$$N^T P = 0$$

$$P' = MP \tag{B.36}$$

$$N'^T P' = 0 \tag{B.37}$$

$$N'^T MP = 0 \tag{B.38}$$

$$N'^T = N^T M^{-1} \tag{B.39}$$

$$N^T M^{-1} MP = N^T P = 0 \tag{B.40}$$

In order to transform a vector (a, b, c), treat it as a normal vector for a plane passing through the origin $[a, b, c, 0]$ and post-multiply it by the inverse of the transformation matrix (Eq. B.39). If it is desirable to keep all vectors as column vectors, then Equation B.41 can be used.

$$N' = \left(N'^T\right)^T = \left(N^T M^{-1}\right)^T = \left(M^{-1}\right)^T N \tag{B.41}$$

B.3.3 Axis-angle rotations

Given an axis of rotation $A = [\ a_x\ \ a_y\ \ a_z]$ of unit length and an angle θ to rotate by (Figure B.26), the rotation matrix M can be formed by Equation B.42. This is a more direct way to rotate a point around an axis, as opposed to implementing the rotation as a series of rotations about the global axes.

$$\hat{A} = \begin{bmatrix} a_x a_x & a_x a_y & a_x a_z \\ a_y a_x & a_y a_y & a_y a_z \\ a_z a_x & a_z a_x & a_z a_z \end{bmatrix}$$

$$A^* = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_z & a_x & 0 \end{bmatrix} \tag{B.42}$$

$$M = \hat{A} + \cos\theta\left(I - \hat{A}\right) + \sin\theta A^*$$
$$p' = MP$$

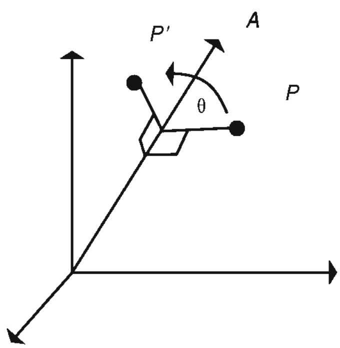

FIGURE B.26

Axis-angle rotation.

B.3.4 Quaternions

Quaternions are discussed in Chapter 2.4 and Chapter 3.1. The equations from those chapters, along with additional equations, are collected here to facilitate the discussion.

Quaternion arithmetic

Quaternions are four-tuples and can be considered as a scalar combined with a vector (Eq. B.43). Addition and multiplication are defined for quaternions by Equations B.44 and B.45, respectively. Quaternion multiplication is associative (Eq. B.46), but it is not commutative (Eq. B.47). The magnitude of a quaternion is computed as the square root of the sum of the squares of its four components (Eq. B.48). Quaternion multiplication has an identity (Eq. B.49) and an inverse (Eq. B.50). The inverse distributes over quaternion multiplication similarly to how the inverse distributes over matrix multiplication (Eq. B.51). A quaternion is normalized by dividing it by its magnitude (Eq. B.52). Equation B.52a is used to compute the quaternion that represents the rotation r such that q represents the half-way rotation between p and r.

$$q = [s, x, y, z] = [s, v] \tag{B.43}$$

$$[s_1, v_1] + [s_2, v_2] = [s_1 + s_2, v_1 + v_2] \tag{B.44}$$

$$[s_1, v_1][s_2, v_2] = [s_1\ s_2 - v_1 \,.\, v_2, s_1 v_2 + s_2 v_1 + v_1 \times v_2] \tag{B.45}$$

$$(q_1 q_2) q_3 = q_1 (q_2 q_3) \tag{B.46}$$

$$q_1 q_2 \neq q_2 q_1 \tag{B.47}$$

$$\|q\| = \sqrt{s^2 + x^2 + y^2 + z^2} \tag{B.48}$$

$$[s, v][1, (0,0,0)] = [s, v] \tag{B.49}$$

$$\begin{gathered} q^{-1} = (1/\|q\|)^2 [s, -v] \\ q^{-1} q = q q^{-1} = [1, (0,0,0)] \end{gathered} \tag{B.50}$$

$$(pq)^{-1} = q^{-1} p^{-1} \tag{B.51}$$

$$qunit = q/(\|q\|) \tag{B.52}$$

$$r = 2(p \cdot q) q - p \tag{B.52a}$$

Rotations by quaternions

A point in space is represented by a vector quantity in quaternion form by using a zero scalar value (Eq. B.53). A quaternion can be used to rotate a vector using quaternion multiplication (Eq. B.54). Compound rotations can be implemented by premultiplying the corresponding quaternions (Eq. B.55), similar to what is routinely done when rotation matrices are used. As should be expected, compounding a rotation with its inverse produces the identity transformation for vectors (Eq. B.56). An axis-angle rotation is represented by a unit quaternion, as shown in Equation B.57. Any scalar multiple of a quaternion represents the same rotation. In particular, the negation of a quaternion (negating each of its four components, $-q = [-s, -x, y, -z]$) represents the same rotation that the original quaternion represents (Eq. B.58).

