Introduction to Algorithm Design

Lecture Notes 5

ROAD MAP

- **Divide And Conquer**
  - Binary Search
  - Maximum Subsequence Sum Problem
  - Merge Sort
  - Quick Sort
  - Binary Tree and Its Properties
  - **Multiplication of Large Integers**
  - **Strassen’s Matrix Multiplication**
  - Closest Pair of Points
  - Convex Hull

Multiplication of Large Integers

- Some applications require manupilation of large integers (over 100 decimal digits long)
  - Such as cryptology
- Such integers are too long to fit in a special word of a modern computer
  - They require special treatment
  - Does not take unit time

Multiplication of Large Integers

- Classical pen-pencil algorithm for multiplying two *n-digit* integer
  - Each of *n* digits of the first number is mulipled by each of *n* digits of second number
- The total is n^2 digit multiplications
- Is it possible to design an algorithm with fewer than n^2 digit multiplication?

Multiplication of Large Integers

Example: multiply 23 and 14

$$23 = 2 \cdot 10^1 + 3 \cdot 10^0 \text{ and } 14 = 1 \cdot 10^1 + 4 \cdot 10^0.$$

$$\begin{aligned} 23 &= (2 \cdot 10^1 + 3 \cdot 10^0) * (1 \cdot 10^1 + 4 \cdot 10^0) \\ &= (2 * 1)10^2 + (3 * 1 + 2 * 4)10^1 + (3 * 4)10^0 \end{aligned}$$

- There are 4 multiplications in total
- The middle term can also be calculated as

$$3 * 1 + 2 * 4 = (2 + 3) * (1 + 4) - (2 * 1) - (3 * 4)$$

- So the result can be obtained by three multiplications only

Multiplication of Large Integers

In general:

For any pair of two-digit integers $a = a_1 a_0$ and $b = b_1 b_0$, their product ***c*** can be computed by the formula

$$c = a*b = c_2 10^2 + c_1 10^1 + c_0$$

where

$c_2 = a_1 * b_1$ → product of their first digits

$c_0 = a_0 * b_0$ → product of their second digits

$c_1 = (a_1 + a_0) * (b_1 + b_0) - (c_2 + c_0)$ → product of the sum of the *a's* digits and the sum of the *b's* digits minus the sum of c_2 and c_0

Multiplication of Large Integers

- **<u>Approach :</u>**

  If we want to multiply two *n-digit* integers a and b where a is positive even number

  - Divide both numbers in the middle
  - Denote first half of the a's digits by a_1 and second half by a_0
    - Same notations for b
  - $a = a_1 a_0$ implies that $a = a_1 10^{n/2} + a_0$ and

    $b = b_1 b_0$ implies that $b = b_1 10^{n/2} + b_0$

Multiplication of Large Integers

- We get

  $c = a*b = (a_1 10^{n/2} + a_0) * (b_1 10^{n/2} + b_0)$

  $c = (a_1*b_1) 10^n + (a_1*b_0 + a_0*b_1) 10^{n/2} + (a_0*b_0)$

  $c = c_2 10^n + c_1 10^{n/2} + c_0$

  where

  $c_2 = a_1*b_1$ → product of their first halves

  $c_0 = a_0*b_0$ → product of their second halves

  $c_1 = (a_1+a_0)*(b_1+b_0)-(c_2+c_0)$ → product of the sum of the *a's* halves and the sum of the *b's* halves minus the sum of c_2 and c_0

Multiplication of Large Integers

- If *n/2* is even, we can apply same method for computing products of c_2, c_1 and c_0.
- Thus we have a recursive algorithm to compute product of two *n-digit* integers
- Recursion is stopped
  - when n becomes 1
  - when we deem *n* small enough to multiply the numbers of that size directly

Multiplication of Large Integers

- <u>**Analysis :**</u>

  How many digit multiplications does this algorithm make?

Multiplication of Large Integers

- **Analysis :**

  Multiplication of n-digit numbers requires three multiplications of *n/2* digit number

  So

  $$M(n) = 3M(n/2) \text{ for } n > 1, \quad M(1) = 1$$

  solving it by backward substitution for n = 2^k yields

  $$M(2^k) = 3M(2^{k-1}) = 3[3M(2^{k-2})] = 3^2 M(2^{k-2})$$
  $$= \cdots = 3^i M(2^{k-i}) = \cdots = 3^k M(2^{k-k}) = 3^k.$$

  since k = $\log_2 n$

  $$M(n) = 3^{\log_2 n} = n^{\log_2 3} \approx n^{1.585},$$

Multiplication of Large Integers

Discussion :

- Used in many problems today
  - Cryptography
  - Security units of mobile devices
- Divide and conquer algorithm outperform the pen-and-pencil method on integers over 600 digits long

Matrix Multiplication

- **Problem Definition :**

Find product *C* of two *n-by-n* matrices *A* and *B*

- We will see that matrix multiplication can be done using less than n^3 scalar multiplications

Matrix Multiplication

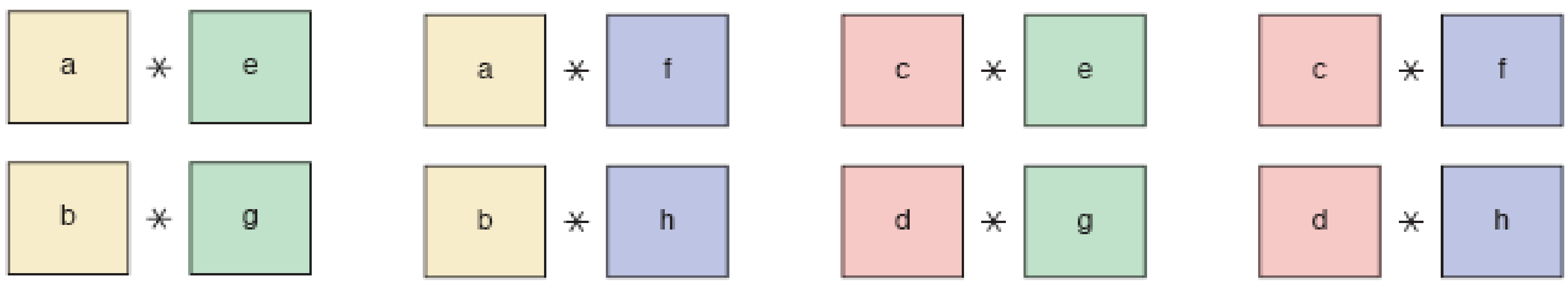
A simple divide and conquer strategy:

- Let *A* and *B* be two *n-by-n* matrices where *n* is a power of *2*
- We can divide *A, B* and their product *C* into four *n/2-by-n/2* submatrices each as follows

$$\begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} * \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

Matrix Multiplication

<u>8 Sub-Problems:</u>



<u>Analysis:</u>

- 8 multiplication operation → (n/2)-by-(n/2) matrix
- 4 addition operation → (n/2)-by-(n/2) matrix

- $T(n)=8*T(n/2)+\Theta(n^2) = \Theta(n^3)$

Strassen's Matrix Multiplication

- To perform matrix multiplication using less than n^3 scalar multiplications
- First lets consider the case of *2-by-2* matrix multiplication
  - We will show that this can be done using 7 multiplications instead of 8 multiplications required by brute-force algorithm.

