CHAPTER 21

The Laplace Transform

DEFINITION

Let $f(x)$ be defined for $0 \le x < \infty$ and let s denote an arbitrary real variable. The *Laplace transform of* $f(x)$, designated by either $\mathscr{L}\{f(x)\}$ or $F(s)$, is

$$\mathscr{L}\{f(x)\} = F(s) = \int_0^\infty e^{-sx} f(x)\,dx \tag{21.1}$$

for all values of s for which the improper integral converges. Convergence occurs when the limit

$$\lim_{R\to\infty} \int_0^R e^{-sx} f(x)\,dx \tag{21.2}$$

exists. If this limit does not exist, the improper integral diverges and $f(x)$ has no Laplace transform. When evaluating the integral in Eq. (*21.1*), the variable s is treated as a constant because the integration is with respect to x.

The Laplace transforms for a number of elementary functions are calculated in Problems 21.4 through 21.8; additional transforms are given in Appendix A.

PROPERTIES OF LAPLACE TRANSFORMS

Property 21.1. ***(Linearity).*** If $\mathscr{L}\{f(x)\} = F(s)$ and $\mathscr{L}\{g(x)\} = G(s)$, then for any two constants c_1 and c_2

$$\mathscr{L}\{c_1 f(x) + c_2 g(x)\} = c_1\mathscr{L}\{f(x)\} + c_2\mathscr{L}\{g(x)\} = c_1F(s) + c_2G(s) \tag{21.3}$$

Property 21.2. If $\mathscr{L}\{f(x)\} = F(s)$, then for any constant a

$$\mathscr{L}\{e^{ax} f(x)\} = F(s-a) \tag{21.4}$$

Property 21.3. If $\mathscr{L}\{f(x)\} = F(s)$, then for any positive integer n

$$\mathscr{L}\{x^n f(x)\} = (-1)^n \frac{d^n}{ds^n}[F(s)] \tag{21.5}$$

Property 21.4. If $\mathscr{L}\{f(x)\} = F(s)$ and if $\lim\limits_{\substack{x\to 0\\ x>0}} \dfrac{f(x)}{x}$ exists, then

$$\mathscr{L}\left\{\frac{1}{x}f(x)\right\}=\int_s^{\infty}F(t)\,dt \tag{21.6}$$

Property 21.5. If $\mathscr{L}\{f(x)\}=F(s)$, then

$$\mathscr{L}\left\{\int_0^x f(t)\,dt\right\}=\frac{1}{s}F(s) \tag{21.7}$$

Property 21.6. If $f(x)$ is periodic with period ω, that is, $f(x+\omega)=f(x)$, then

$$\mathscr{L}\{f(x)\}=\frac{\int_0^{\omega}e^{-sx}f(x)\,dx}{1-e^{-\omega s}} \tag{21.8}$$

FUNCTIONS OF OTHER INDEPENDENT VARIABLES

For consistency only, the definition of the Laplace transform and its properties, Eqs. (*21.1*) through (*21.8*), are presented for functions of x. They are equally applicable for functions of any independent variable and are generated by replacing the variable x in the above equations by any variable of interest. In particular, the counterpart of Eq. (*21.1*) for the Laplace transform of a function of t is

$$\mathscr{L}\{f(t)\}=F(s)=\int_0^{\infty}e^{-st}f(t)\,dt$$

Solved Problems

21.1. Determine whether the improper integral $\int_2^{\infty}\frac{1}{x^2}\,dx$ converges.

Since

$$\lim_{R\to\infty}\int_2^R\frac{1}{x^2}\,dx=\lim_{R\to\infty}\left(-\frac{1}{x}\right)\Bigg|_2^R=\lim_{R\to\infty}\left(-\frac{1}{R}+\frac{1}{2}\right)=\frac{1}{2}$$

the improper integral converges to the value $\frac{1}{2}$.

21.2. Determine whether the improper integral $\int_9^{\infty}\frac{1}{x}\,dx$ converges.

Since

$$\lim_{R\to\infty}\int_9^R\frac{1}{x}\,dx=\lim_{R\to\infty}\ln|x|\Big|_9^R=\lim_{R\to\infty}(\ln R-\ln 9)=\infty$$

the improper integral diverges.

21.3. Determine those values of s for which the improper integral $\int_0^{\infty}e^{-sx}\,dx$ converges.

For $s=0$,

$$\int_0^{\infty}e^{-sx}\,dx=\int_0^{\infty}e^{-(0)(x)}\,dx=\lim_{R\to\infty}\int_0^R(1)\,dx=\lim_{R\to\infty}x\Big|_0^R=\lim_{R\to\infty}R=\infty$$

hence the integral diverges. For $s \neq 0$,

$$\int_0^\infty e^{-sx}dx = \lim_{R\to\infty}\int_0^\infty e^{-sx}dx = \lim_{R\to\infty}\left[-\frac{1}{s}e^{-sx}\right]_{x=0}^{x=R}$$
$$= \lim_{R\to\infty}\left(\frac{-1}{s}e^{-sR} + \frac{1}{s}\right)$$

when $s < 0$, $-sR > 0$; hence the limit is ∞ and the integral diverges. When $s > 0$, $-sR < 0$; hence, the limit is $1/s$ and the integral converges.

