Chapter 2

Linear Time-Invariant Systems

2.1 INTRODUCTION

Two most important attributes of systems are linearity and time-invariance. In this chapter we develop the fundamental input-output relationship for systems having these attributes. It will be shown that the input-output relationship for LTI systems is described in terms of a convolution operation. The importance of the convolution operation in LTI systems stems from the fact that knowledge of the response of an LTI system to the unit impulse input allows us to find its output to any input signals. Specifying the input-output relationships for LTI systems by differential and difference equations will also be discussed.

2.2 RESPONSE OF A CONTINUOUS-TIME LTI SYSTEM AND THE CONVOLUTION INTEGRAL

A. Impulse Response:

The *impulse response* $h(t)$ of a continuous-time LTI system (represented by $\mathbf{T}$) is defined to be the response of the system when the input is $\delta(t)$, that is,

$$h(t) = \mathbf{T}\{\delta(t)\} \tag{2.1}$$

B. Response to an Arbitrary Input:

From Eq. (*1.27*) the input $x(t)$ can be expressed as

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau \tag{2.2}$$

Since the system is linear, the response $y(t)$ of the system to an arbitrary input $x(t)$ can be expressed as

$$\begin{aligned} y(t) = \mathbf{T}\{x(t)\} &= \mathbf{T}\left\{\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau\right\} \\ &= \int_{-\infty}^{\infty} x(\tau)\mathbf{T}\{\delta(t-\tau)\}\,d\tau \end{aligned} \tag{2.3}$$

Since the system is time-invariant, we have

$$h(t-\tau) = \mathbf{T}\{\delta(t-\tau)\} \tag{2.4}$$

Substituting Eq. (*2.4*) into Eq. (*2.3*), we obtain

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \tag{2.5}$$

Equation (*2.5*) indicates that a continuous-time LTI system is completely characterized by its impulse response $h(t)$.

C. Convolution Integral:

Equation (*2.5*) defines the *convolution* of two continuous-time signals $x(t)$ and $h(t)$ denoted by

$$y(t)=x(t)*h(t)=\int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \tag{2.6}$$

Equation (*2.6*) is commonly called the *convolution integral*. Thus, we have the fundamental result that *the output of any continuous-time LTI system is the convolution of the input $x(t)$ with the impulse response $h(t)$ of the system*. Figure 2-1 illustrates the definition of the impulse response $h(t)$ and the relationship of Eq. (*2.6*).

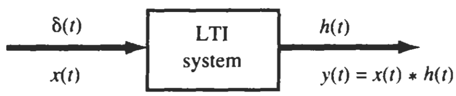

Fig. 2-1 Continuous-time LTI system.

D. Properties of the Convolution Integral:

The convolution integral has the following properties.

1. Commutative:

$$x(t)*h(t)=h(t)*x(t) \tag{2.7}$$

2. Associative:

$$\{x(t)*h_1(t)\}*h_2(t)=x(t)*\{h_1(t)*h_2(t)\} \tag{2.8}$$

3. Distributive:

$$x(t)*\{h_1(t)\}+h_2(t)\}=x(t)*h_1(t)+x(t)*h_2(t) \tag{2.9}$$

E. Convolution Integral Operation:

Applying the commutative property (*2.7*) of convolution to Eq. (*2.6*), we obtain

$$y(t)=h(t)*x(t)=\int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau \tag{2.10}$$

which may at times be easier to evaluate than Eq. (*2.6*). From Eq. (*2.6*) we observe that the convolution integral operation involves the following four steps:

1. The impulse response $h(\tau)$ is time-reversed (that is, reflected about the origin) to obtain $h(-\tau)$ and then shifted by t to form $h(t-\tau)=h[-(\tau-t)]$ which is a function of τ with parameter t.
2. The signal $x(\tau)$ and $h(t-\tau)$ are multiplied together for all values of τ with t fixed at some value.

3. The product $x(\tau)h(t-\tau)$ is integrated over all τ to produce a single output value $y(t)$.
4. Steps 1 to 3 are repeated as t varies over $-\infty$ to ∞ to produce the entire output $y(t)$.

Examples of the above convolution integral operation are given in Probs. 2.4 to 2.6.

F. Step Response:

The *step response* $s(t)$ of a continuous-time LTI system (represented by **T**) is defined to be the response of the system when the input is $u(t)$; that is,

$$s(t) = \mathbf{T}\{u(t)\} \tag{2.11}$$

In many applications, the step response $s(t)$ is also a useful characterization of the system. The step response $s(t)$ can be easily determined by Eq. (2.10); that is,

$$s(t) = h(t) * u(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)\,d\tau = \int_{-\infty}^{t} h(\tau)\,d\tau \tag{2.12}$$

Thus, the step response $s(t)$ can be obtained by integrating the impulse response $h(t)$. Differentiating Eq. (2.12) with respect to t, we get

$$h(t) = s'(t) = \frac{ds(t)}{dt} \tag{2.13}$$

Thus, the impulse response $h(t)$ can be determined by differentiating the step response $s(t)$.

2.3 PROPERTIES OF CONTINUOUS-TIME LTI SYSTEMS

A. Systems with or without Memory:

Since the output $y(t)$ of a memoryless system depends on only the present input $x(t)$, then, if the system is also linear and time-invariant, this relationship can only be of the form

$$y(t) = Kx(t) \tag{2.14}$$

where K is a (gain) constant. Thus, the corresponding impulse response $h(t)$ is simply

$$h(t) = K\delta(t) \tag{2.15}$$

Therefore, if $h(t_0) \neq 0$ for $t_0 \neq 0$, the continuous-time LTI system has memory.