Strassen's Matrix Multiplication

We can use the following formulas

$$\begin{bmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{bmatrix} = \begin{bmatrix} a_{00} & a_{01} \\ b_{10} & b_{11} \end{bmatrix} * \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix}$$

$$= \begin{bmatrix} m_1 + m_4 - m_5 + m_7 & m_3 + m_5 \\ m_2 + m_4 & m_1 + m_3 - m_2 + m_6 \end{bmatrix}$$

where

$$m_1 = (a_{00} + a_{11}) * (b_{00} + b_{11})$$
$$m_2 = (a_{10} + a_{11}) * b_{00}$$
$$m_3 = a_{00} * (b_{01} - b_{11})$$
$$m_4 = a_{11} * (b_{10} - b_{00})$$
$$m_5 = (a_{00} + a_{01}) * b_{11}$$
$$m_6 = (a_{10} - a_{00}) * (b_{00} + b_{01})$$
$$m_7 = (a_{01} - a_{11}) * (b_{10} + b_{11}).$$

Strassen's Matrix Multiplication

- There are 7 multiplications.
- But how many additions are there?
- Is it good idea to use this method for *2-by-2* matrices?

Strassen's Matrix Multiplication

$$\begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

7 multiplication operation

P1=a*(f-h) P2=(a+b)*h P3=(c+d)*e P4=d*(g-e)
P5=(a+d)*(e+h) P6=(b-d)*(g+h) P7=(a-c)*(e+f)

Solution:

a*e+b*g = P5+P4-P2+P6

a*f+b*h = P1+P2

c*e+b*h = P3+P4

c*f + d*h = P5+P1-P3-P7

Strassen's Matrix Multiplication

Approach:

- Let *A* and *B* be two *n-by-n* matrices
  - where *n* is a power of *2*
- Divide *A* and *B* into four *n/2-by-n/2* submatrices
- Calculate 7 submatrix multiplications recursively
- Perform required additions to obtain the matrix C

Strassen's Matrix Multiplication

- **Analysis :**

$$M(n) = 7M(n/2) \text{ for } n > 1, \quad M(1) = 1.$$

Since $n = 2^k$,

$$\begin{aligned} M(2^k) &= 7M(2^{k-1}) = 7[7M(2^{k-2})] = 7^2 M(2^{k-2}) = \cdots \\ &= 7^i M(2^{k-i}) \cdots = 7^k M(2^{k-k}) = 7^k. \end{aligned}$$

Since $k = \log_2 n$,

$$M(n) = 7^{\log_2 n} = n^{\log_2 7} \approx n^{2.807},$$

Strassen's Matrix Multiplication

Discussion :

- Saving in # of multiplications was achieved at the expense of making extra additions
  - We must check # of additions A(n)
  - $A(n) \in \Theta(n^{\log_2 7})$
  - Same order of growth as # of multiplication
- Efficiency is better than brute force
  - Brute force algorithm is n^3
- Is it good for memory efficiency?
- It is not the best algorithm for matrix multiplication
  - Coopersmith and Winogrand algorithm's efficiency is $O(n^{2.376})$

ROAD MAP

- **Divide And Conquer**
  - Binary Search
  - Maximum Subsequence Problem
  - Merge Sort
  - Quick Sort
  - Multiplication of Large Integers
  - Strassen’s Matrix Multiplication
  - **Closest Pair of Points**
  - **Convex Hull**

Closest-Pair Problem

- **Problem Definition :**

  Find the closest points in a set of *n* points.

  - We consider the two dimensional case of the problem
  - We assume that points in question are specified in a standard fashion by their *(x,y)* Cartesian coordinates
  - We assume that distance between two points Pi = (xi, yi) and Pj = (xj,yj) is the standard Euclidean distance

$$d(P_i, P_j) = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$$

Closest-Pair Problem

Approach :

1. Divide points given into two subsets S_1 and S_2 of n/2 points each by drawing a vertical line x=c

- c is be the median μ of x coordinates
- So, n/2 points lie on the left or on the line itself, and n/2 points lie on the right