21.4. Find the Laplace transform of $f(x) = 1$.

Using Eq. (*21.1*) and the results of Problem 21.3, we have

$$F(s) = \mathscr{L}\{1\} = \int_0^\infty e^{-sx}(1)\,dx = \frac{1}{s} \quad (\text{for } s > 0)$$

(See also entry 1 in Appendix A.)

21.5. Find the Laplace transform of $f(x) = x^2$.

Using Eq. (*21.1*) and integration by parts twice, we find that

$$F(s) = \mathscr{L}\{x^2\} = \int_0^\infty e^{-sx}x^2dx = \lim_{R\to\infty}\int_0^R x^2e^{-sx}dx$$
$$= \lim_{R\to\infty}\left[-\frac{x^2}{s}e^{-sx} - \frac{2x}{s^2}e^{-sx} - \frac{2}{s^3}e^{-sx}\right]_{x=0}^{x=R}$$
$$= \lim_{R\to\infty}\left(-\frac{R^2}{s}e^{-sR} - \frac{2R}{s^2}e^{-sR} - \frac{2}{s^3}e^{-sR} + \frac{2}{s^3}\right)$$

For $s < 0$, $\lim_{R\to\infty}[-(R^2/s)e^{-sR}] = \infty$, and the improper integral diverge. For $s > 0$, it follows from repeated use of L'Hôpital's rule that

$$\lim_{R\to\infty}\left(-\frac{R^2}{s}e^{-sR}\right) = \lim_{R\to\infty}\left(\frac{-R^2}{se^{sR}}\right) = \lim_{R\to\infty}\left(\frac{-2R}{s^2e^{sR}}\right)$$
$$= \lim_{R\to\infty}\left(\frac{-2}{s^3e^{sR}}\right) = 0$$
$$\lim_{R\to\infty}\left(-\frac{2R}{s}e^{-sR}\right) = \lim_{R\to\infty}\left(\frac{-2R}{se^{sR}}\right) = \lim_{R\to\infty}\left(\frac{-2}{s^2e^{sR}}\right) = 0$$

Also, $\lim_{R\to\infty}[-(2/s^3)e^{-sR}] = 0$ directly; hence the integral converges, and $F(s) = 2/s^3$. For the special cases $s = 0$, we have

$$\int_0^\infty e^{-sx}x^2dx = \int_0^\infty e^{-s(0)}x^2dx = \lim_{R\to\infty}\int_0^R x^2dx = \lim_{R\to\infty}\frac{R^3}{3} = \infty$$

Finally, combining all cases, we obtain $\mathscr{L}\{x^2\} = 2/s^3$, $s > 0$. (See also entry 3 in Appendix A.)

21.6. Find $\mathscr{L}\{e^{ax}\}$.

Using Eq. (*21.1*), we obtain

$$F(s) = \mathscr{L}\{e^{ax}\} = \int_0^\infty e^{-sx}e^{ax}dx = \lim_{R\to\infty}\int_0^R e^{(a-s)x}dx$$
$$= \lim_{R\to\infty}\left[\frac{e^{(a-s)x}}{a-s}\right]_{x=0}^{x=R} = \lim_{R\to\infty}\left[\frac{e^{(a-s)R}-1}{a-s}\right]$$
$$= \frac{1}{s-a} \quad (\text{for } s > a)$$

Note that when $s \leq a$, the improper integral diverges. (See also entry 7 in Appendix A.)

21.7. Find $\mathcal{L}\{\sin ax\}$.

Using Eq. (*21.1*) and integration by parts twice, we obtain

$$\begin{aligned}
\mathcal{L}\{\sin ax\} &= \int_0^\infty e^{-sx}\sin ax\,dx = \lim_{R\to\infty}\int_0^R e^{-sx}\sin ax\,dx \\
&= \lim_{R\to\infty}\left[\frac{-se^{-sx}\sin ax}{s^2+a^2} - \frac{ae^{-sx}\cos ax}{s^2+a^2}\right]_{x=0}^{x=R} \\
&= \lim_{R\to\infty}\left[\frac{-se^{-sR}\sin aR}{s^2+a^2} - \frac{ae^{-sR}\cos aR}{s^2+a^2} + \frac{a}{s^2+a^2}\right] \\
&= \frac{a}{x^2+a^2} \quad (\text{for } s>0)
\end{aligned}$$

(See also entry 8 in Appendix A.)

21.8. Find the Laplace transform of $f(x) = \begin{cases} e^x & x \le 2 \\ 3 & x > 2 \end{cases}$.

$$\begin{aligned}
\mathcal{L}\{f(x)\} &= \int_0^\infty e^{-sx}f(x)\,dx = \int_0^2 e^{-sx}e^x dx + \int_2^\infty e^{-sx}(3)\,dx \\
&= \int_0^2 e^{(1-s)x}dx + 3\lim_{R\to\infty}\int_2^R e^{-sx}dx = \left.\frac{e^{(1-s)x}}{1-s}\right|_{x=0}^{x=2} - \frac{3}{s}\lim_{R\to\infty}\left. e^{-sx}\right|_{x=2}^{x=R} \\
&= \frac{e^{2(1-s)}}{1-s} - \frac{1}{1-s} - \frac{3}{s}\lim_{R\to\infty}[e^{-Rs}-e^{-2s}] = \frac{1-e^{-2(s-1)}}{s-1} + \frac{3}{s}e^{-2x} \quad (\text{for } s>0)
\end{aligned}$$