B. Causality:

As discussed in Sec. 1.5D, a causal system does not respond to an input event until that event actually occurs. Therefore, for a causal continuous-time LTI system, we have

$$h(t) = 0 \qquad t < 0 \tag{2.16}$$

Applying the causality condition (2.16) to Eq. (2.10), the output of a causal continuous-time

LTI system is expressed as

$$y(t)=\int_0^{\infty} h(\tau)x(t-\tau)\,d\tau \tag{2.17}$$

Alternatively, applying the causality condition (*2.16*) to Eq. (*2.6*), we have

$$y(t)=\int_{-\infty}^{t} x(\tau)h(t-\tau)\,d\tau \tag{2.18}$$

Equation (*2.18*) shows that the only values of the input $x(t)$ used to evaluate the output $y(t)$ are those for $\tau \le t$.

Based on the causality condition (*2.16*), any signal $x(t)$ is called *causal* if

$$x(t)=0 \qquad t<0 \tag{2.19a}$$

and is called *anticausal* if

$$x(t)=0 \qquad t>0 \tag{2.19b}$$

Then, from Eqs. (*2.17*), (*2.18*), and (*2.19a*), when the input $x(t)$ is causal, the output $y(t)$ of a causal continuous-time LTI system is given by

$$y(t)=\int_0^{t} h(\tau)x(t-\tau)\,d\tau=\int_0^{t} x(\tau)h(t-\tau)\,d\tau \tag{2.20}$$

C. Stability:

The BIBO (bounded-input/bounded-output) stability of an LTI system (Sec. 1.5H) is readily ascertained from its impulse response. It can be shown (Prob. 2.13) that a continuous-time LTI system is BIBO stable if its impulse response is absolutely integrable, that is,

$$\int_{-\infty}^{\infty} |h(\tau)|\,d\tau<\infty \tag{2.21}$$

2.4 EIGENFUNCTIONS OF CONTINUOUS-TIME LTI SYSTEMS

In Chap. 1 (Prob. 1.44) we saw that the eigenfunctions of continuous-time LTI systems represented by **T** are the complex exponentials e^{st}, with s a complex variable. That is,

$$\mathbf{T}\{e^{st}\}=\lambda e^{st} \tag{2.22}$$

where λ is the eigenvalue of **T** associated with e^{st}. Setting $x(t)=e^{st}$ in Eq. (*2.10*), we have

$$\begin{aligned} y(t)=\mathbf{T}\{e^{st}\} &= \int_{-\infty}^{\infty} h(\tau)\,e^{s(t-\tau)}\,d\tau=\left[\int_{-\infty}^{\infty} h(\tau)\,e^{-s\tau}\,d\tau\right]e^{st} \\ &= H(s)\,e^{st}=\lambda e^{st} \end{aligned} \tag{2.23}$$

where

$$\lambda=H(s)=\int_{-\infty}^{\infty} h(\tau)\,e^{-s\tau}\,d\tau \tag{2.24}$$

Thus, the eigenvalue of a continuous-time LTI system associated with the eigenfunction e^{st} is given by $H(s)$ which is a complex constant whose value is determined by the value of s via Eq. (*2.24*). Note from Eq. (*2.23*) that $y(0)=H(s)$ (see Prob. 1.44).

The above results underlie the definitions of the Laplace transform and Fourier transform which will be discussed in Chaps. 3 and 5.

2.5 SYSTEMS DESCRIBED BY DIFFERENTIAL EQUATIONS

A. Linear Constant-Coefficient Differential Equations:

A general Nth-order linear constant-coefficient differential equation is given by

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \tag{2.25}$$

where coefficients a_k and b_k are real constants. The order N refers to the highest derivative of $y(t)$ in Eq. (*2.25*). Such differential equations play a central role in describing the input-output relationships of a wide variety of electrical, mechanical, chemical, and biological systems. For instance, in the RC circuit considered in Prob. 1.32, the input $x(t) = v_s(t)$ and the output $y(t) = v_c(t)$ are related by a first-order constant-coefficient differential equation [Eq. (*1.105*)]

$$\frac{dy(t)}{dt} + \frac{1}{RC} y(t) = \frac{1}{RC} x(t)$$

The general solution of Eq. (*2.25*) for a particular input $x(t)$ is given by

$$y(t) = y_p(t) + y_h(t) \tag{2.26}$$

where $y_p(t)$ is a *particular solution* satisfying Eq. (*2.25*) and $y_h(t)$ is a *homogeneous solution* (or *complementary solution*) satisfying the homogeneous differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y_h(t)}{dt^k} = 0 \tag{2.27}$$

The exact form of $y_h(t)$ is determined by N auxiliary conditions. Note that Eq. (*2.25*) does not completely specify the output $y(t)$ in terms of the input $x(t)$ unless auxiliary conditions are specified. In general, a set of auxiliary conditions are the values of

$$y(t), \frac{dy(t)}{dt}, \ldots, \frac{d^{N-1} y(t)}{dt^{N-1}}$$

at some point in time.