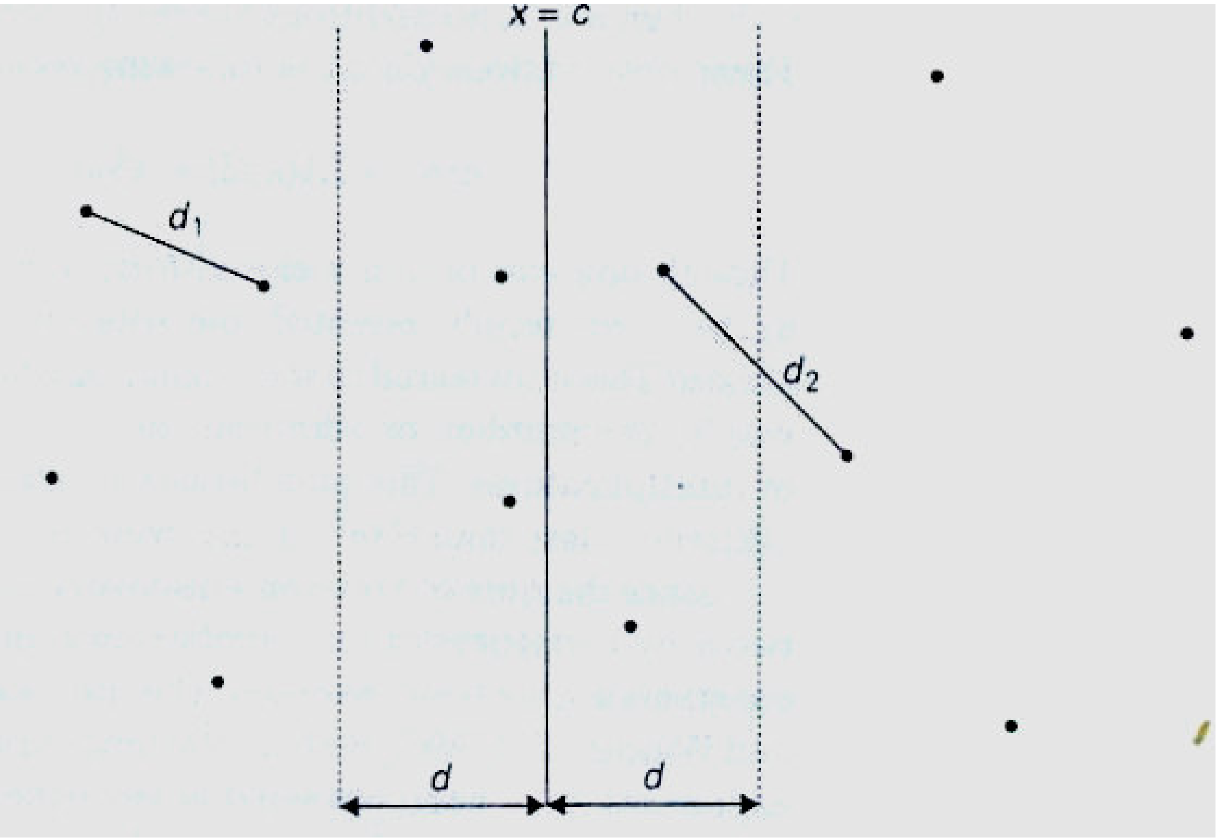
Closest-Pair Problem

Approach :

2. Find recursively the closest pair in the left subset S_1 and the right subset S_2

- let d_1 and d_2 be the smallest distances between pairs of points in S_1 and S_2
- let $d = \min\{d_1, d_2\}$
- *d* is not necessarily the smallest distance between all pairs of points in S_1 and S_2
  - A closer pair of points can lie on the opposite sides of the seperating line
- Should also consider the points in symetric vertical strip of width *2d*
  - the distance for any other pair of points is greater that *d*

Closest-Pair Problem



Idea of the d&c algorithm for the closest-pair problem

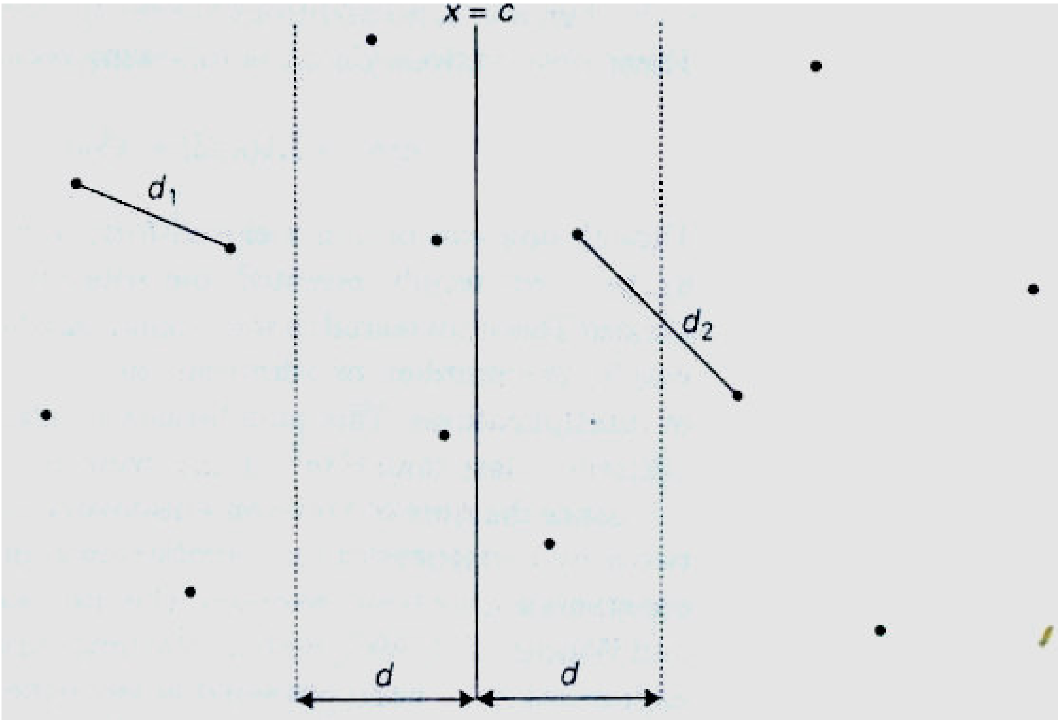
Closest-Pair Problem

Approach :

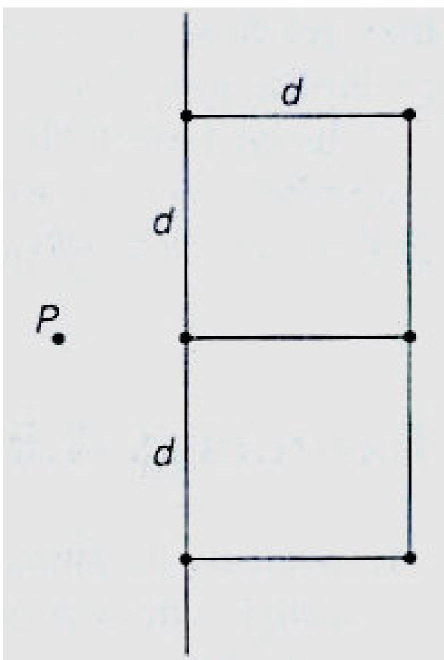
Let C_1 and C_2 be the subsets of points in the left and right parts of the strip respectively

4. For every point P(x,y) in C_1, inspect points in C_2 that may be closer to the *P* than *d*

- Such points must have their y coordinates in the interval [y−d, y+d]
- So there can be no more than such 6 points



(a) Idea of the d&c algorithm for the closest-pair problem



(b) The six points that may need to be examined for point *P* (worst case)

Closest-Pair Problem

Approach :

6. **Maintain list of points in C_1 and C_2 in ascending order of their y coordinates**
   - **This ordering can be maintained by merging two previously sorted lists**
7. **Process C_1 points sequentially while a pointer into the C_2 list scans an interval of width *2d***