21.9. Find the Laplace transform of the function graphed in Fig. 21-1.

$$f(x) = \begin{cases} -1 & x \le 4 \\ 1 & x > 4 \end{cases}$$

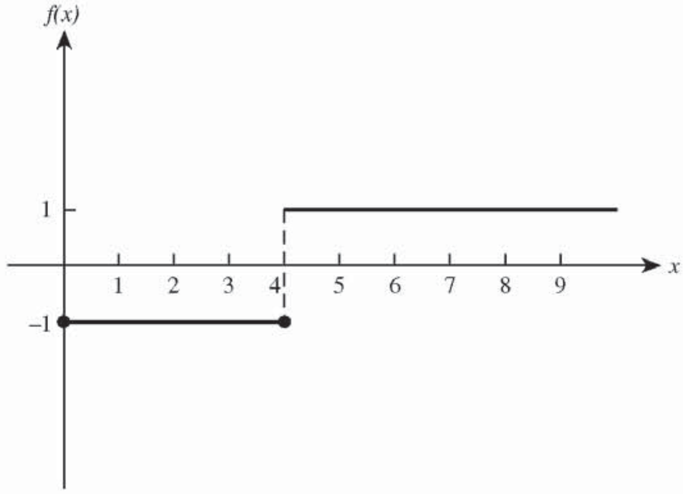

Fig. 21-1

$$\begin{aligned}\mathcal{L}\{f(x)\} &= \int_0^\infty e^{-sx} f(x)dx = \int_0^4 e^{-sx}(-1)dx + \int_4^\infty e^{-sx}(1)\,dx \\ &= \frac{e^{-sx}}{s}\Bigg|_{x=0}^{x=4} + \lim_{R\to\infty}\int_4^R e^{-sx}dx \\ &= \frac{e^{-4s}}{s} - \frac{1}{s} + \lim_{R\to\infty}\left(\frac{-1}{s}e^{-Rs} + \frac{1}{s}e^{-4s}\right) \\ &= \frac{2e^{-4s}}{s} - \frac{1}{s} \quad (\text{for } s > 0)\end{aligned}$$

21.10. Find the Laplace transform of $f(x) = 3 + 2x^2$.

Using Property 21.1 with the results of Problems 21.4 and 21.5, or alternatively, entries 1 and 3 ($n = 3$) of Appendix A, we have

$$\begin{aligned}F(s) &= \mathcal{L}\{3 + 2x^2\} = 3\mathcal{L}\{1\} + 2\mathcal{L}\{x^2\} \\ &= 3\left(\frac{1}{s}\right) + 2\left(\frac{2}{s^3}\right) = \frac{3}{s} + \frac{4}{s^3}\end{aligned}$$

21.11. Find the Laplace transform of $f(x) = 5\sin 3x - 17e^{-2x}$.

Using Property 21.1 with the results of Problems 21.6 ($a = -2$) and 21.7 ($a = 3$), or alternatively, entries 7 and 8 of Appendix A, we have

$$\begin{aligned}F(s) &= \mathcal{L}\{5\sin 3x - 17e^{-2x}\} = 5\mathcal{L}\{\sin 3x\} - 17\mathcal{L}\{e^{-2x}\} \\ &= 5\left(\frac{3}{s^2 + (3)^2}\right) - 17\left(\frac{1}{s - (-2)}\right) = \frac{15}{s^2 + 9} - \frac{17}{s + 2}\end{aligned}$$

21.12. Find the Laplace transform of $f(x) = 2\sin x + 3\cos 2x$.

Using Property 21.1 with entries 8 ($a = 1$) and 9 ($a = 2$) of Appendix A, we have

$$\begin{aligned}F(s) &= \mathcal{L}\{2\sin x + 3\cos 2x\} = 2\mathcal{L}\{\sin x\} + 3\mathcal{L}\{\cos 2x\} \\ &= 2\frac{1}{s^2 + 1} + 3\frac{s}{s^2 + 4} = \frac{2}{s^2 + 1} + \frac{3s}{s^2 + 4}\end{aligned}$$

21.13. Find the Laplace transform of $f(x) = 2x^2 - 3x + 4$.

Using Property 21.1 repeatedly with entries 1, 2 and 3 ($n = 3$) of Appendix A, we have

$$\begin{aligned}F(s) &= \mathcal{L}\{2x^2 - 3x + 4\} = 2\mathcal{L}\{x^2\} - 3\mathcal{L}\{x\} + 4\mathcal{L}\{1\} \\ &= 2\left(\frac{2}{s^3}\right) - 3\left(\frac{1}{s^2}\right) + 4\left(\frac{1}{s}\right) = \frac{4}{s^3} - \frac{3}{s^2} + \frac{4}{s}\end{aligned}$$

21.14. Find $\mathcal{L}\{xe^{4x}\}$.

This problem can be done three ways.

