# Copulas

# Exceedance Correlations

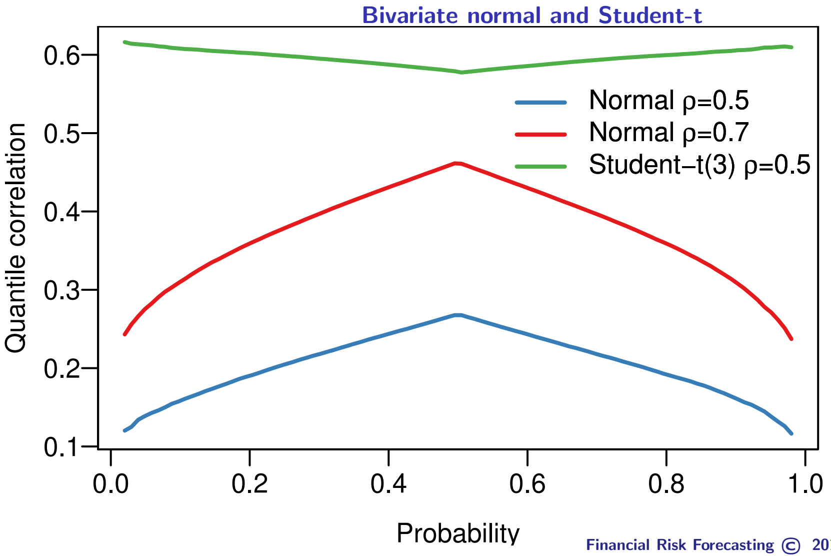
- Exceedance correlations show the correlations of (standardsed) stock returns $X$ and $Y$ as being conditional on exceeding some threshold, that is,

$$\tilde{\rho}(p) = \begin{cases} \mathrm{Corr}[X, Y | X \le Q_X(p) \text{ and } Y \le Q_Y(p)], & \text{for } p \le 0.5 \\ \mathrm{Corr}[X, Y | X > Q_X(p) \text{ and } Y > Q_Y(p)], & \text{for } p > 0.5 \end{cases}$$

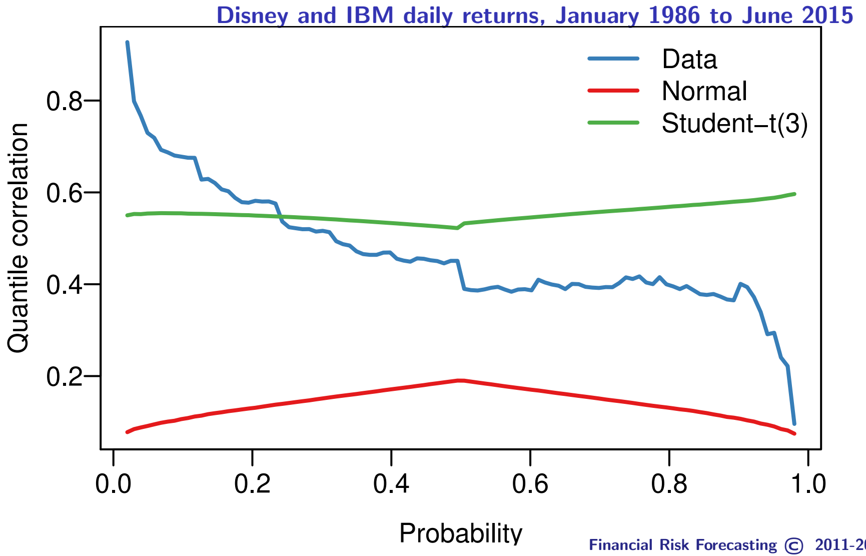
  where $Q_X(p)$ and $Q_Y(p)$ are the $p$-th quantiles of $X$ and $Y$ given a distributional assumption

- Can be used to detect NLD

# Exceedance Plot



# Empirical Exceedance Plot



# Copulas and Non-linear Dependence

- How do we model non-linear dependence more formally?
- One approach is multivariate volatility models (see Chapter 3)
- Alternatively we can use copulas, which allow us to create multivariate distributions with a range of types of dependence

# Intuition Behind Copulas

- A copula is a convenient way to obtain the dependence structure between two or more random variables, taking NLD into account
- We start with the marginal distributions of each random variable and end up with a copula function
- The copula function joins the random variables into a single multivariate distribution by using their correlations

# Intuition Behind Copulas

- The random variables are transformed to uniform distributions using the *probability integral transformation*
- The copula models the dependence structure between these uniforms
- Since the probability integral transform is invertible, the copula also describes the dependence between the original random variables