Closest-Pair Problem

- **Analysis :**

The recurrence for T(n), on *n* presorted points :

$$T(n) = 2\,T(n/2) + M(n)$$

$$T(n) \in O(n \log n)$$

Convex-Hull Problem

- **Problem Definition :**

  Find the smallest convex polygon that contains n given points in a plane

Convex-Hull Problem

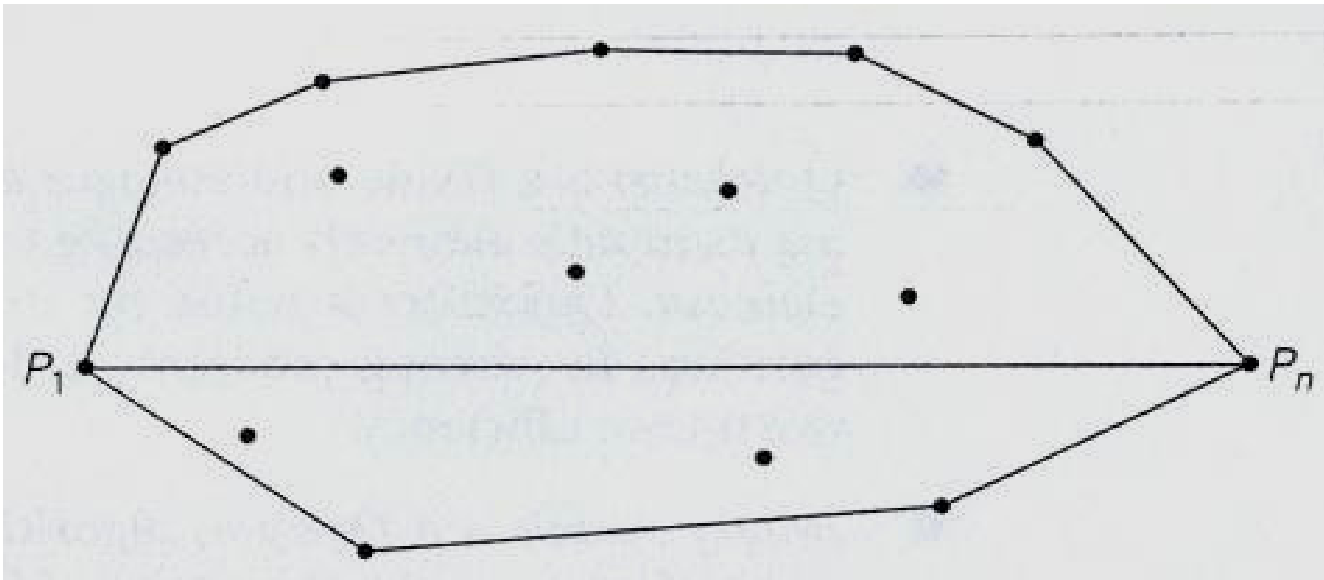
- There are several divide&conquer algorithms for the convex-hull problem.
- We will look at the simplest one !
- This algorithm is sometimes called quickhull
  - its operations resemble those of quicksort

Convex-Hull Problem

Any İdea????

Convex-Hull Problem

- Let $P_1 = (x_1, y_1), \ldots, P_n=(x_n, y_n)$ be a set of $n>1$ points in the plane
- We assume that points are sorted in increasing order of their *x* coordinates
- Leftmost point P_1 and rightmost point P_n are two distinct *extreme points*



Convex-Hull Problem

- Let $\overrightarrow{P_1P_n}$ be the straight line through point P_1 and P_n
- This line seperates the points of S into two sets
  - S_1 is the set of points on the left of this line
  - S_2 is the set of points on the right of this line
- The points of S on the line $\overrightarrow{P_1P_n}$ other than P_1 and P_n can not be extreme points of convex-hull

Convex-Hull Problem

- Boundary of convex hull of S is made of two polygonal chains
  - *Upper* boundary called *upper hull* is a sequence of line segments including
    - Line segment connecting P_1 and P_n
    - Line segments connecting some points in S_1
  - *Lower* boundary called *lower hull* is a sequence of line segments including
    - Line segment connecting P_1 and P_n
    - Line segments connecting some points in S_2
  - Convex-hull of the entire set is composed of the upper and lower hulls

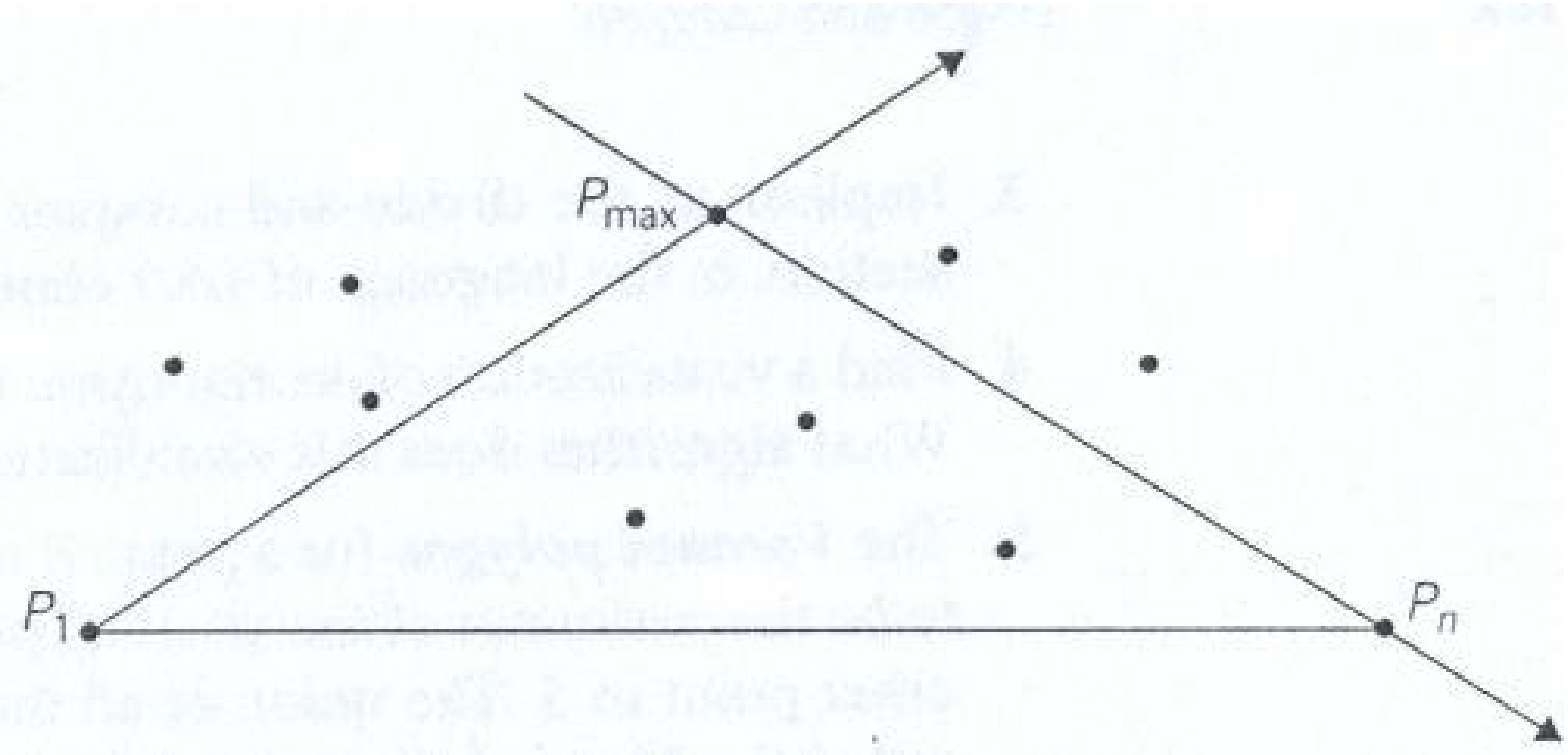
Convex-Hull Problem

- How to construct the upper hull and lower hull?
- Is the problem same as the original convex-hull problem?
- Where is the divide and conquer technique?