(*a*) Using entry 14 of Appendix A with $n = 2$ and $a = 4$, we have directly that

$$\mathcal{L}\{xe^{4x}\} = \frac{1}{(s-4)^2}$$

(*b*) Set $f(x) = x$. Using Property 21.2 with $a = 4$ and entry 2 of Appendix A, we have

$$F(s) = \mathcal{L}\{f(x)\} = \mathcal{L}\{x\} = \frac{1}{s^2}$$

and
$$\mathcal{L}\{e^{4x}x\} = F(s-4) = \frac{1}{(s-4)^2}$$

(*c*) Set $f(x) = e^{4x}$. Using Property 21.3 with $n = 1$ and the results of Problem 21.6, or alternatively, entry 7 of Appendix A with $a = 4$, we find that

$$F(s) = \mathcal{L}\{f(x)\} = \mathcal{L}\{e^{4x}\} = \frac{1}{s-4}$$

and
$$\mathcal{L}\{xe^{4x}\} = -F'(s) = -\frac{d}{ds}\left(\frac{1}{s-4}\right) = \frac{1}{(s-4)^2}$$

21.15. Find $\mathcal{L}\{e^{-2x}\sin 5x\}$.

This problem can be done two ways.

(*a*) Using entry 15 of Appendix A with $b = -2$ and $a = 5$, we have directly that

$$\mathcal{L}\{e^{-2x}\sin 5x\} = \frac{5}{[s-(-2)]^2 + (5)^2} = \frac{5}{(s+2)^2 + 25}$$

(*b*) Set $f(x) = \sin 5x$. Using Property 21.2 with $a = -2$ and the results of Problem 21.7, or alternatively, entry 8 of Appendix A with $a = 5$, we have

$$F(s) = \mathcal{L}\{f(x)\} = \mathcal{L}\{\sin 5x\} = \frac{5}{s^2 + 25}$$

and
$$\mathcal{L}\{e^{-2x}\sin 5x\} = F(s-(-2)) = F(s+2) = \frac{5}{(s+2)^2 + 25}$$

21.16. Find $\mathcal{L}\{x\cos\sqrt{7}x\}$.

This problem can be done two ways.

(*a*) Using entry 13 of Appendix A with $a = \sqrt{7}$, we have directly that

$$\mathcal{L}\{x\cos\sqrt{7}x\} = \frac{s^2 - (\sqrt{7})^2}{[s^2 + (\sqrt{7})^2]^2} = \frac{s^2 - 7}{(s^2 + 7)^2}$$

(*b*) Set $f(x) = \cos\sqrt{7}x$. Using Property 21.3 with $n = 1$ and entry 9 of Appendix A with $a = \sqrt{7}$, we have

$$F(s) = \mathcal{L}\{\cos\sqrt{7}x\} = \frac{s}{s^2 + (\sqrt{7})^2} = \frac{s}{s^2 + 7}$$

and $$\mathcal{L}\{x\cos\sqrt{7}x\} = -\frac{d}{ds}\left(\frac{s}{s^2+7}\right) = \frac{s^2-7}{(s^2+7)^2}$$

21.17. Find $\mathcal{L}\{e^{-x}\cos 2x\}$.

Let $f(x) = x\cos 2x$. From entry 13 of Appendix A with $a = 2$, we obtain

$$F(s) = \frac{s^2-4}{(s^2+4)^2}$$

Then, from Property 21.2 with $a = -1$,

$$\mathcal{L}\{e^{-x}x\cos 2x\} = F(s+1) = \frac{(s+1)^2-4}{[(s+1)^2+4]^2}$$

21.18. Find $\mathcal{L}\{x^{7/2}\}$.

Define $f(x) = \sqrt{x}$. Then $x^{7/2} = x^3\sqrt{x} = x^3 f(x)$ and, from entry 4 of Appendix A, we obtain

$$F(s) = \mathcal{L}\{f(x)\} = \mathcal{L}\{\sqrt{x}\} = \frac{1}{2}\sqrt{\pi}\,s^{-3/2}$$

It now follows from Property 21.3 with $n = 3$ that

$$\mathcal{L}\{x^3\sqrt{x}\} = (-1)^3\frac{d^3}{ds^3}\left(\frac{1}{2}\sqrt{\pi}s^{-3/2}\right) = \frac{105}{16}\sqrt{\pi}\,s^{-9/2}$$

which agrees with entry 6 of Appendix A for $n = 4$.

21.19. Find $\mathcal{L}\left\{\frac{\sin 3x}{x}\right\}$.

Taking $f(x) = \sin 3x$, we find from entry 8 of Appendix A with $a = 3$ that

$$F(s) = \frac{3}{s^2+9} \quad \text{or} \quad F(t) = \frac{3}{t^2+9}$$

Then, using Property 21.4, we obtain

$$\begin{aligned}\mathcal{L}\left\{\frac{\sin 3x}{x}\right\} &= \int_s^\infty \frac{3}{t^2+9}\,dt = \lim_{R\to\infty}\int_s^R \frac{3}{t^2+9}\,dt \\ &= \lim_{R\to\infty}\arctan\frac{t}{3}\bigg|_s^R \\ &= \lim_{R\to\infty}\left(\arctan\frac{R}{3} - \arctan\frac{s}{3}\right) \\ &= \frac{\pi}{2} - \arctan\frac{s}{3}\end{aligned}$$

21.20. Find $\mathcal{L}\left\{\int_0^x \sin 2t\,dt\right\}$.

Taking $f(t) = \sinh 2t$, we have $f(x) = \sinh 2x$. It now follows from entry 10 of Appendix A with $a = 2$ that