B. Linearity:

The system specified by Eq. (*2.25*) will be linear only if all of the auxiliary conditions are zero (see Prob. 2.21). If the auxiliary conditions are not zero, then the response $y(t)$ of a system can be expressed as

$$y(t) = y_{zi}(t) + y_{zs}(t) \tag{2.28}$$

where $y_{zi}(t)$, called the *zero-input response*, is the response to the auxiliary conditions, and $y_{zs}(t)$, called the *zero-state response*, is the response of a linear system with zero auxiliary conditions. This is illustrated in Fig. 2-2.

Note that $y_{zi}(t) \neq y_h(t)$ and $y_{zs}(t) \neq y_p(t)$ and that in general $y_{zi}(t)$ contains $y_h(t)$ and $y_{zs}(t)$ contains both $y_h(t)$ and $y_p(t)$ (see Prob. 2.20).

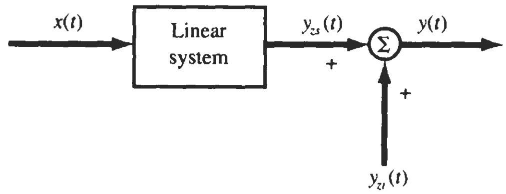

Fig. 2-2 Zero-state and zero-input responses.

C. Causality:

In order for the linear system described by Eq. (*2.25*) to be causal we must assume the condition of *initial rest* (or an *initially relaxed condition*). That is, if $x(t) = 0$ for $t \le t_0$, then assume $y(t) = 0$ for $t \le t_0$ (see Prob. 1.43). Thus, the response for $t > t_0$ can be calculated from Eq. (*2.25*) with the initial conditions

$$y(t_0) = \frac{dy(t_0)}{dt} = \cdots = \frac{d^{N-1}y(t_0)}{dt^{N-1}} = 0$$

where

$$\frac{d^k y(t_0)}{dt^k} = \left.\frac{d^k y(t)}{dt^k}\right|_{t=t_0}.$$

Clearly, at initial rest $y_{zi}(t) = 0$.

D. Time-Invariance:

For a linear causal system, initial rest also implies time-invariance (Prob. 2.22).

E. Impulse Response:

The impulse response $h(t)$ of the continuous-time LTI system described by Eq. (*2.25*) satisfies the differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k h(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k \delta(t)}{dt^k} \tag{2.29}$$

with the initial rest condition. Examples of finding impulse responses are given in Probs. 2.23 to 2.25. In later chapters, we will find the impulse response by using transform techniques.

2.6 RESPONSE OF A DISCRETE-TIME LTI SYSTEM AND CONVOLUTION SUM

A. Impulse Response:

The *impulse response* (or *unit sample response*) $h[n]$ of a discrete-time LTI system (represented by **T**) is defined to be the response of the system when the input is $\delta[n]$, that is,

$$h[n] = \mathbf{T}\{\delta[n]\} \tag{2.30}$$

B. Response to an Arbitrary Input:

From Eq. (*1.51*) the input $x[n]$ can be expressed as

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] \tag{2.31}$$

Since the system is linear, the response $y[n]$ of the system to an arbitrary input $x[n]$ can be expressed as

$$y[n] = \mathbf{T}\{x[n]\} = \mathbf{T}\left\{\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]\right\}$$

$$= \sum_{k=-\infty}^{\infty} x[k]\mathbf{T}\{\delta[n-k]\} \tag{2.32}$$

Since the system is time-invariant, we have

$$h[n-k] = \mathbf{T}\{\delta[n-k]\} \tag{2.33}$$

Substituting Eq. (*2.33*) into Eq. (*2.32*), we obtain

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \tag{2.34}$$

Equation (*2.34*) indicates that a discrete-time LTI system is completely characterized by its impulse response $h[n]$.

C. Convolution Sum:

Equation (*2.34*) defines the *convolution* of two sequences $x[n]$ and $h[n]$ denoted by

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \tag{2.35}$$

Equation (*2.35*) is commonly called the *convolution sum*. Thus, again, we have the fundamental result that *the output of any discrete-time LTI system is the convolution of the input $x[n]$ with the impulse response $h[n]$ of the system.*

Figure 2-3 illustrates the definition of the impulse response $h[n]$ and the relationship of Eq. (*2.35*).

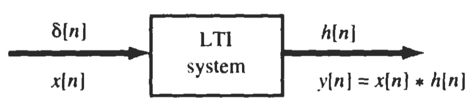

Fig. 2-3 Discrete-time LTI system.

D. Properties of the Convolution Sum:

The following properties of the convolution sum are analogous to the convolution integral properties shown in Sec. 2.3.

1. Commutative:

$$x[n] * h[n] = h[n] * x[n] \tag{2.36}$$

2. Associative:

$$\{x[n] * h_1[n]\} * h_2[n] = x[n] * \{h_1[n] * h_2[n]\} \tag{2.37}$$

3. Distributive:

$$x[n] * \{h_1[n]\} + h_2[n]\} = x[n] * h_1[n] + x[n] * h_2[n] \tag{2.38}$$

E. Convolution Sum Operation:

Again, applying the commutative property (*2.36*) of the convolution sum to Eq. (*2.35*), we obtain

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \tag{2.39}$$

which may at times be easier to evaluate than Eq. (*2.35*). Similar to the continuous-time case, the convolution sum [Eq. (*2.35*)] operation involves the following four steps:

1. The impulse response $h[k]$ is time-reversed (that is, reflected about the origin) to obtain $h[-k]$ and then shifted by n to form $h[n-k] = h[-(k-n)]$ which is a function of k with parameter n.
2. Two sequences $x[k]$ and $h[n-k]$ are multiplied together for all values of k with n fixed at some value.
3. The product $x[k]h[n-k]$ is summed over all k to produce a single output sample $y[n]$.
4. Steps 1 to 3 are repeated as n varies over $-\infty$ to ∞ to produce the entire output $y[n]$.

