 23.12 shows the stability margins defined in the Nyquist diagram.

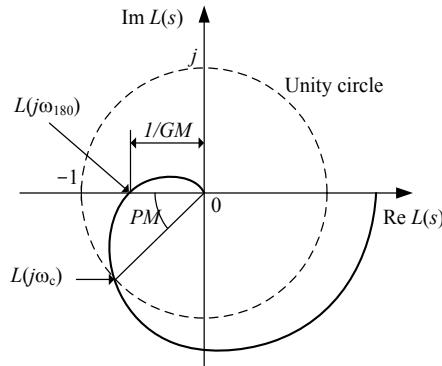


Figure 23.12: Gain margin GM and phase margin PM defined in the Nyquist diagram.

GM is the (multiplicative, not additive) increase of the gain that L can tolerate at ω_{180} before the L curve (in the Nyquist diagram) passes through the critical point. Thus,

$$|L(j\omega_{180})| \cdot GM = 1 \quad (23.38)$$

which gives

$$GM = \frac{1}{|L(j\omega_{180})|} = \frac{1}{|\operatorname{Re} L(j\omega_{180})|} \quad (23.39)$$

(The latter expression in (23.39) is because at ω_{180} , $\operatorname{Im} L = 0$ so that the amplitude is equal to the absolute value of the real part.)

If we use decibel as the unit (like in the Bode diagram which we will soon encounter), then

$$GM [\text{dB}] = -|L(j\omega_{180})| [\text{dB}] \quad (23.40)$$

The phase margin PM is the phase reduction that the L curve can tolerate at ω_c before the L curve passes through the critical point. Thus,

$$\arg L(j\omega_c) - PM = -180^\circ \quad (23.41)$$

which gives

$$PM = 180^\circ + \arg L(j\omega_c) \quad (23.42)$$

We can now state as follows: The feedback (closed) system is asymptotically stable if

$$GM > 0 \text{ dB} = 1 \text{ and } PM > 0^\circ \quad (23.43)$$

This criterion is often denoted the *Bode-Nyquist stability criterion*.

Reasonable ranges of the stability margins are

$$2 \approx 6 \text{ dB} \leq GM \leq 4 \approx 12 \text{ dB} \quad (23.44)$$

and

$$30^\circ \leq PM \leq 60^\circ \quad (23.45)$$

The larger values, the better stability, but at the same time the system becomes more sluggish, dynamically. If you are to use the stability margins as design criterias, you can use the following values (unless you have reasons for specifying other values):

$$GM \geq 2.5 \approx 8 \text{ dB} \text{ and } PM \geq 45^\circ \quad (23.46)$$

For example, the controller gain, K_c , can be adjusted until one of the inequalities becomes an equality.³

It can be shown⁴ that for $PM \leq 70^\circ$, the damping of the feedback system approximately corresponds to that of a second order system with relative damping factor

$$\zeta \approx \frac{PM}{100^\circ} \quad (23.47)$$

For example, $PM = 50^\circ \sim \zeta = 0.5$.

³But you should definitely check the behaviour of the control system by simulation, if possible.

⁴The result is based on the assumption that the loop transfer function is $L(s) = \omega_0^2 / [(s + 2\zeta\omega_0)s]$ which gives tracking transfer function $T(s) = L(s) / [1 + L(s)] = \omega_0^2 / [s^2 + 2\zeta\omega_0 s + \omega_0^2]$. The phase margin PM can be calculated from $L(s)$.

23.3.3.2 Stability margins in terms of maximum sensitivity amplitude

An alternative quantity of a stability margin is the minimum distance from the $L(j\omega)$ curve to the critical point. This distance is $|1 + L(j\omega)|$, see Figure 23.13.

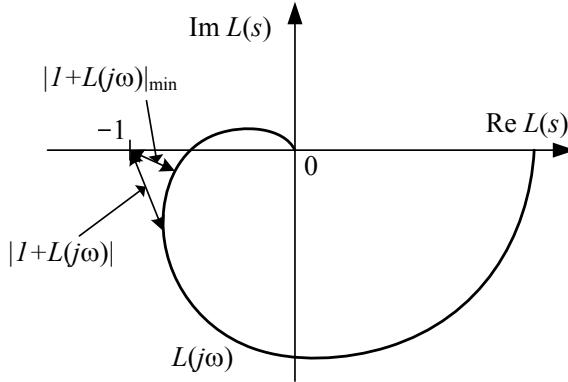


Figure 23.13: The distance between the $L(j\omega)$ curve and the critical point is $|1 + L|$. The minimum of this distance is related to the stability margin.

We can use the minimal value of $|1 + L(j\omega)|$ as a stability margin. However, it is more common to take the inverse of the distance: Thus, a stability margin is the *maximum* value of $1/|1 + L(j\omega)|$. And since $1/[1 + L(s)]$ is the sensitivity transfer function $S(s)$, then $|S(j\omega)|_{\max}$ represents a stability margin. Reasonable values are in the range

$$1.5 \approx 3.5 \text{ dB} \leq |S(j\omega)|_{\max} \leq 3.0 \approx 9.5 \text{ dB} \quad (23.48)$$

If you use $|S(j\omega)|_{\max}$ as a criterion for adjusting controller parameters, you can use the following criterion (unless you have reasons for some other specification):

$$|S(j\omega)|_{\max} = 2.0 \approx 6 \text{ dB} \quad (23.49)$$

23.3.4 Stability analysis in a Bode diagram

It is most common to use a Bode diagram for frequency response based stability analysis of closed loop systems. The Nyquist's Stability Criterion says: The closed loop system is marginally stable if the Nyquist curve (of L) goes through the critical point, which is the point $(-1, 0)$. But where is the critical point in the Bode diagram? The critical point has phase (angle) -180° and amplitude $1 = 0\text{dB}$. The critical point therefore constitutes two lines in a Bode diagram: The 0dB line in the amplitude diagram and the -180° line in the phase diagram. Figure 23.14 shows typical L curves for an asymptotically stable closed loop system. In the figure, GM, PM, ω_c and ω_{180} are indicated.

Example 23.3 Stability analysis of a feedback control system

Given a feedback control system with structure as shown in Figure 23.15.

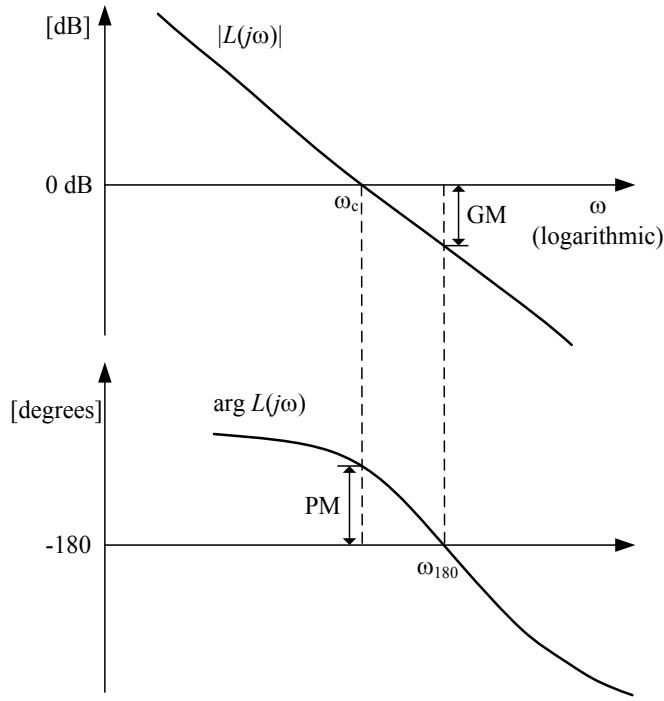


Figure 23.14: Typical L curves of an asymptotically stable closed loop system with GM, PM, ω_c and ω_{180} indicated.

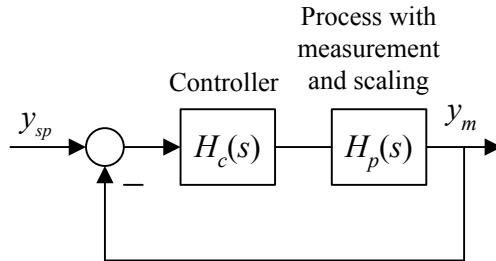


Figure 23.15: Example 23.3: Block diagram of feedback control system.

The loop transfer function is

$$L(s) = H_c(s)H_p(s) = \underbrace{K_c}_{H_c(s)} \underbrace{\frac{1}{(s+1)^2 s}}_{H_p(s)} = \frac{K_c}{(s+1)^2 s} = \frac{n_L(s)}{d_L(s)} \quad (23.50)$$

We will determine the stability property of the control system for different values of the controller gain K_c in three ways: Pole placement, Nyquist's Stability Criterion, and simulation. The tracking transfer function is

$$T(s) = \frac{y_m(s)}{r(s)} = \frac{L(s)}{1 + L(s)} = \frac{n_L(s)}{d_L(s) + n_L(s)} = \frac{K_c}{s^3 + 2s^2 + s + K_c} \quad (23.51)$$

The characteristic polynomial is

$$a(s) = s^3 + 2s^2 + s + K_c \quad (23.52)$$

Figures 23.16 – 23.18 show the step response after a step in the setpoint, the poles, the Bode diagram and Nyquist diagram for three K_c values which result in different stability properties. The detailed results are shown below.

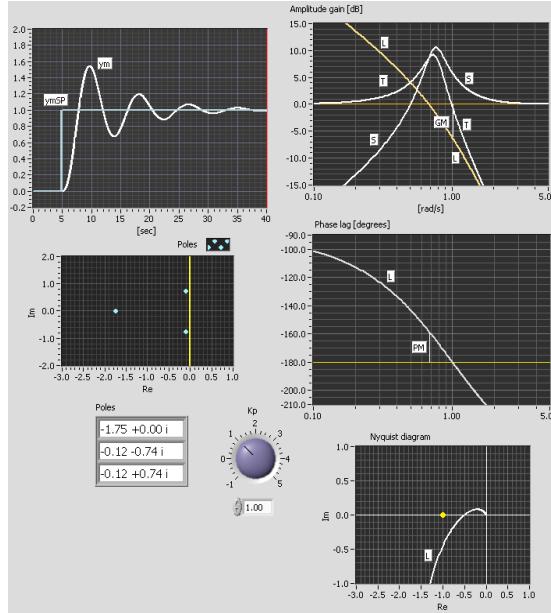


Figure 23.16: Example 23.3: Step response (step in setpoint), poles, Bode diagram and Nyquist diagram with $K_c = 1$. The control system is asymptotically stable.

- $K_c = 1$: Asymptotically stable system, see Figure 23.16. From the Bode diagram we read off stability margins $GM = 6.0\text{dB} = 2.0$ and $PM = 21^\circ$. we see also that $|S(j\omega)|_{\max} = 11 \text{ dB} = 3.5$ (a large value, but it corresponds with the small the phase margin of $PM = 20^\circ$).
- $K_c = 2$: Marginally stable system, see Figure 23.17. From the Bode diagram, $\omega_c = \omega_{180}$. The L curve goes through the critical point in the Nyquist diagram. $|S|_{\max}$ has infinitely large value (since the minimum distance, $1/|S|_{\max}$, between $|L|$ and the critical point is zero).
Let us calculate the period T_p of the undamped oscillations: Since $\omega_{180} = 1.0\text{rad/s}$, the period is $T_p = 2\pi/\omega_{180} = 6.28\text{s}$, which fits well with the simulation shown in Figure 23.17.
- $K_c = 4$: Unstable system, see Figure 23.18. From the Bode diagram, $\omega_c > \omega_{180}$. From the Nyquist diagram we see that the L curve passes outside the critical point. (The frequency response curves of T and S have no physical meaning in this the case.)

[End of Example 23.3]

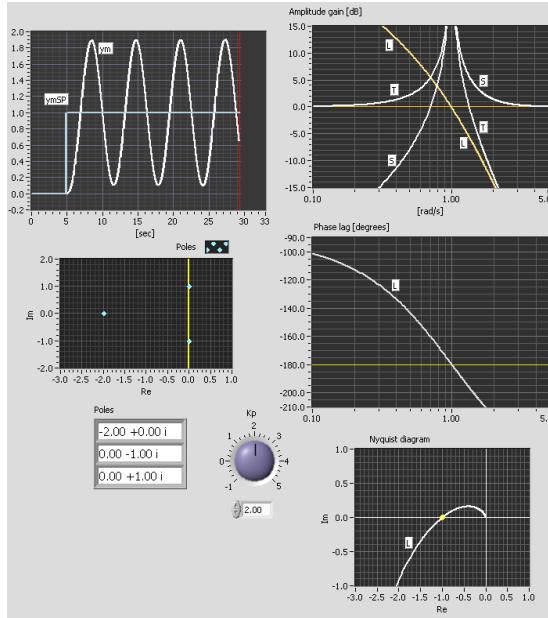


Figure 23.17: Example 23.3: Step response (step in setpoint), poles, Bode diagram and Nyquist diagram with $K_c = 2$. The control system is marginally stable.

23.3.5 Robustness in term of stability margins

Per definition the stability margins expresses the robustness of the feedback control system against certain parameter changes in the loop transfer function:

- *The gain margin, GM*, is how much the loop gain, K , can increase before the system becomes unstable. For example, is $GM = 2$ when $K = 1.5$, the control system becomes unstable for K larger than $1.5 \cdot 2 = 3.0$.
- *The phase margin, PM*, is how much the phase lag function of the loop can be reduced before the loop becomes unstable. One reason of reduced phase is that the time delay in control loop is increased. A change of the time delay by $\Delta\tau$ introduces the factor $e^{-\Delta\tau s}$ in $L(s)$ and contributes to $\arg L$ with $-\Delta\tau \cdot \omega$ [rad] or $-\Delta\tau \cdot \omega \frac{180^\circ}{\pi}$ [deg]. $|L|$ is however not influenced because the amplitude function of $e^{-\tau s}$ is 1, independent of the value of τ . The system becomes unstable if the time delay have increased by $\Delta\tau_{\max}$ such that⁵

$$PM = \Delta\tau_{\max} \cdot \omega_c \frac{180^\circ}{\pi} \text{ [deg]} \quad (23.53)$$

which gives the following maximum change of the time delay:

$$\Delta\tau_{\max} = \frac{PM}{\omega_c} \frac{\pi}{180^\circ} \quad (23.54)$$

If you want to calculate how much the phase margin PM is reduced if the time delay is increased by $\Delta\tau$, you can use the following formula which stems from (23.53):

$$\Delta PM = \Delta\tau \cdot \omega_c \frac{180^\circ}{\pi} \text{ [deg]} \quad (23.55)$$

⁵Remember that PM is found at ω_c .

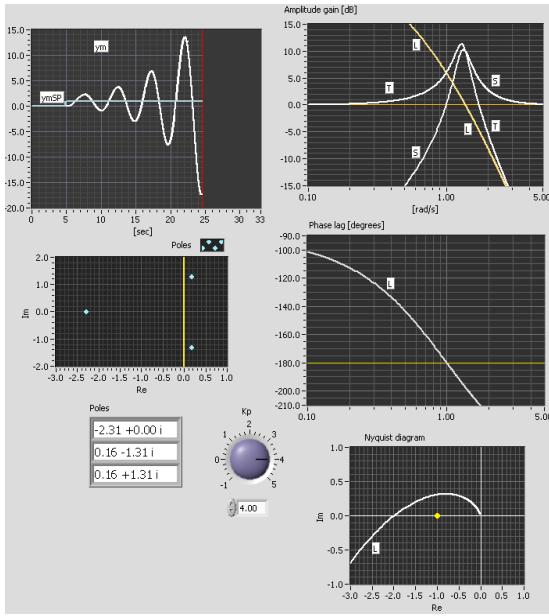


Figure 23.18: Example 23.3: Step response (step in setpoint), poles, Bode diagram and Nyquist diagram with $K_c = 4$. The control system is unstable.

For example, assume that a given control system has $\omega_c = 0.2\text{rad/min}$ and $\text{PM} = 50^\circ$. If the time delay increases by 1min, the phase margin is reduced by $\Delta\text{PM} = 1 \cdot 0.2 \frac{180^\circ}{\pi} = 11.4^\circ$, i.e. from 50° to 38.6° .

23.4 Problems for Chapter 23

Problem 23.1 Frequency response of control system

Figure 23.19 shows the amplitude gain curves of the loop transfer function L , the tracking transfer function T and the sensitivity transfer function S of a feedback control system.

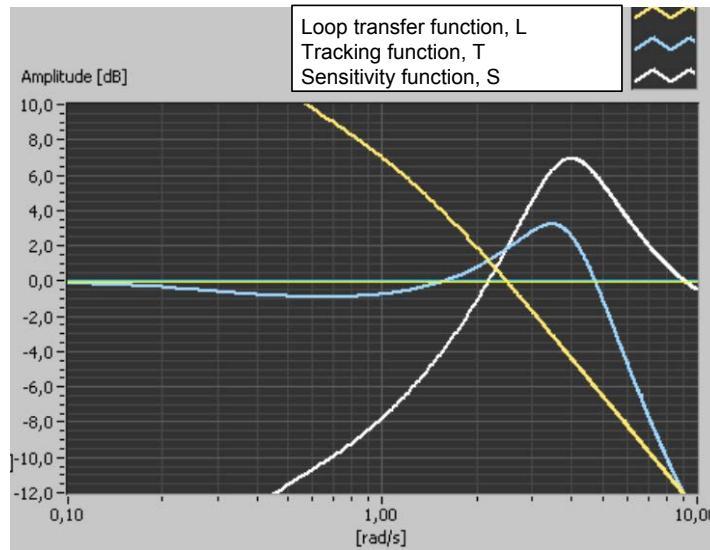


Figure 23.19: Amplitude gain curves of the loop transfer function L , the tracking transfer function T and the sensitivity transfer function S of a feedback control system.

1. Read off from the frequency response curves the following three alternative bandwidths:
 - The crossover frequency ω_c .
 - The -3 dB frequency ω_t of the tracking transfer function
 - The -11 dB frequency ω_s of the sensitivity transfer function
2. Assume that the setpoint is a sinusoid of amplitude $A_r = 4$ and frequency 1 rad/s. What is the amplitude, A_y , of the steady-state sinusoidal process output variable? What is the amplitude, A_e , of the steady-state sinusoidal control error?
3. Assume that the process disturbance is a sinusoid of frequency 1 rad/s. Assume that this disturbance creates a steady-state sinusoidal response in the process output variable of amplitude $A_{y_{OL}} 0.5$ when the process is controlled with a constant control system, i.e. in open loop control. What is the amplitude, $A_{y_{CL}}$, of the process output variable using feedback control, i.e. in closed loop control?
4. Estimate the response time, T_r , of the response on the process output variable due to a step change of the setpoint.

Problem 23.2 Frequency response of temperature control system

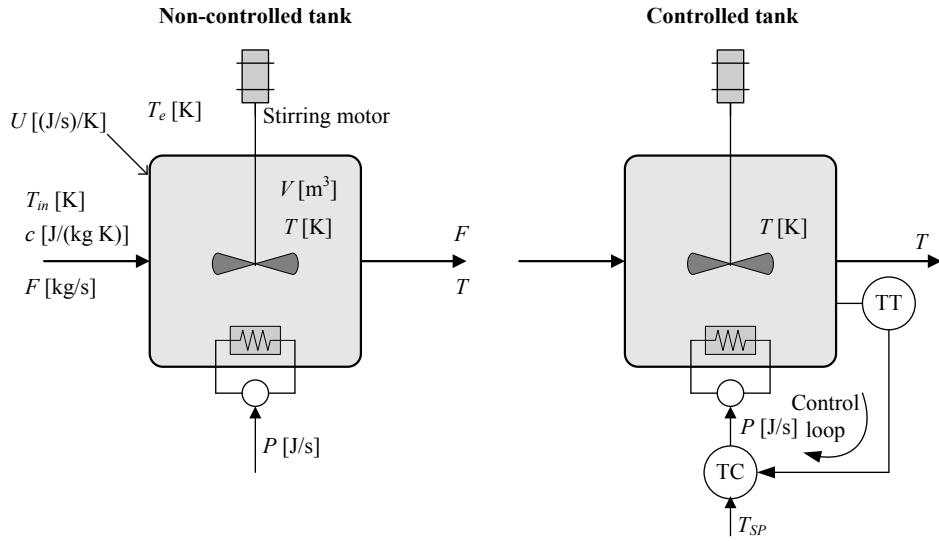


Figure 23.20: To the left: A non-controlled thermal process which is a liquid tank with throughput and heating.

1. The diagram to the left of Figure 23.20 shows a non-controlled thermal process which is a liquid tank with throughput and heating.

Assume that the amplitude gain of the frequency response of the transfer function from inlet temperature T_{in} to outlet temperature T is as shown in the Bode diagram in Figure 23.21.

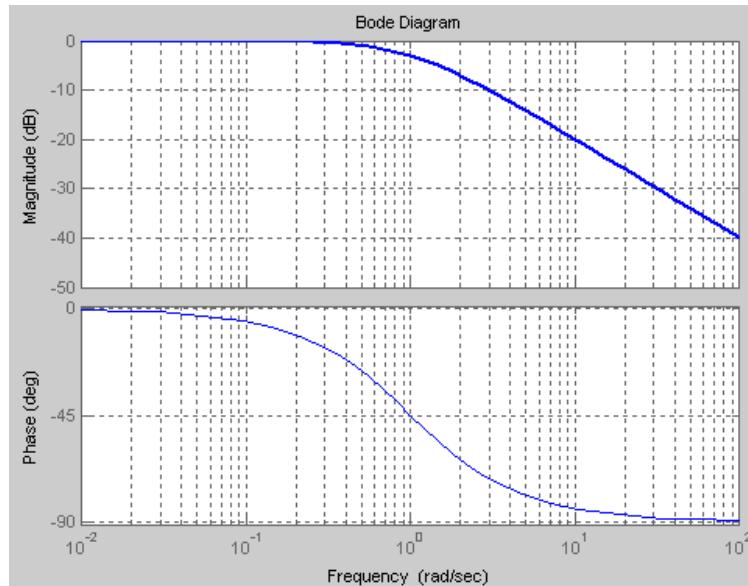


Figure 23.21: Frequency response of open-loop system from T_{in} to T .

Assume that T_{in} contains a frequency component of amplitude $A_{T_{in}}$ of frequency 0.1 rad/s. Calculate the amplitude A_T of the corresponding steady-state response in T .

2. The diagram to the right of Figure 23.20 shows a temperature control system of the

process. Assume that the amplitude gain of the frequency response of the transfer function from inlet temperature T_{in} to outlet temperature T of the control system is as shown in the Bode diagram in Figure 23.22.

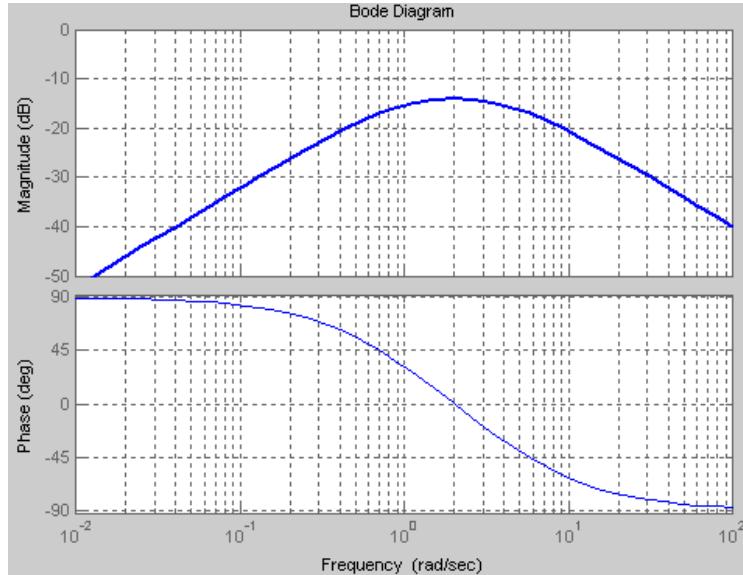


Figure 23.22: Frequency response of closed loop (feedback control) system from T_{in} to T .

Assume that that T_{in} contains a frequency component of amplitude $A_{T_{in}}$ of frequency 0.1 rad/s. What is the amplitude A_T of the corresponding steady-state response in T ? Compare the answer with problem 1 above. Is there any improvement by using control?

Problem 23.3 Stability analysis for various gains in Nyquist diagram

Given a control system with loop transfer function

$$L(s) = \frac{K_c}{(s+1)^3 s} \quad (23.56)$$

Figure 23.23 shows the Nyquist curve of L with $K_c = 0.4$.

(The curve actually encircles the whole right half plane.)

Use Nyquist's stability criterion to calculate the values of K_c that makes the control system become

- Asymptotically stable.
- Marginally stable.
- Unstable. In this case, what is the number of poles in the right half plane?

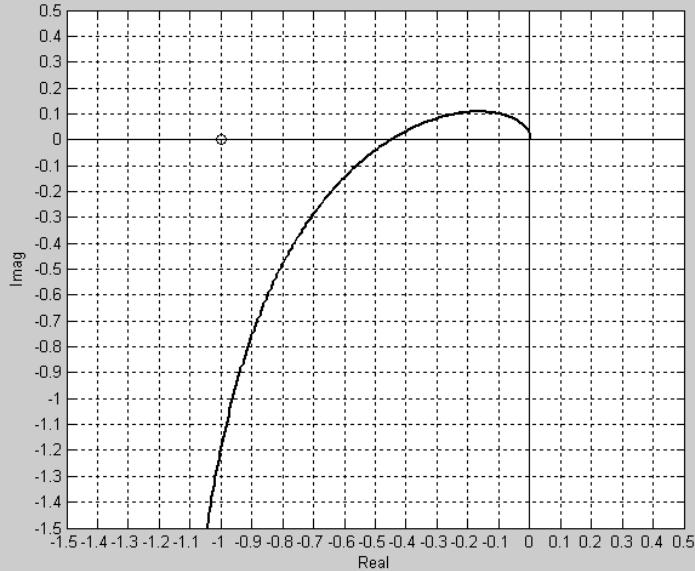


Figure 23.23: Nyquist curve of L with $K_c = 0.4$

Problem 23.4 *Design of stable control system in Nyquist diagram*

Given a closed loop system having the following loop transfer function:

$$L(s) = \frac{K}{s-1} \quad (23.57)$$

1. Show that the corresponding open loop system is unstable by calculating the pole of the system.
2. Figure 23.24 shows the Nyquist curve of L with $K = 2$. Find using the Nyquist stability criterion for which values of K the closed loop system is asymptotically stable. Confirm the answer by calculating the pole of the closed loop system.

Problem 23.5 *Nyquist curve for conditionally stable system*

Figure 23.25 shows the Nyquist curve of $L(j\omega)$ of a feedback system which is open stable.

The loop gain is then $K = 1$. For which values of K is the feedback system asymptotically stable?

Problem 23.6 *Gain margin and sensitivity from Nyquist diagram*

Figure 23.26 shows the Nyquist curve of the loop transfer function L of an asymptotically stable control system.

What is the gain margin GM and the maximum sensitivity gain $|S(j\omega)|_{\max}$?

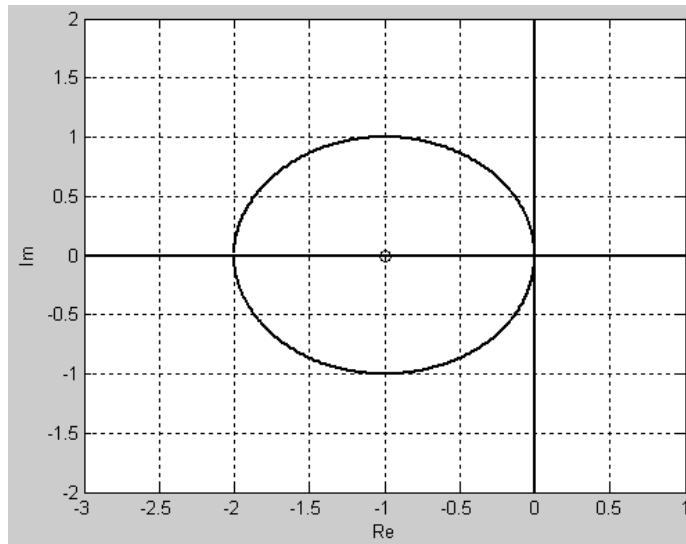
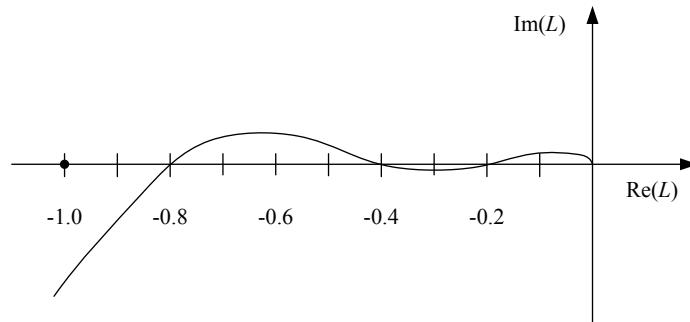

 Figure 23.24: Nyquist curve of L with $K = 2$.


Figure 23.25: Nyquist curve

Problem 23.7 *Stability margin in terms of sensitivity*

Figure 23.27 shows the amplitude gain curves of the loop transfer function L , the tracking transfer function T and the sensitivity transfer function S of a feedback control system.

Determine the stability margin in terms of $|S(j\omega)|_{\max}$. Is the value in the range of reasonable values of this stability margin?

Problem 23.8 *Stability margins in Bode diagram*

Figure 23.28 shows the Bode diagram of the loop transfer function of a given control system.

1. Read off the stability margins GM and PM, and the crossover frequencies ω_c and ω_{180} in the Bode diagram.
2. How large increase of the loop gain will bring the system to the stability limit? What is the period T_p of the steady-state oscillations existing in the system when the system is at the stability limit.

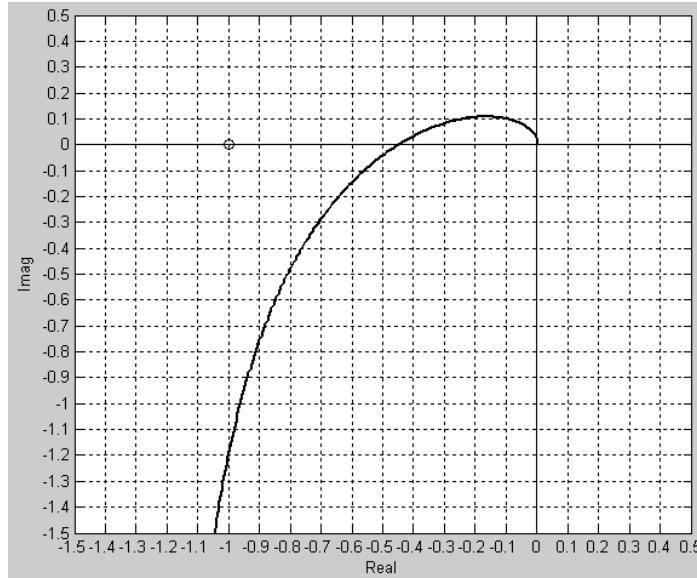


Figure 23.26: Nyquist curve.

Problem 23.9 Marginal stability at increased time delay

Given a feedback control system with time delay $\tau = 4.2$ min. The control system has phase margin

$$\text{PM} = 45^\circ \quad (23.58)$$

and crossover frequency

$$\omega_c = 0.2 \text{ rad/min} \quad (23.59)$$

Assume that the time delay increases, but the controller parameters are not changed. With which value of the time delay τ is the control system marginally stable?

Problem 23.10 When Ziegler-Nichols PID tuning method can not be used

Ziegler-Nichols closed-loop method is based on bringing the closed loop to marginal stability with a P controller with a proper controller gain value. Explain in terms of frequency response why the Ziegler-Nichols closed-loop method can not be used for tuning a PID controller for the following processes. It is assumed that the parameters of the transfer functions have positive values.

$$H_1(s) = \frac{K}{s} \text{ (integrator)} \quad (23.60)$$

$$H_2(s) = \frac{K}{Ts + 1} \text{ (1. order system)} \quad (23.61)$$

$$H_3(s) = \frac{K\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} \text{ (2. order system)} \quad (23.62)$$

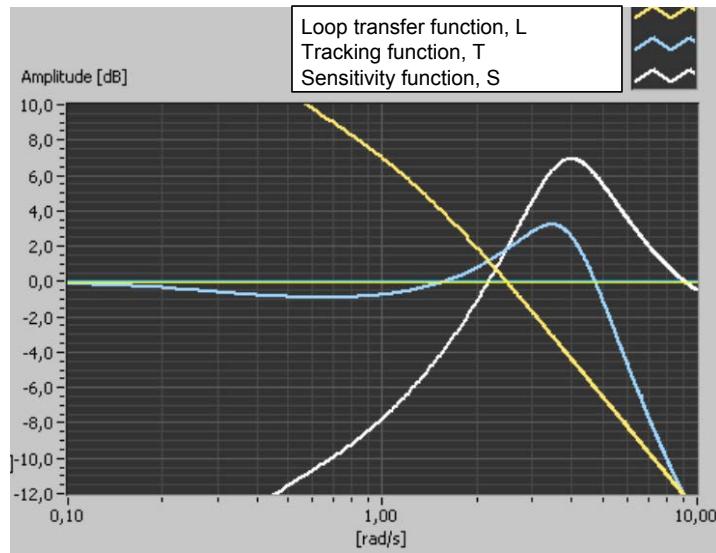


Figure 23.27: Amplitude gain curves

23.5 Solutions to problems for Chapter 23

Solution to Problem 23.1

- From Figure 23.19 we read off:

$$\underline{\omega_c = 2.5 \text{ rad/s}} \quad (23.63)$$

$$\underline{\omega_t = 5.5 \text{ rad/s}} \quad (23.64)$$

$$\underline{\omega_s = 0.55 \text{ rad/s}} \quad (23.65)$$

(Hence, there is quite large difference between the different bandwidths in this example.)

- The calculation from dB is made with the formula $x = 10^{x[\text{dB}]/20}$.

$$\underline{\underline{A_{y_m}}} = \overbrace{|T(j1\text{rad/s})|}^{=-0.8\text{dB}=0.91} A_r = \underline{\underline{0.91A_r}} \quad (23.66)$$

$$\underline{\underline{A_e}} = \overbrace{|S(j1\text{rad/s})|}^{=-8\text{dB}=0.40} A_r = \underline{\underline{0.40A_r}} \quad (23.67)$$

-

$$\underline{\underline{A_{y_{CL}}}} = \overbrace{|S(j1\text{rad/s})|}^{=-8\text{dB}=0.40} A_{y_{OL}} = 0.40 \cdot 0.5 = \underline{\underline{0.2}} \quad (23.68)$$

-

$$\underline{\underline{T_r}} \approx \frac{2}{\omega_t} = \frac{2}{5.5} = \underline{\underline{0.36 \text{ s}}} \quad (23.69)$$

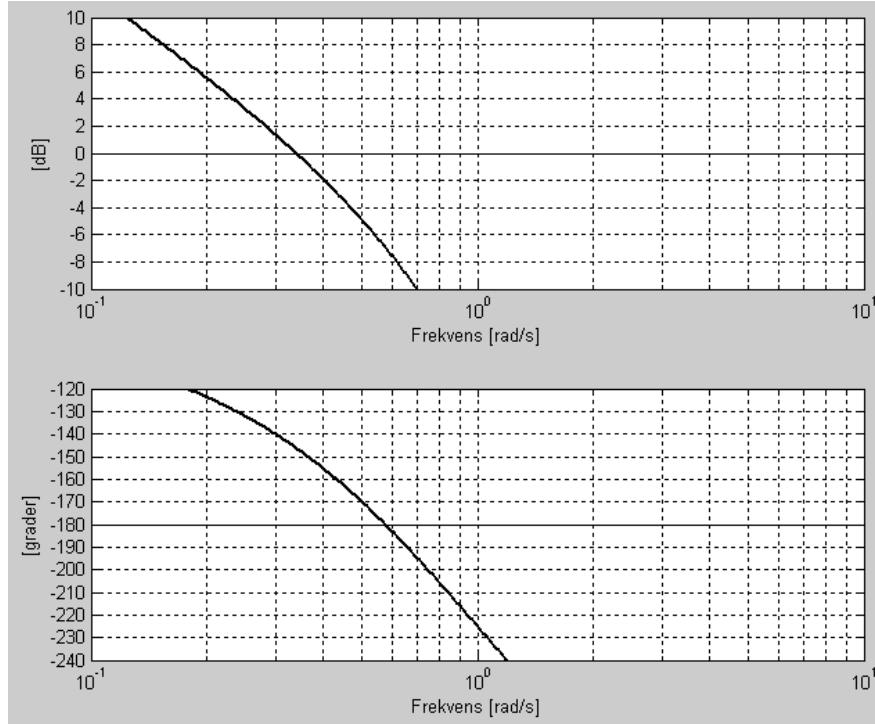


Figure 23.28: Bode diagram

Solution to Problem 23.2

1. In Figure 23.21, which applies to the non-controlled system, we read off that the amplitude gain at frequency 0.1 rad/s is

$$\underline{G_{NC}(0.1) \approx 0 \text{ dB} = 1} \quad (23.70)$$

2. In Figure 23.22, which applies to the controlled system, we read off that the amplitude gain at frequency 0.1 rad/s is

$$\underline{G_C(0.1) \approx -32 \text{ dB} = 0.025} \quad (23.71)$$

Hence, with control the response in T due to T_{in} is about 40 times less than the response in the non-controlled system.

Solution to Problem 23.3

We start by determining the stability property with $K_c = 0.4$. The number of right half plane poles of the control system is

$$P_{CL} = \frac{\arg[1 + L(s)]}{360^\circ} + P_{OL} \quad (23.72)$$

To determine P_{CL} we need to know $\arg L$ and P_{OL} . We have

$$L(s) = \frac{K_c}{(s + 1)^3 s} \quad (23.73)$$

which does not have any poles in the right half plane (the pole in the origin belongs to the left half plane in this context). Hence,

$$P_{OL} = 0 \quad (23.74)$$

$\arg L$ is found from the Nyquist diagram shown in Figure 23.23. $L(j\omega)$ does not encircle the critical point, and therefore the vector L has zero net change of angle, and hence $\arg L = 0$. So, (23.72) becomes

$$P_{CL} = \frac{0}{360^\circ} + 0 = 0 \quad (23.75)$$

Consequently, the control system is asymptotically stable with $K_c = 0.4$.

From Figure 23.23 we see that the L curve with $K_c = 0.4$ passes through the negative real axis at -0.45 . This implies that if K_c is increased by a factor of $1/0.45 = 2.22$, or in other words: if K_c is increased from 0.4 to $0.4 \cdot 2.22 = 0.89$, the L curve will pass through the critical point.

Consequently, the control system is marginally stable with $K_p = 0.89$.

From the results above we can conclude that the control system is asymptotically stable with (positive) $K_p < 0.89$.

If $K_c > 0.89$, the L curve encircles the critical point and $\arg L = 720^\circ$, giving

$$P_{CL} = \frac{720^\circ}{360^\circ} + 0 = 2 \quad (23.76)$$

Therefore, the control system is unstable with $K_p > 0.89$, and it has two poles in the right half plane.

The control system is unstable also with $K_p < 0$. In this case $\arg L = 360^\circ$, and $P_{CL} = 1$, and the control system has one pole in the right half plane.

Solution to Problem 23.4

1. The open loop system is unstable because $L(s)$ has a pole in the right half plane.
(The pole is $p = 1$.)
2. K must be halved to make the L curve pass through the critical point:

$$K_{critical} = \frac{2}{2} = 1 \quad (23.77)$$

Since the open loop system has one pole in the right half plane, $P_{OL} = 1$ in the Nyquist criterion. To make $P_{CL} = 0$ (asymptotically stable closed loop system) $\arg L$ must be 360° , which implies that the critical point must be encircled once. This is achieved with

$$\underline{\underline{K > K_{critical} = 1}} \quad (23.78)$$

The pole of the closed loop is

$$p = 1 - K \quad (23.79)$$

This pole is in the left half plane with $K > 1$, which confirms the result of the Nyquist stability criterion above.

Solution to Problem 23.5

The Nyquist curve passes through the critical point with the following values of K :

$$K = 1/0.8 = 1.25 \quad (23.80)$$

$$K = 1/0.4 = 2.5 \quad (23.81)$$

$$K = 1/0.2 = 5 \quad (23.82)$$

The control system is asymptotically stable with

$$\underline{0 < K < 1.25 \text{ and } 2.5 < K < 5} \quad (23.83)$$

Solution to Problem 23.6

See Figure 23.29.

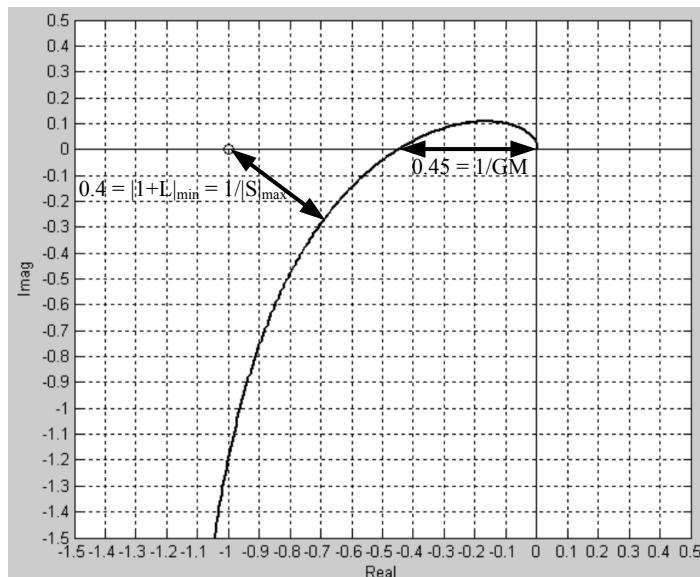


Figure 23.29: Nyquist curve.

As indicated in the figure,

$$0.4 = |1 + L|_{\min} = \frac{1}{|S|_{\max}} \quad (23.84)$$

which gives

$$\underline{|S|_{\max}} = \frac{1}{0.4} = \underline{2.5 = 8.0 \text{ dB}} \quad (23.85)$$

Furthermore,

$$\frac{1}{GM} = 0.45 \quad (23.86)$$

which gives

$$\underline{GM} = \frac{1}{0.45} = \underline{2.22 = 6.9 \text{ dB}} \quad (23.87)$$

Solution to Problem 23.7

From Figure 23.27 we read off

$$|S(j\omega)|_{\max} = 7 \text{ dB} \quad (23.88)$$

which is in the “reasonable” range of

$$3.5 \text{ dB} \leq |S(j\omega)|_{\max} \leq 9.5 \text{ dB} \quad (23.89)$$

Solution to Problem 23.8

1. See Figure 23.30.

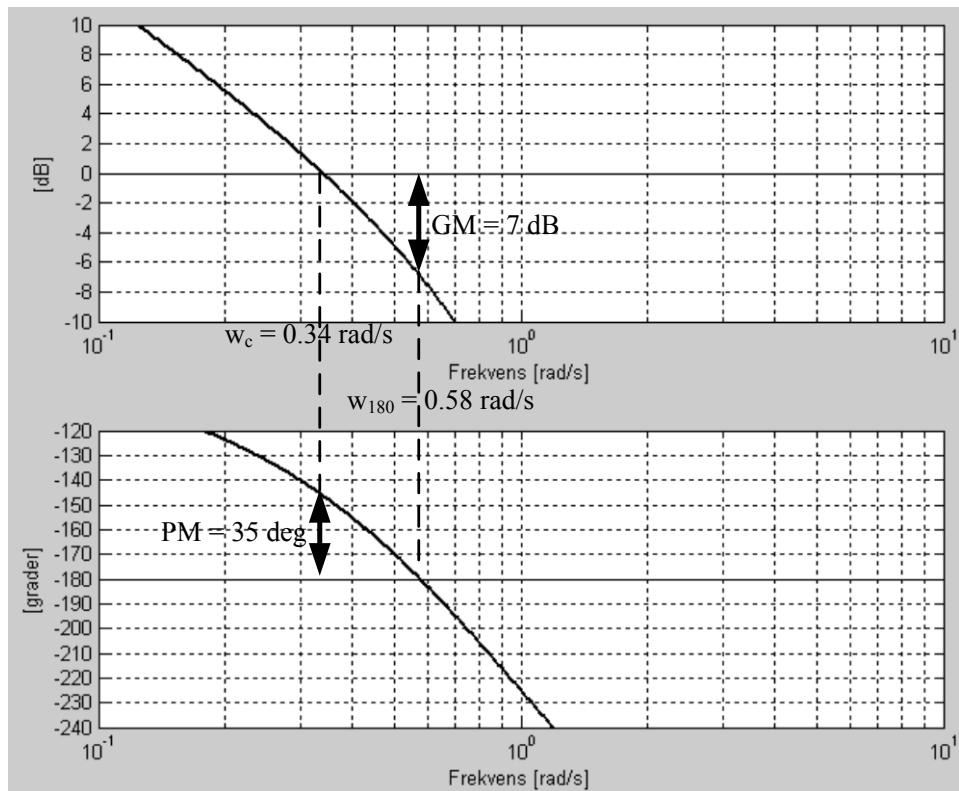


Figure 23.30: Bode plot

From the Bode diagram:

$$\underline{GM = 7 \text{ dB} = 2.2} \quad (23.90)$$

$$\underline{PM = 35^\circ} \quad (23.91)$$

$$\underline{\omega_c = 0.34 \text{ rad/s}} \quad (23.92)$$

$$\underline{\omega_{180} = 0.58 \text{ rad/s}} \quad (23.93)$$

2. An increase of the loop gain by a factor of

$$\underline{GM = 2.2} \quad (23.94)$$

will bring the system to the stability limit.

The period is

$$\underline{\underline{T_p = \frac{2\pi}{\omega_{180}} = \frac{2\pi}{0.58} = 10.8 \text{ s}}}$$
(23.95)

Solution to Problem 23.9

When the control system is marginally stable, the phase margin PM is zero. The maximum change of the time delay that is allowed before the system is marginally stable is

$$\Delta\tau = \frac{\text{PM}}{\omega_c} \cdot \frac{\pi}{180^\circ} = \frac{45^\circ}{0.2} \cdot \frac{\pi}{180^\circ} = 3.9 \text{ min}$$
(23.96)

Since the time delay *before* the change is 4.2 min, the total value of the time delay at marginally stability is

$$\underline{\underline{\tau = 4.2 + 3.9 = 8.1 \text{ min}}}$$
(23.97)

Solution to Problem 23.10

For $H_1(s)$ the loop transfer function with a P controller is

$$L_1(s) = K_c H_1(s) = K_c \frac{K}{s}$$
(23.98)

The phase (angle) of $L_1(j\omega)$ converges to -90° as the frequency goes to infinity. Therefore, the closed loop system will not become marginally stable with any controller gain, and hence, the Ziegler-Nichols method can not be used.

For $H_2(s)$ the loop transfer function with a P controller is

$$L_2(s) = K_c H_2(s) = K_c \frac{K}{Ts + 1}$$
(23.99)

The phase (angle) of $L_2(j\omega)$ converges to -90° as the frequency goes to infinity. Therefore, the closed loop system will not become marginally stable with any controller gain, and hence, the Ziegler-Nichols method can not be used.

For $H_3(s)$ the loop transfer function with a P controller is

$$L_3(s) = K_c H_3(s) = K_c \frac{K\omega_0^2}{s^2 + 2\zeta\omega_0 + \omega_0^2}$$
(23.100)

The phase (angle) of $L_3(j\omega)$ converges to -180° as the frequency goes to infinity. Therefore, the closed loop system will not become marginally stable with any limited controller gain, and hence, the Ziegler-Nichols method can not be used.

Part V

ANALYSIS OF DISCRETE-TIME FEEDBACK SYSTEMS

Chapter 24

Discrete-time signals

Assume that an AD-converter (analog-digital) at discrete points of time converts an analog signal $y_a(t)$, which can be a voltage signal from a temperature or speed sensor, to an equivalent digital signal, $y_d(t_k)$, in the form of a number to be used in operations in the computer, see Figure 24.1.

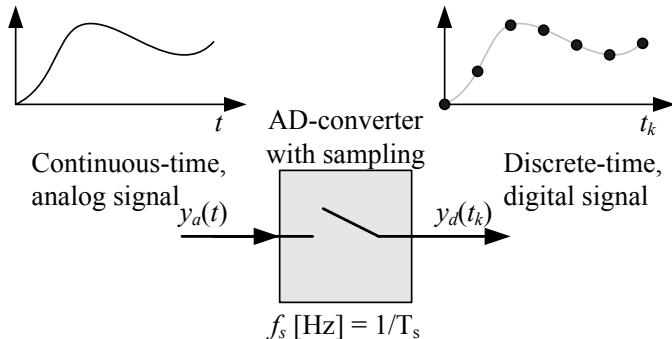


Figure 24.1: Sampling. T_s is the time step between the samplings, or the sampling interval.

(The AD-converter is a part of the interface between the computer and the external equipment, e.g. sensors.) As indicated in Figure 24.1 the resulting *discrete-time signal* is a sequence or a series of signal values defined in discrete points of time. T_s is the time step between the samplings, or the sampling interval. Figure 24.2 shows this signal in more detail.

The discrete points of time may be denoted t_k where k is an integer *time index*. The time series can be written in various ways:

$$\{x(t_k)\} = \{x(kT_s)\} = \{x(k)\} = x(0), x(1), x(2), \dots \quad (24.1)$$

To make the notation simple, we can write the signal in one of the following ways:

$$x(t_k) \quad (24.2)$$

$$x(kT_s) \quad (24.3)$$

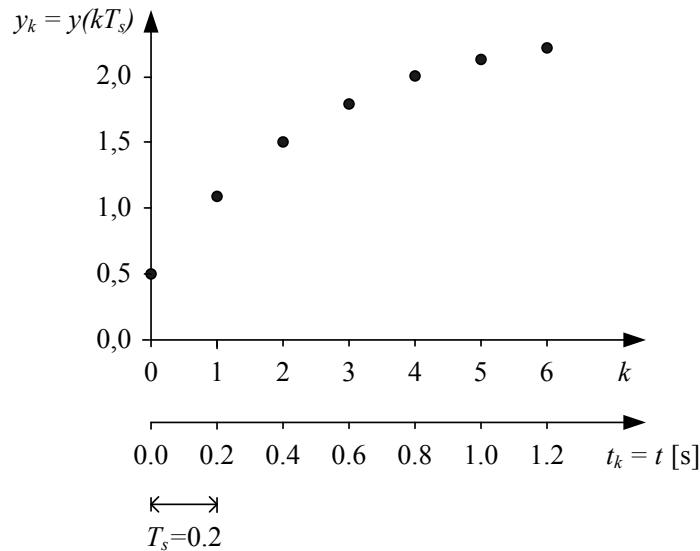


Figure 24.2: Discrete-time signal

$$x(k) \quad (24.4)$$

$$x_k \quad (24.5)$$

In the example above, the discrete-time signal originated from *sampling* of a continuous-time signal. However, discrete-time signals exists in many other circumstances, for example,

- the output signal from a discrete-time (computer-based) *signal filter*, for example a lowpass filter,
- the output from a discrete-time (computer-based) *controller* which controls a physical process,
- the response in a dynamic system as calculated by a (computer-based) *simulator*.

24.1 *Problems for Chapter 24*

Problem 24.1 *Discrete signal*

Given the following continuous-time signal (a ramp):

$$x_c(t) = 2t \quad (24.6)$$

where t is time in seconds.

1. Assume that the signal is sampled with sampling time (time-step) $T_s = 0.5$ s. Express x_d as a function of the discrete time t_k . Write the discrete signal or sequence (time series) x_d from time 0 to 2. Plot x with both discrete time t_k and time index k along the abscissa.
2. Repeat Problem 1 above, but now with $T_s = 0.1$ s.

24.2 *Solutions to problems for Chapter 24*

Solution to Problem 24.1

1.

$$\underline{\underline{x_d(t_k) = 2t_k}} \quad (24.7)$$

$$\underline{\underline{x_d = \{0, 0.5, 1.0, 1.5, 2.0\}}} \quad (24.8)$$

Figure 24.3 shows x_d with $T_s = 0.5$ s.

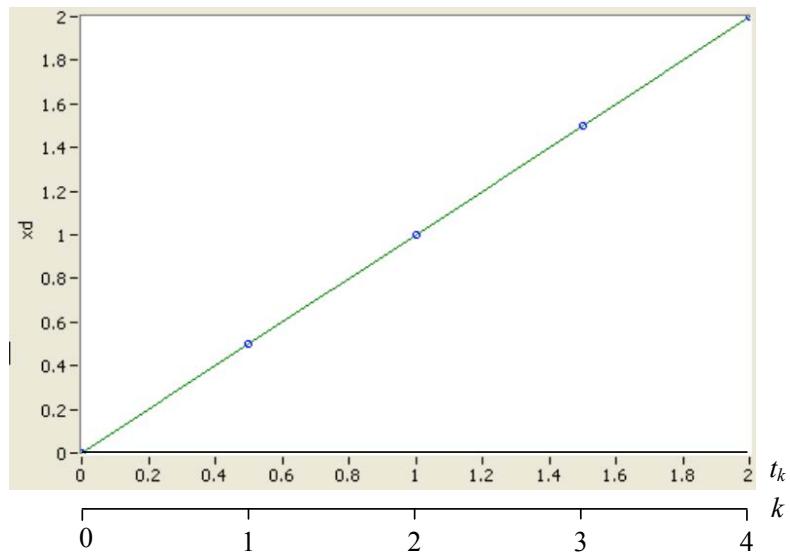


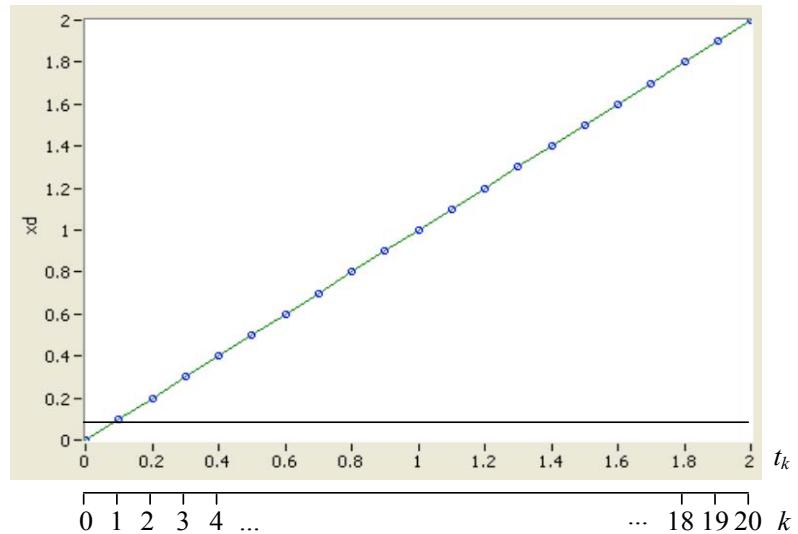
Figure 24.3: x_d with $T_s = 0.5$ s.

2.

$$\underline{\underline{x_d(t_k) = 2t_k}} \quad (24.9)$$

$$\underline{\underline{x_d = \{0, 0.1, 0.2, 0.3, \dots, 1.5, 1.6, 1.7, 1.8, 1.9, 2.0\}}} \quad (24.10)$$

Figure 24.4 shows x_d with $T_s = 0.1$ s.

Figure 24.4: x_d with $T_s = 0.1$ s

Chapter 25

Difference equations

25.1 Difference equation models

The basic model type of continuous-time dynamic systems is the differential equation. Analogously, the basic model type of discrete-time dynamic systems is the *difference equation*. Here is an example of a linear second order difference equation with u as input variable and y as output variable:

$$y(t_{k+2}) + a_1y(t_{k+1}) + a_0y(t_k) = b_0u(t_k) \quad (25.1)$$

which may be written somewhat simpler as

$$y(k+2) + a_1y(k+1) + a_0y(k) = b_0u(k) \quad (25.2)$$

where a_i and b_j are *coefficients* of the difference equation, or model *parameters*. Note that this difference equation has unique coefficients since the coefficient of $y(k+2)$ is 1.

One equivalent form (25.2) is

$$y(k) + a_1y(k-1) + a_0y(k-2) = b_0u(k-2) \quad (25.3)$$

where there are *no time delayed* terms (no negative time indexes), only time advanced terms (positive or zero time indexes). This form can be obtained from (25.2) by increasing each time index in (25.2) by 2.

In most cases we want to write the difference equation as a formula for the output variable. In our example the formula for the output $y(k)$ can be obtained by solving for $y(k)$ from (25.3):

$$y(k) = -a_1y(k-1) - a_0y(k-2) + b_0u(k-2) \quad (25.4)$$

(25.4) says that the output $y(k)$ is given as a linear combination of the output one time step back in time, $y(k-1)$, the output two time steps back in time, $y(k-2)$, and the input two time steps back in time, $u(k-2)$.

25.2 Calculating responses from difference equation models

For example, (25.4) is a formula for calculating dynamic (time-varying) responses in the output, $y(k)$. The formula must be calculated once per time step, and it can be implemented in a While loop or a For loop in a computer program. Assume as an example that $y(1)$, $y(0)$ and $u(0)$ are zero. Then (25.4) gives

$$y(2) = -a_1y(1) - a_0y(0) + b_0u(0) \quad (25.5)$$

$$y(3) = -a_1y(2) - a_0y(1) + b_0u(1) \quad (25.6)$$

$$y(4) = -a_1y(3) - a_0y(2) + b_0u(2) \quad (25.7)$$

and so on.

The *static response* – which is the (steady-state) response of the system when all variables are assumed to have constant values – can be calculated from the static version of the difference equation. The static version is found by neglecting all time-dependencies in the difference equation, and setting $y(k) = y_s$, $y(k-1) = y_s$ etc. where subindex s is for static. For example, the static version of (25.4) is

$$y_s = -a_1y_s - a_0y_s + b_0u_s \quad (25.8)$$

The static response is

$$y_s = \frac{b_0}{1 + a_1 + a_0}u_s \quad (25.9)$$

25.3 Problems for Chapter 25

Problem 25.1 Difference equation form

Given the following difference equation:

$$y(k+3) + ay(k+1) = b_1u(k+2) + b_0u(k) \quad (25.10)$$

Write the corresponding difference equation having only zero or negative time shifts.

Problem 25.2 Block diagram of difference equation

Particularly in the area of discrete-time (digital) signal processing the difference equations constituting the mathematical model of signal filters are represented with mathematical *block diagrams*, in the same way as differential equations of continuous-time systems are represented with block diagrams.

Figure 25.1 shows the most frequently used blocks – or the *elementary blocks* – used in block diagrams of difference equation models.

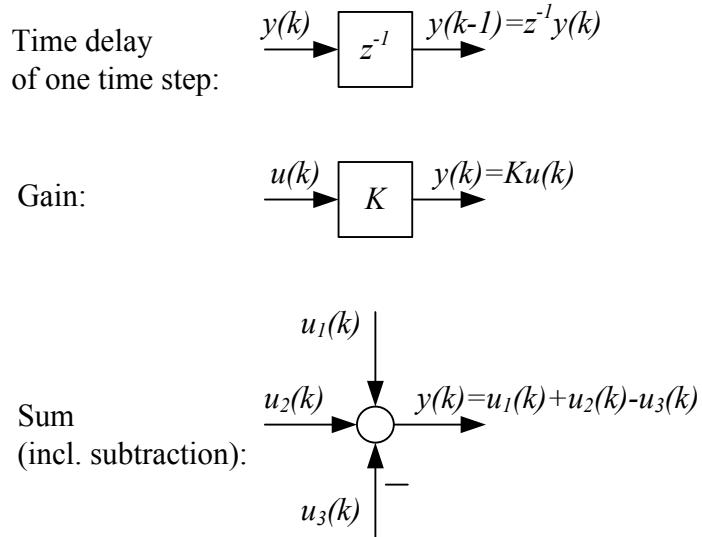


Figure 25.1: Elementary blocks for drawing block diagrams of difference equation models

A comment about the time delay block: The output $y(k)$ is equal to the time delayed input, $y(k-1)$:

$$y(k-1) = z^{-1}y(k) \quad (25.11)$$

Or, equivalently:

$$y(k) = z^{-1}y(k+1) \quad (25.12)$$

The operator z^{-1} is here a time-step delay operator, which is actually the z -transfer function of the time-step delay. (z -transfer functions are described in Chapter 11 of the text-book.)

Draw a block diagram of the following difference equation:

$$y(k+1) = ay(k) + bu(k) \quad (25.13)$$

where a and b are constant parameters. The block diagram shall have $u(k)$ as input and $y(k)$ as output.

Problem 25.3 *Response from discrete time filter*

Given the signal filter

$$y(k) = \frac{1}{3} [u(k) + u(k-1) + u(k-2)]$$

which is a moving average lowpass filter.

1. What is the steady-state response in y when the input u is a constant? Does the filter let a constant input pass unchanged, in steady-state?
2. Assume that the filter input u is a ramp:

$$\{u(k)\} = \{u(0), u(1), u(2), u(3), u(4)\} \quad (25.14)$$

$$= \{0, 0.5, 1.0, 1.5, 2.0\} \quad (25.15)$$

$$= \{0.5k\} \quad (25.16)$$

Calculate the response y at $k = 0..4$. (You can assume that u is zero at negative k .)

Also, calculate the general response $y(k)$. Will there be a constant difference from zero between the output and the input as time index goes to infinity?

25.4 Solutions to problems for Chapter 25

Solution to Problem 25.1

Each of the time indexes is reduced by 3, giving

$$\underline{\underline{y(k) + ay(k-2) = b_1 u(k-1) + b_0 u(k-3)}} \quad (25.17)$$

Solution to Problem 25.2

See Figure 25.2.

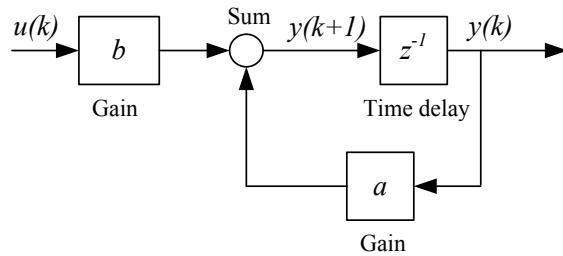


Figure 25.2: Block diagram

Solution to Problem 25.3

- Let us assume that

$$u = U \text{ (constant)} \quad (25.18)$$

In steady-state,

$$\underline{\underline{y_s}} = \frac{1}{3} (U + U + U) = \underline{\underline{U}}$$

Hence, the filter lets a constant input pass unchanged, in steady-state.

- The ramp response is

$$\begin{aligned}
 \underline{\underline{y(0)}} &= \frac{1}{3} [u(0) + u(-1) + u(-2)] = \frac{1}{3} [0 + 0 + 0] = \underline{\underline{0}} \\
 \underline{\underline{y(1)}} &= \frac{1}{3} [u(1) + u(0) + u(-1)] = \frac{1}{3} [0.5 + 0 + 0] = \frac{1}{6} = \underline{\underline{0.17}} \\
 \underline{\underline{y(2)}} &= \frac{1}{3} [u(2) + u(1) + u(0)] = \frac{1}{3} [1.0 + 0.5 + 0] = \underline{\underline{0.5}} \\
 \underline{\underline{y(3)}} &= \frac{1}{3} [u(3) + u(2) + u(1)] = \frac{1}{3} [1.5 + 1.0 + 0.5] = \underline{\underline{1}} \\
 \underline{\underline{y(4)}} &= \frac{1}{3} [u(4) + u(3) + u(2)] = \frac{1}{3} [2.0 + 1.5 + 1.0] = \underline{\underline{1.5}}
 \end{aligned}$$

The general response:

$$\begin{aligned}\underline{\underline{y(k)}} &= \frac{1}{3} [u(k) + u(k-1) + u(k-2)] \\ &= \frac{1}{3} [0.5 \cdot k + 0.5 \cdot (k-1) + 0.5 \cdot (k-2)] \\ &= 0.5 \cdot k + \frac{1}{3} (0 - 0.5 - 1.0) \\ &= \underline{\underline{0.5 \cdot k - 0.5}} \\ &= u(k) - 0.5\end{aligned}$$

Hence, there is a constant difference equal to -0.5 between the output and the input in steady-state.

Chapter 26

Discrete-time state space models

26.1 General form of discrete-time state space models

The general form of a discrete-time state space model is

$$x(k+1) = f[x(k), u(k)] \quad (26.1)$$

$$y(k) = g[x(k), u(k)] \quad (26.2)$$

where x is the state variable, u is the input variable which may consist of control variables and disturbances (in this model definition there is no difference between these two kinds of input variables). y is the output variable. f and g are functions – linear or nonlinear. $x(k+1)$ in (26.1) means the state one time-step ahead (relative to the present state $x(k)$). Thus, the state space model expresses how the systems' state (variables) and output variables evolves along the discrete time axis.

The variables in (26.1) – (26.2) may actually be vectors, e.g.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad (26.3)$$

where x_i is a (scalar) state variable, and if so, f and/or g are vector evaluated functions.

26.2 Linear discrete-time state space models

A special case of the general state space model presented above is the *linear* state space model:

$$x(k+1) = \underbrace{Ax(k) + Bu(k)}_{\equiv f[x(k), u(k)]} \quad (26.4)$$

$$y(k) = \underbrace{Cx(k) + Du(k)}_{\equiv g[x(k), u(k)]} \quad (26.5)$$

where A is the transition matrix, B is the input gain matrix, C is the output gain matrix or measurement gain matrix and D is the direct output gain matrix (in most cases, $D = 0$).

26.3 Discretization of continuous-time state space models

Here are some situations where you need to discretize a continuous-time state space model:

- Creating a simulation algorithm from a process model. This was described in Section 6.2.2.
- Defining the process model to be used as the basis of a state estimator in form of a Kalman Filter, cf. Chapter 34.

The Forward Discretization method is the simplest, most commonly used, and the most flexible method. Only this method will be described here.

Given the following continuous-time state space model, which can be linear or non-linear,

$$\dot{x}(t) = f_c[x(t), u(t)] \quad (26.6)$$

$$y(t) = g_c[x(t), u(t)] \quad (26.7)$$

Approximating the time derivative in (26.6) with Forward differentiation gives

$$\frac{x_{k+1} - x_k}{T_s} = f_c[x_k, u_k] \quad (26.8)$$

Solving for x_{k+1} gives

$$x_{k+1} = x_k + T_s f_c[x_k, u_k] \quad (26.9)$$

The discrete-time version of (26.7) is

$$y(t_k) = g_c[x_k, u_k] = g[x_k, u_k] \quad (26.10)$$

(26.9) and (26.10) constitute a discrete-time version of the original state space model (26.6)–(26.7).

26.4 Problems for Chapter 26

Problem 26.1 Discrete-time state space model

In Problem 5.1 a mathematical model of two coupled liquid tanks is presented. The model written as a continuous-time state space model is

$$h'_1 = \frac{1}{A_1} \left(K_p u_1 - K_{v1} \sqrt{\frac{\rho g h_1}{G}} \right) \quad (26.11)$$

$$h'_2 = \frac{1}{A_2} \left(K_{v1} \sqrt{\frac{\rho g h_1}{G}} - K_{v2} u_2 \sqrt{\frac{\rho g h_2}{G}} \right) \quad (26.12)$$

By applying Forward differentiation approximation to the time-derivatives, we get the following discrete-time model:

$$h'_{1,k} \approx \frac{h_{1,k+1} - h_{1,k}}{T_s} = \frac{1}{A_1} \left(K_p u_{1,k} - K_{v1} \sqrt{\frac{\rho g h_{1,k}}{G}} \right) \quad (26.13)$$

$$h'_{2,k} \approx \frac{h_{2,k+1} - h_{2,k}}{T_s} = \frac{1}{A_2} \left(K_{v1} \sqrt{\frac{\rho g h_{1,k}}{G}} - K_{v2} u_{2,k} \sqrt{\frac{\rho g h_{2,k}}{G}} \right) \quad (26.14)$$

Assume that both levels are output variables. Write this model as a discrete-time state space model on the standard form

$$h_{1,k+1} = f_1 [h_{1,k}, h_{2,k}, \dots] \quad (26.15)$$

$$h_{2,k+1} = f_2 [h_{1,k}, h_{2,k}, \dots] \quad (26.16)$$

(which corresponds to the compact standard form $x_{k+1} = f[x_k, \dots]$). Is the state space model linear or nonlinear?

Problem 26.2 State space model on matrix-vector form

Write the following difference equations model as a state space model on matrix-vector form.

$$x_{1,k+1} = -0.5x_{1,k} \quad (26.17)$$

$$x_{2,k+1} = 2u_k - x_{2,k} - 3x_{1,k} \quad (26.18)$$

$$y_k = x_{2,k} + 4u_k \quad (26.19)$$

26.5 *Solutions to problems for Chapter 26*

Solution to Problem 26.1

Solving (26.13) – (26.14) for $h_{1,k+1}$ and $h_{2,k+1}$, respectively, gives the discrete-time state space model:

$$\underline{\underline{h_{1,k+1} = h_1(t_k) + \overbrace{\frac{T_s}{A_1} \left(K_p u_{1,k} - K_{v_1} \sqrt{\frac{\rho g h_{1,k}}{G}} \right)}^{f_1}} \quad (26.20)}$$

$$\underline{\underline{h_{2,k+1} = h_{2,k} + \overbrace{\frac{T_s}{A_2} \left(K_{v_1} \sqrt{\frac{\rho g h_{1,k}}{G}} - K_{v_2} u_{2,k} \sqrt{\frac{\rho g h_{2,k}}{G}} \right)}^{f_2}} \quad (26.21)}}$$

The outputs variables are

$$\underline{\underline{y_{1,k} = h_{1,k}}} \quad (26.22)$$

$$\underline{\underline{y_{2,k} = h_{2,k}}} \quad (26.23)$$

This state space model is nonlinear due to the square root functions in which the state variables are arguments.

Solution to Problem 26.2

$$\underline{\underline{\begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \end{bmatrix} = \underbrace{\begin{bmatrix} -0.5 & 0 \\ -3 & -1 \end{bmatrix}}_A \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 2 \end{bmatrix}}_B u_k} \quad (26.24)}}$$

$$\underline{\underline{y_k = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_C \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} + \underbrace{[4]}_D u_k}} \quad (26.25)$$

Chapter 27

The z -transform

27.1 Definition of the z -transform

The z -transform of discrete-time signals plays much the same role as the Laplace transform for continuous-time systems.

The z -transform of the discrete-time signal $\{y(k)\}$, or just $y(k)$, is defined as follows:

$$\mathcal{Z}\{y(k)\} = \sum_{k=0}^{\infty} y(k)z^{-k} \quad (27.1)$$

For simplicity, I will use the symbol $y(z)$ for $\mathcal{Z}\{y(k)\}$ when it can not be misunderstood. Strictly, a different variable name should be used, for example $Y(z)$.

Example 27.1 z -transform of a constant

Assume that the signal $y(k)$ has constant value A . This signal can be regarded a step of amplitude A at time-step 0. z -transforming $y(k)$ gives

$$y(z) = \sum_{k=0}^{\infty} y(k)z^{-k} = \sum_{k=0}^{\infty} Az^{-k} = \frac{A}{1-z^{-1}} = \frac{Az}{z-1} \quad (27.2)$$

[End of example 27.1]

27.2 Properties of the z -transform

Below are the most important properties of the z -transform. These properties can be used when calculating the z -transform of composite signals.

- **Linearity:**

$$k_1 y_1(z) + k_2 y_2(z) \iff k_1 y_1(k) + k_2 y_2(k) \quad (27.3)$$

- **Time delay:** Multiplication by z^{-n} means time delay of n time-steps:

$$z^{-n} y(z) \iff y(k - n) \quad (27.4)$$

- **Time advancing:** Multiplication by z^n means time advancing by n time-steps:

$$z^n y(z) \iff y(k + n) \quad (27.5)$$

27.3 z-transform pairs

Below are several important z -transform pairs showing discrete-time functions and their corresponding z -transforms. The time functions are defined for $k \geq 0$.

$$\text{Unity impulse at time-step } k: \delta(k) \iff z^k \quad (27.6)$$

$$\text{Unity impulse at time-step } k = 0: \delta(0) \iff 1 \quad (27.7)$$

$$\text{Unity step at time-step } k = 0: 1 \iff \frac{z}{z - 1} \quad (27.8)$$

$$\text{Time exponential: } a^k \iff \frac{z}{z - a} \quad (27.9)$$

Example 27.2 z -transformation of a composite signal

Given the following discrete-time function:

$$y(k) = Ba^{k-n} \quad (27.10)$$

(which is a time delayed time exponential). The inverse z -transform of $y(k)$ can be calculated using (27.9) together with (27.3) and (27.4). The result becomes

$$y(z) = Bz^{-n} \frac{z}{z - a} = B \frac{z^{1-n}}{z - a} \quad (27.11)$$

[End of example 27.2]

27.4 Inverse z -transform

Inverse z -transformation of a given z evaluated function, say $Y(z)$, is calculating the corresponding time function, say $y(k)$. The inverse transform may be calculated using a complex integral¹, but this method is not very practical. Another method is to find a

¹ $y(k) = \frac{1}{2\pi j} \oint Y(z) z^k \frac{dz}{z}$, where the integration path must be in the area of convergence of $Y(z)$.?

proper combination of precalculated z -transformation pairs, possibly in combination with some of the z -transform properties defined above.

In most cases where you need to calculate a time signal $y(k)$, its z -transform $Y(z)$ stems from a transfer function excited by some discrete-time input signal. You may then calculate $y(k)$ by first transferring the transfer function to a corresponding difference equation, and then calculating $y(k)$ iteratively from this difference equation as explained in Section 25.2.

27.5 Problems for Chapter 27

Problem 27.1 *z-transform of unit impulse*

Calculate the z -transform of $\delta(k)$ which a unit impulse. This is a signal having amplitude 1 at discrete time $k = 0$ and zero at other points of time.

Problem 27.2 *Linearity property*

The Linearity property of the z -transform can be expressed as

$$k_1y_1(z) + k_2y_2(z) \iff k_1y_1(k) + k_2y_2(k) \quad (27.12)$$

Show that the Linearity property holds for this example of a signal:

$$y(k) = A + B = AS(k) + BS(k) \quad (27.13)$$

where A and B are constants, and $S(k)$ is the unity step functions, i.e. a step occurring at time zero. (Hint: Show that the \mathcal{Z} -transform of the right side of (27.12) is equal to the left side of (27.12) for the given signal.)

Problem 27.3 *z-transform of a step signal*

Calculate the z -transform of a step signal of amplitude 4 which occurs at time-index 2. You can use $S(k)$ to represent a unit step function, i.e. a step occurring at time zero.

