

Convert from Cylindrical to Spherical.

$$x = r \cos \theta, \quad x = R \sin \phi \cos \theta \Rightarrow r = R \sin \phi.$$

$$y = r \sin \theta, \quad y = R \sin \phi \sin \theta$$

$$z = z, \quad z = R \cos \phi$$

Now,

$$\frac{\partial u}{\partial R} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial R} + \frac{\partial u}{\partial \theta} \cdot \frac{\partial \theta}{\partial R} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial R}$$

$$= \frac{\partial}{\partial r} (u_x \cos \theta + u_y \sin \theta) \cdot \frac{\partial (R \sin \phi)}{\partial R} + \frac{\partial}{\partial \theta} (u_x \cos \theta + u_y \sin \theta) \cdot \frac{\partial \theta}{\partial R}$$

$$= \left[\frac{\partial u_x}{\partial r} \cos \theta + \frac{\partial u_y}{\partial r} \sin \theta \right] \cdot \sin \phi + \left[\frac{\partial u_x}{\partial \theta} \cos \theta + u_x (-\sin \theta) + \frac{\partial u_y}{\partial \theta} \sin \theta + u_y \cos \theta \right] \frac{\partial \theta}{\partial R}$$

$$\Rightarrow \frac{\partial u}{\partial R} = \left[\frac{\partial u_x}{\partial r} \cos \theta + \frac{\partial u_y}{\partial r} \sin \theta \right] \sin \phi + \left[\frac{\partial u_x}{\partial \theta} \cos \theta - u_x \sin \theta + \frac{\partial u_y}{\partial \theta} \sin \theta + u_y \cos \theta \right] \frac{\partial \theta}{\partial R}$$

$$\begin{aligned}
 \Rightarrow \frac{\partial u_0}{\partial \theta} &= \frac{\partial u_0}{\partial r} \cdot \frac{\partial r}{\partial \theta} + \frac{\partial u_0}{\partial \theta} \cdot \frac{\partial \theta}{\partial \theta} + \frac{\partial u_0}{\partial z} \cdot \frac{\partial z}{\partial \theta} \\
 &= \frac{\partial}{\partial r} (-r u_x \sin \theta + r u_y \cos \theta) \cdot \frac{\partial}{\partial \theta} (R \sin \phi) + \frac{\partial}{\partial \theta} (-r u_x \sin \theta + r u_y \cos \theta) \cdot (1)
 \end{aligned}$$

$$= \left[-ux \sin \theta \frac{\partial r}{\partial x} - r \sin \theta \frac{\partial u_x}{\partial r} - r u_x \frac{\partial \sin \theta}{\partial r} + u_y \cos \theta \frac{\partial r}{\partial y} + r \cos \theta \frac{\partial u_y}{\partial r} + r u_y \frac{\partial \cos \theta}{\partial r} \right] \cdot \sin \phi \frac{\partial R}{\partial \theta} + \left[-u_x \sin \theta \frac{\partial r}{\partial x} - r \sin \theta \frac{\partial u_x}{\partial r} - r u_x \frac{\partial \sin \theta}{\partial r} + u_y \cos \theta \frac{\partial r}{\partial y} + r \cos \theta \frac{\partial u_y}{\partial r} + r u_y \frac{\partial \cos \theta}{\partial r} \right]$$

$$= \left[-u_x \sin \theta - r \sin \theta \frac{\partial u_x}{\partial r} + u_y \cos \theta + r \cos \theta \frac{\partial u_y}{\partial r} \right] \sin \phi \frac{\partial R}{\partial \theta} + \left[(-u_x \sin \theta + u_y \cos \theta) \frac{\partial r}{\partial \theta} - r \sin \theta \frac{\partial u_x}{\partial \theta} + r \cos \theta \frac{\partial u_y}{\partial \theta} - r u_x \cos \theta - r u_y \sin \theta \right]$$

$$\begin{aligned} \frac{\partial u_z}{\partial \phi} &= \frac{\partial u_z}{\partial r} \cdot \frac{\partial r}{\partial \phi} + \frac{\partial u_z}{\partial \theta} \cdot \frac{\partial \theta}{\partial \phi} + \frac{\partial u_z}{\partial \phi} \cdot \frac{\partial \phi}{\partial \phi} \\ &= \frac{\partial u_z}{\partial r} \cdot \frac{\partial}{\partial \phi} (R \sin \phi) + \frac{\partial u_z}{\partial \theta} \cdot \frac{\partial \theta}{\partial \phi} + \frac{\partial u_z}{\partial \phi} \cdot \frac{\partial \phi}{\partial \phi} \\ &= \frac{\partial u_z}{\partial r} \left(\frac{\partial R}{\partial \phi} \sin \phi + R \cos \phi \right) + \frac{\partial u_z}{\partial \theta} \cdot \frac{\partial \theta}{\partial \phi} + \frac{\partial u_z}{\partial \phi} \end{aligned}$$

$$\Rightarrow \frac{\partial u_z}{\partial \phi} = \frac{\partial u_z}{\partial r} \left(\frac{\partial R}{\partial \phi} \sin \phi + R \cos \phi \right) + \left(\frac{\partial R}{\partial \phi} \cos \phi - \sin \phi \cos \phi \right)$$

$$\Rightarrow \frac{\partial u_z}{\partial \phi} = \frac{\partial u_z}{\partial r} \left(\frac{\partial R}{\partial \phi} \sin \phi + R \cos \phi \right) + \frac{\partial u_z}{\partial \theta} \cdot \frac{\partial \theta}{\partial \phi} + \frac{\partial u_z}{\partial \phi} \left(\frac{\partial R}{\partial \phi} \cos \phi - \sin \phi \cos \phi \right)$$