CMPE 556 - Midterm Q3

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1 Perron–Frobenius Theorem

To understand the Perron–Frobenius theorem, first of all we need to understand eigenvalues and eigenvectors well. According to the Equation 1:

$$A \times v = \lambda \times v \tag{1}$$

where $A \in R^{mxm}$, $v \in R^{mx1}$ is an eigenvector and $\lambda \in R^{mxm}$ is a matrix, where the diagonals are the eigenvalues.

Perron–Frobenius theorem has been used in many areas such as Markov chains, population models and economics. We can evaluate the Perron–Frobenius theorem on positive and nonnegative matrices separately. Positive matrices mean that all the elements in the matrix $a_{ij} > 0$ and nonnegative matrices mean that $a_{ij} \geq 0$.

Firstly, I will analyze the theorem for positive matrices. The theorem says that positive r is a Perron-Frobenius eigenvalue of positive matrix A, which for other eigenvalues λ is $|\lambda| \leq \ r$. Moreover, A has right and left positive eigenvectors, whose eigenvalue is r. There is an eigenvector v and eigenvalue λ that all the components of the v are positive [1]. The other name of the r is Perron root [2], which is $r > 0$. The Perron root is simple (alg $mult_A(r) = 1$).

There is a vector p called Perron vector, which satisfies the formulas below:

$$A \times p = \lambda \times p, p > 0, ||p||_1 = 1 \tag{2}$$

A has no nonnegative eigenvectors except for p. The Collatz–Wielandt formula is as follows:

$$r = max_{x \in N} f(x), \tag{3}$$

where $0 \leq x, f(x) = \min\limits_{1 \leq i \leq n} \frac{[Ax]_i}{x_i}$

According to this formula, x represents the non-negative non-zero vectors. The formula is valid for both nonnegative and positive matrices. Perron–Frobenius eigenvalue r is simple and its eigenspaces are one-dimensional. It satisfies the inequality below:

$$\min_i \sum_j a_{ij} \leq r \leq \max_i \sum_j a_{ij}, \tag{4}$$

where a_{ij} is the element of matrix A.

According to [3], Perron-Frobenius theorem states that if A is a non-negative and irreducible matrix, $p(A) > 0$, where $p(A)$ is an eigenvalue of A [4]. Moreover, positive eigenvector v exists such that $A \times v = \lambda \times v$.

References

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