# Theory of Copulas

- Suppose $X$ and $Y$ are two random variables representing returns of two different stocks, with densities $f$ and $g$:

$$X \sim f \text{ and } Y \sim g$$

- Together, the joint distribution and marginal distributions are represented by the joint density $h$:

$$(X, Y) \sim h$$

- We focus separately on the marginal distributions $(F, G)$ and the copula function $C$, which combines them into the joint distribution $H$

# Theory of Copulas

- We want to transform $X$ and $Y$ into random variables that are distributed uniformly between 0 and 1, removing individual information from the bivariate density $h$

**Theorem 1.1** Let a random variable X have a continuous distribution F, and define a new random variable U as:

$$U = F(X)$$

Then, regardless of the original distribution $F$:

$$U \sim \text{Uniform}(0, 1)$$

# Theory of Copulas

- Applying this transformation to $X$ and $Y$ we obtain:

$$U = F(X) \text{ and } V = G(Y)$$

- Using this we arrive at the following theorem

**Theorem 1.2** Let F be the distribution of X, G the distribution of Y and H the joint distribution of (X,Y). Assume that F and G are continuous. Then there exists a unique copula C such that:

$$H(X, Y) = C(F(X), G(Y))$$

# Theory of Copulas

- In applications we are more likely to use densities:

$$h(X, Y) = f(X) \times g(Y) \times C(F(X), G(Y))$$

- The copula contains all dependence information in the original density $h$, but none of the individual information
- Note that we can construct a joint distribution from any two marginal distributions and any copula, and we can also extract the implied copula and marginal distributions from any joint distribution

# The Gaussian Copula

- One example of a copula is the *Gaussian copula*
- Let $\Phi(\cdot)$ denote the normal (Gaussian) distribution and $\Phi^{-1}(\cdot)$ its inverse
- Let $U, V \in [0, 1]$ be uniform random variables and $\Phi_\rho(\cdot)$ the bivariate normal with correlation coefficient $\rho$
- Then the Gaussian copula function can be written as:

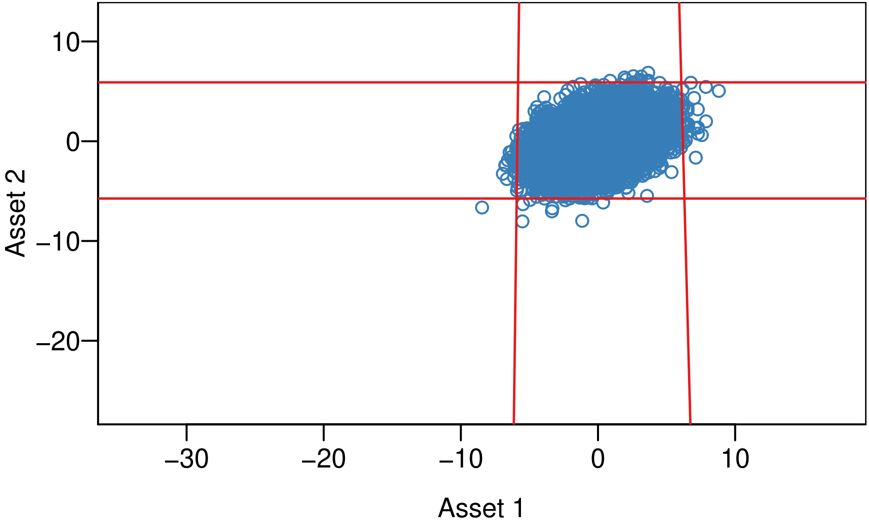
$$C(U, V) = \Phi_\rho(\Phi^{-1}(U), \Phi^{-1}(V))$$

- This function allows us to join the two marginal distributions into a single bivariate distribution