Convex-Hull Problem

- We will use divide-and-conquer technique to construct the upper hull and lower hull.
  - Then the algorithms is called quickhull

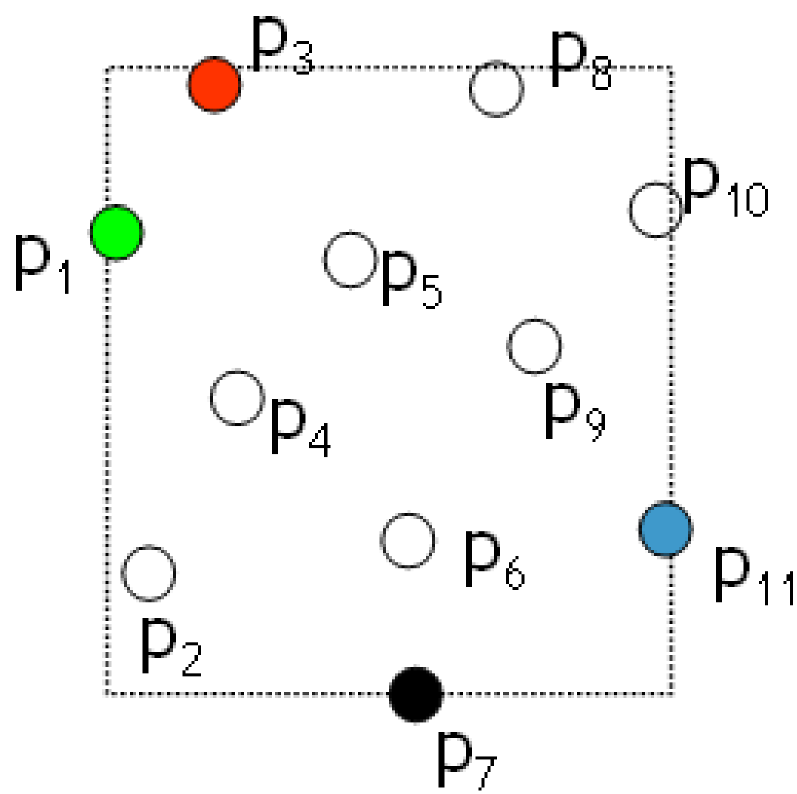
Convex-Hull Problem



Idea of quickhull

Quickhull Example

1. Find extreme points above, below, left and right
2. Define a quadrilateral connecting extreme points, discard internal points
3. Process 4 triangles recursively
4. For each triangle, find vertex with greater distance from the triangle baseline.
5. Define another triangle and discard internal triangles
6. Process 2 triangles recursively



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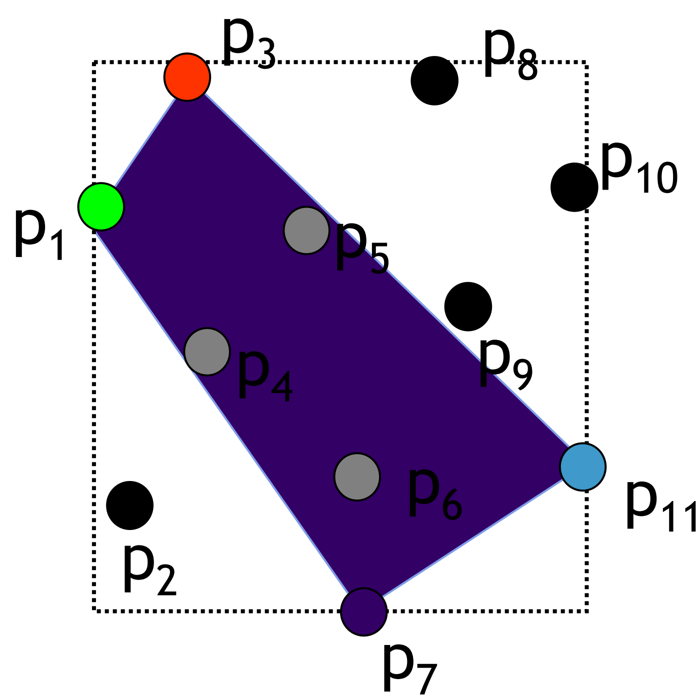

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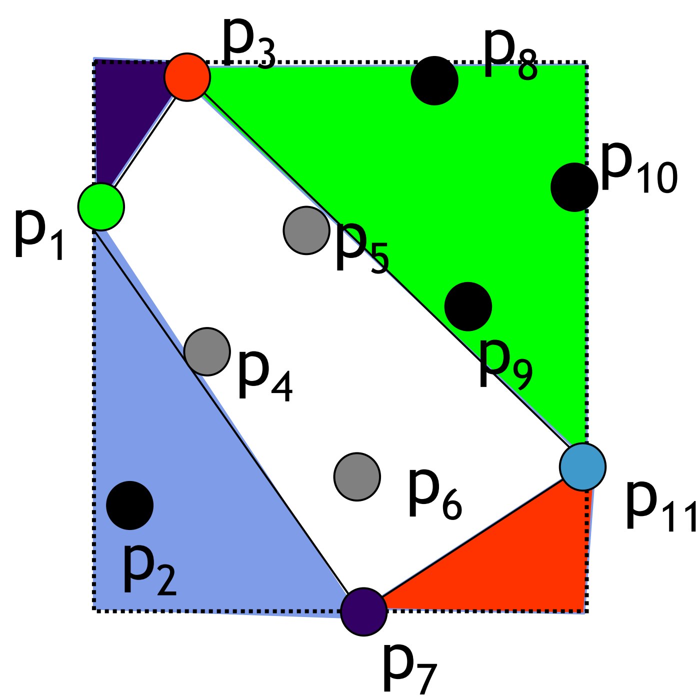