$F(s) = 2/(s^2 - 4)$, and then, from Property 21.5 that

$$\mathcal{L}\left\{\int_0^x \sinh 2t\, dt\right\} = \frac{1}{s}\left(\frac{2}{s^2-4}\right) = \frac{2}{s(s^2-4)}$$

21.21. Prove that if $f(x+\omega) = -f(x)$, then

$$\mathcal{L}\{f(x)\} = \frac{\int_0^{\omega} e^{-sx} f(x)dx}{1+e^{-\omega s}} \tag{1}$$

Since

$$f(x+2\omega) = f[(x+\omega)+\omega] = -f(x+\omega) = -[-f(x)] = f(x)$$

$f(x)$ is periodic with period 2ω. Then, using Property 21.6 with ω replaced by 2ω, we have

$$\mathcal{L}\{f(x)\} = \frac{\int_0^{2\omega} e^{-sx} f(x)dx}{1-e^{-2\omega s}} = \frac{\int_0^{\omega} e^{-sx} f(x)dx + \int_{\omega}^{2\omega} e^{-sx} f(x)dx}{1-e^{-2\omega s}}$$

Substituting $y = x - \omega$ into the second integral, we find that

$$\begin{aligned}\int_{\omega}^{2\omega} e^{-sx} f(x)dx &= \int_0^{\omega} e^{-s(y+\omega)} f(y+\omega)dy = e^{-\omega s}\int_0^{\omega} e^{-sy}[-f(y)]dy \\ &= -e^{-\omega s}\int_0^{\omega} e^{-sy} f(y)dy\end{aligned}$$

The last integral, upon changing the dummy variable of integration back to x, equals

$$-e^{-\omega s}\int_0^{\omega} e^{-sx} f(x)dx$$

Thus,

$$\begin{aligned}\mathcal{L}\{f(x)\} &= \frac{(1-e^{-\omega s})\int_0^{\omega} e^{-sx} f(x)dx}{1-e^{-2\omega s}} \\ &= \frac{(1-e^{-\omega s})\int_0^{\omega} e^{-sx} f(x)dx}{(1-e^{-\omega s})(1+e^{-\omega s})} = \frac{\int_0^{\omega} e^{-sx} f(x)dx}{1+e^{-\omega s}}\end{aligned}$$

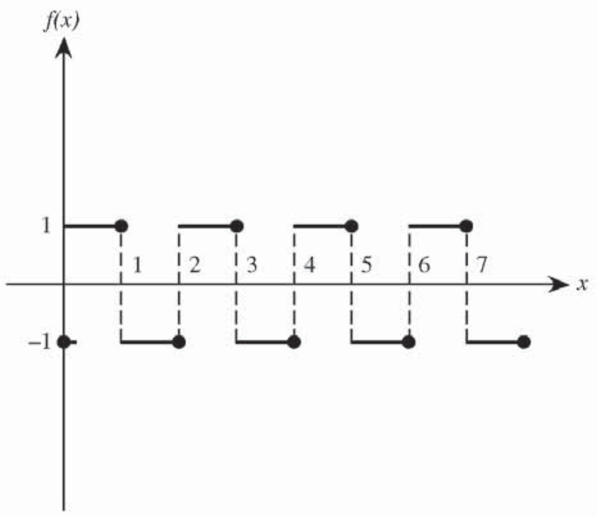

Fig. 21-2

21.22. Find $\mathcal{L}\{f(x)\}$ for the square wave shown in Fig. 21-2.

This problem can be done two ways.

(*a*) Note that $f(x)$ is periodic with period $\omega = 2$, and in the interval $0 < x \le 2$ it can be defined analytically by

$$f(x) = \begin{cases} 1 & 0 < x \le 1 \\ -1 & 1 < x \le 2 \end{cases}$$

From Eq. (*21.8*), we have

$$\mathcal{L}\{f(x)\} = \frac{\int_0^2 e^{-sx} f(x)dx}{1 - e^{-2s}}$$

Since

$$\begin{aligned}\int_0^2 e^{-sx} f(x)\,dx &= \int_0^1 e^{-sx}(1)dx + \int_1^2 e^{-sx}(-1)dx \\ &= \frac{1}{s}(e^{-2s} - 2e^{-s} + 1) = \frac{1}{s}(e^{-s} - 1)^2\end{aligned}$$

it follows that

$$\begin{aligned}F(s) &= \frac{(e^{-s} - 1)^2}{s(1 - e^{-2s})} = \frac{(1 - e^{-s})^2}{s(1 - e^{-s})(1 + e^{-s})} = \frac{1 - e^{-s}}{s(1 + e^{-s})} \\ &= \left[\frac{e^{s/2}}{e^{s/2}}\right]\left[\frac{1 - e^{-s}}{s(1 + e^{-s})}\right] = \frac{e^{s/2} - e^{-s/2}}{s(e^{s/2} + e^{-s/2})} = \frac{1}{s}\tanh\frac{s}{2}\end{aligned}$$

(*b*) The square wave $f(x)$ also satisfies the equation $f(x + 1) = -f(x)$. Thus, using (*1*) of Problem 21.21 with $\omega = 1$, we obtain

$$\begin{aligned}\mathcal{L}\{f(x)\} &= \frac{\int_0^1 e^{-sx} f(x)\,dx}{1 + e^{-s}} = \frac{\int_0^1 e^{-sx}(1)\,dx}{1 + e^{-s}} \\ &= \frac{(1/s)(1 - e^{-s})}{1 + e^{-s}} = \frac{1}{s}\tanh\frac{s}{2}\end{aligned}$$