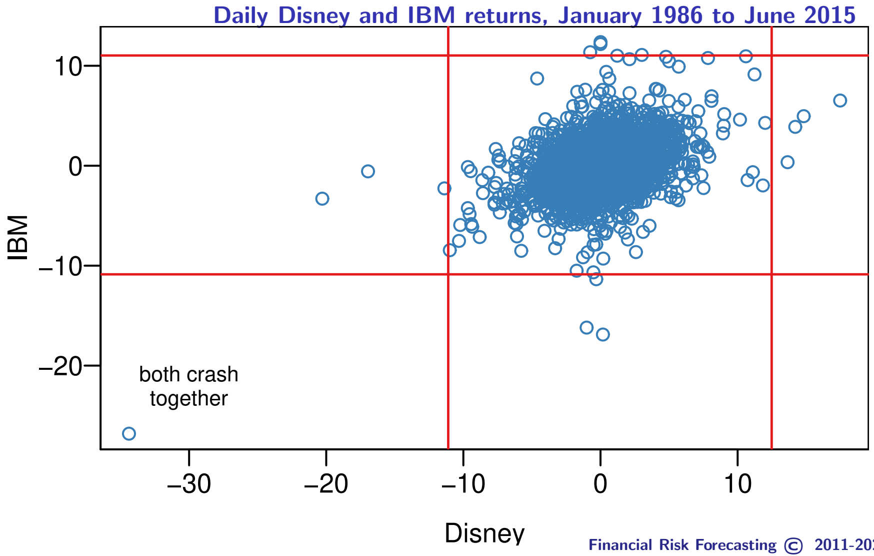
# Application of Copulas

- To illustrate we use the same data on Disney and IBM as used before
- By comparing a scatterplot for simulated bivariate normal data with one for the empirical data, we see that the two do not have the same joint extremes

# Gaussian Scatterplot



# Empirical Scatterplot

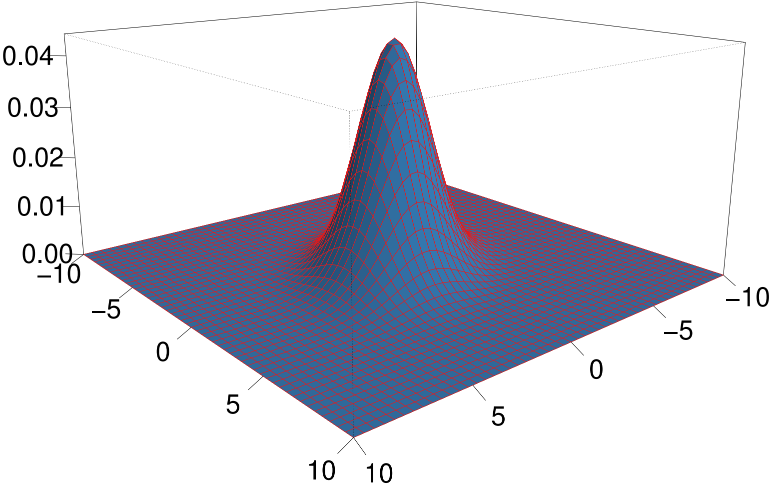


# Application of Copulas

- We estimate two copulas for the data, a Gaussian copula and a Student-t copula
- The copulas can be drawn in three dimensions