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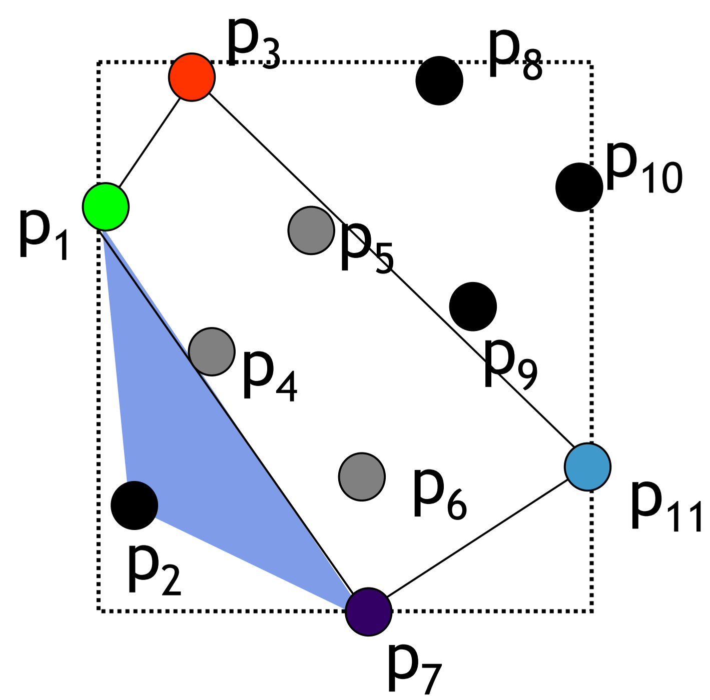

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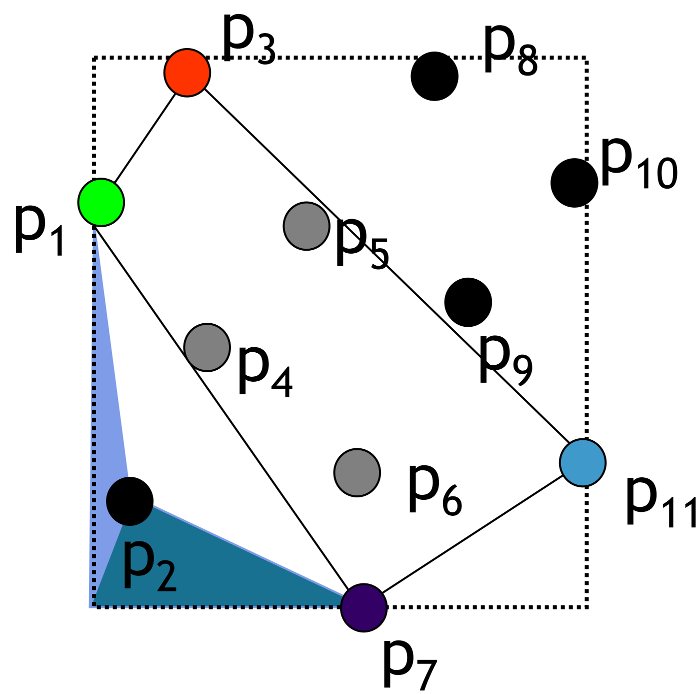

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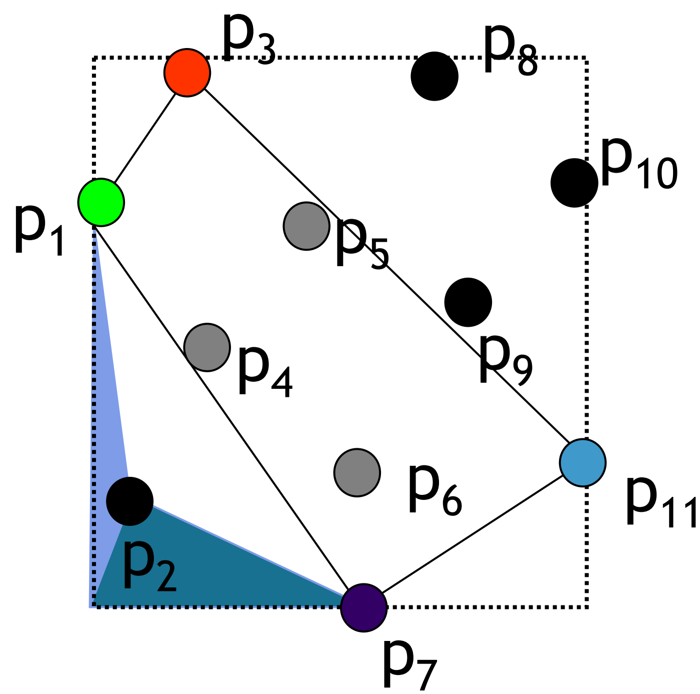

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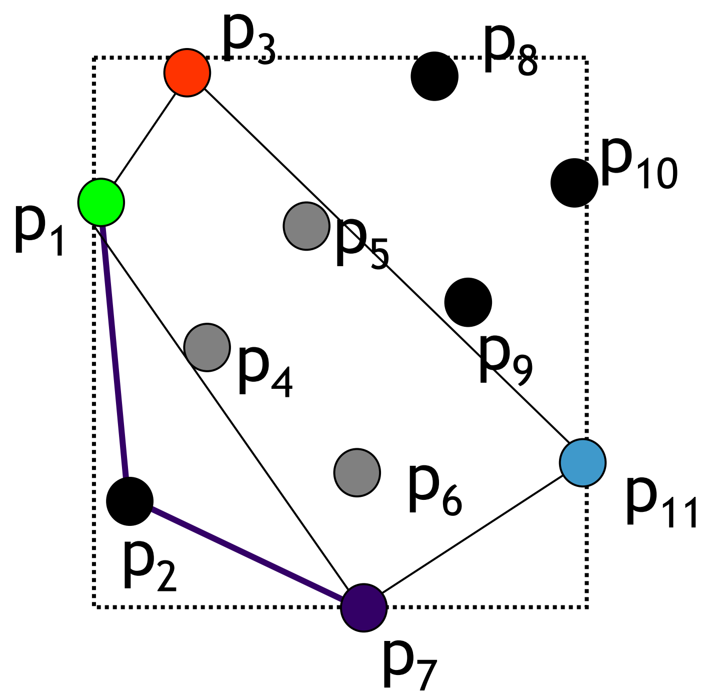