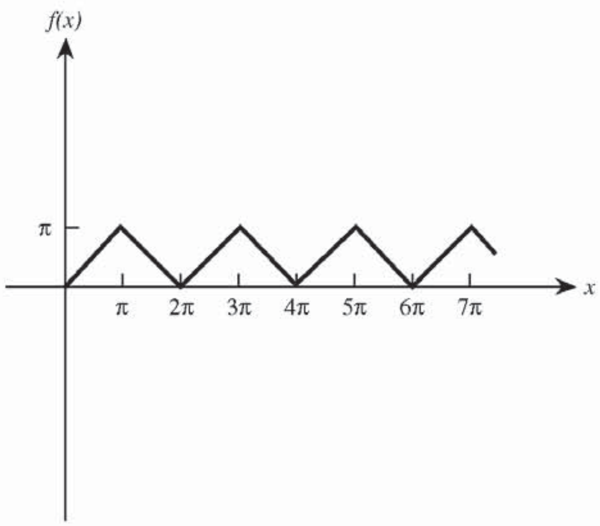

Fig. 21-3

21.23. Find the Laplace transform of the function graphed in Fig. 21-3.

Note that $f(x)$ is periodic with period $\omega = 2\pi$, and in the interval $0 \le x < 2\pi$ it can be defined analytically by

$$f(x) = \begin{cases} x & 0 \le x \le \pi \\ 2\pi - x & \pi \le x < 2\pi \end{cases}$$

From Eq. (*21.8*), we have

$$\mathscr{L}\{f(x)\} = \frac{\int_0^{2\pi} e^{-sx} f(x)dx}{1 - e^{-2\pi s}}$$

Since
$$\int_0^{2\pi} e^{-sx} f(x)dx = \int_0^{\pi} e^{-sx} x\,dx + \int_0^{2\pi} e^{-sx}(2\pi - x)dx$$
$$= \frac{1}{s^2}(e^{-2\pi s} - 2e^{-\pi s} + 1) = \frac{1}{s^2}(e^{-\pi s} - 1)^2$$

it follows that

$$\mathscr{L}\{f(x)\} = \frac{(1/s^2)(e^{-\pi s} - 1)^2}{1 - e^{-2\pi s}} = \frac{(1/s^2)(e^{-\pi s} - 1)^2}{(1 - e^{-\pi s})(1 + e^{-\pi s})}$$
$$= \frac{1}{s^2}\left(\frac{1 - e^{-\pi s}}{1 + e^{-\pi s}}\right) = \frac{1}{s^2}\tanh\frac{\pi s}{2}$$

21.24. Find $\mathscr{L}\left\{e^{4x} x \int_0^x \frac{1}{t} e^{-4t} \sin 3t\, dt\right\}$.

Using Eq. (*21.4*) with $a = -4$ on the results of Problem 21.19, we obtain

$$\mathscr{L}\left\{\frac{1}{x} e^{-4x} \sin 3x\right\} = \frac{\pi}{2} - \arctan\frac{s+4}{3}$$

It now follows from Eq. (*21.7*) that

$$\mathscr{L}\left\{\int_0^x \frac{1}{t} e^{-4t} \sin 3t\, dt\right\} = \frac{\pi}{2s} - \frac{1}{s}\arctan\frac{s+4}{3}$$

and then from Property 21.3 with $n = 1$,

$$\mathscr{L}\left\{x\int_0^x \frac{1}{t} e^{-4t} \sin 3t\, dt\right\} = \frac{\pi}{2s^2} - \frac{1}{s^2}\arctan\frac{s+4}{3} + \frac{3}{s[9 + (s+4)^2]}$$

Finally, using Eq. (*21.4*) with $a = 4$, we conclude that the required transform is

$$\frac{\pi}{2(s-4)^2} - \frac{1}{(s-4)^2}\arctan\frac{s}{3} + \frac{3}{(s-4)(s^2+9)}$$

21.25. Find the Laplace transforms at (*a*) t, (*b*) e^{at}, and (*c*) $\sin at$, where a denotes a constant.

Using entries 2, 7, and 8 of Appendix A with x replaced by t, we find the Laplace transforms to be, respectively,

$$(a)\quad \mathscr{L}\{t\} = \frac{1}{s^2} \qquad (b)\quad \mathscr{L}\{e^{at}\} = \frac{1}{s-a} \qquad (c)\quad \mathscr{L}\{\sin at\} = \frac{a}{s^2+a^2}$$

21.26. Find the Laplace transforms of (*a*) θ^2, (*b*) $\cos a\theta$, (*c*) $e^{b\theta}\sin a\theta$, where a and b denote constants.

Using entries 3 (with $n = 3$), 9, and 15 of Appendix A with x replaced by θ, we find the Laplace transforms to be, respectively.

$$(a)\quad \mathscr{L}\{\theta^2\} = \frac{2}{s^3} \qquad (b)\quad \mathscr{L}\{\cos a\theta\} = \frac{s}{s^2+a^2} \qquad (c)\quad \mathscr{L}\{e^{b\theta}\sin a\theta\} = \frac{a}{(s-b)^2+a^2}$$

Supplementary Problems

In Problems 21.27 and 21.42, find the Laplace transforms of the given function using Eq. (*21.1*).