Problem 27.4 *Inverse z-transform*

Calculate the inverse z -transform of

$$y(z) = \frac{z}{z - 0.5} \quad (27.14)$$

27.6 Solutions to problems for Chapter 27

Solution to Problem 27.1

$$\underline{\underline{\mathcal{Z}\{y(k)\} = y(z)}} = \sum_{k=0}^{\infty} y(k)z^{-k} = \sum_{k=0}^{\infty} \delta(k)z^{-k} \quad (27.15)$$

$$= 1 \cdot z^{-0} + 0 \cdot z^{-1} + 0 \cdot z^{-2} + \cdots + 0 \cdot z^{-n} + \cdots \quad (27.16)$$

$$= \underline{\underline{1}} \quad (27.17)$$

Solution to Problem 27.2

The \mathcal{Z} -transform of the right side of (27.12) becomes

$$\mathcal{Z}\{AS(k) + BS_2(k)\} = \mathcal{Z}\{(A + B)S(k)\} = (A + B)\frac{z}{z - 1} \quad (27.18)$$

The left side of (27.12) becomes

$$A \cdot \mathcal{Z}\{S(k)\} + B \cdot \mathcal{Z}\{S(k)\} = A\frac{z}{z - 1} + B\frac{z}{z - 1} = (A + B)\frac{z}{z - 1} \quad (27.19)$$

Hence, the \mathcal{Z} -transform of the right side of (27.12) is equal to the left side of (27.12), and consequently the linear property holds.

Solution to Problem 27.3

The signal is

$$y(k) = 4 \cdot S(k - 2) \quad (27.20)$$

Using the Linearity property and the Time delay property of the \mathcal{Z} -transform, we get

$$\underline{\underline{\mathcal{Z}\{y(k)\} = y(z)}} = 4z^{-2}\mathcal{Z}\{S(k)\} = 4z^{-2}\frac{z}{z - 1} = \frac{4}{z^2 - z} \quad (27.21)$$

Solution to Problem 27.4

Using the \mathcal{Z} -transform pair denoted “Time exponential” we get

$$\underline{\underline{y(k) = 0.5^k}} \quad (27.22)$$

Chapter 28

Discrete-time (or z -) transfer functions

28.1 Introduction

Models in the form of difference equations can be z -transformed to z -transfer *functions*, which plays the same role in discrete-time systems theory as s transfer functions do in continuous-time systems theory. More specific:

- The combined model of systems in a serial connection can be found by simply multiplying the individual z -transfer functions.
- The frequency response can be calculated from the transfer function.
- The transfer function can be used to represent the system in a simulator or in computer tools for analysis and design (as SIMULINK, MATLAB or LabVIEW)

28.2 From difference equation to transfer function

As an example we will derive the discrete-time or z -transfer function from input u to output y from the difference equation (25.4), which is repeated here:

$$y(k) = -a_1y(k-1) - a_0y(k-2) + b_0u(k-2) \quad (28.1)$$

First, we take the z -transform of both sides of the difference equation:

$$\mathcal{Z}\{y(k)\} = \mathcal{Z}\{-a_1y(k-1) - a_0y(k-2) + b_0u(k-2)\} \quad (28.2)$$

Using the linearity property (27.3) and the time delay property (27.4) (28.2) can be written as

$$\mathcal{Z}\{y(k)\} = -\mathcal{Z}\{a_1y(k-1)\} - \mathcal{Z}\{a_0y(k-2)\} + \mathcal{Z}\{b_0u(k-2)\} \quad (28.3)$$

and

$$y(z) = -a_1 z^{-1} y(z) - a_0 z^{-2} y(z) + b_0 z^{-2} u(z) \quad (28.4)$$

which can be written as

$$y(z) + a_1 z^{-1} y(z) + a_0 z^{-2} y(z) = b_0 z^{-2} u(z) \quad (28.5)$$

or

$$[1 + a_1 z^{-1} + a_0 z^{-2}] y(z) = b_0 z^{-2} u(z) \quad (28.6)$$

$$y(z) = \underbrace{\frac{b_0 z^{-2}}{1 + a_1 z^{-1} + a_0 z^{-2}}}_{H(z)} u(z) \quad (28.7)$$

$$= \underbrace{\frac{b_0}{z^2 + a_1 z^1 + a_0}}_{H(z)} u(z) \quad (28.8)$$

where $H(z)$ is the z -transfer function from u to y . Hence, z -transfer functions can be written both with positive and negative exponents of z .¹

28.3 From transfer function to difference equation

In the above Section we derived a z -transfer function from a difference equation. We may go the opposite way – to derive a difference equation from a given z -transfer function. Some applications of this are

- Deriving a filtering algorithm from a filtering transfer function
- Deriving a control function from a given controller transfer function
- Deriving a simulation algorithm from the transfer function of the system to be simulated

The procedure will be illustrated via a concrete example. Assume given the following transfer function:

$$H(z) = \frac{b_0}{z^2 + a_1 z + a_0} = \frac{y(z)}{u(z)} \quad (28.9)$$

We start by cross multiplying (28.9):

$$(z^2 + a_1 z + a_0) y(z) = b_0 u(z) \quad (28.10)$$

which can be written as

$$z^2 y(z) + a_1 z y(z) + a_0 y(z) = b_0 u(z) \quad (28.11)$$

¹In signal processing theory transfer functions are usually written with negative exponents of z , while in control theory they are usually written with positive exponents.

Taking the inverse transform of the above expression gives

$$\underbrace{z^2 y(z)}_{y(k+2)} + \underbrace{a_1 z y(z)}_{a_1 y(k+1)} + \underbrace{a_0 y(z)}_{a_0 y(k)} = \underbrace{b_0 u(z)}_{b_0 u(k)} \quad (28.12)$$

Reducing each of the time indexes by 2 yields

$$y(k) + a_1 y(k-1) + a_0 y(k-2) = b_0 u(k-2) \quad (28.13)$$

Usually it is practical to have the output variable alone on the left side:

$$y(k) = -a_1 y(k-1) - a_0 y(k-2) + b_0 u(k-2) \quad (28.14)$$

28.4 Calculating time responses for discrete-time transfer functions

Assume given a transfer function, say $H(z)$, with input variable u and output variable y . Then,

$$y(z) = H(z)u(z) \quad (28.15)$$

If $u(z)$ is given, the corresponding time response in y can be calculated in several ways:

1. By finding a proper transformation pair in Section 27.3, possibly combined with some of the z -transform properties in Section 27.2.
2. By deriving a differential equation corresponding to the transfer function and then calculating $y(k)$ iteratively according to the difference equation. The procedure of deriving a differential equation corresponding to a given transfer function is explained in Section 28.3, and the calculation of time responses for a difference equation is described in Section 25.2.

28.5 Static transfer function and static response

The static version H_s of a given transfer function $H(z)$ will now be derived. Using the static transfer function the static response can easily be calculated. Assume that the input variable u is a step of amplitude U . The stationary response can be found using the final value theorem:

$$\lim_{k \rightarrow \infty} y(k) = y_s = \lim_{z \rightarrow 1} (z-1)y(z) \quad (28.16)$$

$$= \lim_{z \rightarrow 1} (z-1)H(z)u(z) \quad (28.17)$$

$$= \lim_{z \rightarrow 1} (z-1)H(z) \frac{zU}{z-1} \quad (28.18)$$

$$= H(1)U \quad (28.19)$$

Thus, we have the following static transfer function:

$$H_s = \frac{y_s}{u_s} = \lim_{z \rightarrow 1} H(z) = H(1) \quad (28.20)$$

Using the static transfer function the static response can be calculated by

$$y_s = H_s U \quad (28.21)$$

Example 28.1 Static transfer function

Let us consider the following transfer function:

$$H(z) = \frac{y(z)}{u(z)} = \frac{az}{z - (1 - a)} \quad (28.22)$$

which is the transfer function of the lowpass filter presented in Ch. 3.4.8.6.4, repeated here:

$$y(k) = (1 - a)y(k - 1) + au(k) \quad (28.23)$$

The corresponding static transfer function is

$$H_s = \frac{y_s}{u_s} = \lim_{z \rightarrow 1} H(z) = \lim_{z \rightarrow 1} \frac{a}{1 - (1 - a)z^{-1}} = \frac{a}{1 - (1 - a) \cdot 1} = 1 \quad (28.24)$$

Thus,

$$y_s = H_s u_s = u_s \quad (28.25)$$

Can we find the same correspondence between u_s and y_s from the difference equation (28.23)? Setting $y(k) = y(k - 1) = y_s$ and $u(k) = u_s$ gives

$$y_s = (1 - a)y_s + au_s \quad (28.26)$$

giving

$$\frac{y_s}{u_s} = \frac{a}{1 - (1 - a)} = 1 \quad (28.27)$$

which is the same as (28.24).

[End of Example 28.1]

28.6 Poles and zeros

Poles and zeros of z -transfer functions are defined in the same way as for s transfer functions: The zeros of the transfer function are the z -roots of numerator polynomial, and the poles are the z -roots of the denominator polynomial.

One important application of poles is stability analysis, cf. Section 30.

Example 28.2 Poles and zeros

Given the following z -transfer function:

$$H(z) = \frac{(z - b)}{(z - a_1)(z - a_2)} \quad (28.28)$$

The poles are a_1 and a_2 , and the zero is b .

[End of Example 28.2]

28.7 From s -transfer functions to z -transfer functions

In some cases you need to find a discrete-time z -transfer function from a given continuous-time s transfer function:

- In accurate model based design of a discrete controller for a process originally in the form of a continuous-time s transfer function, $H_p(s)$. The latter should be discretized to get a discrete-time process model before the design is started.
- Implementation of continuous-time control and filtering functions in a computer program.

There are several methods for discretization of an s transfer function. The methods can be categorized as follows, and they are described in the following sections:

1. **Discretization based on having a zero order hold (ZOH) element on the input of the system.** This method should be used in controller design of a process which has a sample and hold element on its input, as when a physical process is controlled by a computer via a DA converter (digital to analog). Zero order hold means that the input signal is held constant during the time-step or sampling interval. Figure 28.1 shows a block diagram of a continuous-time process with transfer function model $H(s)$ having a zero order hold element on its input.

ZOH discretization gives a perfect z -transfer function in the sense that it produces exactly the same response as produced by the s -transfer function at the discrete points of time. (If we discretize using e.g. the Forward differentiation the responses will differ a little.) The ZOH discretization method is actually complicated to implement manually, but tools as Matlab and LabVIEW have functions that perform the discretization easily, and in most practical applications, you will be using such tools.

2. **Using an appropriate approximation to time-derivatives**, as Forward Difference method, or Backward Difference method, or Tustin's method. In such cases the input signal is a discrete-time signal with no holding (no ZOH element is assumed). The procedure has the following steps:

- From the given the continuous-time s -transfer function $H_c(s)$, derive the corresponding differential equation.

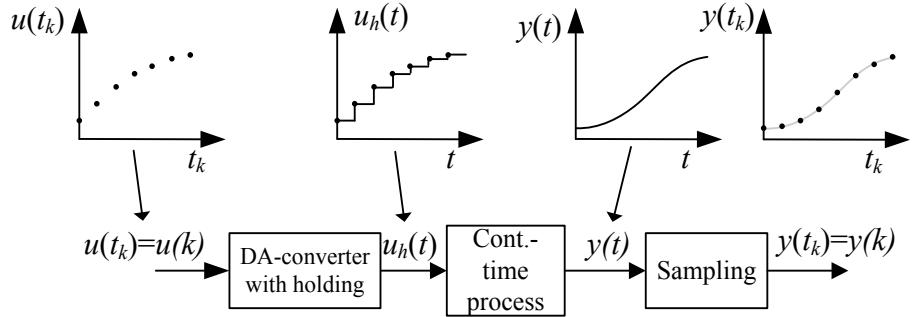


Figure 28.1: Block diagram of a process having a zero order hold element on its input. u_h is the piecewise constant (held) input signal.

- Apply some approximation to the time-derivatives of the differential equation. If you do not have any other preferences, use the Backward difference method. The result is a difference equation.
- Calculate the z -transfer function from the difference equation, cf. Section 28.2.

Here is an example of discretization using the `c2d` function in MATLAB:

Example 28.3 Discretization using the `c2d` function in MATLAB

The MATLAB code shown below discretizes the s -transfer function

$$H_{cont}(s) = \frac{2}{3 + 4s} \quad (28.29)$$

with sampling time $Ts = 0.1$.

```
Hcont = tf([2],[3,4]);
Ts=0.1;
Hdisc=c2d(Hcont,Ts,'zoh')
```

The result as shown in MATLAB is

Transfer function:

0.06241

$z - 0.8752$

Sampling time: 0.1

[End of Example 28.3]

Here is an example of discretizing a s -transfer function manually using the Backward differentiation approximation:

Example 28.4 Discretizing a first order transfer function

We will discretize the following continuous-time transfer function:

$$H_c(s) = \frac{K}{Ts + 1} = \frac{y(s)}{u(s)} \quad (28.30)$$

- Deriving the corresponding differential equation: Cross-multiplying gives

$$(Ts + 1)y(s) = Ku(s) \quad (28.31)$$

Resolving the parenthesis gives

$$Tsy(s) + y(s) = Ku(s) \quad (28.32)$$

Taking the inverse Laplace transform of both sides of this equation gives the following differential equation (because multiplying by s means time-differentiation in the time-domain):

$$Ty'(t) + y(t) = Ku(t) \quad (28.33)$$

Let us use t_k to represent the present point of time – or discrete time:

$$Ty'_k + y_k = Ku_k \quad (28.34)$$

- Applying the Backward differentiation approximation:

$$T \frac{y_k - y_{k-1}}{T_s} + y_k = Ku_k \quad (28.35)$$

Solving for y_k gives the following difference equation:

$$y_k = \frac{T}{T + T_s} y_{k-1} + \frac{T_s K}{T + T_s} u_k \quad (28.36)$$

- Taking the z -transform of the difference equation:

$$y(z) = \frac{T}{T + T_s} z^{-1} y(z) + \frac{T_s K}{T + T_s} u(z) \quad (28.37)$$

from which we obtain the following z -transfer function:

$$H(z) = \frac{y(z)}{u(z)} = \left(z - \frac{T}{T + T_s} \right)^{-1} \frac{z T_s K}{T + T_s} \quad (28.38)$$

[End of Example 28.4]

28.8 Problems for Chapter 28

Problem 28.1 Find the z-transfer function

Given the signal filter

$$y(k) = \frac{1}{3} [u(k) + u(k-1) + u(k-2)]$$

which is a moving average lowpass filter. Find the z-transfer function from u to y .

Problem 28.2 From z-transfer function to difference equation

Assume that the following transfer function from input signal u to output signal y of a physical system is found by some system identification method:

$$H(z) = \frac{y(z)}{u(z)} = \frac{a}{bz^2 + cz + d}$$

What is the corresponding difference equation relating u and y ?

Problem 28.3 Output response from a z-transfer function

Assume that the input signal u to the following transfer function is an impulse of amplitude A .

$$H(z) = \frac{y(z)}{u(z)} = \frac{z}{z-1} \quad (28.39)$$

Calculate the output response $y(k)$.

Problem 28.4 Static z-transfer function

In Problem 28.1 the following transfer function of a moving average lowpass filter was found:

$$H(z) = \frac{y(z)}{u(z)} = \frac{1}{3} [1 + z^{-1} + z^{-2}] \quad (28.40)$$

1. Calculate the corresponding static transfer function H_s .
2. Assume that the filter input is a constant, $u(k) = U$. Calculate the corresponding steady-state filter output y_s from H_s .

Problem 28.5 Poles and the zeros of z-transfer function

Given the following transfer function:

$$H(z) = \frac{bz^{-2} + z^{-1}}{1 - az^{-1}} \quad (28.41)$$

Calculate the poles and the zeros of the transfer function.

Problem 28.6 *From s-transfer function to z-transfer function*

The s -transfer function of a (continuous-time) integrator is

$$H_{\text{cont}}(s) = \frac{y(s)}{u(s)} = \frac{1}{s} \quad (28.42)$$

Derive a corresponding z -transfer function $H_{\text{disc}}(z)$ assuming Backward discretization. The time-step is T_s .

28.9 *Solutions to problems for Chapter 28*

Solution to Problem 28.1

z transformation gives

$$y(z) = \frac{1}{3} [u(z) + z^{-1}u(z) + z^{-2}u(z)] = \frac{1}{3} [1 + z^{-1} + z^{-2}] u(z) \quad (28.43)$$

which gives the transfer function

$$\underline{\underline{\frac{y(z)}{u(z)}}} = \frac{1}{3} [1 + z^{-1} + z^{-2}] \quad (28.44)$$

Solution to Problem 28.2

Cross-multiplication gives

$$[bz^2 + cz + d] y(z) = au(z)$$

or

$$bz^2 y(z) + c z y(z) + d y(z) = au(z) \quad (28.45)$$

which, inverse-transformed, gives this difference equation:

$$\underline{\underline{by(k+2) + cy(k+1) + dy(k) = au(k)}}$$

Solution to Problem 28.3

The \mathcal{Z} -transform of y becomes

$$y(z) = H(z)u(z) = \frac{z}{z-1} \cdot A \quad (28.46)$$

$y(k)$ is given by the inverse transform of $y(z)$:

$$\underline{\underline{y(k)}} = \mathcal{Z}^{-1}\{y(z)\} = \mathcal{Z}^{-1}\left\{A \frac{z}{z-1}\right\} = \underline{\underline{A}} \quad (28.47)$$

(which is a step of amplitude A at time zero).

Solution to Problem 28.4

1. The static transfer function is

$$\underline{\underline{H_s}} = H(z=1) = \frac{1}{3} [1 + 1^{-1} + 1^{-2}] = \underline{\underline{1}} \quad (28.48)$$

2. The steady-state filter output is

$$\underline{\underline{y_s}} = H_s U = 1 \cdot U = \underline{\underline{U}} \quad (28.49)$$

Solution to Problem 28.5

It is convenient to start by rewriting the transfer function as follows:

$$H(z) = \frac{z^{-2}b + z^{-1}}{1 - az^{-1}} \cdot \frac{z^2}{z^2} = \frac{z + b}{z^2 - az} = \frac{z + b}{(z - a)z} \quad (28.50)$$

Thus, the zero z is

$$\underline{\underline{z = -b}} \quad (28.51)$$

and the poles p_i are

$$\underline{\underline{p_1 = a; p_2 = 0}} \quad (28.52)$$

Solution to Problem 28.6

(28.42) can be written

$$sy(s) = u(s) \quad (28.53)$$

Inverse Laplace transform gives

$$\dot{y}(t) = u(t) \quad (28.54)$$

Applying Backward discretization to the time-derivative and introducing discrete time notation:

$$\dot{y}(t_k) \approx \frac{y(t_k) - y(t_{k-1})}{T_s} = u(t_k) \quad (28.55)$$

Solving for $y(t_k)$:

$$y(t_k) = y(t_{k-1}) + T_s u(t_k) \quad (28.56)$$

Taking the \mathcal{Z} -transform:

$$y(z) = z^{-1}y(z) + T_s u(z) \quad (28.57)$$

The transfer function becomes

$$\underline{\underline{H_{\text{disc}}(z) = \frac{y(z)}{u(z)} = \frac{T_s}{1 - z^{-1}} = \frac{zT_s}{z - 1}}} \quad (28.58)$$

Chapter 29

Frequency response of discrete-time systems

As for continuous-time systems, the frequency response of a discrete-time system can be calculated from the transfer function: Given a system with z -transfer function $H(z)$. Assume that input signal exciting the system is the sinusoid

$$u(t_k) = U \sin(\omega t_k) = U \sin(\omega k T_s) \quad (29.1)$$

where ω is the signal frequency in rad/s. The time-step is T_s .

It can be shown that the stationary response on the output of the system is

$$y(t_k) = Y \sin(\omega k T_s + \phi) \quad (29.2)$$

$$= U A \sin(\omega k T_s + \phi) \quad (29.3)$$

$$= \underbrace{U |H(e^{j\omega T_s})|}_Y \sin \left[\omega t_k + \underbrace{\arg H(e^{j\omega T_s})}_{\phi} \right] \quad (29.4)$$

where $H(e^{j\omega T_s})$ is the *frequency response* which is calculated with the following substitution:

$$H(e^{j\omega T_s}) = H(z)|_{z=e^{j\omega T_s}} \quad (29.5)$$

The *amplitude gain function* is

$$A(\omega) = |H(e^{j\omega T_s})| \quad (29.6)$$

The *phase lag function* is

$$\phi(\omega) = \arg H(e^{j\omega T_s}) \quad (29.7)$$

$A(\omega)$ and $\phi(\omega)$ can be plotted in a Bode diagram.

Figure 29.1 shows as an example the Bode plot of the frequency response of the following transfer function (time-step is 0.1s):

$$H(z) = \frac{b}{z - a} = \frac{0.0952}{z - 0.9048} \quad (29.8)$$

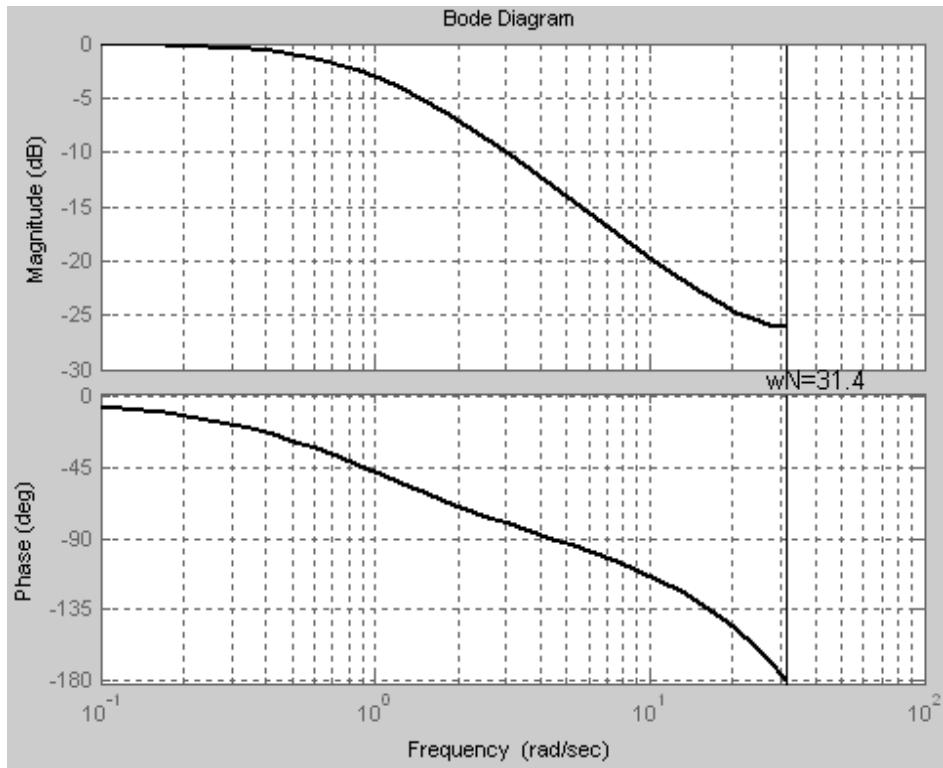


Figure 29.1: Bode plot of the transfer function (29.8). $\omega_N = 31.4$ rad/s is the Nyquist frequency (sampling time h is 0.1s).

Note that the plots in Figure 29.1 are drawn only up to the Nyquist frequency which in this case is

$$\omega_N = \frac{\omega_s}{2} = \frac{2\pi/T_s}{2} = \frac{\pi}{T_s} = \frac{\pi}{0.1} = 10\pi \approx 31.4 \text{ rad/s} \quad (29.9)$$

The plots are not drawn (but they exist!) above the Nyquist frequency because of symmetry of the frequency response, as explained in the following section.

Example 29.1 Calculating the frequency response manually from the z-transfer function

Given the z -transfer function

$$H(z) = \frac{b}{z - a} \quad (29.10)$$

The frequency response becomes

$$H(e^{j\omega T_s}) = \frac{b}{e^{j\omega T_s} - a} \quad (29.11)$$

$$= \frac{b}{\cos \omega T_s + j \sin \omega T_s - a} \quad (29.12)$$

$$= \frac{b}{\underbrace{(\cos \omega T_s - a)}_{\text{Re}} + j \underbrace{\sin \omega T_s}_{\text{Im}}} \quad (29.13)$$

$$= \frac{b}{\sqrt{(\cos \omega T_s - a)^2 + (\sin \omega T_s)^2} e^{j \arctan[(\sin \omega T_s)/(\cos \omega T_s - a)]}} \quad (29.14)$$

$$= \frac{b}{\sqrt{(\cos \omega T_s - a)^2 + (\sin \omega T_s)^2}} e^{j[-\arctan(\frac{\sin \omega T_s}{\cos \omega T_s - a})]} \quad (29.15)$$

The amplitude gain function is

$$A(\omega) = |H(e^{j\omega T_s})| = \frac{b}{\sqrt{(\cos \omega T_s - a)^2 + (\sin \omega T_s)^2}} \quad (29.16)$$

and the phase lag function is

$$\phi(\omega) = \arg H(e^{j\omega T_s}) = -\arctan\left(\frac{\sin \omega T_s}{\cos \omega T_s - a}\right) [\text{rad}] \quad (29.17)$$

[End of Example 29.1]

Even for the simple example above, the calculations are cumbersome, and prone to errors. Therefore you should use some computer tool for calculating the frequency response, as MATLAB's Control System Toolbox or LabVIEW's Control Design Toolkit.

Symmetry of frequency response

It can be shown that the frequency response is symmetric as follows: $|H(e^{j\omega T_s})|$ and $\arg H(e^{j\omega T_s})$ are unique functions in the frequency interval $[0, \omega_N]$ where ω_N is the Nyquist frequency. In the following intervals $[m\omega_s, (m+1)\omega_s]$ (m is an integer) the functions are mirrored as indicated in Figure 29.2 which has a logarithmic frequency axis. (The Bode plots in this section are for the transfer function (29.8).)

The symmetry appears clearer in the Bode plots in Figure 29.3 where the frequency axis is linear.

Due to the symmetry of the frequency response, it is strictly not necessary to draw more of frequency response plots than of the frequency interval $[0, \omega_N]$.

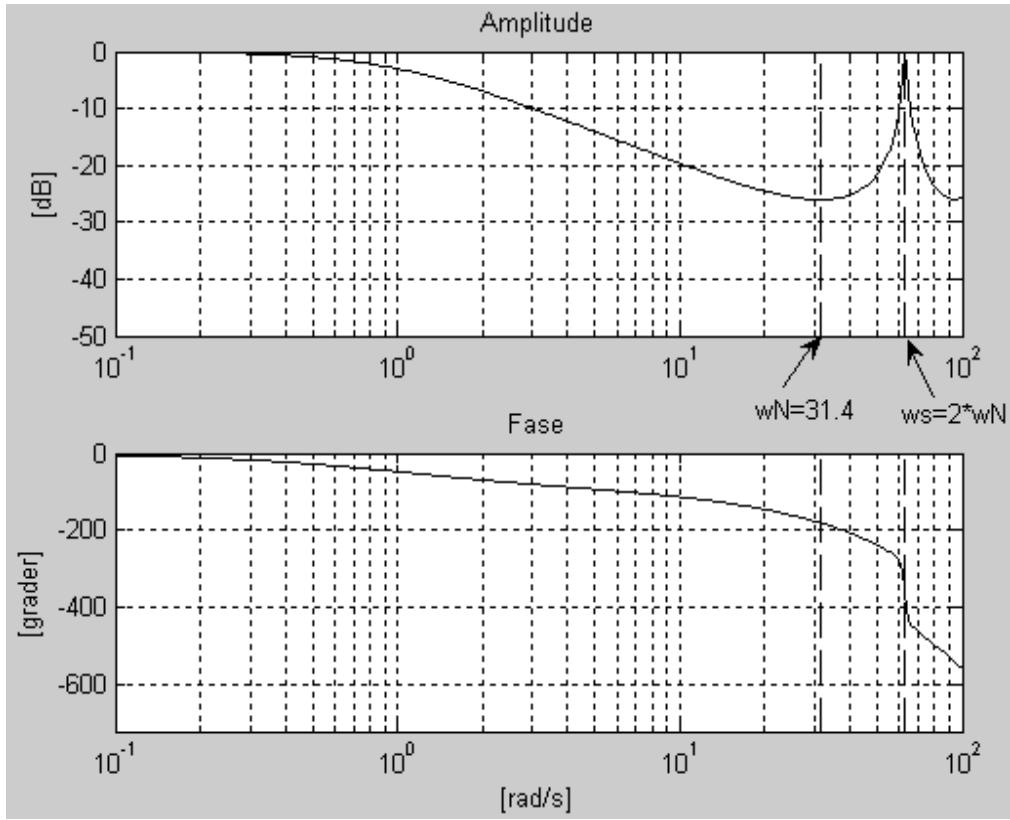


Figure 29.2: Bode plots of frequency response of (29.16). The frequency axis is logarithmic.

29.1 Problems for Chapter 29

Problem 29.1 Gain and amplitude functions

Given the following transfer function:

$$H(z) = \frac{1}{z} \quad (29.18)$$

What is the amplitude gain function and the phase lag function?

Problem 29.2 Frequency response

Given a continuous-time filter with transfer function

$$H_{\text{cont}}(s) = \frac{1}{T_f s + 1} \quad (29.19)$$

with

$$T_f = 1 \text{ sec} \quad (29.20)$$

Discretization of $H_{\text{cont}}(s)$ using the Backward method of discretization with time-step T_s gives the following discrete-time filter:

$$H_{\text{disc}}(z) = \frac{az}{z - (1 - a)} \quad (29.21)$$

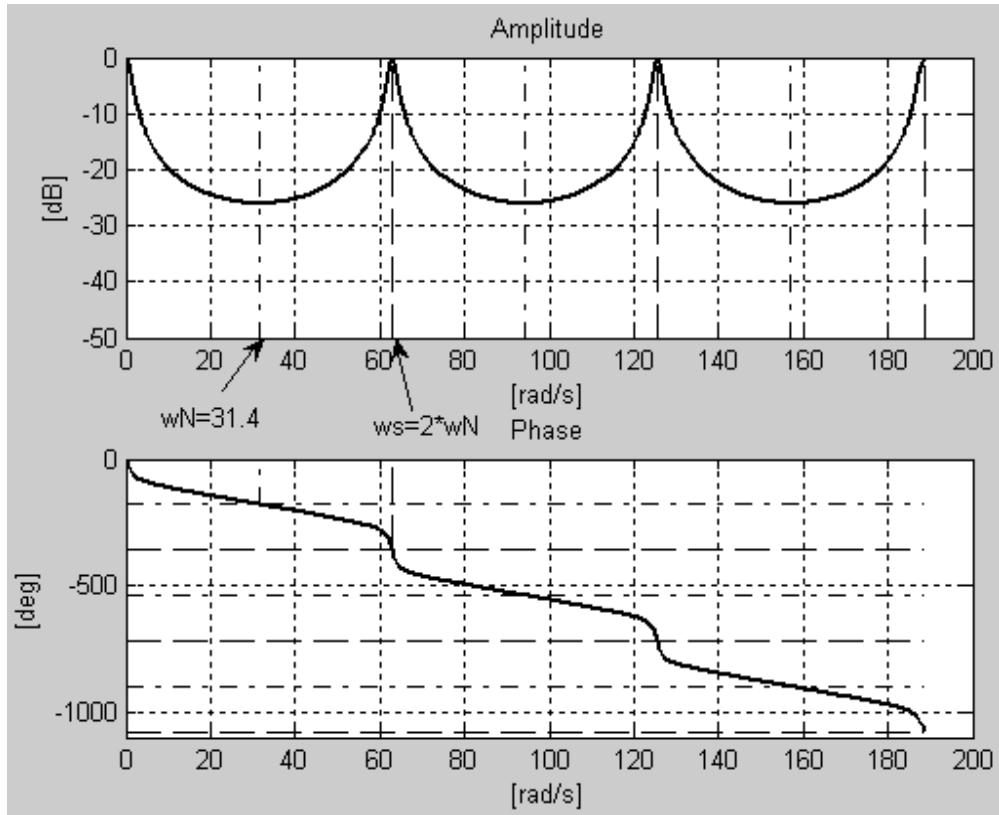


Figure 29.3: Bode plots of frequency response of (29.17). The frequency axis is linear to make the symmetries of the frequency responses clearer.

where

$$a = \frac{T_s}{T_f + T_s} \quad (29.22)$$

Figure 29.4 shows the frequency responses of $H_{\text{cont}}(s)$ and $H_{\text{disc}}(z)$ with time-step

$$T_s = 0.2 \text{ sec} \quad (29.23)$$

1. Why is there a difference between the frequency responses of $H_{\text{cont}}(s)$ and $H_{\text{disc}}(z)$, and why is the difference more apparent at higher frequencies than at lower frequencies? (Qualitative answers are ok.)
2. The frequency response curves of $H_{\text{disc}}(z)$ are unique up to a certain frequency – which frequency? Express this frequency in Hz and rad/s. Is that frequency indicated in Figure 29.4?

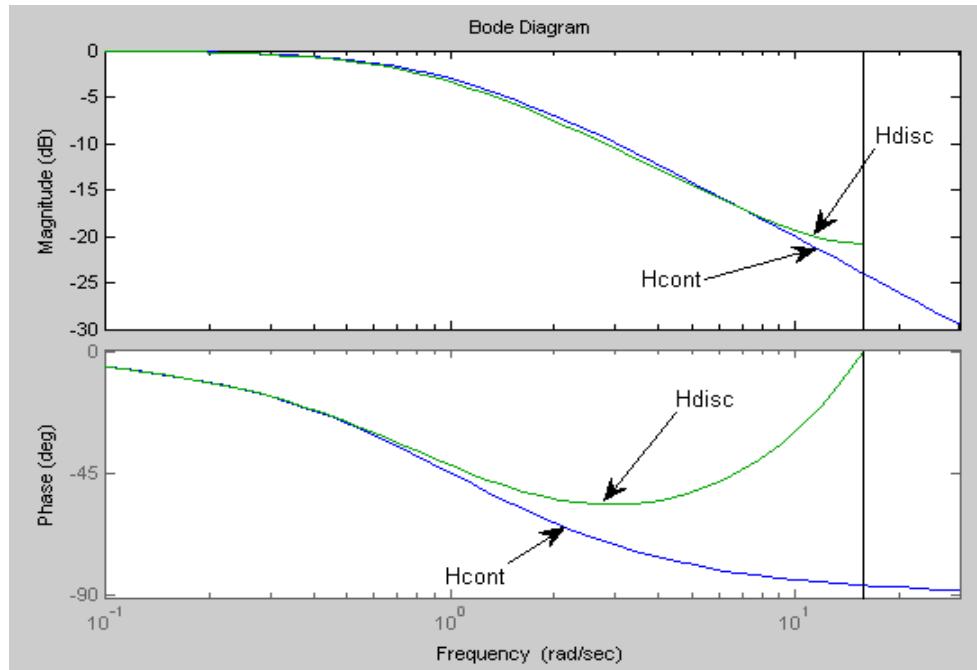


Figure 29.4: Frequency response

29.2 Solutions to problems for Chapter 29

Solution to Problem 29.1

The amplitude gain function is

$$\underline{\underline{A(\omega)}} = |H(e^{j\omega T_s})| = \left| \frac{1}{e^{j\omega T_s}} \right| = \frac{1}{|e^{j\omega T_s}|} = \frac{1}{1} = \underline{\underline{1}} \quad (29.24)$$

The phase lag function is

$$\underline{\underline{\phi(\omega)}} = \arg H(e^{j\omega T_s}) = \arg \frac{1}{e^{j\omega T_s}} = \arg e^{-j\omega T_s} = \underline{\underline{-\omega T_s}} = \underline{\underline{-\omega \cdot 0.05 \text{ [rad]}}} \quad (29.25)$$

Solution to Problem 29.2

- There is a difference between the frequency responses of $H_{cont}(s)$ and $H_{disc}(z)$ because the discrete-time filter is derived from the continuous-time filter using an *approximation* of the time-derivatives in the filter model when that model is written as a differential equation.

An explanation of why the difference increases with increasing frequency is that the approximation of the time-derivative becomes less accurate if signals vary faster (i.e. have higher frequency).

- The frequency response curves of $H_{disc}(z)$ are unique up to the Nyquist frequency:

$$\underline{\underline{f_N}} = \frac{1}{T_s} = \frac{1}{0.2} = \underline{\underline{5 \text{ Hz}}} \quad (29.26)$$

$$\underline{\underline{\omega_N}} = \frac{\pi}{T_s} = \frac{\pi}{0.2} = \underline{\underline{5\pi \text{ rad/s}}} = 15.7 \text{ rad/s} \quad (29.27)$$

which *is* the frequency indicated with a vertical line in Figure 29.4.

Chapter 30

Stability analysis of discrete-time dynamic systems

30.1 Definition of stability properties

Assume given a dynamic system with input u and output y . The stability property of a dynamic system can be defined from the *impulse response*¹ of a system as follows:

- **Asymptotic stable system:** The steady state impulse response is zero:

$$\lim_{k \rightarrow \infty} y_\delta(k) = 0 \quad (30.1)$$

- **Marginally stable system:** The steady state impulse response is different from zero, but limited:

$$0 < \lim_{k \rightarrow \infty} y_\delta(k) < \infty \quad (30.2)$$

- **Unstable system:** The steady state impulse response is unlimited:

$$\lim_{k \rightarrow \infty} y_\delta(k) = \infty \quad (30.3)$$

The impulse response for the different stability properties are illustrated in Figure 30.1.

(The simulated system is defined in Example 30.1.)

30.2 Stability analysis of transfer function models

In the following we will base the analysis on the following fact: *The transfer function is the z-transformed impulse response.* Here is the proof of this fact: Given a system with transfer

¹An impulse $\delta(0)$ is applied at the input.

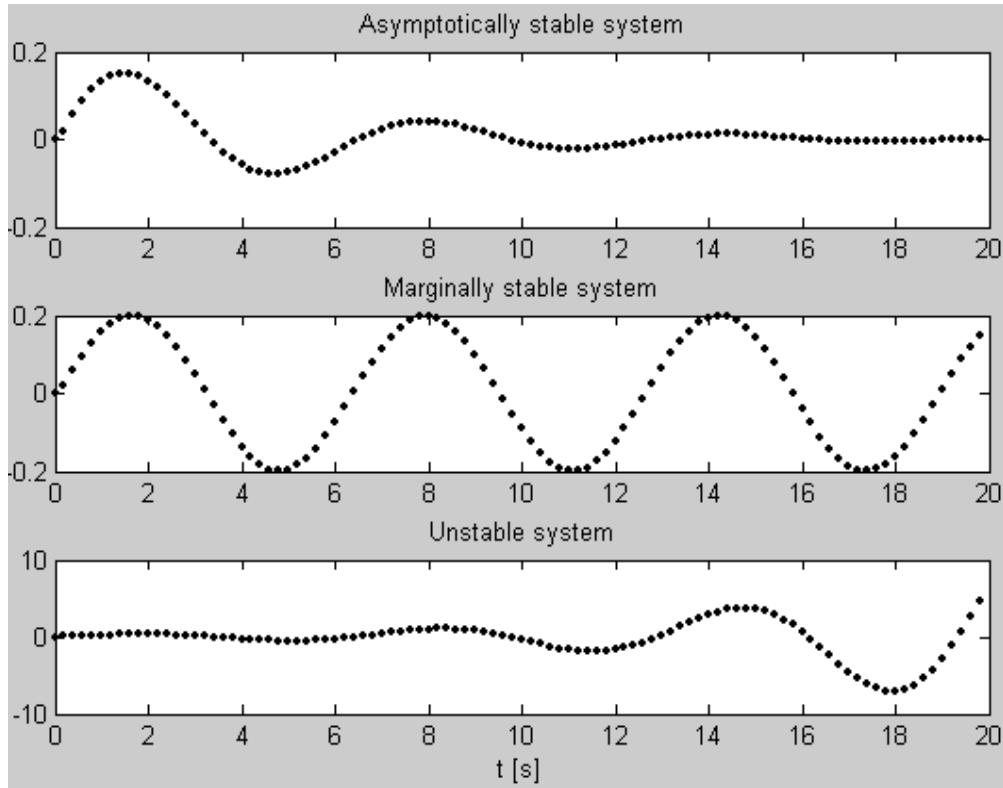


Figure 30.1: Impulse response and stability properties

function $H(z)$. Assume that the input u is an impulse, which is a signal having value 1 at time index $k = 0$ and value zero at other points of time. According to (27.7) $u(z) = 1$. Then the z -transformed impulse response is

$$y(z) = H(z)u(z) = H(z) \cdot 1 = H(z) \quad (30.4)$$

(as stated).

Now, we proceed with the stability analysis of transfer functions. The impulse response $y_\delta(k)$, which defines the stability property of the system, is determined by the poles of the system's poles and zeros since the impulse responses is the inverse z -transform of the transfer function:

$$y_\delta(k) = \mathcal{Z}^{-1}\{H(z)\} \quad (30.5)$$

Consequently, the stability property is determined by the poles and zeros of $H(z)$. However, we will soon see that only the poles determine the stability.

We will now derive the relation between the stability and the poles by studying the impulse response of the following system:

$$H(z) = \frac{y(z)}{u(z)} = \frac{bz}{z-p} \quad (30.6)$$

The pole is p . Do you think that this system is too simple as a basis for deriving general conditions for stability analysis? Actually, it is sufficient because we can always think that a

given z -transfer function can be partial fractionated in a sum of partial transfer functions or terms each having one pole. Using the superposition principle we can conclude about the stability of the original transfer function.

In the following, cases having of multiple (coinciding) poles will be discussed, but the results regarding stability analysis will be given.

The system given by (30.6) has the following impulse response calculated below. It is assumed that the pole in general is a complex number which may be written on polar form as

$$p = me^{j\theta} \quad (30.7)$$

where m is the magnitude and θ the phase. The impulse response is

$$y_\delta(k) = \mathcal{Z}^{-1} \left\{ \frac{bz}{z - p} \right\} \quad (30.8)$$

$$= \mathcal{Z}^{-1} \left\{ \frac{p}{1 - pz^{-1}} \right\} \quad (30.9)$$

$$= \mathcal{Z}^{-1} \left\{ b \sum_{k=0}^{\infty} p^k z^{-k} \right\} \quad (30.10)$$

$$= bp^k \quad (30.11)$$

$$= b|m|^k e^{jk\theta} \quad (30.12)$$

From (30.12) we see that it is the *magnitude* m which determines if the steady state impulse response converges towards zero or not. From (30.12) we can now state the following relations between stability and pole placement (the statements about multiple poles have however not been derived here):

- **Asymptotic stable system:** All poles lie inside (none is on) the unit circle, or what is the same: all poles have magnitude less than 1.
- **Marginally stable system:** One or more poles – but no multiple poles – are on the unit circle.
- **Unstable system:** At least one pole is outside the unit circle. Or: There are multiple poles on the unit circle.

The “stability areas” in the complex plane are shown in Figure 30.2.

Let us return to the question about the relation between the *zeros* and the stability. We consider the following system:

$$H_1(z) = \frac{y(z)}{u(z)} = \frac{b(z - c)}{z - p} = (z - c)H(z) \quad (30.13)$$

where $H(z)$ is it the “original” system (without zero) which were analyzed above. The zero is c . $H_1(z)$ can be written as

$$H_1(z) = \frac{bz}{z - p} + \frac{-bc}{z - p} \quad (30.14)$$

$$= H(z) - cz^{-1}H(z) \quad (30.15)$$

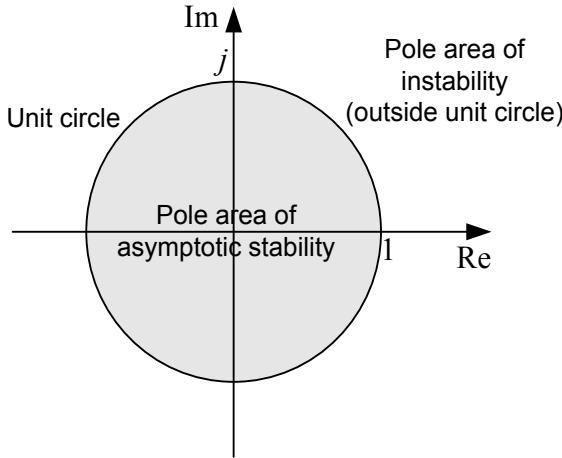


Figure 30.2: The different stability property areas of the complex plane

The impulse response of $H_1(z)$ becomes

$$y_{\delta_1}(k) = y_{\delta}(k) - cy_{\delta}(k-1) \quad (30.16)$$

where $y_{\delta}(k)$ is the impulse response of $H(z)$. We see that the zero does not influence whether the steady state impulse response converges towards zero or not. We draw the conclusion that the zeros of the transfer function do not influence the stability of the system.

Example 30.1 Stability analysis of discrete-time system

The three responses shown in Figure 30.1 are actually the impulse responses in three systems each having a transfer function on the form

$$\frac{y(z)}{u(z)} = H(z) = \frac{b_1 z + b_0}{z^2 + a_1 z + a_0} \quad (30.17)$$

The parameters of the systems are given below:

1. Asymptotically stable system: $b_1 = 0.019$, $b_0 = 0.0190$, $a_1 = -1.885$ and $a_0 = 0.923$.
The poles are

$$z_{1,2} = 0.94 \pm j0.19 \quad (30.18)$$

They are shown in Figure 30.3 (the zero is indicated by a circle). The poles are inside the unity circle.

2. Marginally stable system: $b_1 = 0.020$, $b_0 = 0.020$, $a_1 = -1.96$ and $a_0 = 1.00$. The poles are

$$z_{1,2} = 0.98 \pm j0.20 \quad (30.19)$$

They are shown in Figure 30.3. The poles are on the unity circle.

3. Unstable system: $b_1 = 0.021$, $b_0 = 0.021$, $a_1 = -2.04$ and $a_0 = 1.08$. The poles are

$$z_{1,2} = 1.21 \pm j0.20 \quad (30.20)$$

They are shown in Figure 30.3. The poles are outside the unity circle.

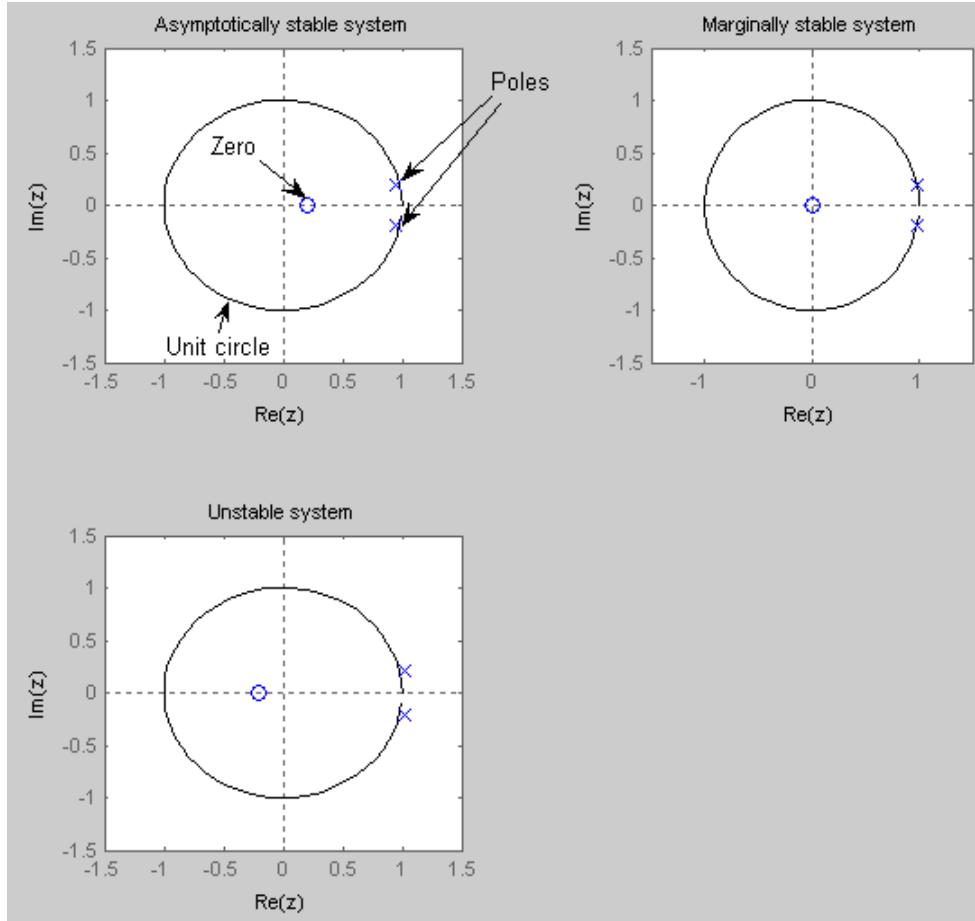


Figure 30.3: Example 30.1: Poles (and zeros) for the three systems each having different stability property

[End of Example 30.1]

30.3 Stability analysis of state space models

Assume that the system has the following state space model:

$$x_{k+1} = Ax_k + Bu_k \quad (30.21)$$

$$y_k = Cx_k + Du_k \quad (30.22)$$

We can determine the stability by finding the corresponding transfer function from u to y , and then calculating the poles from the transfer function, as we did in the previous section. Let us derive the transfer function: Take the \mathcal{Z} -transform of eqs. (30.21) – (30.22) to get (I is the identity matrix of equal dimension as of A)

$$zIx(z) = Ax(z) + Bu(z) \quad (30.23)$$

$$y(z) = Cx(z) + Du(z) \quad (30.24)$$

Solving (30.23) for $x(z)$ gives

$$x(z) = (zI - A)^{-1}Bu(z) \quad (30.25)$$

Inserting this $x(z)$ into (30.24) gives

$$y(z) = [C(zI - A)^{-1}B + D] u(z) \quad (30.26)$$

So, the transfer function is

$$H(z) = \frac{y(z)}{u(z)} = C(zI - A)^{-1}B + D \equiv C \frac{\text{adj}(zI - A)}{\det(zI - A)} B + D \quad (30.27)$$

The stability property can now be determined from the poles of this transfer function. The poles are the roots of the characteristic equation:

$$\det(zI - A) = 0 \quad (30.28)$$

But (30.64) actually defines the *eigenvalues* of A , $\text{eig}(A)$! The eigenvalues are the z -solutions to eq. (30.28). Therefore, *the poles are equal to the eigenvalues*, and the relation between stability properties and eigenvalues are the same relation as between stability properties and poles, cf. Section 30.2. To make it clear:

- **Asymptotic stable system:** All eigenvalues (poles) lie inside (none is on) the unit circle, or what is the same: All eigenvalues have magnitude less than 1.
- **Marginally stable system:** One or more eigenvalues – but no multiple eigenvalues – are on the unit circle.
- **Unstable system:** At least one eigenvalue is outside the unit circle. Or: There are multiple eigenvalues on the unit circle.

The “stability areas” in the complex plane are as shown in Figure 30.2.

Example 30.2 Stability analysis of a state space model

Given the following state space model:

$$x_{k+1} = \underbrace{\begin{bmatrix} 0.7 & 0.2 \\ 0 & 0.8 \end{bmatrix}}_A x_k + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_k \quad (30.29)$$

$$y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k + \begin{bmatrix} 0 \end{bmatrix} u_k \quad (30.30)$$

It can be shown that the eigenvalues of A are 0.7 and 0.8. Both lies inside the unit circle, and hence the system is asymptotically stable.

[End of Example 30.2]

30.4 Problems for Chapter 30

Problem 30.1 Impulse response

Given the following transfer function:

$$H(z) = \frac{y(z)}{u(z)} = \frac{1}{z - 0.5} \quad (30.31)$$

Calculate the first four values of the impulse response. Determine the stability property from the impulse response.

Problem 30.2 Properties of z -transfer functions

Determine the stability property of the following transfer functions:

$$H_1(s) = \frac{1}{z - 0.5} \quad (30.32)$$

$$H_2(s) = \frac{1}{z + 0.5} \quad (30.33)$$

$$H_3(s) = \frac{z - 2}{z - 0.5} \quad (30.34)$$

$$H_4(s) = \frac{1}{z - 1} \quad (30.35)$$

$$H_5(s) = \frac{1}{z - 2} \quad (30.36)$$

$$H_6(s) = \frac{1}{(z - 1)^2} \quad (30.37)$$

$$H_7(s) = \frac{1}{z} \quad (30.38)$$

$$H_8(z) = \frac{1}{z^2 - 2.5z + 1} \quad (30.39)$$

Problem 30.3 Stability of z -transfer function

Discretizing the continuous-time transfer function

$$H_{\text{con}}(s) = \frac{y(s)}{u(s)} = \frac{K}{Ts + 1} \quad (30.40)$$

using the Forward discretization method with time-step T_s yields the following discrete-time transfer function:

$$H_{\text{dis}}(z) = \frac{y(z)}{u(z)} = \frac{\frac{KT_s}{T}}{z - \left(1 - \frac{T_s}{T}\right)} \quad (30.41)$$

For which (positive) values of T_s is the discrete-time system asymptotically stable? (You can assume that T is positive.)

Problem 30.4 *Stability of state space model*

Determine the stability property of the following state space model.

$$x_{k+1} = \begin{bmatrix} 1 & 0.5 \\ 0 & 0.9 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k \quad (30.42)$$

30.5 *Solutions to problems for Chapter 30*

Solution to Problem 30.1

The impulse response is equal to the transfer function:

$$y_\delta(z) = H(z) = \frac{1}{z - 0.5} \quad (30.43)$$

$$y_\delta(z) = H(z) \quad (30.44)$$

From the z transform pair (27.9),

$$y_\delta(k) = 0.5^k \quad (30.45)$$

The first four values of the impulse response is

$$\{0.5^0, 0.5^1, 0.5^2, 0.5^3, 0.5^4\} = \underline{\underline{\{1.0, 0.5, 0.25, 0.125, 0.0625\}}} \quad (30.46)$$

Obviously, the impulse response converges towards zero as time goes to infinity. Hence, the transfer function is asymptotically stable.

Solution of Problem 30.2

The transfer function

$$H_1(s) = \frac{1}{z - 0.5} \quad (30.47)$$

is asymptotically stable since the pole $p = 0.5$ is inside the unit circle.

The transfer function

$$H_2(s) = \frac{1}{z + 0.5} \quad (30.48)$$

is asymptotically stable since the pole $p = -0.5$ is inside the unit circle.

The transfer function

$$H_3(s) = \frac{z - 2}{z - 0.5} \quad (30.49)$$

is asymptotically stable since the pole $p = 0.5$ is inside the unit circle. (The zero is 2, but the stability property is independent of the value of the zero.)

The transfer function

$$H_4(s) = \frac{1}{z - 1} \quad (30.50)$$

is marginally stable since the pole $p = 1$ is on the unit circle, and that pole is single.

The transfer function

$$H_5(s) = \frac{1}{z - 2} \quad (30.51)$$

is unstable since the pole $p = 2$ is outside the unit circle.

The transfer function

$$H_6(s) = \frac{1}{(z - 1)^2} \quad (30.52)$$

is unstable since there are multiple (two) poles, namely $p_{1,2} = 1$, on the unit circle.

The transfer function

$$H_7(s) = \frac{1}{z} \quad (30.53)$$

is asymptotically stable since the pole $p = 0$ is inside the unit circle.

The transfer function

$$H_8(z) = \frac{1}{z^2 - 2.5z + 1} \quad (30.54)$$

is unstable since one of the poles is outside the unit circle. The poles are $p_{1,2} = 0.5, 2.0$.

Solution to Problem 30.3

The pole of (30.41) is

$$p = 1 - \frac{T_s}{T} \quad (30.55)$$

The system is asymptotically stable if

$$\left| p = 1 - \frac{T_s}{T} \right| < 1 \quad (30.56)$$

If

$$1 - \frac{T_s}{T} > 0 \quad (30.57)$$

(30.56) becomes

$$1 - \frac{T_s}{T} < 1 \quad (30.58)$$

which gives

$$\underline{T_s > 0} \quad (30.59)$$

which is always satisfied.

If

$$1 - \frac{T_s}{T} < 0 \quad (30.60)$$

(30.56) becomes

$$-\left(1 - \frac{T_s}{T}\right) < 1 \quad (30.61)$$

which gives

$$\underline{\frac{T_s}{2} < \frac{T}{2}} \quad (30.62)$$

So, the system is asymptotically stable if

$$\underline{\underline{T_s < \frac{T}{2}}}$$
 (30.63)

Solution to Problem 30.4

The stability property is determined by the system eigenvalues, which are the roots of the characteristic equation:

$$\det(zI - A) = \det \left(z \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0.5 \\ 0 & 0.9 \end{bmatrix} \right) \quad (30.64)$$

$$= \det \left(\begin{bmatrix} z-1 & 0.5 \\ 0 & z-0.9 \end{bmatrix} \right) \quad (30.65)$$

$$= (z-1)(z-0.9) - 0 \cdot 0.5 \quad (30.66)$$

$$= (z-1)(z-0.9) \quad (30.67)$$

The roots are

$$z_{1,2} = 1, 0.9 \quad (30.68)$$

One pole is on the unit circle, and the other pole is inside the unit circle. Thererfore, the system is marginally stable.

Chapter 31

Stability analysis of discrete-time feedback systems

You can analyze the dynamics (frequency response) and the stability of discrete-time feedback systems in the same way as you can analyze the dynamics and stability of continuous-time feedback systems, cf. Chapters 23 and 21, respectively. I assume that you already have knowledge about these topics.

Here is a summary of the relevant differences between analysis of continuous-time and discrete-time feedback systems:

- In the block diagrams etc. every s -transfer function is replaced by an equivalent z -transfer function, using a proper discretization method, e.g, the ZOH method.
- The stability property of any discrete-time system is given by the placement of the z -poles (or eigenvalues) in the complex plane. A discrete-time feedback system is asymptotically stable if all the poles of the closed-loop system lie inside the unit circle. These closed-loop poles are the poles of the tracking transfer function, $T(z)$.
- The stability property of a discrete-time feedback system can be analyzed in a Nyquist diagram or a Bode diagram stability based on the frequency response of the loop transfer function $L(z)$, which is the product of all the individual transfer functions of the feedback loop. Definitions of crossover frequencies and stability margins are as for continuous-time systems.

Here are some examples:

Example 31.1 Pole based stability analysis of feedback system

Assume given a control system where the P controller

$$H_c(z) = K_c \quad (31.1)$$

controls the process (which is actually an integrating process)

$$H_p(z) = \frac{K_i T_s}{z - 1} \quad (31.2)$$

We assume that $K_i = 1$ and $T_s = 1$. The loop transfer function becomes

$$L(z) = H_c(z)H_p(z) = \frac{K_c}{z - 1} = \frac{n_L(z)}{d_L(z)} \quad (31.3)$$

We will calculate the range of values of K_c that ensures asymptotic stability of the control system.

The characteristic polynomial is, cf. (21.34),

$$c(z) = d_L(z) + n_L(z) = z - 1 + K_c \quad (31.4)$$

The pole is

$$p = 1 - K_c \quad (31.5)$$

The feedback system is asymptotically stable if p is inside the unity circle or has magnitude less than one:

$$|p| = |1 - K_c| < 1 \quad (31.6)$$

which is satisfied with

$$0 < K_c < 2 \quad (31.7)$$

Assume as an example that $K_c = 1.5$. Figure 31.1 shows the step response in y_m for this value of K_c .

[End of Example 31.1]

Example 31.2 Stability analysis in Nyquist diagram

Given the following continuous-time process transfer function:

$$H_p(s) = \frac{y_m(z)}{u(z)} = \frac{K}{\left(\frac{s}{\omega_0}\right)^2 + 2\zeta\frac{s}{\omega_0} + 1} e^{-\tau s} \quad (31.8)$$

with parameter values

$$K = 1; \zeta = 1; \omega_0 = 0.5 \text{ rad/s}; \tau = 1 \text{ s} \quad (31.9)$$

The process is controlled by a discrete-time PI-controller having the following z -transfer function, which can be derived by taking the z -transform of the PI control function presented in Ch. 12.5:

$$H_c(z) = \frac{K_p \left(1 + \frac{T_s}{T_i}\right) z - K_p}{z - 1} \quad (31.10)$$

where the time-step (or sampling interval) is

$$T_s = 0.2 \text{ s} \quad (31.11)$$

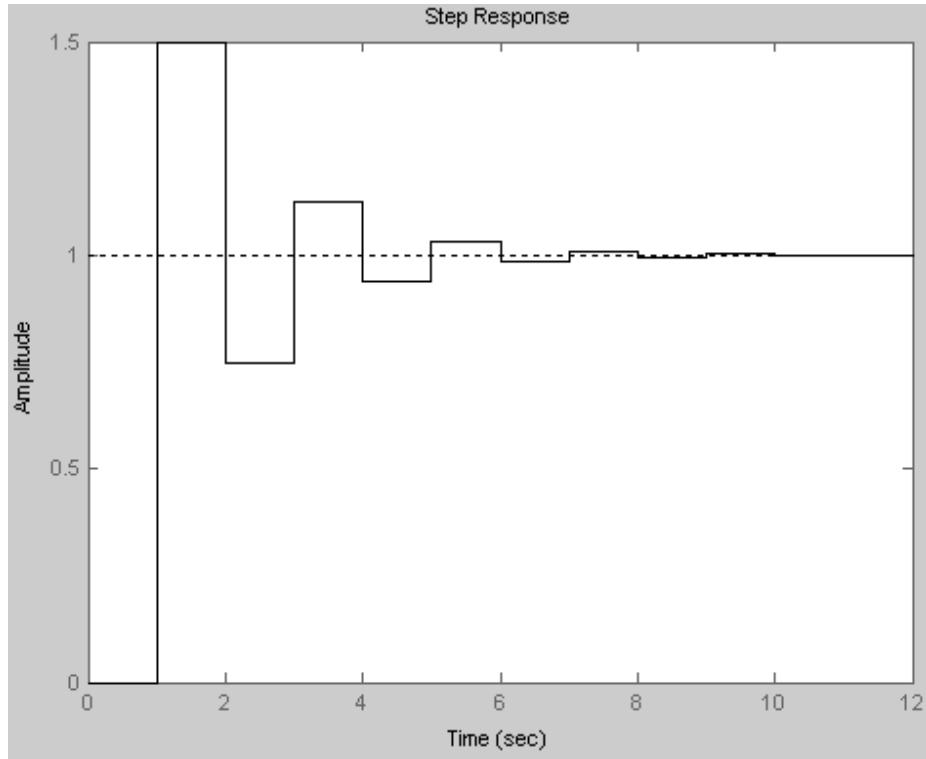


Figure 31.1: Example 31.1: Step response in y_m . There is a step in $y_{m_{SP}}$.

Tuning the controller with the Ziegler-Nichols closed-loop method ? in a simulator gave the following controller parameter settings:

$$K_p = 2.0; T_i = 5.6 \text{ s} \quad (31.12)$$

To perform the stability analysis of the discrete-time control system $H_p(s)$ is discretized assuming zero order hold (using MATLAB or LabVIEW). The result is

$$H_{pd}(z) = \frac{0.001209z + 0.001169}{z^2 - 1.902z + 0.9048} z^{-10} \quad (31.13)$$

The loop transfer function is

$$L(z) = H_c(z)H_{pd}(z) \quad (31.14)$$

Figure 31.2 shows the Nyquist plot of $L(z)$.

From the Nyquist diagram we read off

$$\omega_{180} = 0.835 \text{ rad/s} \quad (31.15)$$

and

$$\operatorname{Re} L(e^{j\omega_{180}T_s}) = -0.558 \quad (31.16)$$

which gives the following gain margin, cf. (23.39),

$$GM = \frac{1}{|\operatorname{Re} L(e^{j\omega_{180}T_s})|} = \frac{1}{|-0.558|} = 1.79 = 5.1 \text{ dB} \quad (31.17)$$

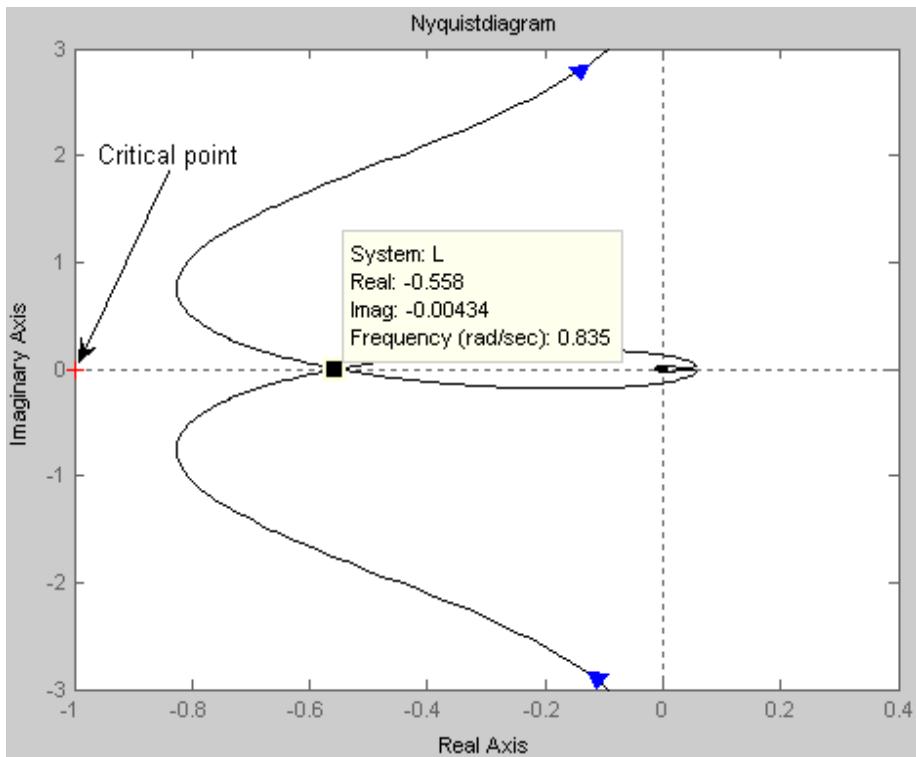


Figure 31.2: Example 31.2: Nyquist diagram of $L(z)$

The phase margin can be found to be

$$\text{PM} = 35^\circ \quad (31.18)$$

Figure 31.3 shows the step response in y_m (unity step in setpoint $y_{m_{SP}}$).

[End of Example 31.2]

Example 31.3 Stability analysis in Bode diagram

See Example 31.2. Figure 31.4 shows a Bode plot of $L(e^{j\omega T_s})$.

The stability margins are shown in the figure. They are

$$\text{GM} = 5.12 \text{dB} = 1.80 \quad (31.19)$$

$$\text{PM} = 35.3^\circ \quad (31.20)$$

which is in accordance with Example 31.2.

[End of Example 31.3]

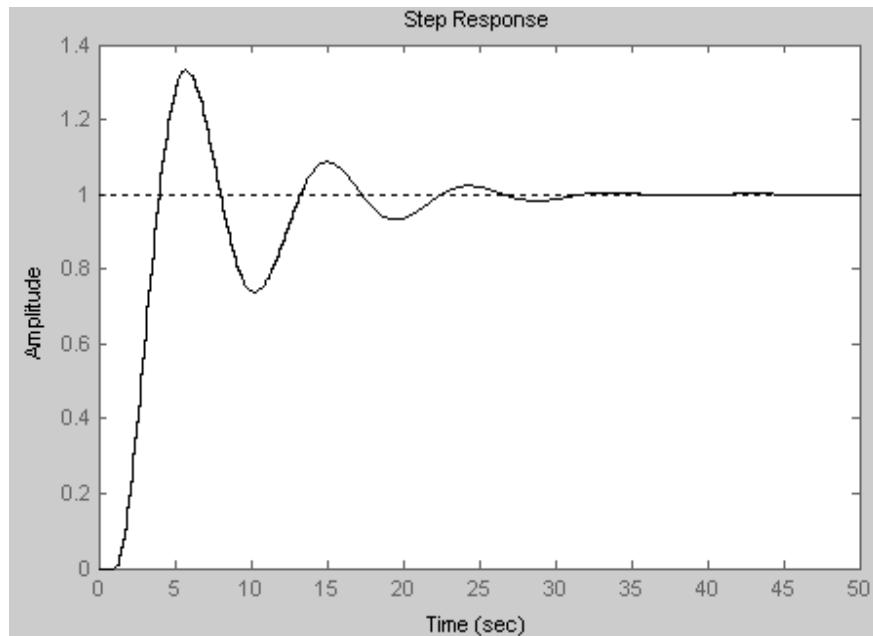


Figure 31.3: Example 31.2: Step response in y_m (unity step in setpoint $y_{m_{SP}}$)

31.1 *Problems for Chapter 31*

Problem 31.1

Given a feedback control system where the controller (a P controller)

$$H_c(z) = K_p \quad (31.21)$$

controls a process. The transfer function of the combined process and sensor (these systems are in series) is

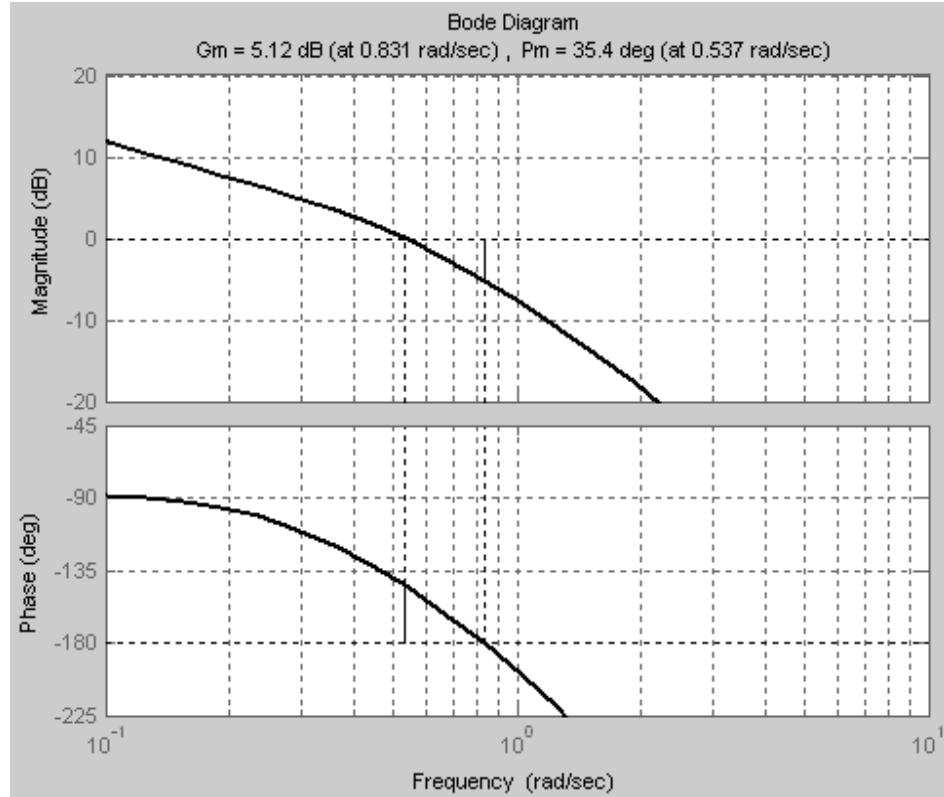
$$H_{pm}(z) = \frac{z^{-1}}{1 - z^{-1}}$$

Calculate for which values of the controller gain K_c the control system is asymptotically stable.

Problem 31.2

Assume that Figure 31.5 shows the Bode diagram of the loop transfer function of some discrete-time control system.

Read off the stability margins GM and PM , and the crossover frequencies ω_c and ω_{180} in the Bode diagram.


 Figure 31.4: Example 31.3: Bode plot of L

31.2 Solutions to problems for Chapter 31

Solution to Problem 31.1

The loop transfer function is

$$L(z) = H_c(z)H_{pm}(z) = K_p \frac{z^{-1}}{1 - z^{-1}} \quad (31.22)$$

The tracking transfer function is

$$T(z) = \frac{L(z)}{1 + L(z)} = \frac{K_p \frac{z^{-1}}{1 - z^{-1}}}{1 + K_p \frac{z^{-1}}{1 - z^{-1}}} = \frac{K_p}{z - 1 + K_p} = \frac{K_p}{z - (1 - K_p)} \quad (31.23)$$

The pole of $T(z)$ is

$$p = 1 - K_p \quad (31.24)$$

The control system is asymptotically stable if the absolute value of the pole is less than one:

$$|p| = |1 - K_p| < 1 \quad (31.25)$$

which is obtained with

$$\underline{\underline{0 < K_p < 2}} \quad (31.26)$$

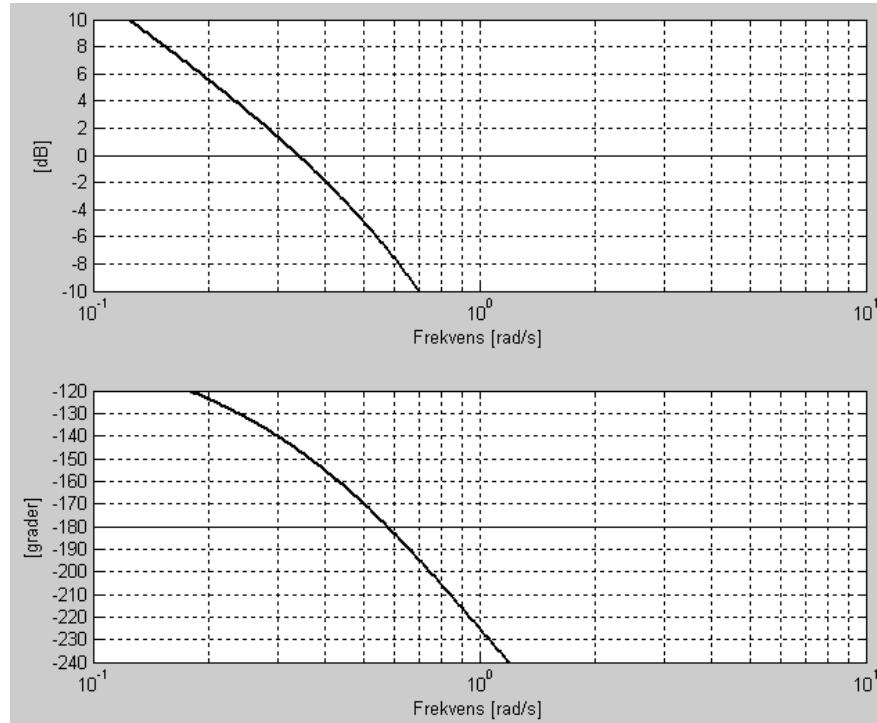


Figure 31.5: Bode diagram

Solution to Problem 31.2

From the Bode diagram:

$$\underline{\underline{GM = 7 \text{ dB} = 2.2}} \quad (31.27)$$

$$\underline{\underline{PM = 35^\circ}} \quad (31.28)$$

$$\underline{\underline{\omega_c = 0.34 \text{ rad/s}}} \quad (31.29)$$

$$\underline{\underline{\omega_{180} = 0.58 \text{ rad/s}}} \quad (31.30)$$

Chapter 32

Subspace identification of blackbox discrete-time models

A black-box model is a dynamic model in the form of a state space model or a transfer function with *non-physical* parameters. A black-box model represents the dynamic properties of the system and does not contain explicit information about the physics of the system. Such models are useful in many situations, for example in model-based analysis and design (as controller tuning) of control systems and in signal modeling.

It is common to have black-box models in the form of *discrete-time models* – not continuous-time models. This is because discrete-time models are more directly related to the discrete-time nature of the sampled data (time-series) from experiments from which the black-box model is developed.

Once you have a discrete-time model – a transfer function model or a state space model, you can analyze it in many ways:

- Run simulations
- Plot frequency response
- Plot poles or eigenvalues
- Transform to equivalent continuous-time model to find e.g. time constants or other dynamic characteristics.