Examples of the above convolution sum operation are given in Probs. 2.28 and 2.30.

F. Step Response:

The *step response* $s[n]$ of a discrete-time LTI system with the impulse response $h[n]$ is readily obtained from Eq. (*2.39*) as

$$s[n] = h[n] * u[n] = \sum_{k=-\infty}^{\infty} h[k]u[n-k] = \sum_{k=-\infty}^{n} h[k] \tag{2.40}$$

From Eq. (*2.40*) we have

$$h[n] = s[n] - s[n-1] \tag{2.41}$$

Equations (*2.40*) and (*2.41*) are the discrete-time counterparts of Eqs. (*2.12*) and (*2.13*), respectively.

2.7 PROPERTIES OF DISCRETE-TIME LTI SYSTEMS

A. Systems with or without Memory:

Since the output $y[n]$ of a memoryless system depends on only the present input $x[n]$, then, if the system is also linear and time-invariant, this relationship can only be of the form

$$y[n] = Kx[n] \tag{2.42}$$

where K is a (gain) constant. Thus, the corresponding impulse response is simply

$$h[n] = K\delta[n] \tag{2.43}$$

Therefore, if $h[n_0] \neq 0$ for $n_0 \neq 0$, the discrete-time LTI system has memory.

B. Causality:

Similar to the continuous-time case, the causality condition for a discrete-time LTI system is

$$h[n] = 0 \qquad n < 0 \tag{2.44}$$

Applying the causality condition (*2.44*) to Eq. (*2.39*), the output of a causal discrete-time LTI system is expressed as

$$y[n] = \sum_{k=0}^{\infty} h[k]x[n-k] \tag{2.45}$$

Alternatively, applying the causality condition (*2.44*) to Eq. (*2.35*), we have

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k] \tag{2.46}$$

Equation (*2.46*) shows that the only values of the input $x[n]$ used to evaluate the output $y[n]$ are those for $k \leq n$.

As in the continuous-time case, we say that any sequence $x[n]$ is called *causal* if

$$x[n] = 0 \qquad n < 0 \tag{2.47a}$$

and is called *anticausal* if

$$x[n] = 0 \qquad n \geq 0 \tag{2.47b}$$

Then, when the input $x[n]$ is causal, the output $y[n]$ of a causal discrete-time LTI system is given by

$$y[n] = \sum_{k=0}^{n} h[k]x[n-k] = \sum_{k=0}^{n} x[k]h[n-k] \tag{2.48}$$

C. Stability:

It can be shown (Prob. 2.37) that a discrete-time LTI system is BIBO stable if its impulse response is absolutely summable, that is,

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty \tag{2.49}$$

2.8 EIGENFUNCTIONS OF DISCRETE-TIME LTI SYSTEMS

In Chap. 1 (Prob. 1.45) we saw that the eigenfunctions of discrete-time LTI systems represented by $\mathbf{T}$ are the complex exponentials z^n, with z a complex variable. That is,

$$\mathbf{T}\{z^n\} = \lambda z^n \tag{2.50}$$

where λ is the eigenvalue of $\mathbf{T}$ associated with z^n. Setting $x[n]=z^n$ in Eq. (2.39), we have

$$y[n]=\mathbf{T}\{z^n\}=\sum_{k=-\infty}^{\infty} h[k]z^{n-k}=\left[\sum_{k=-\infty}^{\infty} h[k]z^{-k}\right]z^n$$

$$=H(z)z^n=\lambda z^n \tag{2.51}$$

where

$$\lambda=H(z)=\sum_{k=-\infty}^{\infty} h[k]z^{-k} \tag{2.52}$$

Thus, the eigenvalue of a discrete-time LTI system associated with the eigenfunction z^n is given by $H(z)$ which is a complex constant whose value is determined by the value of z via Eq. (2.52). Note from Eq. (2.51) that $y[0]=H(z)$ (see Prob. 1.45).

The above results underlie the definitions of the z-transform and discrete-time Fourier transform which will be discussed in Chaps. 4 and 6.

2.9 SYSTEMS DESCRIBED BY DIFFERENCE EQUATIONS

The role of differential equations in describing continuous-time systems is played by *difference equations* for discrete-time systems.

A. Linear Constant-Coefficient Difference Equations:

The discrete-time counterpart of the general differential equation (2.25) is the Nth-order linear constant-coefficient difference equation given by

$$\sum_{k=0}^{N} a_k y[n-k]=\sum_{k=0}^{M} b_k x[n-k] \tag{2.53}$$

where coefficients a_k and b_k are real constants. The order N refers to the largest delay of $y[n]$ in Eq. (2.53). An example of the class of linear constant-coefficient difference equations is given in Chap. 1 (Prob. 1.37). Analogous to the continuous-time case, the solution of Eq. (2.53) and all properties of systems, such as linearity, causality, and time-invariance, can be developed following an approach that directly parallels the discussion for differential equations. Again we emphasize that the system described by Eq. (2.53) will be causal and LTI if the system is initially at rest.