$$v = [0, x, y, z] \tag{B.53}$$

$$v' = \mathrm{Rot}(v) = q v q^{-1} \tag{B.54}$$

$$\begin{aligned}\mathrm{Rot}_q\big(\mathrm{Rot}_p(v)\big) &= q(pvp^{-1})q^{-1} \\ &= ((qp)v(p^{-1}q^{-1})) \\ &= ((qp)v(qp^{-1})) \\ &= \mathrm{Rot}_{qp}(v)\end{aligned} \tag{B.55}$$

$$\begin{aligned}\mathrm{Rot}^{-1}(\mathrm{Rot}(v)) &= q^{-1}(qvq^{-1})q \\ &= (q^{-1}q)v(q^{-1}q) = v\end{aligned} \tag{B.56}$$

$$\mathrm{Rot}_{[\theta,x,y,z]} \equiv [\cos(\theta/2),\ \sin(\theta/2)(x,y,z)] \tag{B.57}$$

$$\begin{aligned}-q &\equiv \mathrm{Rot}_{[-\theta,-x,-y,-z]} \\ &= [\cos(-\theta/2),\ \sin(-\theta/2)(-(x,y,z))] \\ &= [\cos(\theta/2), -\sin(\theta/2)(-(x,y,z))] \\ &= [\cos(\theta/2),\ \sin(\theta/2)(x,y,z)] \\ &\equiv \mathrm{Rot}_{[\theta,x,y,z]} \\ &\equiv q\end{aligned} \tag{B.58}$$

Conversions

It is often useful to convert back and forth between rotation matrices and quaternions. Often, quaternions are used to interpolate between orientations, and the result is converted to a rotation matrix so as to combine it with other matrices in the display pipeline.

Given a unit quaternion ($q = [s, x, y, z]$, $s^2 + x^2 + y^2 + z^2 = 1$), one can easily determine the corresponding rotation matrix by rotating the three unit vectors that correspond to the principal axes. The rotated vectors are the columns of the equivalent rotation matrix (Eq. B.59).

$$M_q = \begin{bmatrix} 1-2y^2-2z^2 & 2xy-2sz & 2xz+2sy \\ 2xy+2sz & 1-2x^2-2z^2 & 2yz-2sx \\ 2xy-2sy & 2yz+2sx & 1-2x^2-2y^2 \end{bmatrix} \tag{B.59}$$

Given a rotation matrix, one can use the definitions for the terms of the matrix in Equation B.59 to solve for the elements of the equivalent unit quaternion. The fact that the unit quaternion has a magnitude of one ($s^2 + x^2 + y^2 + z^2 = 1$) makes it easy to see that the diagonal elements sum to $4 \cdot s^2 - 1$. Summing the diagonal elements of the matrix in Equation B.60 results in Equation B.61. The diagonal elements can also be used to solve for the remaining terms (Eq. B.62). The square roots of these last equations can be avoided if the off-diagonal elements are used to solve for x, y, and z at the expense of testing for a divide by an s that is equal to zero (in which case Eq. B.62 can be used).

$$\begin{bmatrix} m_{0,0} & m_{0,1} & m_{0,2} \\ m_{1,0} & m_{1,1} & m_{1,2} \\ m_{2,0} & m_{2,1} & m_{2,2} \end{bmatrix} \tag{B.60}$$

$$s = \frac{\sqrt{m_{0,0} + m_{1,1} + m_{2,2} + 1}}{2} \tag{B.61}$$

$$\begin{aligned}m_{0,0} &= 1 - 2y^2 - 2z^2 \\ &= 1 - 2(y^2 + z^2) \\ &= 1 - 2(1 - x^2 - s^2) \\ &= -1 + 2x^2 + 2s\end{aligned} \tag{B.62}$$