How can you estimate a black-box model in practice? Several estimation methods are available?, as the least squares method described earlier in this chapter, the prediction error method, and a subspace method. I suggest you using a *subspace method* . These are effective, and easy to use in practice. Subspace methods estimate a discrete-time black-box state space model on the form of a linear – possibly multivariable – state space model

$$x_{k+1} = Ax_k + Bu_k + Ke_k \quad (32.1)$$

$$y_k = Cx_k + Du_k + e_k \quad (32.2)$$

with initial state $x(0)$. x_k is the state vector, which is actually calculated during the estimation. u_k is the known input vector, y_k is the known output vector, e_k is the (white or random) noise-vector, and A , B , C , D , and K are coefficient matrices. t_{k+1} means the time at timestep no. k which is an integer time-index. $t_k = kT_s$ where T_s [sec] is the timestep or sampling time. The order n of this model is the number of states, which is the number of elements in the state-vector.

From this state space model a z -transfer function from u til y can be calculated with the following formula:

$$H_{y,u}(z) = C(zI - A)^{-1}B + D \quad (32.3)$$

This formula can be derived by taking the \mathcal{Z} -transform of (32.1) – (32.2) and then solving for output $y(z)$, cf. Section 30.3. However, in practice you will probably use a proper function in e.g. MATLAB or LabVIEW, cf. the examples below.

All the five matrices in (32.1) – (32.2) are estimated. These matrices are assumed having some special canonical forms. Even the initial state is estimated, from known time-series of the input u and the corresponding output y . Once a state space model is estimated, a transfer function may be calculated using (32.3). You must select the model order n so that you are content with the accuracy of the model. This can be done by running simulations with the model for different orders n , see Figure 11.4.

Note that if you assume that the system contains a time delay of T_d [s], you need at least the following model order to include the time delay in the model properly:

$$n_{\min} = \frac{T_d}{T_s} \quad (32.4)$$

where T_s is the sampling time.

In the following are two examples about estimating a black-box model of an electrical motor using subspace estimation, with MATLAB (System Identification Toolbox) and LabVIEW (System Identification Toolkit), respectively.

Example 32.1 Subspace model estimation in MATLAB

Figure 32.1 shows an electrical DC motor. It is manipulated with an input voltage signal, u , and the rotational speed is measured with a tachometer which produces a output voltage signal, y , which is proportional to the speed.

During one experiment lasting for about 17 seconds, u was adjusted manually, and both the sequences (time series) of u_k and y_k were saved to a file. The sampling time was 0.02 s. Figure 32.2 shows a small extract of the log file as displayed with Notepad.¹

The first column contains time stamps, but they are not used in this application. The second column contains the input sequence $u(t_k)$, and the third column contains the output sequence, $y(t_k)$.

Actually, the input and output sequences were divided into two parts:

¹The whole logfile, named logfile1.lvm, is available from the home page of this book at <http://techteach.no>.

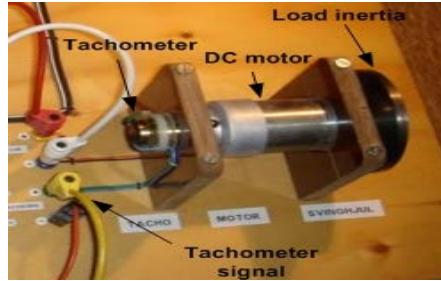


Figure 32.1: Example 32.1: Electrical (DC) motor for which an s -transfer function is estimated.

logfile1.lvm - Notisblokk			
Fil	Rediger	Format	Vis
33.460	2.200	1.761	
33.480	2.200	1.826	
33.500	2.200	1.773	
33.520	2.200	1.814	
33.540	2.200	1.785	
33.560	2.200	1.826	
33.580	-2.410	1.814	
33.600	-2.410	1.802	
33.620	-2.410	1.843	
33.640	-2.410	1.415	
33.660	-2.410	1.093	

Figure 32.2: Example 32.1: An extract of the log file.

- The first half, named u_{estim} and y_{estim} , were used to estimate a transfer function.
- The second half, named u_{valid} and y_{valid} , were used to check that the model is a good model. This is done by simulating the model with u_{valid} as input signal and comparing the simulated response, y_{sim} , with y_{valid} . If y_{valid} is quite similar to y_{sim} we can conclude that the model is good.

The estimation is made with the **n4sid**-function in MATLAB's System Identification Toolbox.

Below is the MATLAB-code which accomplishes the task.

```
%Loads data from file into workspace.
load logfile1.lvm;
Ts=0.02; %Sampling interval
L=length(logfile1);%(Matrix name becomes same as logfile name.)
N=round(L/2);
%Generates proper time signal, and extracts data from logfile:
```

```

t_estim=Ts*[1:N]';
u_estim=logfile1(1:N,2);
y_estim=logfile1(1:N,3);
t_valid=Ts*[N+1:L]';
u_valid=logfile1(N+1:L,2);
y_valid=logfile1(N+1:L,3);
modelorder=1;%Defines order of estimated model.

%Estimation of model. Model is on internal theta-format:
model_est=n4sid([y_estim u_estim],modelorder);
%th2tf-function calculates numerator and denominator coeff. arrays
%in z-transfer function:
[num,den]=th2tf(model_est);
H_disc=tf(num,den,Ts); %Generates an LTI-model from z-transf func.

%Simulation:
y_sim=lsim(H_disc,u_valid,t_valid);%Simulates with u_valid as input.
figure(1)

%Plots y_estim and u_estim:
plot(t_estim,[y_estim,u_estim]);
ylabel(' [V]'); xlabel('t [s]')
figure(2)
%Plots y_sim, y_valid, and u_valid.
plot(t_valid,[y_sim,y_valid,u_valid]);
ylabel(' [V]'); xlabel('t [s]')

%Converts to s-transfer function:
H_cont=d2c(H_disc,'zoh')

```

Figure 32.3 shows the input u_{estim} and output y_{estim} used for the estimation.

I selected order $n = 1$ for the model because it seemed not to be any large improvement in using a higher order, and according to the parsimony principle of system identification, you should select the simplest model among proper models. Figure 32.4 shows the input u_{valid} and output y_{valid} used for validating the model, together with the simulated output y_{sim} .

Since y_{valid} is quite similar to y_{sim} we can conclude that the model is good.

The resulting discrete-time transfer function model was

$$H_{disc}(z) = \frac{0.05788}{z - 0.9344} \quad (32.5)$$

This model was converted to the following continuous-time transfer function using the **d2c** function:

$$H_{cont}(s) = \frac{2.993}{s + 3.393} \quad (32.6)$$

$$= \frac{2.993/3.393}{(1/3.393)s + 1} = \frac{0.88}{0.29s + 1} = \frac{K}{Ts + 1} \quad (32.7)$$

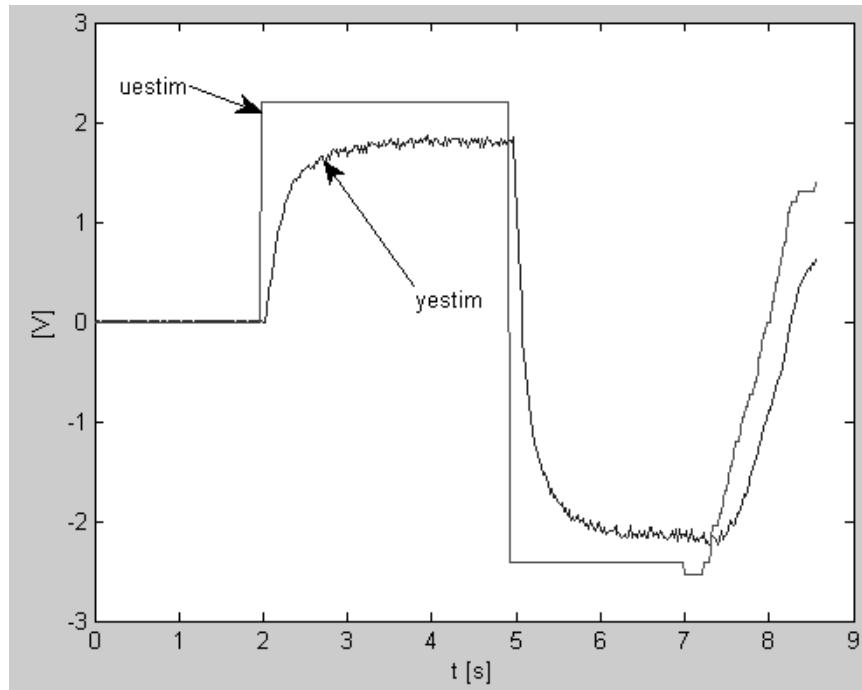


Figure 32.3: Example 32.1: The input $uestim$ and output $yestim$ used for the estimation.

Thus, the gain is 0.88, and the time constant is 0.29 sec. I know from experience with the motor that these are good values!

[End of Example 32.1]

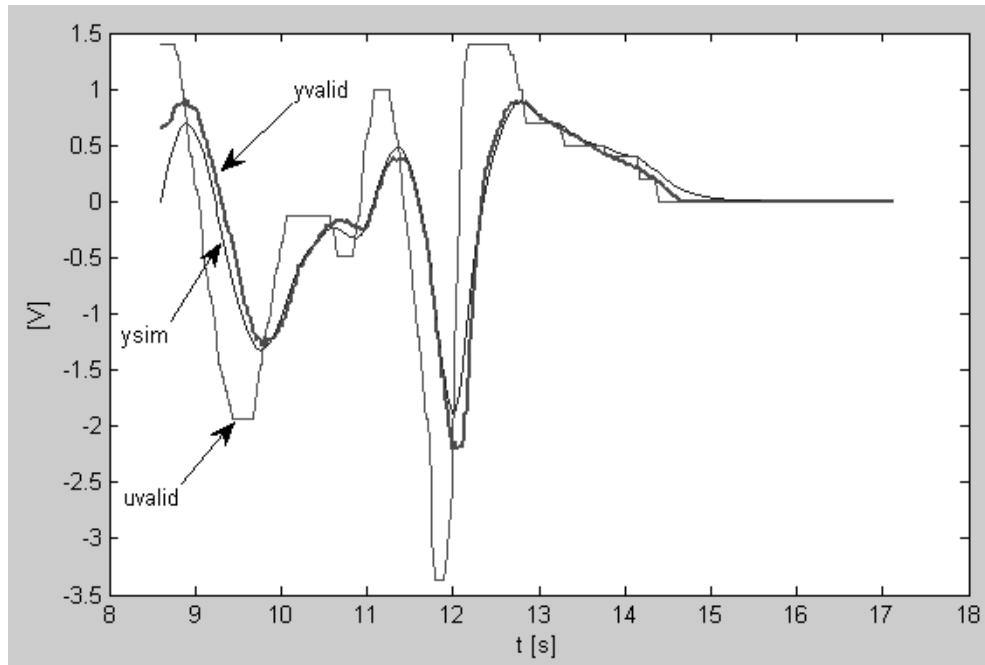


Figure 32.4: Example 32.1: The input u_{valid} and output y_{valid} used for validating the model, together with the simulated output y_{sim} .

32.1 *Problems for Chapter 32*

To appear.

32.2 *Solutions to problems for Chapter 32*

To appear.

Part VI

STATE ESTIMATION

Chapter 33

Stochastic signals

33.1 Introduction

In practical systems there are signals that vary more or less randomly. For example, measurement signals contain random noise, and process disturbances have some random component. Consequently, control signals and controlled variables, i.e. process output variables, have some random behaviour. The future value of a random signal can not be predicted precisely, i.e. such signals are non-deterministic, while steps, ramps and sinusoids are deterministic. Instead, random signals can be described with statistical measures, typically *expectation (or mean) value* and *standard deviation or variance (standard deviation is square root of variance)*.

Random signals may be denoted *stochastic signals*. Characteristics of assumed random process disturbances and random measurement noise are used in design of state estimators with Kalman Filters in Chapter 34.

33.2 How to characterize stochastic signals

33.2.1 Realizations of stochastic processes

A *stochastic process* may be characterized by its mean and standard deviation or variance. The stochastic process can be observed via one or more *realizations* of the process in the form of a *sequence* or time-series of samples, say $\{x_0, x_1, x_2 \dots\}$. Another realization of the same stochastic process will certainly show different sample values, but the mean value and the variance will be almost the same (the longer the realization sequence is, the more equal the mean values and the variances will be). Figure 33.1 shows as an example *two different* realizations (sequences) of the same stochastic process, which in this case is Gaussian (normally) distributed with expectation (mean) value 0 and standard deviation 1. We see that the sequences are not equal.

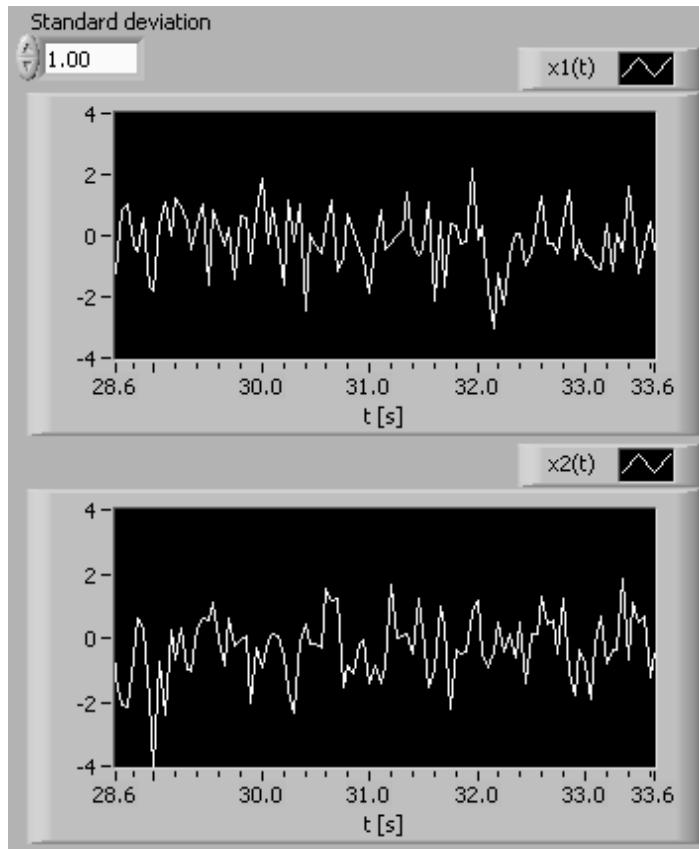


Figure 33.1: Two different realizations of the same stochastic process, which in this case is Gaussian (normally) distributed with expectation value 0 and standard deviation 1 . (Created with the Gaussian White Noise function in LabVIEW.)

33.2.2 Probability distribution of a stochastic variable

As known from statistics a stochastic variable can be described by its *probability distribution function*, PDF. Figure 33.2 shows two commonly used PDFs, the *Normal* (Gaussian) PDF and the *Uniform* PDF.

With the Normal PDF the probability that the variable has value in the range $\{-\sigma, +\sigma\}$ where σ is the standard deviation is approximately 68% (the standard deviation is defined below). With the Uniform PDF the probability that the variable has a value in the $\{-A, +A\}$ range is uniform (constant) and the variable can not take any value outside this range.

A stochastic process is *stationary* if the PDF is time independent (constant), or in other words, if the statistical properties are time independent.

33.2.3 The expectation value and the mean value

The *expectation value* , $E(x)$, of the stochastic variable x is the mean (or average) value of x calculated from an infinite number of samples of x . For a limited number N of samples

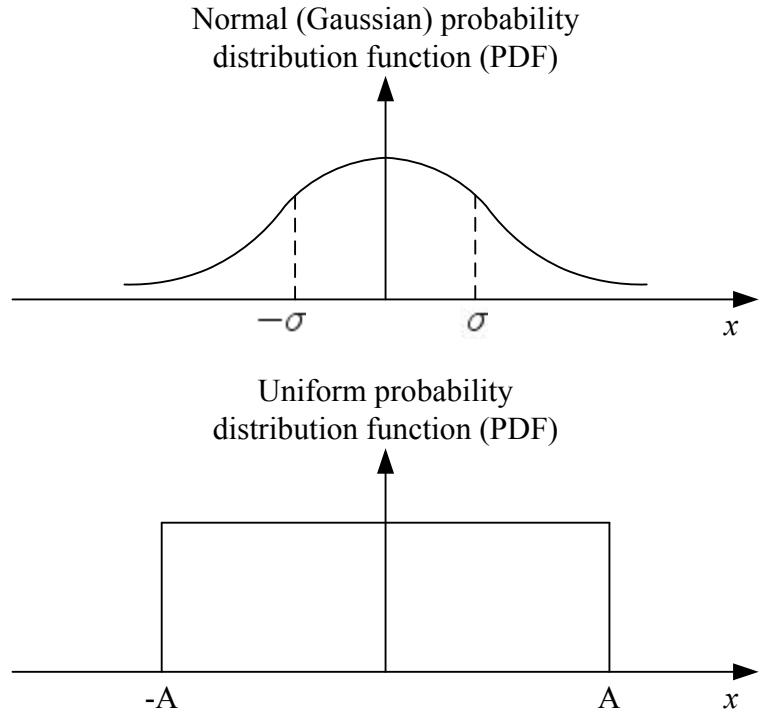


Figure 33.2: The Normal (Gaussian) PDF and the Uniform PDF

the mean value can be calculated from

Mean value:

$$m_x = \frac{1}{N} \sum_{k=0}^{N-1} x_k \quad (33.1)$$

Often these two terms (expectation value and mean value) are used interchangeably.

If x is a vector, say

$$x_k = \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} \quad (33.2)$$

then the mean value of x has the form

$$m_x = \begin{bmatrix} m_{x_1} \\ m_{x_2} \end{bmatrix} = \begin{bmatrix} \frac{1}{N} \sum_{k=0}^{N-1} x_{1,k} \\ \frac{1}{N} \sum_{k=0}^{N-1} x_{2,k} \end{bmatrix} \quad (33.3)$$

33.2.4 Variance. Standard deviation

The *variance* of a stochastic variable is the mean or expected value of the squared difference between the value and its mean value:

$$\text{Var}(x) = E \left\{ [x_k - m_x]^2 \right\} \quad (33.4)$$

The variance can be calculated from a sequence of samples as follows:

$$\text{Var}(x) = \frac{1}{N-1} \sum_{k=0}^{N-1} [x_k - m_x]^2 \quad (33.5)$$

(Statistically it is better to divide by $N - 1$ and not by N since the estimate of the statistical variance becomes unbiased using $N - 1$.) The variance is sometimes denoted the *power* of the signal.

The *standard deviation* may give a more meaningful value of the variation of a signal. The standard deviation is the square root of the variance:

$$\sigma = \sqrt{\text{Var}(x)} \quad (33.6)$$

In many situations σ^2 is used as a symbol for the variance.

33.2.5 Auto-covariance. Cross-covariance

Sometimes it is useful to express how a stochastic variable, say x_k , varies along the time-index axis (k). This type of variance can be expressed by the *auto-covariance*:

Auto-covariance:

$$R_x(L) = E\{[x_{k+L} - m_x][x_k - m_x]\} \quad (33.7)$$

where L is the *lag*. Note that the argument of the auto-covariance function is the lag L . Figure 33.3 shows $R_x(L)$ for a signal x where the covariance decreased as the lag increases (this is typical). As indicated in Figure 33.3 the auto-covariance usually has a peak value at lag $L = 0$.

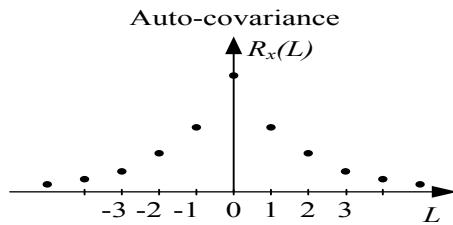


Figure 33.3: The auto-covariance for a signal x where the covariance decreased as the lag increases (this is typical).

If $L = 0$ the auto-covariance becomes the variance:

$$R_x(0) = E\{[x_{k+0} - m_x][x_k - m_x]\} \quad (33.8)$$

$$= E\{[x_k - m_x]^2\} = \text{Var}(x) = \sigma^2 \quad (33.9)$$

In some applications x is a *vector*, say

$$x_k = \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} \quad (33.10)$$

What does the auto-covariance look like in this case? For simplicity, assume that each of the four variables above have zero mean. The auto-covariance then becomes

$$R_x(L) = E\{[x_{k+L}][x_k]^T\} \quad (33.11)$$

$$= E \left\{ \begin{bmatrix} x_{1,k+L} \\ x_{2,k+L} \end{bmatrix} \begin{bmatrix} x_{1,k} & x_{2,k} \end{bmatrix} \right\} \quad (33.12)$$

$$= \begin{bmatrix} E[x_{1,k+L}x_{1,k}] & E[x_{1,k+L}x_{2,k}] \\ E[x_{2,k+L}x_{1,k}] & E[x_{2,k+L}x_{2,k}] \end{bmatrix} \quad (33.13)$$

If $L = 0$, the auto-covariance becomes

$$R_x(0) = \begin{bmatrix} \underbrace{E\{[x_{1,k}]^2\}}_{=\text{Var}(x_1)} & E[x_{1,k}x_{2,k}] \\ E[x_{2,k}x_{1,k}] & \underbrace{E\{[x_{2,k}]^2\}}_{=\text{Var}(x_2)} \end{bmatrix} \quad (33.14)$$

Hence, the *variances are on the diagonal*.

The *cross-covariance* between two *different* scalar signals, say x and y , is

Cross-covariance:

$$R_{xy}(L) = E\{[x_{k+L} - m_x][y_k - m_y]\} \quad (33.15)$$

The cross-covariance can be *estimated* from sequences of sample values of x and y of length N with

$$\begin{aligned} R_{xy}(L) \\ = S \sum_{k=0}^{N-1-|L|} [x_{k+L} - m_x][y_k - m_y], \quad L = 0, 1, 2, \dots \end{aligned} \quad (33.16)$$

$$= S \sum_{k=1}^{N-1-|L|} [y_{k-L} - m_y][x_k - m_x] = R_{yx}(-L), \quad L = -1, -2, \dots \quad (33.17)$$

where S is a scaling factor which is defined below:

- **Raw estimate:**

$$S = 1 \quad (33.18)$$

- **Normalized estimate:**

$$S = \frac{1}{R_{xy}(0) \text{ in the raw estimate}} \quad (33.19)$$

This gives $R_{xy}(0) = 1$.

- **Unbiased estimate:**

$$S = \frac{1}{N-L} \quad (33.20)$$

This calculates $R_{xy}(L)$ as an average value of the product $[x(k+L) - m_x][y(k) - m_y]$. However, $R_{xy}(L)$ may be very “noisy” if L is large since then the summation is calculated with only a few additive terms (it is assumed that there are noise or random components in x and/or y).

- **Biased estimate:**

$$S = \frac{1}{N} \quad (33.21)$$

With this option $R_{xy}(L)$ is not an average value of the product $[x(k+L) - m_x][y(k) - m_y]$ since the sum of terms is divided by N no matter how many additive terms there are in the summation. Although this makes $R_{xy}(L)$ become “biased” it reduces the “noise” in $R_{xy}(L)$ because the “noisy” terms are weighed by $1/N$ instead of $1/(N-L)$. Unless you have reasons for some other selection, you may use biased estimate as the default option.

Correlation (auto/cross) is the same as covariance (auto/cross) except that the mean value, as m_x , is removed from the formulas. Hence, the cross-correlation is, cf. (33.15),

$$r_{xy}(L) = E\{x_{k+L}y_k\} \quad (33.22)$$

33.3 White and coloured noise

33.3.1 White noise

An important type of stochastic signals are the so-called *white noise signals (or processes)*. “White” is because in some sense white noise contains equally much of all frequency components, analogously to white light which contains all colours. White noise has zero mean value:

Mean value of white noise:

$$m_x = 0 \quad (33.23)$$

There is no co-variance or relation between sample values at different time-indexes, and hence the auto-covariance is zero for all lags L except for $L = 0$. Thus, the auto-covariance is the pulse function shown in Figure 33.4.

Mathematically the auto-covariance function of white noise is

$$R_x(L) = \text{Var}(x)\delta(L) = \sigma^2\delta(L) = V\delta(L) \quad (33.24)$$

Here, the short-hand symbol V has been introduced for the variance. $\delta(L)$ is the *unit pulse* defined as follows:

Unit pulse:

$$\delta(L) = \begin{cases} 1 & \text{when } L = 0 \\ 0 & \text{when } L \neq 0 \end{cases} \quad (33.25)$$

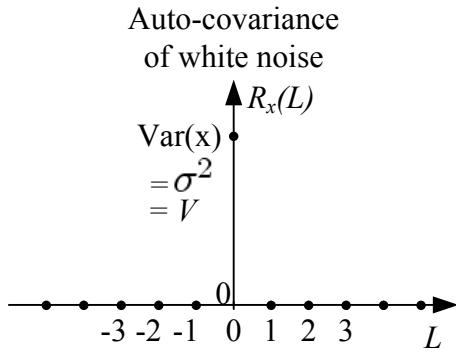


Figure 33.4: White noise has auto-correlation function like a pulse function.

White noise is an important signal in estimation theory because the random noise which is always present in measurements, can be represented by white noise. For example, the variance of the assumed white measurement noise is used as an input parameter in the Kalman Filter design, cf. Chapter 34.

If you calculate the auto-covariance of a white noise sequence of finite length, the auto-covariance function will not be exactly as the ideal function shown in Figure 33.4, but the main characteristic showing a relatively large value at lag $L = 0$ is there.

Example 33.1 White noise

Figure 33.5 shows a simulated white noise signal x and its auto-covariance $R_x(L)$ (normalized) calculated from the most recent $N = 50$ samples of x .¹

The white noise characteristic of the signal is clearly indicated by $R_x(L)$.

[End of Example 33.1]

33.3.2 Coloured noise

As opposite to white noise, *coloured noise* does not vary completely randomly. In other words, there is a co-variance between the sample values at different time-indexes. As a consequence, the auto-covariance $R_x(L)$ is non-zero for lags $L \neq 0$. $R_x(L)$ will have a maximum value at $L = 0$, and $R_x(L)$ will decrease for increasing L .

You may generate coloured noise from white noise by sending the white noise through a dynamic system, typically a lowpass filter. Such a system is denoted *shaping filter*. The output signal of the shaping filter will be coloured noise. You can tune the colour of the coloured noise by adjusting the parameters of the shaping filter.

Example 33.2 Coloured noise

¹Implemented in LabVIEW.

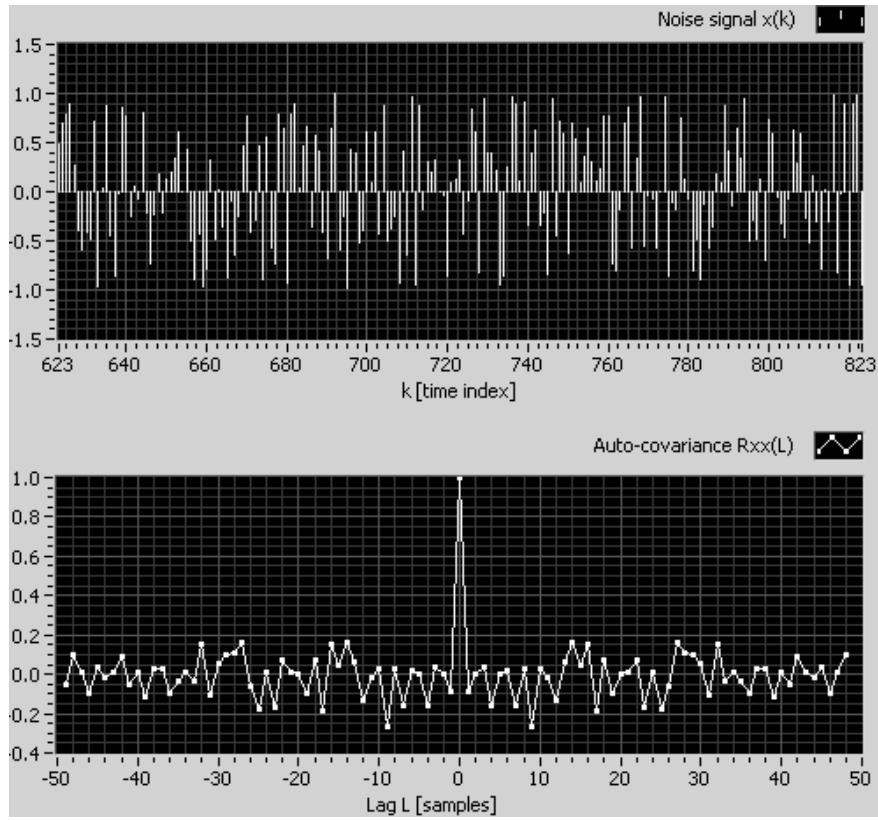


Figure 33.5: Example 33.1: Simulated white noise signal x and its auto-covariance $R_x(L)$ (normalized) calculated from the most recent $N = 50$ samples of x .

Figure 33.6 shows a simulated coloured noise signal x and its auto-covariance $R_x(L)$ (normalized) calculated from the most recent $N = 50$ samples of x .²

²Implemented in LabVIEW.

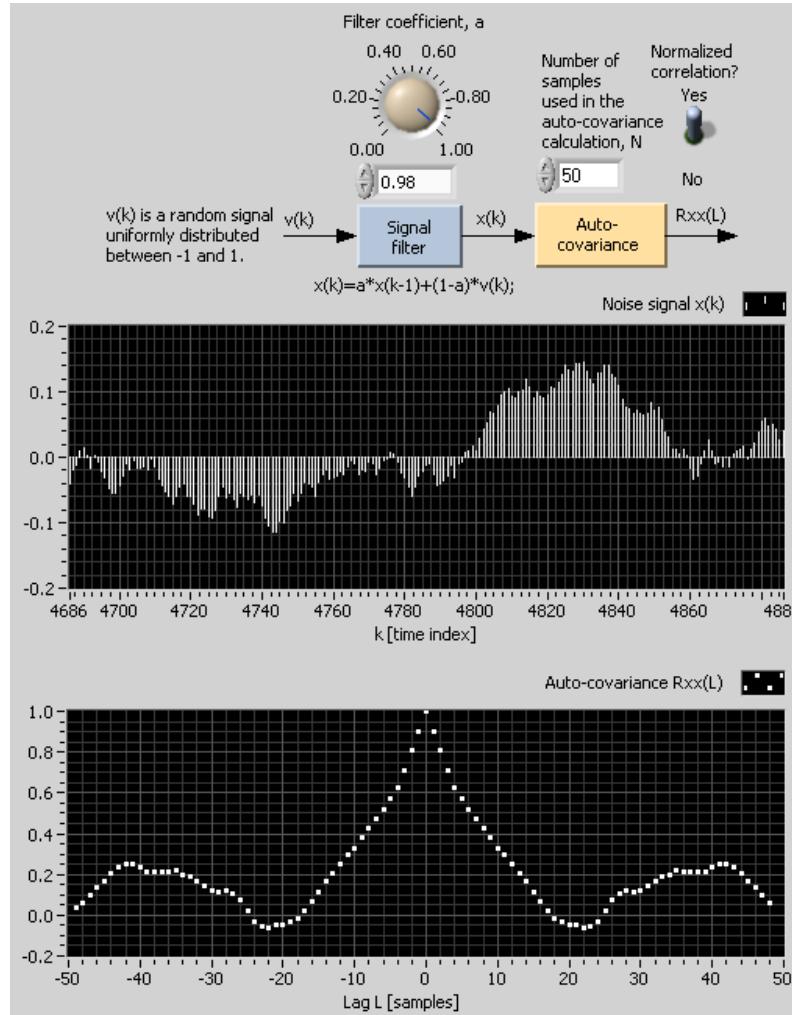


Figure 33.6: Example 33.1: Simulated coloured noise signal x and its auto-covariance $R_x(L)$ (normalized) calculated from the most recent $N = 50$ samples of x .

The coloured noise is the output of this shaping filter:

$$x_k = ax_{k-1} + (1 - a)v_k \quad (33.26)$$

which is a discrete-time first order lowpass filter. The filter input v_k is white noise. The filter parameter is $a = 0.98$. (If the filter parameter is 0 the filter performs no filtering, and the output is just white noise.)

The coloured noise characteristic of the signal is shown both in the plot of the signal x_k in the upper diagram of Figure 33.6 and in the auto-covariance $R_x(L)$ shown in the lower diagram of Figure 33.6.

[End of Example 33.2]

33.4 Propagation of mean value and co-variance through static systems

If a stochastic (“random”) signals excites a static or dynamic system, the mean value and the co-variance of the output signal is different from those of the input. In this section we will concentrate on *static* systems. The results are useful e.g. in calculating the system gain needed to obtain a random signal of a specified variance when the source signal is a random signal of fixed variance.

The theory of the propagation of mean value and co-variance through *dynamic* systems is certainly important if you are going to analyze and design signal filters, controllers and state estimators assuming they are excited by random signals. However, it is my experience that this theory is not needed to be able to use the tools that exist for such applications (e.g. the Kalman Filter for state estimation). Therefore, I have omitted the topic of propagation of mean value and co-variance through dynamic systems in this book.

Assume given the following static linear system:

$$y_k = Gv_k + C \quad (33.27)$$

where v is a stationary stochastic input signal with mean value m_v and co-variance $R_v(L)$. y is the output of the system. G is the gain of the system, and C is a constant. In a multivariable system G is a matrix and C is a vector, but in the following we will assume that G and C are scalars, which is the most usual case.

Let us calculate the mean value and the auto-covariance of the output y . The mean value becomes

$$m_y = E[y_k] = GE[v_k] + C \quad (33.28)$$

$$= Gm_v + C \quad (33.29)$$

The auto-covariance of the output becomes

$$R_y(L) = E\{[y_{k+L} - m_y][y_k - m_y]\} \quad (33.30)$$

$$= E\{([Gv_{k+L} + C] - [Gm_v + C])([Gv_k + C] - [Gm_v + C])\} \quad (33.31)$$

$$= E\{(Gv_{k+L} - Gm_v)(Gv_k - Gm_v)\} \quad (33.32)$$

$$= E\{(G[v_{k+L} - m_v])(G[v_k - m_v])\} \quad (33.33)$$

$$= G^2 \underbrace{E\{([v_{k+L} - m_v])([v_k - m_v])\}}_{= R_v(L)} \quad (33.34)$$

$$= G^2 R_v(L) \quad (33.35)$$

If the system (33.27) is multivariable, that is, if v and y are vectors, G is a matrix and C is a vector. In this case we will get

$$R_y(L) = GR_v(L)G^T \quad (33.36)$$

Let us sum it up: For a scalar system (33.27):

- Mean of output of stochastic static system:

$$m_y = Gm_v + C \quad (33.37)$$

- Co-variance of output of stochastic static system:

$$R_y(L) = G^2 R_v(L) \quad (33.38)$$

- The variance, which is equal to $R_y(0)$:

$$\sigma_y^2 = G^2 \sigma_v^2 \quad (33.39)$$

- The standard deviation:

$$\sigma_y = G\sigma_v \quad (33.40)$$

Example 33.3 Mathematical operations to achieve specified output mean and variance

Assume that you have signal generator available in a computer tool that can generate a white noise signal v having mean $m_v = 0$ and variance $\sigma_v^2 = 1$, and that you want to generate a signal y of mean $m_y = M$ and variance $\sigma_y^2 = V$. Find proper mathematical operations on v that create this y .

The gain G can be calculated from (33.39):

$$G = \sqrt{\frac{\sigma_y^2}{\sigma_v^2}} = \sqrt{\frac{V}{1}} = \sqrt{V} = \sigma_y \quad (33.41)$$

The constant C can be calculated from (33.37):

$$C = m_y - Gm_v = M - G \cdot 0 = M \quad (33.42)$$

So, the mathematical operation is

$$y_k = Gv_k + C = \sqrt{V} \cdot v_k + M = \sigma_y v_k + M \quad (33.43)$$

In words: Multiply the input by the specified standard deviation and add the specified mean value to this result.

[End of Example 33.3]

33.5 Problems for Chapter 33

Problem 33.1 Statistical measures of a signal

Given the following sequence of measurement values:

$$\{x_k\} = \{x_0, x_1, x_2\} = \{0.73, 1.23, 0.89\} \quad (33.44)$$

1. Calculate the mean value, m_x .
2. Calculate the variance, σ_x^2 .
3. Calculate the standard deviation, σ_x .
4. Calculate the auto-covariance, $R_x(L)$ with $L = 0$ and $L = 1$ for the following options:
 - (a) Raw estimate
 - (b) Normalized estimate
 - (c) Unbiased estimate
 - (d) Biased estimate

Problem 33.2 Characteristics of uniformly distributed noise

Given a random sequence, $\{x_k\}$, uniformly distributed between $-A$ and $+A$.

1. Calculate the mean value or expectation value m_x using this formula:

$$m_x = \int_{-\infty}^{\infty} xP(x)dx = \int_{-A}^{A} xP(x)dx \quad (33.45)$$

where $P(x)$ is the probability density. For a uniformly distributed signal,

$$P(x) = \frac{1}{2A} \quad (33.46)$$

since the area under the $P(x)$ -curve must be 1 (the width of the curve is $2A$, so the height is $1/(2A)$ for the area to be 1).

2. Show that the standard deviation of the sequence is

$$\sigma_x = \frac{A}{\sqrt{3}} \quad (33.47)$$

Problem 33.3 Auto covariance of white noise

Draw by hand the principal auto covariance of white noise with variance 4. What is the standard deviation σ_x of this signal?

Problem 33.4 *Shaping filter*

The following discrete-time first order lowpass filter is one example of a shaping filter:

$$x_k = ax_{k-1} + (1 - a)v_k \quad (33.48)$$

which is a discrete-time first order lowpass filter. x is the filter output, and v_k is the filter input, which is assumed to be white noise.

1. Explain (no calculations are needed) why $a = 0$ makes the output become white, and hence, there is no “shaping” through the filter.
2. Explain (no calculations are needed) why a between 0 and 1 makes the output become “coloured”.

Problem 33.5 *Generating a random signal*

Assume that you in a computer tool are to generate a random signal y having mean value 3 and variance 4, and that the tool has a function that generates a random signal u of mean value 0 and variance 1. How can you obtain y from u ? (Express y as a mathematical function of u .)

Problem 33.6 *Programming a uniformly distributed random signal in Python*

In Python, the random generator `numpy.random.uniform()` can be used to generate a signal as a sequence of n uniformly distributed random values between amplitude limits $\pm A$ with zero mean value:

```
signal = numpy.random.uniform(-A, A, n)
```

The relation between the standard deviation and the amplitude is given by (33.47).

Make a Python program that generates and plots a uniformly distributed random signal, x , with the following specifications:

- Sampling time: 0.1 s.
- Start time: 0 s.
- Stop time: 10 s.
- Standard deviation: 0.1.
- Mean value: 1.

In the same diagram, plot the mean(x) and mean(x) $\pm A$.

What is the value of A ?

33.6 *Solutions to problems for Chapter 33*

Solution to Problem 33.1

$$\{x_k\} = \{x_0, x_1, x_2\} = \{0.73, 1.23, 0.89\} \quad (33.49)$$

1. Mean value:

$$\underline{\underline{m_x}} = \frac{1}{N} \sum_{k=0}^{N-1} x_k \quad (33.50)$$

$$= \frac{1}{3} [x_0 + x_1 + x_2] \quad (33.51)$$

$$= \frac{1}{3} [0.73 + 1.23 + 0.89] \quad (33.52)$$

$$= \underline{\underline{0.95}} \quad (33.53)$$

2. Variance:

$$\underline{\underline{\sigma_x^2}} = \frac{1}{N-1} \sum_{k=0}^{N-1} [x_k - m_x]^2 \quad (33.54)$$

$$\frac{1}{3-1} \sum_{k=0}^{3-1} [x_k - m_x]^2 \quad (33.55)$$

$$= \frac{1}{2} \left\{ \begin{array}{l} [x_0 - m_x]^2 \\ + [x_1 - m_x]^2 \\ + [x_2 - m_x]^2 \end{array} \right\} \quad (33.56)$$

$$= \frac{1}{2} \left\{ \begin{array}{l} [0.73 - 0.95]^2 \\ + [1.23 - 0.95]^2 \\ + [1.13 - 0.95]^2 \end{array} \right\} \quad (33.57)$$

$$= \underline{\underline{0.0652}} \quad (33.58)$$

3. Standard deviation:

$$\underline{\underline{\sigma_x}} = \sqrt{\sigma_x^2} = \sqrt{0.0652} = \underline{\underline{0.255}} \quad (33.59)$$

4. Calculate the auto-covariance, $R_x(L)$ with $L = 0$ and $L = 1$ for the following options:

(a) Raw estimate of auto-covariance: General formula:

$$R_x(L) = \sum_{k=0}^{N-1-|L|} [x_{k+L} - m_x][x_k - m_x] = R_x(L)_{\text{Raw}} \quad (33.60)$$

$L = 0$:

$$\underline{\underline{R_x(0)}} = \sum_{k=0}^{3-1-|0|=2} [x_{k+0} - m_x][x_k - m_x] \quad (33.61)$$

$$= \sum_{k=0}^2 [x_k - m_x]^2 \quad (33.62)$$

$$= [x_0 - m_x]^2 \quad (33.63)$$

$$+ [x_1 - m_x]^2 \quad (33.64)$$

$$+ [x_2 - m_x]^2 \quad (33.65)$$

$$= [0.73 - 0.95]^2 \quad (33.66)$$

$$+ [1.23 - 0.95]^2 \quad (33.67)$$

$$+ [0.89 - 0.95]^2 \quad (33.68)$$

$$= \underline{\underline{0.1304}} \quad (33.69)$$

$$= R_x(0)_{\text{Raw}} \quad (33.70)$$

$L = 1$:

$$\underline{\underline{R_x(1)}} = \sum_{k=0}^{3-1-|1|=1} [x_{k+1} - m_x][x_k - m_x] \quad (33.71)$$

$$= [x_1 - m_x][x_0 - m_x] \quad (33.72)$$

$$+ [x_2 - m_x][x_1 - m_x] \quad (33.73)$$

$$= [1.23 - 0.95][0.73 - 0.95] \quad (33.74)$$

$$+ [0.89 - 0.95][1.23 - 0.95] \quad (33.75)$$

$$= \underline{\underline{-0.0784}} \quad (33.76)$$

$$= R_x(1)_{\text{Raw}} \quad (33.77)$$

(b) Normalized estimate: General formula:

$$\begin{aligned} R_x(L) &= \frac{1}{R_x(0)_{\text{Raw}}} \sum_{k=0}^{N-1-|L|} [x_{k+L} - m_x][x_k - m_x] \\ &= \frac{1}{R_x(0)_{\text{Raw}}} R_x(L)_{\text{Raw}} \end{aligned} \quad (33.78)$$

$L = 0$:

$$\underline{\underline{R_x(0)}} = \frac{1}{R_x(0)_{\text{Raw}}} R_x(0)_{\text{Raw}} = \underline{\underline{1}} \quad (33.79)$$

$L = 1$:

$$\underline{\underline{R_x(1)}} = \frac{1}{R_x(0)_{\text{Raw}}} R_x(1)_{\text{Raw}} = \frac{1}{0.1304} (-0.0784) = \underline{\underline{-0.6012}} \quad (33.80)$$

(c) Unbiased estimate: General formula:

$$R_x(L) = \frac{1}{N-L} R_x(L)_{\text{Raw}} \quad (33.81)$$

$L = 0$:

$$\underline{\underline{R_x(0)}} = \frac{1}{3-0} R_x(0)_{\text{Raw}} = \frac{1}{3} \cdot 0.1304 = \underline{\underline{0.0435}} \quad (33.82)$$

$L = 1$:

$$\underline{\underline{R_x(1)}} = \frac{1}{3-1} R_x(1)_{\text{Raw}} = \frac{1}{2} (-0.0784) = \underline{\underline{-0.0392}} \quad (33.83)$$

(d) Biased estimate: General formula:

$$R_x(L) = \frac{1}{N} R_x(L)_{\text{Raw}} \quad (33.84)$$

$L = 0$:

$$\underline{\underline{R_x(0)}} = \frac{1}{3} R_x(0)_{\text{Raw}} = \frac{1}{3} \cdot 0.1304 = \underline{\underline{0.0435}} \quad (33.85)$$

$L = 1$:

$$\underline{\underline{R_x(1)}} = \frac{1}{3} R_x(1)_{\text{Raw}} = \frac{1}{3} (-0.0784) = \underline{\underline{-0.0261}} \quad (33.86)$$

Solution to Problem 33.2

1. Mean value:

$$\underline{\underline{m_x}} = \int_{-A}^A x P(x) dx \quad (33.87)$$

$$= \int_{-A}^A x \cdot \frac{1}{2A} \cdot dx \quad (33.88)$$

$$= \frac{1}{2A} \left[\frac{x^2}{2} \right]_{-A}^A \quad (33.89)$$

$$= \frac{1}{2A} \left[\frac{A^2}{2} - \frac{(-A)^2}{2} \right] \quad (33.90)$$

$$= \underline{\underline{0}} \quad (33.91)$$

2. Variance:

$$\sigma_x^2 = \int_{-A}^A (x - m_x)^2 P(x) dx \quad (33.92)$$

$$= \int_{-A}^A (x - 0)^2 \frac{1}{2A} dx \quad (33.93)$$

$$= \frac{1}{2A} \int_{-A}^A x^2 dx \quad (33.94)$$

$$= \frac{1}{2A} \left[\frac{x^3}{3} \right]_{-A}^A \quad (33.95)$$

$$= \frac{1}{2A} \left[\frac{A^3}{3} - \frac{(-A)^3}{3} \right] \quad (33.96)$$

$$= \frac{A^2}{3} \quad (33.97)$$

Standard deviation:

$$\underline{\underline{\sigma_x}} = \frac{A}{\sqrt{3}} \quad (33.98)$$

Solution to Problem 33.3

See Figure 33.7.

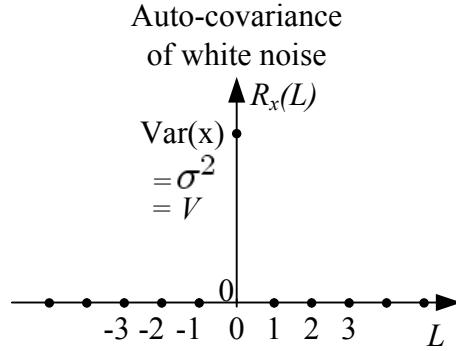


Figure 33.7: Auto-covariance

The standard deviation is

$$\underline{\underline{\sigma_x}} = \sqrt{\sigma_x^2} = \sqrt{4} = \underline{\underline{2}} \quad (33.99)$$

Solution to Problem 33.4

- With $a = 0$ the filter model is

$$x_k = v_k \quad (33.100)$$

So, if the input is white, the output is white.

- If a is between 0 and 1, the filter output $x(k)$ depends not only on the input $v(k)$ but also on $x(k-1)$ which is x at the previous time step. Therefore, $x(k)$ will not vary purely randomly (it will not become purely white) – it is “coloured”.

Solution to Problem 33.5

y expressed as a function of u :

$$\underline{\underline{y = Gu + C}} \quad (33.101)$$

where G and C are calculated from the following formulas:

$$m_y = Gm_u + C \quad (33.102)$$

and

$$\sigma_y^2 = G^2 \sigma_u^2 \quad (33.103)$$

where

$$m_u = 0 \quad (33.104)$$

$$m_y = 3 \quad (33.105)$$

$$\sigma_u^2 = 1 \quad (33.106)$$

$$\sigma_y^2 = 4 \quad (33.107)$$

Now, from (33.103) we get

$$\underline{\underline{G}} = \sqrt{\frac{\sigma_y^2}{\sigma_u^2}} = \sqrt{\frac{4}{1}} = \underline{\underline{2}} \quad (33.108)$$

and from (33.102) we get

$$\underline{\underline{C}} = m_y - Gm_u = 3 - 2 \cdot 0 = \underline{\underline{3}} \quad (33.109)$$

Solution to Problem 33.6

From (33.47) we get amplitude:

$$A = \sqrt{3}\sigma_x = \sqrt{3} \cdot 0.1 = 0.173 \quad (33.110)$$

Program 33.1 generates and plots x .

http://techteach.no/control/python/prog_plot_sim_uniform.py

Listing 33.1: prog_plot_sim_uniform.py

```
import numpy as np
import matplotlib.pyplot as plt

# %% Generating time signal:

Ts = 0.1
t_start = 0
t_stop = 10
t_array = np.arange(t_start, t_stop+Ts, Ts)
n = len(t_array)

# %% Parameters of uniform():

mean_x = 1
sigma_x = 0.1
A = np.sqrt(3)*sigma_x

# %% Generating array of noise:

x = mean_x + np.random.uniform(-A, A, n)

# %% Plotting:

plt.close('all')
```

```
fig1 = plt.figure(num=1, figsize=(12, 9))

plt.plot(t_array, x, 'b-o')
plt.plot(t_array, np.zeros(n) + mean_x, 'r--')
plt.plot(t_array, np.zeros(n) + mean_x + A, 'g--')
plt.plot(t_array, np.zeros(n) + mean_x - A, 'g--')
plt.grid()
plt.xlabel('t [s]')
plt.legend(labels=('x', 'mean_x',
                   'mean_x - A', 'mean_x + A'))
plt.show()

plt.savefig('plot_unif_noise.pdf')
```

The signal x is plotted in Figure 33.8

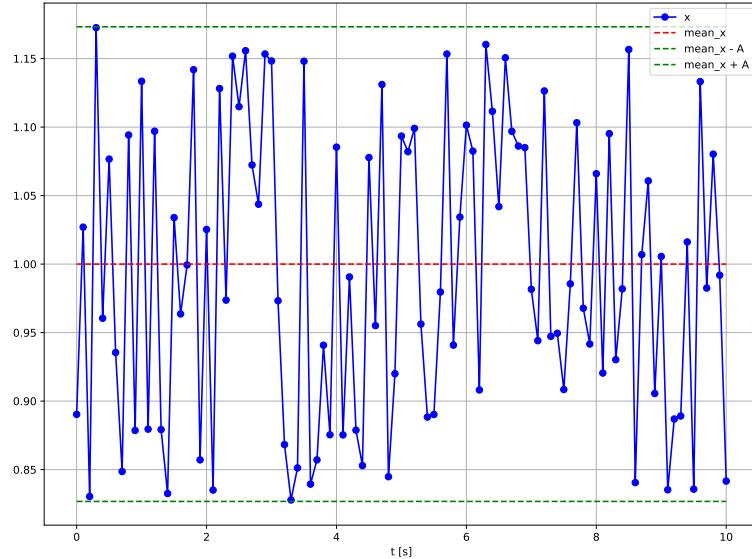


Figure 33.8: Uniformly distributed random signal, x .

Chapter 34

State estimation with Kalman Filter

34.1 Introduction

The Kalman Filter is a commonly used algorithm to estimate the values of unknown state variables of a dynamic system. Figure 34.1 illustrates the principle of the Kalman Filter. The Kalman Filter is basically a process simulator and a sensor simulator which uses a process model and a sensor model to estimate the states and the measurements. The difference between the real measurement and the simulated (estimated) measurement is used to correct the present estimate. That difference is called the innovation variable, or the innovation “process”. The correction is proportional to the innovation variable. The proportional gain, which is actually a matrix of time-varying elements, is called the Kalman Filter gain, K . Overall, the Kalman Filter uses process knowledge in terms of model and measurements (“m & m”) to calculate the state estimate.

Although the Kalman Filter is a *state* estimator, also process disturbances and process parameters can be estimated. The clue is to model the disturbances or parameters as state variables, and then the Kalman Filter algorithm can be used to estimate them.

Since a Kalman Filter is a software algorithm that produces a “soft measurement”, the Kalman Filter is (an example) of a soft sensor.

Why can state estimates be useful?

- **Monitoring:** State estimates can provide valuable information about important variables in a physical process, for example feed composition to a reactor, environmental forces acting on a ship, load torques acting on a motor, etc.
- **Control:** The estimated states can be used, as if they were real measurements, by a controller, see Figure . The controller can be e.g. a feedforward controller implementing feedforward from estimated disturbances, or a model predictive controller (MPC) using estimated states and estimated disturbances. Actually, it is

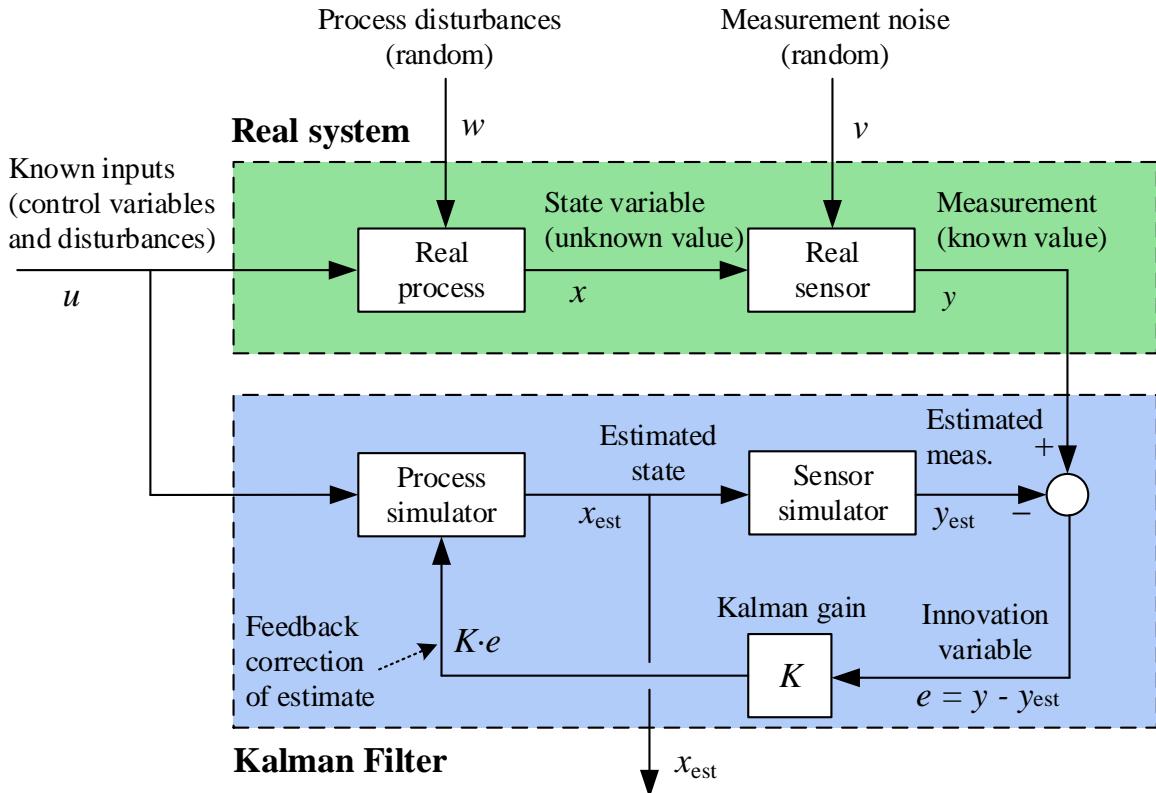


Figure 34.1: The principle of the Kalman Filter.

common that MPCs have an “integrated” Kalman Filter.

The Kalman Filter algorithm was developed by Rudolf E. Kalman around 1960 ?. There is a continuous-time version of the Kalman Filter and several discrete-time versions. Here the *predictor-corrector* version of the discrete-time Kalman Filter will be described. This version seems to be the most commonly used version.

The Kalman Filter algorithm was originally developed for systems assumed to be represented with a *linear* state space model. However, in many applications the system model is *nonlinear*. Furthermore the linear model is just a special case of a nonlinear model. Therefore, I have decided to present the Kalman Filter for nonlinear models, but comments are given about the linear case. The Kalman Filter for nonlinear models is denoted the *Extended Kalman Filter* because it is an extended use of the original Kalman Filter. However, for simplicity we can just denote it the Kalman Filter, dropping “extended” in the name. I will present the Kalman Filter without detailed derivation.

As with every model-based algorithm you should *test* your Kalman Filter with a simulated process before applying it to the real system. In the testing, you should start with testing the Kalman Filter with the nominal model, i.e. assumed correct model, in the simulator, including process and measurement noise. This is the model on which you are basing the Kalman Filter. If you have designed and implemented your Kalman Filter correctly, the estimated states should reproduce the simulated states (but with less noise in the

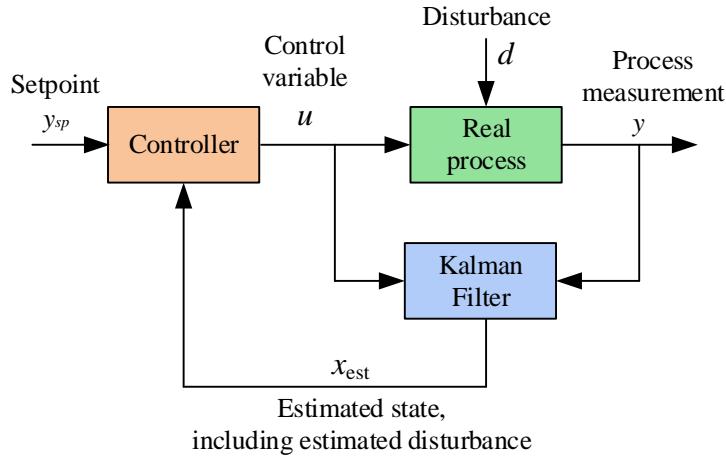


Figure 34.2: Control system where the controller uses estimated state variables, x_{est} , for control.

estimates).

This chapter is meant to provide sufficient information about how to design and implement a standard Kalman Filter. More theory about state estimation and about various Kalman Filter forms can be found in e.g. ([Simon 2006](#)).

34.2 Observability of discrete-time systems

A necessary condition for the Kalman Filter to work correctly is that the system for which the states are to be estimated, is *observable*, which can be checked numerically. You should check for observability before applying the Kalman Filter. (There may still be other problems that prevent the Kalman Filter from producing accurate state estimates, as a faulty or inaccurate mathematical model.)

The observability check presented here applies only to *linear* state space models, which may stem from a linearized nonlinear model.

Observability of discrete-time systems can be defined as follows: The discrete-time system

$$x_{k+1} = Ax_k + Bu_k \quad (34.1)$$

$$y_k = Cx_k + Du_k \quad (34.2)$$

is observable if there is a finite number of time steps k so that knowledge about the input sequence u_0, \dots, u_{k-1} and the output sequence y_0, \dots, y_{k-1} is sufficient to determine the initial state state of the system, x_0 .

Let us derive a criterion for the system to be observable. Since the influence of input u on state x is known from the model, let us for simplicity assume that $u_k = 0$. From the model

(34.1) – (34.2) we get

$$y_0 = Cx_0 \quad (34.3)$$

$$y_1 = Cx_1 = CAx_0 \quad (34.4)$$

⋮

$$y_{n-1} = CA^{n-1}x_0 \quad (34.5)$$

which can be expressed compactly as

$$\underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{M_{\text{obs}}} x_0 = \underbrace{\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix}}_Y \quad (34.6)$$

Let us make a definition:

Observability matrix:

$$M_{\text{obs}} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \quad (34.7)$$

(34.6) has a unique solution only if the rank of M_{obs} is n . Therefore:

Observability Criterion:

The system (34.1) – (34.2) is observable if and only if the observability matrix has rank equal to n where n is the order of the system model (the number state variables).

The rank can be checked by calculating the determinant of M_{obs} . If the determinant is non-zero, the rank is full, and hence, the system is observable. If the determinant is zero, the system is non-observable.

Non-observability has several consequences:

- There are state variables or linear combinations of state variables which do not make any response in the estimated measurement, and therefore, their estimates can not be corrected by the innovation process.
- The value of the Kalman Filter gain may diverge.
- The transfer function from the input variable u to the output variable y has an order that is less than the number of state variables (n).

Observability may be checked manually or with a pertinent function in e.g. Python (Control Package), Matlab, and LabVIEW. In Example 34.1, observability is checked manually and with Python.

Example 34.1 Observability

Given the following state space model:

$$\begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}}_A \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_B u_k \quad (34.8)$$

$$y_k = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_C \begin{bmatrix} x_{1,k} \\ x_{2,k} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_D u_k \quad (34.9)$$

The observability matrix is ($n = 2$)

$$M_{\text{obs}} = \begin{bmatrix} C \\ CA^{2-1} = CA \end{bmatrix} = \left[\begin{array}{c|cc} & [1 & 0] \\ \hline [1 & 0] & [1 & a] \\ & [0 & 1] \end{array} \right] = \begin{bmatrix} 1 & 0 \\ 1 & a \end{bmatrix} \quad (34.10)$$

The determinant of M_{obs} is

$$\det(M_{\text{obs}}) = 1 \cdot a - 1 \cdot 0 = a \quad (34.11)$$

From (34.11) we can conclude that the system is observable only if $a \neq 0$.

Does this result make sense?

- Assume that $a \neq 0$ which means that the first state variable, x_1 , contains some non-zero information about the second state variable, x_2 . Hence, x_2 can be “observed” from the observed (measured) x_1 , and the system is observable.
- Assume that $a = 0$ which means that x_1 contains no information about x_2 . In this case the system (or specifically x_2) is non-observable despite x_1 being observed (measured), and we say that the system is non-observable.

The Python program 34.1 implements an observability check both with the rank and with the determinant of the observability matrix.

http://techteach.no/control/python/observability_check.py

Listing 34.1: observability_check.py

```
import numpy as np
import control

# %% Model matrices as 2D arrays:
```

```

a = 0 # Model param

A = np.array([[1, a],
              [0, 1]])
B = np.array([[0],
              [1]])
C = np.array([[1, 0]])
D = np.array([[0]])

# %% Creating and checking the observability matrix:

M_obs = control.obsv(A, C)

#Rank check:
rank_M_obs = np.linalg.matrix_rank(M_obs)
print('rank_M_obs = ', rank_M_obs)

# Determinant check:
det_M_obs = np.linalg.det(M_obs)
print('det_M_obs = ', det_M_obs)

```

With $a = 0$ (non-observability) the results are:

rank_M_obs = 1
det_M_obs = 0.0

[End of Example 34.1]

34.3 The Kalman Filter algorithm

34.3.1 The assumed process model

The Kalman Filter presented below assumes that the system model consists of the following discrete-time state space model:

$$x_{k+1} = f(x_k, u_k) + Gw_k \quad (34.12)$$

and this measurement model:

$$y_k = g(x_k, u_k) + v_k \quad (34.13)$$

Some times, but rarely, the additive term Hw_k is included in (34.13), but I skip this term here, which is equivalent to setting $H = [0]$, a matrix of zeros.

Typically, the discrete-time model (34.12) – (34.13) is the discretized version of the following continuous-time model, where we for simplicity have disregarded random disturbance and random measurement noise (but such terms are included in the discrete-time model):

$$x' = f_{\text{cont}}(x, u) \quad (34.14)$$

$$y = g(x, u) \quad (34.15)$$

Often, Euler Forward discretization is used to discretize (34.14). In that case:

$$x_{k+1} = \underbrace{x_k + T_s f_{\text{cont}}(x_k, u_k)}_{f(x_k, u_k)} \quad (34.16)$$

(34.13) is just the discrete-time version of (34.15).

A *linear* model is just a special case:

$$x_{k+1} = \underbrace{Ax_k + Bu_k}_{=f} + Gw_k \quad (34.17)$$

and

$$y_k = \underbrace{Cx_k + Du_k}_{=g} + v_k \quad (34.18)$$

The models above contains the following variables and functions:

- x is the state vector of n state variables:

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad (34.19)$$

- u is the input vector of m input variables:

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} \quad (34.20)$$

It is assumed that the value of u is known. u includes control variables and known disturbances.

- f is the system vector function:

$$f = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \quad (34.21)$$

where f is any nonlinear or linear function.

- w is random (white) disturbance (or process noise) vector:

$$w = \begin{bmatrix} w_1 \\ \vdots \\ w_q \end{bmatrix} \quad (34.22)$$

with auto-covariance matrix Q which is typically assumed diagonal:

$$Q = \begin{bmatrix} Q_{11} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & Q_{nn} \end{bmatrix} \quad (34.23)$$

Hence, the number q of process disturbances is assumed to be equal to the number n of state variables.

- G is the process noise gain matrix relating the process noise to the state variables. It is common to assume that $q = n$, making G square:

$$G = \begin{bmatrix} G_{11} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & G_{nn} \end{bmatrix} \quad (34.24)$$

It is actually common to set the elements of G equal to one:

$$G_{ii} = 1 \quad (34.25)$$

making G an identity matrix:

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_n \quad (34.26)$$

- y is the measurement vector of r measurement variables:

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_r \end{bmatrix} \quad (34.27)$$

- $g(\cdot)$ is the measurement vector function:

$$g = \begin{bmatrix} g_1 \\ \vdots \\ g_r \end{bmatrix} \quad (34.28)$$

where g_i is any nonlinear or linear function. Typically, g is a linear function on the form

$$g = Cx \quad (34.29)$$

where C is the measurement gain matrix. (34.29) implies that the D matrix in (34.18) is a matrix of zeros.

- v is a random (white) measurement noise vector:

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_r \end{bmatrix} \quad (34.30)$$

with auto-covariance R which is typically assumed diagonal:

$$R = \begin{bmatrix} R_{11} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & R_{rr} \end{bmatrix} \quad (34.31)$$

34.3.2 The result of Kalman Filtering: an optimal state estimate

We define the state estimation error vector:

$$e_{x,k} = x_{\text{est},k} - x_k \quad (34.32)$$

where: x_k is the assumed true state vector, and $x_{\text{est},k}$ is the state estimate. The Kalman Filter produces an optimal estimate in the sense that the expectation value of the sum (actually of any linear combination) of the estimation errors gets a minimal value. In other words, The Kalman Filter minimizes the following sum of squared errors:

$$E[e_{x,k}^T e_k] = E[e_{x_{1,k}}^2 + \dots + e_{x_{n,k}}^2] \quad (34.33)$$

The Kalman Filter estimate is therefore sometimes denoted the “least mean-square estimate”. This result actually assumes that the model is linear, so for nonlinear models, this result is approximate.

34.3.3 The Kalman Filter algorithm – step by step

I present here the *Kalman Filter algorithm*, in the same order as it may be implemented in a program. The mathematical block diagram shown in Figure 34.3 gives a graphical representation of the Kalman Filter, except that the formulas for the Kalman Filter Gain, K_k , are not shown in the block diagram.

Initialization

This step is the initial step, and the operations here are executed only once, before the estimation loop starts.

The initial value $x_{p,0}$ of the predicted state estimate x_p is set equal to this initial (guessed) value of the state estimate:

$$x_{p,0} = x_{\text{init}} \quad (34.34)$$

Also the auto-covariance matrix of predicted state estimate error must be given an initial (a guessed) value. This auto-covariance is defined as:

$$P_{p,k} = E[(x_k - m_{x_{p,k}})(x - m_{x_{p,k}})^T] \quad (34.35)$$

Its initial value is set to:

$$P_{p,k} = P_{p,\text{init}} \quad (34.36)$$

A typical initial value is a matrix of ones (the identity matrix):

$$P_{p,\text{init}} = \text{diag}(1, \dots, 1) = I_n \quad (34.37)$$

The subsequent actions are implemented in the estimation loop in the given order.

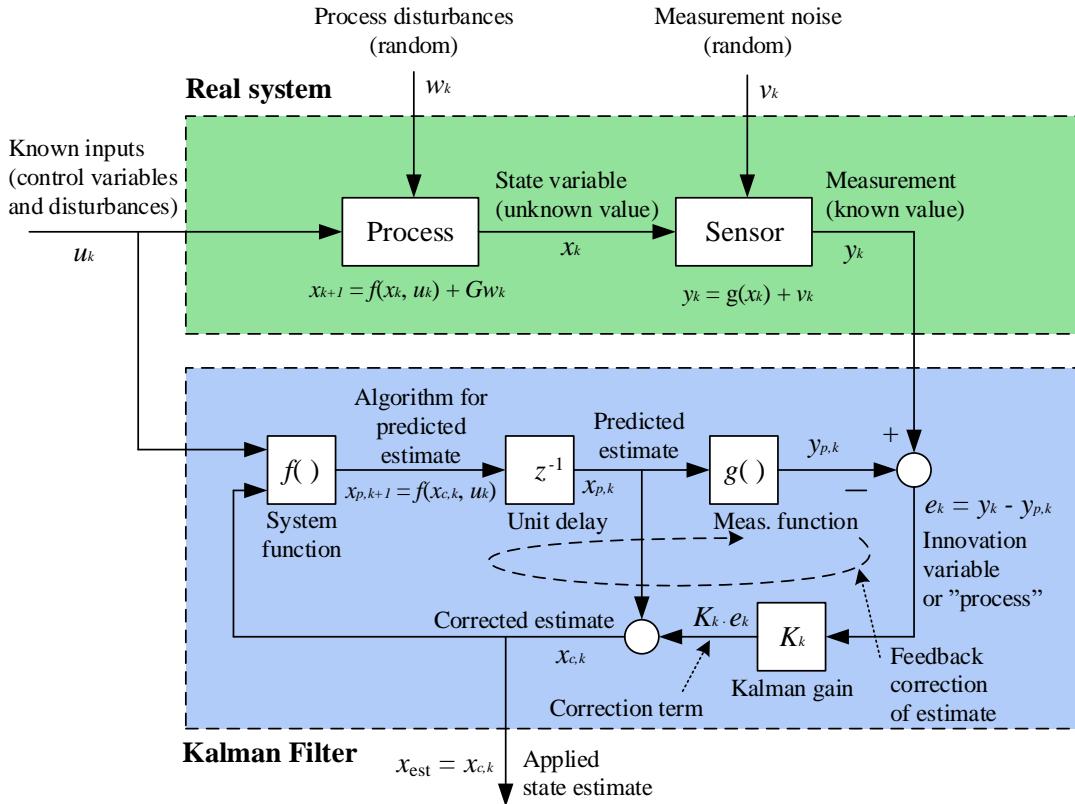


Figure 34.3: The Kalman Filter algorithm (34.43) – (34.44) represented by a block diagram.

Real measurement

The real measurement, $y_{m,k}$, is read from the sensor.

Predicted measurement

The predicted measurement is calculated from the predicted state according to the sensor model:

$$y_{p,k} = g(x_{p,k}) \quad (34.38)$$

In most cases, y_p is just one of the (predicted) state variables. If it is, say equal to x_i , we have: $y_{p,k} = x_{i,p,k}$.

Note: In the sensor model (34.18), the measurement noise w_k is an additive term. However, we can not include w_k in (34.38) since w_k is not known or not predictable (since it is assumed being white noise).

Innovation variable

The innovation variable is the difference between the real measurement and the predicted measurement:

$$e_k = y_{m,k} - y_{p,k} \quad (34.39)$$

Kalman Filter gain

The Kalman Filter gain is calculated as:

$$K_k = P_{p,k} C^T [C P_{p,k} C^T + R]^{-1} \quad (34.40)$$

Here, C is the measurement gain matrix of a linearized model of the original nonlinear model (34.13) calculated at the most recent operating point, which is $(x_{p,k}, u_k)$:

$$C = \left. \frac{\partial g}{\partial x} \right|_{\text{op} = (x_{p,k}, u_k)} \quad (34.41)$$

It is common that $g() = C$, and if so, no linearization is actually needed.

Steady-state Kalman Filter gain: If the model is linear and time invariant (i.e. system matrices are not varying with time), the Kalman Filter gain will converge towards a steady-state value value, K_s , if the number of iterations goes to infinity:

$$K_k \rightarrow K_\infty = K_s \quad (34.42)$$

You can pre-calculate K_s by running the algorithm for K_k for a sufficiently large number of iterations. In this book, I stick to the time-varying Kalman Filter gain since it is relatively easy to calculate with matrix-vector operations.

Corrected state estimate

Calculate the corrected state estimate $x_{c,k}$ by adding the corrective term $K_k e_k$ to the predicted state estimate $x_{p,k}$:

$$x_{c,k} = x_{p,k} + K_k e_k \quad (34.43)$$

Note 1: $x_{c,k}$, and not $x_{p,k}$, is used as the state estimate in applications.

Note 2: Formula (34.43) is an *error-driven* correction term of the state estimates. You can expect the errors of the state estimates to be smaller than if there were no such correction term. This correction can also be regarded as a feedback correction of the estimates – the feedback is from the predicted measurement back to the rate of changes of the state estimates. This feedback is indicated in Figure 34.3. It is well known from control systems theory that error-driven or feedback correction (or control) generally reduces errors.

Note 3: The corrected estimate $x_{c,k}$ is also denoted the a posteriori estimate because it is calculated after the present measurement is taken. It is also denoted the measurement-updated estimate.

Note 4: The state estimates should be prevented from getting unrealistic values. For example, the estimate of a liquid level should not be negative. The following Python code implements such limitation:

```
:  
x_c_k = ...  
if (x_c_k > x_max):  
    x_c_k = x_max  
elif (x_c_k < x_min):  
    x_c_k = x_min  
:  
:
```

Such a limitation may also be implemented with some built-in limitation functions, e.g. with `numpy.clip()` in Python:

```
:  
x_c_k = ...  
x_c_k = np.clip(x_c_k, x_min, x_max)  
:  
:
```

Predicted state estimate for the next time step

The prediction, $x_{p,k+1}$, is calculated using the present state estimate $x_{c,k}$ and the known input u_k in process model:

$$x_{p,k+1} = f(x_{c,k}, u_k) \quad (34.44)$$

Note 1: In the process model (34.12), the process disturbance v_k is an additive term. However, we can not include v_k in (34.44) since v_k is not known or not predictable (since we have assumed it is “white noise”).

Note 2: The predicted estimate is also denoted the a priori estimate because it is calculated before the present measurement is taken. It is also denoted the time-updated estimate.

Note 3: The state estimates should be prevented from getting unrealistic values. For example, the estimate of a liquid level should not be negative. The following Python code implements such limitation:

```

⋮
x_p_kp1 = ...
if (x_p_kp1 > x_max):
    x_p_kp1 = x_max
elif (x_p_kp1 < x_min):
    x_p_kp1 = x_min
⋮

```

Such a limitation may also be implemented with some built-in limitation functions, e.g. with `numpy.clip()` in Python:

```

⋮
x_p_kp1 = ...
x_p_kp1 = np.clip(x_p_kp1, x_min, x_max)
⋮

```

Auto-covariance of corrected state estimate error

The auto-covariance matrix of the estimation error of the corrected estimate is:

$$P_{c,k} = E \left[(x_k - m_{x_{c,k}}) (x_k - m_{x_{c,k}})^T \right] \quad (34.45)$$

where $m_{x_{c,k}}$ is mean value.

A formula for $P_{c,k}$ is:

$$P_{c,k} = [I - K_k C] P_{p,k} \quad (34.46)$$

where: K_k is the Kalman Gain given by (34.40). C is given by (34.41). In the first iteration of the estimation loop, $P_{p,k}$ is known from the initialization, (34.36). In subsequent iterations, $P_{p,k}$, is known from its predicted value, see below.

Auto-covariance of predicted state estimate error in the next iteration

The auto-covariance of the state estimate error is defined by (34.35). Its value in the next iteration is:

$$P_{p,k+1} = A P_{c,k} A^T + G Q G^T \quad (34.47)$$

Here, A is the transition matrix of a linearized model of the original discrete-time nonlinear model (34.12) calculated at the most recent operating point, which is $(x_{c,k}, u_k)$. Although

we can obtain A by linearizing $f(\cdot)$ in that model, I suggest that A is obtained by first linearizing the continuous-time model (34.14) and then discretizing that model¹:

$$A = I + T_s A_{\text{cont}} = I + T_s \cdot \frac{\partial f_{\text{cont}}(\cdot)}{\partial x} \Big|_{\text{op} = (x_{c,k}, u_k)} \quad (34.48)$$

Index-shift (or time-shift) to prepare for the next iteration

$$x_{p,k} = x_{p,k+1}$$

$$P_{p,k} = P_{p,k+1}$$

Example 34.2 shows the design and implementation of a Kalman Filter for estimation of the inflow (and the level) of a liquid tank using the level measurement.