B. Recursive Formulation:

An alternate and simpler approach is available for the solution of Eq. (2.53). Rearranging Eq. (2.53) in the form

$$y[n]=\frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k x[n-k]-\sum_{k=1}^{N} a_k y[n-k]\right\} \tag{2.54}$$

we obtain a formula to compute the output at time n in terms of the present input and the previous values of the input and output. From Eq. (2.54) we see that the need for auxiliary conditions is obvious and that to calculate $y[n]$ starting at $n=n_0$, we must be given the values of $y[n_0-1]$, $y[n_0-2], \ldots, y[n_0-N]$ as well as the input $x[n]$ for $n \geq n_0-M$. The general form of Eq. (2.54) is called a *recursive equation* since it specifies a recursive procedure for determining the output in terms of the input and previous outputs. In the special case when $N = 0$, from Eq. (2.53) we have

$$y[n] = \frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k x[n-k]\right\} \tag{2.55}$$

which is a *nonrecursive equation* since previous output values are not required to compute the present output. Thus, in this case, auxiliary conditions are not needed to determine $y[n]$.

C. Impulse Response:

Unlike the continuous-time case, the impulse response $h[n]$ of a discrete-time LTI system described by Eq. (2.53) or, equivalently, by Eq. (2.54) can be determined easily as

$$h[n] = \frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k \delta[n-k] - \sum_{k=1}^{N} a_k h[n-k]\right\} \tag{2.56}$$

For the system described by Eq. (2.55) the impulse response $h[n]$ is given by

$$h[n] = \frac{1}{a_0}\sum_{k=0}^{M} b_k \delta[n-k] = \begin{cases} b_n/a_0 & 0 \le n \le M \\ 0 & \text{otherwise} \end{cases} \tag{2.57}$$

Note that the impulse response for this system has finite terms; that is, it is nonzero for only a finite time duration. Because of this property, the system specified by Eq. (2.55) is known as a *finite impulse response* (FIR) system. On the other hand, a system whose impulse response is nonzero for an infinite time duration is said to be an *infinite impulse response* (IIR) system. Examples of finding impulse responses are given in Probs. 2.44 and 2.45. In Chap. 4, we will find the impulse response by using transform techniques.

Solved Problems

RESPONSES OF A CONTINUOUS-TIME LTI SYSTEM AND CONVOLUTION

2.1. Verify Eqs. (2.7) and (2.8), that is,

(a) $x(t) * h(t) = h(t) * x(t)$

(b) $\{x(t) * h_1(t)\} * h_2(t) = x(t) * \{h_1(t) * h_2(t)\}$

(a) By definition (2.6)

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

By changing the variable $t - \tau = \lambda$, we have

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(t-\lambda)h(\lambda)\,d\lambda = \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)\,d\lambda = h(t) * x(t)$$

(b) Let $x(t) * h_1(t) = f_1(t)$ and $h_1(t) * h_2(t) = f_2(t)$. Then

$$f_1(t) = \int_{-\infty}^{\infty} x(\tau) h_1(t-\tau)\, d\tau$$

and
$$\{x(t) * h_1(t)\} * h_2(t) = f_1(t) * h_2(t) = \int_{-\infty}^{\infty} f_1(\sigma) h_2(t-\sigma)\, d\sigma$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau) h_1(\sigma-\tau)\, d\tau \right] h_2(t-\sigma)\, d\sigma$$

Substituting $\lambda = \sigma - \tau$ and interchanging the order of integration, we have

$$\{x(t) * h_1(t)\} * h_2(t) = \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} h_1(\lambda) h_2(t-\tau-\lambda)\, d\lambda \right] d\tau$$

Now, since

$$f_2(t) = \int_{-\infty}^{\infty} h_1(\lambda) h_2(t-\lambda)\, d\lambda$$

we have

$$f_2(t-\tau) = \int_{-\infty}^{\infty} h_1(\lambda) h_2(t-\tau-\lambda)\, d\lambda$$

Thus,
$$\{x(t) * h_1(t)\} * h_2(t) = \int_{-\infty}^{\infty} x(\tau) f_2(t-\tau)\, d\tau$$

$$= x(t) * f_2(t) = x(t) * \{h_1(t) * h_2(t)\}$$

2.2. Show that

(a) $x(t) * \delta(t) = x(t)$ (2.58)

(b) $x(t) * \delta(t-t_0) = x(t-t_0)$ (2.59)

(c) $x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$ (2.60)

(d) $x(t) * u(t-t_0) = \int_{-\infty}^{t-t_0} x(\tau)\, d\tau$ (2.61)

(a) By definition (2.6) and Eq. (1.22) we have

$$x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau = x(\tau)|_{\tau=t} = x(t)$$

(b) By Eqs. (2.7) and (1.22) we have

$$x(t) * \delta(t-t_0) = \delta(t-t_0) * x(t) = \int_{-\infty}^{\infty} \delta(\tau-t_0) x(t-\tau)\, d\tau$$

$$= x(t-\tau)|_{\tau=t_0} = x(t-t_0)$$

(c) By Eqs. (2.6) and (1.19) we have

$$x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau) u(t-\tau)\, d\tau = \int_{-\infty}^{t} x(\tau)\, d\tau$$

since $u(t-\tau) = \begin{cases} 1 & \tau < t \\ 0 & \tau > t \end{cases}$.