21.27. $f(x) = 3$

21.28. $f(x) = \sqrt{5}$

21.29. $f(x) = e^{2x}$

21.30. $f(x) = e^{-6x}$

21.31. $f(x) = x$

21.32. $f(x) = -8x$

21.33. $f(x) = \cos 3x$

21.34. $f(x) = \cos 4x$

21.35. $f(x) = \cos bx$, where b denotes a constant

21.36. $f(x) = xe^{-8x}$

21.37. $f(x) = xe^{bx}$, where b denotes a constant

21.38. $f(x) = x^3$

21.39. $f(x) = \begin{cases} x & 0 \le x \le 2 \\ 2 & x > 2 \end{cases}$

21.40. $f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ e^x & 1 < x \le 4 \\ 0 & x > 4 \end{cases}$

21.41. $f(x)$ in Fig. 21-4

21.42. $f(x)$ in Fig. 21-5

In Problems 21.43 and 21.76, use Appendix A and the Properties 21.1 through 21.6, where appropriate, to find the Laplace transforms of the given functions.

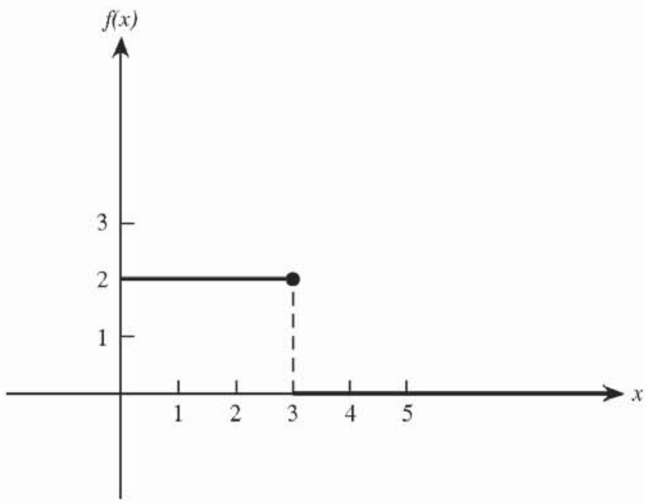

Fig. 21-4

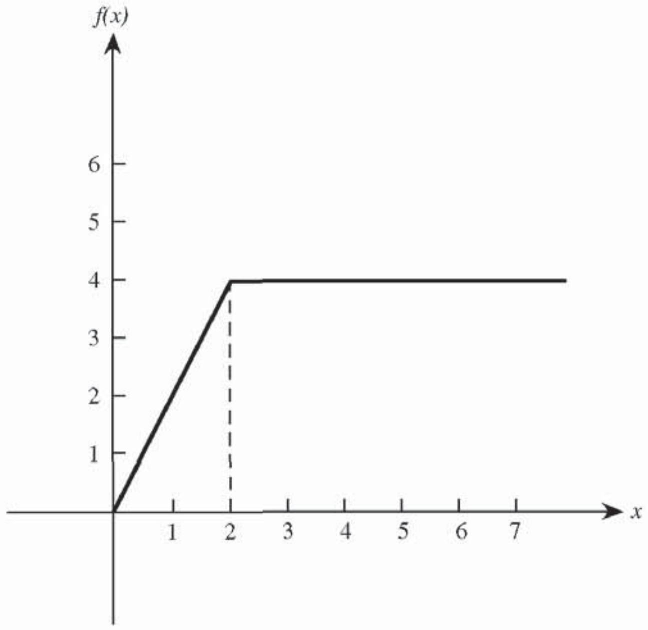

Fig. 21-5

21.43. $f(x) = x^7$

21.44. $f(x) = x\cos 3x$

21.45. $f(x) = x^5e^{-x}$

21.46. $f(x) = \dfrac{1}{\sqrt{x}}$

21.47. $f(x) = \dfrac{1}{3}e^{-x/3}$

21.48. $f(x) = 5e^{-x/3}$

21.49. $f(x) = 2\sin^2\sqrt{3}x$

21.50. $f(x) = 8e^{-5x}$

21.51. $f(x) = 3\sin\dfrac{x}{2}$

21.52. $f(x) = -\cos\sqrt{19}x$

21.53. $f(x) = -1.$

21.54. $f(x) = e^{-x}\sin 2x$

21.55. $f(x) = e^{x}\sin 2x$

21.56. $f(x) = e^{-x}\cos 2x$

21.57. $f(x) = e^{3x}\cos 2x$

21.58. $f(x) = e^{3x}\cos 5x$

21.59. $f(x) = e^{5x}\sqrt{x}$

21.60. $f(x) = e^{-5x}\sqrt{x}$

21.61. $f(x) = e^{-2x}\sin^2 x$

21.62. $x^3 + 3\cos 2x$

21.63. $5e^{2x} + 7e^{-x}$

21.64. $f(x) = 2 + 3x$

21.65. $f(x) = 3 - 4x^2$

21.66. $f(x) = 2x + 5\sin 3x$

21.67. $f(x) = 2\cos 3x - \sin 3x$

21.68. $2x^2\cosh x$

21.69. $2x^2e^{-x}\cosh x$

21.70. $x^2\sin 4x$

21.71. $\sqrt{x}\,e^{2x}$

21.72. $\int_0^x t\sinh t\,dt$

21.73. $\int_0^x e^{3t}\cos t\,dt$

21.74. $f(x)$ in Fig. 21-6

21.75. $f(x)$ in Fig. 21-7