34.3.4 Features of the Kalman Filter

34.3.4.1 The error-model

Assuming that the system model is linear and that the model is correct (giving a correct representation of the real system), it can be shown that the behaviour of the error of the corrected state estimation, $e_{x_{c,k}}$, cf. (34.32), is given by the following *error-model*:²

$$e_{x_{c,k+1}} = (I - K_k C) A e_{x_{c,k}} + (I - K_k C) G v_k - K_k w_{k+1} \quad (34.49)$$

This model can be used to analyze the Kalman Filter in terms of dynamics, stability and steady state behaviour.

Note: (34.32) is not identical to the auto-covariance of the estimation error which is

$$P_{c,k} = E \left\{ [e_{x_{c,k}} - m_{x_{c,k}}] [e_{x_{c,k}} - m_{x_{c,k}}]^T \right\} \quad (34.50)$$

But (34.32) is the *trace* of P_c (the trace is the sum of the diagonal elements):

$$e_{x_c} = \text{trace}[P_{c,k}] \quad (34.51)$$

34.3.4.2 The dynamics of the Kalman Filter

The error-model (34.49) of the Kalman Filter represents a dynamic system. The dynamics of the Kalman Filter can be analyzed by calculating the eigenvalues of the system matrix of (34.49). These eigenvalues are:

$$\{\lambda_1, \lambda_2, \dots, \lambda_n\} = \text{eig}[(I - K_k C) A] \quad (34.52)$$

¹because then you can exploit a pertinent function in e.g. Python or Matlab for discretization

²You can derive this model by subtracting the model describing the corrected state estimate from the model that describes the real state (the latter is simply the process model).

34.3.4.3 The stability of the Kalman Filter

It can be shown that the Kalman Filter always is an asymptotically stable dynamic system (otherwise it could not give an optimal estimate). In other words, the eigenvalues defined by (34.52) are always inside the unity circle.

34.4 Tuning the Kalman Filter

The main tuning factor of the Kalman Filter is the matrix Q – the process disturbance auto-covariance. Remember that the process disturbance influences the state variables, cf. (34.13). Therefore, a large Q tells the Kalman Filter that the variations in the real state variables are relatively large. Hence, the larger Q , the larger the Kalman Gain K_k will be, to provide a stronger updating of the estimates. However, this strong estimate update causes more measurement noise to be added to the estimates because the measurement noise is a term in the innovation process e which is multiplied by K_k :

$$x_{c,k} = x_{p,k} + K_k e_k \quad (34.53)$$

$$= x_{p,k} + K_k [g(x_k) + v_k - g(x_{p,k})] \quad (34.54)$$

where v is real measurement noise.

Consequently, we can state the main rule for tuning the Kalman Filter as follows:

Select elements of Q as large as possible without the state estimates becoming too noisy.

But Q is a matrix! How to select it “large” or “small”? Since each of the process disturbances typically are assumed to act on their respective state independently, Q can be set as a diagonal matrix:

$$Q = \begin{bmatrix} Q_{11} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & Q_{nn} \end{bmatrix} = \text{diag}(Q_{11}, \dots, Q_{nn}) \quad (34.55)$$

where each of the diagonal elements can be adjusted independently. If you do not have any idea about numerical values, you can start by setting all the diagonal elements to one, and hence Q is

$$Q = Q_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (34.56)$$

where Q_0 is the only tuning parameter. If you do not have any idea about a proper value of Q_0 you may initially try

$$Q_0 = 0.01 \quad (34.57)$$

Then you may adjust Q_0 or try to fine tune each of the diagonal elements individually.

Example 34.2 demonstrates the effect of tuning the Kalman Filter with the Q matrix.

34.5 Estimating parameters and disturbances with Kalman Filter

34.5.1 Introduction

Note that you can try to estimate model errors with a Kalman Filter by defining such errors as *augmentation state variables* and estimating them as ordinary states, in the same way as you can estimate disturbances by defining them as augmentation state variables, cf. Section 34.5.

In many applications the Kalman Filter is used to estimate parameters and/or disturbances in addition to the “ordinary” state variables. Examples:

- Parameter: The fraction of biodegradable organic material in organic feed to a biogas reactor.
- Disturbance: Environmental forces acting on a ship. (The estimate can then be used for feedforward control in the position controller of the ship.)

These parameters and/or disturbances must be represented as state variables to be estimated with a Kalman Filter. They represent additional state variables. The original state vector is augmented with these new state variables which we may denote the augmentative states. The Kalman Filter is used to estimate the augmented state vector which consists of both the original state variables and the augmentative state variables.

How can you model these augmentative state variables? The augmentative model must be in the form of a difference equation which in the form of a state space model so that it can be augmented to the original state space model. To set up such an augmentative model you must make an assumption about the behaviour of the augmentative state. Let us look at some augmentative models.

34.5.2 The augmentative state (x_a) is constant

The most common augmentative model is based on the assumption that the augmentative state variable x_a is constant. The corresponding differential equation is:

$$(x_a)' = 0 \quad (34.58)$$

Discretizing this differential equation with the Euler Forward method gives:

$$x_{a,k+1} = x_{a,k} \quad (34.59)$$

It is common to assume that the state is driven by some random disturbance, hence the augmentative model becomes:

$$x_{a,k+1} = x_{a,k} + w_{a,k} \quad (34.60)$$

where w_a is random process disturbance with assumed auto-covariance matrix Q_a . As pointed out in Section 34.3, the variance Q_a can be used as a tuning parameter of the estimate of x_a .

34.5.3 The augmentative state (x_a) has constant rate

The corresponding differential equation is:

$$(x_a)'' = 0 \quad (34.61)$$

or, in state space form, with $x_{a_1} \equiv x_a$:

$$(x_{a_1})' = x_{a_2} \quad (34.62)$$

$$(x_{a_2})' = 0 \quad (34.63)$$

where x_{a_2} is another augmentative state variable. Applying Euler Forward discretization with sampling interval h [sec] to (34.62) – (34.63) and including white process noise to the resulting difference equations gives

$$x_{a1,k+1} = x_{a1,k} + T_s x_{a2,k} + w_{a1,k} \quad (34.64)$$

$$x_{a2,k} = x_{a2,k} + w_{a2,k} \quad (34.65)$$

Now follows a comprehensive example which covers:

- Kalman Filter algorithm development
- Kalman Filter tuning
- Kalman Filter implementation in a Python program

Example 34.2 Kalman Filter for estimation of inflow of a simulated buffer tank

The system

A buffer tank is presented in Ch. 41.3. In this example, we will design and implement (in Python) a Kalman Filter for a simulated tank which has a level control system. The Kalman Filter will estimate F_{in} and h , hence, it will be a soft sensor for F_{in} and h . We may here regard F_{in} as a disturbance that we want to estimate.

The Kalman Filter needs at least one measurement. In this example, the measurement is the simulated level, h .

The time-step of the simulator and of the Kalman Filter is:

$$T_s = 10 \text{ s} \quad (34.66)$$

Figure 34.4 shows the tank with a level control system.

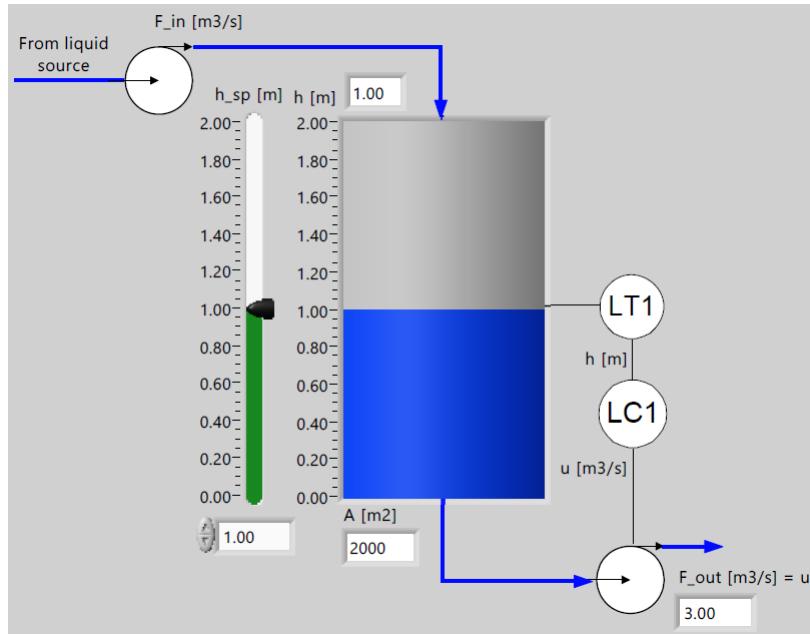


Figure 34.4: Buffer tank with level control system.

The level is controlled with a PI controller tuned with the Skogestad method, cf. Ch. 18.4. The level PI controller is tuned with the Skogestad method with $T_c = 1000$ s, as in Example 18.4, giving:

$$K_c = -2 \quad (34.67)$$

$$T_i = 2000 \text{ s} \quad (34.68)$$

The process model

Process model parameters:

$$A_{\text{area}} = 2000 \text{ m}^2 \quad (34.69)$$

The mathematical model of the process is based on mass balance of the liquid in the tank. The model is:

$$(Ah)' = F_{\text{in}} - F_{\text{out}} \quad (34.70)$$

Mass balance of the liquid in the tank is (mass is ρAh):

$$(\rho A_{\text{area}} h)' = \rho F_{\text{in}} - \rho F_{\text{out}} \quad (34.71)$$

F_{out} is the outflow demanded by the level controller. Therefore,

$$F_{\text{out}} = u \quad (34.72)$$

where u is the control signal. By inserting u for F_{out} and cancelling out density ρ , (34.71) becomes:

$$h' = \frac{1}{A_{\text{area}}} (F_{\text{in}} - u) \equiv f_1(\cdot) \quad (34.73)$$

We assume that F_{in} is unknown, but slowly changing – almost constant. Therefore, we define the following augmentative model for F_{in} :

$$(F_{\text{in}})' = 0 \equiv f_2(\cdot) \quad (34.74)$$

h is measured. Hence,

$$h_m = h \equiv g(\cdot) \quad (34.75)$$

To summarize: The continuous-time model of the system is given by the differential equations (34.73) – (34.74) with (34.75) as the output or measurement equation.

In the context of linearization, it can be convenient to collect f_1 and f_2 in one vector function:

$$f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \quad (34.76)$$

In some contexts, it is convenient to introduce standard names of the state variables:

$$x_1 = h \quad (34.77)$$

$$x_2 = F_{\text{in}} \quad (34.78)$$

However, we will in this example continue using the original symbols h and F_{in} .

For the Kalman Filter we need a discrete-time state space model. Applying Euler Forward discretization with time step T_s and including white disturbance noises w_1 and w_2 in the resulting difference equations, give:

$$h_{k+1} = h_k + \frac{T_s}{A_{\text{area}}} (F_{\text{in},k} - u_k) + w_{1,k} \quad (34.79)$$

$$F_{\text{in},k+1} = F_{\text{in},k} + w_{2,k} \quad (34.80)$$

w_1 and w_2 are independent (uncorrelated) white process disturbances with assumed variances Q_{11} and Q_{22} , respectively. The process disturbance covariance matrix is then:

$$Q = \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{bmatrix} \quad (34.81)$$

As mentioned above, h is measured. The measurement is here denoted h_{meas} . Assuming white measurement noise v_k with variance R_{11} (a scalar), the measurement equation is:

$$h_{m,k} = h_k + v_k \quad (34.82)$$

The measurement noise covariance matrix is:

$$R = [R_{11}] \quad (34.83)$$

where:

$$R_{11} = (\sigma_{\text{meas,noise}})^2$$

where:

$$\sigma_{\text{meas,noise}} = \frac{a_{\text{meas,noise}}}{\sqrt{3}} \quad (34.84)$$

where:

$$a_{\text{meas,noise}} = 0.01 \text{ m} \quad (34.85)$$

$a_{\text{meas,noise}}$ is the amplitude of an assumed uniformly distributed random noise signal of amplitude 0.01 m (i.e. the noise is between ± 0.01 m. The relation between the standard deviation $\sigma_{\text{meas,noise}}$ and the amplitude $a_{\text{meas,noise}}$ is derived in Solution 33.6.

The PI level controller

The tank level, h , is controlled with a PI controller which is tuned with the Skogestad method. The PI controller is in the Python program presented below in this example, but I do not show details about the controller here.

The Kalman Filter algorithm

The Kalman Filter algorithm is as follows (numerical values are given later in this example).

Initialization:

$$\begin{aligned} h_{p,\text{init}} &= h_{m,\text{init}} = 2.0 \text{ m} \\ F_{\text{in},p,k} &= F_{\text{in},p,\text{init}} = 3.0 \text{ m}^3/\text{s} \\ P_{p,k} &= P_{p,\text{init}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

In the simulation/estimation loop:

- The innovation variable:

$$e_k = h_{m,k} - h_{p,k} \quad (34.86)$$

where: $h_{m,k}$ is the simulated measurement. $h_{p,k}$ is the predicted measurement.

- The Kalman Gain:

$$K_k = P_{p,k} C [C P_{p,k} C^T + R]^{-1} \quad (34.87)$$

where:

$$C = C_{\text{cont}} = \left. \frac{\partial g}{\partial x} \right|_{\text{op}} = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad (34.88)$$

- Measurement-corrected estimates, which are used as the applied estimates:

$$h_{c,k} = h_{p,k} + K_{0,0,k} \cdot e_k \quad (34.89)$$

$$F_{\text{in},c,k} = F_{\text{in},p,k} + K_{1,0,k} \cdot e_k \quad (34.90)$$

- Model-predicted estimates as one Euler Forward step:

$$h_{p,k+1} = h_{c,k} + \frac{T_s}{A_{\text{area}}} (F_{\text{in},c,k} - u_k) \quad (34.91)$$

$$F_{\text{in},p,k+1} = F_{\text{in},c,k} \quad (34.92)$$

- Auto-covariance of meas-corrected state estimate error, (34.47):

$$P_{c,k} = [I - K_k C] P_{p,k} \quad (34.93)$$

- Auto-covariance of predicted state estimate error in the next iteration:

$$P_{p,k+1} = AP_{c,k}A^T + GQG^T \quad (34.94)$$

where:

- Disturbance gain:

$$G = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Discrete-time system transition matrix:

$$A = I + T_s A_{\text{cont}} \quad (34.95)$$

$$= I + T_s \left. \frac{\partial f}{\partial x} \right|_{x_{p,k}, u_k} \quad (34.96)$$

$$= I + T_s \left[\begin{array}{cc} \frac{\partial f_1}{\partial x_1} = 0 & \frac{\partial f_1}{\partial x_2} = \frac{1}{A_{\text{area}}} \\ \frac{\partial f_2}{\partial x_1} = 0 & \frac{\partial f_2}{\partial x_2} = 0 \end{array} \right] \Bigg|_{x_{p,k}, u_k} \quad (34.97)$$

$$= \begin{bmatrix} 1 & \frac{T_s}{A_{\text{area}}} \\ 0 & 1 \end{bmatrix} \quad (34.98)$$

- Disturbance co-variance:

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & Q_{22} \end{bmatrix} \quad (34.99)$$

where:

$$Q_{22} = 10^2 = 100$$

which is set by “trial-and-error” to give reasonably fast and smooth estimate of F_{in} . Q_{22} is used as the main tuning factor of the Kalman Filter.

- Index-shift (or time-shift):

$$h_{p,k} = h_{p,k+1} \quad (34.100)$$

$$F_{\text{in},p,k} = F_{\text{in},p,k+1} \quad (34.101)$$

$$P_{p,k} = P_{p,k+1} \quad (34.102)$$

Python program

The Python program 'kalman_pi_buffertank_sim.py' available via the link below implements the tank simulator with the Kalman Filter and the level PI controller.

http://techteach.no/control/python/kalman_pi_buffertank_sim.py

Results

Figure 34.5 shows the responses from the following simulation scenario:

- $h_{sp} = 2.0$ m (constant).
- F_{in} is changed from 3.0 to 4.0 m^3/s at $t = 1000$ s.
- The level measurement noise is uniformly distributed random noise, cf. (34.85), is included in the level measurement.

Figure 34.5 shows simulated responses. The Kalman Filter estimates F_{in} with zero error in steady state, but there is a transient estimation error.

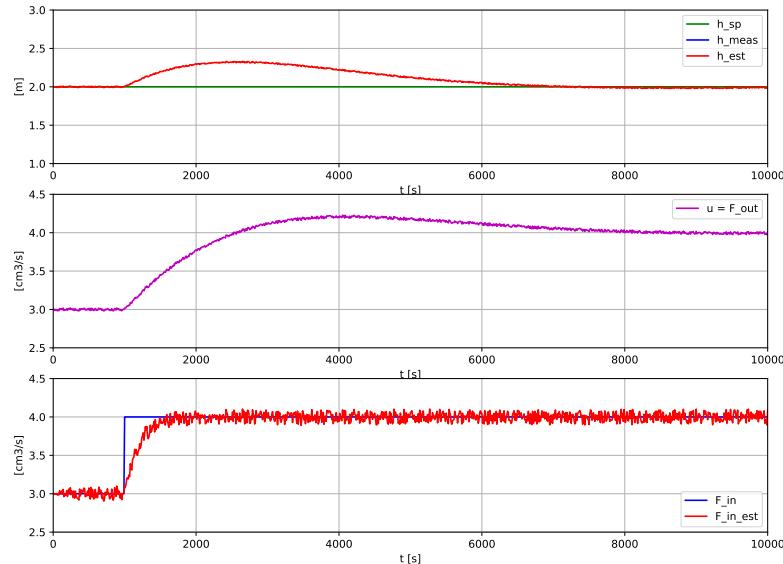


Figure 34.5: Simulated tank with PI level control and Kalman Filter with $Q_{22} = 1$.

It is interesting to see how the estimate of F_{in} is influenced by the tuning factor Q_{22} , while keeping the disturbance variance unchanged (namely keeping it zero) in the simulations:

- Figure 34.6 shows the estimates with $Q_{22} = 1$, which is set with the code `std_Q_22 = 1` in the Python program. As expected, the estimation of F_{in} is less noisy and slower.
- Figure 34.7 shows the estimates with $Q_{22} = 100$. As expected, the estimation of F_{in} is more noisy and faster.

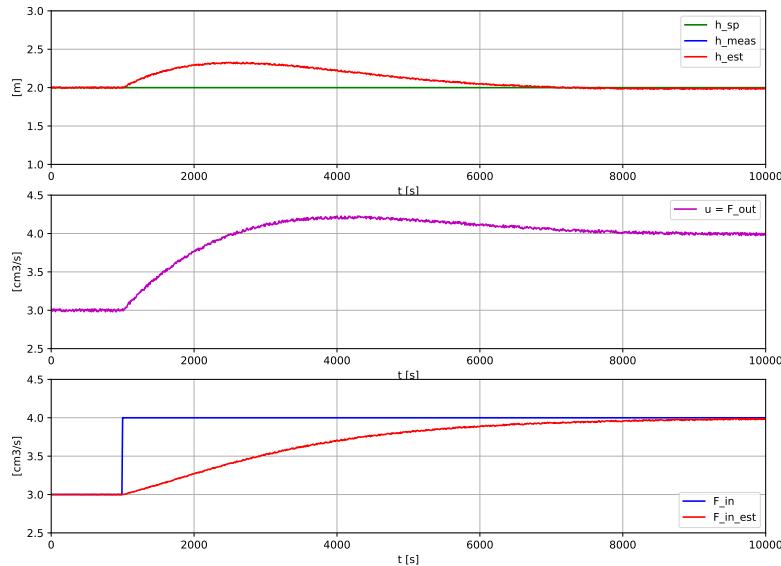


Figure 34.6: Kalman Filter estimates with $Q_{22} = 1$.

[End of Example 34.2]

34.6 Kalman Filtering when process measurement is absent

It may happen that the process measurement, y_m , is absent for some time while the Kalman Filter runs. The reason for the absence may be:

- Sensor failure
- Communication failure between the sensor and the Kalman Filter algorithm
- Low-rate sampling (large time step) of the measurement comparing with a high-rate (small time step) Kalman Filter algorithm

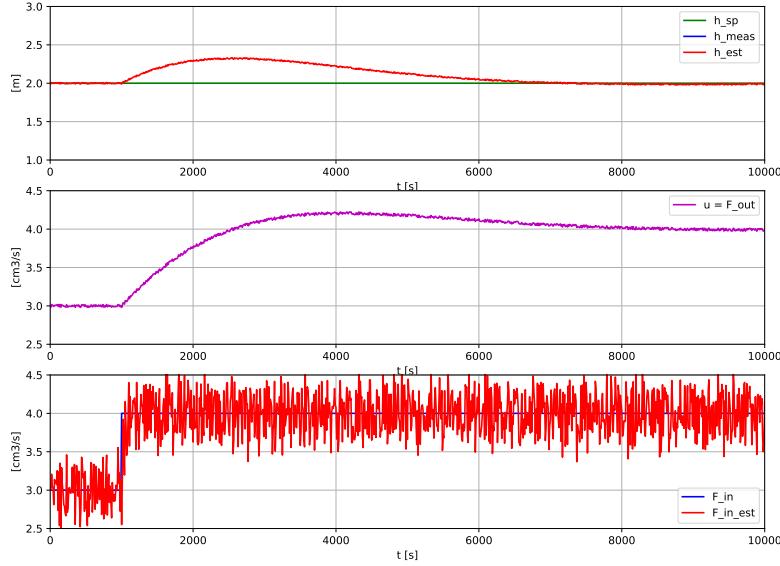


Figure 34.7: Kalman Filter estimates with $Q_{22} = 100$.

We assume that the occurrence of the absence is known. How can the Kalman Filter cope with an absent measurement? An absent measurement corresponds to the measurement being irrelevant. An irrelevant measurement can be modelled as a measurement with an infinitely large measurement noise, i.e.:

$$R = [\infty] \quad (34.103)$$

Let us use (34.103) in the formula for the Kalman Gain, (34.40):

$$K_k = P_{p,k} C^T [C P_{p,k} C^T + R]^{-1} = [0] \quad (34.104)$$

So, the Kalman Filter gain becomes a matrix of zeros. This implies that the corrected estimate, (34.43), becomes:

$$x_{c,k} = x_{p,k} + \underbrace{K_k e_k}_0 = x_{p,k} \quad (34.105)$$

In other words: The corrected estimate is equal to the predicted estimate (which was calculated in the previous iteration of the Kalman Filter algorithm). Or: There is no measurement-based update of the state estimate, which makes sense: The measurement should be disregarded.

Another consequence of (34.104) is that the auto-covariance of corrected state estimate error, (34.46), becomes equal to the auto-covariance of predicted state estimate error:

$$P_{c,k} = [I - K_k C] P_{p,k} = P_{p,k} \quad (34.106)$$

Such a Kalman Filter, i.e. a Kalman Filter which runs without the measurement-based, corrected update of the state estimate, only with the model-based, predicted estimated, is

sometimes denoted a ballistic Kalman Filter³, or an open loop-Kalman Filter. In other words: The Kalman Filter is just a *process simulator*.

In a Kalman Filter running in open loop, as described above, the estimated states will certainly eventually drift from the real states since there is no correction of the estimates. Still, the estimates, and in particular the estimate of the process measurement, may be useful, for example for control purposes where the controller can continue operating despite the absent real measurement. The alternative is turning the controller off since there is no real measurement. In other words: A Kalman Filter (when running in absence of real measurements) can increase the robustness of a control system.

³“Ballistic” because the estimated state develops uncorrected or unmanipulated, like a ball thrown into space.

34.7 *Problems for Chapter 34*

Problem 34.1 *Kalman Filter for a DC motor*

Ch. 41.6 describes a DC motor. A mathematical model is given by (41.10).

1. The load torque L is to be estimated with a Kalman Filter. The speed S is measured. L is assumed constant. Write a mathematical model for the Kalman Filter.
2. In the Kalman Filter algorithm, the measurement covariance R is needed. How can you in practice obtain an estimate of R ?
3. Here is a Python program implementing a Kalman Filter for the DC motor:

http://techteach.no/control/python/kalman_filter_sim_dc_motor.py

L is changed from 0 to -1 at $t = 5$ s. Run the simulator. Do you think that the Kalman Filter is working well?

4. With the Python program available above: Play with the parameter `std_w_L` (increase, decrease) in the program. What is the impact of this parameter?

34.8 *Solutions to problems for Chapter 34*

Solution to Problem 34.1

1. Model for Kalman Filter:

$$S' = [K(u + L) - S] / T + w_1 \quad (34.107)$$

$$L' = 0 + w_2 \quad (34.108)$$

$$S_m = S + v_1 \quad (34.109)$$

w_1 and w_2 are random process disturbances. v_1 is random measurement noise.

Alternatively, using standard symbols:

$$x_1' = [K(u + x_2) - x_1] / T + w_1 \quad (34.110)$$

$$x_2' = 0 + w_2 \quad (34.111)$$

$$y = x_1 + v_1 \quad (34.112)$$

where:

$$x_1 = S \quad (34.113)$$

$$x_2 = L \quad (34.114)$$

$$y = S_m \quad (34.115)$$

2. R can be calculated (estimated) as the variance of a data series of real speed measurements.

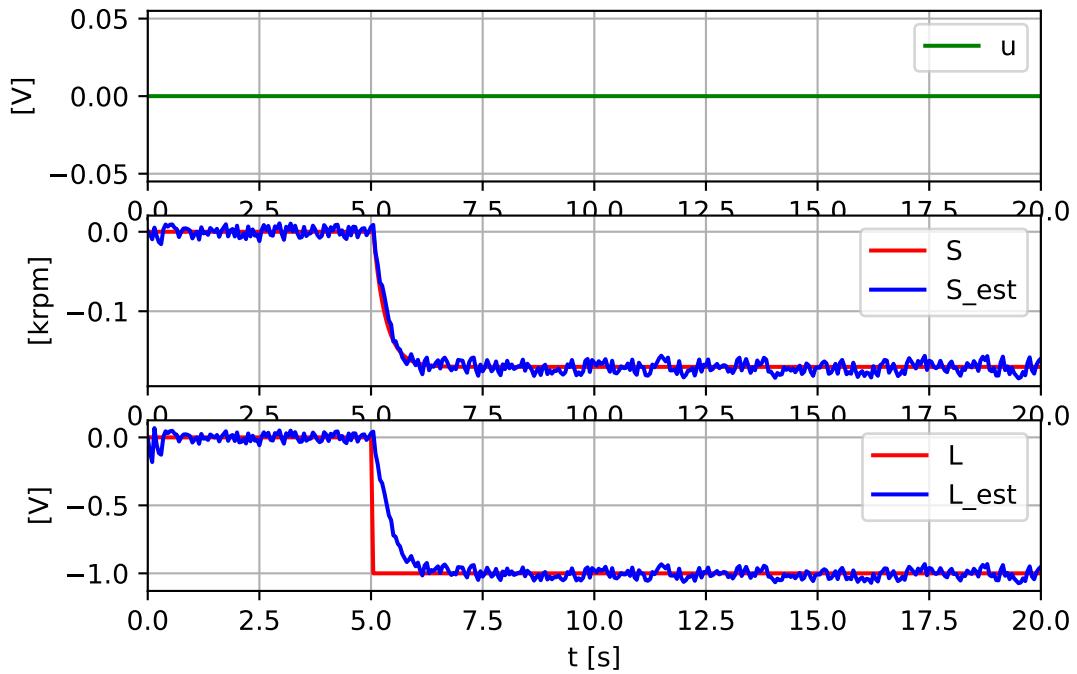
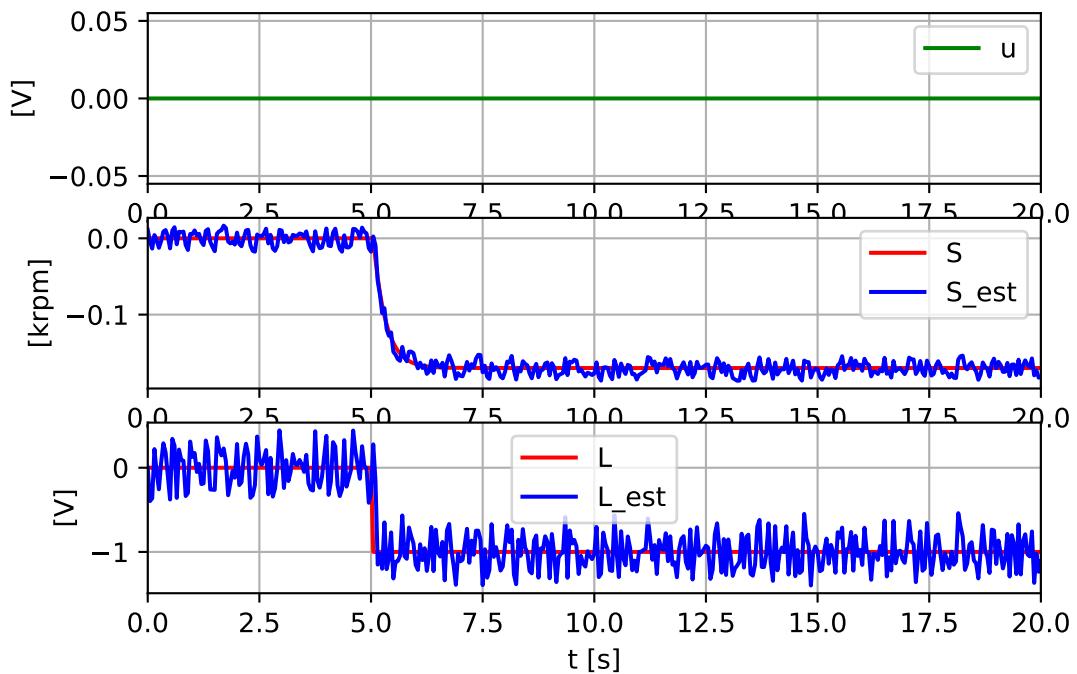
3. Figure 34.8 shows the simulated responses. It seems that the Kalman Filter is working well. The load torque L is estimated well in the sense that the constant L is estimated correctly.

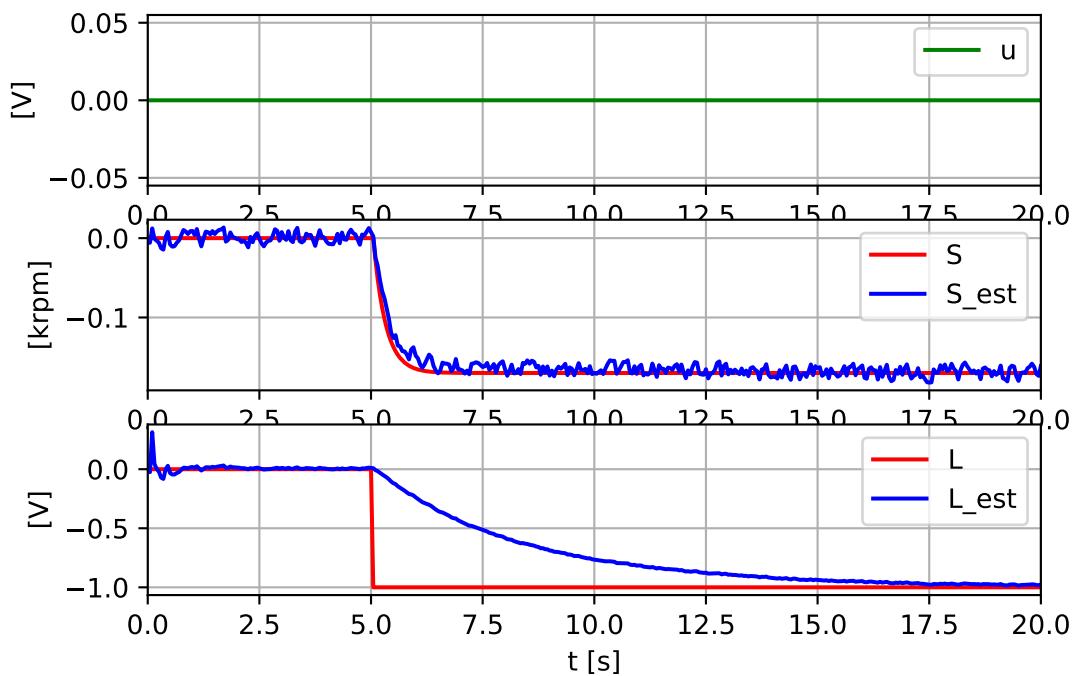
4. The parameter std_w_L is the standard deviation of the assumed process random disturbance acting on L . The variance Q_{22} used in the Q matrix of the Kalman Filter is therefore $(\text{std_w_L})^2$.

If you *increase* std_w_L (i.e., Q_{22} is increased), you tell the Kalman Filter that L is varying much, and the Kalman Filter will increase the speed of the estimate (good), while the estimate is more noisy (bad). This is confirmed in Figure 34.9 where std_w_L has been increased from 0.05 to 0.5.

If you *decrease* std_w_L (i.e., Q_{22} is decreased), you tell the Kalman Filter that L is varying little, and the Kalman Filter will decrease the speed of the estimate (bad), while the estimate is less noisy (good). This is confirmed in Figure 34.10 where std_w_L has been decreased from 0.05 to 0.005.

So, you should tune std_w_L (or Q_{22}) as a compromise between fast & noisy estimation (std_w_L large) and slow & smooth estimation (std_w_L small).

Figure 34.8: Problem 34.1: Kalman Filter with $Q_{22} = 0.05$.Figure 34.9: Problem 34.1: Kalman Filter with $Q_{22} = 0.5$.

Figure 34.10: Problem 34.1: Kalman Filter with $Q_{22} = 0.005$.

Chapter 35

Moving Horizon Estimation

35.1 Introduction

Moving horizon estimation (MHE) Robertson et al. (1996)¹ is a method for state estimation of dynamic systems that can be regarded as an alternative to Kalman Filtering. The principle is to continuously use the present and the previous measurements of the system and known inputs to the system over a fixed-length *historical time horizon*, together with an assumed mathematical model of the system to calculate the optimal (best) present state of the system. This is illustrated in Figure 35.1.

35.2 The assumed process model

It is assumed that the model of the pertinent system is given as a discrete time state space model on the following form:

$$x_{k+1} = f(x_k, \cdot) + w_k \quad (35.1)$$

$$y_k = g(x_k, \cdot) + v_k \quad (35.2)$$

where:

- k is the discrete time index, so that the actual time is $t_k = kT_s$ where T_s is the time-step or sampling time.
- x is the state vector to be estimated. More precisely, it is the value of x at time (index) k that we want to estimate. x is a vector of n scalar state variables:

$$x = \begin{bmatrix} x(1) \\ x(2) \\ \vdots \\ x(n) \end{bmatrix} \quad (35.3)$$

¹A good presentation of MHE and several other topics in state estimation is given in Boegli (2014).

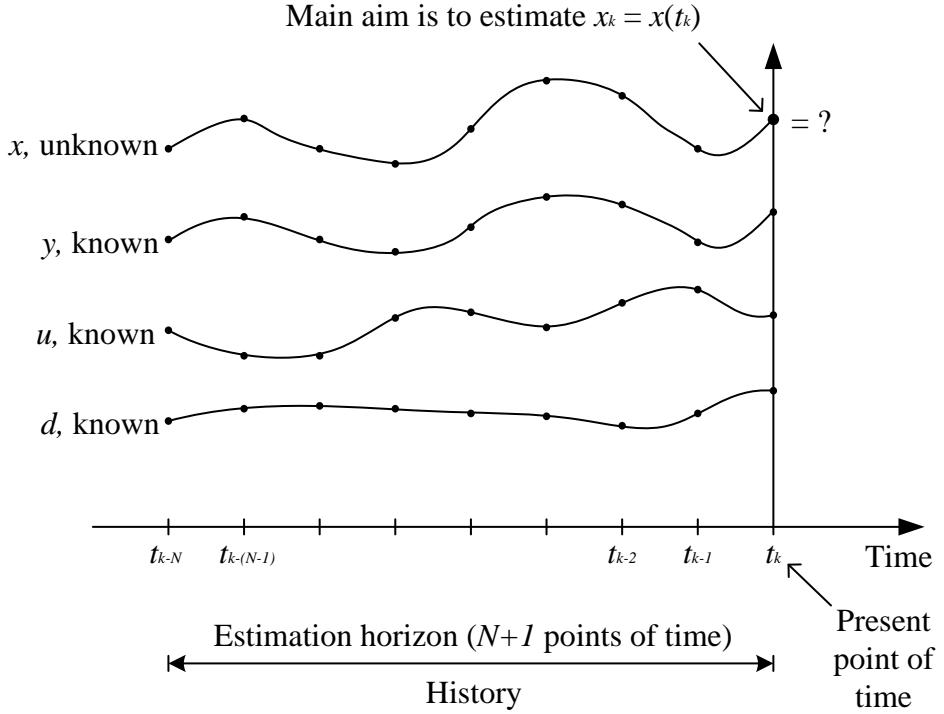


Figure 35.1: Principle of Moving horizon estimation (MHE). x is state. y is measurement. u is control signal. d is disturbance.

- f is a vectorial function. It is a vector of n nonlinear, or linear, functions of x_k :

$$f = \begin{bmatrix} f_1(x_k, \cdot) \\ f_2(x_k, \cdot) \\ \vdots \\ f_n(x_k, \cdot) \end{bmatrix} \quad (35.4)$$

There may be additional arguments of f , as the control variable (vector) u_k , the process disturbance (vector), d_k , and parameters (vector), p . These additional arguments are represented by the dots in Eq. (35.1), and we assume they have *known* values. (Estimation of unknown disturbances or parameters is described in Section 35.4.)

- w_k represents the *non-modeled or unknown process disturbance*, acting on the state, at time index k . w_k may alternatively be interpreted as a *model error* since w_k represents the error of the prediction of the state at the next time-step. Any known disturbance is represented by d_k , cf. the above item. It is not necessary to assume any particular statistical properties of w_k , however, it may be reasonable to assume it is a random signal – “white noise” – with a specified covariance matrix, say Q , just as in the Kalman Filter. We may then use Q in the objective function of MHE, as described below.
- y is the system output vector of m elements assumed being measured and therefore having known value.

- g is a vectorial function. It is a vector of m nonlinear, or linear, functions of x and possibly of additional variables and parameters, represented by dots:

$$g = \begin{bmatrix} g_1(x_k, \cdot) \\ g_2(x_k, \cdot) \\ \vdots \\ g_m(x_k, \cdot) \end{bmatrix} \quad (35.5)$$

- v_k is the *measurement error*. It is not necessary to assume any particular statistical properties of v_k , however, it may be reasonable to assume it is a random signal – “white noise” – with a specified covariance matrix, say R , just as in the Kalman Filter. We may then use R in the objective function of MHE, as described below.

35.3 The optimization problem of MHE

The optimization problem of MHE to be solved continuously, to continuously calculate the state estimate, is:

$$\min_X J \quad (35.6)$$

J is the objective function. It is defined below. X is a matrix containing the state at each point of time of the estimation horizon:

$$\begin{aligned} X &= [x_{k-N}, x_{k-(N-1)}, x_{k-1}, x_k] \\ &= \begin{bmatrix} x(1)_{k-N} & x(1)_{k-(N-1)} & \cdots & x(1)_{k-1} & x(1)_k \\ x(2)_{k-N} & x(2)_{k-(N-1)} & \cdots & x(2)_{k-1} & x(2)_k \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ x(n)_{k-N} & x(n)_{k-(N-1)} & \cdots & x(n)_{k-1} & x(n)_k \end{bmatrix} \end{aligned} \quad (35.7)$$

X can be denoted the total state matrix. X will be the solution of the MHE optimization problem. From this X ,

$$x_k = \begin{bmatrix} x(1)_k \\ x(2)_k \\ \vdots \\ x(n)_k \end{bmatrix}$$

is used as the applied present state estimate. As an example, assume the model has $n = 3$ state variables and the horizon length is 5, the number of optimization variables $n(N + 1) = 3 \cdot 6 = 18$.

Before we look at the details of J , the optimization function to be minimized, let's review the mathematical term *norms* since J contain norms.

35.3.0.0.1 About norms. A norm is a measure of the length of a vector. There are various kinds of norms, but the most common one is the quadratic norm and variations of such. For a given vector z , the expression $\|z\|_M^2$ is the square of the M -quadratic norm of z .² In detail, $\|z\|_M^2$ is

$$\begin{aligned}\|z\|_M^2 &= z^T M z \\ &= [z(1), \dots, z(r)] \begin{bmatrix} M_{11} & & \\ & \ddots & \\ & & M_{rr} \end{bmatrix} \begin{bmatrix} z(1) \\ \vdots \\ z(r) \end{bmatrix} \\ &= M_{11}z(1)^2 + \dots + M_{rr}z(r)^2\end{aligned}$$

If $M = I$, the identity matrix, the M -quadratic norm is the square of the well-known length of the vector, since the length is $\sqrt{z^T z} = \sqrt{z(1)^2 + \dots + z(r)^2}$.

35.3.0.0.2 The optimization function. The optimization function to be minimized, cf. Eq. (35.6), is

$$J = \sum_{i=k-N}^{k-1} \|x_{i+1} - f(x_i, \cdot)\|_{Q^{-1}}^2 + \sum_{i=k-N}^k \|y_i - g(x_i, \cdot)\|_{R^{-1}}^2 \quad (35.8)$$

Based on Eqs. (35.1) and (35.2), J can also be written as

$$J = \sum_{i=k-N}^{k-1} \|w_i\|_{Q^{-1}}^2 + \sum_{i=k-N}^k \|v_i\|_{R^{-1}}^2 \quad (35.9)$$

or

$$J = \sum_{i=k-N}^{k-1} w_i^T Q^{-1} w_i + \sum_{i=k-N}^k v_i^T R^{-1} v_i \quad (35.10)$$

where:

The process disturbance vector:

$$w_i = \begin{bmatrix} w(1)_i \\ \vdots \\ w(n)_i \end{bmatrix}$$

The measurement vector:

$$v_i = \begin{bmatrix} v(1)_i \\ \vdots \\ v(m)_i \end{bmatrix}$$

²Quadratic norms are also denoted l^2 -norms or Euclidian distance.

Q^{-1} is a weight matrix:

$$Q^{-1} = \begin{bmatrix} \frac{1}{Q_{11}} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{Q_{nn}} \end{bmatrix}$$

Assuming that w is random, Q can be interpreted as the inverse of the process disturbance covariance matrix.

R^{-1} is a weight matrix:

$$R^{-1} = \begin{bmatrix} \frac{1}{R_{11}} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{R_{mm}} \end{bmatrix}$$

Assuming that v is random, R can be interpreted as the inverse of the measurement error covariance matrix.

Now, Eq. (35.10) can be written in detail as

$$J = \sum_{i=k-N}^{k-1} \left[\frac{w(1)_i^2}{Q_{11}} + \cdots + \frac{w(n)_i^2}{Q_{nn}} \right] + \sum_{i=k-N}^k \left[\frac{v(1)_i^2}{R_{11}} + \cdots + \frac{v(m)_i^2}{R_{mm}} \right]$$

Roughly said, MHE minimizes, in a least squares sense, the measurement errors and the non-measured process disturbances (or model errors) over the estimation horizon. In other words, MHE utilizes *the measurements* (by minimizing the influence of the measurement errors) and *the model* (by minimizing the model errors) to calculate the state estimate.

Typically, the objective function is presented with an additive term denoted the arrival cost³, a :

$$J = \sum_{i=k-N}^{k-1} \|w_i\|_{Q^{-1}}^2 + \sum_{i=k-N}^k \|v_i\|_{R^{-1}}^2 + a \quad (35.11)$$

where

$$a = \|x_{k-N} - \bar{x}_{k-N}\|_P^2 \quad (35.12)$$

where \bar{x}_{k-N} is the actual state at the start of the horizon, and P^{-1} is the corresponding weight or cost matrix. The actual state is not known, so it is problematic to give it a value. This uncertainty can be expressed with a very large P , causing a to vanish. Furthermore, an estimate of x_{k-N} will actually be provided since x_{k-N} is included in the first term of Eq. (35.8). Consequently, it is probably ok to omit the arrival cost term from the objective function, i.e. a can be set to zero in Eq. (35.11).

35.3.0.0.3 Constraints. The MHE optimization problem can also include constraints on X (the optimization variables). For example, if the liquid level in a tank is one state variable, it is natural to define the maximum possible level as an upper bound and the minimum possible level as the lower bound on X .

³Maybe a better name would have been the departure cost?

35.3.0.0.4 Guessed value of X . When solving the optimization problem, it is necessary that the optimizer is supplied with a good guess of the optimization variable, X . As a good value of X_{guess} at time index k , here denoted X_{guess_k} , we can use the pertinent portion of $X_{\text{opt}_{k-1}}$, the optimal solution found at time index $k-1$ (the previous point of time). However, in $X_{\text{opt}_{k-1}}$, we must leave out the state $x_{\text{opt}_{k-1}}$ while we insert a guessed value for time index k . Lets us denote the latter guessed state by x_{guess_k} . How to select x_{guess_k} ? It is here suggested that a model-predicted value of x based on $x_{\text{opt}_{k-1}}$ is used to calculate x_{guess_k} , as follows:

$$x_{\text{guess}_k} = x_{\text{pred}_k} = f(x_{\text{opt}_{k-1}}, u_k, d_k)$$

To summarize, X_{guess_k} can be selected as

$$X_{\text{guess}_k} = \left[X_{\text{opt}(2:k-1)}, x_{\text{pred}_k} \right] \quad (35.13)$$

where (2:k-1) is Matlab-like notation.

35.3.0.0.5 No linearization. In MHE, no linearization of the state space model is needed. Hence, it is a “nonlinear” state estimation method. This is contrary to Extended Kalman Filtering, which requires linearization of the state space model to obtain the Kalman gain.

35.4 Estimation of model parameters and state disturbances

In state estimation, it is often a wish to estimate model parameters and/or process disturbances in addition to the “ordinary” states that stem from the principles of mechanistic modeling, i.e. material balances, energy balances, impulse balances (laws of motion) etc. Such parameters or disturbances can be estimated in a straightforward way with MHE (as with a Kalman Filter) if we model them as state variables. Remember that a state variable is represented by its time derivative, i.e. a differential equation, in a continuous-time state space model. It is common to assume that a parameter or a disturbance to be estimated is constant. What is the time derivative of a constant? Zero! Therefore, a model parameter p can be represented by the differential equation

$$\dot{p} = 0 \quad (35.14)$$

The corresponding difference equation to be included in the discrete time state space model of MHE, is

$$p_{k+1} = p_k + w_p \quad (35.15)$$

where w is model error pertinent to this parameter state variable. The original state space model has now been *augmented* with this difference equation. p is an *augmentation* state variable. The original state vector has been augmented, to become:

$$x_{\text{aug}} = \begin{bmatrix} x_{\text{original}} \\ p \end{bmatrix} \quad (35.16)$$

Similary, a state disturbance can be estimated by augmenting the original model with the following difference equation

$$d_{k+1} = d_k + w_d \quad (35.17)$$

35.5 Tuning factors of MHE

The main tuning factors of MHE are:

- *The estimation horizon length, N .* The larger N , the “safer” optimal solution can be expected, but on the expense of more computational demand. A typical value of N seems to be between 5 and 20, assuming an appropriate time step length (which may be e.g. 1/5 of smallest time constant-like dynamics represented by the model).
- *The covariance matrix, R , of the measurement error.* Increasing the value of R_{jj} , which is the variance of measurement error of variable $y(j)$, implies higher influence of $y(j)$ on the state estimate, but at the expense of measurement noise $v(j)$ also influencing more on the estimate. Although you may use R as a tuning factor, it may be reasonable to fix it to the variance of an appropriate measurement series of the real process measurement. If you do not have any measurement series at hand, you may set it to the square of 1/100 of an assumed value of the measurement (which corresponds to the standard deviation of the measurement error assumed as 1/100 of the measurement value), i.e.

$$R(j, j) = \left[\frac{y(j)_{\text{assumed}}}{100} \right]^2 \quad (35.18)$$

- *The covariance matrix, Q , of the non-modelled process disturbance.* It is not easy to set a proper value of Q , but the following starting value may be selected for $Q(j, j)$ – the covariance of the non-modelled process disturbance acting on state $x(j)$:

$$Q(j, j) = \left[\frac{x(j)_{\text{assumed}}}{1000} \right]^2 \quad (35.19)$$

If you increase $Q(j, j)$, the estimate of $x(j)$ may approach to its “true” value faster, but at the expense of the estimate becoming more noisy. This can be seen from Eq. (35.9): If element $Q(j, j)$ is increased, it is implicitly assumed that the pertinent non-modelled process disturbance w_j is increased, which calls for a more “aggressive” update of the state estimate by the MHE.

Example 35.1 MHE with the fmincon optimizer of Optimization Toolbox of Matlab

In this example, I have implemented an MHE in Matlab for the following simulated model:

$$\dot{x}(1) = x(2) \quad (35.20)$$

$$\dot{x}(2) = [-x(2) + Ku]/T + d \quad (35.21)$$

$$y = x(1) \quad (35.22)$$

where K is the gain and T is the time constant. d is a process disturbance which is to be estimated along with $x(1)$ and $x(2)$. The measurement is $y = x(1)$. u is the control signal.

The Matlab program, which is named mhe.m, is available via this link:

<http://techteach.no/control/matlab/mhe.m>

Figure 35.2 shows the results of a simulation. The MHE is started when the simulation has run for as long as the length of the estimation horizon, which is here 10 time-steps corresponding to 5 s. We see that the MHE estimates the states, including the disturbance, well.

Notes about the Matlab implementation:

- The optimization variable in the MHE optimization problem is the total state matrix, X , cf. Eq. (35.7). You can represent X with a Matlab *matrix* in fmincon! Thus, it is not necessary to transform this matrix to an array for fmincon.
- The objective function and the constraints function are defined as local functions within the script, and they exist only within the script.

[End of Example 35.1]

35.6 *Problems for Chapter 35*

To appear

35.7 *Solutions to problems for Chapter 35*

To appear

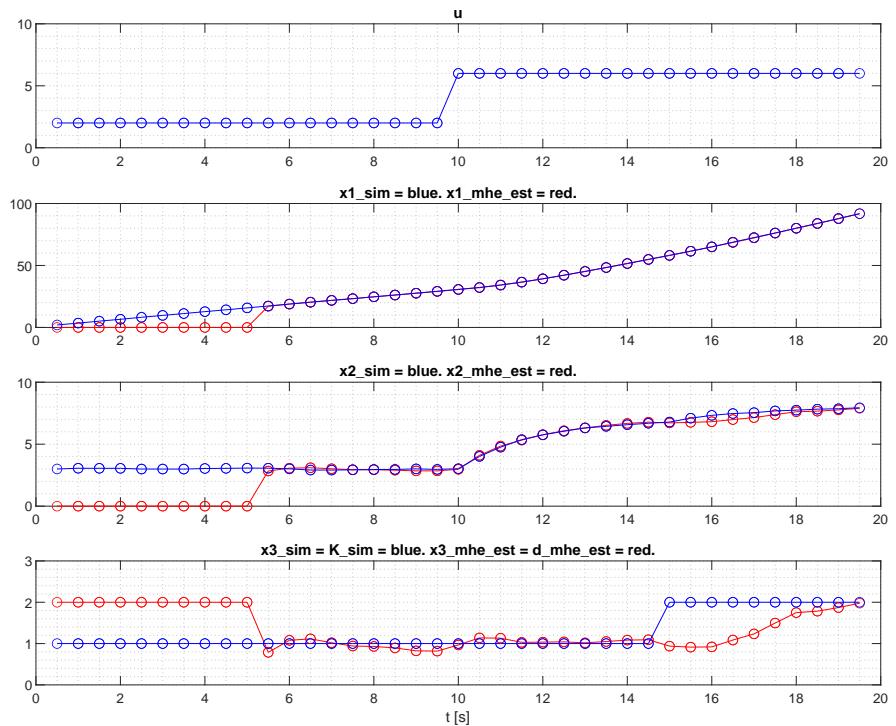


Figure 35.2: Example 35.1: MHE estimation.

Part VII

MODEL-BASED CONTROL

Chapter 36

How to test robustness with simulations

This part of the book describes several *model-based* controllers. A model-based controller contains the mathematical model of the process to be controlled. Of course, a process model can never give a perfect description of a physical process. Hence, there are *model errors*. The model errors are in the form of

- erroneous model structure, and/or
- erroneous parameter values.

So, the controller is based on more or less erroneous information about the process. If the model errors are large, the real control system may behave quite different from what is specified during the design. The control system may even become unstable.

A control system that is supposed to work in real life must be sufficiently *robust*. How can you check if the system is robust (before implementation)? You can *simulate the control system*. In the simulation you include reasonable model errors. How do you include model errors in a simulator? By using *different* models in the control function and in the process in the simulator. This is illustrated in Figure 36.1.

You may use the initial model, M_0 , in the control function, while you use a changed model, M_1 , for the process. You must thoroughly plan which model errors (changes) that you will make, and whether the changes are *additive* or *multiplicative*. A parameter, say K , is changed additively if it is changed as

$$K_1 = K_0 + \Delta K \quad (36.1)$$

A multiplicative change is implemented with

$$K_1 = FK_0 \quad (36.2)$$

where the factor F may be set to e.g. 1.2 (a 20% increase) or 0.8 (a 20% decrease).

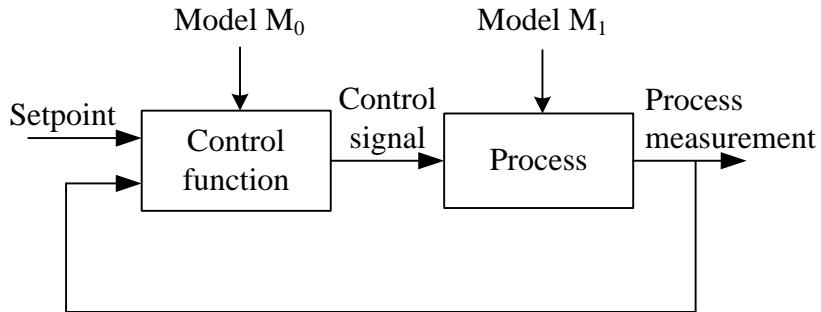


Figure 36.1: Testing the control system with model errors. Models M_0 and M_1 are made different by purpose.

Even if the process model is accurate, the behaviour of the controller can be largely influenced by measurement noise. Therefore, you get a more real picture if you also include measurement noise in the simulator. Typically the measurement noise is a random signal¹.

Finally, I will mention that there are control system design methods which ensure robustness of the control system. In the design phase you specify assumed maximum model errors, together with performance specifications. The control function is typically a non-standard controller, which have to be simplified before implementation. It is however beyond the scope of this compendium to describe these design methods. More information can be found in e.g. ?.

¹Simulation tools as LabVIEW and Simulink contains signal generators for random signals.

36.1 Problems for Chapter 36

Problem 36.1 Model-based control of DC motor

Figure 36.2 shows a model-based speed control system of an electric motor.

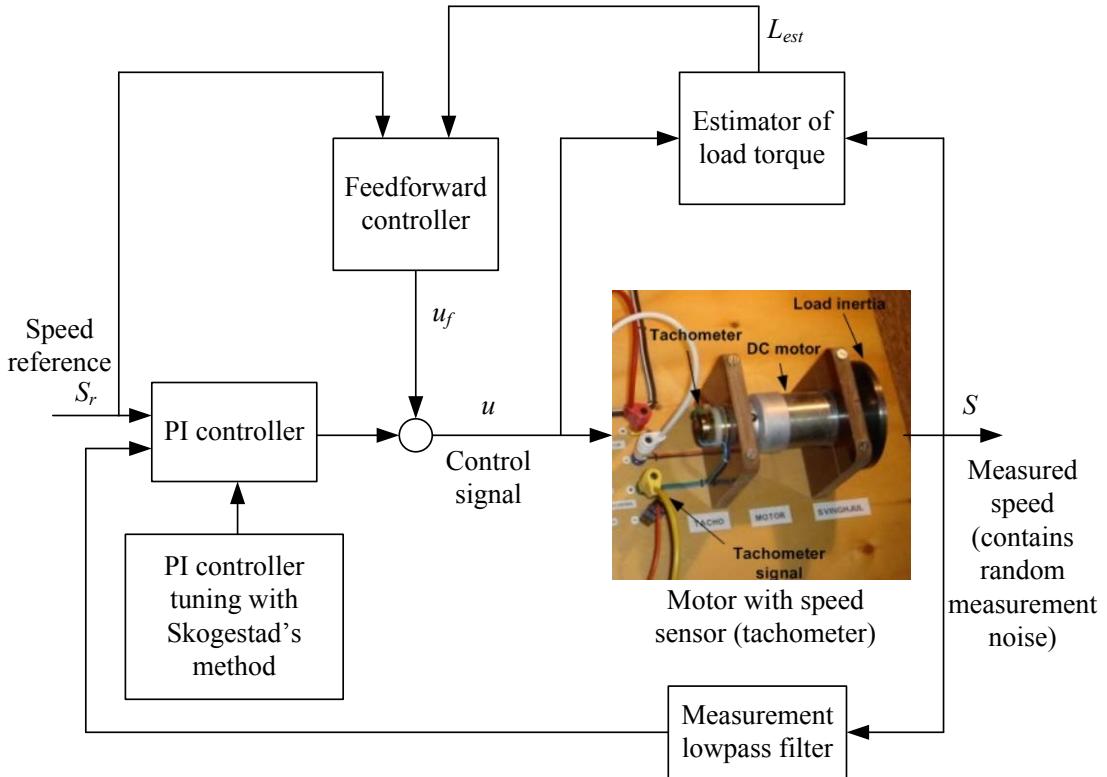


Figure 36.2: A model-based speed control system of an electric motor

The motor is controlled with an input voltage signal, u , and the rotational speed, S , is measured with a tachometer which produces a voltage being proportional to the speed.

A proper mathematical model of the motor is the following differential equation:

$$S' = \frac{1}{T_m} \{-S + K_m[u + L]\} \quad (36.3)$$

L is equivalent load torque (represented in the same unit as the control variable, namely voltage). L can be regarded as a process disturbance. K_m is gain. T_m is time constant. (Parameter values are given in Appendix 41.6, but these values are not needed in the present problem.)

The control system is based on feedback control and feedforward control. The feedback controller is a PI controller which is tuned using the Skogestad model-based tuning formulas for a “time constant process”:

$$K_p = \frac{T_m}{K_m T_C} \quad (36.4)$$

$$T_i = \min [T_m, cT_C] \quad (36.5)$$

where T_C is the specified closed-loop time constant. We set $c = 2$. Assume (for simplicity) that it decided to use

$$T_i = T_m \quad (36.6)$$

The feedforward controller is based on the motor model (36.3) from estimated load torque L_{est} and speed reference S_r :

$$u_f = \frac{1}{K_m} T_m S'_r + S_r - L_{est} \quad (36.7)$$

L_{est} is estimated with an estimator (an observer or a Kalman Filter – which one of these does not matter here), which uses the mathematical model of the motor to calculate the estimate.

The feedback controller, the feedforward controller, and the estimator are model-based. Hence, the whole control system is *model-based*.

Assume that you intend to test the robustness of the model-based control system against model errors, and also that you want to see how the measurement noise is influencing the behaviour of the control system. Unfortunately, you can not perform experiments on the real motor. Instead you must use a simulated motor. Explain how you can do this simulated experiments. Draw a block diagram similar to the one shown in Figure 36.2, where you indicate which model parameters to use in the individual blocks. You can assume that it is interesting to see if the control system is robust against model parameter variations of $\pm 20\%$.

36.2 Solutions to problems for Chapter 36

Solution to Problem 36.1

Figure 36.3 shows the control system that can be used in a simulator (in e.g. Simulink or LabVIEW).

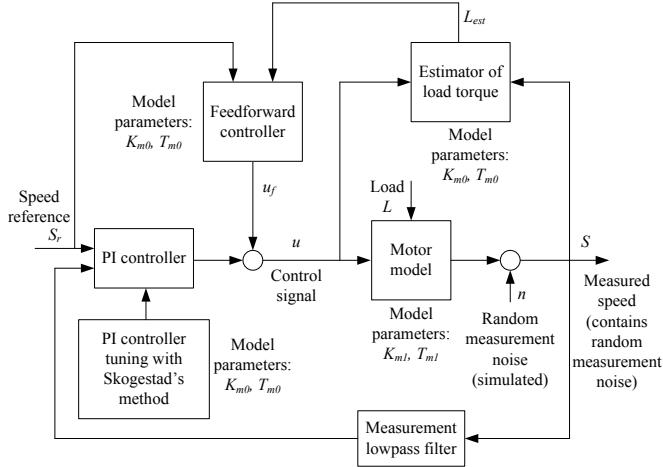


Figure 36.3: Control system

You can run the simulations with model parameters (indexed with 0 in the figure) in the simulated motor that are *different* from the model parameters (indexed with 1 in the figure) used in the feedback controller, the feedforward controller and the estimator. Measurement noise is included in the simulator using a proper random signal generator (such signal generators exist in Simulink and LabVIEW).

Parameters K_{m_0} and T_{m_0} are the parameters you assume are accurately known and therefore use in the feedback controller, the feedforward controller and the estimator.

Parameters K_{m_1} and T_{m_1} are parameters you can use in the simulated motor when you want to introduce model errors. You can for example systematically vary K_{m_1} between

$$0.8K_{m_0} \leq K_{m_1} \leq 1.2K_{m_0} \quad (36.8)$$

and vary T_{m_1} between

$$0.8T_{m_0} \leq T_{m_1} \leq 1.2T_{m_0} \quad (36.9)$$

and run simulations with each of the different parameter sets.

Chapter 37

Linear Quadratic (LQ) optimal control

37.1 Introduction

Optimal control of a process means that the control function is designed so that a given *optimization criterion* or *performance index* gets a minimal value. It is assumed that the process can be described by a *linear* model, and that the criterion contains a *quadratic* function of the state variables and the control variables.¹ This type of optimal control is therefore denoted *Linear Quadratic control* – or *LQ control*. The reference (setpoint) is assumed to be zero in the basic LQ control problem, and hence the term *LQ regulation* or LQR is also used. We will however consider a non-zero reference in this chapter, cf. Section 37.3.

A particular feature of the LQ controller is that it will stabilize any linear process! However, you do not have a guarantee that it will stabilize any nonlinear process, even if this process is linearizable. A simulation study should be made to check if the control system works well under varying conditions, including model errors.

LQ control can be applied to both monovariable and multivariable processes. It turns out that the control function is based on feedback from all the states of the process. If not all the states can be measured, an observer or a Kalman Filter can be used to estimate the states, and the controller then uses the estimated states as if they were measured. This principle is denoted the *certainty equivalence principle*. It turns out that the control function and the state estimator can be designed independently as long as the process is linear. The principle of separate design of the controller and the estimator is denoted the *separation principle*.

LQ controllers can be designed for continuous-time and for discrete-time systems, and for stochastic systems (systems excited by random disturbances) and deterministic systems (random noise is not taken into account in the controller design). I have chosen to describe

¹The main reason why a quadratic criterion is used is that the control function is relatively easy to derive and easy to implement :-)

LQ control for deterministic continuous-time systems. Of course, in a practical implementation you will (probably) need a discrete-time implementation of the controller, and this will be described in this chapter.

LQ control is quite similar to Model-based Predictive Control (MPC), which has become an important control function the last decades. Also MPC is based on a quadratic criterion. However, MPC takes into account limitations in the control variables and the state variables, hence making it somewhat more useful than LQ controller, but also much more computational demanding to implement. MPC is described in Chapter 38.

37.2 The basic LQ controller

In basic LQ control it is assumed that the process to be controlled is given by the following linear state space model

$$x'(t) = Ax(t) + Bu(t) \quad (37.1)$$

The LQ controller brings the state x from any initial state $x(0)$ to zero in an optimal way. What is “optimal”? It is defined by the optimization criterion:

$$J = \int_{t=0}^{t=\infty} [x^T(t)Qx(t) + u^T(t)Ru(t) + 2x^T(t)Nu(t)] dt \quad (37.2)$$

It is very common that the weight matrix N is a zero matrix (of proper dimension), and in these cases the criterion is

$$J = \int_{t=0}^{t=\infty} [x^T(t)Qx(t) + u^T(t)Ru(t)] dt \quad (37.3)$$

N is assumed to be zero in the following.

Q and R are *weight or cost matrices* of the states and the control signal, respectively. Q and R are selected by the user, and they are the tuning parameters of the LQ controller. Q is a symmetric positive semidefinite matrix, and R is a symmetric positive definite matrix.

The criterion (37.2) gives you (the user) the possibility to punish large variations in the states (by selecting a large Q) or to punish large variations in the control variable u (by selecting a large R). It is fair to say that the LQ controller is a user-friendly controller because the tuning parameters (Q and R) are meaningful, at least in the qualitative sense.

As an example, assume that the system has two state variables, x_1 and x_2 , hence

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (37.4)$$

and one (scalar) control variable, u , and that the weight matrices are:

$$Q = \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{bmatrix} \quad (37.5)$$

$$R = [R_{11}] \quad (37.6)$$

The criterion J becomes

$$J = \int_{t=0}^{t=\infty} [x^T Q x + u^T R u] dt \quad (37.7)$$

$$= \int_{t=0}^{t=\infty} \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u^T R u \right\} dt \quad (37.8)$$

$$= \int_{t=0}^{t=\infty} \{ Q_{11}x_1^2 + Q_{22}x_2^2 + R_{11}u^2 \} dt \quad (37.9)$$

Thus, J is a sum of quadratic terms of the state variables and the control variable.

It can be shown that the control function that gives J a minimum value is as follows

LQ controller:

$$u(t) = -G(t)x(t) \quad (37.10)$$

In other words, *the control signal is based on feedback from a linear combination of the state variables*. The controller gain $G(t)$ (a matrix) is given by the Riccati equation which will not be shown here. Figure 37.1 shows a block diagram of the control system.

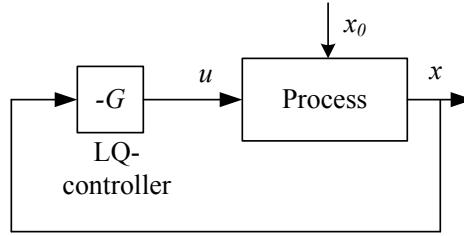


Figure 37.1: Control system with optimal LQ controller

In the above example the controller becomes

$$u(t) = -G(t)x(t) = - \begin{bmatrix} G_{11}(t) & G_{12}(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad (37.11)$$

$$= -[G_{11}(t)x_1(t) + G_{12}(t)x_2(t)] \quad (37.12)$$

It is common to implement the *steady-state value* $G = G(t = \infty)$ of the controller gain:

Steady-state LQ controller:

$$u(t) = -Gx(t) \quad (37.13)$$

G can be calculated offline, and in advance (before the control system is started).

G can be calculated from the following formulas:

$$G = R^{-1} (B^T P + N^T) \quad (37.14)$$

where P is the solution of the steady-state Riccati equation:

$$A^T P + PA - (PB + N)R^{-1}(B^T P + N^T) + Q = 0 \quad (37.15)$$

The Python Control Package (a guide to the package is in Appendix 43) has the function lqr to calculate G :

$$(G, S, E) = \text{control.lqr}(A, B, Q, R, [N])$$

where:

- A and B are the model matrices in (37.1) in the form of 2D-arrays.
- Q , R , and N are the weight matrices in (37.2) in the form of 2D-arrays. It is common that N is zero, and in that case, N can be omitted from the input argument list of the lqr() function.
- G is the array of steady state LQ controller gain.
- S is the array of steady-state solution of the Riccati equation.
- E is the array of eigenvalues of the closed loop (feedback) control system.

Figure 37.2 illustrates the information what information is needed to compute the steady-state LQ controller gain G .

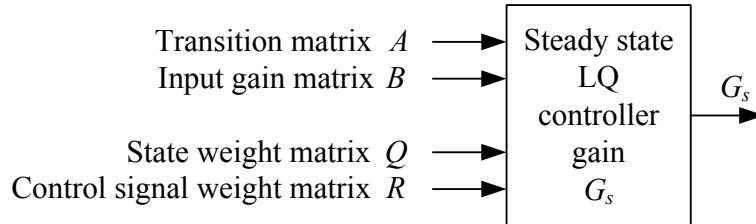


Figure 37.2: Information needed to compute the steady-state LQ controller gain, G .

Here are comments about the LQ controller:

- **The reference is zero, and there are no process disturbances:** These assumptions seem somewhat unrealistic, because in real control systems the reference is typically non-zero, and the disturbances are non-zero. Therefore, for the controller to represent realistic control problems, the variables in (37.1) should actually be regarded as deviation variables about some operating point. Then, how do you bring the states to the operating point, so that the mean value control error is zero? By enforcing *integrators* into the controller. This is described in detail in Section 37.3.
- **Controllability:** The process to be controlled has to be *controllable*. If it is not controllable, there exists no finite steady-state value of the gain G . Controllability means that there exists a control signal $u(t)$ so that any state can be reached from any initial state in finite time. It can be shown that a system is controllable if the rank of the controllability matrix

$$M_{\text{control}} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \quad (37.16)$$

is n (the order of A , which is the number of state variables).

- **Non-measured states:** If not all the states are measured, you can use estimated states instead of measured states in the controller:

$$u(t) = -Gx_{\text{est}}(t) \quad (37.17)$$

where x_{est} is the estimated state vector from a state estimator (observer or Kalman Filter). Figure 37.3 shows a block diagram of the control system with state estimator.

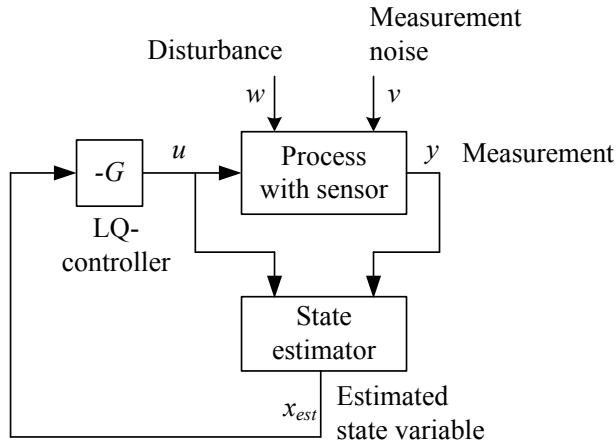


Figure 37.3: Control system with state estimator

- **The eigenvalues of the control system:** Assume that the controller is (37.17). By combining this controller with the process model (37.1) we get the following model of the *control system* (the closed-loop system):

$$\dot{x} = Ax + B(-Gx) \quad (37.18)$$

$$= (A - BG)x \quad (37.19)$$

The eigenvalues $\{s_1, s_2, \dots, s_n\}$ of the control system are the eigenvalues of the transition matrix $(A - BG)$ in (37.19):

$$0 = \det[sI - (A - BG)] \quad (37.20)$$

$$= (s - s_1)(s - s_2) \cdots (s - s_n) \quad (37.21)$$

$$= s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0 \quad (37.22)$$

- **Stability of the control system:** It can be shown that a LQ control system is asymptotically stable.² In other words, the eigenvalues of the control system are in the left half of the complex plane.
- **Tuning of the LQ controller:** From the criterion (37.2) we can conclude that a larger value of the weight of one particular state variable causes the time response of that state variable to become smaller, and hence the *control error* (deviation from

²Not a big surprise, since the controller minimizes the criterion J .

zero) is smaller. But what should be the initial values of the weight matrices, before they are tuned? One possibility is

$$Q = \text{diag} \left\{ \frac{1}{|x_{i_{\max}}|^2} \right\} \quad (37.23)$$

where $x_{i_{\max}}$ is the assumed maximum value of state variable x_i , and

$$R = \text{diag} \left\{ \frac{1}{|u_{j_{\max}}|^2} \right\} \quad (37.24)$$

where $u_{j_{\max}}$ is the assumed maximum value of control variable u_j .

- **Pole placement design of the control system:** Above it was stated that the eigenvalues of the control system are the roots of the characteristic equation

$$0 = \det [sI - (A - BG)] \quad (37.25)$$

For most systems, the poles, $\{s_i\}$, of the system are the same as the eigenvalues. In the following the term poles are used instead of eigenvalues. In pole placement design of control systems the poles are specified by the user, and, assuming the controller has the linear feedback structure as given by (37.13), it is usually possible to solve (37.25) for the controller gain matrix G .

Example 37.1 LQ control of pendulum on cart

This example is about stabilization of a pendulum on a cart using LQ optimal control. A mathematical model of the pendulum, and linearized models for the standing position and for the hanging position, are given in Ch. 41.8.

Controller

Assume that the setpoint or reference of the cart is position 0 m, and the reference of the pendulum is vertically up, i.e. at angle 0. A steady-state LQ controller designed at this operating point is:

$$u = -Gx = -[G_{11}x_1 + G_{12}x_2 + G_{13}x_3 + G_{14}x_4] \quad (37.26)$$

The controller output, u , is applied as the force F acting on the cart. Hence, the force is calculated as a linear combination of the states of the system. The states are assumed to be available via measurements. (Thus, there is a feedback from the measured states to the process via the controller.)

If the reference of the cart is non-zero, r_{x_1} , and the reference of the pendulum angle is non-zero, r_{x_3} (=180° if vertically down), we can modify the controller (37.26) to become:

$$u = -Gx - [G_{11}(x_1 - r_{x_1}) + G_{12}x_2 + G_{13}(x_3 - r_{x_3}) + G_{14}x_4] \quad (37.27)$$

The controller gain,

$$G = [G_{11}, G_{12}, G_{13}, G_{14}]$$

will be calculated using the `lqr()` function of the Python Control Package described earlier:

$$(G, S, E) = \text{control.lqr}(A, B, Q, R, [N])$$

where A and B are the matrices in (41.43) corresponding to the pertinent operating point (pendulum up; pendulum down), and Q is the state weight matrix and R is the control weight matrix of the LQ(R) optimization criterion (37.3).

Q has the following form:

$$Q = \begin{bmatrix} Q_{11} & 0 & 0 & 0 \\ 0 & Q_{22} & 0 & 0 \\ 0 & 0 & Q_{33} & 0 \\ 0 & 0 & 0 & Q_{44} \end{bmatrix} = \text{diag}(Q_{11}, Q_{12}, Q_{13}, Q_{14}) \quad (37.28)$$

Q_{ii} can be used as controller tuning parameters. R is

$$R = [R_{11}] \quad (37.29)$$

and is also used as tuning parameter. However, in this example, the scalar R_{11} can be set to 1 (because (37.3) can be scaled by dividing it by R_{11}), leaving Q_{ii} as the only tuning parameters.

Simulations

The following Python program implements the controller tuning, i.e. calculating the G matrix in (37.27), and simulates the cart with pendulum including the controller.

http://techteach.no/control/python/sim_pendulum_on_cart_lqr.py

Note: In the simulations, the non-linear model (41.28)-(41.31) is used to represent the real cart with pendulum. The linear model (41.43) is used only for controller tuning.