(*d*) In a similar manner, we have

$$x(t) * u(t-t_0) = \int_{-\infty}^{\infty} x(\tau) u(t-\tau-t_0)\, d\tau = \int_{-\infty}^{t-t_0} x(\tau)\, d\tau$$

since $u(t-\tau-t_0) = \begin{cases} 1 & \tau < t-t_0 \\ 0 & \tau > t-t_0 \end{cases}$.

2.3. Let $y(t) = x(t) * h(t)$. Then show that

$$x(t-t_1) * h(t-t_2) = y(t-t_1-t_2) \tag{2.62}$$

By Eq. (*2.6*) we have

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau \tag{2.63a}$$

and
$$x(t-t_1) * h(t-t_2) = \int_{-\infty}^{\infty} x(\tau-t_1) h(t-\tau-t_2)\, d\tau \tag{2.63b}$$

Let $\tau - t_1 = \lambda$. Then $\tau = \lambda + t_1$ and Eq. (*2.63b*) becomes

$$x(t-t_1) * h(t-t_2) = \int_{-\infty}^{\infty} x(\lambda) h(t-t_1-t_2-\lambda)\, d\lambda \tag{2.63c}$$

Comparing Eqs. (*2.63a*) and (*2.63c*), we see that replacing t in Eq. (*2.63a*) by $t-t_1-t_2$, we obtain Eq. (*2.63c*). Thus, we conclude that

$$x(t-t_1) * h(t-t_2) = y(t-t_1-t_2)$$

2.4. The input $x(t)$ and the impulse response $h(t)$ of a continuous time LTI system are given by

$$x(t) = u(t) \qquad h(t) = e^{-\alpha t} u(t),\ \alpha > 0$$

(*a*) Compute the output $y(t)$ by Eq. (*2.6*).

(*b*) Compute the output $y(t)$ by Eq. (*2.10*).

(*a*) By Eq. (*2.6*)

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau$$

Functions $x(\tau)$ and $h(t-\tau)$ are shown in Fig. 2-4(*a*) for $t < 0$ and $t > 0$. From Fig. 2-4(*a*) we see that for $t < 0$, $x(\tau)$ and $h(t-\tau)$ do not overlap, while for $t > 0$, they overlap from $\tau = 0$ to $\tau = t$. Hence, for $t < 0$, $y(t) = 0$. For $t > 0$, we have

$$\begin{aligned} y(t) &= \int_0^t e^{-\alpha(t-\tau)}\, d\tau = e^{-\alpha t} \int_0^t e^{\alpha\tau}\, d\tau \\ &= e^{-\alpha t} \frac{1}{\alpha}(e^{\alpha t} - 1) = \frac{1}{\alpha}(1 - e^{-\alpha t}) \end{aligned}$$

Thus, we can write the output $y(t)$ as

$$y(t) = \frac{1}{\alpha}(1 - e^{-\alpha t}) u(t) \tag{2.64}$$

(*b*) By Eq. (*2.10*)

$$y(t)=h(t)*x(t)=\int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau$$

Functions $h(\tau)$ and $x(t-\tau)$ are shown in Fig. 2-4(*b*) for $t<0$ and $t>0$. Again from Fig. 2-4(*b*) we see that for $t<0$, $h(\tau)$ and $x(t-\tau)$ do not overlap, while for $t>0$, they overlap from $\tau=0$ to $\tau=t$. Hence, for $t<0$, $y(t)=0$. For $t>0$, we have

$$y(t)=\int_0^t e^{-\alpha\tau}\,d\tau=\frac{1}{\alpha}(1-e^{-\alpha t})$$

Thus, we can write the output $y(t)$ as

$$y(t)=\frac{1}{\alpha}(1-e^{-\alpha t})u(t) \tag{2.65}$$

which is the same as Eq. (*2.64*).