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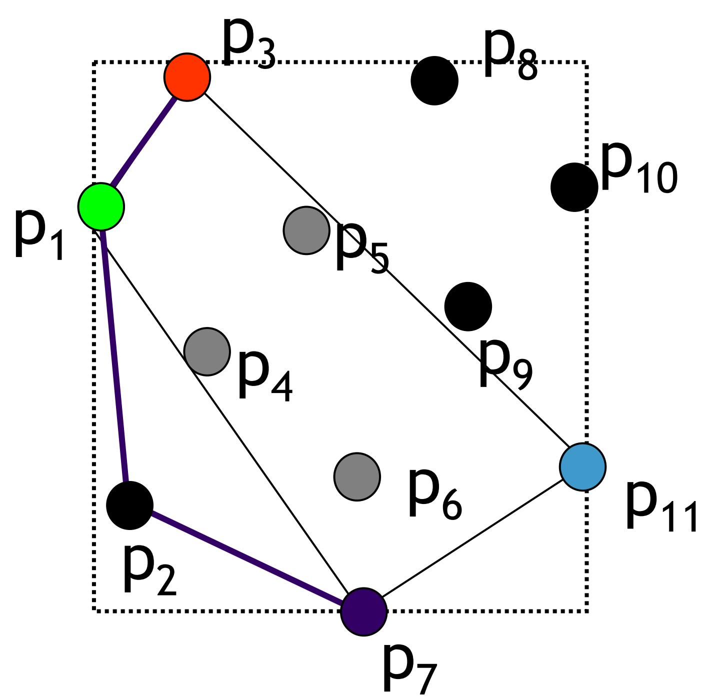

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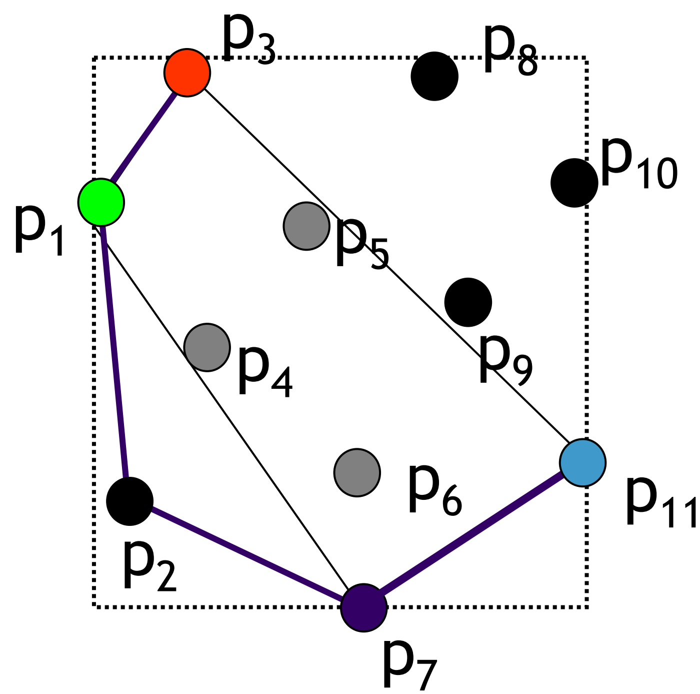

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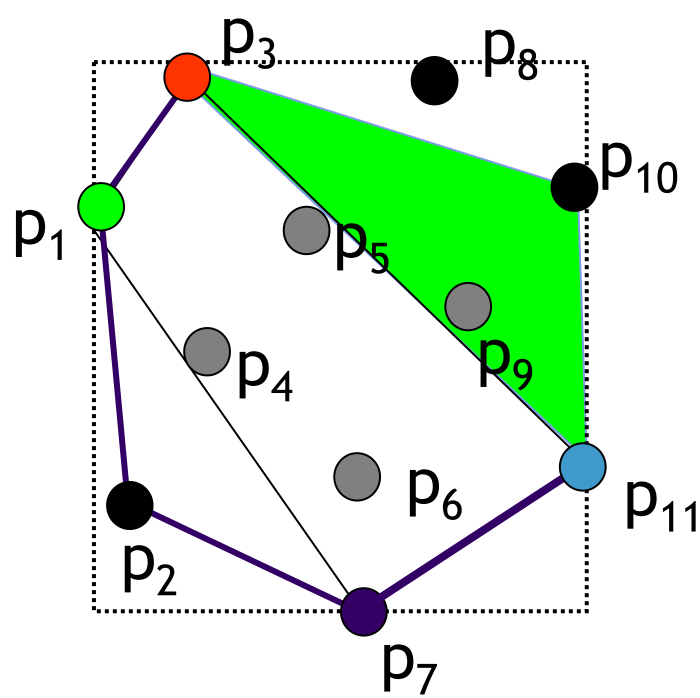

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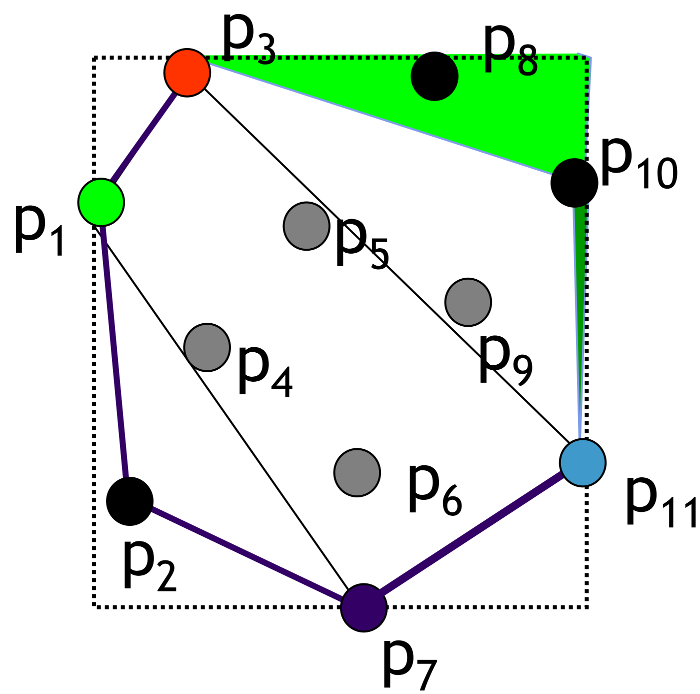