In the program, the code cell entitled *Selection of mode* contains the variable `op_mode` which can be set to select between standing up and hanging down position of the pendulum.

Pendulum up:

Figure 37.4 shows simulated responses where the pendulum angle reference is $r_{x_3} = 0$ deg = 0 rad, and the cart position reference r_{x_1} is varying. The initial pendulum angle is 10 deg. The weight matrix Q is set to $Q = \text{diag}(100, 0, 100, 0)$, and $R_{11} = 1$. The control system works well: The cart follows well the varying position reference, and the pendulum is stabilized upwards.

Pendulum down:

Figure 37.5 shows simulated responses where the pendulum angle reference is $r_{x_3} = 180$ deg = π rad, and the cart position reference r_{x_1} is varying.

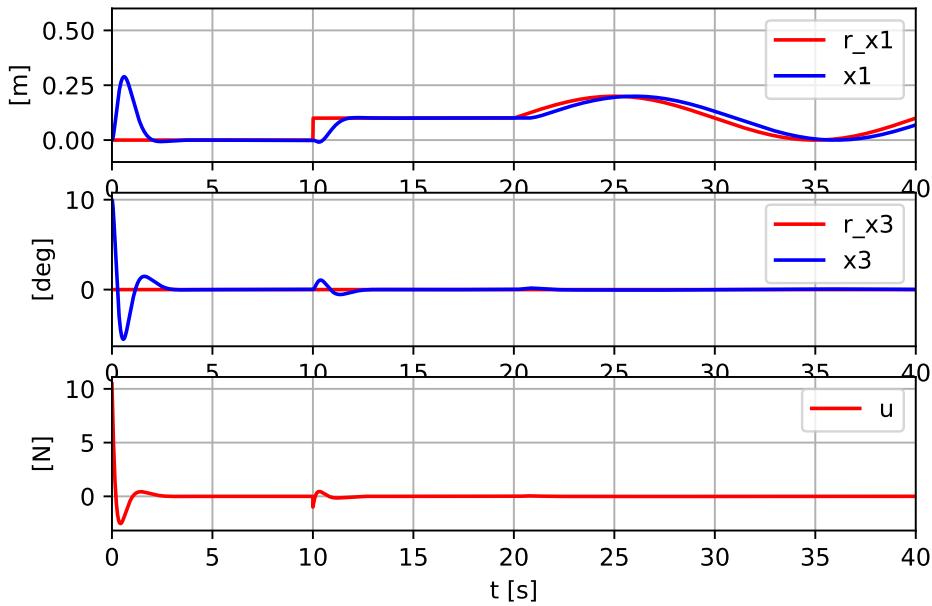


Figure 37.4: Simulated responses for the cart with pendulum with LQ control. The pendulum is controlled to a standing position, while the cart position is varied.

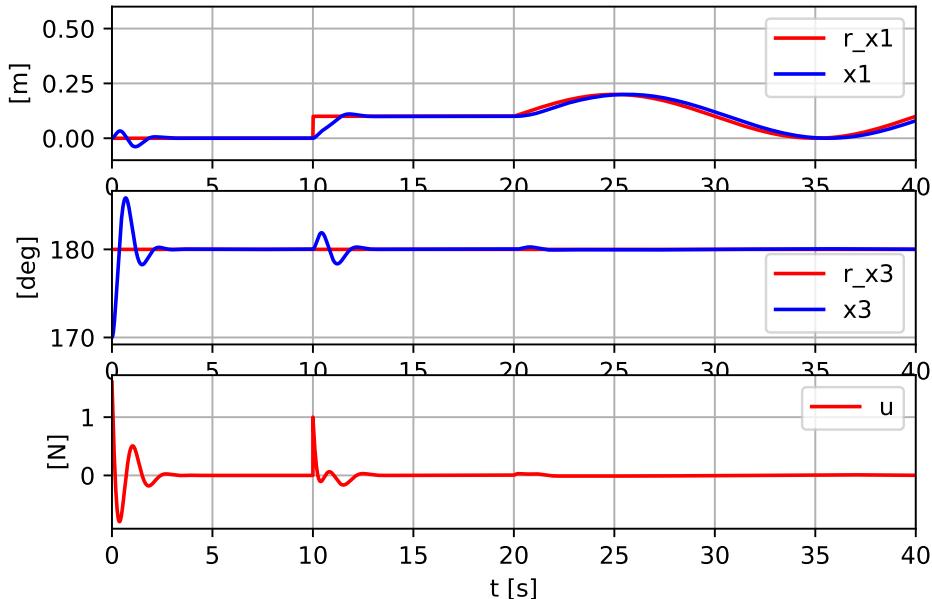


Figure 37.5: Simulated responses for the cart with pendulum with LQ control. The pendulum is controlled to a hanging position, while the cart position is varied.

The initial pendulum angle is 170 deg. The weight matrix Q is set to

$Q = \text{diag}(100, 0, 100, 0)$, and $R_{11} = 1$. The control system works well: The cart follows well the varying position reference, and the pendulum is stabilized upwards.

[End of Example 37.1]

37.3 LQ controller with integral action

37.3.1 Introduction

The basic LQ controller described in Section 37.2 does not have integral action. If process disturbances are prevalent, the process variables may not reach their setpoints, due to the lacking integral action of the controller. Let us include integral action in the controller!

37.3.2 Including integrators in the controller

Figure 37.6 shows a block diagram of the control system with integrators included in the controller.

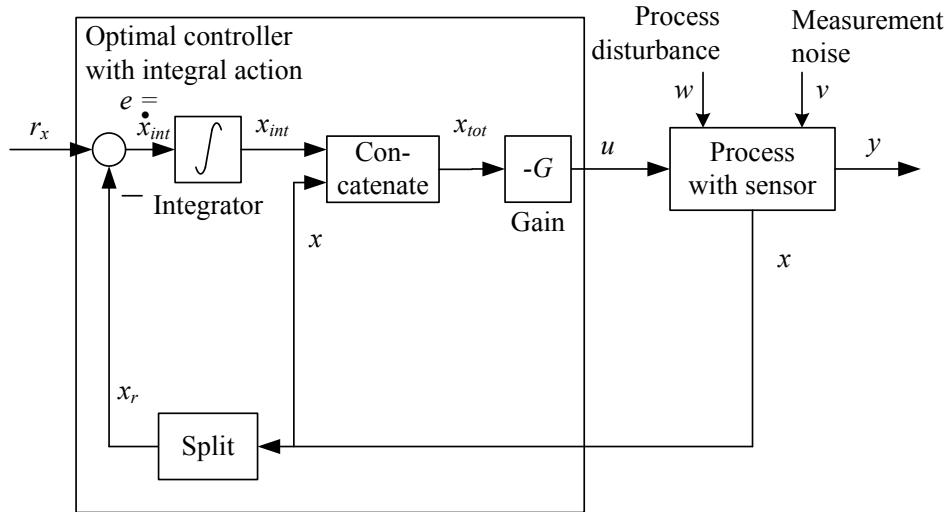


Figure 37.6: Optimal control system with integrators in the controller. e is the control error.

The integrator block actually represents a number of single integrators, as many as there are reference variables.

Example 37.2 LQ controller with integrator

Figure 37.7 shows the detailed structure of a process with two state variables being controlled by a LQ controller with integrator.

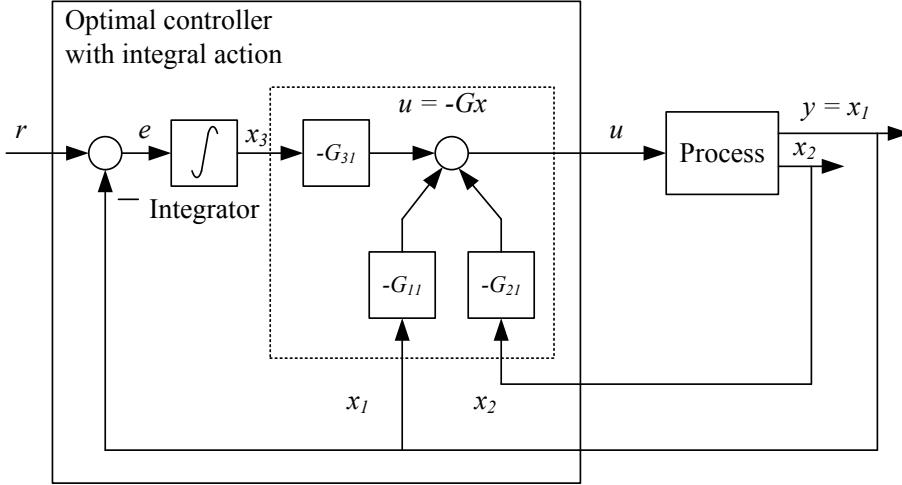


Figure 37.7: Example 37.2: A process with two state variables being controlled by a LQ controller with integrator

[End of Example 37.2]

The output of the integrators are regarded as augmentative state variables, x_{int} . These state variables are given by the following differential equation(s):

$$x'_{int} = r_x - x_r \quad (37.30)$$

which corresponds to this integral equation:

$$x_{int}(t) = \int_0^t (r_x - x_r) d\tau \quad (37.31)$$

Here, r_x is the reference vector for the state vector x_r , which consists of those state variables among the process state vector x that are to track a reference. (In the above equations it is assumed that x_r is directly available from measurements. If x_r is taken from a state estimator, $x_{r,est}$ is used instead of x_r , of course.)

The total state vector that is used to design the LQ controller is the state vector consisting of the original process state vector augmented with the integrator state vector:

$$x_{tot} = \begin{bmatrix} x \\ \dots \\ x_{int} \end{bmatrix} \quad (37.32)$$

The control variable u is given by the control function

$$u = -Gx_{tot} = -G \begin{bmatrix} x \\ \dots \\ x_{int} \end{bmatrix} \quad (37.33)$$

Note: When writing up the state space model that is used for designing the LQ controller, you can disregard the reference r_x , i.e. you set it to zero because it is not taken into account when calculating the controller gain G . But of course it must be included in the implemented controller which is given by (37.33), with x_{int} given by (37.35).

37.3.3 Discrete-time implementation of the LQ controller

In a computer-based implementation of a LQ controller you will probably need to discretize the continuous-time integrator (37.30). This can be done using Backward or Forward discretization. The Backward method is the best with respect to numerical accuracy, and it can be applied to (37.30) without any problems because it is a linear differential equation. Applying Backward discretization on (37.30) gives

$$x'_{\text{int},k} \approx \frac{x_{\text{int},k} - x_{\text{int},k-1}}{T_s} = r_{x,k} - x_{r,k} \quad (37.34)$$

Solving for $x_{\text{int},k}$ gives the final integrator algorithm ready for being programmed:

$$x_{\text{int},k} = x_{\text{int},k-1} + T_s [r_{x,k} - x_{r,k}] \quad (37.35)$$

A practical issue of any controller having integral action is *anti windup*, which is a feature to prevent the integrator to “wind up” – or increasing its output continually – while the total control signal is at its saturation limit, either the maximum or the minimum limit. If anti windup is not implemented, the control error may become unnecessarily large for an unnecessarily long time.

37.4 Problems for Chapter 37

Problem 37.1 LQ control of ship

See Figure 4.20 which shows a ship. In this problem we concentrate on the so-called surge (forward-backward) direction, i.e., the movements in the other directions are disregarded. The wind acts on the ship with the force F_w which is a function of the wind attack angle ϕ and the wind speed V_w . This function is assumed to be known for a given ship. The hydrodynamic damping force F_h (damping from the water) is proportional to the square of the difference between the ship speed, \dot{y} , and the water current speed u_c . The proportionality constant is D .

Applying Newtons's Law of Motion we obtain the following mathematical model of the surge motion:

$$my'' = \underbrace{-D|y' - u_c| (y' - u_c)}_{F_h} + F_w(\phi, V_w) + F_t \quad (37.36)$$

(Model parameter values are given in Appendix 41.2, but these values are not needed in the present problem.)

This problem is about ship position control using LQ optimal control. Assume that the water current u_c has a known value at any instant of time (in a practical application it may have been estimated with a state estimator, e.g. a Kalman Filter). Also, assume that the wind force F_w has a known value at any instant of time (with a mathematical wind model the wind force can be calculate from information about the wind attack angle ϕ and the wind speed V_w provided by the sensor). We also assume that the ship position y and speed \dot{y} are known at any instant of time.

1. Write the ship model as a (nonlinear) state space model using $x_1 = y$ and $x_2 = y'$ as state variables. F_t is control variable.
2. The ship position will be controlled with LQ control with integral action. Figure 37.8 shows a block diagram of the ship.

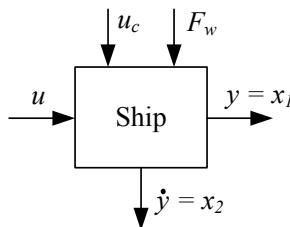


Figure 37.8: Block diagram of ship

Enhance this block diagram so that it shows the control system in detail, including the feedbacks and the integrator of the controller.³ You can assume that the controller gains have known values (calculation of these gains is the focus of a following task).

³Feedforward from wind force F_w and water current u_c is an enhancement of the control system, but we will not include feedforward in this problem.

3. Augment the state space model found in Problem 1 above with the state variable of the integral part of the controller. (This augmented state space model is needed in the following problem.)
4. Figure 37.9 illustrates the matrices needed to compute the steady-state LQ controller gain G_s (using a proper function in e.g. Matlab or LabVIEW).

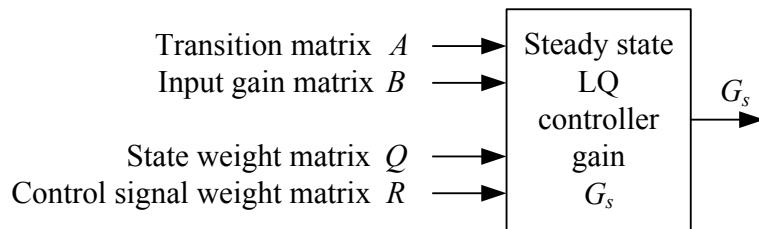


Figure 37.9: Information needed to compute the steady-state LQ controller gain, G_s .

Matrices A and B are found by linearizing the augmented state space model found in the problem above. The elements of the weight matrices Q and R – at least their initial values since these values may be adjusted later – shall be expressed as functions of the allowable maximum values of the proper variable in the model.

5. Suppose you want to reduce the fuel consumption (less aggressive control). How can you adjust some of the weights of the LQ criterion to obtain this?
6. Suppose you want to increase the damping in the control system. In other words: You want to limit (reduce) the speed of the ship. How can you adjust some of the weights of the LQ criterion to obtain this?

37.5 Solutions to problems for Chapter 37

Solution to Problem 37.1

1.

$$\underline{\underline{x'_1 = \overbrace{x_2}^{f_1}} \quad (37.37)}$$

$$\underline{\underline{x'_2 = \overbrace{\frac{1}{m} [-D|x_2 - u_c|(x_2 - u_c) + F_w(\phi, V_w) + F_t]}^{f_2}} \quad (37.38)$$

2. See Figure 37.10.

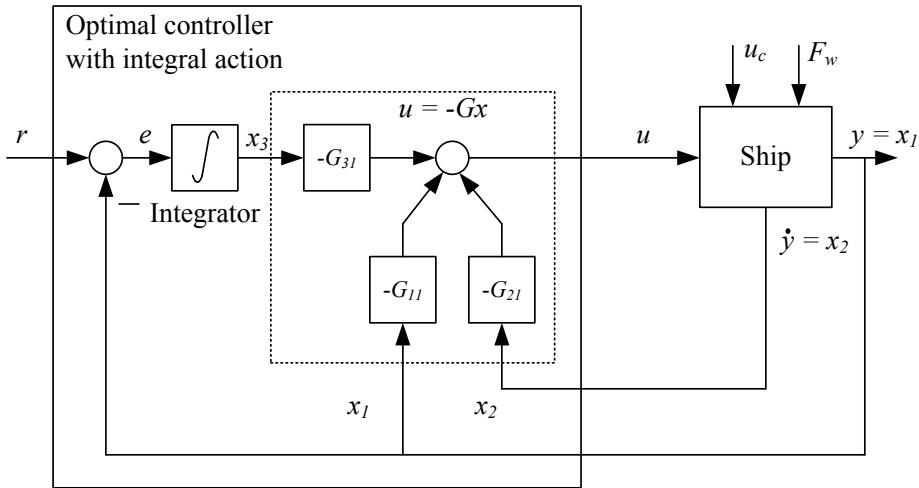


Figure 37.10: State feedback control system

3. See Figure 37.10. The state-variable of the integrator is defined by

$$x'_3 = r - x_1 \quad (37.39)$$

The augmented (total) state space model becomes

$$\underline{\underline{x'_1 = \overbrace{x_2}^{f_1}} \quad (37.40)}$$

$$\underline{\underline{x'_2 = \overbrace{\frac{1}{m} [-D|x_2 - u_c|(x_2 - u_c) + F_w(\phi, V_w) + F_t]}^{f_2}} \quad (37.41)}$$

$$\underline{\underline{x'_3 = \overbrace{r - x_1}^{f_3}}} \quad (37.42)$$

4. Matrices A and B are found by linearization:

$$\underline{\underline{A = \left[\begin{array}{ccc} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{array} \right] = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & -\frac{2D}{m}|x_2 - u_c| & 0 \\ 0 & 0 & -1 \end{array} \right]}} \quad (37.43)$$

$$B = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \\ \frac{\partial f_3}{\partial u} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{m} \\ 0 \end{bmatrix} \quad (37.44)$$

(Element $A(2, 2)$ can be found by resolving the absolute value by first assuming $(x_2 - u_c)$ is positive and then assuming $(x_2 - u_c)$ is negative, then taking the partial derivative, and finally expressing the results of the partial differentiations compactly.)

State weight matrix:

$$Q = \begin{bmatrix} Q_{11} & 0 & 0 \\ 0 & Q_{22} & 0 \\ 0 & 0 & Q_{33} \end{bmatrix} = \begin{bmatrix} \frac{1}{|x_{1\max}|^2} & 0 & 0 \\ 0 & \frac{1}{|x_{2\max}|^2} & 0 \\ 0 & 0 & \frac{1}{|x_{3\max}|^2} \end{bmatrix} \quad (37.45)$$

State weight matrix:

$$R = [R_{11}] = \left[\frac{1}{|u_{\max}|^2} \right] \quad (37.46)$$

5. To reduce the fuel consumption (less aggressive control) you can increase the cost (weight) of the control signal, i.e. increase R_{11} .
6. To reduce the speed of the ship you can increase the cost (weight) of the speed, i.e. increase Q_{22} .

Chapter 38

Model Predictive Control (MPC)

38.1 Introduction

Model predictive control (MPC) is the dominant model-based control method. In year 2002, Maciejowski stated that “MPC is the only advanced control technique that is more advanced than standard PID to have a significant and widespread impact on industrial process control” [Maciejowski \(2002\)](#). It is fair to say that this statement still holds. However, it will be exciting to see what will be the role of artificial intelligence (AI) in control of technical and industrial systems. Will it replace MPC and PID control?

The history of MPC may be traced back to Dynamic Matrix Control (DMC) method implemented by Cutler and Ramaker at Shell Oil in 1973 [Cutler & Ramaker \(1980\)](#). A standard overview over MPC technology is given in [Qin & Badgwell \(2003\)](#). A more recent overview is given in [Lee \(2011\)](#).

MPC is available in various professional and industrial software tools, e.g. DeltaV Predict (Emerson Process), 800xA APC (ABB), PCS7 (Siemens), MPC Toolbox of Matlab and Simulink (Mathworks), and Control Design Toolkit of LabVIEW (National Instruments).

As a mathematical problem, MPC and MHE are almost identical: Both exploit a mathematical model which is run (basically simulated) over a time horizon. However, they can also be regarded as opposites of each other: MPC looks into the future, while MHE looks into the past. And, in MPC, the process measurement is an input (to the MPC), and the control signal is an output, while in MHE, the situation is opposite – the process measurement is an output (from the MHE), and the control signal is an input.

MPC exists in different versions. Here, nonlinear MPC is presented. The term “nonlinear” is used because the underlying mathematical model of the process to be controlled, is a *nonlinear* state space model. The model may be multivariable and may contain time delays. Nonlinear MPC can of course be applied to linear models, too, since linear state space models are just a special case of nonlinear state space models.

38.2 The MPC method

38.2.1 The principle of MPC

The principle of MPC is continuously calculation of the optimal (“best”) control signal sequence over a future or prediction time horizon using the following information:

- A process model. The model is used by the optimizer to simulate the process over the prediction horizon.
- The current process state as obtained from measurements and/or state estimates from a state estimator which typically is in the form of a Kalman Filter.
- Setpoint values and process disturbance values known over the prediction horizon.
- Constraints (maximum and minimum values) on the control signal, the process variable, and state variables.

It can be claimed that MPC resembles closely how a human controls a process, like driving a car: The driver looks ahead to take into account future disturbances like other cars, pedestrians and other obstacles, and future speed setpoints as shown on the signs ahead, while manipulating the various actuators (throttle, break, steering wheel, gear).

The predictions/simulations in the MPC can be based on any model that is representative of the process to be controlled. In the MPC presented in this chapter, the following discrete time nonlinear state space model is used.

$$x_{k+1} = f(x_k, u_k, d_k, \cdot) \quad (38.1)$$

$$y_k = g(x_k, \cdot) \quad (38.2)$$

x is the state vector, y is the process output variable vector, u is the control signal vector, and d is the process disturbance vector. f and g are nonlinear (or linear) vectorial functions.

In the MPC presented here, no linearization of the model is needed. So, it is a “nonlinear MPC”. There are MPC algorithms assuming a linear model – “linear MPC”, for example the MPC available in LabVIEW, which requires linearization of the process model used by the MPC. The linear model assumed is on the the following form:

$$x_{k+1} = Ax_k + B_1 u_k + B_2 d_k \quad (38.3)$$

$$y_k = Cx_k + Du_k \quad (38.4)$$

Nonlinear MPC can be used with linear models too, since a linear model (38.3)-(38.4) is just a special case of a nonlinear model (38.1)-(38.2).

Figure 38.1 illustrates the principle of MPC. The predicted values are found by successive simulations over the prediction horizon performed by the optimizer until the optimal solution (optimal control signal sequence) has been found. The optimal control sequence or array (or matrix in the multivariable case), u_{opt} is calculated as the solution of an

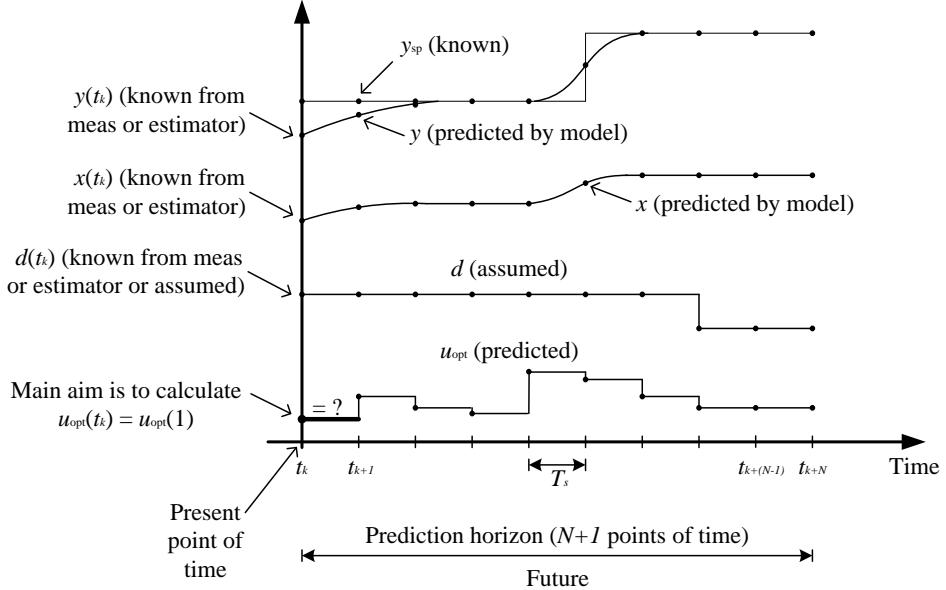


Figure 38.1: The principle of MPC.

optimization problem where typically the future (predicted) control errors and control signal changes are minimized in a least squares sense. And from this optimal future control sequence, the first element is picked out and applied as control signal to the process, i.e.

$$u_k = u_{\text{opt}}(1) \quad (38.5)$$

38.2.2 The optimization objective function of MPC

The objective function, or optimization criterion, to be minimized in MPC may be stated as follows:

$$\min_U J \quad (38.6)$$

J is the objective function. It is defined below. U is a matrix containing the r control signals at each point of time of the prediction horizon:

$$\begin{aligned} U &= [u_k, u_{k+1}, \dots, u_{k+(N-1)}, u_N] \\ &= \begin{bmatrix} u(1)_k & u(1)_{k+1} & \dots & u(1)_{k+(N-1)} & u(1)_{k+N} \\ u(2)_k & u(2)_{k+1} & \dots & u(2)_{k+(N-1)} & u(2)_{k+N} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ u(r)_k & u(r)_{k+1} & \dots & u(r)_{k+(N-1)} & u(r)_{k+N} \end{bmatrix} \end{aligned} \quad (38.7)$$

The number of optimization variables is the number of elements of U . The number is $r(N + 1)$.

U can be denoted the total control signal matrix. U will be the solution of the MPC

optimization problem. From this U ,

$$u_k = \begin{bmatrix} u(1)_k \\ u(2)_k \\ \vdots \\ u(r)_k \end{bmatrix}$$

is used as the control signal applied to the process actuator.¹

The objective function to be minimized, cf. Eq. (38.6), is

$$J = \sum_{i=k}^{k+N} (\|e\|_{C_e}^2 + \|du\|_{C_{du}}^2) \quad (38.8)$$

where expressions like $\|\cdot\|_M$ means M -quadratic norm, see Page 606. In more detail, Eq. (38.8) is

$$J = \sum_{i=k}^{k+N} \left[e_i^T C_e e_i + (du)_i^T C_{du} (du)_i \right] \quad (38.9)$$

The symbols in Eq. (38.9) are described below.

The control error vector:

$$e_i = \begin{bmatrix} e(1)_i \\ \vdots \\ e(m)_i \end{bmatrix} \quad (38.10)$$

where $e(j)_i$ is the control error related to process output no. j at time-index i :

$$e(j)_i = y(j)_{\text{sp}_i} - y(j)_i \quad (38.11)$$

The control signal change vector:

$$du_i = \begin{bmatrix} du(1)_i \\ \vdots \\ du(r)_i \end{bmatrix} \quad (38.12)$$

where $du(j)_i$ is the control signal change relative to the control signal at the previous point of time:

$$du(j)_i = u(j)_i - u(j)_{i-1} \quad (38.13)$$

The matrixes C_e and C_{du} in Eq. (38.8) are cost (or weight) matrixes which typically are set as constant matrixes:

$$C_e = \begin{bmatrix} C_e(1,1) & & 0 \\ & \ddots & \\ 0 & & C_e(m,m) \end{bmatrix} \quad (38.14)$$

$$C_{du} = \begin{bmatrix} C_{du}(1,1) & & 0 \\ & \ddots & \\ 0 & & C_{du}(r,r) \end{bmatrix} \quad (38.15)$$

¹You can represent U with a Matlab *matrix* in fmincon. So it is not necessary to transform this matrix to an array for fmincon.

C_e and C_{du} are tuning factors in MPC.

Now, Eq. (38.9) can be written in detail as

$$J = \sum_{i=k}^{k+N} \left\{ \left[C_e(1,1) e(1)_i^2 + \cdots + C_e(m,m) e(m)_i^2 \right] \right. \quad (38.16)$$

$$\left. + \left[C_{du}(1,1) \left(\frac{du(1)}{dt} \right)_i^2 + \cdots + C_{du}(r,r) \left(\frac{du(r)}{dt} \right)_i^2 \right] \right\} \quad (38.17)$$

Roughly said, MPC produces the control signal that gives the optimal compromise between control errors and control signal changes. It is not possible to obtain both very small control errors and very small control signal changes. Hence, a compromise will always exist.

Constraints

You may include constraints in the MPC optimization problem: Typically, upper and lower bounds are set for the control signal. Furthermore, you can set limits on the process output variable and on certain state variables. For example, if the liquid level in a tank is one state variable, it is natural to define a maximum level limit and a minimum level limit.

Guessed value of U

When solving the optimization problem, it is necessary that the optimizer is supplied with a good guess of the optimization variable, U . As a good value of U_{guess} at time index k , here denoted U_{guess_k} , we can use the optimal solution found at time index $k - 1$ (the previous point of time):

$$U_{\text{guess}_k} = U_{\text{opt}_{k-1}} \quad (38.18)$$

38.2.3 Control signal blocking

Control blocking, or control grouping, can be used to reduce the number of optimization variables. Control blocking is to fix the control signal in time-blocks in the prediction horizon, see Figure 38.2. My experience from is that using as small number as 3 intervals may not deteriorate the performance of the MPC. I have even tried using only one interval (i.e. constant u throughout the horizon), with acceptable performance. Control blocking, as other settings, should be tested in simulations before being applied to a real process.

38.2.4 Tuning factors of MPC

The main tuning factors of MPC are:

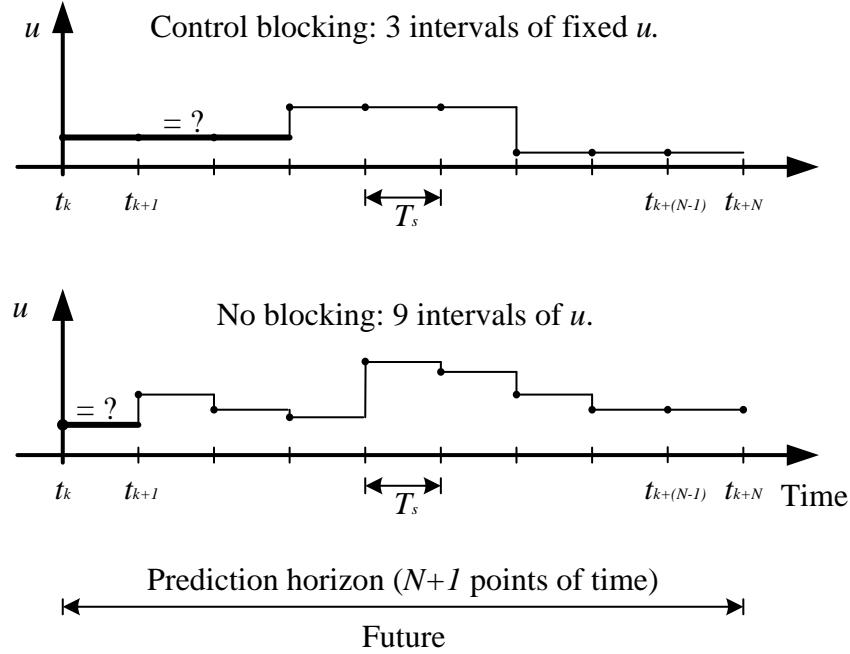


Figure 38.2: Control blocking

- *The prediction horizon length, N .* The larger N , the better ability to take into account future setpoints and disturbances. A drawback of selecting a large N , is the increase of the computational demand which is due to the more challenging optimization problem (more optimization variables to be optimized) and longer simulations. A typical value of N for simple applications seems to be between 5 and 50, assuming an appropriate time step length (which may be e.g. 1/5 of smallest time constant-like dynamics represented by the model). To reduce the computational burden, control signal blocking can be considered, as explained earlier in this section.
- *The control error cost matrix, C_e .* Increasing the value of $C_e(j, j)$, forces the pertinent control error, $e(j)$, to become smaller², but at the expense of larger variation in the control signal. Initially, you may try setting $C_e(j, j)$ equal to the square of inverse of the maximum expected absolute value of the control error:

$$C_e(j, j) = \frac{1}{[|e(j)|_{\max}]^2} \quad (38.19)$$

This implies a normalization of the error terms in the objective function. For example, the first error term in Eq. (38.16) becomes

$$C_e(1, 1)e(1)_i^2 = \frac{e(1)_i^2}{[|e(1)|_{\max}]^2}$$

Then, you may try using only C_{du} as a tuning parameter (cf. next item). In the scalar case, i.e. $m = 1$ and $r = 1$, you may simply set $C_e = 1$ since it is only the ratio between C_e and C_{du} that counts for the tuning.

²The higher cost of something, the less of it is bought/used.

- The control signal rate of change cost matrix, C_{du} . Also the terms in C_{du} may be normalized before the tuning. Initially, you may try setting $C_{du}(j,j)$ equal to the square of inverse of the maximum rate of change of the control signal:

$$C_{du}(j,j) = \frac{1}{\left[\left| \frac{du(j)}{dt} \right|_{\max} \right]^2} \quad (38.20)$$

Increasing the cost gives smoother control signal (less change). Decreasing gives more abrupt changes in the control signal.

38.2.5 The need for a state estimator

The optimizer in MPC uses successive simulations for the prediction. For the simulations to become accurate, it is necessary that the initial state of the simulations are close to the present state, x_k , of the process to be controlled. Typically, not all the states are measured, and if so, a state estimator – most often a Kalman Filter – is used to provide an estimate of x_k . Even if all the states are estimated, state estimator can be useful for several reasons:

- The estimates are typically less noisy than the (raw) measurements.
- If a process sensor fails, and this failure is detected, the state estimator may be configured to continue providing a representative state estimate despite the lack of measurement-based update or correction of the estimate. This enhances the robustness of the MPC.
- A state estimator may be used to estimate disturbances and/or model parameters. This increases the robustness of the MPC as the process model underlying the MPC becomes more accurate. Disturbances and model parameters can be estimated as augmented state variables modelled as constants, i.e. as state variables having time-derivatives equal to zero but with an additive random disturbance/noise. This augmentation is explained in detail on Page 639.

Example 38.1 Model-predictive control with the slsqp optimizer of Python/Scipy

This example is about MPC of a simulated air heater lab station. The air heater and a mathematical model is presented in Appendix 41.5. The dynamics of the air heater is “time constant with time delay”.

The MPC is implemented in a Python program with the slsqp optimizer of the Scipy package of Python. The MPC has the following features:

- Time step 0.5 s.
- Prediction horizon: 8.0 s.
- Number of blocks (groups) of the control signal (u): 3.

- Constraint on the temperature (state): $27 \leq T \leq 31$ °C, implemented in the constraints function which is an argument of slsqp().
- Constraints on the control signal u : $0 \leq u \leq 5$ V, defined as bounds on the optimization variable (u). These bounds are arguments of slsqp().
- Constraint on the rate of change of the control signal: $-0.25 \leq du/dt \leq 0.25$ V/s, implemented in the constraints function which is an argument of slsqp().
- Cost: $C_e = 1$, and $C_{du} = 20$.
- An Extended Kalman Filter which estimates an input disturbance d , which is used by the MPC.
- Simulation start time = 0 s. Stop time = 300 s.

The simulation scenario is:

- T_{sp} is initially constant, then a step, then a ramp, then a sinusoid, and finally constant. During a simulation time interval, the setpoint is given a value *above* the constraint of T , to demonstrate the ability of the MPC to take the constraints of T into account.
- d is changed as a step during the simulation, to demonstrate the ability of the Kalman Filter to estimate a changing disturbance. The estimate, d_{est} , is used as a “soft measurement” of d by the MPC.

The Python program below implements the MPC.

http://techteach.no/control/python/mpc_airheater.py

In the program, the user can set the value of the variable cont_plot to select between continuous (online) plot (cont_plot = 1) and batch (offline) plot (cont_plot = 0).

Figure 38.3 shows the results with the MPC applied to the simulated air heater.

Comments to the results shown in Figure 38.3:

- *Diagram upper left:* The tracking of T_{sp} is excellent where there are no change in d and no active constraints on T . The constraint on T is respected by the MPC, except a small disrespect after the step change of d , which is due to the fact that the MPC uses d_{est} and not the true value d when calculating the optimal u .
- *Diagram upper right:* The upper bound on u is respected by the MPC, see around $t = 60$ s. Note that the MPC starts increasing the control signal *before* the point of time of the setpoint change, see around $t = 100$ s. This demonstrates the predictive nature of the MPC.

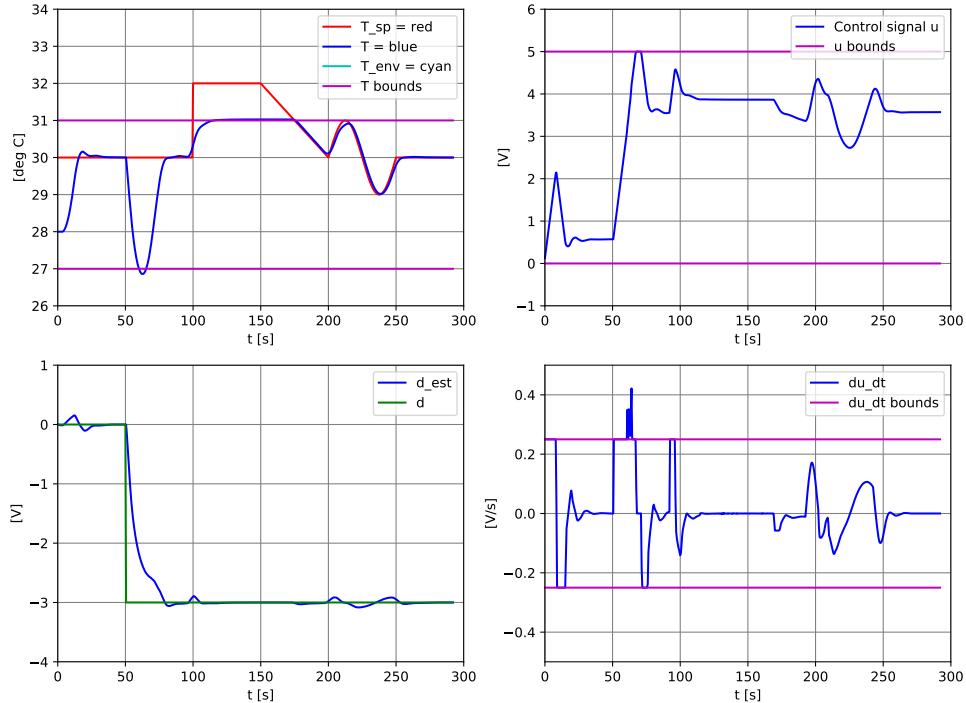


Figure 38.3: Example 38.1: Control of the simulated air heater temperature with MPC.

- *Diagram lower left:* The Kalman Filter estimates d correctly in steady state, but there is (of course) an estimation error in the transient phase.
- *Diagram lower right:* The constraints on du/dt are respected by the MPC, except a disrespect around $t = 60$ s, which is which is due to the fact that the MPC uses d_{est} and not the true value d when calculating the optimal u .

[End of Example 38.1]

38.3 Problems for Chapter 38

Problem 38.1 MPC for ship position control

See Problem 37.1 which is about positional control of a ship. In that problem the position is controlled with LQ control. Now, assume MPC instead of LQ control. The state variables are $x_1 = y$ and $x_2 = \dot{y}$. The control variable is F_t .

1. Assume that you want to make the control system more sluggish so that the positional control error is allowed to become larger. (One possible motivation may be that tight position control is not required for a period, for example if the ship is parked, waiting for a certain operation to be initiated.) Which parameter of the criterion is proper for adjustment, and should you increase or decrease that parameter?
2. The thruster force F_t has of course a positive limit and a negative limit (assuming that the thruster can rotate both directions). Can the MPC algorithm take these limits into account when it calculates the optimal thruster force to be applied to the ship?

Problem 38.2 Hierarchical control with MPC and PID

MPC may be used for direct manipulation of the control variables. MPC can also be used in a hierarchic control system. Figure 38.4 shows using MPC in outer control loop while PID controllers are used in inner control loops. This control structure is similar to a conventional control structure frequently used in industry – which?

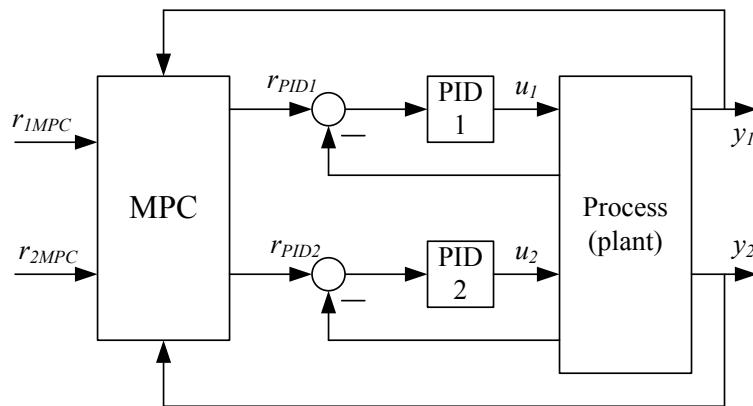


Figure 38.4: MPC in outer control loop while PID controllers are used in inner control loops

Suggest concrete examples of r_{1MPC} , r_{PID_1} , and u_1 (select any application you want).

38.4 *Solutions to problems for Chapter 38*

Solution to Problem 38.1

1. The optimization criterion is

$$J = \sum_{k=0}^{N_p} \left\{ Q_1 [e_{1,k}]^2 + Q_2 [e_{2,k}]^2 + \dots + Q_n [e_{n,k}]^2 \right\} \\ + \sum_{k=1}^{N_c} R_1 \left\{ [\Delta u_{1,k}]^2 + R_2 [\Delta u_{2,k}]^2 + \dots + R_r [\Delta u_{r,k}]^2 \right\}$$

(38.21)

The state variables are $x_1 = y$ and $x_2 = y'$. Thererfore, the error e_1 is the position control error. To allow for a larger e_1 the cost or weight of e_1 – which is Q_1 – should be increased.

2. Yes, MPC takes into account specific limits of control variable(s) when calculating the optimal control signal.

Solution to Problem 38.2

Cascade control.

$r_{1\text{MPC}}$ may be level of a tank. r_{PID_1} may be flow reference (setpoint) to a flow PI(D) controller. u_1 may be control signal to a control valve.

Chapter 39

Inverse dynamics control

39.1 Introduction

Inverse dynamics control is a model-based control method which can be applied to nonlinear multivariable processes.¹ It is assumed that the model is a (nonlinear) state space model. The value of each of the state variables and process disturbances must be available at any instance of time – either from measurements or from an estimator (as a Kalman Filter).

Inverse dynamics control has some alternative names, namely *feedback linearization*, *nonlinear decoupling*, and *computed torque control*. The latter is used when the control method is used for position control of robotic systems, cf. Section 39.3.

Note: The version of the inverse dynamics control presented in this chapter assumes that the process has no time delay.

The control function consists of three main parts, cf. Figure 39.1.

- A *decoupler and linearizer* which is based on the process model and the instantaneous values of the states and the disturbances.
- A *multiloop PID controller* which is designed for the decoupled linear process.
- A *feedforward controller* which is designed for the decoupled linear process.

The following sections present inverse dynamics control for two different kinds of processes:

- First order processes, Section 39.2.
- Second order processes, Section 39.3.

¹The method can, of course, be applied also to linear processes and monovariable processes.

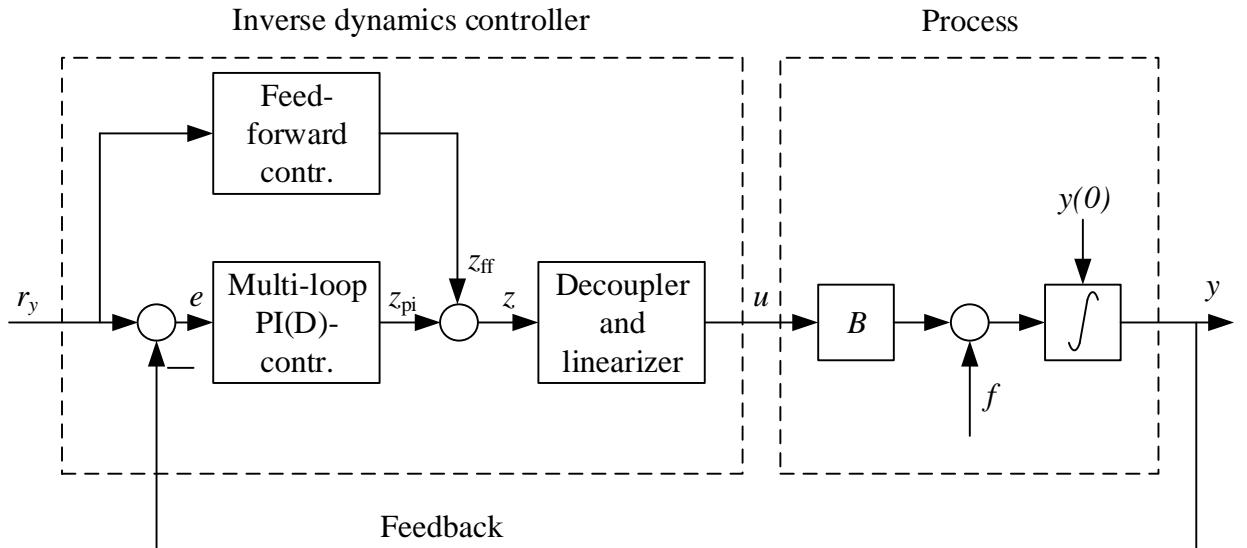


Figure 39.1: An overall block diagram of the inverse dynamics controller.

39.2 Inverse dynamics control of first order processes

It is assumed that the process model is a state space model on the following form:

$$y' = f(y, d) + B(y, d) \cdot u \quad (39.1)$$

or, simpler:

$$y' = f + Bu \quad (39.2)$$

x is the state vector, d is the disturbance vector, and u is the control vector. f is a vector of scalar functions, and B is a matrix of scalar functions. Note that the control vector u is assumed to appear *linearly* in the model.

Figure 39.2 shows a block diagram representation of the process model (39.2).

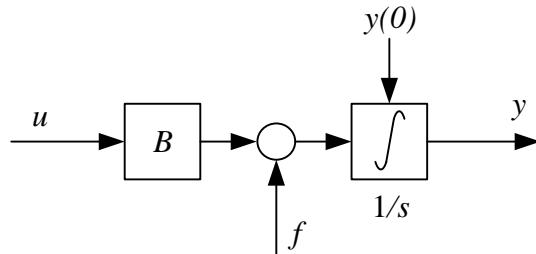


Figure 39.2: A block diagram representation of the process model (39.2).

The reference (or setpoint) of y is denoted r_y .

We can derive the control function “graphically” as shown in Figure 39.3. The decoupler

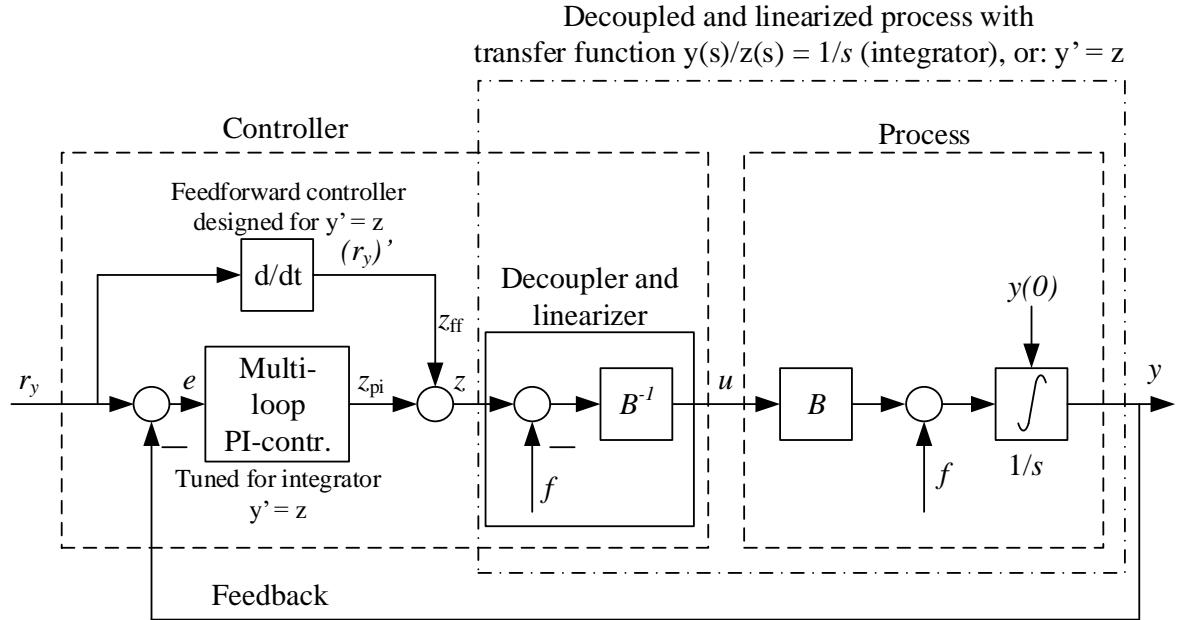


Figure 39.3: Block diagram of the control system based on feedback linearization.

and linearizer creates the following model relating the transformed control signal z and the process output y :

$$y' = z \quad (39.3)$$

which are actually n decoupled or independent *integrators* (n is the number of state variables), because $y(t) = \int_0^t z d\tau$. The transfer function from z to y is

$$\frac{y(s)}{z(s)} = \frac{1}{s} \quad (39.4)$$

We can denote the integrator (39.3) as the *transformed process*.

We will now derive the control function for this transformed process, and thereafter derive the final control function. How can you control an integrator? With PI feedback and feedforward!

PI tuning

The (multiloop) feedback PI controller is

$$z_{fb} = K_c e + \frac{K_c}{T_i} \int_0^t e d\tau \quad (39.5)$$

where e is the control error:

$$e \stackrel{\text{def}}{=} r_y - y \quad (39.6)$$

In (39.5), the coefficients are actually matrices:

$$K_c = \begin{bmatrix} K_{c_1} & 0 & \cdots & 0 \\ 0 & K_{c_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & K_{c_n} \end{bmatrix} \quad (39.7)$$

$$\frac{K_c}{T_i} = \begin{bmatrix} \frac{K_{c_1}}{T_{i_1}} & 0 & \cdots & 0 \\ 0 & \frac{K_{c_2}}{T_{i_2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{K_{c_n}}{T_{i_n}} \end{bmatrix} \quad (39.8)$$

We can use the Skogestad method to tune K_{c_j} and T_{i_j} . The process to tune for is an integrator process with integrator gain $K_i = 1$ and zero time delay, cf. Ch. 16.8.2.2, giving the following PI settings:

$$K_{c_j} = \frac{1}{T_{C_j}} \quad (39.9)$$

$$T_{i_j} = 2T_{C_j} \quad (39.10)$$

where T_{C_j} is the specified time constant of feedback loop no. j .

Feedforward controller

In addition to the PI feedback action the controller should contain *feedforward* from the reference r_y to get fast reference tracking when needed (assuming the reference is varying). The feedforward control function can be derived by substituting the process output y in the process model (39.3) by r_y and then solving for y , giving:

$$z_{\text{ff}} = r_y' \quad (39.11)$$

If the setpoint is varied abruptly (but you may vary it smoothly), a lowpass filtered can be used before the time derivative is calculated.

The resulting control signal

From Figure 39.3 we see that the controller function for the transformed process (39.3) is the sum of the feedback control function and the feedforward control function derived above:

$$z = z_{\text{pi}} + z_{\text{ff}} \quad (39.12)$$

$$= \underbrace{K_c e + K_i \int_0^t ed\tau}_{z_{\text{pi}}} + \underbrace{r_y'}_{z_{\text{ff}}} \quad (39.13)$$

Also from Figure 39.3, we see that the resulting controller function, that is, the formula for the control vector u , is:

$$u = B^{-1}(z - f) = B^{-1}(z_{\text{pi}} + z_{\text{ff}} - f) \quad (39.14)$$

z_{pi} and z_{ff} are shown in (39.13).

About the resulting control system

Here are some characteristics of the control system:

- The controller is *model based* since it contains f and B from the process model.
- Since the process disturbance is an argument of f and/or B the controller implements *feedforward* from the disturbance. (It also implements feedforward from the reference, due to the term $(r_y)'$ in the controller.)
- The control system consists of n *decoupled single-loop* control systems. This is illustrated in Figure 39.4.

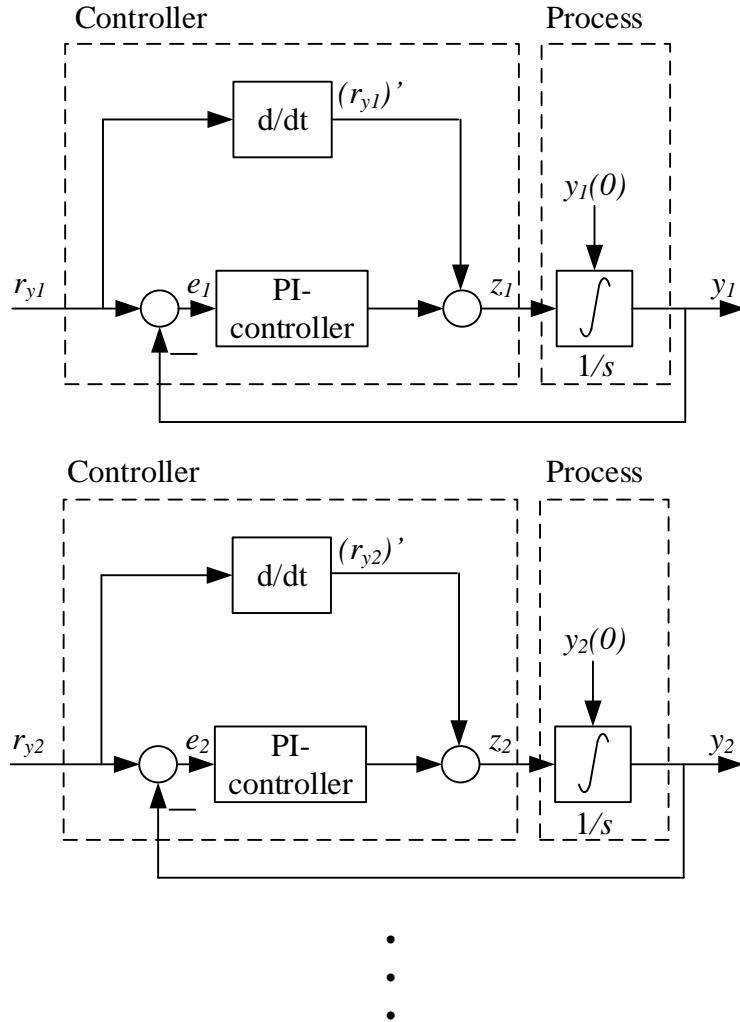


Figure 39.4: The inverse dynamics control system consists of n decoupled single-loop control systems.

Example 39.1 Inverse dynamics control applied to level control

In this example, inverse dynamics control will be applied to a level control system. Figure 39.5 shows the control system.

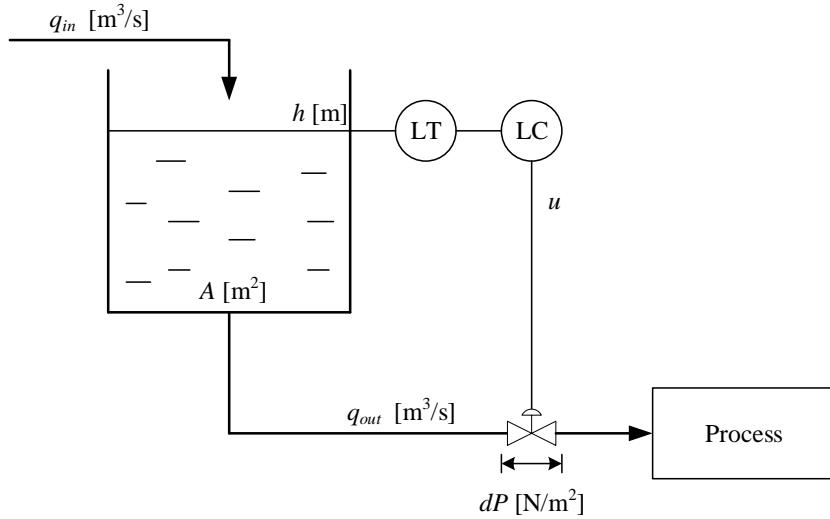


Figure 39.5: Level control system.

It is assumed that the outflow is proportional to the control signal u and to the square root of the pressure drop along the control valve. The process model based on mass balance is (ρ is density)

$$\rho Ah' = \rho q_{in} - \rho K_v u \sqrt{dP} \quad (39.15)$$

or

$$h' = \underbrace{\frac{q_{in}}{A}}_f + \underbrace{\left(-\frac{K_v \sqrt{dP}}{A} \right)}_B u \quad (39.16)$$

The control function becomes, cf. (39.14),

$$u = B^{-1} \left[K_c e + K_i \int_0^t e d\tau + r_y' - f \right] \quad (39.17)$$

$$= \left(-\frac{K_v \sqrt{dP}}{A} \right)^{-1} \left[K_c e + K_i \int_0^t e d\tau + r_y' - \frac{q_{in}}{A} \right] \quad (39.18)$$

$$= -\frac{A}{K_v \sqrt{dP}} \left[K_c e + K_i \int_0^t e d\tau + r_y' - \frac{q_{in}}{A} \right] \quad (39.19)$$

This control function requires that the differential pressure dP the inflow q_{in} are measured.

[End of Example 39.1]

39.3 Inverse dynamics control of second order processes

Mechanical processes with position as the process output variable (to be controlled) typically have a model on the form of a set of second order differential equations (due to Newton's Second Law):

$$y'' = f(y, y', d) + B(y, y', d) \cdot u \quad (39.20)$$

or simply:

$$y'' = f + Bu \quad (39.21)$$

where y is the position, y' is the speed, d is a disturbance (e.g. a load force or torque), and u is the control variable.

We assume that r_y is the reference of y (position).

We can derive the control function "graphically" as shown in Figure 39.6. The decoupler

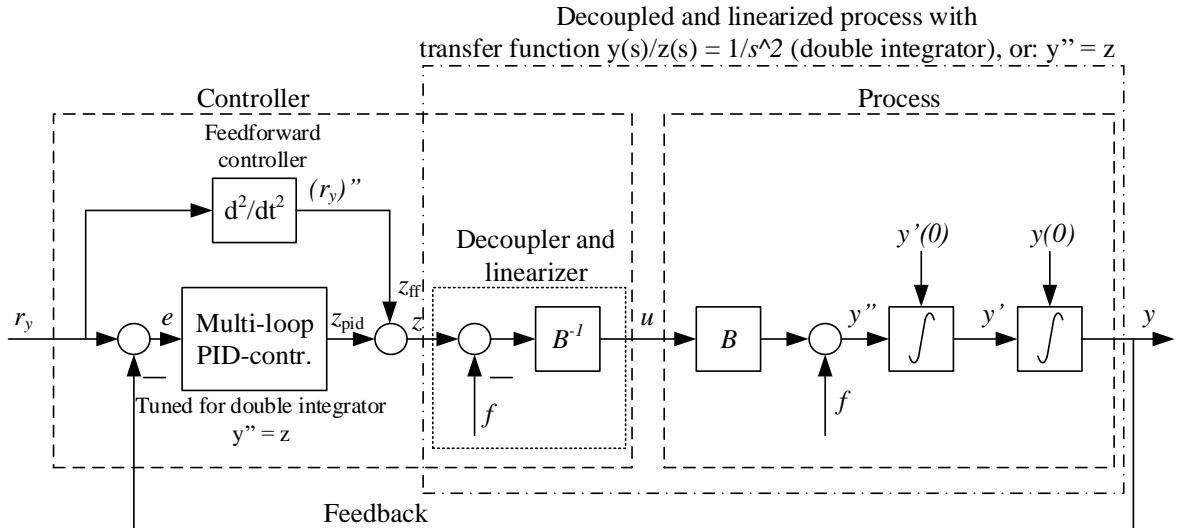


Figure 39.6: Block diagram of the control system based on feedback linearization.

and linearizer creates the following model relating the transformed control signal z and the process output y :

$$y'' = z \quad (39.22)$$

which are n decoupled (independent) *double integrators*. The transfer function from z to y is

$$\frac{y(s)}{z(s)} = \frac{1}{s^2} \quad (39.23)$$

These double integrators can be controlled with feedback with PID controllers plus feedforward, as described in the following.

PID tuning

The PID double integrators (39.23) can be controlled with feedback with PID controller:

$$z_{\text{pid}} = K_c e + \frac{K_c}{T_i} \int_0^t e d\tau + K_c T_d e' \quad (39.24)$$

where e is the control error:

$$e \stackrel{\text{def}}{=} r_y - y \quad (39.25)$$

In (39.24), K_c , K_c/T_i and $K_c T_d$ are actually diagonal matrices:

$$K_c = \begin{bmatrix} K_{c1} & 0 & \cdots & 0 \\ 0 & K_{c2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & K_{cn} \end{bmatrix} \quad (39.26)$$

$$\frac{K_c}{T_i} = \begin{bmatrix} \frac{K_{c1}}{T_{i1}} & 0 & \cdots & 0 \\ 0 & \frac{K_{c2}}{T_{i2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{K_{cn}}{T_{in}} \end{bmatrix} \quad (39.27)$$

$$K_c T_d = \begin{bmatrix} K_{c1} T_{d1} & 0 & \cdots & 0 \\ 0 & K_{c2} T_{d2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & K_{cn} T_{dn} \end{bmatrix} \quad (39.28)$$

K_{cj} , T_{ij} and T_{dj} can be calculated with the Skogestad method for a double integrator process having double integrator gain $K_{ii} = 1$, cf. Ch. 16.8.4. Assuming that the PID controller has the parallel form, the PID settings are (16.93)-(16.95), which are repeated here:

$$K_{cj} = \frac{2}{(T_{Cj})^2} \quad (39.29)$$

$$T_{ij} = 4T_{Cj} \quad (39.30)$$

$$T_{dj} = T_{Cj} \quad (39.31)$$

where T_{Cj} is the specified time constant of feedback loop no. j .

Feedforward controller

In addition to the PID feedback control, the controller should contain *feedforward* from the reference y_r to get fast reference tracking (assuming the reference is varying). The feedforward controller can be derived by substituting the process output y in the process model (39.3) by y_r and then solving for y , giving:

$$z_{\text{ff}} = r_y'' \quad (39.32)$$

The resulting control signal

Now, we have the following control function for the process (39.22) consisting of the sum of the feedback control function and the feedforward control function:

$$z = z_{\text{pid}} + z_{\text{ff}} \quad (39.33)$$

$$= \underbrace{K_c e + \frac{K_c}{T_i} \int_0^t e d\tau + K_c T_d \frac{de_f}{dt}}_{z_{\text{pid}}} + \underbrace{r_y''}_{z_{\text{ff}}} \quad (39.34)$$

Also from Figure 39.6, we see that the resulting controller function, that is, the formula for the control vector u , is:

$$u = B^{-1} (z - f) = B^{-1} (z_{\text{pid}} + z_{\text{ff}} - f) \quad (39.35)$$

z_{pid} and z_{ff} are as in (39.34).

About the resulting control system

The control system consists of n decoupled single-loop control systems. This is illustrated in Figure 39.7.

Example 39.2 Inverse dynamics control applied to motion control

Given the following mathematical model of a body:

$$my'' = F_c + F_d \quad (39.36)$$

where y [m] is position, $m = 10$ kg is mass, F_c [F] is the force demanded by the controller, and F_d [N] is net disturbance force (sum of damping, friction, gravitation, etc).

Assume that the position reference of y is r_y , and that the specified time-constant of the control system is 1 s. Also, assume that the PID controller is a parallel PID controller.

To derive the control function we first write the process model on the standard form (39.21):

$$y'' = \underbrace{\frac{1}{m} F_d}_f + \underbrace{\frac{1}{m} F_c}_B \underbrace{u}_u \quad (39.37)$$

The control function becomes

$$\underline{u} = B^{-1} [z_{\text{pid}} + z_{\text{ff}} - f] \quad (39.38)$$

$$= \underbrace{\left(\frac{1}{m} \right)^{-1} \left[z_{\text{pid}} + z_{\text{ff}} - \left(\frac{1}{m} F_d \right) \right]}_{\underline{u}} \quad (39.39)$$

where z_{pid} is given by (39.24), and z_{ff} is given by (39.32).

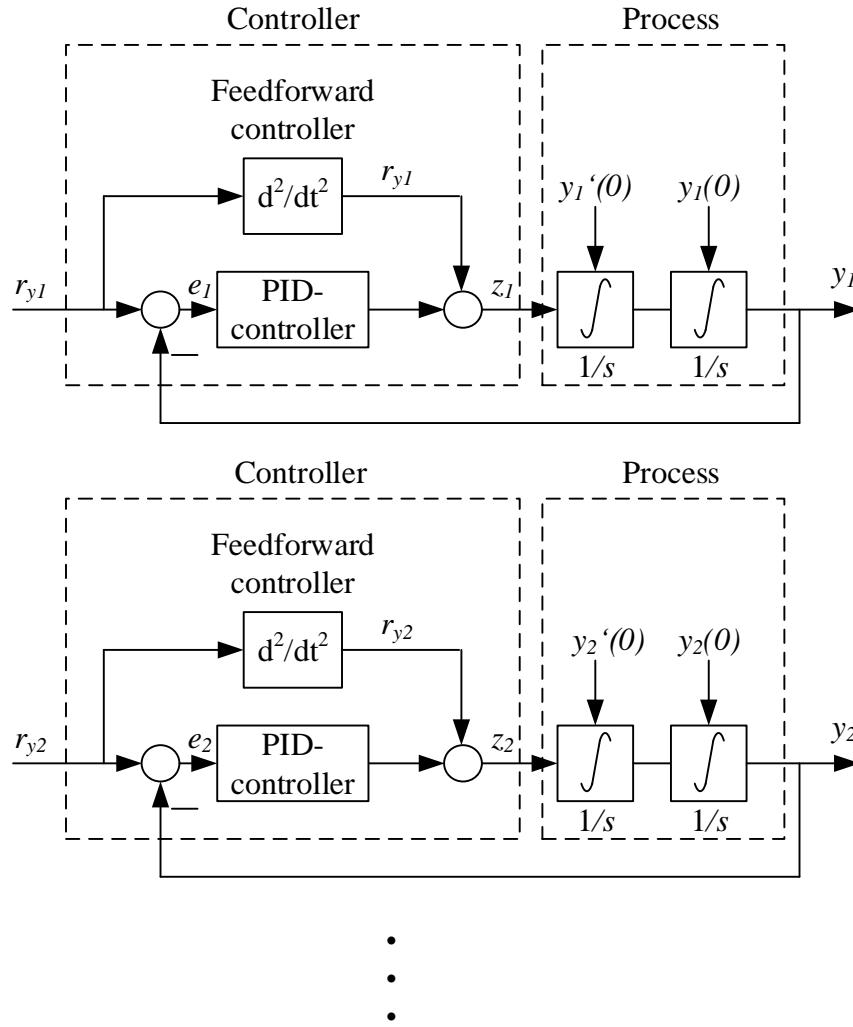


Figure 39.7: The control system consists of n decoupled single-loop control systems.

Let us, as an example, specify the closed-loop time constant T_C as:

$$T_C = 1 \text{ s} \quad (39.40)$$

The PID settings become:

$$K_c = \frac{2}{(T_C)^2} = 2 \quad (39.41)$$

$$T_i = 4T_C = 4 \text{ s} \quad (39.42)$$

$$T_d = T_C = 1 \text{ s} \quad (39.43)$$

[End of Example 39.2]

Computed torque control

In robotics, the so-called computed torque controller is well-known. This controller is the same as the inverse dynamics controller derived in the present section. I will now derive the

computed torque controller from the inverse dynamics controller (39.35).

The computed torque controller assumes a process model of the mechanical system (robot) on the following form:

$$M(q)q'' + C(q, q') + F(q') + G(q) + L = Q \quad (39.44)$$

or simply:

$$Mq'' + C + F + G + L = Q$$

where q is generalized position, i.e. a set of translational position and/or rotational positions of the robot arms, q' is generalized speed, M is the inertia matrix, C is the coriolis and centripetal force matrix, F is the friction force matrix, G is the gravity loading matrix, and L is the generalized load (force and/or torque) matrix, and Q is the generalized applied force (“translation” force and/or “rotational” force (torque)), assumed being the control (manipulating) variable. Often, $C(q, q')$ is written as $C_1(q, q') \cdot q'$ where C_1 is a matrix, but here I use the simpler form $C(q, q')$.

(39.44) written on the form (39.21) is:

$$q'' = M^{-1}(-C - F - G - L) + M^{-1}Q \quad (39.45)$$

which is on the form

$$y'' = f + Bu \quad (39.46)$$

with

$$y = q \quad (39.47)$$

$$u = Q \quad (39.48)$$

$$f = M^{-1}(-C - F - G - L) \quad (39.49)$$

$$B = M^{-1}$$

The inverse dynamics controller given by (39.35) becomes

$$u = B^{-1}(z_{\text{pid}} + z_{\text{ff}} - f) \quad (39.50)$$

$$= M[z_{\text{pid}} + r_q'' - M^{-1}(-C - F - G - L)] \quad (39.51)$$

$$= M(z_{\text{pid}} + r_q'') + (C + F + G + L) \quad (39.52)$$

where z_{pid} is the PID controller (39.34). r_q is the reference or setpoint of q . (39.52) is the computed torque controller, derived as an inverse dynamics controller.

In some literature, a PD controller is used instead of a PID controller in (39.52). With a PD controller (i.e. lacking integral action), non-modelled forces may give non-zero steady-state control errors.

39.4 Problems for Chapter 39

Problem 39.1 Temperature and level control of a tank

Figure 39.8 shows a tank where continuous flows of cold liquid and hot liquid are mixed in a tank.² The liquid in the tank is assumed being homogeneous.

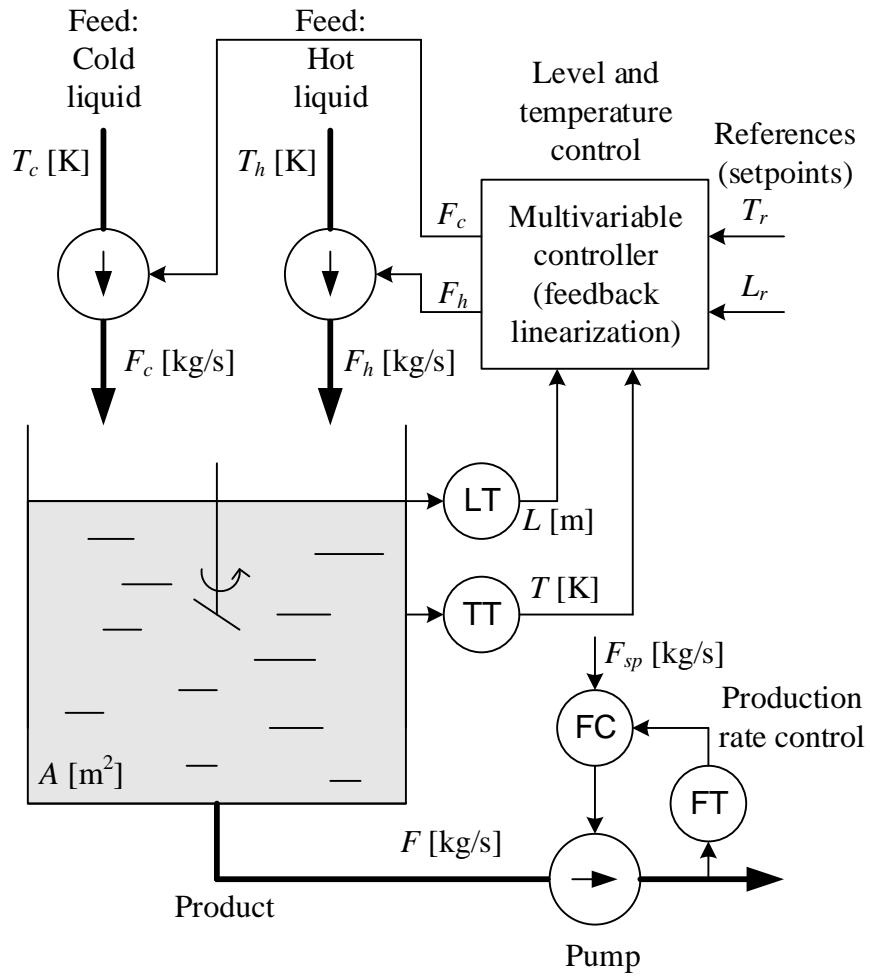


Figure 39.8: A tank.

The product flow rate out of the tank is controlled with a ordinary flow control loop.

The level and the temperature of the tank shall be controlled to follow (track) their reference values. The cold liquid and the hot liquid flows can be manipulated, so they are control variables. (To make the actual flows become equal to the demanded flow as calculated by the multivariable controller, local flow control loops around the pumps may be needed, but these control loops are not shown in the figure.)

It is not obvious how to control the cold flow and the hot flow to obtain the reference level

²This example is not very realistic, but it is assumed to be relatively simple and easy to understand.

and temperature because both flows affect both the level and the temperature. The control problem may be solved with a “traditional” control structure where e.g. the level controller adjusts the cold flow, and the temperature controller adjusts the ratio between hot and cold flow. But, instead of such a traditional control structure, you will design a model-based multivariable controller based on inverse dynamics control.

The mathematical model of the process is as follows. The model is based on the following assumptions:

- The density ρ and the specific heat capacity c are the same in all flows and in the tank.
- The temperature is homogeneous in the liquid in the tank.
- There is no heat transfer through the walls of the tank.
- Energy dependent on pressure and kinetics is disregarded.

Mass balance of the mixed liquid of the tank is

$$\rho Ah' = F_c + F_h - F \quad (39.53)$$

An energy balance of the liquid in the tank is (it is assumed that both the level and the temperature can vary):

$$c\rho A(LT)' = cF_c T_c + cF_h T_h - cFT \quad (39.54)$$

In (39.54),

$$c\rho A(LT)' = c\rho A(L')T + c\rho AL(T') \quad (39.55)$$

With (39.55) inserted into (39.54) and then cancelling c , (39.54) becomes

$$\rho ALT' = F_c(T_c - T) + F_h(T_h - T) \quad (39.56)$$

The process model is now (39.53) and (39.56).

1. Write the process model (39.53) and (39.56) on the standard form to be used in an inverse dynamics controller where F_c and F_h are control variables, and L and T are state variables to be controlled (to track their respective references).
2. Design the inverse dynamics controller, but you do not have to give the formulas for tuning the gains and the integral times of the internal PI controllers, as this is the topic of the following task. The references are L_r and T_r .
3. Calculate the gains, K_{cL} and K_L , and the integral times, T_{iL} and T_{iT} , of the internal PI controllers. It is specified that the control loops of the decoupled processes (which are just integrators) shall have response times (time constants) T_{CL} and T_{CT} , respectively.
4. Which parameters and variables must be known at any instant of time to make the control function implementable?

5. Assume that there is a change in the level reference. Will this change cause any change in the temperature?

Will a change in the temperature reference cause any change in the level?

6. Assume for example that any change in the temperature of the cold inflow, T_c , is regarded as a disturbance to the control system. Does the control function implement *feedforward* from this disturbance?

Problem 39.2 Position control of a ship

Figure 39.9 shows a ship.