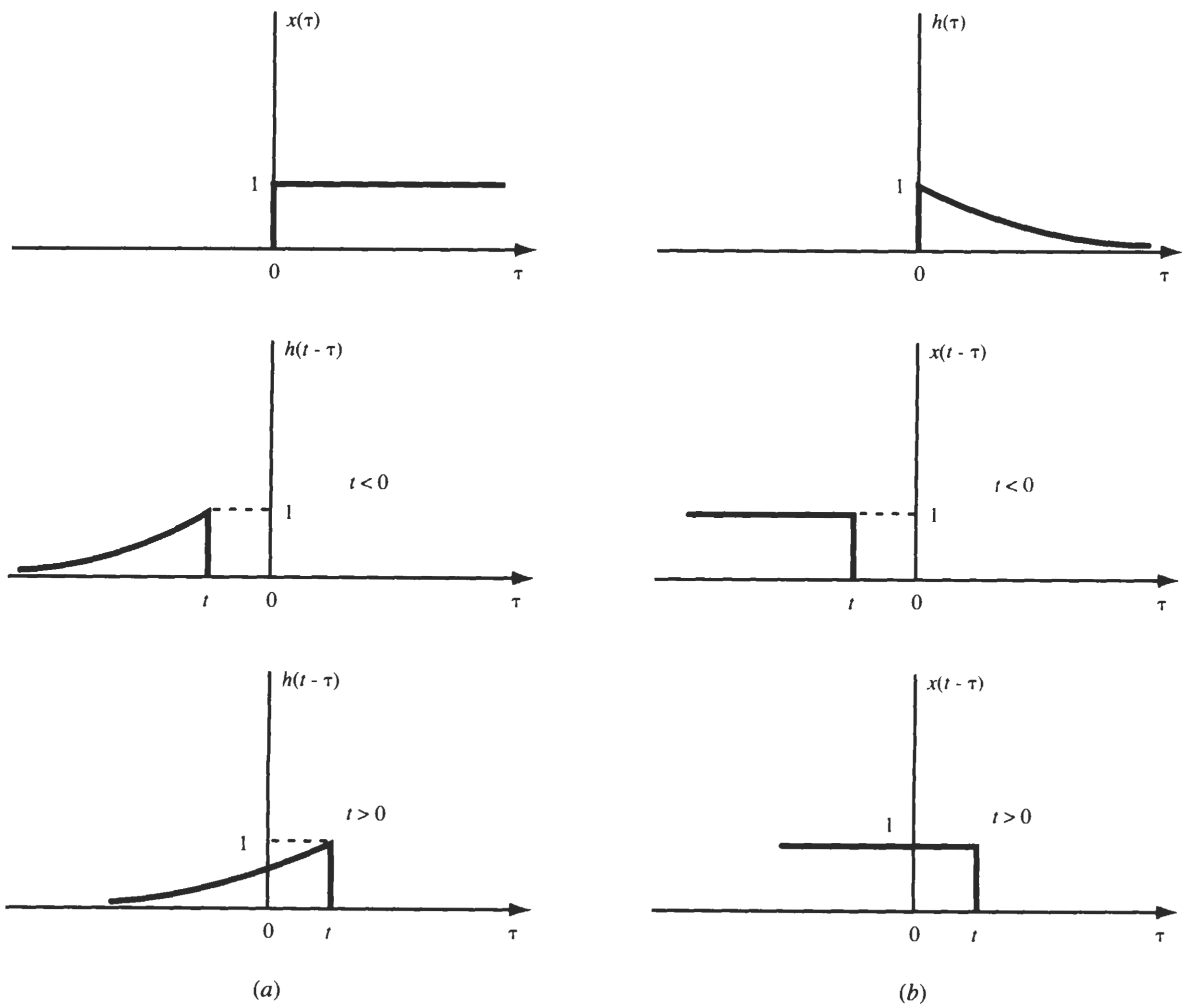

Fig. 2-4

2.5. Compute the output $y(t)$ for a continuous-time LTI system whose impulse response $h(t)$ and the input $x(t)$ are given by

$$h(t) = e^{-\alpha t}u(t) \qquad x(t) = e^{\alpha t}u(-t) \qquad \alpha > 0$$

By Eq. (2.6)

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

Functions $x(\tau)$ and $h(t-\tau)$ are shown in Fig. 2-5(a) for $t < 0$ and $t > 0$. From Fig. 2-5(a) we see that for $t < 0$, $x(\tau)$ and $h(t-\tau)$ overlap from $\tau = -\infty$ to $\tau = t$, while for $t > 0$, they overlap from $\tau = -\infty$ to $\tau = 0$. Hence, for $t < 0$, we have

$$y(t) = \int_{-\infty}^{t} e^{\alpha\tau}e^{-\alpha(t-\tau)}\,d\tau = e^{-\alpha t}\int_{-\infty}^{t} e^{2\alpha\tau}\,d\tau = \frac{1}{2\alpha}e^{\alpha t} \tag{2.66a}$$

For $t > 0$, we have

$$y(t) = \int_{-\infty}^{0} e^{\alpha\tau}e^{-\alpha(t-\tau)}\,d\tau = e^{-\alpha t}\int_{-\infty}^{0} e^{2\alpha\tau}\,d\tau = \frac{1}{2\alpha}e^{-\alpha t} \tag{2.66b}$$

