Systèmes linéaires

$Ax = b$

$A \in \mathbb{C}^{N \times N}$

$b \in \mathbb{C}^N$

Methode de Newton

$F(x) = 0 \qquad F : \mathbb{R}^N \to \mathbb{R}^N$

$$x_{n+1} = x_n - F'(x_n)^{-1} F(x_n)$$

Si $A \in \mathbb{C}^{N \times N}$

$\exists ! \, Ax = b \ \forall b \iff A$ bijective

thm du rang $\left(\begin{array}{l} \iff A \text{ injective} \\ \iff A \text{ surjective} \end{array} \right.$

$\iff \det A \neq 0$

$\iff 0$ n'est pas v.p. de A.

Soit A carré non-inversible,

Sol pas unique.

$\exists$ sol $\iff b \in \underset{\underset{\operatorname{Im} A}{\shortparallel}}{\operatorname{Ran} A} = \underset{\underset{\operatorname{Span}}{\shortparallel}}{\operatorname{Vect}} (A_1 \dots A_n)$ si $A = (A_1 \dots A_n)$

$2x + 4y = \alpha$

$4x + 8y = \beta$

$2\alpha = \beta$

$\operatorname{Im} A = \left\{ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \text{ tq } 2\alpha - \beta = 0 \right\} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}^\perp$

Prop : si $A \in \mathbb{C}^{N \times M}$

$\mathbb{C}^N = \operatorname{Im} A \underset{\text{somme direct}}{\oplus} \operatorname{Ker}(A^*)$

$\mathbb{C}^M = \operatorname{Im} A^* \oplus \operatorname{Ker}(A)$

$A = \begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix} \qquad A^T = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix} \qquad A^T y = 0 \iff y^T A = 0$

$24 - L_2 = 0$

Demo:

- $\ker A \subset \operatorname{Im} A^{*\perp}$

Soit $x \in \ker A$. Soit $y \in \operatorname{Im} A^*$. $\langle x, y \rangle = 0$.

$y = A^* z$

$\langle x, y \rangle = \langle x, A^* z \rangle = \langle Ax, z \rangle = 0$.

- $\operatorname{Im} A^{*\perp} \subset \ker A$

Soit $x \in \operatorname{Im}(A^*)^\perp$ $\quad (\forall y \in \operatorname{Im} A^*, \langle x, y \rangle = 0)$

$\forall z ; \langle x, A^* z \rangle = 0$

$\forall z \; \langle Ax, z \rangle = 0$

$Ax = 0$

Supp. A inversible.

- Pivot de Gauss

$$\begin{cases} x + 2y + 3z = 1 \\ 4x + 5y + 6z = 2 \\ 7x + 8y + 10z = 3 \end{cases} \qquad \begin{array}{l} L_2 \leftarrow L_2 - 4L_1 \\ L_3 \leftarrow L_3 - 7L_1 \end{array}$$

$$\Rightarrow \begin{cases} x + 2y + 3z = 1 \\ 0x + \dots \\ 0x + \dots \end{cases} \rightarrow \begin{cases} \dots \\ \dots \\ 0x + 0y + z = \dots \end{cases}$$

$O(N^3)$

Pivot de Gauss (sans pivot)

$A = L\,U$ ← Tri sup

L ← Tri inf

$$\underset{A_1}{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{pmatrix}} \to \underset{A_2}{\begin{pmatrix} 1 & 2 & 3 \\ 0 & x & x \\ 0 & x & x \end{pmatrix}} \to \underset{A_3}{\begin{pmatrix} 1 & 2 & 3 \\ 0 & x & x \\ 0 & 0 & x \end{pmatrix}}$$

Multiplication à gauche.

$$A_2 = \begin{pmatrix} 1 & & \\ -4 & 1 & \\ -7 & 0 & 1 \end{pmatrix} A_1$$

$$A_{n+1} = \underbrace{B_n}_{\text{Tri inf.}} A_n$$

$$U = B_n B_{n-1} \dots B_1 \underbrace{A_1}_{= A}$$

$$A = \underbrace{B_1^{-1} B_2^{-1} \dots B_N}_{L} U$$

$$\underset{\text{perm}}{P} A = LU \quad \text{Pratique}$$

$$\underbrace{L\,U x}_{y} = b$$

Gram-Schmidt "$\Leftrightarrow$" $A = \underset{\text{orth}}{\overset{\mathbb{C}^{N \times M}}{Q}} \underset{\mathbb{C}^{M \times M}}{\overset{\text{tri sup}}{R}}$

$U_1 \dots U_M \in \mathbb{C}^N \quad M \leq N$

$$Q^* Q = \underbrace{1}_{I_M}$$

$q_1 = \frac{U_1}{\|U_1\|}$

$q_2 = \frac{U_2 - \langle q_1, U_2 \rangle q_1}{\| \quad \|}$

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$A = \begin{pmatrix} U_1 & \dots & U_M \end{pmatrix} \qquad Q = \begin{pmatrix} q_1 & \dots & q_m \end{pmatrix}$

q_n est obtenu par c.l. de $U_1 \dots U_n$

$Q = A R^{-1}$

Cholesky

$A = L L^T$ A symm. pos-défini

$\underbrace{QR_x}_{y} = b \Rightarrow y = Q^* b$

Moindre carrés

$y = \alpha X + \beta$, chercher $(\alpha, \beta) \in \mathbb{R}^2$ qui minimize. $\sum_{i=1}^{N} |y_i - (\alpha x_i + \beta)|^2$

Soit $e_i = y_i (\alpha x_i + \beta) \quad \forall i = 1 \dots N$

$\sum_{i=1}^{N} |y_i - (\alpha x_i + \beta)|^2 = \|e\|^2 = \langle e, e \rangle$

$\alpha, \beta \to e$

$e = y - A \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad A = \begin{pmatrix} & \end{pmatrix}$

$A = N \begin{pmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_N & \vdots \end{pmatrix}^{2} \begin{pmatrix} & \end{pmatrix}$

$\sum \| y - A \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \|^2$

De façon generale $A \in \mathbb{R}^{N \times M}$, $b \in \mathbb{R}^{N}$ | $x = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

$b = y$

chercher $\min_{x \in \mathbb{R}^M} \| Ax - b \|^2$ $N >> M$

$$= \min \langle Ax-b, Ax-b \rangle = (Ax)^T (Ax) - b^T Ax - (Ax^T b) + \|b\|^2$$

$$= x^T A^T A x - 2 (A^T b)^T x + \|b\|^2$$

$$2 A^T A x - 2 (A^T b)^T$$

$$2 A^T A x = 2 A^T b$$ formes normales