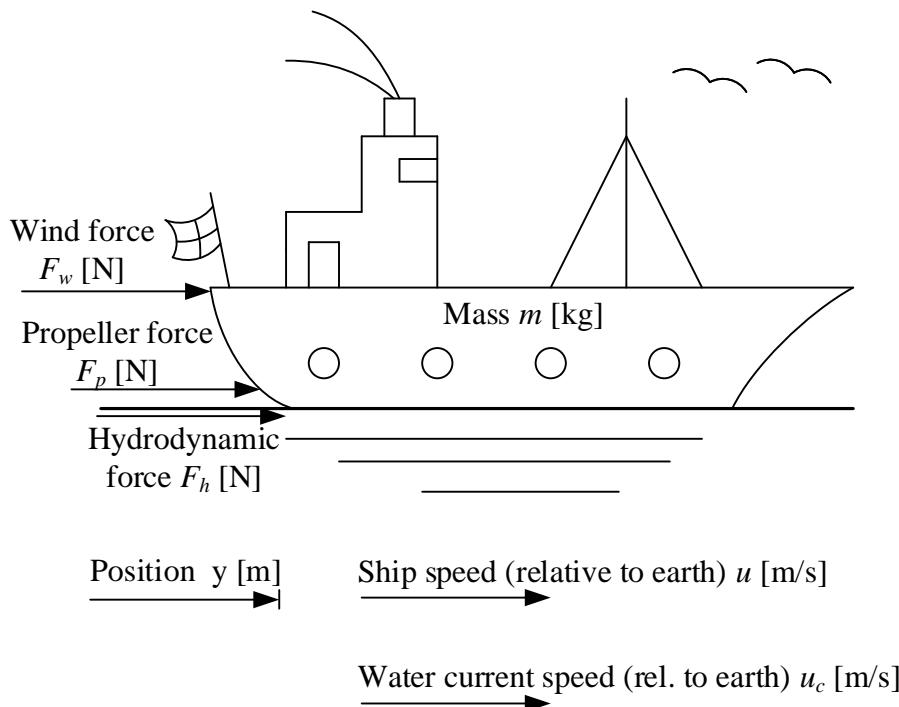


Figure 39.9: Ship.

In this problem we concentrate on the so-called surge (forward-backward) direction, i.e., the motions in the other directions are disregarded. The wind acts on the ship with the force F_w which is a function of the wind attack angle ϕ and the wind speed V_w . This function is assumed to be known for a given ship. The hydrodynamic damping force F_h (damping from the water) is proportional to the square of the difference between the ship speed, y' , and the water current speed u_c . The proportionality constant is D .

Applying Newton's Law of Motion we obtain the following mathematical model of the surge motion:

$$my'' = \underbrace{-D |y' - u_c| (y' - u_c)}_{F_h} + F_w(\phi, V_w) + F_t \quad (39.57)$$

(Model parameter values are given in Appendix 41.2, but these values are not needed in the present exercise.)

1. Design the position control function as an inverse dynamics controller. The position reference is r [m]. The closed loop time-constant is specified as T_C . Express the PID parameters as a function of T_C .
2. Which variables and parameters must have known values (from measurements or estimators) to make the controller function implementable.
3. Let's define the wind force F_w as a disturbance. Explain how the inverse dynamics controller implements feedforward from this disturbance. Assuming that the ship model is correct and that F_w is perfectly known, what is then the impact that F_w will have on the ship position y ?

39.5 *Solutions to problems for Chapter 39*

Solution to Problem 39.1

1. The standard process model form is

$$x' = f + Bu \quad (39.58)$$

We have

$$\underline{x} = \begin{bmatrix} L \\ T \end{bmatrix} \quad (39.59)$$

and

$$\underline{u} = \begin{bmatrix} F_c \\ F_h \end{bmatrix} \quad (39.60)$$

We get

$$\underline{B} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{\rho A} & \frac{1}{\rho A} \\ \frac{T_c - T}{\rho AL} & \frac{T_h - T}{\rho AL} \end{bmatrix} \quad (39.61)$$

and

$$\underline{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} -\frac{F}{\rho A} \\ 0 \end{bmatrix} \quad (39.62)$$

2. The control function is generally

$$u = B^{-1} \left[K_p e + K_i \int_0^t e d\tau + r' - f \right] \quad (39.63)$$

where (I use indices L and T instead of 1 and 2 as in the standard form)

$$\underline{u} = \begin{bmatrix} F_c \\ F_h \end{bmatrix} \quad (39.64)$$

$$\underline{B} = \begin{bmatrix} \frac{1}{\rho A} & \frac{1}{\rho A} \\ \frac{T_c - T}{\rho AL} & \frac{T_h - T}{\rho AL} \end{bmatrix} \quad (39.65)$$

$$\underline{e} = \begin{bmatrix} L_r - L \\ T_r - T \end{bmatrix} \quad (39.66)$$

$$\underline{K}_p = \begin{bmatrix} K_{pL} & 0 \\ 0 & K_{pT} \end{bmatrix} \quad (39.67)$$

$$\underline{K}_i = \begin{bmatrix} K_{iL} & 0 \\ 0 & K_{iT} \end{bmatrix} = \begin{bmatrix} \frac{K_{pL}}{T_{iL}} & 0 \\ 0 & \frac{K_{pT}}{T_{iT}} \end{bmatrix} \quad (39.68)$$

$$\underline{r}' = \begin{bmatrix} L_r' \\ T_r' \end{bmatrix} \quad (39.69)$$

$$\underline{f} = \begin{bmatrix} -\frac{F}{\rho A} \\ 0 \end{bmatrix} \quad (39.70)$$

3.

$$\underline{\underline{K_{pL} = \frac{1}{T_{CL}}}} \quad (39.71)$$

$$\underline{\underline{T_{iL} = 2T_{CL}}} \quad (39.72)$$

$$\underline{\underline{K_{pT} = \frac{1}{T_{CT}}}} \quad (39.73)$$

$$\underline{\underline{T_{iT} = 2T_{CT}}} \quad (39.74)$$

4. The following parameters and variables must be known:

- ρ
- A
- L (using a level sensor)
- T (temperature sensor)
- T_c (temperature sensor)
- T_h (temperature sensor)
- F (flow sensor)

5. No

No

6. Yes

Solution to Problem 39.2

1. To derive the control function we first write the process model on the standard form:

$$y'' = \underbrace{\frac{1}{m} [-D |y' - u_c| (y' - u_c) + F_w(\phi, V_w)]}_f + \underbrace{\frac{1}{m} F_t}_B \underbrace{u}_u \quad (39.75)$$

The control function becomes

$$\underline{\underline{F_t = B^{-1} \left[K_p e + \frac{K_p}{T_i} \int_0^t ed\tau + K_p T_d \frac{de_f}{dt} + r'' - f \right]}} \quad (39.76)$$

$$\begin{aligned} &= \left(\frac{1}{m} \right)^{-1} \left[K_p e + \frac{K_p}{T_i} \int_0^t ed\tau + K_p T_d \frac{de_f}{dt} + r'' \right. \\ &\quad \left. + \frac{1}{m} [-D |y' - u_c| (y' - u_c) + F_w(\phi, V_w)] \right] \end{aligned} \quad (39.77)$$

where e is the control error:

$$e = r - y \quad (39.78)$$

The Skogestad PID settings become:

$$\underline{\underline{K_p = \frac{2}{T_C^2}}} \quad (39.79)$$

$$\underline{\underline{T_i = 4T_C}} \quad (39.80)$$

$$\underline{\underline{T_d = T_C}} \quad (39.81)$$

2. Of course, all variables and parameters of the control function (39.77) must be known to make the controller implementable. In more details:
 - The parameters m and D must be known.
 - The ship position y must be measured using e.g. GPS measurement.
 - The ship speed y' can be calculated as the time-derivative of y , or it can be estimated with a Kalman Filter or an observer.
 - The water speed u_c must either be measured or estimated with a Kalman Filter or an observer.³
 - The wind angle ϕ and the wind speed V_w must be measured with a wind sensor (mounted on the ship). The wind force function F_w is assumed to be known for a given ship.⁴
3. In (39.77) the control variable F_t is a function of the disturbance $F_w(\phi, V_w)$. This dependency implements feedforward. The controller will compensate for the disturbance, so that the ship position y will not be influenced by F_w , whatever value of F_w .

³Kongsberg Maritime (Norway) use a Kalman Filter in their ship positioning systems – or DP (Dynamic Positioning) systems.

⁴ F_w is calculated from the geometry of the ship.

Part VIII

APPENDICES

Chapter 40

Some good control questions

Below are some good questions you should pose to yourself if you get involved in control system design. Many terms in the answers below are explained during this book, so don't expect to understand all of it at this moment. You may read this section again after you have completed the book!

- **Is there really a need for control?** There *is* a need for control if there is a chance that the process output variable can drift too far away from its desired value with just constant control. Such a drift can be caused by severe variations of environmental variables (process disturbances). For *unstable* processes, like water tanks and exothermal reactors and motion systems like robots and ships which must be positioned, there will always be a need for control to keep the process output variable (level; temperature; position, respectively) at a setpoint or reference value.
- **Which process variable(s) needs to be controlled (to make it become equal to a setpoint or reference value)?** Liquid level in a given tank? Pressure of vapour in the tank? Temperature? Flow? Composition?
- **How to measure that process variable?** Select an appropriate sensor, and make sure it detects the value of the process variable with as little time delay and sluggishness as possible! In other words, measure as directly as you can.
- **Is the measurement signal noisy?** It probably is. Use a lowpass filter to filter or smooth out the noise, but don't make the filtering too strong – or you will also filter out significant contents of the measurement signal, causing the controller to react on erroneous information.
- **How to manipulate the process variable?** Select an actuator that gives a strong impact on the process variable (to be controlled)! Avoid time delays if possible, because time delays in a control loop will limit the speed of the control, and you may get a sluggish control loop, causing the control error to become large after disturbance variations.
- **Which controller function (in the feedback controller)?** Try the PI controller. Most PID controllers actually operate as PI controllers since the derivative (D) term

is deactivated because it amplifies measurement noise through the controller, causing noisy control signal which may cause excessive wear of a mechanical actuator.

However, there are certain processes which requires an active D term, e.g. the double integrator which is an approximate model of a kinetic system (a “body”) to be position controlled with force or torques as control variable.

- **How to tune the controller?**

If you don't have mathematical model of the process to be controlled, try the Good Gain method, which is a simple, experimental tuning method which does not require the control loop to become marginally stable, and therefore oscillate, during the tuning. (The Ziegler-Nichols closed loop tuning method, on the other hand, requires marginal stability.)

If you do have a mathematical process model, try the Skogestad model-based tuning method to get the controller parameters directly from the process model parameters. Alternatively, with a model, you can create a simulator of your control system in e.g. LabVIEW or Simulink or Scicos, and then apply the Good Gain method on the simulator.

If your controller has an Auto-tune button, try it!

- **Is the stability of the control system good?** The most important requirement to a control system is that it has acceptable stability: The responses due to abrupt excitations may oscillate somewhat, but they should be well damped.

If you don't have a mathematical model of the system, apply a (small) step change of the setpoint and observe whether the stability of the control system is ok. If you think that the stability is not good enough (too little damping of oscillations), try reducing the controller gain and increasing the integral time somewhat, say by a factor of two. Also, derivative control action can improve stability of the control loop, but remember the drawback of the D-term related to amplification of random measurement noise.

If you do have a mathematical model of the control system, create a simulator, and simulate responses in the process variable and the control variable due to step changes in the setpoint and the disturbances (load variables), e.g. environmental temperature, feed composition, external forces, etc. With a simulator, you may also want to make realistic changes of certain parameters of the process model, for example time delays or some other physical parameters, to see if the control system behaves well despite these parameter changes (assuming the controller is not tuned again). In this way you can test the robustness of the control system against process variations. If the control system does not behave well, for example having too poor stability, after the parameter changes, consider Gain scheduling which is based on changing the PID parameters automatically as functions of certain process parameters.

- **Is the control error small enough after disturbance changes, and after setpoint changes?**

If you do not have a simulator of the control system, it may be difficult to generate disturbance changes yourself, but setpoint changes can of course be applied easily.

If you have a simulator, you can apply both setpoint changes and disturbance changes.

If – using either experiments or simulator – the control error is too large after the changes of the setpoint and the disturbance, try to tune the controller again to obtain faster control.

- **Still not happy with control system performance after retuning the controller?** Then, look for other control structures or methods based on exploiting *more process information*:
 - *Feedforward control*: This requires that you measure one or more of the disturbances (load variables) acting on the process, and using these measurements to directly adjust the control signal to compensate for the disturbance(s).
 - *Cascade control*: This requires that you measure some internal process variable which is influenced by the disturbance, and construct an inner control loop (inside the main control loop) based on this internal measurement to quickly compensate for the disturbance.
 - *Model-based control*: Consider for example optimal control with state-variable feedback (LQ (Linear Quadratic) optimal control), or model-based predictive control (MPC).

Chapter 41

Process models

In the following sections, mathematical models of various physical processes are presented. Some of these models are used in examples and in problems in the book. They may be used in additional problems, and as a basis of dynamic simulators.

41.1 Wood chips tank

41.1.1 System description

Figure 41.1 shows a wood chips tank with a feed screw with continuous feed of wood chips, conveyor belt, which runs with a fixed speed. There is a continuous outflow of wood chips.¹

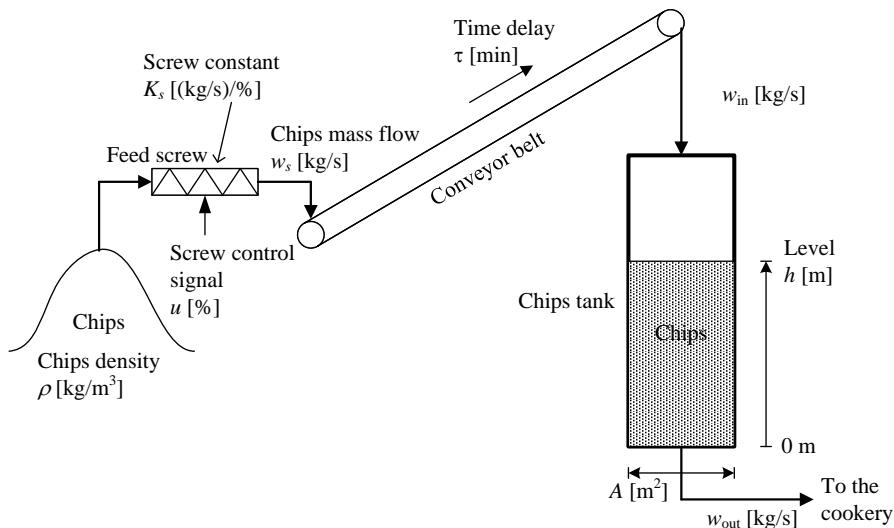


Figure 41.1: Wood chips tank.

The conveyor belt makes up a time delay or transport delay from the screw to the tank.

41.1.2 Variables and parameters

Variables and parameters of the wood chip tank are defined in Table 41.1.² We assume that there is an overflow if the level exceeds the maximum level.

Table 41.1: Wood chips tank: Variables and parameters.

Symbol	Value (default)	Unit	Description
h	10	m	Wood chips level
-	[0, 15]	m	Range of level
u	50	%	Control signal to feed screw
w_s	25	kg/s	Feed screw flow (flow into conveyor belt)
w_{in}	25	kg/s	Wood chips flow into tank (from belt)
w_{out}	25	kg/s	Wood chips outflow from tank
ρ	145	kg/m ³	Wood chips density
A	13.4	m ²	Tank cross sectional area
K_s	0.5	(kg/s)/%	Feed screw gain (capacity)
τ	250 s	s	Transport time (time delay) on conveyor belt

41.1.3 Overall block diagram

Figure 41.2 shows a block diagram of the wood chips tank.

41.1.4 Mathematical model

The chips flow through the feed screw, w_s , is the inflow to the conveyor belt. w_s is assumed being proportional to the control signal, u :

$$w_s = K_s u \quad (41.1)$$

The outflow from the belt, which is also the inflow to the tank, is the same as the inflow to the belt and the screw flow, but time delayed:

$$w_{in}(t) = w_s(t - \tau) = K_s u(t - \tau)$$

¹Typically, there is such a wood chips tank in the beginning of the production line of a paper and pulp factory.

²Courtesy of earlier Sødra Cell, and even earlier Norske Skog, Tofte, Norway.

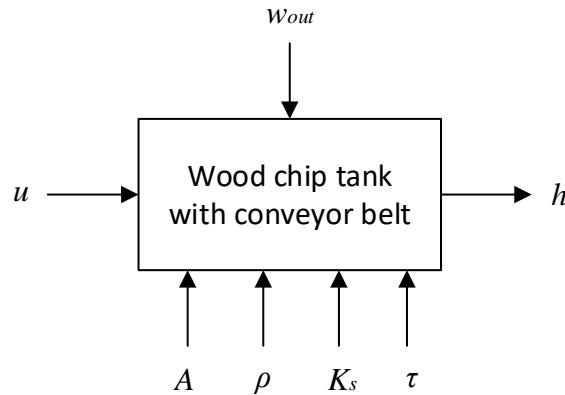


Figure 41.2: Block diagram of the wood chips tank.

The outflow from the tank is w_{out} .

A mathematical model of the tank based on material balance of the wood chips in the tank is:

$$\rho Ah'(t) = w_{\text{in}}(t) - w_{\text{out}}(t) = K_s u(t - \tau) - w_{\text{out}}(t) \quad (41.2)$$

41.2 Ship

41.2.1 System description

Figure 41.3 shows a ship. In this example we will only take the longitudinal motion relative to the ship, also denoted the surge motion, into account.³

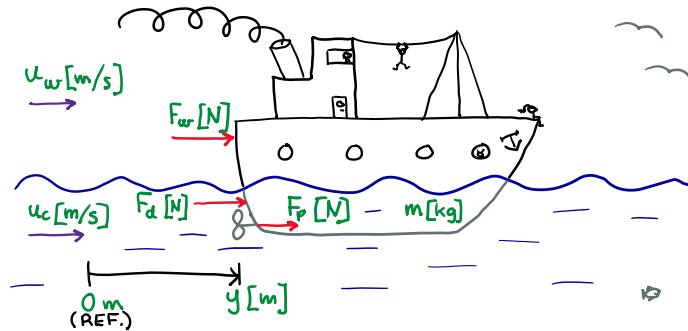


Figure 41.3: Ship.

³Courtesy of Kongsberg Maritime AS, Norway for providing realistic parameter values.

41.2.2 Variables and parameters

Variables and parameters are defined in Table 41.2. The parameter values are realistic.⁴

Table 41.2: Ship: Variables and parameters

Symbol	Value (default)	Unit	Description
$y = x_1$	-	m	Longitudinal or surge position of the ship
$\dot{y} = x_2$	-	m/s	Ship speed
F_p	$[-467 \cdot 10^3, +552 \cdot 10^3]$	N	Propeller force applied to move the ship
F_h	-	N	Hydrodynamic force on the ship
F_w	-	N	Wind force on the ship
u_c	$[-3, +3]$	m/s	Water current speed
V_w	Cf. windscale in Fig. 41.4	m/s	Wind speed
m	$71164 \cdot 10^3$	kg	Mass of ship
D_h	$8.4 \cdot 10^3$	$N/(m/s)^2$	Hydrodynamic force constant
D_w	$0.177 \cdot 10^3$	$N/(m/s)^2$	Wind force constant

Figure 41.4 shows the wind scale which characterizes various ranges of wind speed V_w .

m/s	Description
0.5 - 1.8 m/sec	light air
1.9 - 3.3 m/sec	light breeze
3.4 - 5.4 m/sec	gentle breeze
5.5 - 7.9 m/sec	breeze
8.0 - 11.0 m/sec	fresh breeze
11.1 - 14.1 m/sec	strong breeze
14.2 - 17.2 m/sec	near gale
17.3 - 20.8 m/sec	gale
20.9 - 24.4 m/sec	strong gale
24.5 - 28.5 m/sec	storm
28.6 - 32.6 m/sec	violent storm
> 32.6 m/sec	hurricane

Figure 41.4: Wind scale

⁴Courtesy of Kongsberg Maritime AS, Norway.

41.2.3 Overall block diagram

Figure 41.5 shows a block diagram of the ship.

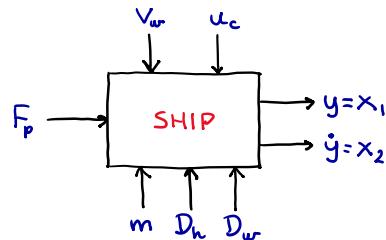


Figure 41.5: Block diagram of the ship

41.2.4 Mathematical model

We use the Newton's Second Law to model the motion of the ship:

$$\text{mass times acceleration} = \text{sum of forces}$$

In mathematical terms:

$$my'' = F_p + F_h + F_w \quad (41.3)$$

where:

- F_h is proportional to the difference between the water speed and the ship speed:

$$F_h = D_h (u_c - y') |u_c - y'| \quad (41.4)$$

The absolute value in the last term ensures that the sign of F_h correct.

- F_w is proportional to the square of the difference between the wind speed and the ship speed:

$$F_w = D_w (V_w - y') |V_w - y'| \quad (41.5)$$

The absolute value in the last term ensures that the sign of F_w correct.

When the ship moves, there is also a motion of an amount of water. The mass of the water is denoted the added mass. However, we disregard the added mass here.

41.3 Buffer tank

41.3.1 System description

Figure 41.6 shows a water tank. The tank may represent a buffer tank, or an equalization magazine, at the inlet of a plant, e.g. a water resource recovery facility (WWRF). The geometrical design of such a magazine may not have straight walls as in Figure 41.6, see Figure 1.17. However, if relatively small variations in the level are assumed, a tank with straight walls approximates the magazine with non-straight walls.

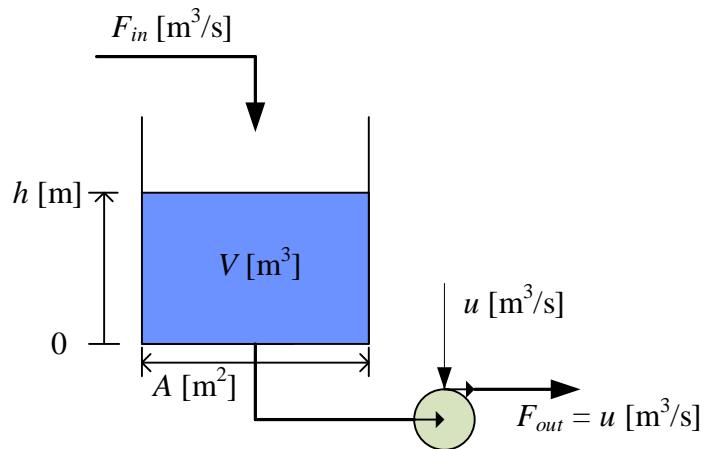


Figure 41.6: Water tank with pump outlet

41.3.2 Variables and parameters

Variables and parameters of the water tank are defined in Table 41.3. The numerical values resembles a typical operating point of the VEAS WWRF, Slemmestad, Norway.

Table 41.3: Water tank: Variables and parameters

Symbol	Value (default)	Unit	Description
h	2.0	m	Water level
F_{in}	3.0	m^3/s	Inflow
F_{out}	3.0	m^3/s	Outflow through pump
u	3.0	m^3/s	Control signal to pump
A	2000	m^2	Inner cross sectional area of tank
V	4000	m^3	Volume of water in tank

41.3.3 Overall block diagram

Figure 41.7 shows an overall block diagram of the water tank.

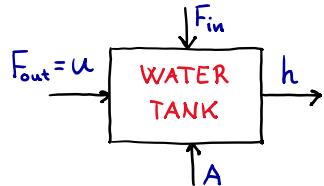


Figure 41.7: Block diagram of the water tank

41.3.4 Mathematical model

A mathematical model expressing the level variations can be derived from material balance of the water in the tank. The model is:

$$Ah' = F_{\text{in}} - F_{\text{out}} \quad (41.6)$$

where

$$F_{\text{out}} = u \quad (41.7)$$

41.4 Heated liquid tank

41.4.1 System description

Figure 41.8 shows a heated tank. Liquid (assumed water) flows into and out of the tank. The volume of liquid is constant (this can be realized with overflow or level of regulation). The inflow and outflow are thus equal. There is a heat transfer between the liquid and the air outside the tank. In the tank there are homogeneous conditions thanks to a mixer (there is thus no spatial variations in temperature). It is assumed that the mixer does not add power to the liquid. It is assumed that there is a time delay in the response in the temperature if there is a change in the supplied power.

41.4.2 Variables and parameters

Variables and parameters of the heated tank are defined in Table 41.4.

41.4.3 Overall block diagram

Figure 41.9 shows an overall block diagram of the heated tank.

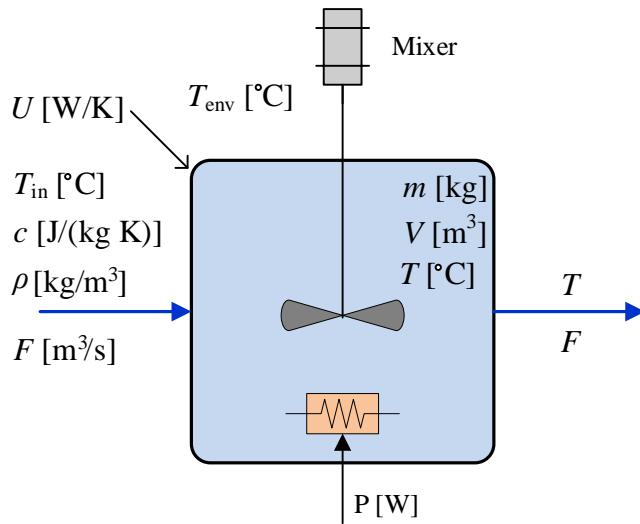


Figure 41.8: Heated tank.

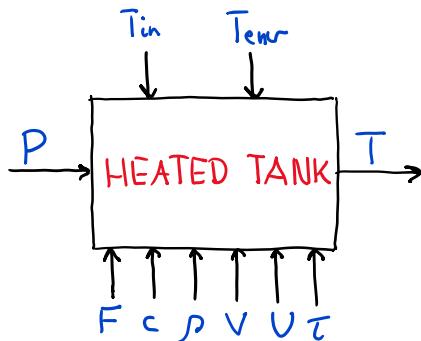


Figure 41.9: Block diagram of the heated tank

41.4.4 Mathematical model

A mathematical model of the temperature variation, based on energy balance of the liquid in the tank, is:

$$c\rho V T'(t) = P(t - \tau) + c\rho F [T_{in}(t) - T(t)] + U [T_{env}(t) - T(t)] \quad (41.8)$$

The term $P(t - \tau)$ expresses that the supplied power has time delay τ . This time delay represents the time delay in the temperature that will be observed in any practical tank due to the unavoidable imperfect mixing.

Table 41.4: Heated tank: Variables and parameters

Symbol	Value (default)	Unit	Description
P	0	W	Supplied power
T	-	°C	Temperature of liquid
T_{init}	20	°C	Initial temperature
T_{env}	20	°C	Environmental temperature
T_{in}	20	°C	Temperature of liquid inflow
F	$0.25 \cdot 10^{-3}$	m^3/s	Liquid flow
c	4200	$\text{J}/(\text{kg}\cdot\text{K})$	Specific heat capacity of liquid
ρ	1000	kg/m^3	Density of liquid
V	0.2	m^3	Liquid volume in tank
U	1000	W/K	Heat transfer coefficient of tank
τ	60 s	s	Time delay in the temperature response

41.5 Air heater

41.5.1 System description

The physical laboratory rig air heater is described on http://techteach.no/air_heater. Figure 41.10 shows the air heater.⁵

41.5.2 Variables and parameters

Variables and parameters and assumed parameter values are defined in Table 41.5.

41.5.3 Overall block diagram

Figure 41.11 shows an overall block diagram of the air heater.

41.5.4 Mathematical model

A mathematical model that has proven to describe quite well the dynamic behaviour of the outlet air temperature is given by the following differential equation representing “time constant with time delay” dynamics from control signal u to outlet temperature T :

$$\theta T'(t) = K_h [u(t - \tau)] + [T_{\text{env}}(t) - T(t)] \quad (41.9)$$

⁵University South-Eastern Norway, campus Porsgrunn, has 26 of identical units of this lab station, being used in several control courses in both bachelor and master programmes in technology.

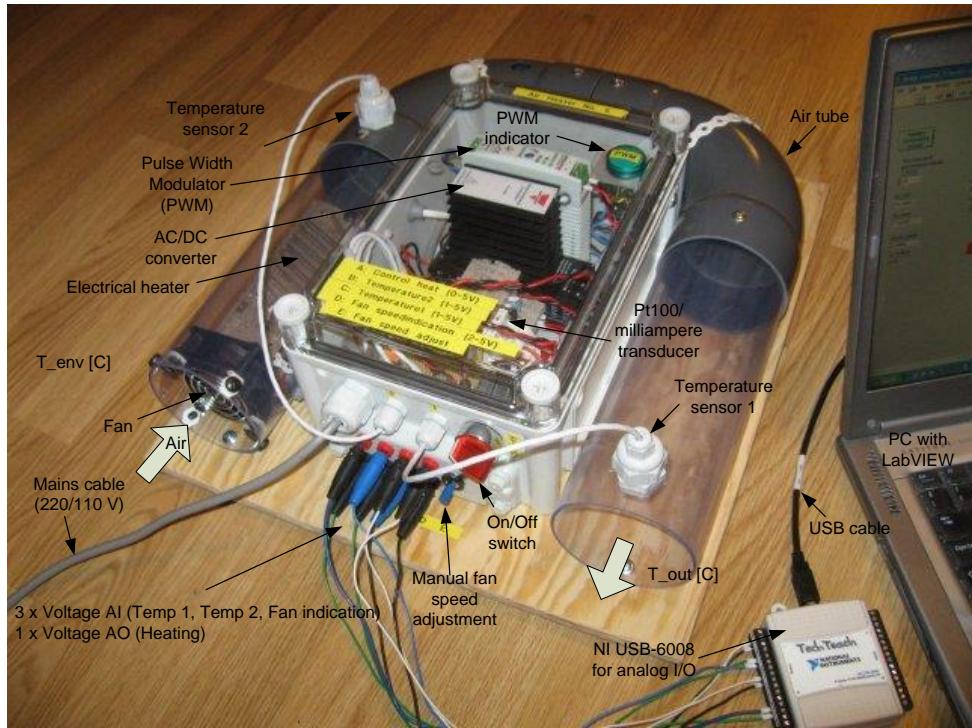


Figure 41.10: Air heater

This model may be derived from mechanistic (first-principles) modeling principles, i.e. a simple energy balance of the air, if we make the idealized assumption that the tube is a so-called CSTR (continuous stirred tank reactor) with air inflow and outflow and heat transfer with the environment through the “reactor” (tube) walls, and – in addition – we include the time delay as described above. In reality, the idealized CSTR conditions are not satisfied, but they lead to a useful model structure with parameter values that may be estimated from experimental data.

Many other stable physical processes show “time constant with time delay” dynamics, and such processes may be reasonably well represented with models similar to (41.9).

41.5.5 Data file

A datafile from an experiment with the air heater is available on:

http://techteach.no/control/python/airheater_logfile.txt

The time step (sampling time) is 0.1 s.

The datafile contain the following columns (from left in the file):

- Time t [s]

Table 41.5: Air heater: Variables and parameters

Symbol	Value (default)	Unit	Description
T	-	°C	Temperature of the air flowing out of tube. Measured by a sensor.
T_{init}	25	°C	Initial temperature
T_{env}	25	°C	The environmental, or ambient, temperature. It is the temperature in the outlet air of the air tube when the control signal to the heater has been set to zero for relatively long time (some minutes).
u	-	V	Control signal to heater
K_h	3.5	°C/V	Heater gain
θ	23.0	s	Time constant representing sluggishness of heater
τ	3.0	s	Time delay representing air transportation and sluggishness of heater

- Control signal u [V]
- Temperature measurement signal T [°C]

41.6 DC-motor

41.6.1 System description

Figure 41.12 shows a DC-motor with tachometer. The motor can be manipulated with an input voltage signal, u .

The rotational speed is measured with a tachometer which produces a output voltage signal which is proportional to the speed. The speed, S , is calculated continuously from the tachometer voltage, and hence the speed is assumed to be known at any instant of time. The motor and the tachometer are regarded as one unit, but we refer to it as “motor”.

41.6.2 Overall block diagram

Figure 41.13 shows an overall block diagram of the motor with tachometer.

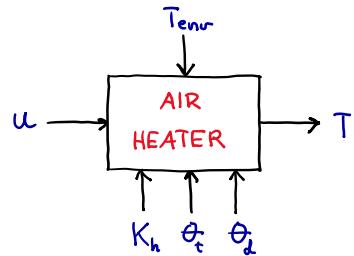


Figure 41.11: Block diagram of the air heater

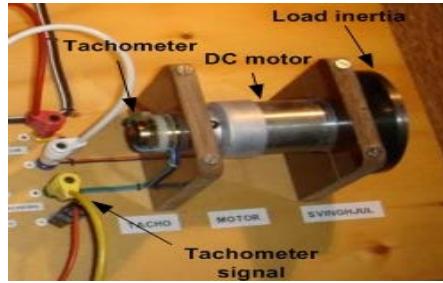


Figure 41.12: DC motor

41.6.3 Variables and parameters

Variables and parameters with assumed values are defined in Table 41.6.

41.6.4 Mathematical model

A mathematical model of the motor is (omitting the time argument for simplicity):

$$TS' = K(u + L) - S \quad (41.10)$$

The speed measurement signal generated by the tachometer is

$$S_m = K_t S \quad (41.11)$$

From (41.11) you can calculate the speed in krpm from the measurement in voltge:

$$S = S_m / K_t \quad (41.12)$$

41.6.5 Datafile

A datafile from an experiment is available on:

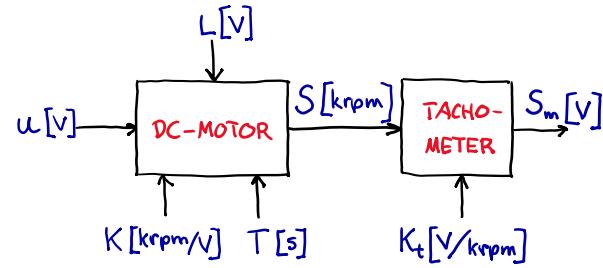


Figure 41.13: Block diagram of DC-motor with tachometer

Table 41.6: DC motor with tachometer: Variables and parameters

Symbol	Value (default)	Unit	Description
S	-	krpm	Rotational speed of the motor
S_{init}	0	krpm	Initial speed
S_m		V	Speed measurement signal generated by the tachometer
u	[-10 V, +10 V]	V	Control signal to the motor
L	0	V	Equivalent load torque, represented in the same unit as the control variable. L can be regarded as environmental variable or process disturbance.
K	0.17	krpm/V	Gain of motor with tachometer
T	0.30	s	Time constant of motor with tachometer
K_t	5.0	V/krpm	Tachometer gain

http://techteach.no/control/python/data_dc_motor.txt

The time step (sampling time) is 0.02 s.

The datafile contain the following columns (from left in the file):

- Time t [s]
- Control signal u [V]
- Tachometer measurement signal S_m [V]

41.7 Biogas reactor

41.7.1 System description

A biogas reactor is a vessel or tank in which organic matter, like food waste, slaughter waste, livestock manure, sewage sediments, etc., is converted by various cultures of microorganisms into energy-rich, combustable methane (CH_4) gas, which is the most important product, and carbondioxide (CO_2) gas. The two cultures assumed in the model presented here, are acidogens, which generate volatile fatty acids (VFAs), and methanogens which generate methane. The biological conversion process takes place without oxygen, and is often called an anaerobic digestion (AD) process. The AD process is assumed continuous, not “plug-flow”.

Biogas contains roughly 65% CH_4 and 35% CO_2 . Combustion of the CH_4 gas results in CO_2 and H_2O . Biogas, raw or upgraded into approx. 98% CH_4 , can be used in combustion motors to produce mechanical power to vehicles, or electrical power through a generator connected to the motor, or just heat in gas burners. The liquid phase can be used as fertilizer. The AD process is a part of a closed carbon cycle, opposite to processing and utilizing fossil fuel. The climate footprint is favourable as the combustion converts CH_4 , which has (gives) a relatively high climate footprint, into CO_2 which has a considerably lower footprint, and H_2O , which has no footprint.

One of the results of my own research in model-based monitoring and control of biogas reactors ?, is a mathematical model adapted to a pilot reactor⁶ at Foss Biolab at Foss farm in Skien, Norway, using online measurements and laboratory analyses. The model may be referred to as the (modified) Hill model [Hill \(1983\)](#). Figure 41.14 shows the principal construction of the reactor.

41.7.2 Variables and parameters

Nomenclature of variables are shown in Table 41.7.

Abbreviations are defined in Table 41.9.

41.7.3 Overall block diagram

Figure 41.15 shows an overall block diagram of the mathetmatical model of the biogas reactor.

⁶Reactor feed is filtered cow manure.

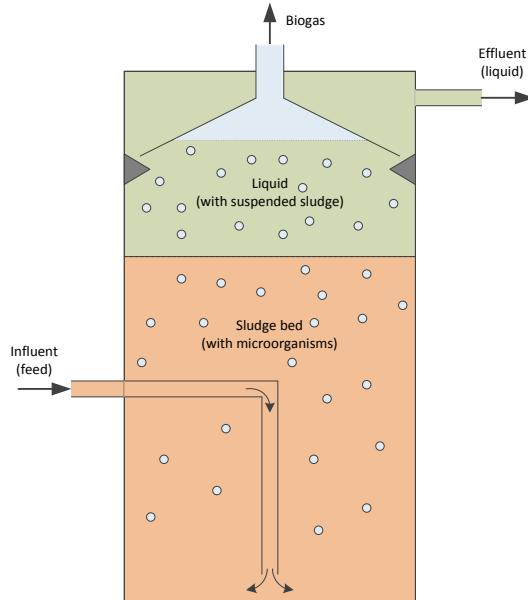


Figure 41.14: Principal construction of the biogas reactor.

41.7.4 Mathematical model

The mathematical model is as presented below. Figure 41.15 shows an overall block diagram of the mathematical model. Nomenclature of variables is given in Table 41.7. Parameters with assumed values are defined in Table 41.8. Abbreviations are defined in Table 41.9.

- Definition of the portion of the raw waste which can serve as substrate for the conversion done by microorganisms:

$$S_{\text{bvs,in}} = B_0 S_{\text{vs,in}} \quad (41.13)$$

- Definition of the portion of the biodegradable material which is initially in acid form:

$$S_{\text{vfa,in}} = A_f S_{\text{bvs,in}} \quad (41.14)$$

- Material balance of biodegradable volatile solids:

$$S_{\text{bvs}}' = (S_{\text{bvs,in}} - S_{\text{bvs}}) D - \mu k_1 X_{\text{acid}} \quad (41.15)$$

- Material balance of total VFA:

$$S_{\text{vfa}}' = (S_{\text{vfa,in}} - S_{\text{vfa}}) D + \mu k_2 X_{\text{acid}} - \mu_c k_3 X_{\text{meth}} \quad (41.16)$$

- Material balance of acidogens:

$$X_{\text{acid}}' = \left(\mu - K_d - \frac{D}{b} \right) X_{\text{acid}} \quad (41.17)$$

Table 41.7: Biogas reactor: Variables.

Symbol	Unit	Description
F_{feed}	$\frac{\text{L}}{\text{d}}$	Flow, or load rate
$D = \frac{F_{\text{feed}}}{V}$	d^{-1}	Dilution rate (normalized flow)
F_{meth}	$\frac{\text{L CH}_4}{\text{d}}$	Methane gas flow
$q_{\text{meth}} = \frac{F_{\text{meth}}}{V}$	$(\frac{\text{L CH}_4}{\text{d}})/\text{d}$	Volume-specific (normalized) methane gas flow
μ	d^{-1}	Reaction (growth) rate of acidogens
μ_c	d^{-1}	Reaction (growth) rate of methanogens
μ_m	d^{-1}	Maximum reaction rate for acidogens
μ_{mc}	d^{-1}	Maximum reaction rate for methanogens
$S_{\text{vs,in}}$	$\frac{\text{g VS}}{\text{L}}$	Concentration of VS in influent
$S_{\text{bvs,in}}$	$\frac{\text{g BVS}}{\text{L}}$	Concentration of BVS in influent
S_{bvs}	$\frac{\text{g BVS}}{\text{L}}$	Concentration of BVS in reactor
$S_{\text{vfa,in}}$	$\frac{\text{g VFA}}{\text{L}}$	Concentration of VFA in biodegradable part of influent
S_{vfa}	$\frac{\text{g VFA}}{\text{L}}$	Concentration of VFA acids in reactor
X_{acid}	$\frac{\text{g acidogens}}{\text{L}}$	Concentration of acidogens
X_{meth}	$\frac{\text{g methanogens}}{\text{L}}$	Concentration of methanogens
T_{reac}	$^{\circ}\text{C}$	Reactor temperature

- Material balance of methanogens:

$$X_{\text{meth}}' = \left(\mu_c - K_{dc} - \frac{D}{b} \right) X_{\text{meth}} \quad (41.18)$$

- Volume-specific (normalized) methane gas flow (production):

$$q_{\text{meth}} = \mu_c k_5 X_{\text{meth}} \quad (41.19)$$

Reaction rates, assuming Monod kinetics, are:

$$\mu = \mu_m \frac{S_{\text{bvs}}}{K_s + S_{\text{bvs}}} \quad (41.20)$$

$$\mu_c = \mu_{mc} \frac{S_{\text{vfa}}}{K_{sc} + S_{\text{vfa}}} \quad (41.21)$$

Table 41.8: Biogas reactor: Parameters.

Symbol	Value (default)	Unit	Description
B_0	0.25	$\frac{\text{g BVS}}{\text{g VS/L}}$	Biodegradability constant
A_f	0.69	$\frac{\text{g VFA}}{\text{g BVS/L}}$	Acidity constant
b	2.90	d/d	Retention time ratio
k_1	3.89	$\frac{\text{g BVS}}{\text{g acidogens}}$	Yield constant
k_2	1.76	$\frac{\text{g VFA}}{\text{g acidogens}}$	Yield constant
k_3	31.7	$\frac{\text{g VFA}}{\text{g methanogens}}$	Yield constant
k_5	26.3	$\frac{\text{L meth/L reac}}{\text{g methanogens/L reac}}$	Yield constant
K_d	0.02	d^{-1}	Specific death rate of acidogens
K_{dc}	0.02	d^{-1}	Specific death rate of methanogens
K_s	15.5	$\frac{\text{g BVS}}{\text{L}}$	Monod half-velocity constant for acidogens
K_{sc}	3	$\frac{\text{g VFA}}{\text{L}}$	Monod half-velocity constant for methanogens
V	250	L	Reactor volume of Foss pilot reactor

Table 41.9: Abbreviations.

VS	Volatile solids (“organic matter”)
BVS	Biodegradable volatile solids
VFA	Volatile fatty acid

where the maximum reaction rates are linear functions of the reactor temperature (the Hashimoto function):

$$\mu_m(T_{\text{reac}}) = \mu_{mc}(T_{\text{reac}}) = 0.013T_{\text{reac}} - 0.129 \quad (41.22)$$

$$(20^\circ\text{C} < T_{\text{reac}} < 60^\circ\text{C})$$

The reactor temperature may be kept at a specified temperature setpoint, e.g. 35°C (as for mesophilic conditions), with an automatic temperature control system.

Model of temperature

In the above model, the temperature, T_{reac} , is regarded as a model parameter with a set value. However, T_{reac} is actually a dynamic variable, and it can be modeled with an energy balance, cf. Sec. 41.4. The dynamic behaviour of T_{reac} can be simulated using that model.

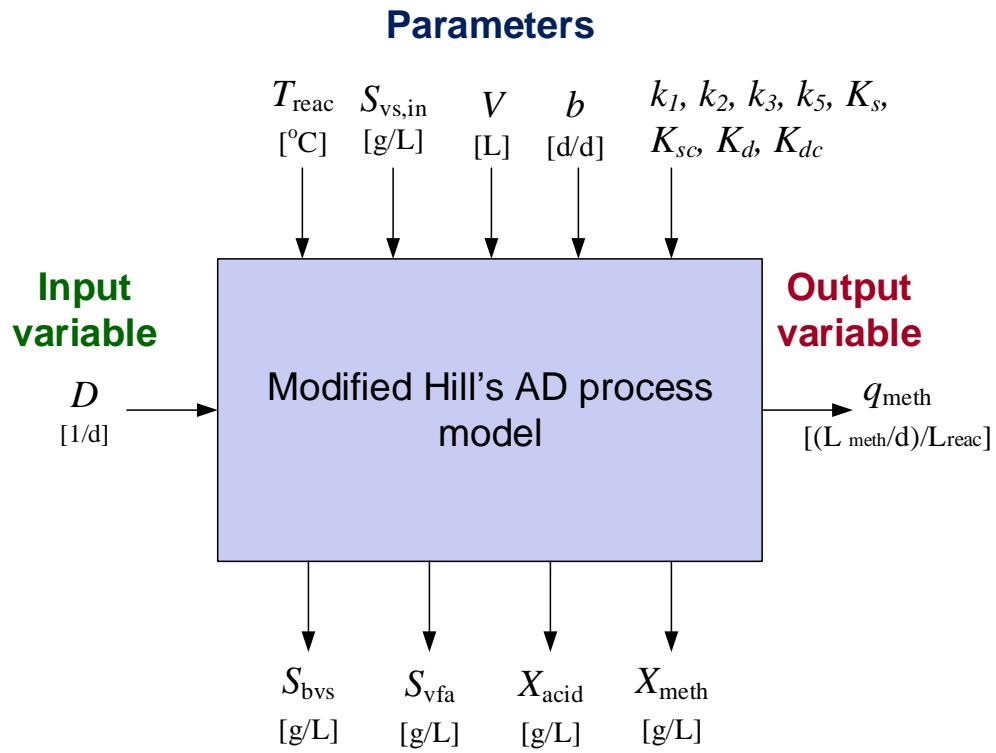


Figure 41.15: Overall block diagram of the mathematical model of the biogas reactor.

41.7.5 Operating point

In analysis of reactor dynamics and stability and in design of some types of state estimators and controllers it may be necessary to define a proper steady-state operating point. A steady-state operating point can be found from e.g. a simulation by reading off the value of the state variables at steady-state. One example of a steady-state operating point is given in Table 41.10. (On the Foss pilot reactor, $D = 0.18 \text{ d}^{-1}$ corresponds to $F_{\text{feed}} = VD = 250 \text{ L} \cdot 0.18 \text{ d}^{-1} = 45 \text{ L/d}$)

Table 41.10: Values of inputs and states in one example of a steady-state operation point.

$D = 0.18 \text{ d}^{-1}$
$T_{\text{reac}} = 35 \text{ °C}$
$S_{\text{vs,in}} = 30.2 \text{ g/L}$
$S_{\text{bvs}} = 5.2149 \text{ g/L}$
$S_{\text{vfa}} = 1.0093 \text{ g/L}$
$X_{\text{acid}} = 1.3166 \text{ g/L}$
$X_{\text{meth}} = 0.3637 \text{ g/L}$

41.8 Pendulum on cart

41.8.1 System description

Figure 41.16 shows the cart with the pendulum.

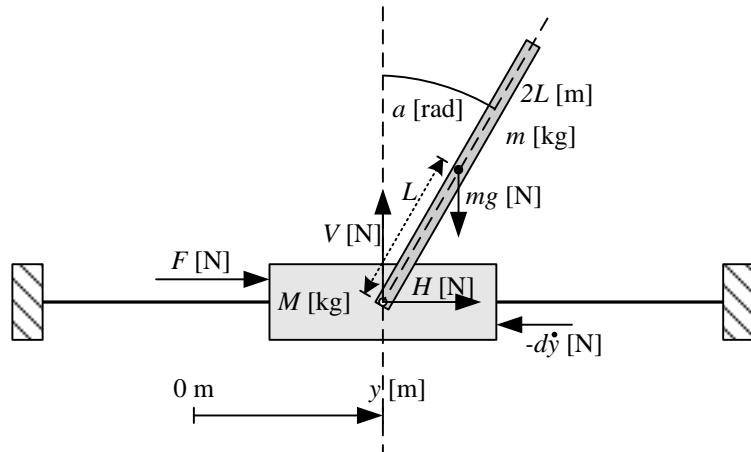


Figure 41.16: Cart with pendulum.

A motor attached to the cart acts on the cart with a force F . The mass of the motor is assumed included in the mass of the cart. This force is manipulated by the controller to stabilize the pendulum in an standing position (similar to a rocket) or in a hanging position (similar to a crane) at a specified position of the cart.

41.8.2 Variables and parameters

Variables and parameters and assumed parameter values are defined in Table 41.11.

Table 41.11: Pendulum on cart: Variables and parameters.

Symbol	Value (default)	Unit	Description
y	0	m	Cart position.
a	0 (standing) or π (hanging)	rad	Pendulum angle
L	0.5	m	Half length of pendulum (from fixed point on cart to center of gravity)
M	1	kg	Mass of cart
m	0.1	kg	Mass of pendulum
d	0	N/(m/s)	Damping coefficient
g	9.81	m/s ²	Gravity
F	0	N	Force applied to cart

41.8.3 Overall block diagram

Figure 41.17 shows an overall block diagram of the pendulum.

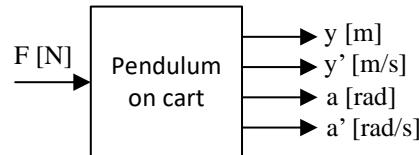


Figure 41.17: Block diagram of the pendulum.

41.8.4 Mathematical model

The mathematical model is based on the following principles:

1. Force balance applied to the cart:

$$My'' = F - H - dy' \quad (41.23)$$

2. Force balance (Newton's Second Law) applied to the horizontal movement of the center of gravity of the pendulum:

$$m(y + L \sin a)'' = H \quad (41.24)$$

The second order differentiation must be carried out, but the result of it is not shown here.

3. Force balance applied to the vertical movement of the center of gravity of the pendulum:

$$m(L \cos a)'' = V - mg \quad (41.25)$$

The second order differentiation must be carried out, but the result of the differentiation is not shown here.

4. Torque balance (the rotational version of Newton's Second Law) applied to the center of gravity of the pendulum:

$$Ia'' = VL \sin a - HL \cos a \quad (41.26)$$

In the above equations,

- I is the moment of inertia of the pendulum about its center of gravity. For the pendulum shown in Figure 1,

$$I = \frac{mL^2}{3} \quad (41.27)$$

- V and H are vertical and horizontal forces, respectively, in the pivot.

- d is a damping coefficient.

From Eq. (41.24)-(41.23), the internal forces V and H can be eliminated, resulting in two second order differential equations containing y'' and a'' . These differential equations are, however, not shown here.

Let us define the following state variables:

- $x_1 = y$ (cart horizontal position)
- $x_2 = y'$ (cart horizontal speed)
- $x_3 = a$ (pendulum angular position)
- $x_4 = a'$ (pendulum angular speed)

The two differential equations mentioned above can then be written as the following non-linear state space model:

$$x_1' = f_1 \quad (41.28)$$

$$x_2' = f_2 \quad (41.29)$$

$$x_3' = f_3 \quad (41.30)$$

$$x_4' = f_4 \quad (41.31)$$

where:

$$f_1 = x_2 \quad (41.32)$$

$$f_2 = \frac{-m^2 L^2 g \cos x_3 \sin x_3 + (I + mL^2) [mL(x_4)^2 \sin x_3 - dx_2]}{D_1} + \frac{(I + mL^2)}{D_1} u \quad (41.33)$$

$$f_3 = x_4 \quad (41.34)$$

$$f_4 = \frac{(m + M)(mgL \sin x_3) - mL \cos x_3 [mL(x_4)^2 \sin x_3 - dx_2]}{D_1} + \frac{-mL \cos x_3}{D_1} u \quad (41.35)$$

where:

$$D_1 = (I + mL^2)(m + M) - m^2 L^2 \cos^2 x_3 \quad (41.36)$$

Linearized model

A control system can be used to stabilize the pendulum at two alternative operating points, namely vertically up or vertically down. The matrices A and B of a linear model are derived for each of these operating points. Instead of linearizing using the general linearization formulas presented in Ch. 5.4.2, we will use a simpler approach where we assume small variations around the pertinent operating point.

Pendulum up:

Linearization of (41.28)-(41.31) about $x_3 = a = 0$ is based on the following assumptions:

$$\cos x_3 \approx 1 \quad (41.37)$$

$$\sin x_3 \approx x_3 \text{ [rad]} \quad (41.38)$$

$$(x_4)^2 = (\dot{a})^2 \approx 0 \quad (41.39)$$

Pendulum down:

Linearization of (41.28)-(41.31) about $x_3 = a = \pi$ is based on the following assumptions:

$$\cos x_3 \approx -1 \quad (41.40)$$

$$\sin x_3 \approx \pi - x_3 \text{ [rad]} \quad (41.41)$$

$$(x_4)^2 = (\dot{a})^2 \approx 0 \quad (41.42)$$

Using these assumptions, the linearized model becomes:

$$\begin{bmatrix} \Delta x_1' \\ \Delta x_2' \\ \Delta x_3' \\ \Delta x_4' \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{(I+mL^2)d}{D_2} & -\frac{m^2L^2g}{D_2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{K_{\cos}mLd}{D_2} & \frac{K_{\sin}(m+M)mgL}{D_2} & 0 \end{bmatrix}}_A \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \\ \Delta x_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{I+mL^2}{D_2} \\ 0 \\ -\frac{K_{\cos}mL}{D_2} \end{bmatrix}}_B [\Delta u] \quad (41.43)$$

where:

$$D_2 = (I + mL^2)(m + M) - m^2L^2 \quad (41.44)$$

The values of K_{\cos} and K_{\sin} in (41.43) are:

- Pendulum up:

$$K_{\cos} = 1 \quad (41.45)$$

$$K_{\sin} = 1 \quad (41.46)$$

- Pendulum down:

$$K_{\cos} = -1 \quad (41.47)$$

$$K_{\sin} = -1 \quad (41.48)$$

Chapter 42

Optimization

42.1 The optimization problem

42.1.1 Introduction

Optimization is about to find the best solution, for example:

- Which model parameter values makes a mathematical model represent a given real system most accurately?
- Which PI controller settings gives the best performance of a given control system?
- Which are the best future control signals – or control moves – by a model-predictive controller?
- Which are the best estimates calculated by a state estimator?
- Which is the feed rate to a biogas reactor that maximizes the biogas production?

Typically, optimization problems are stated as *minimization* problems:

Find the value of the optimization variable x that
minimizes the *objective function* $f(x)$,
taking into account any constraints on x or on some functions of x .
The solution is denoted the *optimal solution*, x_{opt} .

Figure 42.1 illustrates a minimization problem. Here, the optimization variable, x , is a vector of two elements. The problem is to calculate the combined values of $x(1)$ and $x(2)$ so that f is minimized.

Some well-known references for optimization theory are [Edgar et al. \(2001\)](#) and [Nocedal & Wright \(2006\)](#).

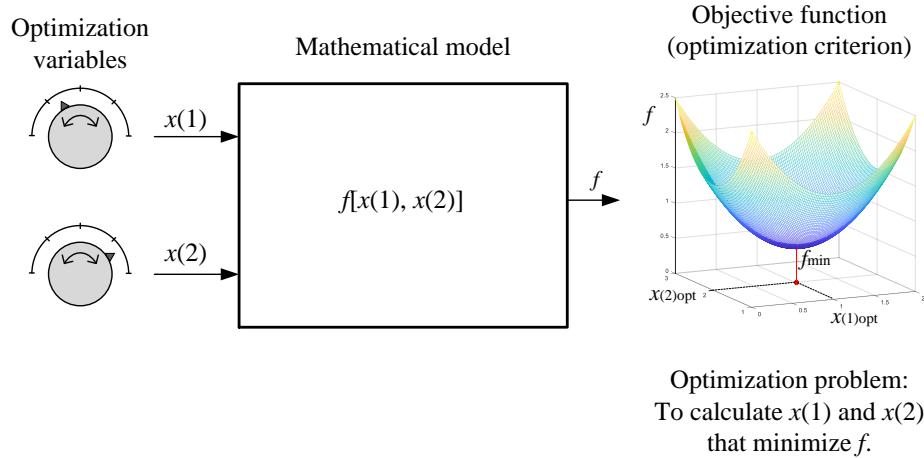


Figure 42.1: The optimization problem

42.1.2 Mathematical formulation of the optimization problem

There are many different ways of formulating mathematically optimization (minimization) problems. The following formulation is quite general. (It complies with the formulation required by the optimization function `fmincon` in Matlab.)

For a given mathematical model M , find the value of x that minimizes some objective function $f(x)$, that is,

$$\min_x f(x) \quad (42.1)$$

subject to (often denoted “s.t.”) *constraints*, which may be in the form of:

- *Inequality constraints:*

$$g(x) \geq 0 \quad (42.2)$$

where g is a linear or nonlinear function.

- *Equality constraints:*

$$h(x) = 0 \quad (42.3)$$

where h is a linear or nonlinear function of x .

- *Lower bounds and upper bounds:*

$$x_{\text{lb}} \leq x \leq x_{\text{ub}} \quad (42.4)$$

(42.2) and (42.3) define constraints on the relation between the optimization variables, while (42.4) define constraints or bounds on the values of the optimization variables. Equality constraints, (42.3), are not so common in optimization problems.

What if the inequality constraint is not a “larger than or equal” constraint, but a “smaller than or equal” constraint? No problem, since we can just multiply both sides of (42.2) by -1 to get the following “larger than or equal” constraint:

$$\underbrace{-g(x)}_{g_1(x)} \geq 0 \quad (42.5)$$

which has the same form as (42.2).

42.1.3 Feasibility region

The constraints define the *feasible region* of the optimization problem. The optimal solution can only be found within the feasible region. In many cases it the optimal solution is at the border of the feasibility region.

As an example, Figure 42.2 shows the feasible region based on the following constraints:

$$1 \leq x_1 \leq 3 \quad (42.6)$$

$$0.5 \leq x_2 \leq 1.5 \quad (42.7)$$

$$x_2 \leq \frac{x_1}{2} \quad (42.8)$$

which is equivalent to

$$\frac{x_1}{2} - x_2 \geq 0 \quad (42.9)$$

which is on the standard form of (42.2).

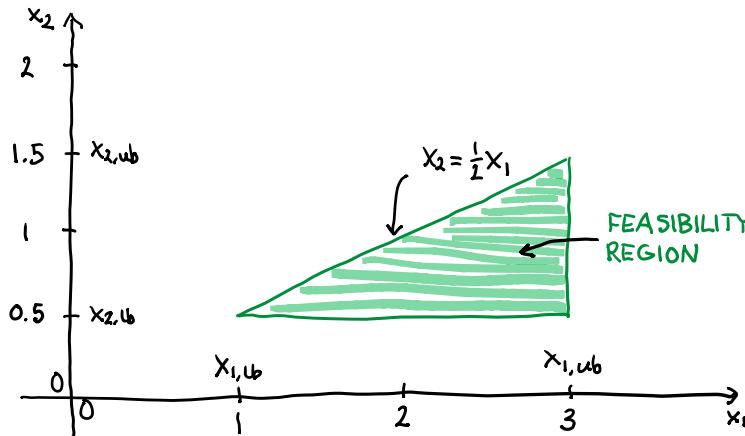


Figure 42.2: Illustration of the feasible region

42.1.4 Some characteristics of the optimal solution

In this section we assume that the objective function is some nonlinear – e.g. a square – function of the optimization variables. If the objective function is linear, the results described here do not apply.

Some typical characteristics of the objective function are:

- Whether we have found a local or global minimum.
- Whether the function is convex or concave.

Figure 42.3 illustrates these characteristics. It is assumed there that the objective function has one optimization variable, x .

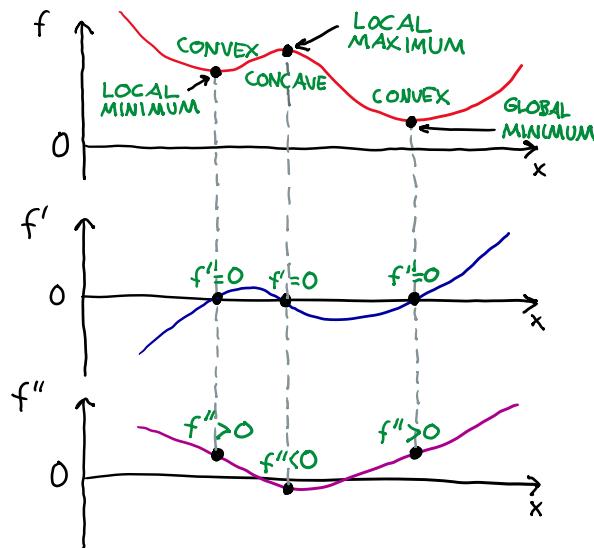


Figure 42.3: Some characteristics of the objective function

Comments to Figure 42.3:

- The objective function f has a global minimum, a local minimum, and a local maximum. Usually, it is the global minimum we are looking for.
- At minimum and maximum, the derivative is off is zero:

$$f'(x) = 0$$

- In the region of x where f is convex, i.e. where f curves upward, the second-derivative of f is positive:

$$f''(x) > 0$$

- At the minimum, f is convex.
- In the region of x where f is concave, i.e. where f curves downward, the second-derivative of f is negative:

$$f''(x) < 0$$

- At the maximum, f er f concave.

In Section 42.2.2 we will use the derivatives to solve an optimization problem analytically (“with hand calculation”).

42.1.5 What about maximization problems?

Suppose the optimization problem is to *maximize* a function, h :

$$\max_x h(x) \quad (42.10)$$

You can turn that maximization problem into a minimization problem as follows: Define a function f as $-h$:

$$f = -h \quad (42.11)$$

Figure 42.4 shows a plot of both h and f . Obviously, $\max h$ and $\min f$ occurs at the same x . Now, you can calculate $x_{\text{opt},\min}$ as the solution to following minimization problem:

$$\min_x [f(x) = -h(x)] \quad (42.12)$$

$x_{\text{opt},\min}$ also maximizes (42.10). Thus,

$$x_{\text{opt},\min} = x_{\text{opt},\max} \quad (42.13)$$

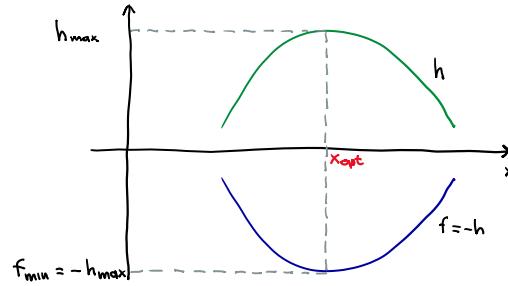


Figure 42.4: The same x_{opt} maximizes f_1 and minimizes $-f_1$.

So, if you can solve minimization problems, you can also solve maximization problems.

42.2 How to solve optimization problems

42.2.1 Introduction

There are several ways to solve optimization problems (minimization problems):

- **Analytical method:** Here we use the properties of the optimal solution as described in Section 42.1.4. We compute x_{opt} from $f'(x_{\text{opt}}) = 0$. And then we check that

$f''(x_{\text{opt}}) > 0$ to be sure that x_{opt} really gives the minimum of f , and not the maximum. The analytical method is applicable only for fairly simple and “well-formed” optimization problems. We will study an example of the use of the analytical method in Chapter 42.2.2.

- **Iterative method:** In such methods we start with a guess of the optimal solution, x_{guess} , and then we use an iterative method or algorithm to keep getting closer to the optimal solution, x_{opt} . The most well-known iterative methods are the gradient method (where in each iteration we go down “the steepest path” and finally we are in the “valley floor”) and Newton-Raphson’s method (which assumes that the objective function resembles a parabola, which has a minimum point which can be found with relatively few iterations). Figure 42.5 illustrates the principle of iterative methods, where it is assumed that f has only one optimization variable, x . At

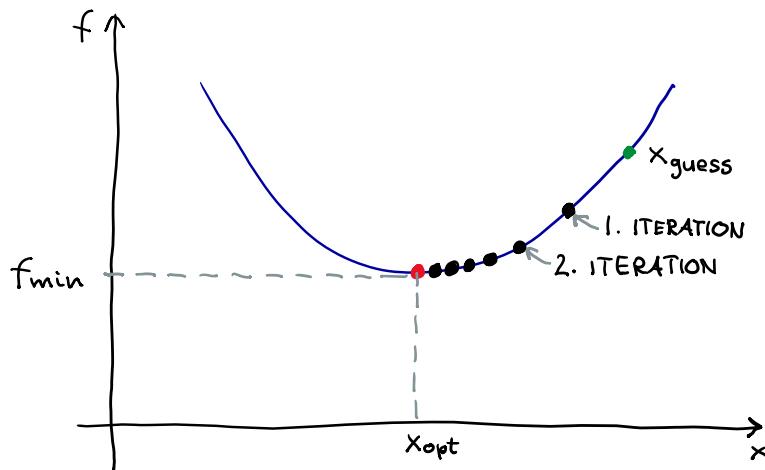


Figure 42.5: Principle of iterative methods of optimization

unfortunate choices of x_{guess} , such iterative methods may be deceived to end up at a local minimum (a local valley) or even a maximum (a mountain peak), cf. Figure 42.3. We will look into two iterative methods later in this chapter, namely the steepest descent method in Section 42.2.5, and the Newton-Raphson method in Section 42.2.6. However, in relatively simple optimization problems¹ we do well with the grid method:

- **The grid method, also denoted the brute force method:**
- The principle is to calculate the objective function, f , for all possible combinations of the optimization variables, x_1, x_2, \dots, x_n – within their value ranges, of course. The optimal solution is simply the combination of the optimization variables that gives the least value of f . Figure 42.6 illustrates the principle of the grid method, where it is assumed that f has only one optimization variable, x . As indicated in the figure, the

¹We can say that we have a relatively simple optimization problem if the number of optimization variables is 5 or less.

grid method may provide an optimal solution that deviates somewhat from the true optimal solution, but with good resolution of x (i.e. many x values as candidates for x_{opt}), the deviation can be insignificant. In the figure, there are 8 candidates for x_{opt} , but usually we choose a larger number, e.g. 100 for better resolution and thus a more accurate f_{\min} . We take a closer look at the grid method in Chapter 42.2.²

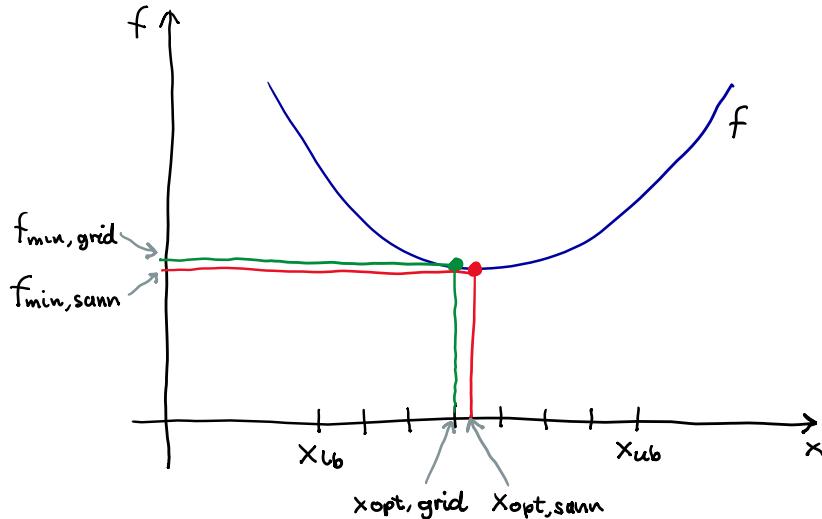


Figure 42.6: The principle of the grid method for optimization

The grid method has several good qualities:

- It is easy to implement in a program.
- It finds the global minimum, or rather: approximately the global minimum, without being fooled to stop at the local minimum.
- It can be applied to all kinds of optimization problems.
- It requires no advanced knowledge of optimization theory.

And some less good ones:

- Limited accuracy of the optimal solution.
- A large computational burden if the number of optimization variables are large and/or the objective function is computationally demanding. Imagine 5 optimization

²Python has a built-in optimization function based on the grid method (brute force method): `scipy.optimize.brute()`. You may be able to save some programming time using this feature compared to programming the grid method from scratch, but hardly much. I have experienced that the execution time is almost the same. A good feature of `scipy.optimize.brute()` is that it can be configured so that an accurate iterative optimization is started automatically based on the optimal solution found with the grid method.

variables with 100 grid points each. The total number of grid points (combinations of the optimization variables) with each function call will then be $100^5 = 10^{10}$, which is a large number. The program may then run for a long time.

42.2.2 Analytical solution

If the optimization problem is simple enough, we can calculate the optimal solution x_{opt} analytically, i.e. with “hand calculation”, based on the properties associated with the derivatives of the objective function f , cf. Section 42.1.4. The procedure below can be extended to optimization problems with more than one optimization variable, but I am content here to assume only one optimization variable, x .

1. Calculate a candidate for x_{opt} by solving the equation for x :

$$f'(x) = 0 \quad (42.14)$$

Let us call the candidate $x_{\text{opt,cand}}$.

2. Check that

$$f''(x_{\text{opt,cand}}) > 0 \quad (42.15)$$

to be sure that $x_{\text{opt,cand}}$ really gives the minimum, and not the maximum, of f . If $f''(x_{\text{opt,cand}}) < 0$, you have found the maximum! (If it turns out that $f''(x_{\text{opt,cand}}) = 0$, you have actually found a so-called saddle point, which is neither a maximum nor a minimum.)

Let us look at a simple example, which demonstrates the analytical method.

Example 42.1 Analytical solution of optimization problem

Given the optimization problem

$$\min_x f(x)$$

where

$$f(x) = (x - 2)^2 + 10 \quad (42.16)$$

Figure 42.7 shows a plot of $f(x)$.

I am sure you can see, both from (42.2) and from Figure 42.2, that the solution is

$$x_{\text{opt}} = 2 \quad (42.17)$$

$$f_{\text{min}} = f(x_{\text{opt}}) = f(0) = 10 \quad (42.18)$$

But we have to calculate the solution too, so that we can try the method explained above:

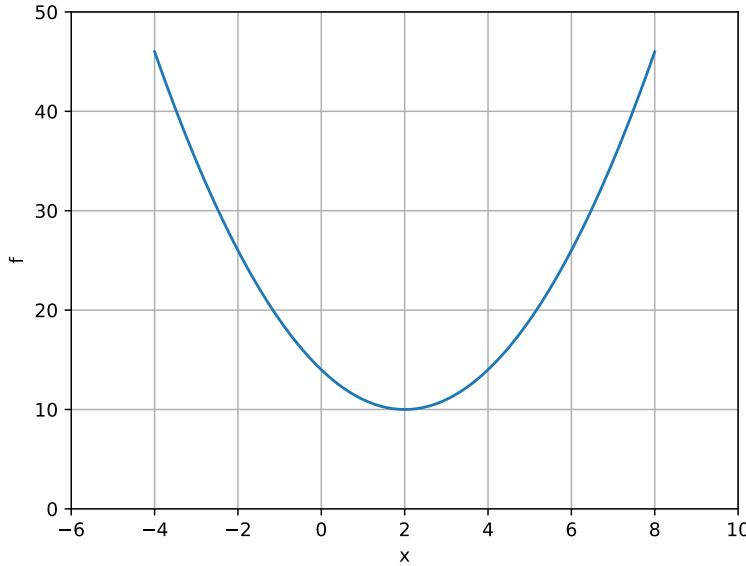


Figure 42.7: Plot of the objective function $f(x) = (x - 2)^2 + 10$

1. We set $f'(x)$ equal to zero:

$$f'(x) = 2(x - 2) \stackrel{!}{=} 0$$

where the exclamation mark means "set equal". We get

$$x = 2 = x_{\text{opt,cand}}$$

2. We check the sign of $f''(x_{\text{opt,cand}})$: We have

$$f''(x) = 2$$

which gives

$$f''(x_{\text{opt,cand}}) = f''(2) = 2$$

which is positive. Therefore, $x_{\text{opt,cand}} = 2$ is a minimum of f .

So, also the analytical result is $x_{\text{opt}} = 2$ and $f_{\min} = 10$.

[End of Example 42.1]

In Chapter 42.2 we need to solve the optimization problem in Example 42.2 with the grid method.

42.2.3 The grid or brute force method of optimization

Many optimization problems have relatively few optimization variables, e.g. 5 or less. In such cases, I will recommend that you consider using the grid method , which is a simple,

straight-forward method that can be implemented from scratch in any programming language.

The principle of the grid method is to calculate the objective function, f , for all possible combinations of the optimization variables, x_1, x_2, \dots, x_n – within their value ranges, of course. Each of these value ranges must be divided into a number of values of the relevant optimization variable, for example 100 values for each variable. The optimal solution is the combination of the optimization variables that gives the least value of f .

We shall now assume two optimization variables, x_1 and x_2 . I think you can see how the methodology can be simplified into cases with one optimization variable, and also how to extend to cases with more than two optimization variables. With two optimization variables, the value ranges can be presented as a grid or grid, see Figure 42.2, where for simplicity I have assumed 10 values for both x_1 and x_2 . The grid then consists of $N_{x_1} \times N_{x_2} = 10 \times 10 = 100$ grid points.

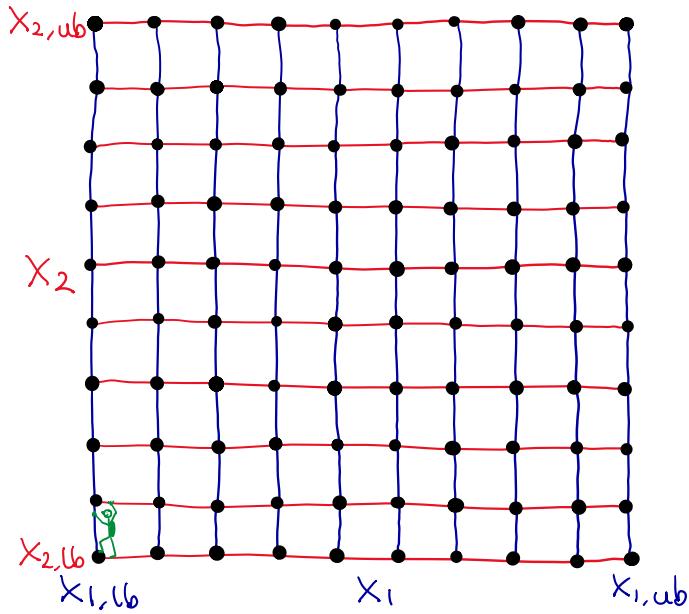


Figure 42.8: Grid with two optimization variables, x_1 and x_2 , with $10 \times 10 = 100$ grid points. $x_{1,\text{lb}}$ and $x_{1,\text{ub}}$ are lower bound and upper bound, respectively, of x_1 . The same applies for x_2 .

How do we implement the grid method in a program? With nested for-loops! The outer loop goes through all the x_1 values, and for each of the x_1 values, the inner loop goes through all the x_2 values. In the inner loop, the objective function f is calculated, which is thus calculated for all possible combinations of x_1 and x_2 . With e.g. 10 values for x_1 and 10 values for x_2 , f is calculated for all the 100 possible combinations. In Figure 42.2 this is illustrated with the climber starting at the bottom left and then working upwards in the first column (i.e. all x_2 values for the current x_1 value). Then she or he comes down in

some way³, and then work the way up the next column, etc., etc. – until all the columns are climbed through.

Figure 42.9 illustrates this. The optimal solution is the combination of x_1 and x_2 that gives the least value of f .