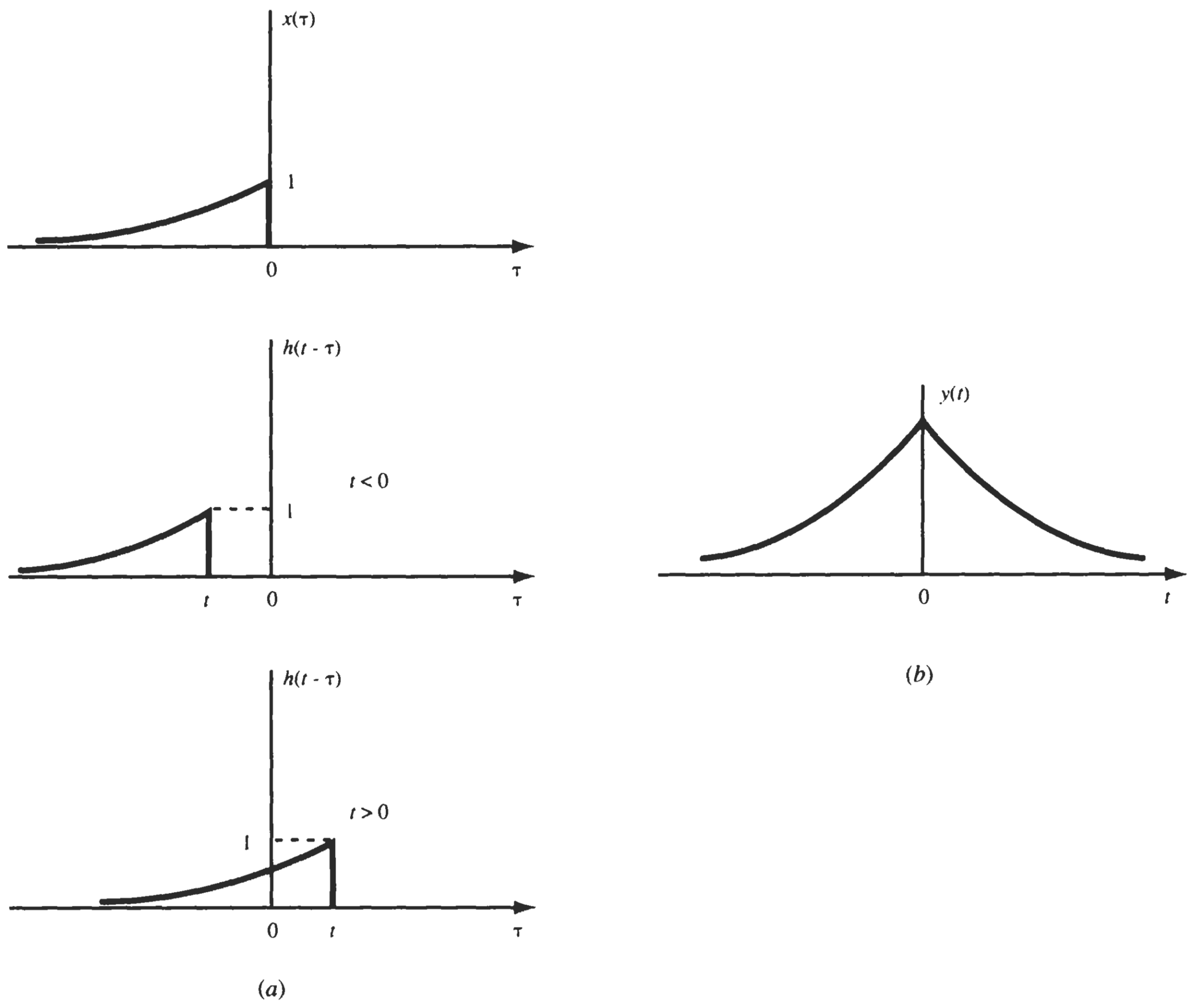

Fig. 2-5

Combining Eqs. (*2.66a*) and (*2.66b*), we can write $y(t)$ as

$$y(t)=\frac{1}{2\alpha}e^{-\alpha|t|} \qquad \alpha>0 \tag{2.67}$$

which is shown in Fig. 2-5(b).

2.6. Evaluate $y(t)=x(t)*h(t)$, where $x(t)$ and $h(t)$ are shown in Fig. 2-6, (a) by an analytical technique, and (b) by a graphical method.

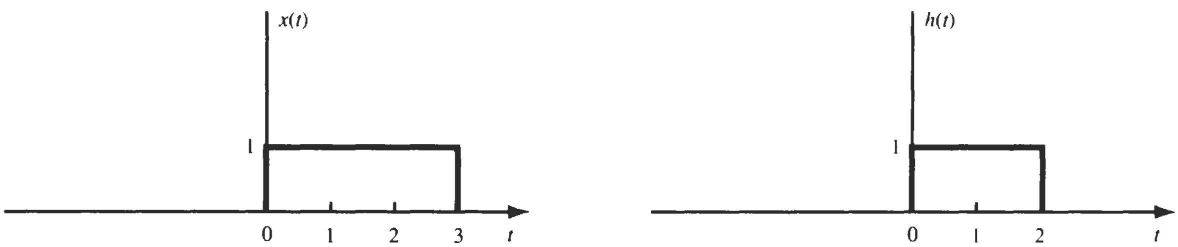

Fig. 2-6

(a) We first express $x(t)$ and $h(t)$ in functional form:

$$x(t)=u(t)-u(t-3) \qquad h(t)=u(t)-u(t-2)$$

Then, by Eq. (*2.6*) we have

$$\begin{aligned} y(t) &= x(t)*h(t)=\int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \\ &= \int_{-\infty}^{\infty}[u(\tau)-u(\tau-3)][u(t-\tau)-u(t-\tau-2)]\,d\tau \\ &= \int_{-\infty}^{\infty}u(\tau)u(t-\tau)\,d\tau-\int_{-\infty}^{\infty}u(\tau)u(t-2-\tau)\,d\tau \\ &\quad -\int_{-\infty}^{\infty}u(\tau-3)u(t-\tau)\,d\tau+\int_{-\infty}^{\infty}u(\tau-3)u(t-2-\tau)\,d\tau \end{aligned}$$

Since

$$u(\tau)u(t-\tau)=\begin{cases}1 & 0<\tau<t,\ t>0\\ 0 & \text{otherwise}\end{cases}$$

$$u(\tau)u(t-2-\tau)=\begin{cases}1 & 0<\tau<t-2,\ t>2\\ 0 & \text{otherwise}\end{cases}$$

$$u(\tau-3)u(t-\tau)=\begin{cases}1 & 3<\tau<t,\ t>3\\ 0 & \text{otherwise}\end{cases}$$

$$u(\tau-3)u(t-2-\tau)=\begin{cases}1 & 3<\tau<t-2,\ t>5\\ 0 & \text{otherwise}\end{cases}$$