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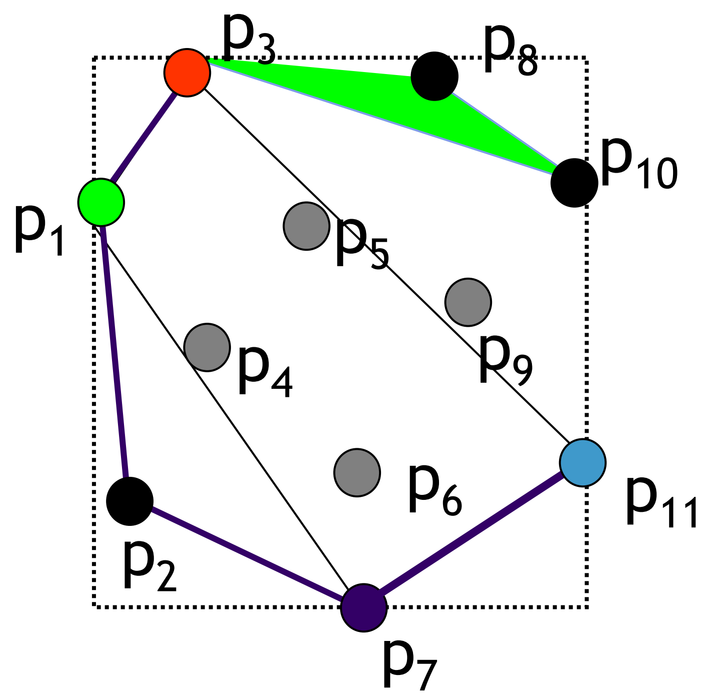

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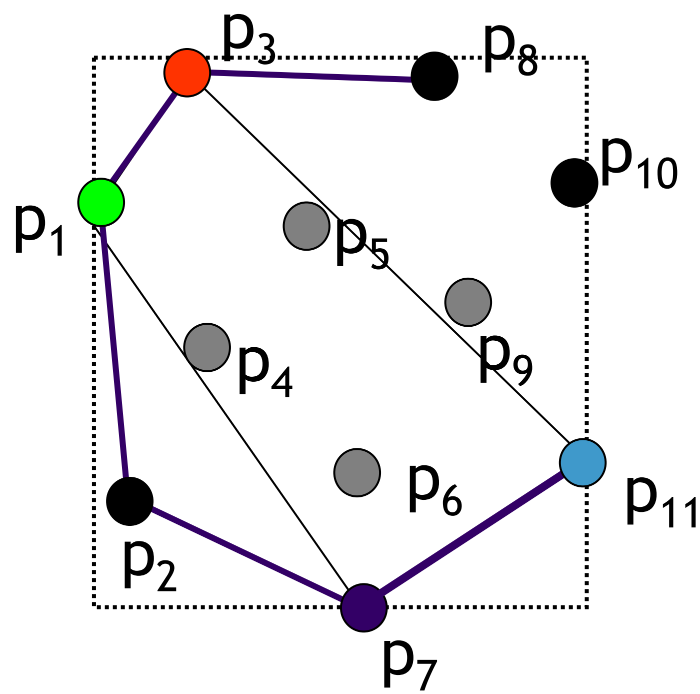

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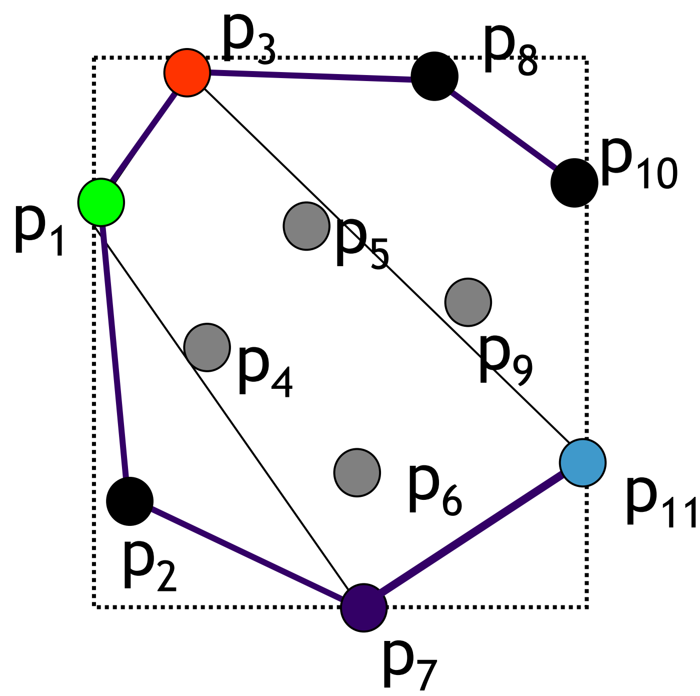

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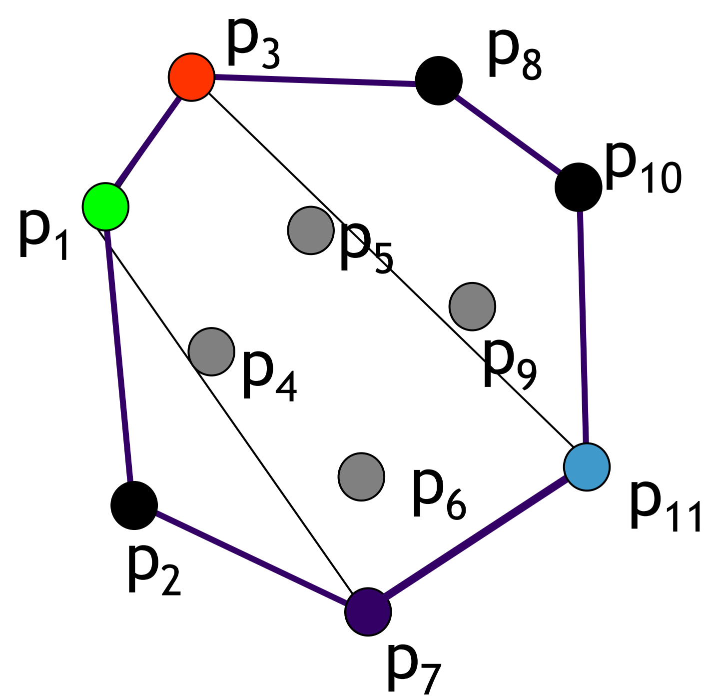

Convex-Hull Problem

- **Analysis :**

  - Worst case efficiency is $\Theta(n^2)$
    - As quicksort

  - In average case we expect a better performance
    - Under a natural assumption that points given are chosen randomly from a uniform distribution over some convex region, average case turns to be **Θ(nlogn)**

Convex-Hull Problem

Discussion :

- Quickhull has the same efficiency with quicksort
  - Θ`(nlogn)` in average case but $\Theta(n^2)$ in worst case
- There exists more sophisticated d&c algorithms for this problem with the worst case efficiency Θ`(nlogn)`

Divide & Conquer

- **Discussion :**

  There are 3 criterias for efficiency of D&C algorithms

  - # of subproblems
  - Proportion of the main problem and subproblem
  - Time to divide the problem and combine the sub-solutions