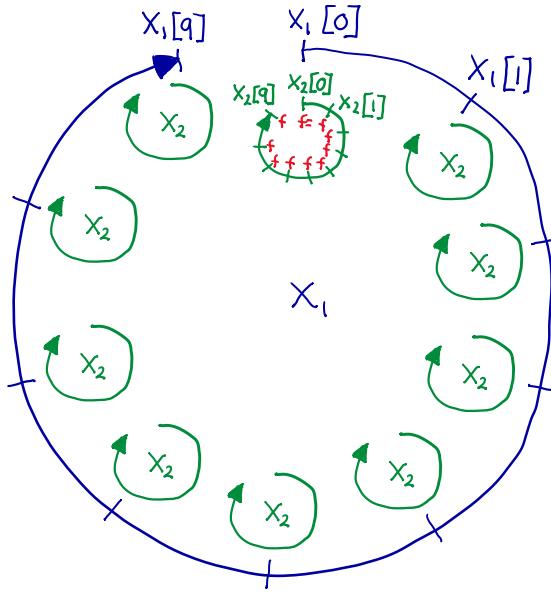


Figure 42.9: Implementation of the grid method with nested front loops. The outer loop goes through all x_1 values, and for each x_1 value, the inner loop goes through all the x_2 values and calculates the objective function f .

So we compute the objective function f for all combinations of x_1 and x_2 . But how do we find the combination of x_1 and x_2 that gives the least value of f ? Two alternative ways are:

- Check-and-replace method:** We start by giving f , which is to be minimized, value infinity (∞), which is the worst possible value f can have since f should be minimized. For each new combination of x_1 and x_2 , f is checked against the value of f we have so far. If the new value of f is less than the previous value of f , we replace the previous value with the new one, and at the same time we store x_1 and x_2 , which are thus associated with the new value of f . When all the combinations are tried, we are left with f_{\min} and $x_{1,\text{opt}}$ and $x_{2,\text{opt}}$.
- Search-method:** For each combination of x_1 and x_2 , both x_1 and x_2 are computed and stored in a 2D array, or an nD array if it is n optimization variables. Then we can search for the smallest value of f in the array. In this context, it may be useful to convert the array to a list with Python's built-in `list()` function, and then use the index method of the list object combined with Python's built-in `min()` function.

³Maybe there is a parachute in the bag?

Although there are functions in the numpy library for array objects that correspond to the `index()` method for list objects, in my opinion the `index()` method for list objects is much easier to use.

Of the two methods mentioned above, I tend to use the check-and-replace method, but I will soon show examples of both methods.

How to handle inequality constraints in the grid method

In Section 42.1.2 we saw that optimization problems can have inequality constraints. Inequality constraints are easy to take into account in the grid method: We punish violations of inequality by giving f value infinity, ∞ .

We shall now look at some examples with the grid method.

Example 42.2 Grid optimization with the check-and-replace method

In Example 42.2, we solved the following optimization problem analytically:

$$\min_x f(x)$$

where

$$f(x) = (x - 2)^2 + 10 \quad (42.19)$$

We will now solve this optimization problem with the grid method with the check-and-replace method implemented in Python.

Let us assume the following value constraint of x :

$$x_{lb} = -4 \leq x \leq x_{ub} = 8 \quad (42.20)$$

We can start with $N_x = 100$ values of x .

The program shown below implements the grid method with the check-and-replace method.

Program name: `prog_optim_grid_one_optimvar.py`.

```
# Import of package:  
import numpy as np  
  
# Definition of objective function:  
def fun_f(x):  
    f = (x - 2)**2 + 10  
    return f  
  
# Initialization:  
x_lb = -4  
x_ub = 8  
N_x = 100
```

```
x_array = np.linspace(x_lb, x_ub, N_x)

f_min = np.inf
x_opt = 0

# Grid method:
for x in x_array:
    f = fun_f(x)
    if (f < f_min): # Improvement of solution
        f_min = f
        x_opt = x

# Presenting results:
print('Optimal solution:')
print(f'x_opt = {x_opt:.8f}')
print(f'f_min = {f_min:.8f}')
print(Check: x_opt used in f():')
print(f'f(x_opt) = {fun_f(x_opt):.8f}')
```

Comments to the program:

- The code import numpy as np imports the package numpy. We need some of the features of numpy in the program.
- The code def fun_f (x): is the beginning of the definition of the function that calculates the objective function.
- Under # Initialization, the array x_array is defined which consists of all the 100 x values for which the objective function is to be calculated.
- The code f_min = np.inf defines the variable f_min, which must have the least value of the objective function f at any time, and give it the initial value infinite, ∞ . When the program is running, hopefully f_min will get a smaller value.
- The code x_opt = 0 gives the variable that should have finally obtained the optimal x value, starting value 0, but strictly speaking we could have dropped this code line.
- The code for x in x_array: is the start of the loop that goes through all the 100 x values in x_array. For each x value, the objective function is calculated.
- The code if (f < f_min), which is inside the for loop and is therefore run for each of the 100 x values, checks if f has a smaller value than the present smallest f value, which is f_min. If so, the code line f_min = f ensures that f_min is updated with the new, smaller f value. In addition, the x_opt value is given to the corresponding x value (thus giving the new, smaller f value).
- When the pre-loop has been run the 100 times, f_min has the smallest value of f , and x_opt has the corresponding optimal value of x .
- Then follow some call of the print () function to get the value of the optimal solution, i.e. x_opt and f_min, written to the console.

- The code print ($f'f(x_{\text{opt}}) = \{\text{fun_f}(x_{\text{opt}}):. 8f\} '$) is for control purposes only. I call the objective function, `fun_f`, with `x_opt` as the input argument, i.e. compute $f(x_{\text{opt}})$, which should get a value equal to f_{min} as calculated by the grid method. If these two values are different, the program is not working properly, and we need to start troubleshooting. The formatting code `\{\text{fun_f}(x_{\text{opt}}): 8f\}` is f-string formatting, cf. Chapter 42.2.

The result of the program run is shown in the box below.

```
Optimal solution:  
x_opt = 1.93939394  
f_min = 10.00367309  
Check: x_opt used in f():  
f(x_opt) = 10.003673
```

We know from the analytical calculation of the solution that the exact answer is $x_{\text{opt}} = 2$ and $f_{\text{min}} = 10$. The relative error in f_{min} is

$$\frac{10.00367309 - 10}{10} \cdot 100 = 0.0367309 \%$$

The grid method solution is somewhat inaccurate, as we generally have to expect it to be. Only when we have such luck with our definition of `x_array` that one of the x values is identical to the true x_{opt} , the grid solution is identical to the true optimal solution. In this example, none of the x values have a value of exactly 2. But what if we had chosen eg. $N_x = 101$?⁴

Actually, there are *two* x values that give $f = 10.003673$, namely $x = 1.93939394$ and $x = 2.06060606$. The if expression in our program will give us the first of these x values.

Figure 42.10 shows a plot of f as a function of the 100 grid values for x (but I do not show the program code for the plot here).

Above we used $N_x = 100$. The result with $N_x = 1000$ is:

```
Optimal solution:  
x_opt = 1.99399399  
f_min = 10.00003607
```

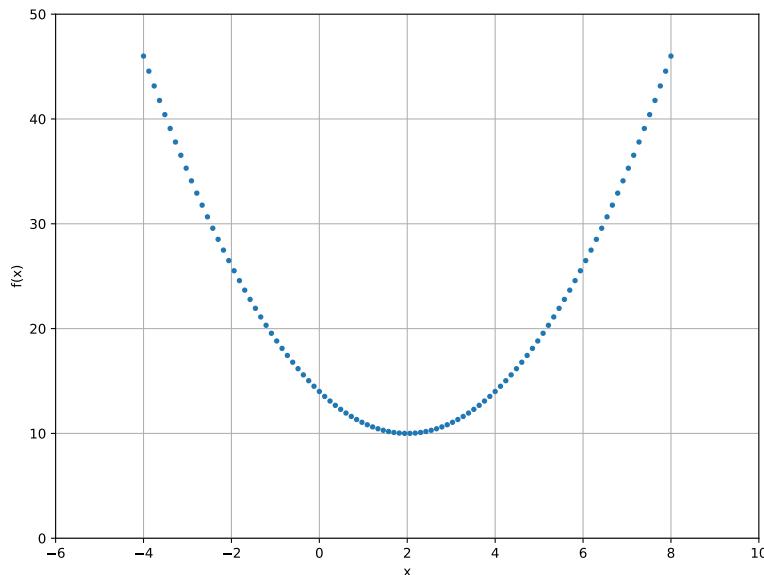
What is now the relative error in f_{min} ? It is 0.003607% which is approx. 1/10 of the error with $N_x = 100$, which is a pretty big improvement, but there are also many more calculations, of course.

[End of Example 42.2]

The following example shows how the optimization problem solved by the check-and-replace method with in Example 42.2 can be solved with the search method.

Example 42.3 Grid optimization with the search method

⁴Then one of the x values would have a value of exactly 2.

Figure 42.10: Plot of f as a function of the 100 grid values for x

The optimization problem is already defined in Example 42.2. The program shown below implements the search method. (To save space, I didn't post comments in the code itself.)

Program name: prog_optim_grid_array.py.

```
import numpy as np

def fun_f(x):
    f = (x - 2)**2 + 10
    return f

x_lb = -4
x_ub = 8
N_x = 100
x_array = np.linspace(x_lb, x_ub, N_x)

f_array = fun_f(x_array)
f_list = list(f_array)
f_min = min(f_list)
index_optim = f_list.index(f_min)
x_opt = x_array[index_optim]

print('Optimal solution:')
print(f'x_opt = {x_opt:.8f}')
print(f'f_min = {f_min:.8f}')
```

Comments on the most interesting parts of the program:

- The code `f_array = fun_f(x_array)` calculates the object function f for all elements of the `x_array`. The array `f_array` contains all the values of f . (This is a vectorized calculation of `f_array`.)
- The code `f_list = list(f_array)` converts the `f_array` array to list `f_list` with Python's built-in `list()` function. (Strictly speaking, there is a copy of `f_array` that is converted since `f_array` exists even after the conversion.)
- The code `f_min = min(f_list)` uses Python's built-in `min()` function to find the smallest of the values of the elements in `f_list`, and the variable `f_min` is assigned to these values. (Here also `f_min = min(f_array)` would work.)
- The code `index_optim = f_list.index(f_min)` finds the index of the element in `f_list` that has element value `f_min`, and assigns the `index_optim` variable to this index value. Actually, there are two values of x that give $f = 10.003673$, namely $x = 1.93939394$ and $x = 2.06060606$. The code `index_optim = f_list.index(f_min)` returns the first of these values.
- The code `x_opt = x_array[index_optim]` finds the element of `x_array` that has an index equal to `index_optim`, and assigns `x_opt` this element value.
- The `print()` expressions print the results of the console with f-string formatted print.

The result of the program run is shown in the box below.

```
Optimal solution:  
x_opt = 1.93939394  
f_min = 10.00367309
```

The result is the same as in the Example 42.2.

[End of Example 42.3]

So far, we have looked at examples where there is only one optimization variable and no inequality constraint. Now let's go a little further and look at three more advanced examples:

- Example 42.4 with two optimization variables *without* inequality constraint on the optimization variables.
- Example 42.5 with two optimization variables *with* inequality constraint on the optimization variables.
- Example 42.6 with so-called *linear optimization* (maximization) where the object function is linear and there is more than one inequality constraint.

Following these examples, I believe you are well prepared for solving various types of optimization problems.

Example 42.4 Grid optimization with two optimization variables – without limiting inequality

The optimization problem is:

$$\min_x f(x) \quad (42.21)$$

where

$$f(x) = (x_1 - 1)^2 + (x_2 - 2)^2 + 0.5 \quad (42.22)$$

From (42.21) we see immediately that the optimal solution is:

$$f_{\min} = 0.5 \quad (42.23)$$

$$x_{1,\text{opt}} = 1 \quad (42.24)$$

$$x_{2,\text{opt}} = 2 \quad (42.25)$$

Figure 42.2 shows a plot of f , where the optimal solution is plotted.⁵

But let's pretend that we do not know the solution. We will find it with the grid method with the check-and-replace method programmed in Python. The program is shown in the box below.

Program name: prog_optim_x1_x2_grid_no_ineq_constr.py.

```
# Import packages:
import numpy as np
import matplotlib as plt

# Definition of objective function:
def fun_f(x1, x2):
    f = (x1 - 1)**2 + (x2 - 2)**2 + 0.5
    return f

# Initialization:
x1_lb = 0
x1_ub = 2
N_x1 = 100
x1_array = np.linspace(x1_lb, x1_ub, N_x1)

x2_lb = 1
x2_ub = 3
N_x2 = 100
x2_array = np.linspace(x2_lb, x2_ub, N_x2)

f_min = np.inf
```

⁵This plot is generated in MATLAB, but a similar plot can be generated in Python with the plot_surface() function in the Matplotlib package, cf. https://matplotlib.org/mpl_toolkits/mplot3d/tutorial.html.

```

x1_opt = 0
x2_opt = 0

# Grid method:
for x1 in x1_array:
    for x2 in x2_array:
        # Calculation of objective function:
        f = fun_f(x1, x2)
        # Improvement of solution:
        if (f < f_min):
            f_min = f
            x1_opt = x1
            x2_opt = x2

# Presentation of results:
print(f'f_min = {f_min:.8f}')
print(f'x1_opt = {x1_opt:.8f}')
print(f'x2_opt = {x2_opt:.8f}')

```

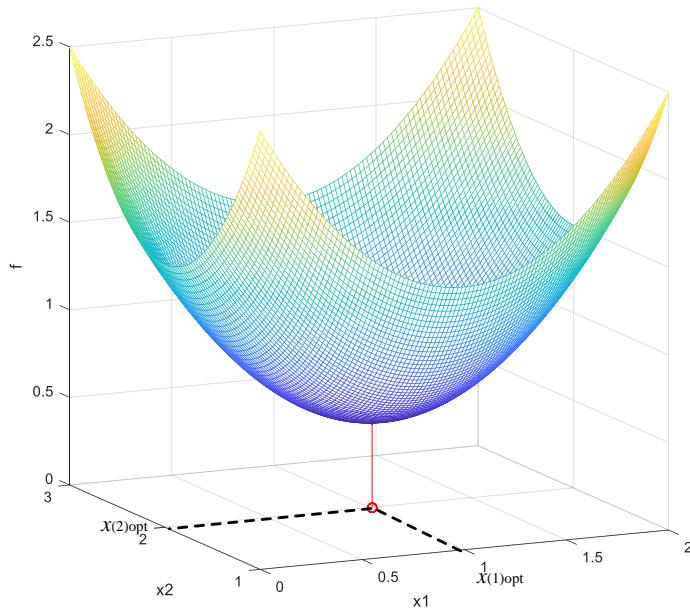


Figure 42.11: Plot the object function f with the optimal solution plotted

Comments on the program:

- The program is in principle as in Example 42.4, but it now contains two nested for-loops since we now have two optimization variables, x_1 and x_2 . The nested loops make sure that the objective function is calculated for all combinations of current values of x_1 and x_2 .

The result as shown in the console:

```
f_min = 0.50020406
x1_opt = 0.98989899
x2_opt = 1.98989899
```

which is quite similar to the exact solution given above. By choosing larger N_{x_1} and N_{x_2} , we will get even closer to the exact solution, but with more calculations (more throughput of the loops).

[End of Example 42.4]

The following example is an extension of Example 42.4 in that the optimization problem now includes an inequality constraint.

Example 42.5 *Grid optimization with two optimization variables – with inequality limitation*

The optimization problem is:

$$\min_x f(x) \quad (42.26)$$

where

$$f(x) = (x_1 - 1)^2 + (x_2 - 2)^2 + 0.5 \quad (42.27)$$

with the following value constraints on the optimization variables:

$$0 \leq x_1 \leq 2 \quad (42.28)$$

$$1 \leq x_2 \leq 3 \quad (42.29)$$

and the following inequality constraint on the optimization variables:

$$x_2 - x_1 - 1.5 \geq 0 \quad (42.30)$$

The feasible region of the optimization problem is now given by inequality (42.30) along with the intervals (42.28)–(42.29). Figure 42.12 shows the feasibility region. I have also drawn the optimal solution – in advance; the result is calculated in the program shown below. The optimal solution lies on a boundary of the feasibility region, as is often the case with optimization problems with inequality constraints.

The program is identical to the program in Example 42.4, except that inequality constraint code is included in the objective function definition. This new feature of the program is shown in the box below.

Program name: prog_optim_x1_x2_grid_with_ineq_constr.py.

```
def fun_f(x1, x2):
    f = (x1 - 1)**2 + (x2 - 2)**2 + 0.5
```

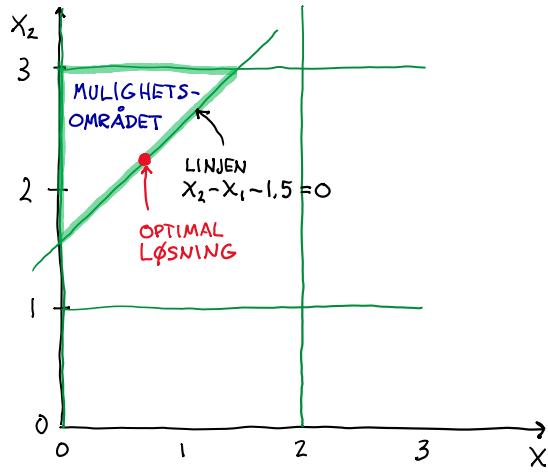


Figure 42.12: The feasibility region defined by the value constraints (42.28) – (42.29) and the inequality constraint (42.30).

```
# Inequality constraint:
if not(x2 - x1 - 1.5 >= 0): f = np.inf
return f
```

The result as shown in the console is:

```
f_min = 0.62753801
x1_opt = 0.74747475
x2_opt = 2.25252525
```

which is a completely different result than in Example 42.4. The inequality limitation has thus strongly influenced the solution.

Figure 42.13 shows a plot of f given by (42.27) in the feasibility region. The optimal solution lies on a boundary of the feasibility region.

[End of Example 42.5]

We now consider an example with a linear objective function with inequality constraints, which is typical within e.g. production optimization.

Example 42.6 Grid optimization with linear objective function with inequality constraints

The optimization problem is:

$$\max_{x,y} h(x) = \min_{x,y} [-h(x)] = \min_{x,y} [f(x)] \quad (42.31)$$

where

$$h(x) = 820x + 465y \quad (42.32)$$

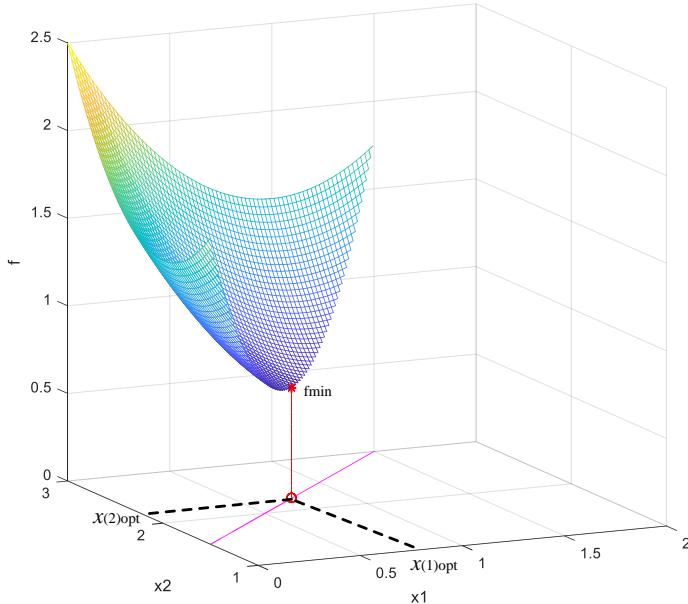


Figure 42.13: Plot of f given by (42.27) in the feasibility region

and

$$f(x) = -h(x) = -(820x + 465y) \quad (42.33)$$

Value constraints of the optimization variables are:

$$0 \leq x \leq 150 \quad (42.34)$$

$$60 \leq y \leq 400 \quad (42.35)$$

Inequality constraints are:

$$y \leq -\frac{x}{3} + 250 \quad (42.36)$$

$$y \leq -\frac{4x}{3} + 360 \quad (42.37)$$

The program below calculates the optimal solution, i.e. the values of x and y that maximizes (42.31), or minimizes (42.33), using the grid method.

Program name: prog_linoptim_x_y_with_plot.py.

```
# Import of packages:
import numpy as np

# Definition of objective function:
def fun_f(x, y):
    # Objective function (to be minimized):
    f = -(820*x + 465*y)
    # Inequality constraint:
    return f
```

```

if not(y <= -x/3 + 250):  f = np.inf
if not(y <= -4*x/3 + 360):  f = np.inf
return f

# Initialization:
x_lb = 0
x_ub = 150
N_x = 1000
x_array = np.linspace(x_lb, x_ub, N_x)

y_lb = 60
y_ub = 400
N_y = 1000
y_array = np.linspace(y_lb, y_ub, N_y)

f_min = np.inf
x_opt = 0
y_opt = 0

# Grid optim (check-and-replace):
for x in x_array:
    for y in y_array:
        # Calculation of objective function:
        f = fun_f(x, y)
        # Improvement of solution:
        if (f <= f_min):
            f_min = f
            x_opt = x
            y_opt = y

h_max = -f_min

# Presenting result:
print(f'h_max = {h_max:.0f}')
print(f'x_opt = {x_opt:.0f}')
print(f'y_opt = {y_opt:.0f}')

#Plotting: y_obj_array = (-820*x_array + h_max)/465
y_constr_1_array = -x_array/3 + 250
y_constr_2_array = -4*x_array/3 + 360

plt.close('all') # Closes all open figure windows
fig_width_inch = 24/2.54
fig_height_inch = 24/2.54
plt.figure(num='Linear optimization', figsize=(fig_width_inch, fig_height_inch))
plt.grid(which='both', color='grey')

plt.plot(x_opt, y_opt, 'ro',

```

```

x_array, y_obj_array, 'r',
x_array, y_constr_1_array, 'b',
x_array, y_constr_2_array, 'g')

plt.xlabel('x')
plt.ylabel('y')
plt.legend(labels=('y_opt_point', 'y_obj_curve', 'y_constr_1', 'y_constr_2'),
           loc='best', handlelength=2, fontsize=10)

plt.show()

plt.savefig('prog_optim_lin.pdf') #pdf-fil av plott

```

The result (the optimal solution) as shown in the console:

```

h_max = 197340
x_opt = 150
y_opt = 160

```

which is very close to the correct solution, which can be calculated analytically (not shown here):

```

h_max = 197400
x_opt = 150
y_opt = 160

```

Figure 42.14 shows:

- Inequality constraint curves:
 - (42.36), which gives the following inequality constraint curve (upper constraint):

$$y_{\text{constr1}} = -\frac{x}{3} + 250 \quad (42.38)$$
 - and (42.37) which gives the following inequality constraint curve (lower constraint):

$$y_{\text{constr2}} = -\frac{4x}{3} + 360 \quad (42.39)$$
- Curve of the objective function (42.32):

$$h_{\max} = 820x + 465y \quad (42.40)$$

which gives

$$y = (h_{\max} - 820x) / 465 \quad (42.41)$$

with $h = 197340$.

- The optimal solution ($x_{\text{opt}} = 150$, $y_{\text{opt}} = 160$), which is plotted as a point.
Note that the optimal solution lies on one of the constraint curves, which is typical of optimization problems in general.

[End of Example 42.6]

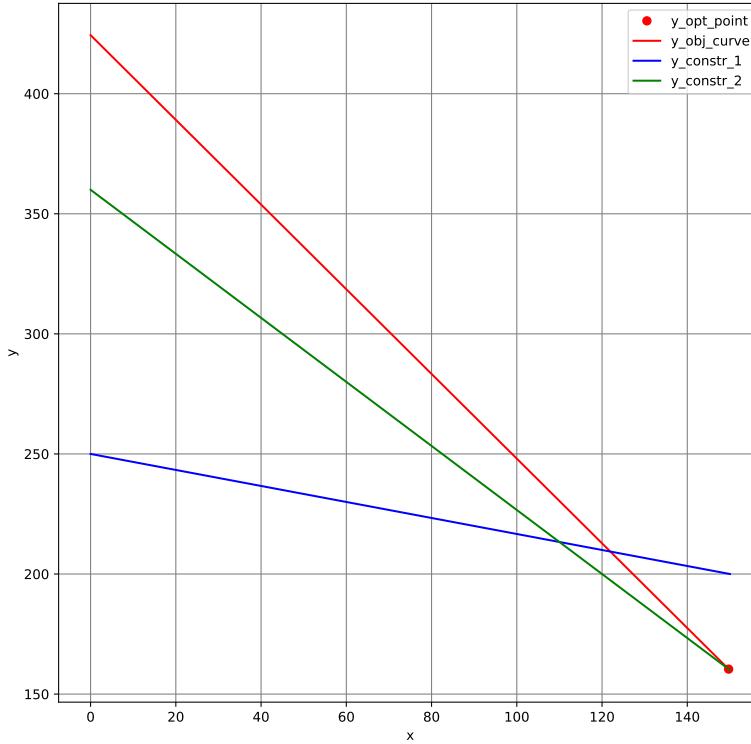


Figure 42.14: Plots of constraint functions (42.38) and (42.39) and the objective function (42.41).

42.2.4 Testing: Have you actually found the minimum?

How can you be sure that the program you created has really found the optimal solution? A simple test method test is to perturbate (disrupt) the optimal solution with small changes – both increase and decrease. The changes should be small, perhaps 1% of the optimal solution. Figure 42.15 illustrates this for the case of one optimization variable. If the object function f gets a greater value for each of the perturbations in the optimization variables, you may conclude that you have found the minimum.

Example 42.7 shows how a test program can be created.

Example 42.7 Testing the optimal solution

Below is a program for testing the optimal solution found in Example 42.5.

In the program, $kx1$ and $kx2$ are used as perturbation factors for $x1$ and $x2$, respectively. For example, $kx1 = 1.01$ means that $x1_{\text{opt}}$ is increased by 1%, while $kx1 = 0.99$ means that $x1_{\text{opt}}$ is reduced by 1%. The same applies to $x2$.

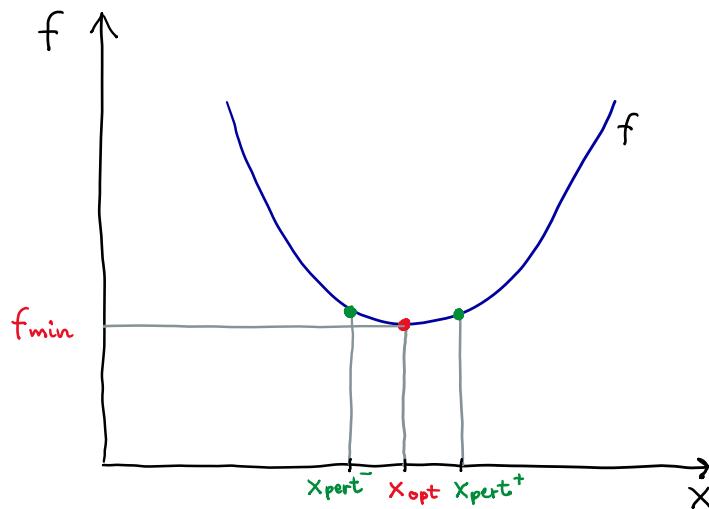


Figure 42.15: Perturbation around the optimal solution

Program name: prog_optim_test.py.

```
# Import av pakker:
import numpy as np

# Def of objective function:
def fun_f(x1, x2):
    f = (x1 - 1)**2 + (x2 - 2)**2 + 0.5
    # Ineq constraints:
    if not(x2 - x1 - 1.5 >= 0):
        f = np.inf

    return f

# Optimal solution:
x1_opt = 0.74747475
x2_opt = 2.25252525
f_min = fun_f(x1_opt, x2_opt)

print('-----')
print(f'x1_opt = {x1_opt:.8f}')
print(f'x2_opt = {x2_opt:.8f}')
print(f'f_min = {f_min:.8f}')

# Perturbation of optim variables:
kx1 = 1.01
kx2 = 1.00
```

```

x1_pert = x1_opt * kx1
x2_pert = x2_opt * kx2
f_pert = fun_f(x1_pert, x2_pert)
df = f_pert - f_min
print('-----')
print(f'x1_pert = x1_opt*{kx1:.2f}')
print(f'x2_pert = x2_opt*{kx2:.2f}')
print(f'f_pert - f_min = {df:.8f}')
print('-----')
print('Test ok?', (df > 0))

```

Below are the results for both positive and negative perturbation of x1_opt. I'm not showing the test result for and x2_opt here.

kx1 = 1.01 (positive perturbation of x1_opt):

```

-----
x1_opt = 0.74747475
x2_opt = 2.25252525
f_min = 0.62753800
-----
x1_pert = x1_opt*1.01
x2_pert = x2_opt*1.00
f_pert - f_min = inf
-----
Test ok? True

```

kx1 = 0.99 (negative perturbation of x1_opt):

```

-----
x1_opt = 0.74747475
x2_opt = 2.25252525
f_min = 0.62753800
-----
x1_pert = x1_opt*0.99
x2_pert = x2_opt*1.00
f_pert - f_min = 0.00383100
-----
Test ok? True

```

Thus, the tests for x1_opt are successful.

[End of Example 42.7]

42.2.5 Steepest decent method

42.2.5.1 Introduction

The steepest descent method is for solving *unconstrained* optimization problems. In the steepest descend method, the optimization variable, x , is moved so that the value of the objective function is the largest reduction possible. It is like trying to get to the bottom of a valley by always walking down as steeply as possible.

42.2.5.2 Scalar x

In the scalar case (x scalar), the steepest descend iteration is:

$$x_{k+1} = x_k + \Delta x_k \quad (42.42)$$

where the step is

$$\Delta x_k = -K f'(x_k) \quad (42.43)$$

where K is a factor which can be used to determine the size or length of the step, and $f'(x_k)$ is the derivative, or the gradient, of the objective function. In the standard steepest descend search, the numerical value of K is 1.

Δx_k is proportional to $f'(x_k)$. This is illustrated in Figure 42.16 where two different values of x are considered. This proportionality implies that the closer to the optimum (minimum of f), the smaller the step. This sounds like the minimum will be found. However, it may happen that Δx_k becomes too large, so that x_{k+1} will pass x_{opt} , causing the search to “jump” to other side of the “valley”. Consequently, there may be oscillations in the search. There are several methods to improve (optimize) the step size, e.g. conjugate gradient methods [Edgar et al. \(2001\)](#), and variants of Newton’s method. It is also possible to just manually set K to a value less than the default of 1. These are methods that apply also for the vectorial case (x a vector).

42.2.5.3 Vectorial x

In the vectorial case, the steepest descend iteration is:

$$x_{k+1} = x_k + \Delta x_k \quad (42.44)$$

where the search step is

$$\Delta x_k = -K \nabla f(x_k) \quad (42.45)$$

where $\nabla f(x_k)$ is the gradient of f , calculated at x_k . So, the step, Δx_k , is taken in the negative direction of the gradient, or – in other words – along the steepest descent. In the standard steepest descend search, the numerical value of K is 1, or, which gives the equivalent result, $K = I$, the identity matrix. There are several methods to improve (optimize) the step size and the direction, but we will not discuss these improvements here, except we will look at Newton’s method in Section 42.2.6.

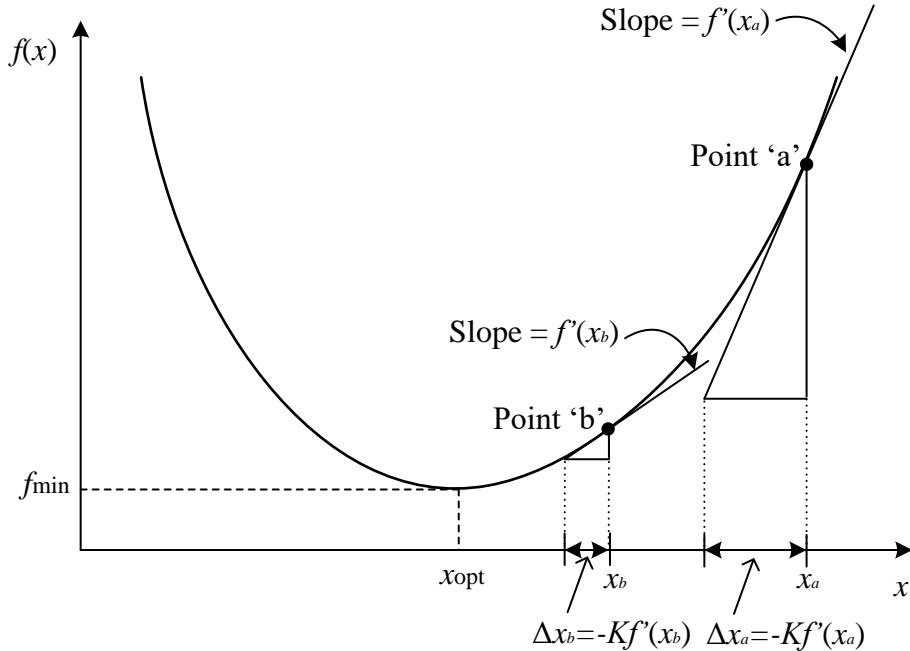


Figure 42.16: In the steepest descent method the search step size, Δx_k is proportional to the derivative of the objective function, $f'(x_k)$.

42.2.5.4 Numerical calculation of the derivative

The gradient in (42.43) may be calculated analytically or numerically. For numerical calculation, the center difference approximation is probably appropriate:

$$\nabla f(x_k) \approx \begin{bmatrix} \frac{f[x(1)_k+h, x(2)_k, \dots, x(n)_k] - f[x(1)_k-h, x(2)_k, \dots, x(n)_k]}{2h} \\ \frac{f[x(1)_k, x(2)_k+h, \dots, x(n)_k] - f[x(1)_k, x(2)_k-h, \dots, x(n)_k]}{2h} \\ \vdots \\ \frac{f[x(1)_k, x(2)_k, \dots, x(n)_k+h] - f[x(1)_k, x(2)_k, \dots, x(n)_k-h]}{2h} \end{bmatrix} \quad (42.46)$$

where the increment size h can be selected as a very small number, for example 10^{-4} (independently of the search step size).

To summarize:

Steepest descent search method for minimization:

1. Make a good guess, x_{guess} , of the optimal solution, and set

$$x_0 = x_{\text{guess}} \quad (42.47)$$

2. Iterate with

$$x_{k+1} = x_k + \Delta x_k \quad (42.48)$$

where the increment $\Delta x(x_k)$ is

$$\Delta x_k = -K \nabla f(x_k) \quad (42.49)$$

The standard value of K is 1, but it can be reduced to obtain a smoother, but slower, search.

Continue the iterations until an appropriate stop condition is satisfied, e.g.

$$|f(x_{k+1}) - f(x_k)| \leq df \quad (42.50)$$

When the stop condition is met,

$$x_{\text{opt}} = x_{k+1} \quad (42.51)$$

Example 42.8 Steepest descent search - scalar x

We will use Newton search to find the global optimal solution of the function plotted in Figure 42.17. The function is⁶:

$$f(x) = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \quad (42.52)$$

where

$$a_4 = 0.00232, a_3 = -0.111, a_2 = 1.80, a_1 = -11.6, a_0 = 34.4 \quad (42.53)$$

The guessed value is selected as

$$x_{\text{guess}} = 12$$

Below is a Matlab script that implements the search.

Figure 42.17 shows f , x_{guess} , and x_{opt} .

The results as shown in Matlab are:

```
x_guess = 12
x_opt = 18.7476
f_min = 4.7662
abs_df = 1.7186e-05
```

⁶This function is “home-made” for illustration.

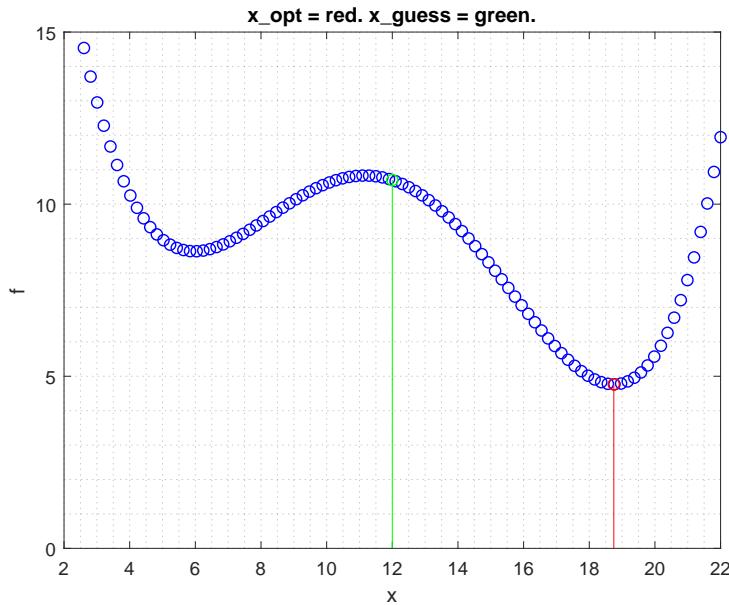


Figure 42.17: Example 42.8: Steepest descent search with $x_{\text{guess}} = 12$. x_{opt} is at global minimum.

```
[k,grad,dx,x,abs_df]
1.0000 -0.3162 0.3162 12.3162 0.1191
2.0000 -0.4369 0.4369 12.7530 0.2267
3.0000 -0.6000 0.6000 13.3530 0.4240
4.0000 -0.8095 0.8095 14.1625 0.7552
5.0000 -1.0456 1.0456 15.2081 1.2028
6.0000 -1.2275 1.2275 16.4356 1.5132
7.0000 -1.1841 1.1841 17.6196 1.2064
8.0000 -0.7877 0.7877 18.4073 0.4362
9.0000 -0.2850 0.2850 18.6923 0.0484
10.0000 -0.0497 0.0497 18.7420 0.0014
11.0000 -0.0056 0.0056 18.7476 0.0000
```

Note that $x_{\text{guess}} = 12$ is to the right of the maximum point, see Figure 42.17. Since f' is monotonically decreasing between the starting point of the search and the global minimum, the search will approach that minimum.

To demonstrate that the starting point (the guessed value) is crucial for the result, let us select

$$x_{\text{guess}} = 10$$

which is to the left of the maximum point, see Figure 42.17. Now, the result is:

```
x_guess = 10
x_opt = 5.9775
f_min = 8.6305
```

which is a *local* minimum, different from the global minimum. So, the starting point is

crucial for the result of the search.

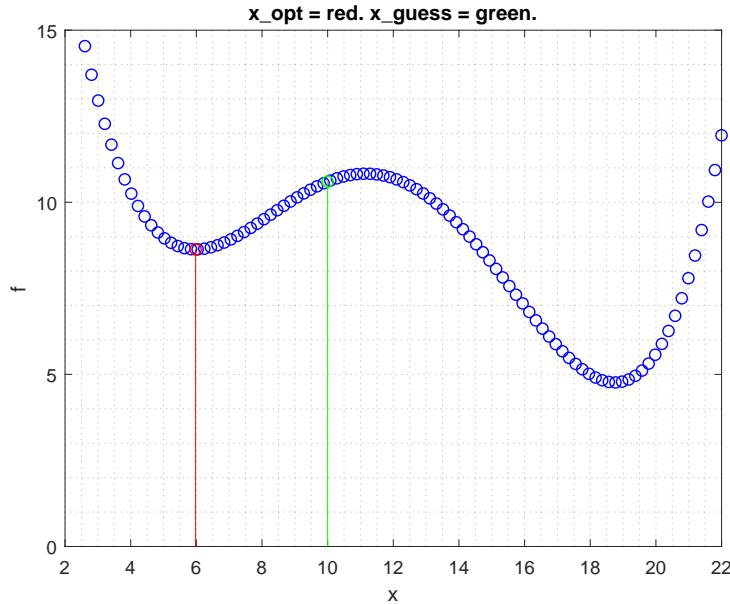


Figure 42.18: Example 42.8: Steepest descent search with $x_{\text{guess}} = 10$. x_{opt} is at a *local* minimum.

Script name: `matlab_script_steepest_descent_scalar.m`.

```
clear all, close all, format compact
a4=0.00232; a3=-0.111; a2=1.80; a1=-11.6; a0=34.4;
%Creating anonymous function for the objective function:
f_obj=@(x) a4*x.^4 + a3*x.^3 + a2*x.^2 + a1*x + a0;
x_guess=[10]
x_k=x_guess;
h=1e-4; %Step size in center difference method
N=1000;%Preset max number of iterations
abs_df_spec=1e-4;%Stopping criterion
for k=1:N-1
    gradient_num_k=(f_obj(x_k+h)-f_obj(x_k-h))/(2*h);
    dx_k=-gradient_num_k;
    x_kp1=x_k+dx_k;
    f_k=f_obj(x_k);
    f_kp1=f_obj(x_kp1);
    abs_df=abs(f_kp1-f_k);
    x_k=x_kp1;
    if abs_df < abs_df_spec
        break
    end %if
end %for loop
disp('Result:')
x_opt=x_kp1
```

```
f_min=f_obj(x_opt)
abs_df
```

[End of Example 42.8]

In the following example, x is vectorial.

Example 42.9 Steepest descent search - vectorial x

We will now make a steepest descent search to solve the optimization problem already presented in Example 42.4. For convenience, the problem formulation is repeated here:

$$\min_x f(x) \quad (42.54)$$

where

$$f(x) = [x(1) - 1]^2 + [x(2) - 2]^2 + 0.5 \quad (42.55)$$

The stop criterion is selected as

$$|f(x_{k+1}) - f(x_k)| \leq df = 10^{-4}$$

The gradient is:

$$\nabla f(x_k) = \begin{bmatrix} 2x(1)_k - 2 \\ 2x(2)_k - 4 \end{bmatrix}$$

Obviously, the optimal solution is

$$f_{\min} = 0.5$$

at

$$x_{\text{opt}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

We calculate the first iteration:

$$x_1 = x_0 - \nabla f(x_0) = x_0 - \begin{bmatrix} 2x(1)_0 - 2 \\ 2x(2)_0 - 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Checking the stop condition:

$$|f(x_1) - f(x_0)| = \left| f \left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} \right) - f \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \right| = |2.5 - 2.5| = 0 \leq 10^{-4} \text{ (Yes!)}$$

So, the stop condition is met after the first iteration since $f(x_1) = f(x_0)$. However, optimum is not found! One explanation is that the step size is too large, as a consequence of the gradient being relatively large.

To approach (much) closer to the optimum, the step size must be reduced. Although rigorous methods exist (as pointed out in the beginning of this section), we will here just observe the effect of a reduction of the step size. Let us take the iteration as

$$x_{k+1} = x_k - K \nabla f(x_k)$$

with

$$K = 0.1$$

The result, as calculated with the Matlab script shown below, is now:

$$f_{\min} = 0.5002$$

$$x_{\text{opt}} = \begin{bmatrix} 0.9908 \\ 1.9908 \end{bmatrix}$$

which is very close to the correct (analytical) result presented above.

The number of iteration until the stop condition is met, is 21.

A Matlab script implementing the above is shown below.

Script name: `matlab_script_steepest_descent_2_vars.m`.

```
%Creating anonymous function for the objective function:
f_obj=@(x) (x(1)-1)^2+(x(2)-2)^2+0.5;
x_guess=[0,1]';
%Guess x_k=x_guess;
N=10000;%Preset max number of iterations
abs_df=1e-4;%Stopping criterion
for k=1:N-1
    G_k=[2*(x_k(1)-1), 2*(x_k(2)-2)]'; %Gradient:
    K=0.1;
    dx_k=-K*G_k
    x_kp1=x_k+dx_k;
    f_k=f_obj(x_k);
    f_kp1=f_obj(x_kp1);
    df=f_kp1-f_k
    x_k=x_kp1;
    if abs(df) < abs_df
        break
    end %if
end %for loop
disp('Result:')
k
x_opt=x_kp1
f_min=f_obj(x_opt)
```

The result is as shown above.

[End of Example 42.9]

42.2.6 The Newton search method

42.2.6.1 Introduction

The Newton search method is an iterative method for solving *unconstrained* optimization problems. The method does not take into account any constraints on the form of (42.3) and (42.2). Optimization methods that do take constraints into account are often denoted Nonlinear Programming (NLP) methods. Section 42.2.8 introduces two professional NLP solvers, namely `fmincon` in Matlab and `sqp` in Octave. Concepts that are central in this method are central also in solvers for constrained problems (NLP solvers).

The Newton search method will here be introduced assuming only a scalar optimization variable, x . Thereafter, the Newton search method with a vectorial optimization variable, $x = [x(1), x(2), \dots, x(n)]^T$, is presented.

42.2.6.2 Scalar x

See Figure 42.3. At the minimum of f , f is flat, that is,

$$f'(x_{\text{opt}}) = 0 \quad (42.56)$$

Generally, Newton's method is an iterative method for solving equations on the form $F(x) = 0$. In the context of optimization, Newton's method is used to solve (42.56) for x . But since minimization points as well as maximizing points are characterized by $f' = 0$, as illustrated in Figure 42.3, we can not just take the solution of (42.56) as the minimizing solution. By solving $f'(x) = 0$ for x , we only have a *candidate* of the optimal x , say x_{cand} . To ensure that x_{cand} is actually the optimal (minimizing) solution, we must check that f is *convex* where $f'(x_{\text{cand}}) = 0$. f is convex at x_{cand} if the second order derivative of f is strictly positive at x_{cand} , that is, if

$$f''(x_{\text{cand}}) > 0 \quad (42.57)$$

x_{cand} is the results of a number of Newton iterations. Figure 42.19 illustrates one iteration.

We assume that an estimate, or a candidate, of the optimal x exists at iteration number k , namely x_k , and that both $f'(x_k)$ and $f''(x_k)$ are known (at x_k). An improved estimate can be obtained graphically as shown in Figure 42.19. We will now find the formula of x_{k+1} . From Figure 42.19 we find that the slope at x_k is

$$f''(x_k) = \frac{f'(x_k) - 0}{x_k - x_{k+1}} \quad (42.58)$$

Solving for x_{k+1} gives the Newton iteration expressed as a formula:

$$x_{k+1} = x_k - [f''(x_k)]^{-1} f'(x_k) \quad (42.59)$$

How many Newton iterations should be calculated? In a computer program, the Newton iteration can be implemented in a While Loop with the stop condition of the loop being, for example,

$$|f(x_{k+1}) - f(x_k)| \leq df \quad (42.60)$$

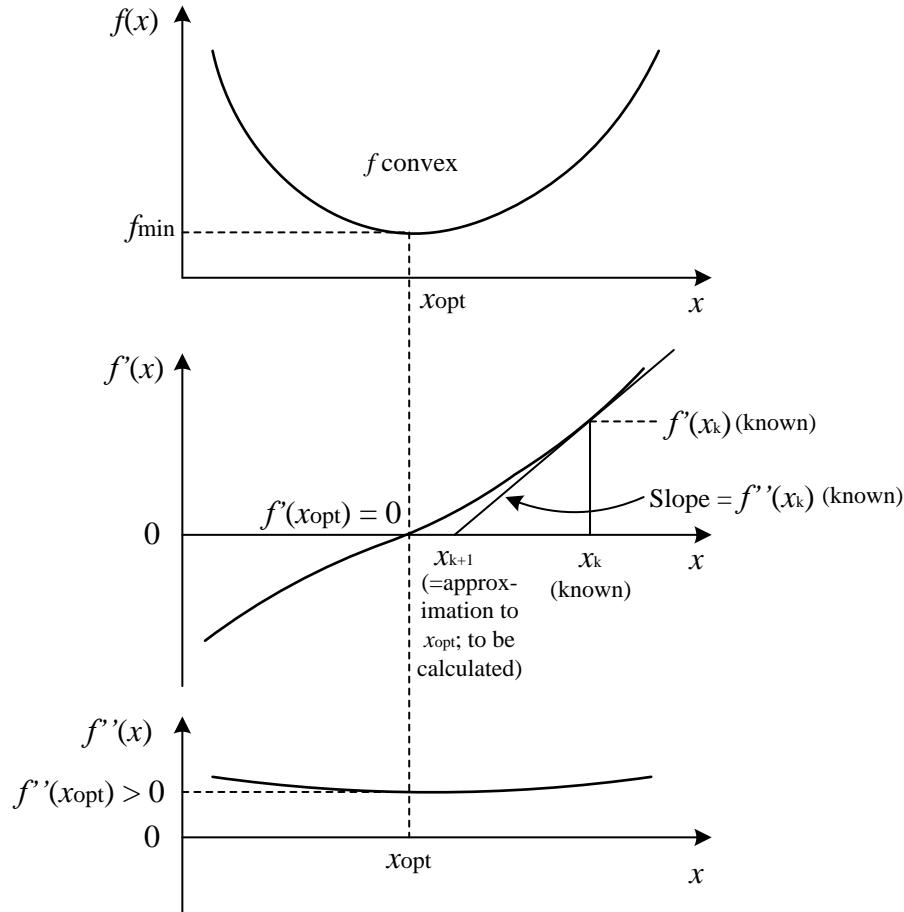


Figure 42.19: One iteration in Newton's method for searching for x_{opt} .

where df is a constant of an appropriate value. Alternatively, a For Loop can be used with a fixed, maximum number of iterations (Newton iterations) and with the possibility to break the loop if (42.60) is satisfied before the maximum number of iterations is reached.

Once we have found a candidate, $x_{\text{cand}} = x_{k+1}$, of the optimal solution, we must check that $f''(x_{\text{cand}}) > 0$. If this test is passed,

$$x_{\text{opt}} = x_{\text{cand}} \quad (42.61)$$

Note that the Newton's search method is a *local optimizer (minimizer)*. It is only if x_{guess} is sufficiently close to the global optimum, that the Newton search will arrive at the *global* optimum. As an example, see 42.3. If $x_{\text{guess}} = 4$, a Newton search will arrive at a local optimum. If $x_{\text{guess}} = 16$, the Newton search will arrive at the global optimum.

In the Newton's method, x will be moved towards a point where

$$f'(x_{\max}) = 0$$

However, this point may be a (local) maximum if the initial value of x in the search is where f is concave. Figure 42.20 illustrates this situation. The Newton iteration, (42.59),

will move x towards x_{\max} . So, it is crucial that the guessed value of x is where f is not concave, i.e., is convex. This situation can be avoided by using the steepest descent method instead of Newton's method where f is not concave, cf. Section 42.2.7.

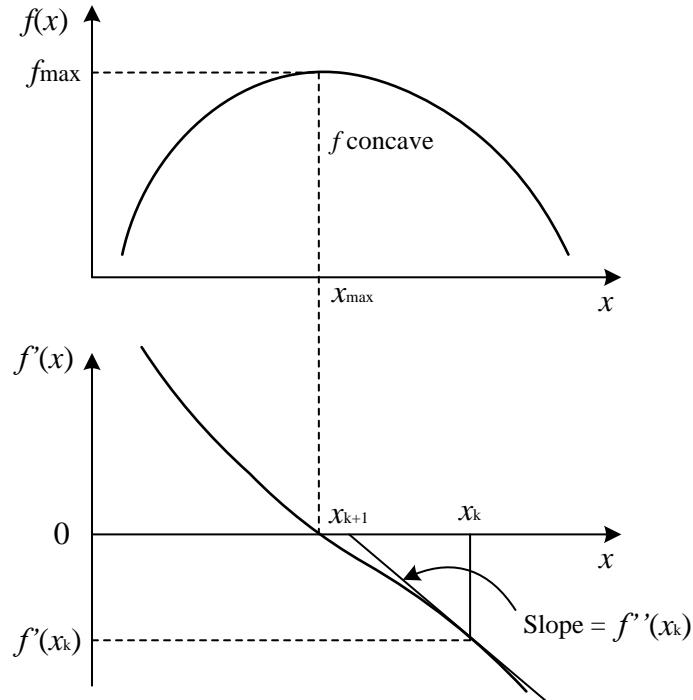


Figure 42.20: One iteration in Newton's method when f is concave.

42.2.6.3 An explanation of why the Newton search is fast

Let us consider the steepest descent algorithm for the scalar case, (42.42) – (42.43), which is repeated here for convenience:

$$x_{k+1} = x_k - Kf'(x_k) \quad (42.62)$$

where the factor K is included. K can be regarded as the step size. In the standard steepest descent method, $K = 1$ (fixed). Now, let's open for other values of K . (42.62) can be regarded as a nonlinear discrete-time difference equation or model with x as the state-variable. It is nonlinear because it can be assumed that the derivative, or gradient, $f'(x)$, is a nonlinear function of x . To analyze the dynamic properties of this model, we can consider the linearized model. Let us define dx as the deviation variable corresponding to x . Linearization of (42.62), which is based on a first order Taylor series of the nonlinear term, gives the following linear model:

$$dx_{k+1} = dx_k - Kf''(x_k)dx = [1 - Kf''(x_k)] dx_k \quad (42.63)$$

According to systems theory of discrete-time systems, the fastest dynamic response in dx is obtained with the next state, dx_{k+1} , is assumed zero, independent of the present state, dx_k .⁷

⁷An alternative design based on z -plane theory is as follows: The fastest dynamics of a discrete-time system is when the z -eigenvalue(s) of the system being zero, i.e. in the origin of the complex z -plane. The z -eigenvalue is $z = 1 - Kf''(x_k)$, which is set to 0.

This dynamics is denoted dead-beat response. So, dead-beat dynamics is obtained with

$$1 - K f''(x_k) = 0$$

which gives

$$K = [f''(x_k)]^{-1}$$

Inserting this K into (42.63), gives the “dead-beat steepest descent” algorithm:

$$x_{k+1} = x_k - [f''(x_k)]^{-1} f'(x_k)$$

which is the Newton search algorithm, (42.59)!

42.2.6.4 For a quadratic f , the optimum is found in one search iteration!

Assume that $f(x)$ is quadratic, say

$$f(x) = a_1 + a_2 x^2$$

This implies:

$$f'(x) = 2a_2 x$$

and

$$f''(x) = 2a_2$$

Inserting these functions in the Newton algorithm, gives:

$$x_{k+1} = x_k - [f''(x_k)]^{-1} f'(x_k) = x_k - [2a_2]^{-1} 2a_2 x_k = 0x_k$$

Thus, the Newton algorithm exhibits dead-beat dynamics, and this result is obtained directly, without any approximate analysis involving linearization. This implies that for quadratic functions, the search will arrive at the minimum in just one iteration, whatever is selected as the starting or guessed value of x of the search. This holds also for optimization problems where x is vectorial. Example 42.11 gives a demonstration.

42.2.6.5 Vectorial x

Assume x a vector:

$$x = \begin{bmatrix} x(1) \\ x(2) \\ \vdots \\ x(n) \end{bmatrix}$$

It can be shown Edgar et al. (2001) that the Newton iteration, (42.59), now takes the form

$$x_{k+1} = x_k - [\nabla^2 f(x_k)]^{-1} \nabla f(x_k) \quad (42.64)$$

In (42.64),

$$\nabla f(x_k) = \begin{bmatrix} \frac{\partial f(x_k)}{\partial x(1)} \\ \frac{\partial f(x_k)}{\partial x(2)} \\ \vdots \\ \frac{\partial f(x_k)}{\partial x(n)} \end{bmatrix} \quad (42.65)$$

which is the *gradient* of f with respect to x . In words, the gradient of f is a vector of first order partial derivatives of f . In (42.65), $\partial f(x_k)/\partial x(1)$ means the partial derivative of f with respect to $x(1)$, calculated at $x = x_k$.

Furthermore in (42.64),

$$\nabla^2 f(x_k) \equiv H(x_k) = \begin{bmatrix} \frac{\partial^2 f}{\partial x(1)^2} & \frac{\partial^2 f}{\partial x(1)\partial x(2)} & \cdots & \frac{\partial^2 f}{\partial x(1)\partial x(n)} \\ \frac{\partial^2 f}{\partial x(2)\partial x(1)} & \frac{\partial^2 f}{\partial x(2)^2} & \cdots & \frac{\partial^2 f}{\partial x(2)\partial x(n)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x(n)\partial x(1)} & \frac{\partial^2 f}{\partial x(n)\partial x(2)} & \cdots & \frac{\partial^2 f}{\partial x(n)^2} \end{bmatrix} \quad (42.66)$$

which is the *Hessian* (matrix) of f with respect to x . In words, the Hessian of f is a matrix of second order partial derivatives of f .

As for the scalar case discussed above, the Newton iteration can be implemented in a While Loop or For Loop (the latter with a preset maximum number of iterations) with the stop or break condition of the loop being

$$|f(x_{k+1}) - f(x_k)| \leq df \quad (42.67)$$

where df is a constant of an appropriate value.

To verify that x_{opt} minimizes and not maximizes f , we must check that f is convex at x_{opt} . f is convex if the Hessian is positive definite, which is ensured if all the eigenvalues of the Hessian are strictly positive [Edgar et al. \(2001\)](#). Hence, x_{opt} minimizes f if

$$\forall \text{eig } \nabla^2 f(x_{\text{cand}}) > 0 \quad (42.68)$$

(The symbol \forall means “for each”.)

42.2.6.6 Avoiding the inversion of the Hessian

The Newton iteration, 42.64, can be written as

$$x_{k+1} = x_k + \Delta x_k \quad (42.69)$$

where

$$\Delta x_k = - [\nabla^2 f(x_k)]^{-1} \nabla f(x_k) \quad (42.70)$$

Generally, the inverse of matrices should not be calculated because the inverse may be mathematically ill-conditioned, i.e. sensitive to numerical errors. If possible, you should instead express the unknown as the solution of an equivalent systems of linear equations, and solve for the unknown, which in our case is Δx_k , using robust algorithms or functions for solving systems of linear equations, for example using the “\” operator in Matlab. By premultiplying (42.70) by the $-\nabla^2 f(x_k)$ (minus one times the Hessian), we get the following system of linear equations with Δx_k as the unknown:

$$-\nabla^2 f(x_k) \Delta x_k = \nabla f(x_k) \quad (42.71)$$

In Matlab, we can solve (42.71) for $\Delta x(x_k)$ with code like

```
dx_k = -H_k\grad_k
```

which is numerically better than

```
dx_k = -inv(H_k)*grad_k
```

42.2.6.7 Numerical calculation of the gradient and the Hessian

In Example 42.11, both the gradient and the Hessian of f was calculated analytically. Alternatively, they can be calculated numerically from calculations of the objective function f only. The center difference approximation to the derivative, (42.46), may be appropriate.

To sum it up:

Newton's search method method for minimization:

1. Make a good guess, x_{guess} , of the optimal solution, and set

$$x_0 = x_{\text{guess}} \quad (42.72)$$

2. Iterate with

$$x_{k+1} = x_k + \Delta x_k$$

where the increment $\Delta x(x_k)$ may be calculated directly by

$$\Delta x_k = -[\nabla^2 f(x_k)]^{-1} \nabla f(x_k) \quad (42.73)$$

or, preferably, indirectly by solving the following system of linear equation for $\Delta x(x_k)$:

$$-\nabla^2 f(x_k) \Delta x_k = \nabla f(x_k) \quad (42.74)$$

Continue the iterations until an appropriate stop condition is satisfied, e.g.

$$|f(x_{k+1}) - f(x_k)| \leq df \quad (42.75)$$

The candidate of the optimal solution is then

$$x_{\text{cand}} = x_{k+1} \quad (42.76)$$

3. If f is convex at x_{cand} , the optimal solution has been found, that is,

$$x_{\text{opt}} = x_{\text{cand}} = x_{k+1} \quad (42.77)$$

To check for convexity: f is convex at x_{cand} if the Hessian $\nabla^2 f(x_{\text{cand}})$ is positive definite, that is, if

$$\forall \text{eig } \nabla^2 f(x_{\text{cand}}) > 0 \quad (42.78)$$

Example 42.10 Newton search - scalar x

We will use Newton search to find the global minimization solution of the objective function plotted in Figure 42.21. The function is given in Example 42.8, but is repeated here for convenience:

$$f(x) = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0 \quad (42.79)$$

where

$$a_4 = 0.00232, a_3 = -0.111, a_2 = 1.80, a_1 = -11.6, a_0 = 34.4 \quad (42.80)$$

The Matlab script below implements a Newton search to find the global optimal solution.

Let us at first try the guessed value as

$$x_{\text{guess}} = 12$$

where f is *concave*, see Figure 42.21. The figure also shows the result of the search. The result as shown in Matlab is:

$$f_{\max} = 10.8309$$

at

$$x_{\max} = 11.1624$$

So, the Newton search has – unfortunately – arrived where f is at maximum. This illustrates that Newton's method fails if the search starts where f is concave.

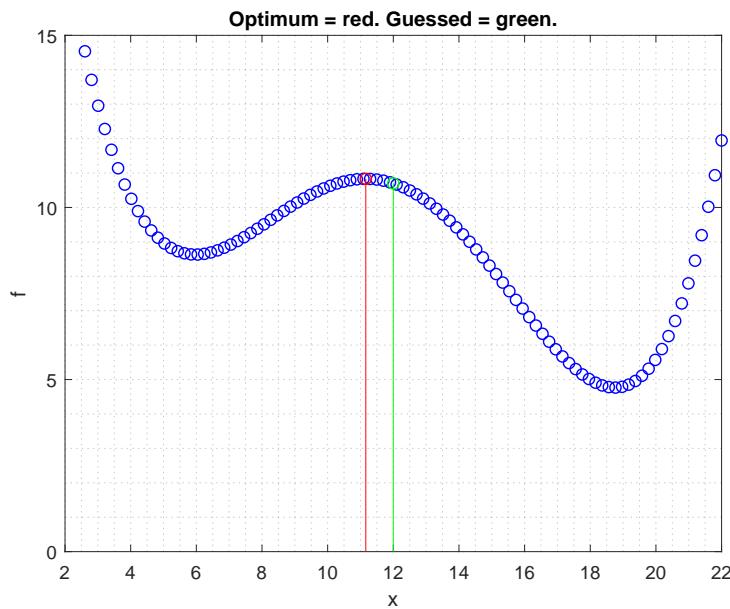


Figure 42.21: Example 42.10: Newton search with $x_{\text{guess}} = 12$. x_{opt} arrives (unfortunately) at f_{\max} .

We will now set

$$x_{\text{guess}} = 16$$

where f is *convex*, see Figure 42.22. The figure also shows the result of the search. The result as shown in Matlab is:

$$f_{\min} = 4.7662$$

at

$$x_{\text{opt}} = 18.7483$$

Now, the Newton search has – correctly – arrived at f_{\min} . This illustrates that Newton's method succeeds (only) if the search starts where f is convex.

Script name: `matlab_script_newton_search_scalar.m`.

```
clear all, close all, format compact
a4=0.00232; a3=-0.111; a2=1.80; a1=-11.6; a0=34.4;
```

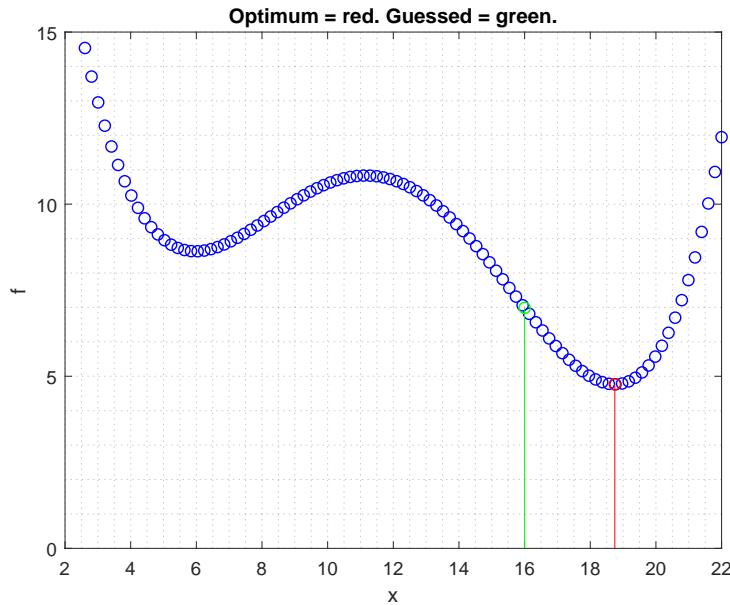


Figure 42.22: Example 42.10: Newton search with $x_{\text{guess}} = 16$. x_{opt} arrives at f_{\min} .

```
%Creating anonymous function for the objective function:
f_obj=@(x) a4*x.^4 + a3*x.^3 + a2*x.^2 + a1*x + a0;
%Anonymous function for gradient:
grad=@(x) 4*a4*x.^3 + 3*a3*x.^2 + 2*a2*x + a1;
%Anonymous function for Hessian:
hessian=@(x) 12*a4*x.^2 + 6*a3*x + 2*a2;
x_guess=12;
x_k=x_guess;
N=1000;%Preset max number of iterations
abs_df_spec=1e-4;%Stopping criterion
for k=1:N-1
    dx_k=-inv(hessian(x_k))*grad(x_k);
    %A numerically better alternative to calc dx_k:
    %dx_k=-hessian(x_k)\grad(x_k);
    x_kp1=x_k+dx_k;
    f_k=f_obj(x_k);
    f_kp1=f_obj(x_kp1);
    abs_df=abs(f_kp1-f_k);
    x_k=x_kp1;
    if abs_df < abs_df_spec
        break
    end %if
end %for loop

disp('Result:')
k
x_opt=x_kp1
```

```
f_min=f_obj(x_opt)
abs_df
```

[End of Example 42.10]

Example 42.11 Newton search - vectorial x

We will now make a Newton search to solve the optimization problem already presented in Section 42.2.3:

$$\min_x f(x) \quad (42.81)$$

where

$$f(x) = [x(1) - 1]^2 + [x(2) - 2]^2 + 0.5 \quad (42.82)$$

which is a quadratic objective function.

Let the stop criterion be

$$|f(x_{k+1}) - f(x_k)| \leq df = 10^{-4}$$

We follow the procedure given above:

1. A guess:

$$x_0 = x_{\text{guess}} = \begin{bmatrix} x(1)_{\text{guess}} \\ x(2)_{\text{guess}} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

2. The first Newton iteration:

$$x_1 = x_0 + \Delta x_0$$

where Δx_0 for simplicity is calculated directly from Eq, (42.73)

$$\Delta x_0 = -[\nabla^2 f(x_0)]^{-1} \nabla f(x_0)$$

Here:

$$\nabla f(x_0) = \begin{bmatrix} \frac{\partial f(x_0)}{\partial x(1)} \\ \frac{\partial f(x_0)}{\partial x(2)} \end{bmatrix} = \begin{bmatrix} 2(x(1)_0 - 1) \\ 2(x(2)_0 - 2) \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

and

$$\nabla^2 f(x_0) = \begin{bmatrix} \frac{\partial^2 f(x_0)}{\partial x(1)^2} & \frac{\partial^2 f(x_0)}{\partial x(1)\partial x(2)} \\ \frac{\partial^2 f(x_0)}{\partial x(2)\partial x(1)} & \frac{\partial^2 f(x_0)}{\partial x(2)^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

giving

$$\Delta x_0 = -\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So, the first Newton iteration is

$$x_1 = x_0 + \Delta x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Checking the stop condition:

$$|f(x_1) - f(x_0)| = \left| f\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) - f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \right| = |0.5 - 2.5| = 2 \leq 10^{-4} \text{ (No!)}$$

The stop condition is not satisfied, so we make the second Newton iteration:

$$x_1 = x_0 + \Delta x_0$$

where

$$\Delta x_1 = -[\nabla^2 f(x_1)]^{-1} \nabla f(x_1)$$

where

$$\nabla f(x_1) = \begin{bmatrix} 2(x(1)_1 - 1) \\ 2(x(2)_1 - 2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and

$$\nabla^2 f(x_1) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

giving

$$\Delta x_1 = -\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus, the second Newton iteration is

$$x_2 = x_1 + \Delta x_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Checking the stop condition:

$$|f(x_2) - f(x_1)| = \left| f\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) - f\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) \right| = |0.5 - 0.5| = 0 \leq 10^{-4}$$

Now, the stop condition is satisfied, so the candidate of the optimal solution is

$$x_{\text{cand}} = x_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

3. Checking for convexity:

$$\text{eig } \nabla^2 f(x_{\text{cand}}) = \text{eig } \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Both eigenvalues are strictly positive, so f is convex at x_{cand} . Consequently, the optimal solution is

$$x_{\text{opt}} = x_{\text{cand}} = x_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

giving

$$f_{\min}(x_{\text{opt}}) = 0.5$$

Actually, the optimum was found in just one Newton iteration. This confirms that the Newton algorithm is exactly a “dead-beat” algorithm for quadratic objective function, as was explained earlier in this section.

A Matlab script implementing the above is shown below.

Script name: `matlab_script_newton_search.m`.

```
clear all, close all, format compact

%Creating anonymous function for the objective function: f_obj=@(x)
(x(1)-1)^2+(x(2)-2)^2+0.5;

x_k=[0,1]';
%Init N=100; %Preset max number of iterations
abs_df=1e-4;%Stopping criterion
for k=1:N-1
    %Gradient: G_k=[2*(x_k(1)-1); 2*(x_k(2)-2)];
    %Hessian: H_k=[2,0; 0,2];
    %Calculation of increment of x (used in x_kp1=x_k+dx_k):
    dx_k=-inv(H_k)*G_k;
    %A numerically better alternative to calc dx_k:
    %dx_k=-H_k\G_k;
    x_kp1=x_k+dx_k;
    f_k=f_obj(x_k);
    f_kp1=f_obj(x_kp1);
    df=f_kp1-f_k;
    x_k=x_kp1;
    if abs(df) < abs_df
        break
    end %if
end %for loop
disp('Result:')
k x_opt=x_kp1
f_min=f_obj(x_opt)
```

The result as shown in Matlab is:

```
k = 2
x_opt = 1 2
f_min = 0.5000
```

[End of Example 42.11]

42.2.7 Combining steepest descent with Newton for robust search

The steepest descent method, cf. Section 42.2.5, has these typical benefits and drawbacks:

- Benefit: It always moves towards a minimum of f , even if f is concave ($f'' > 0$) at the

starting point of the search.

- Drawback: It may behave poorly as the search approaches the minimum, with oscillations and/or slow convergence.

The Newton's method, cf. Section 42.2.6, has these typical benefits and drawbacks:

- Benefit: It converges fast and accurately to the minimum when f is close to the minimum.
- Drawback: It moves towards a maximum (away from the desired minimum) if f is concave ($f'' < 0$) at the starting point of the search.

By combining the steepest descent method with Newton's method, the benefits of both methods are retained while the drawbacks are omitted. This is summarized in the following.⁸

Combined steepest descent search and Newton search:

1. Start the search with the steepest descent method:

$$x_{k+1} = x_k - \nabla f(x_k)$$

which should eventually bring f into a region where it is convex (if it is not already convex from the start of the search).

2. When f has become convex, continue the search with Newton's method to arrive quickly and smoothly at f_{\min} :

$$x_{k+1} = x_k - [\nabla^2 f(x_k)]^{-1} \nabla f(x_k)$$

To check for f being convex, calculate the eigenvalues of the Hessian, $\nabla^2 f(x_k)$. If all of these eigenvalues have strictly positive real parts, the Hessian is positive definite, and f is convex.

Example 42.12 Combined steepest descend search and Newton search

The combined method is here applied to the optimization problem presented in Example 42.8. x_{guess} is set as 12, which is a concave region of f . The Matlab script below shows an implementation of the method.

Script name: `matlab_script_combined_steepest_and_newton_search.m`.

```
clear all, close all, format compact
a4=0.00232; a3=-0.111; a2=1.80; a1=-11.6; a0=34.4;
```

⁸This is one of two strategies to overcome the problem about the Newton method in a concave region suggested in e.g. https://courses.maths.ox.ac.uk/node/view_material/18818. The second strategy suggested is to reverse the sign of the Newton step, from minus to plus, when f is concave.

```
%Creating anonymous function for the objective function:
f_obj=@(x) a4*x.^4 + a3*x.^3 + a2*x.^2 + a1*x + a0;
%Anonymous function for gradient:
grad=@(x) 4*a4*x.^3 + 3*a3*x.^2 + 2*a2*x + a1;
%Anonymous function for Hessian:
hessian=@(x) 12*a4*x.^2 + 6*a3*x + 2*a2;
x_guess=12;
x_k=x_guess;
N=1000;%Preset max number of iterations
abs_df_spec=1e-4;%Stopping criterion
for k=1:N-1
    if sum((eig(hessian(x_k)))<=0)>0,
        %convex=0
        dx_k=-grad(x_k)
    else
        %convex=1
        dx_k=-inv(hessian(x_k))*grad(x_k);
        %A numerically better alternative to calc dx_k:
        %dx_k=-hessian(x_k)\grad(x_k);
    end %if
    x_kp1=x_k+dx_k;
    f_k=f_obj(x_k);
    f_kp1=f_obj(x_kp1);
    abs_df=abs(f_kp1-f_k);
    x_k=x_kp1;
    if abs_df < abs_df_spec
        break
    end %if
end %for loop

disp('Result:')
k
x_opt=x_kp1
f_min=f_obj(x_opt)
abs_df
```

The result as shown in Matlab, is:

```
k = 12
x_opt = 18.7483
f_min = 4.7662
abs_df = 1.7086e-08
```

With the steepest descent search in Example, `abs_df = 1.7186e-05`. So, the combined method gives here a more accurate final result which illustrates the benefit of using Newton search in the final stage (that is, after f has become convex) of the search.

Figure 42.23 illustrates the result.

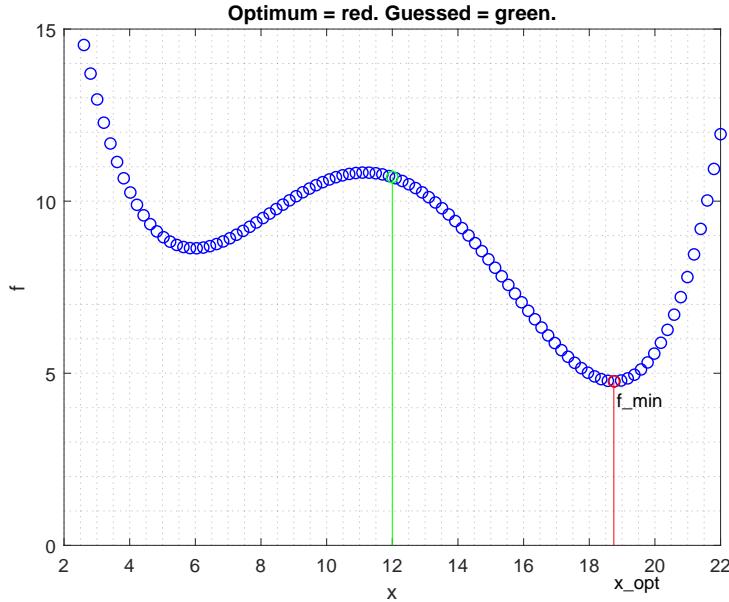


Figure 42.23: Example 42.12: Steepest descent search with $x_{\text{guess}} = 12$. x_{opt} is at global minimum.