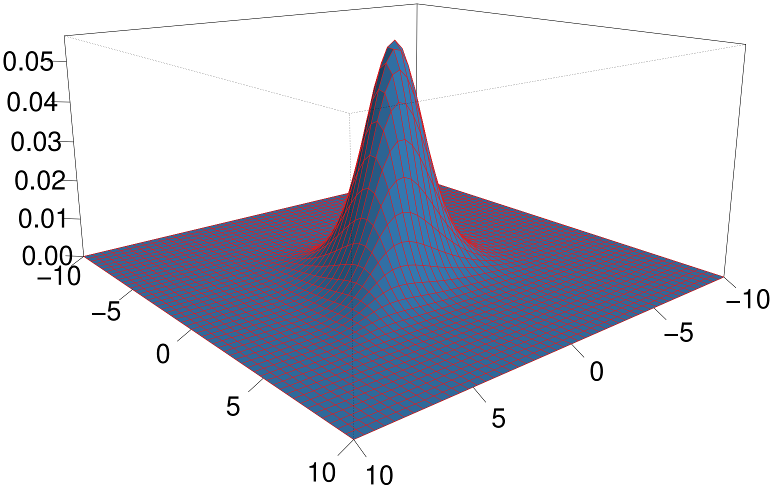
# Fitted Gaussian Copula

## Daily Disney and IBM returns, January 1986 to June 2015



# Fitted Student-t Copula

## Daily Disney and IBM returns, January 1986 to June 2015

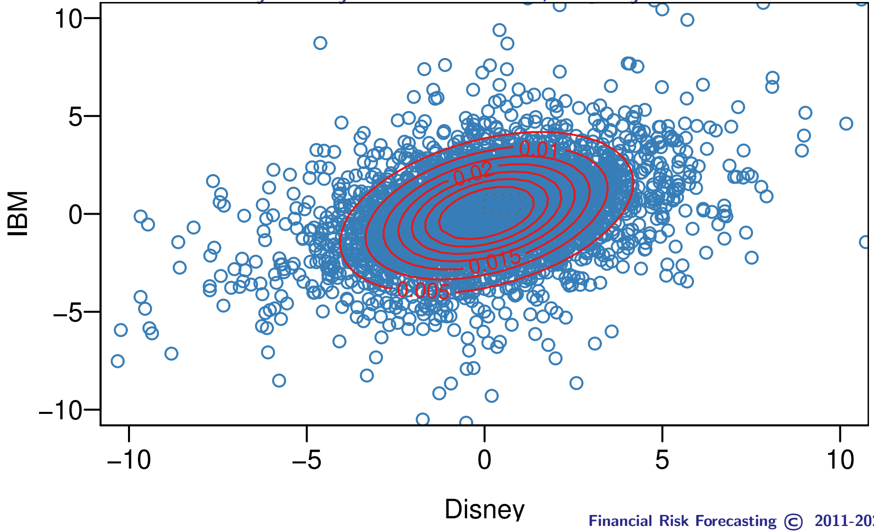


# Application of Copulas

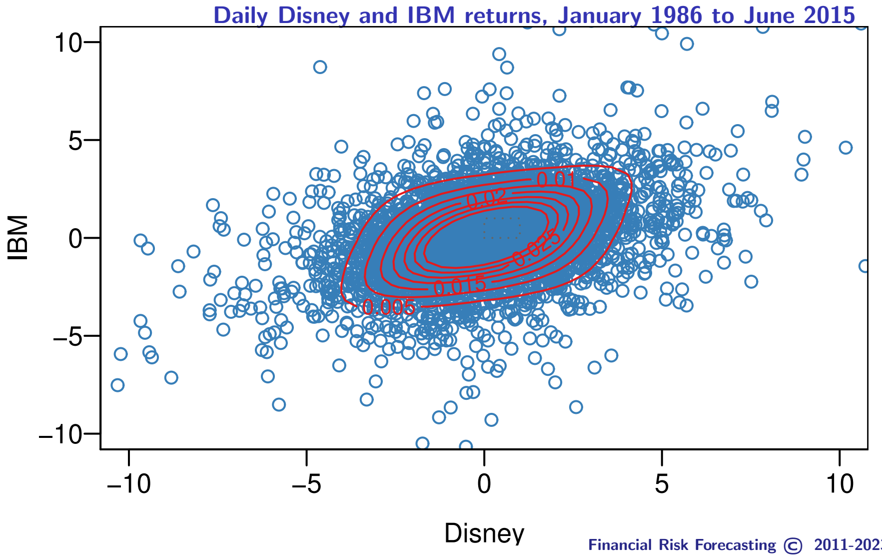
- It can be difficult to compare distributions by looking at three-dimensional graphs
- Contour plots may give a better comparison

# Contours of Gaussian Copula

**Daily Disney and IBM returns, January 1986 to June 2015**



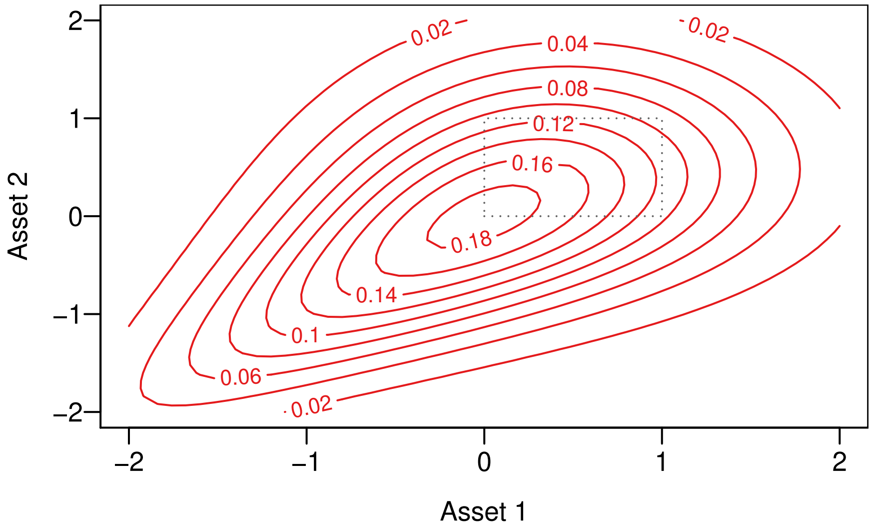
# Contours of Student-t Copula



# Clayton's Copula

- As noted earlier, there are a number of copulas available
- One widely used is the Clayton copula, which allows for asymmetric dependence
- Parameter $\theta$ measures the strength of dependence
- We estimate a Clayton copula for the same data as before

# Contours of Clayton's Copula, $\theta = 1$



# Contours of Clayton's Copula, $\theta = 0.483$