CSE 211: Discrete Mathematics **(Due: 01/12/20)**

Problem Session #6

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Problem 1: Functions

(0 points)

Let f be a function from S to T. Show that:
(a) f (S) ∪ f (T) = f (S ∪ T)
(Solution) When we check the equality of two functions, we have to prove that:

1. f (S) ∪ f (T) ⊆ f (S ∪ T) (→)
2. f (S ∪ T) ⊆ f (S) ∪ f (T) (←)

- f (S) ∪ f (T) ⊆ f (S ∪ T) (→)
  - Let y ∈ f(S) ∪ f(T), then y ∈ f(S) **OR** y ∈ f(T).
  - y ∈ f(S) **OR** y ∈ f(T) implies ∃ x ∈ S **OR** ∃ x ∈ T, such that f(x) = y.
  - ∃ x ∈ S **OR** ∃ x ∈ T = x ∈ S ∪ T.
  - x ∈ S ∪ T → y = f(x) ∈ f(S ∪ T) → y ∈ f(S ∪ T).
  - Recall that we accept y ∈ f(S) ∪ f(T) at the beginning.
  - Hence, f (S) ∪ f (T) ⊆ f (S ∪ T).
- f (S ∪ T) ⊆ f (S) ∪ f (T) (←)
  - Let y ∈ f(S ∪ T), then ∃ x ∈ S ∪ T s.t. f(x) = y.
  - ∃ x ∈ S ∪ T implies x ∈ S OR x ∈ T then f(x) ∈ f(S) OR f(x) ∈ f(T).
  - f(x) ∈ f(S) OR f(x) ∈ f(T) = f(x) ∈ f(S) ∪ f(T).
  - Recall that we accept y ∈ f(S ∪ T) at the beginning.
  - Hence, f (S ∪ T) ⊆ f (S) ∪ f (T).

(b) f (S ∩ T) ⊆ f (S) ∩ f (T)

(Solution) Let y ∈ f(S ∩ T) then ∃ x ∈ S ∩ T s.t. f(x) = y.

∃ x ∈ S ∩ T → x ∈ S AND x ∈ T then f(x) ∈ f(S) AND f(x) ∈ f(T).

We combine y = f(x) ∈ f(S) AND y = f(x) ∈ f(T), then y = f(x) ∈ f(S) ∩ f(T).

Hence, f (S ∩ T) ⊆ f (S) ∩ f (T).

Problem 2: Hasse Diagram

(0 points)

Draw Hasse diagram for ({3, 4, 12, 24, 48, 72}, ÷).

(Solution) To draw a Hasse diagram, provided set must be a poset.
A poset or partially ordered set A is a pair, (B, $\leq$) of a set B whose elements are called the vertices of A and obeys following rules:

- Reflexivity : $p \leq p \ \forall \ p \in B$.
- Anti-symmetric : $p \leq q$ and $q \leq p$ iff $p = q$.
- Transitivity : if $p \leq q$ and $q \leq r$ then $p \leq r$.

We have to find the poset for the divisibility. "÷" says "a relation (a, b) where a, b ∈ B, a divides b."
Let the set is A. Then,
$A = \{(3 \prec 3), (4 \prec 4), (12 \prec 12), (24 \prec 24), (48 \prec 48), (72 \prec 72), (3 \prec 12), (3 \prec 24), (3 \prec 48), (3 \prec 72), (4 \prec 12), (4 \prec 24), (4 \prec 48), (4 \prec 72), (12 \prec 24), (12 \prec 48), (12 \prec 72), (24 \prec 48), (24 \prec 72)\}$
Let's draw the directed graph the relation above in Figure 1:

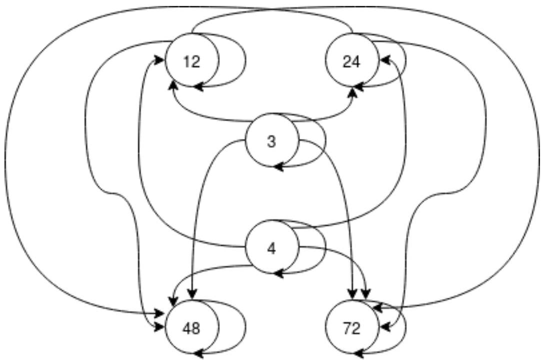

Figure 1: The original graph

Since we know that a poset MUST provide reflexivity, we also do not need the reflexive relations in A. Hence A can be updated as:
$A = \{(3 \prec 12), (3 \prec 24), (3 \prec 48), (3 \prec 72), (4 \prec 12), (4 \prec 24), (4 \prec 48), (4 \prec 72), (12 \prec 24), (12 \prec 48), (12 \prec 72), (24 \prec 48), (24 \prec 72)\}$
In the next step, remove the self-loops in Figure 2:

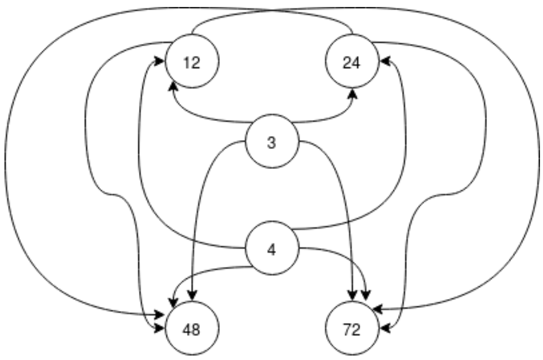

Figure 2: The graph without self-loops

Since we know that a poset MUST provide transivity, we also do not need the transitive relations in A. Hence A can be updated as:
A = $\{(3 \prec 12), (4 \prec 12), (12 \prec 24), (24 \prec 48), (24 \prec 72)\}$
Remove the transivite edges and the hasse diagram is obtained in Figure 3:

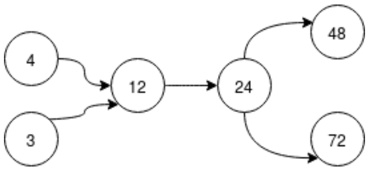

Figure 3: The hasse diagram of ({3, 4, 12, 24, 48, 72}, ÷)

Problem 3: Special elements of POSETs (0 points)

Consider the poset of ({3, 4, 12, 24, 48, 72}, ÷) in Problem 2. Remember that A is a poset and it is shown as:

A = $\{(3 \prec 12), (4 \prec 12), (12 \prec 24), (24 \prec 48), (24 \prec 72)\}$

(a) Find the maximal elements.
(Solution)
Definition: Let (A,$\prec$) be a partially ordered set (poset). An element a $\in$ (A, $\prec$) is called **maximal** if there is no other element b $\in$ A such that a $\prec$ b. That is, an element a is maximal if it has no immediate successor.

In a Hasse diagram, a vertex corresponds to a maximal element if there is no edge leaving the vertex.

In Figure 3, we can see that 48 and 72 are maximal elements of the poset.

(b) Find the minimal elements.
(Solution)
Definition: An element a $\in$ (A, $\prec$) is called minimal if there is no other element b $\in$ A such that b $\prec$ a. In other words, an element a is minimal if it has no immediate predecessor.

In a Hasse diagram, a vertex corresponds to a minimal element if there is no edge entering the vertex.

In Figure 3, we can see that 3 and 4 are minimal elements of the poset.

(c) Is there a greatest element?
(Solution)
Definition: An element a $\in$ (A, $\prec$) is called the greatest (maximum) element if it is greater than every other element of the poset: b $\prec$ a, $\forall$ b $\in$ A.

In the example, the greatest element does not exist since there is no any one element that succeeds all the elements.
(d) Is there a least element?
(Solution)
Definition: An element a $\in$ (A, $\prec$) is called the least (minimum) element if it is less than every other element of the poset: a $\prec$ b, $\forall$ b $\in$ A.

In the example, the least element does not exist since there is no any one element that precedes all the elements.

(e) Let S = $\{3, 4, 12\}$ be a subset of $\{3, 4, 12, 24, 48, 72\}$. What is the upper bounds of S?
(Solution)

Definition: Let (S, $\prec$) be a poset of a set A. For elements x and y $\in$ A, if there is an element z $\in$ A such that (x $\prec$ z) and (y $\prec$ z), then z is called an upper bound of x and y.

Considering the definition, the upper bounds of S is 12 and 24 since (3 $\prec$ 12), (4 $\prec$ 12), (12 $\prec$ 24) in Figure 3.

(f) Let S = $\{12, 48\}$ be a subset of $\{3, 4, 12, 24, 48, 72\}$. What is the lower bounds of S?
(Solution)

Definition: Let (S, $\prec$) be a poset of a set A. For elements x and y $\in$ A, if there is an element z $\in$ A such that (z $\prec$ x) and (z $\prec$ y), then z is called an lower bound of x and y.

Considering the definition, the lower bounds of S is 3, 4 and 24 since (3 $\prec$ 12), (4 $\prec$ 12), (24 $\prec$ 48) in Figure 3.

Problem 4: Functions (0 points)

Find these values.

(a) $\lceil \frac{3}{4} \rceil$

(Solution) 1

(b) $\lfloor \frac{7}{8} \rfloor$

(Solution) 0

(c) $\lceil \frac{-3}{4} \rceil$

(Solution) 0

(d) $\lfloor \frac{-7}{8} \rfloor$

(Solution) -1

(e) $\lceil 3 \rceil$

(Solution) 3

(f) $\lfloor -1 \rfloor$

(Solution) -1

(g) $\lfloor \frac{1}{2} + \lceil \frac{3}{2} \rceil \rfloor$

(Solution) 2

(h) $\lfloor \frac{1}{2} \lfloor \frac{5}{2} \rfloor \rfloor$

(Solution) 1