[End of Example 42.12]

42.2.8 Some professional NLP optimizers

42.2.8.1 Introduction

In some applications, you need more powerful and flexible optimizers than one you can implement yourself with e.g. the grid search method and/or the Newton search method. Many optimization problems can be solved with Nonlinear Programming (NLP) optimizers, for example, parameter estimation of nonlinear models, model-predictive control, etc.⁹ In these optimizers, constraints on the form of (42.3) and (42.2) are, in principle, included as “penalty” terms in a modified objective function Edgar et al. (2001):

$$L(x, \lambda, u) = f(x) + \sum_{j=1}^r u_j g_j(x) + \sum_{i=1}^m \lambda_i h_i(x) \quad (42.83)$$

where L is denoted the *Lagrangian*, and λ_i and u_j are the Lagrange multipliers of the inequality and equality functions, respectively. The modified optimization problem is solved from $\nabla_x L = 0$.

In the following sections, the practical use of the following two professional Nonlinear Programming (NLP) optimizers are presented:

⁹“Nonlinear programming” is a traditional term used on optimization algorithms for solving nonlinear optimization problems.

- fmincon in Matlab's Optimization Toolbox. (fmincon = “finds a constrained minimum of a function of several variables”, cf. Matlab documentation.)
- slsqp in Python... To appear.

Complete examples with Python and Matlab code are given at the end of the respective sections.

42.2.8.2 slsqp (Python)

To appear.

42.2.8.3 fmincon (Matlab)

The information below is compiled from the documentation about fmincon in Matlab.¹⁰

fmincon attempts to solve optimization problems on the form of

$$\min_x f(x)$$

subject to:

- Linear constraints¹¹:

$$A \cdot x \leq B$$

$$A_{eq} \cdot x = B_{eq}$$

- Nonlinear inequality (less-than-or-equal) constraints:

$$g_{leq}(x) \leq 0$$

- Nonlinear equality constraints:

$$h_{eq}(x) = 0$$

- Bounds on the optimization variables:

$$lb \leq x \leq ub$$

fmincon offers a number of alternative algorithms: interior point, sqp, active set (default), and trust region reflective. You can choose one different from the default selection via the option input parameter, see below.

¹⁰Where appropriate, I have modified the nomenclature.

¹¹Alternatively, these linear constraints may be defined with the functions g_{leq} and h_{eq} .

There are several optional input arguments and output arguments of fmincon, and they can be omitted in the function call.¹² Below is the function call with a complete set of arguments:

```
[x,fval,exitflag,output,lambda,grad,hessian] = ...
fmincon(fun,x_guess,A,B,Aeq,Beq,lb,ub,nonlcon,options)
```

The input arguments of fmincon are:

- fun, a user-defined function with x as input argument and the scalar value of the objective function, $f(x)$, as output. You can pass any model parameters to fun while fun is invoked by fmincon by using an anonymous function call in fmincon. This is demonstrated in the example below.
- x_guess, your guessed value of x_{opt} (the optimal solution).
- A, the matrix in the linear equality constraints.
- B, the vector in the linear equality constraints.
- Aeq, the matrix in the linear inequality constraints.
- Beq, the vector in the linear equality constraints.
- lb, the lower bound on x . For example, in the case of three optimization variables: $lb = [x1_lb, x2_lb, x3_lb]$ (assuming numerical values of $x1_lb$ etc. are already set). Inf (infinity) can be used as a value of a bound.
- ub, the upper bound on x .
- nonlcon, a user-defined function with x as input argument and vectors g_leq and h_eq as output vectors calculated (in the function) as the left-hand part of the nonlinear inequalities and equalities, respectively. fmincon calculates the minimum such that each of the elements of the vector g_leq is ≤ 0 , and similarly $h_{\text{eq}} = 0$. If no bounds exists, set g_leq = [] and/or h_eq = [].) You can pass any model parameters to nonlcon by using an anonymous function call in fmincon. This is demonstrated in the example below.
- options: Various options can be set as pairs of properties and values. Example: To set the solver algorithm to sqp and the maximum number of iterations to 500:

```
options = optimoptions(@fmincon,'Algorithm','sqp','MaxIterations',500);
```

The default settings of the above two properties corresponds to:

```
options = optimoptions(@fmincon,'Algorithm','interior-point','MaxIterations',1000);
```

Default settings apply if the options input argument is omitted in the fmincon call, or if options is set as:

```
options = optimoptions(@fmincon);
```

The output arguments of fmincon are:

¹²Personally, I use the complete list of arguments, but typically setting some of the arguments as empty, that is, '[]'.

- $x = x_{\text{opt}}$ (the optimal solution).
- $fval = f(x_{\text{opt}})$.
- exitflag , which is an integer expressing various exit conditions.
- output which is a Matlab struct with information about the number of iterations, the number of function calls, etc.
- lambda , which are the Langrange multipliers at x_{opt} .
- $\text{grad} = \nabla f(x_{\text{opt}})$, which is a vector of zeros at (the exact) optimum.
- $\text{hessian} = \nabla^2 f(x_{\text{opt}})$, which is positive definite matrix at optimum.

Example 42.13 *fmincon* (Matlab)

We will see how fmincon can be used to solve the following optimization problem, which is the same problem that we solved with the grid search method in Section 42.2.3.

$$\min_x f(x) \quad (42.84)$$

where

$$f(x) = (x_1 - p_1)^2 + (x_2 - p_2)^2 + p_3 \quad (42.85)$$

where $p_1 = 1$, $p_2 = 2$, $p_3 = 0.5$ are model parameters.

Inequality (less-than-or-equal) constraint:

$$g_{\text{leq}}(x) = x(1) - x(2) + 1.5 \leq 0 \quad (42.86)$$

Bounds on the optimization variables:

$$0 \leq x_1 \leq 2 \quad (42.87)$$

$$1 \leq x_2 \leq 3 \quad (42.88)$$

The guess of the optimal solution is

$$x_{\text{guess}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Below is a Matlab script that finds the optimum. Note: The objective function and the constraints function are defined as local functions within the script, and they exist only within the script. Local functions are supported from Matlab version R2016b.

Script name: `script_fmincon_matlab.m`.

```
clear all;
format compact;
```

```
%Model params:
p1 = 1;
p2 = 2;
p3 = 0.5;
params_model.p1 = p1;%params_model is a struct. p1 is a field.
params_model.p2 = p2;
params_model.p3 = p3;

%Inputs to fmincon:
x_guess = [0,1]';
Aineq = [];
Bineq = [];
Aeq = [];
Beq = [];
x_lb = [0,2]';
x_ub = [1,3]';

fun_objective_handle = @(x)fun_objective(x,params_model);
    %fun_objective is def as a local function at end of script.
fun_constraints_handle = @(x)fun_constraints(x,params_model);
    %fun_constraints is def as a local function at end of script.
my_optim_options = optimoptions('fmincon');
options.display = 'off';
%options.algorithm = 'sqp';

%Executing fmincon:
[x_opt,fval,exitflag,output,lambda,grad,hessian] =...
    fmincon(fun_objective_handle,x_guess,Aineq,Bineq,Aeq,Beq,x_lb,x_ub, ...
        fun_constraints_handle,my_optim_options);

%Results:
disp('Results:')
fval
x_opt

%Defining local functions:

function f = fun_objective(x,params_model)
p1 = params_model.p1;
p2 = params_model.p2;
p3 = params_model.p3;
f = (x(1)-p1)^2+(x(2)-p2)^2+p3;
end %function

function [g_leq,h_eq] = fun_constraints(x,params_model)
p1 = params_model.p1;
p2 = params_model.p2;
p3 = params_model.p3;
g_leq = x(1)-x(2)+1.5; %Left side of "less than or equal" nonlin in
h_eq = []; %Left side of nonlinear equalities.
```

```
end %function
```

The result shown in Matlab is:

```
Local minimum found that satisfies the constraints.
```

```
Optimization completed because the objective function is non-decreasing in feasible directions, to within the default value of the optimality tolerance, and constraints are satisfied to within the default value of the constraint tolerance.
```

```
<stopping criteria details>
```

```
Optimal solution:  
fval =  
    0.6250  
x_opt =  
    0.7500  
    2.2500
```

which is very similar to the result as found with the grid search method in Section 42.2.3. (With a finer grid in the grid method, the grid solution will become even closer to the fmincon solution.)

[End of Example 42.13]

42.2.9 Global optimization

The grid search method with a sufficiently small resolution (search step) is a global optimizer, but the method may arrive at an inaccurate value of the global optimum. The Newton search method will arrive at an accurate value of the local optimum. These two methods can be combined into an accurate global optimizer as follows:

1. Do a grid search. The optimal solution from this search is here denoted x_{grid} .
2. Use the optimum from the grid search as the optimal guess in the Newton search, that is,

$$x_{\text{guess}} = x_{\text{grid}}$$

As an alternative to the second item above (a Newton search), a new grid search can be made around the solution from the first grid search and with a (much) smaller resolution. For example, the range of $x(1)$ can be set to

$$[x(1)_{\text{grid}} - \Delta x(1), x(1)_{\text{grid}} + \Delta x(1)]$$

where $x(1)_{\text{grid}}$ is the solution for x_1 from the first grid search, and $\Delta x(1)$ is the resolution. The same applies to $x(2)$.

Dedicated global optimization methods exist, for example the so-called genetic algorithms, cf. the Global Optimization Toolbox for Matlab, and/or [Edgar et al. \(2001\)](#).

Chapter 43

A guide to the Python Control package

43.1 Introduction

43.1.1 About this guide

The Python Control Package is for analysis and design of dynamic systems in general and feedback control systems in particular. The package resembles the Control System Toolbox in MATLAB.

The package is developed at California Institute of Technology (Caltech), USA, by prof. Richard M. Murray and coworkers.

The package requires Numpy, Scipy, and Matplotlib (these packages are installed with the Anaconda distribution of Python tools).

The guide covers only some of the functions in the Python Control Package. However, these function are basic, and if you master these functions, you should be well prepared for using other functions in the package.

Most of the tutorial is about continuous-time models, i.e. transfer functions based on the Laplace transform and state space models based on differential equations. Discrete-time models are briefly covered in one chapter at the end of the tutorial. That coverage is brief because the basic functions for continuous-time models can be used also for discrete-time models, i.e. with the same syntax, however with the sampling time (period) as an extra input argument in the functions.

43.1.2 Information about Python Control package on the Internet

The home page of the Python Control package is

<https://pypi.org/project/control/>

A complete list of the functions in the package is available via the link named Homepage on the above home page.

43.1.3 Installing the Python Control package

You can install the package with the command

```
pip install control
```

executed e.g. at the Anaconda prompt (in the Anaconda command window)¹.

Some functions in the Python Control package, for example the lqr function for calculating the stationary controller gain G in LQ control, requires that the package slycot is installed. You can install it with the following command at the Anaconda prompt:

```
conda install -c conda-forge slycot
```

(The straightforward “pip install slycot” may not work.)

43.1.4 Importing the Python Control package into Python

The following command (in Python) imports the Python Control package into Python:

```
import control
```

43.1.5 Using arrays for numerical data

In Python, tuples, lists, dictionaries, and arrays can be used to store numerical data. However, only arrays are practical for mathematical operations on the data, like addition and multiplication. Therefore, I use arrays as the numerical data type consistently in this book.

To use arrays, you must import the numpy package. It has become a tradition to rename the numpy package as np. Thus, to import numpy, include the following command in the beginning of your program:

```
import numpy as np
```

¹In Windows: Start menu / Anaconda / Anaconda prompt.

43.2 Transfer functions

This section is about Laplace transform based transfer functions, which may be referred to as s -transfer functions (s is the “Laplace variable”). Discrete time transfer functions, or z -transfer functions, are covered by Section 43.5.1.2.

43.2.1 How to create transfer functions

The `control.tf()` function is used to create transfer functions with the following syntax:

$$H = \text{control}.tf(\text{num}, \text{den})$$

where H is the resulting transfer function (object). `num` (representing the numerator) and `den` (representing the denominator) are arrays where the elements are the coefficients of the s -polynomials in descending order from left to right.

Of course, you can use any other names than H , `num`, and `den` in your own programs.

To illustrate the syntax, assume that the transfer function is

$$H(s) = \frac{b_1 s + b_0}{a_1 s + a_0} \quad (43.1)$$

In this case,

$$\text{num} = \text{np.array}([b_1, b_0])$$

and

$$\text{den} = \text{np.array}([a_1, a_0])$$

where, of course, the values of b_1 , b_0 , a_1 and a_0 have been defined earlier.

Example 43.1 Creating a transfer function

In this example, we create the following transfer function:

$$H(s) = \frac{2}{5s + 1} \quad (43.2)$$

The Python program 43.1 creates this transfer function. The code `print('H(s) = ', H)` is used to present the transfer function in the console (of Spyder).

http://techteach.no/control/python/create_tf.py

Listing 43.1: create_tf.py

```
import numpy as np
import control

# %% Creating the transfer function:

num = np.array([2])
den = np.array([5, 1])
H = control.tf(num, den)

# %% Displaying the transfer function:

print('H(s) =', H)
```

The result of the code above is shown as follows in the console:

$$\begin{aligned} H(s) = \\ 2 \\ \hline \\ 5 s + 1 \end{aligned}$$

If you execute “H” (+ enter) in the Spyder console, the transfer function is more nicely displayed, see Figure 43.1.

Figure 43.1: The transfer function nicely displayed with H (+ enter) executed in the console.

[End of Example 43.1]

43.2.2 Combinations of transfer functions

43.2.2.1 Series combination

Figure 43.2 illustrates a series combination of two transfer functions.

The resulting transfer function is

$$\frac{y(s)}{u(s)} = H(s) = H_2(s)H_1(s) \quad (43.3)$$

If you are to calculate the combined transfer function manually using (43.3), the order of

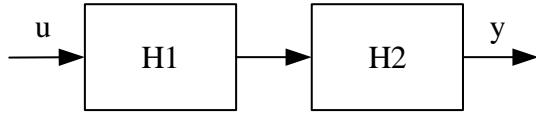


Figure 43.2: A series combination of two transfer functions, $H_1(s)$ and $H_2(s)$.

the factors in (43.3) is of no importance for SISO² transfer functions. But for MIMO³ transfer functions, the order in (43.3) is crucial.

Whether SISO or MIMO, you create a series combination with the `series()` function of the Python Control package:

```
H = control.series(H1, H2)
```

Example 43.2 Series combination of transfer functions

Assume a series combination,

$$H(s) = H_1(s)H_2(s)$$

of the following two transfer functions:

$$H_1(s) = \frac{K_1}{s} \tag{43.4}$$

$$H_2(s) = \frac{K_2}{T_1 s + 1} \tag{43.5}$$

where $K_1 = 2$, $K_2 = 3$, and $T = 4$.

Manual calculation gives:

$$H(s) = \frac{K_1}{s} \cdot \frac{K_2}{T s + 1} = \frac{K_1 K_2}{T s^2 + s} = \frac{6}{4 s^2 + s}$$

Program 43.2 shows how the calculations can be done with the `control.series()` function.

http://techteach.no/control/python/series_tf.py

Listing 43.2: `series_tf.py`

```

import numpy as np
import control

K1 = 2
K2 = 3
T = 4

```

²SISO = Single Input Single Output

³MIMO = Multiple Input Multiple Output

```

num1 = np.array([K1])
den1 = np.array([1, 0])

num2 = np.array([K2])
den2 = np.array([T, 1])

H1 = control.tf(num1, den1)
H2 = control.tf(num2, den2)

H = control.series(H1, H2)

print('H = ', H)

```

The result of the code above as shown in the console is:

```

H =
 6
-----
4 s^2 + s

```

[End of Example 43.2]

43.2.2.2 Parallel combination

Figure 43.3 illustrates a parallel combination of two transfer functions.

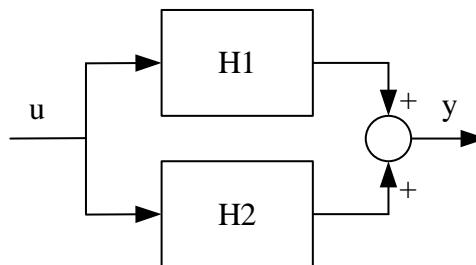


Figure 43.3: A parallel combination of two transfer functions, $H_1(s)$ and $H_2(s)$.

The resulting transfer function is

$$\frac{y(s)}{u(s)} = H(s) = H_2(s) + H_1(s) \quad (43.6)$$

The `control.parallel()` function calculates parallel combinations:

```
H = control.parallel(H1, H2)
```

Example 43.3 Parallel combination of transfer functions

Given the transfer functions, $H_1(s)$ and $H_2(s)$, as in Example 43.2.

Manual calculation of their parallel combination gives⁴:

$$H(s) = \frac{2}{s} + \frac{3}{4s+1} = \frac{2(4s+1) + 3s}{s(4s+1)} = \frac{11s+2}{4s^2+s}$$

Program 43.3 shows how the calculations can be done with the control.parallel() function.

http://techteach.no/control/python/parallel_tf.py

Listing 43.3: parallel_tf.py

```
import numpy as np
import control

K1 = 2
K2 = 3
T = 4

num1 = np.array([K1])
den1 = np.array([1, 0])

num2 = np.array([K2])
den2 = np.array([T, 1])

H1 = control.tf(num1, den1)
H2 = control.tf(num2, den2)

H = control.parallel(H1, H2)

print('H =', H)
```

The result of the code above as shown in the console is:

```
H =
 11 s + 2
-----
 4 s^2 + s
```

[End of Example 43.3]

43.2.2.3 Feedback combination

Figure 43.4 illustrates a feedback combination of two transfer functions.

⁴For simplicity, I insert here the numbers directly instead of the symbolic parameters, but in general I recommend using symbolic parameters.

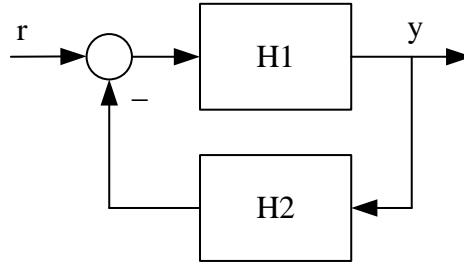


Figure 43.4: A feedback combination of two transfer functions, $H_1(s)$ and $H_2(s)$.

The resulting transfer function, from r (reference) to y , which can be denoted the closed loop transfer function, can be calculated from the following expression defining y (for simplicity, I drop the argument s here):

$$y = H_1 \cdot (r - H_2 y) = H_1 r - H_1 H_2 y$$

which gives

$$y = \frac{H_1}{1 + H_1 H_2} r$$

Thus, the resulting transfer function is

$$\frac{y(s)}{r(s)} = H(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)} \quad (43.7)$$

The `control.feedback()` function calculates the resulting transfer function of a negative feedback combination:

```
H = control.feedback(H1, H2, sign=-1)
```

You may drop the argument $\text{sign} = -1$ if there is negative feedback since negative feedback is the default setting.

You must use $\text{sign} = 1$ if there is a positive feedback instead of a negative feedback in Figure 43.4.

In most cases – at least in feedback control systems – a negative feedback with $H_2(s) = 1$ in the feedback path is assumed. Then, $H_1()$ is the open loop transfer function, $L(s)$, and (43.7) becomes

$$\frac{y(s)}{r(s)} = H(s) = \frac{L(s)}{1 + L(s)} \quad (43.8)$$

In such cases, you can write

```
H = control.feedback(L, 1)
```

$L(s)$ may be the series combination (i.e. the product) of the controller, the process, the sensor, and the measurement filter:

$$L(s) = C(s) \cdot P(s) \cdot S(s) \cdot F(s) \quad (43.9)$$

Series combination using `control.series()` is described in Section 43.2.2.1.

Example 43.4 *The closed loop transfer function*

Given a negative feedback loop with the following open loop transfer function:

$$L(s) = \frac{2}{s} \quad (43.10)$$

Manual calculation of the closed loop transfer function with (43.8) gives

$$H(s) = \frac{L(s)}{1 + L(s)} = \frac{\frac{2}{s}}{1 + \frac{2}{s}} = \frac{2}{s + 2} \quad (43.11)$$

Program 43.4 shows how the calculations can be done with the `control.feedback()` function.

http://techteach.no/control/python/programs/feedback_tf.py

Listing 43.4: `feedback_tf.py`

```
import numpy as np
import control

num = np.array([2])
den = np.array([1, 0])

L = control.tf(num, den)

H = control.feedback(L, 1)

print('H = ', H)
```

The result:

H =
2

s + 2

[End of Example 43.4]

43.2.3 How to get the numerator and denominator of a transfer function

You can get (read) the numerator coefficients and denominator coefficients of a transfer function, say `H`, with the `control.tfdata()` function:

$$(\text{num_list}, \text{den_list}) = \text{control.tfdata}(H)$$

where num_list and den_list are *lists* (not arrays) containing the coefficients.

To convert the lists to arrays, you can use the np.array() function:

```
num_array = np.array(num_list)
```

and

```
den_array = np.array(den_list)
```

Example 43.5 Getting the numerator and denominator of a transfer function

See Program 43.5.

http://techteach.no/control/python/get_tf_num_den.py

Listing 43.5: get_tf_num_den.py

```
import numpy as np
import control

# %% Creating a transfer function:
num = np.array([2])
den = np.array([5, 1])
H = control.tf(num, den)

# %% Getting the num and den coeffs as lists and then as arrays:
(num_list, den_list) = control.tfdata(H)
num_array = np.array(num_list)
den_array = np.array(den_list)

# %% Displaying the num and den arrays:
print('num_array =', num_array)
print('den_array =', den_array)
```

The result:

```
num_array = [[2]]
den_array = [[[5 1]]]
```

To “get rid of” the two inner pairs of square brackets, i.e. to reduce the dimensions of the arrays:

```
num_array = num_array[0,0,:]
den_array = den_array[0,0,:]
```

producing:

[2]
[5 1]

[End of Example 43.5]

43.2.4 Simulation with transfer functions

The function `control.forced_response()` is a function for simulation with transfer function and state space models. Here, we focus on simulation with transfer functions.

`control.forced_response()` can simulated with any user-defined input signal. Some alternative simulation functions assuming special input signals are:

- `control.step_response()`
- `control.impulse_response()`
- `control.initial_response()`

`control.forced_response()` may be used in any of these cases. Therefore, I limit the presentation in this document to the `control.forced_response()` function.

The syntax of `control.forced_response()` is:

$$(t, y, x) = \text{control.forced_response}(\text{sys}, t, u, X0)$$

where:

- Input arguments:
 - `sys` is the system to be simulated – a transfer function or a state space model.
 - `t` is the user-defined array of points of simulation time.
 - `u` is the user-defined array of values of the input signal of same length at the simulation time array.
 - `X0` is the initial state. For SISO transfer functions, you can set `X0 = 0`.
- Output (return) arguments:
 - `t` is the returned array of time – the same as the input argument.
 - `y` is the returned array of output values.
 - `x` is the returned array of state values. For transfer functions, you are probably not interested in `x`, only in `y`.

To plot the simulated output (`y` above), and maybe the input (`u` above), you can use the plotting function in the `matplotlib.pyplot` module which requires import of this module. The common way to import the module is:

```
import matplotlib.pyplot as plt
```

Example 43.6 *Simulation with a transfer function*

We will simulate the response of the transfer function

$$\frac{y(s)}{u(s)} = \frac{2}{5s + 1}$$

with the following conditions:

- Input u is a step of amplitude 4, with step time $t = 0$.
- Simulation start time is $t_0 = 0$ sec.
- Simulation stop time is $t_1 = 20$ sec.
- Simulation time step, or sampling time, is $dt = 0.01$ s.
- Initial state is 0.

Program 43.6 implements this simulation.

http://techteach.no/control/python/sim_tf.py

Listing 43.6: sim_tf.py

```
# %% Import:
import numpy as np
import control
import matplotlib.pyplot as plt

# %% Creating model:
num = np.array([2])
den = np.array([5, 1])
H = control.tf(num, den)

# %% Defining signals:
t0 = 0
t1 = 20
dt = 0.01
nt = int(t1/dt) + 1 # Number of points of sim time
t = np.linspace(t0, t1, nt)
u = 2*np.ones(nt)

# %% Simulation:
(t, y, x) = control.forced_response(H, t, u, X0=0)

# %% Plotting:
plt.close('all')
fig_width_cm = 24
fig_height_cm = 18
```

```
plt.figure(1, figsize=(fig_width_cm/2.54, fig_height_cm/2.54))

plt.subplot(2, 1, 1)
plt.plot(t, y, 'blue')
#plt.xlabel('t [s]')
plt.grid()
plt.legend(labels=('y',))

plt.subplot(2, 1, 2)
plt.plot(t, u, 'green')
plt.xlabel('t [s]')
plt.grid()
plt.legend(labels=('u',))

plt.savefig('sim_tf.pdf')
```

Figure 43.5 shows plots of the output y and the input u .

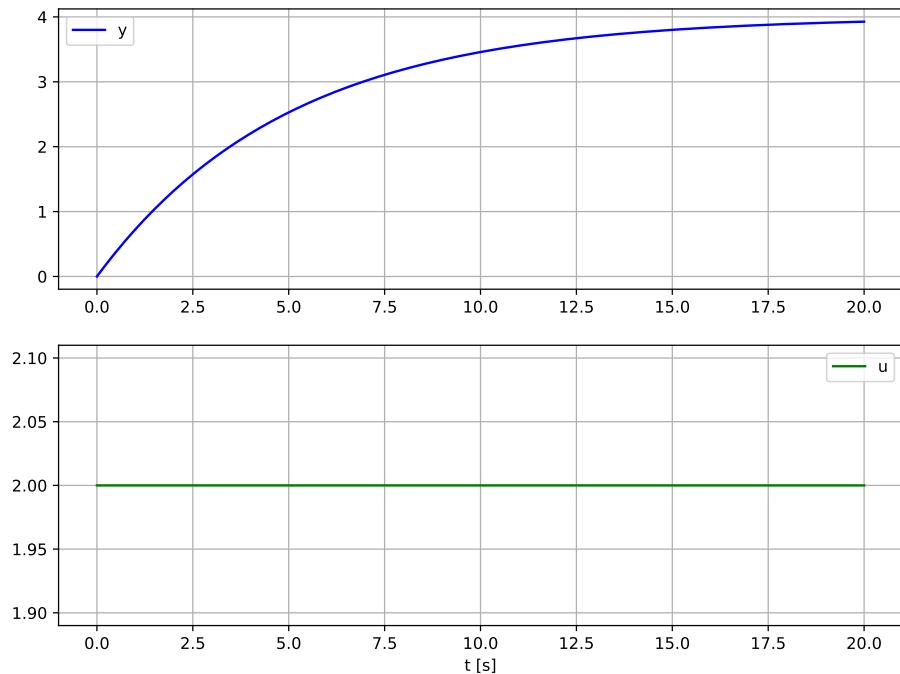


Figure 43.5: Plots of the output y and the input u .

[End of Example 43.6]

43.2.5 Poles and zeros of transfer functions

Poles and zeros of a transfer function, H , can be calculated and plotted in a cartesian diagram with

$$(p, z) = \text{control.pzmap}(H)$$

Example 43.7 Poles and zeros of a transfer function

Given the following transfer function:

$$H(s) = \frac{s + 2}{s^2 + 4}$$

Manual calculations gives:

- Poles:

$$p_{1,2} = \pm 2j$$

- Zero:

$$z = -2$$

Program 43.7 calculates the poles and the zero and plots them with the `control.pzmap()` function. The `plt.savefig()` function is used to generate a pdf file of the diagram.

http://techteach.no/control/python/poles_tf.py

Listing 43.7: poles_tf.py

```
import numpy as np
import control
import matplotlib.pyplot as plt

num = np.array([1, 2])
den = np.array([1, 0, 4])

H = control.tf(num, den)

(p, z) = control.pzmap(H)

print('poles = ', p)
print('zeros = ', z)

plt.savefig('poles_zeros.pdf')
```

The result:

```
poles = [-0.+2.j 0.-2.j]
zeros = [-2.]
```

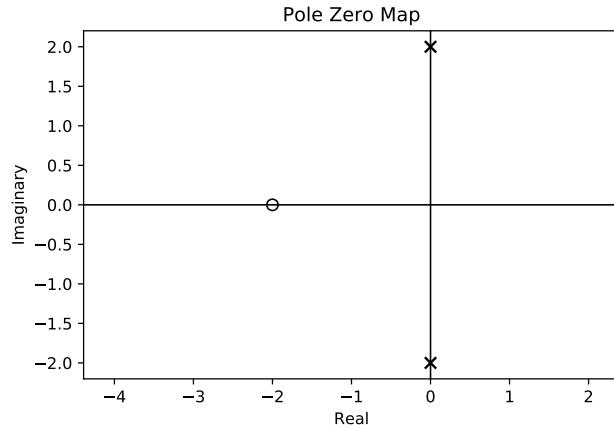


Figure 43.6: Pole-zero plot.

Figure 43.6 shows the pole-zero plot.

[End of Example 43.7]

43.2.6 The Padé-approximation of a time delay

The transfer function of a time delay is

$$e^{-T_d s} \quad (43.12)$$

where T_d is the time delay. In the Python Control Package, there is no function to you can not define this s -transfer function (while this is straightforward for z -transfer functions, cf. Ch. 43.5.1.2). However, you can use the control.pade() function to generate an Padé-approximation of the time delay (43.12).

Once you have a Padé-approximation of the time delay, you may use the control.series() function to combine it with the transfer function having no time delay:

$$H_{\text{with_delay}}(s) = H_{\text{without_delay}}(s) \cdot H_{\text{pade}}(s) \quad (43.13)$$

Example 43.8 Padé-approximation

Given the following transfer function with time constant of 10 s and *no* time delay:

$$H_{\text{without_delay}}(s) = \frac{1}{10s + 1} \quad (43.14)$$

Assume that this transfer function is combined in series with a transfer function, $H_{\text{pade}}(s)$, of a 10th order Padé-approximation representing a time delay of 10 s. The resulting transfer function is:

$$H_{\text{with_delay}}(s) = H_{\text{without_delay}}(s) \cdot H_{\text{pade}}(s) = \frac{1}{10s + 1} \cdot H_{\text{pade}}(s) \quad (43.15)$$

Program 43.8 generates these transfer functions and simulated a step response of $H_{\text{with_delay}}(s)$.

http://techteach.no/control/python/pade_approx.py

Listing 43.8: pade_approx.py

```
import numpy as np
import control
import matplotlib.pyplot as plt

# %% Generating transfer function of Pade approx:
T_delay = 5
n_pade = 10
(num_pade, den_pade) = control.pade(T_delay, n_pade)
H_pade = control.tf(num_pade, den_pade)

# %% Generating transfer function without time delay:
num = np.array([1])
den = np.array([10, 1])
H_without_delay = control.tf(num, den)

# %% Generating transfer function with time delay:
H_with_delay = control.series(H_pade, H_without_delay)

# %% Simulation of step response:
t = np.linspace(0, 40, 100)
(t, y) = control.step_response(H_with_delay, t)
plt.plot(t, y)
plt.xlabel('t [s]')
plt.grid()

# %% Generating pdf file of the plotting figure:
plt.savefig('pade_approx.pdf')
```

Figure 43.7 shows the step response of $H_{\text{with_delay}}(s)$.

[End of Example 43.8]

43.3 Frequency response

43.3.1 Frequency response of transfer functions

The function `control.bode_plot()` generates frequency response data in terms of magnitude and phase. The function may also plot the data in a Bode diagram. However, in the following example, I have instead used the `plt.plot()` function to plot the data as this gives more freedom to configure the plot.

Example 43.9 *Frequency response*

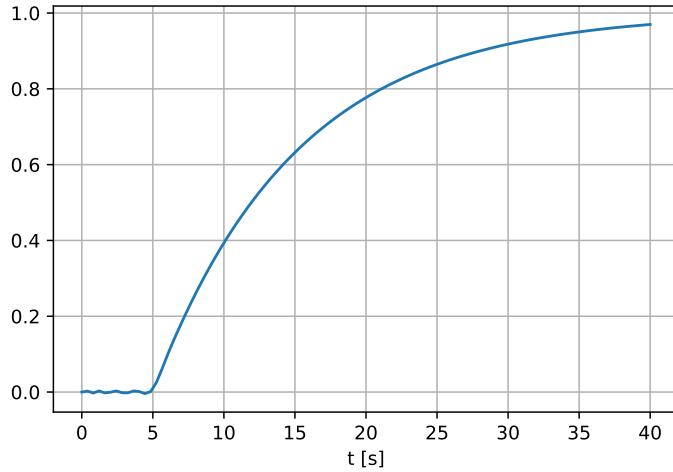


Figure 43.7: Step response of $H_{\text{with_delay}}(s)$ where the time delay is approximated with a Padé-approximation.

A first order lowpass filter has the following transfer function:

$$H(s) = \frac{1}{\frac{s}{\omega_b} + 1} \quad (43.16)$$

where $\omega_b = 1$ rad/s, which is the bandwidth.

Program 43.9 generates and plots frequency response of $H(s)$ in terms of magnitude and phase.

http://techteach.no/control/python/bode_plot_lowpass_filter.py

Listing 43.9: bode_plot_lowpass_filter.py

```
import numpy as np
import control
import matplotlib.pyplot as plt

# %% Generating Bode plot:

wb = 1 # Bandwidth [rad/s]
H = control.tf([1], [1/wb, 1])

w0 = 0.1
w1 = 10
dw = 0.001
nw = int((w1-w0)/dw) + 1 # Number of points of freq
w = np.linspace(w0, w1, nw)

(mag, phase_rad, w) = control.bode_plot(H, w)

# %% Plotting:
```

```
plt.close('all')
plt.figure(1, figsize=(12, 9))

plt.subplot(2, 1, 1)
plt.plot(np.log10(w), mag, 'blue')
#plt.xlabel('w [rad/s]')
plt.grid()
plt.legend(labels=('mag',))

plt.subplot(2, 1, 2)
plt.plot(np.log10(w), phase_rad*180/np.pi, 'green')
plt.xlabel('w [rad/s]')
plt.grid()
plt.legend(labels=('phase [deg]',))

# %% Generating pdf file of the plotting figure:
plt.savefig('bode_plot_filter.pdf')
```

Figure 43.8 shows the Bode plot. In the plot we can see that bandwidth is indeed 1 rad/s (which is at $0 = \log_{10}(1)$ rad/s in the figure).

[End of Example 43.9]

43.3.2 Frequency response and stability analysis of feedback loops

Figure 43.9 shows a feedback loop with its loop transfer function, $L(s)$.

control.bode_plot()

We can use the function `control.bode_plot()` to calculate the magnitude and phase of L , and to plot the Bode plot of L .

The syntax of `control.bode_plot()` is:

$$(mag, phase_rad, w) = control.bode_plot()$$

Several input arguments can be set, cf. Example 43.10.

In addition to calculating the three return arguments above, `control.bode_plot()` can show the following analysis values in the plot:

- The amplitude cross-over frequency, ω_b [rad/s], which is also often regarded as the bandwidth of the feedback system.
- The phase cross-over frequency, ω_{180} [rad/s].
- The gain margin, GM, which is found at $\omega_{180} \equiv \omega_g$ [rad/s] (g for gain margin).

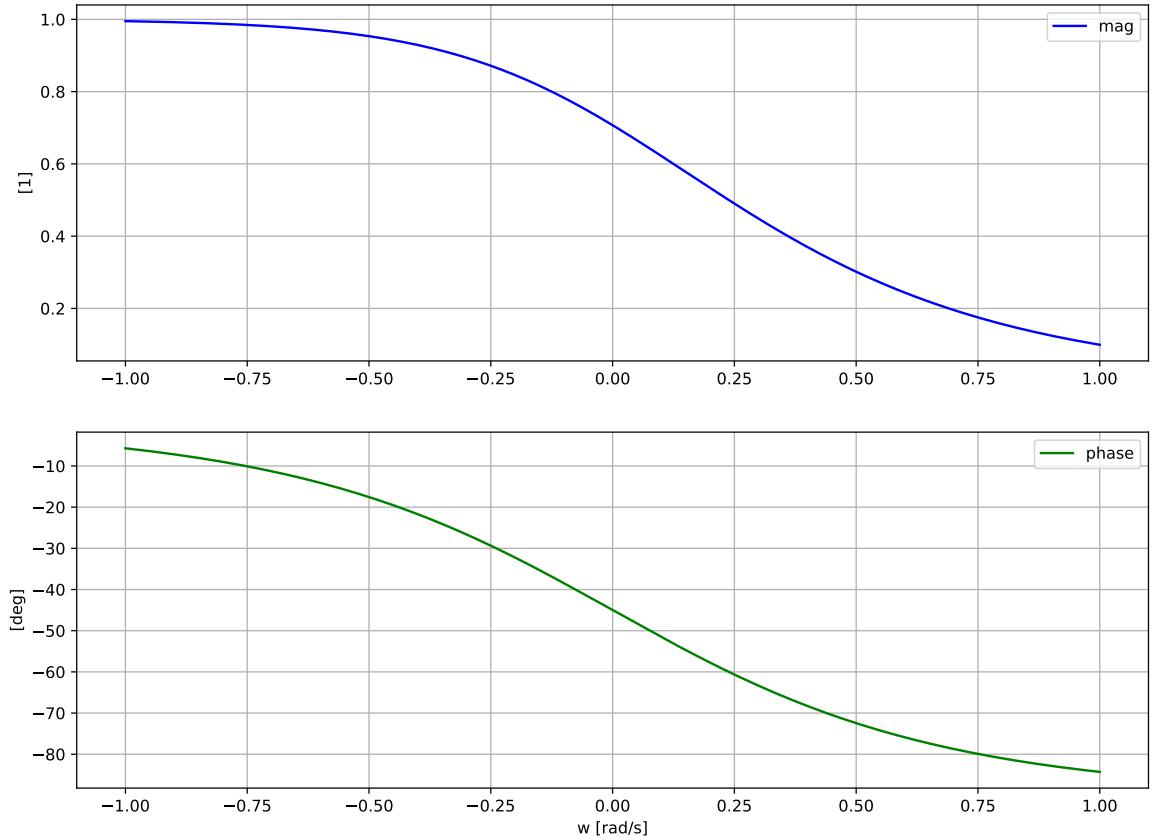


Figure 43.8: Bode plot.

- The phase margin, PM, which is found at $\omega_b \equiv \omega_p$ [rad/s] (p for phase margin).

`control.margin()`

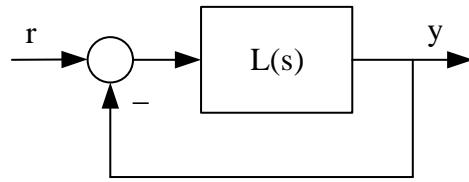
The `control.bode_plot()` does *not* return the above four analysis values to the workspace (although it shows them in the Bode plot). Fortunately, we can use the `control.margin()` function to calculate these analysis values. `control.margin()` can be used as follows:

$$(GM, PM, wg, wp) = \text{control.margin}(L)$$

where L is the loop transfer function, and the four return arguments are as in the list above. Note that GM has unit one; *not* dB, and that PM is in degrees.

Example 43.10 demonstrates the use of `control.bode_plot()` and `control.margin()`.

Example 43.10 Frequency response

Figure 43.9: A feedback loop with its loop transfer function, $L(s)$

Given a control loop where the process to be controlled has the following transfer function (an integrator and two time constants in series):

$$P(s) = \frac{1}{(s+1)^2 s}$$

The controller is a P controller:

$$C(s) = K_c$$

where $K_c = 2$ is the controller gain.

The loop transfer function becomes:

$$L(s) = P(s) \cdot C(s) = \frac{K_c}{(s+1)^2 s} = \frac{K_c}{s^3 + 2s + s} \quad (43.17)$$

Program 43.10 generates and plots frequency response of $H(s)$, and shows the stability margins and the cross-over frequencies.

http://techteach.no/control/python/bode_plot_with_stab_margins.py

Listing 43.10: bode_plot_with_stab_margins.py

```

import numpy as np
import control
import matplotlib.pyplot as plt

# %% Creating the loop transfer function:

Kp = 1
C = control.tf([Kp], [1])
P = control.tf([1], [1, 2, 1, 0])
L = control.series(C, P)

# %% Frequencies:

w0 = 0.1
w1 = 10
dw = 0.001
nw = int((w1-w0)/dw) + 1 # Number of points of freq
w = np.linspace(w0, w1, nw)

# %% Plotting:
  
```

```

plt.close('all')
plt.figure(1, figsize=(12, 9))
(mag, phase_rad, w) = control.bode_plot(
    L, w, dB=True, deg=True, margins=True)

# %% Calculating stability margins and crossover frequencies:
(GM, PM, wg, wp) = control.margin(L)

# %% Printing:
print('GM [1 (not dB)] =', f'{GM:.2f}')
print('PM [deg] =', f'{PM:.2f}')
print('wg [rad/s] =', f'{wg:.2f}')
print('wp [rad/s] =', f'{wp:.2f}')

# %% Generating pdf file of the plotting figure:
plt.savefig('bode_with_stab_margins.pdf')

```

Below are the results of `control.margin()` as shown in the console. The values are the same as shown in the Bode plot in Figure 43.10 (2 dB \approx 6).

GM [1 (not dB)] = 2.00
PM [deg] = 21.39
wg [rad/s] = 1.00
wp [rad/s] = 0.68

[End of Example 43.10]

43.4 State space models

43.4.1 How to create state space models

The function `control.ss()` creates a *linear* state space model with the following form:

$$\dot{x} = Ax + Bu \quad (43.18)$$

$$y = Cx + Du \quad (43.19)$$

where A, B, C, D are the model matrices.

The syntax of `control.ss()` is:

$$S = control.ss(A, B, C, D)$$

$G_m = 6.02 \text{ dB}$ (at 1.00 rad/s), $P_m = 21.39 \text{ deg}$ (at 0.68 rad/s)

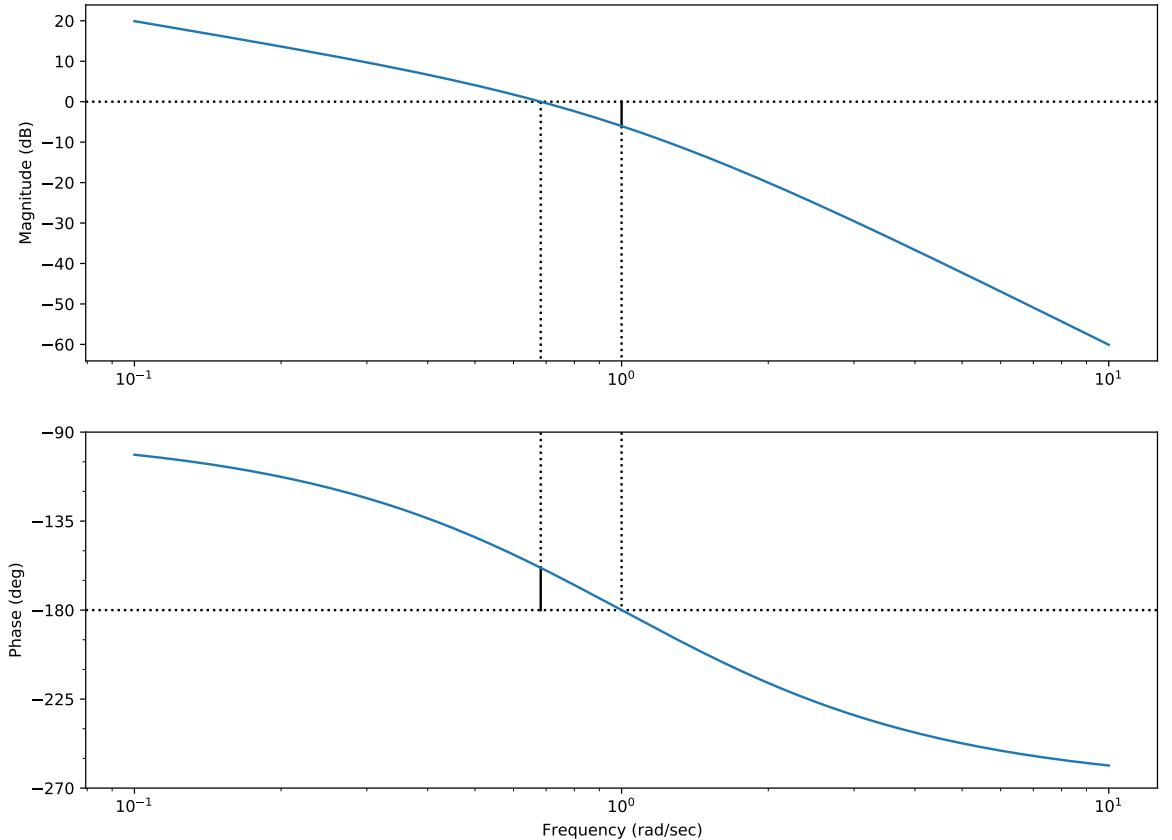


Figure 43.10: Bode plot including the stability margins and the crossover frequencies.

where S is the resulting state space model, and the matrices A , B , C , D are in the form of 2D arrays in Python. (Actually, they may be of the list data type, but I recommend using arrays, cf. Section 43.1.5.)

Example 43.11 Creating a state space model

Figure 43.11 shows a mass-spring-damper-system.

z is position. F is applied force. d is damping constant. k is spring constant. Newton's 2. Law gives the following mathematical model:

$$m\ddot{z}(t) = F(t) - d\dot{z}(t) - kz(t) \quad (43.20)$$

Let us define the following state variables:

- Position:

$$x_1 = z$$

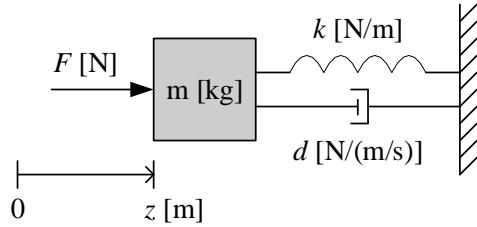


Figure 43.11: Mass-spring-damper system.

- Speed:

$$x_2 = \dot{z} = \dot{x}_1$$

Let us define the position x_1 as the output variable:

$$y = x_1$$

Eq. (43.20) can now be expressed with the following equivalent state space model:

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{d}{m} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_x + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_B F \quad (43.21)$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_C \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_x + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_D F \quad (43.22)$$

Assume following parameter values:

$$m = 10 \text{ kg}$$

$$k = 4 \text{ N/m}$$

$$d = 2 \text{ N/(m/s)}$$

Program 43.11 creates the above state space model with the control.ss() function.

http://techteach.no/control/python/create_ss.py

Listing 43.11: create_ss.py

```
import numpy as np
import control

m = 10 # [kg]
k = 4 # [N/m]
d = 2 # [N/(m/s)]

# %% System matrices as 2D arrays:
A = np.array([[0, 1], [-k/m, -d/m]])
B = np.array([[0], [1/m]])
```

```
C = np.array([[1, 0]])
D = np.array([[0]])

# %% Creating the state space model:
S = control.ss(A, B, C, D)

# %% Displaying S:
print('S =', S)
```

The results as shown in the console of Spyder:

```
A = [[ 0.  1.]
      [-0.4 -0.2]]

B = [[0. ]
      [0.1]]

C = [[1.  0.]]]

D = [[0.]]
```

43.4.2 How to get the model matrices of a state space model

You can get (read) the model matrices of a given state space model, say S , with the `control.ssdata()` function:

```
(A_list, B_list, C_list, D_list) = control.ssdata(S)
```

where the matrices are in the form of *lists* (not arrays).

To convert the lists to arrays, you can use the `np.array()` function, e.g.

```
A_array = np.array(A_list)
```

Example 43.12 Getting the model matrices of a given state space model

Program 43.12 creates a state space model and gets its matrices with the `control.ssdata()` function.

http://techteach.no/control/python/get_ss_matrices.py

Listing 43.12: `get_ss_matrices.py`

```
import numpy as np
import control
```

```
# %% Creating a state space model:  
A = np.array([[0, 1], [2, 3]])  
B = np.array([[4], [5]])  
C = np.array([[6, 7]])  
D = np.array([[8]])  
  
S = control.ss(A, B, C, D)  
  
# %% Getting the model matrices as lists and then as arrays:  
(A_list, B_list, C_list, D_list) = control.ssdata(S)  
A_array = np.array(A_list)  
B_array = np.array(B_list)  
C_array = np.array(C_list)  
D_array = np.array(D_list)  
  
# %% Displaying the matrices as arrays:  
print('A_array =', A_array)  
print('B_array =', B_array)  
print('C_array =', C_array)  
print('D_array =', D_array)
```

The results as shown in the console:

```
A_array = [[0. 1.] [2. 3.]]  
B_array = [[4.] [5.]]  
C_array = [[6. 7.]]  
D_array = [[8.]]
```

[End of Example 43.12]

43.4.3 Simulation with state space models

Simulation with state space models can be done with the `control.forced_response()` function, cf. Section 43.2.4, with the following syntax:

The syntax of `control.forced_response()` is:

$$(t, y, x) = \text{control.forced_response}(\text{sys}, t, u, x0)$$

where `sys` is the state space model, cf. Section 43.4.1.

Example 43.13 *Simulation with a state space model*

The program shown below runs a simulation with the state space model presented in Example 43.11 with the following conditions:

- Force (input signal) F is a step of amplitude 10 N, with step time $t = 0$.
- Simulation start time: $t_0 = 0$ s.
- Simulation stop time: $t_1 = 50$ s.
- Simulation time step, or sampling time: $dt = 0.01$ s.
- Initial states: $x_{1,0} = 1$ m, $x_{2,0} = 0$ m/s.

Program 43.13 implements the simulation.

http://techteach.no/control/python/sim_ss.py

Listing 43.13: sim_ss.py

```
# %% Import:
import numpy as np
import control
import matplotlib.pyplot as plt

# %% Model parameters:
m = 10 # [kg]
k = 4 # [N/m]
d = 2 # [N/(m/s)]

# %% System matrices as 2D arrays:
A = np.array([[0, 1], [-k/m, -d/m]])
B = np.array([[0], [1/m]])
C = np.array([[1, 0]])
D = np.array([[0]])

# %% Creating the state space model:
S = control.ss(A, B, C, D)

# %% Defining signals:
t0 = 0 # [s]
t1 = 50 # [s]
dt = 0.01 # [s]
nt = int(t1/dt) + 1 # Number of points of sim time
t = np.linspace(t0, t1, nt)
F = 10*np.ones(nt) # [N]

# %% Initial state:
x1_0 = 1 # [m]
x2_0 = 0 # [m/s]
x0 = np.array([x1_0, x2_0])
```

```
# %% Simulation:  
  
(t, y, x) = control.forced_response(S, t, F, x0)  
  
# %% Extracting individual states:  
  
x1 = x[0,:]  
x2 = x[1,:]  
  
# %% Plotting:  
  
plt.close('all')  
plt.figure(1, figsize=(12, 9))  
  
plt.subplot(3, 1, 1)  
plt.plot(t, x1, 'blue')  
plt.grid()  
plt.legend(labels=('x1 [m]',))  
  
plt.subplot(3, 1, 2)  
plt.plot(t, x2, 'green')  
plt.grid()  
plt.legend(labels=('x2 [m/s]',))  
  
plt.subplot(3, 1, 3)  
plt.plot(t, F, 'red')  
plt.grid()  
plt.legend(labels=('F [N]',))  
plt.xlabel('t [s]')  
  
# %% Generating pdf file of the plotting figure:  
  
plt.savefig('sim_ss.pdf')
```

Figure 43.12 shows the simulated signals.

[End of Example 43.13]

43.4.4 From state space model to transfer function

The function `control.ss2tf()` derives a transfer function from a given state space model. The syntax is:

$$H = \text{control.ss2tf}(S)$$

where H is the transfer function and S is the state space model.

Example 43.14 *From state space model to transfer function*

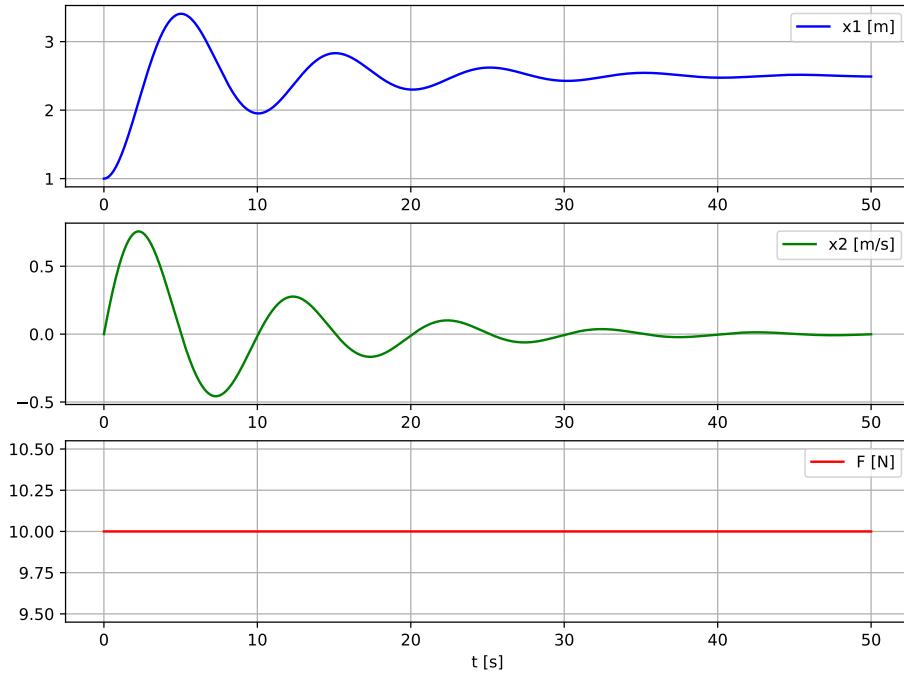


Figure 43.12: Plots of the simulated signals of the mass-spring-damper system.

In Example 43.11 a state space model of a mass-spring-damper system is created with the control.ss() function. The program shown below derives the following two transfer functions from this model:

- The transfer function, H_1 , from force F to position x_1 . To obtain H_1 , the output matrix use is set as

$$C = [1, 0]$$

- The transfer function, H_2 , from force F to position x_2 . To obtain H_2 , the output matrix is set as

$$C = [0, 1]$$

Program 43.14 derives the two transfer functions from a state space model.

http://techteach.no/control/python/from_ss_to_tf.py

Listing 43.14: from_ss_to_tf.py

```
import numpy as np
import control

# %% Model params:
```

```

m = 10 # [kg]
k = 4 # [N/m]
d = 2 # [N/(m/s)]

# %% System matrices as 2D arrays:

A = np.array([[0, 1], [-k/m, -d/m]])
B = np.array([[0], [1/m]])
D = np.array([[0]])

# %% Creating the state space model with x1 as output:

C1 = np.array([[1, 0]])
S1 = control.ss(A, B, C1, D)

# %% Deriving transfer function H1 from S1:

H1 = control.ss2tf(S1)

# %% Displaying H1:

print('H1 =', H1)

# %% Creating the state space model with x2 as output:

C2 = np.array([[0, 1]])
S2 = control.ss(A, B, C2, D)

# %% Deriving transfer function H2 from S2:

H2 = control.ss2tf(S2)

# %% Displaying H1:

print('H2 =', H2)

```

The result of the code above, as shown in the console of Spyder, is shown below. The very small numbers – virtually zeros – in the numerators of H_1 and H_2 are due to numerical inaccuracies in the `control.ss2tf()` function.

```

H1 =
 0.1
-----
s^2 + 0.2 s + 0.4

H2 =
0.1 s + 1.665e-16
-----
s^2 + 0.2 s + 0.4

```

[End of Example 43.14]

43.5 Discrete-time models

43.5.1 Transfer functions

43.5.1.1 Introduction

Many functions in the Python Control Package are used in the same way for discrete-time transfer functions, or z -transfer functions, as for continuous-time transfer function, or s -transfer function, except that for z -transfer functions, you must include the sampling time Ts as an additional parameter. For example, to create a z -transfer function, the control.tf() is used in this way:

$$H_d = \text{control.tf}(num_d, den_d, Ts)$$

where Ts is the sampling time. H_d is the resulting z -transfer function.

Thus, the descriptions in Ch. 43.2 gives you a basis for using these functions for z -transfer functions as well. Therefore, the descriptions are not repeated here. Still there are some specialities related to z -transfer function, and they are presented in the subsequent sections.

43.5.1.2 How to create transfer functions

The control.tf() function is used to create z -transfer functions with the following syntax:

$$H = \text{control.tf}(num, den, Ts)$$

where H is the resulting transfer function (object). num (representing the numerator) and den (representing the denominator) are arrays where the elements are the coefficients of the z -polynomials of the numerator and denominator, respectively, in descending order from left to right, with positive exponentials of z . Ts is the sampling time (time step).

Note that control.tf() assumes *positive* exponents of z . Here is one example of such a transfer function:

$$H(z) = \frac{0.1z}{z - 1} \tag{43.23}$$

(which is used in Example 43.15). However, in e.g. signal processing, we may see negative exponents in transfer functions. $H(z)$ given by (43.23) and written in terms of negative exponents of z , are:

$$H(z) = \frac{0.1}{1 - z^{-1}} \tag{43.24}$$

(43.23) and (43.24) are equivalent. But, in the Python Control Package, we must use only positive exponents of z in transfer functions.

Example 43.15 Creating a z -transfer function

Given the following transfer function⁵:

$$H(z) = \frac{0.1z}{z - 1} \quad (43.25)$$

Program 43.15 creates $H(z)$. The code `print('H(z) = ', H)` is used to present the transfer function in the console (of Spyder).

http://techteach.no/control/python/create_tf_z.py

Listing 43.15: create_tf_z.py

```
import numpy as np
import control

# %% Creating the z-transfer function:
Ts = 0.1
num = np.array([0.1, 0])
den = np.array([1, -1])
H = control.tf(num, den, Ts)

# %% Displaying the transfer function:
print('H(z) = ', H)
```

The result as shown in the console:

```
H(z) =
 0.1 z
-----
z - 1
dt = 0.1
```

[End of Example 43.15]

43.5.1.3 Discretizing an s -transfer function

The `control.sample_system()` function can be used to discretize given continuous-time models, including s -transfer functions:

```
sys_disc = control.sample_system(sys_cont, Ts, method='zoh')
```

where:

- `sys_cont` is the continuous-time model – a transfer function, or a state space model.

⁵This is the transfer function of an integrator based on the Euler Backward method of discretization: $y_k = y_{k-1} + Ts \cdot u_k$ with sampling time $Ts = 0.1$ s.

- T_s is the sampling time.
- The discretization method is 'zoh' (zero order hold) by default, but you can alternatively use 'matched' or 'tustin'. (No other methods are supported.)
- sys_disc is the resulting discrete-time model – a transfer function, or a state space model.

Example 43.16 *Discretizing an s-transfer function*

Given the following s -transfer function:

$$H_c(s) = \frac{3}{2s + 1} \quad (43.26)$$

Program 43.16 discretizes this transfer function using the zoh method with sampling time 0.1 s.

http://techteach.no/control/python/discretize_tf.py

Listing 43.16: discretize_tf.py

```
import numpy as np
import control

# %% Creating the s-transfer function:
num_cont = np.array([3])
den_cont = np.array([2, 1])
H_cont = control.tf(num_cont, den_cont)

# %% Discretizing:
Ts = 0.1
H_disc = control.sample_system(H_cont, Ts, method='zoh')

# %% Displaying the z-transfer function:
print('H_disc(z) =', H_disc)
```

The result as shown in the console:

```
H_disc(z) =
 0.1463
-----
z - 0.9512
dt = 0.1
```

[End of Example 43.16]

43.5.1.4 Exact representation of a time delay with a z -transfer function

In Section 43.2.6 we saw how to use the `control.pade()` function to generate a transfer function which is an Padé-approximation of the true transfer function of the time delay, $e^{-T_d s}$. As alternative to the Padé-approximation, you can generate an exact representation of the time delay in terms of a z -transfer function.

The z -transfer function of a time delay is:

$$H_d(z) = \frac{1}{z^{n_d}} \quad (43.27)$$

where

$$n_d = \frac{T_d}{T_s} \quad (43.28)$$

Example 43.17 Creating a z -transfer function of a time delay

Assume the time delay is

$$T_d = 5 \text{ s}$$

and the sampling time is

$$T_s = 0.1 \text{ s}$$

So, the transfer function of the time delay becomes

$$H_{\text{delay}}(z) = \frac{1}{z^{n_d}}$$

with

$$n_d = \frac{T_d}{T_s} = \frac{5}{0.1} = 50$$

Python program 43.17 creates $H_{\text{delay}}(z)$, which represents this time delay exactly. The program also simulates the step response of $H_{\text{delay}}(z)$.⁶

http://techteach.no/control/python/time_delay_hz.py

Listing 43.17: `time_delay_hz.py`

```
import numpy as np
import control
import matplotlib.pyplot as plt

# %% Generating a z-transfer function of a time delay:

Ts = 0.1
Td = 5
nd = int(Td/Ts)
denom_tf = np.append([1], np.zeros(nd))
```

⁶For some reason, the returned simulation array, y , becomes a 2D array. I turn it into a 1D array with $y = y[0,:]$ for the plotting.

```

H_delay = control.tf([1], denom_tf, Ts)

# %% Displaying the z-transfer function:
print('H_delay(z) =', H_delay)

# %% Sim of step response of time delay transfer function:
t = np.arange(0, 10+Ts, Ts)
(t, y) = control.step_response(H_delay, t)
y = y[0,:] # Turning 2D array into 1D array for plotting
plt.plot(t, y)
plt.xlabel('t [s]')
plt.grid()

# %% Generating pdf file of the plotting figure:
plt.savefig('step_response_hz_time_delay.pdf')

```

The result as shown in the console:

```

H_delay(z) =
 1
-----
z^50
dt = 0.1

```

Figure 43.13 shows the step response of $H_{\text{delay}}(z)$.

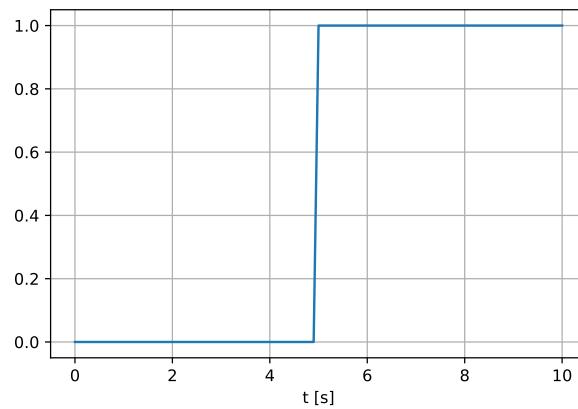


Figure 43.13: The step response of $H_{\text{delay}}(z)$

[End of Example 43.17]

43.5.2 Frequency response

Frequency response analysis of z -transfer functions is accomplished with the same functions as for s -transfer function. Therefore, I assume it is sufficient that I refer you to Ch. 43.3.

However, note the following comment in the manual of the Python Control Package: “If a discrete time model is given, the frequency response is plotted along the upper branch of the unit circle, using the mapping $z = \exp(j \omega dt)$ where ω ranges from 0 to π/dt and dt is the discrete timebase. If not timebase is specified ($dt = \text{True}$), dt is set to 1.”

43.5.3 State space models

In the Python Control Package, discrete-time linear state space models have the following form:

$$x_{k+1} = A_d x_k + B_d u_k \quad (43.29)$$

$$y_k = C_d x_k + D_d u_k \quad (43.30)$$

where A_d, B_d, C_d, D_d are the model matrices.

Many functions in the package are used in the same way for both discrete-time linear state space models and for continuous-time state space models, except that for discrete-time state space models, you must include the sampling time Ts as an additional parameter. For example, to create a discrete-time state space model, the control.ss() is used in this way:

```
S_d = control.ss(A_d, B_d, C_d, D_d, Ts)
```

where Ts is the sampling time. S_d is the resulting discrete time state space model.

Thus, the descriptions in Ch. 43.4 gives you a basis for using these functions for continuous-time state space models as well. Therefore, the descriptions are not repeated here.

Chapter 44

Selected mathematical formulas

44.1 Differentiation of vector functions

In the formulas below, it is assumed that v is a column vector:

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \quad (44.1)$$

Assuming s is scalar, then

$$\frac{ds}{dv} = \begin{bmatrix} \frac{\partial s}{\partial v_1} \\ \vdots \\ \frac{\partial s}{\partial v_n} \end{bmatrix} \quad (44.2)$$

Assuming v is a vector and c is a *constant* vector:

$$\frac{d}{dv} (c^T v) = c \quad (44.3)$$

$$\frac{d}{dv} (v^T c) = c \quad (44.4)$$

Assuming M is a matrix:

$$\frac{d}{dv} (v^T M v) = 2Mv \quad (44.5)$$

Chapter 45

Abbreviations

Table 45.1 defines abbreviations used in this document.

Table 45.1: Abbreviations

Abbreviation	Meaning
DC	Direct Current
DMC	Dynamic Matrix Control
LS	Least Squares (method)
MHE	Moving Horizon Estimation (or -Estimator)
MPC	Model Predictive Control (or -Controller)
NLS	Nonlinear Least Squares (method)
P&I D	Piping and Instrumentation Diagram
PID	Proportional + Integral + Derivative
SSPE	Sum of Squared Prediction Errors

Bibliography

- Åstrøm, K. J. & Hägglund, T. (1995), *PID Controllers: Theory, Design and Tuning*, ISA.
- Blickley, G. J. (1990), ‘Modern control started with ziegler-nichols tuning’, *Control Engineering* **2**.
- Boegli, M. (2014), *Real-Time Moving Horizon Estimation for Advanced Motion Control Application to Friction State and Parameter Estimation*, PhD thesis. Arenberg Doctoral School, KU Leuven, Belgium.
- Cutler, C. R. & Ramaker, B. L. (1980), ‘Dynamic matrix control - a computer control algorithm’, *Proc. Joint Automatic Control Conference, USA-CA* .
- Edgar, T. F., Himmelblau, D. & Lasdon, L. (2001), *Optimization of Chemical Processes*, McGraw-Hill.
- Haugen, F. & Lie, B. (2013), ‘Relaxed Ziegler-Nichols Closed Loop Tuning of PI Controllers’, *Modeling, Identification and Control* **34**(2), 83–97.
- Hill, D. T. (1983), ‘Simplified monod kinetics of methane fermentation of animal wastes’, *Agric. Wastes* **5**.
- Lee, J. H. (2011), ‘Model predictive control: Review of the three decades of development’, *International Journal of Control, Automation, and Systems* **9**(3), 415–424.
- Maciejowski, J. (2002), *Predictive Control with Constraints*, Prentice-Hall.
- Nocedal, J. & Wright, S. J. (2006), *Numerical Optimization*, 2nd ed., Springer.
- Qin, S. J. & Badgwell, T. A. (2003), ‘A survey of industrial model predictive control technology’, *Control Engineering Practice* **11**, 733–764.
- Robertson, D., Lee, J. & Rawlings, J. (1996), ‘A moving horizon-based approach for least-squares estimation’, *AIChE Journal* **8**, 2209–2224.
- Seborg, D. E., Edgar, T. F. & Mellichamp, D. A. (2004), *Process Dynamics and Control*, John Wiley and Sons.
- Simon, D. (2006), *Optimal State Estimation*, Wiley.
- Skogestad, S. (2003), ‘Simple analytic rules for model reduction and PID controller tuning’, *Journal of Process Control* **14**.
- Ziegler, J. & Nichols, N. (1942), ‘Optimum settings for automatic controllers’, *Trans. ASME* **64**(3), 759–768.

Index

- Åström and Hägglund, 318
Åström and Hägglund Relay tuning method, 318
- a posteriori estimate, 585
a priori estimate, 585
academic PID, 268
Acc, 75
Accumulation, 75
adaptive control, 346
additive PID, 268
air heater, 676
amplitude crossover frequency, 470
amplitude gain, 443
analysis of control systems, 457
anti wind-up, 279
anti windup, 630
Argument variation principle, 466
asymptotically stable, 425
asymptotically stable system, 428
augmentative states, 589
augmented Kalman Filter, 589
auto-covariance, 558
auto-tuning, 320
automatic mode, 35
averaging level control, 41, 380
- Balance Law, 75
bandpass filters, 448
bandstop filters, 448
bandwidth, 450, 459, 760
bandwidth
 1. ordens lavpassfilter, 450
batch LS formula, 233
biogas reactor, 681
black-box model, 547
black-box modeling, 73
- block diagram, 37
block diagram manipulation, 178
block diagrams, 76, 157
Bode diagram, 444
Bode-Nyquist stability criterion, 472
brute force method, 695
brute(), 225
buffer tank, 673
bumpless transfer, 283
- capasitor, 88
cascade control, 370
center difference method, 254
certainty equivalence principle, 620
characteristic equation, 427
characteristic polynomial, 178, 427, 433
chip tank, 377, 668
closed loop control, 34
closed loop transfer function, 750
coloured noise, 561
compensation properties, 300
computed torque control, 646
computed torque controller, 655
concave, 693
confidence intervals, 237
constant control, 34
control blocking, 639
control grouping, 639
control loop, 34
controllability, 623
controller canonical block diagram, 182
convergence of LS-estimate, 237
correlation, 560
cross-covariance, 559
crossover frequency, 460
- D-kick, 278

dB, 445
dead-time, 205
decibel, 445
decoupling, 646
dependent variables, 76
Derivative gain, 269
derivative kick, 278
difference equation, 496
differential equations, 73
digital twin, 117
disturbance, 76, 103
downhill simplex algorithm, 225
downhill simplex method, 229
dynamic system, 73
dynamics, 191
eigenvalues, 184, 432
elektriske systemer, 86
elementary mathematical blocks, 160
energy balance, 79
environmental variable, 103
environmental variables, 76
equation-error vector, 232
equilibrium point, 111
error-model, 587
error-model of Kalman Filter, 587
Euler Backward, 131
Euler Explicit, 132
Euler Forward method, 119
Euler Implicit, 132
excitation in parameter estimation, 218
Expectation value, 556
exponentially weighted moving average (EWMA) filter, 71
extended Kalman Filter, 575
Fail Closed (FC) valve, 60
Fail Open (FO) valve, 60
feasible region, 692
feedback, 749
feedback linearization, 646
feedback loops, 34
feedforward control, 46, 405
filter, 69, 448
filter in the D-term, 278
First principles modeling, 73
force balance, 81
Fourier series approximation, 319
frequency components, 442
frequency response, 442, 444, 445
frequency response analysis, 458
frequency response and transfer function, 445
front panel, 129
gain, 193
for second order system, 196
gain function, 444
gain margin, 471
gain scheduling, 341
global minimum, 693
grid method, 698
highpass filters, 448
impulse response, 424
independent variables, 76
inductor, 88
input disturbance, 325
input variable, 103
Integral gain, 269
integrator, 191
integrator anti wind-up, 279
interlocks, 62
Inverse dynamics control, 646
Kalman Filter, 574
Kalman Filter algorithm, 582
kinetic system, 81
Kirchhoff's current law, 86
Kirchhoffs spenningslov, 87
LabVIEW, 129
Laplace transform, 166
least squares, 216, 249
least squares estimation, 236
linear motion, 81
linear MPC, 636
linearization of non-linear models, 108
local maximum, 693
local minimum, 693
local model, 109
loop transfer function, 302, 433
lowpass filter, 69
2. ordens, 449
lowpass filters, 448
lowpassfilter in the D-term, 278
LQ control, 620

-
- LQR, 620
 lstsq(), 236

 Manipulating variables, 76
 manual control, 34
 manual mode, 35
 marginally stable, 425
 marginally stable system, 428
 mass balance, 77
 mass system, 77
 mass-spring-damper, 82, 104, 185, 429, 764
 dynamics, 197
 static response, 106
 mathematical block diagram, 157
 mathematical model, 75
 mathematical modeling, 73
 mean value, 556
 measurement filter, 69
 measurement-updated estimate, 585
 Mechanistic modeling, 73
 model, 73
 model errors, 73, 615
 model predictive control, 635
 model uncertainty, 73
 modeling, 73
 momentum, 81, 83
 momentum balance, 81, 83
 motion systems, 81
 Moving averaging filter, 71
 MPC, 635
 multiloop PID controller, 646

 Nelder-Mead, 246
 Nelder-Mead algorithm, 225
 Nelder-Mead optimization, 226
 Nelder-Mead solver, 229
 Newton's second law, 82
 NLP, 229
 nonlinear decoupling, 646
 nonlinear MPC, 636
 nonlinear programming, 229, 246
 nonlinear state space model, 104
 normal equation, 233
 Nyquist's spesial stability criterion, 470
 Nyquist's stability criterion, 466
 Nyquist's stability criterion for open
stable systems, 470

 observability, 576
 observability matrix, 577
 observed variable, 231
 On-off tuning method, 318
 one quarter decay ratio, 309
 open loop control, 34
 open loop transfer function, 750
 operating point, 109
 optimal control, 620
 order
 transfer function, 178
 output variable, 103
 overdamped systems, 198
 overshoot factor, 200

 P&I D, 36, 55
 P-kick, 278
 parallel, 748
 parallel PID, 268
 parameter, 103
 parameter vector, 231
 passband, 449
 performance, 300
 phase crossover frequency, 470
 phase function, 444
 phase lag, 443
 phase margin, 471
 Piping & Instrumentation Diagram, 36
 pole, 178
 poles, 184, 432
 poles and stability, 426
 positioner, 378
 power, 90
 power of a signal, 558
 prediction-error vector, 232
 primary loop, 370
 Process & Instrumentation Diagram, 55
 process gain, 276
 proportional band, 268
 proportional kick, 278

 ratio control, 379
 RC-circuit, 88, 210, 450
 Recursive least squares method, 250
 regression, 220
 regression model, 231
 regression variable, 231
 regression vector, 231
 relative damping factor, 196

Relaxed Ziegler-Nichols settings, 315
relay tuner, 321
Relay tuning method, 318
repeats per minute, 269
resistor, 88
resistors, 87
response-time, 207
RHP (right half plane), 467
robust control, 615
robustness, 476
rotational motion, 83
Routh's stability criterion, 465

scipy.optimize.brute(), 225
second order systems, 196
secondary controller, 370
secondary loop, 370
self regulation, 464
sensitivity bandwidth, 460
sensitivity transfer function, 302
separation principle, 620
sequential control, 417
serial PID controller, 337
series, 746
shaping filter, 561
ship, 670
signal filter, 448
simulation, 753
Skogestad method, 324
slsqp, 641
soft sensor, 574
specifications of control system, 309
split-range control, 379
stability, 358, 424
stability margin, 363
stability margins, 471
standard deviation, 558
state, 105
state feedback, 373
state machine, 418
state space, 105
state space model, 102
state space models, 76
state trajectory, 105
state variable, 103
static response, 106, 497
statisk transferfunksjon, 180
steady state Kalman Filter gain, 584
steepest descent optimization method, 716
stochastic signals, 555
stopband, 449
subspace methods, 547
System and Control Diagram, 62
system matrix, 432

temperature control, 343
testing, 575
time constant, 193
time delay, 136, 205
time-updated estimate, 585
torque balance, 83
tracking properties, 300
tracking transfer function, 433
transfer function and frequency response,
 445
transfer functions, 175, 176
transfer functions, 76
transfer functions (s), 745
transfer functions (z), 772
transformed process, 648
transition matrix, 586
transportation time, 136

undamped resonance frequency, 196
undamped system, 201
underdamped system, 199
unit pulse, 560
unstable, 425
unstable system, 428

Water resource recovery facilities, 42
weight matrices, 621
white noise, 560
White-box modeling, 73
wood chip tank, 668
wood-chip tank, 377
WWRF, 42

zero, 178
zero order hold (zoh), 515
zero-pole gain form of transfer functions,
 178
Ziegler-Nichols closed loop method, 311
Ziegler-Nichols open loop method, 339
Ziegler-Nichols process reaction curve
 method, 339
ZOH discretization, 515