21.76. $f(x)$ in Fig. 21-8

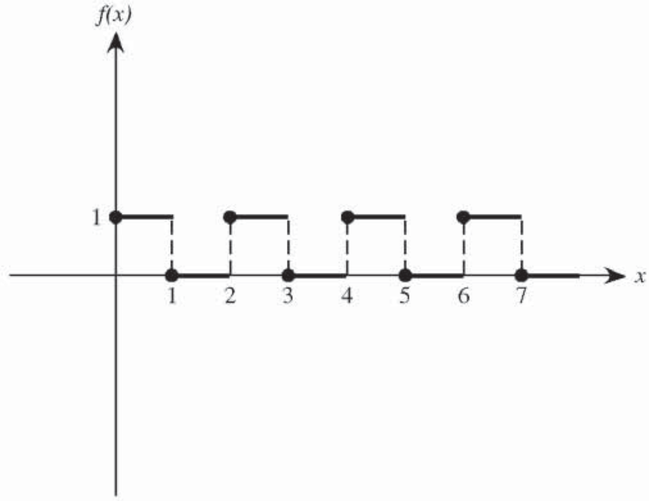

Fig. 21-6

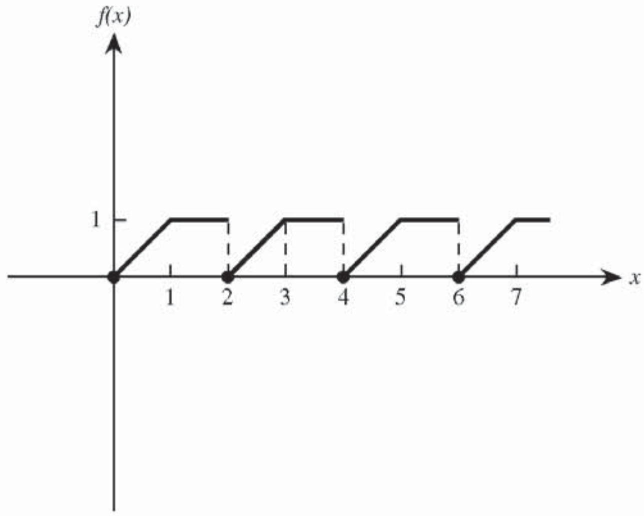

Fig. 21-7

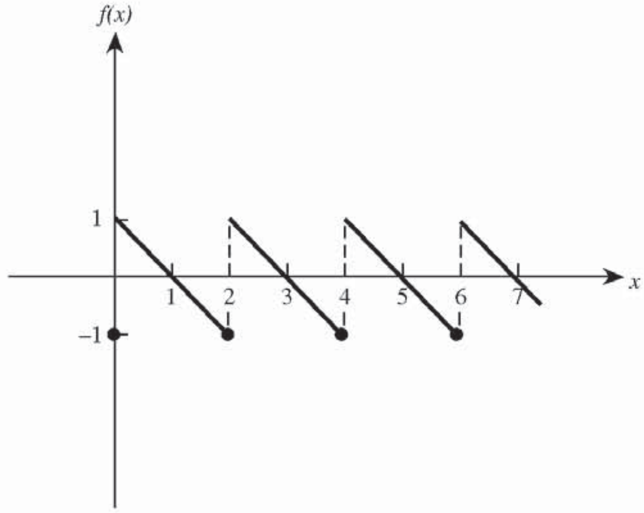

Fig. 21-8

CHAPTER 22

Inverse Laplace Transforms

DEFINITION

An *inverse Laplace transform* of $F(s)$, designated by $\mathscr{L}^{-1}\{F(s)\}$, is another function $f(x)$ having the property that $\mathscr{L}\{f(x)\} = F(s)$. This presumes that the independent variable of interest is x. If the independent variable of interest is t instead, then an inverse Laplace transform of $F(s)$ if $f(t)$ where $\mathscr{L}\{f(t)\} = F(s)$.

The simplest technique for identifying inverse Laplace transforms is to recognize them, either from memory or from a table such as Appendix A (see Problems 22.1 through 22.3). If $F(s)$ is not in a recognizable form, then occasionally it can be transformed into such a form by algebraic manipulation. Observe from Appendix A that almost all Laplace transforms are quotients. The recommended procedure is to first convert the denominator to a form that appears in Appendix A and then the numerator.

MANIPULATING DENOMINATORS

The method of *completing the square* converts a quadratic polynomial into the sum of squares, a form that appears in many of the denominators in Appendix A. In particular, for the quadratic $as^2 + bs + c$, where a, b, and c denote constants,

$$\begin{aligned} as^2 + bs + c &= a\left(s^2 + \frac{b}{a}s\right) + c \\ &= a\left[s^2 + \frac{b}{a}s + \left(\frac{b}{2a}\right)^2\right] + \left[c - \frac{b^2}{4a}\right] \\ &= a\left(s + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right) \\ &= a(s+k)^2 + h^2 \end{aligned}$$

where $k = b/2a$ and $h = \sqrt{c - (b^2/4a)}$. (See Problems 22.8 through 22.10.)

The method of *partial fractions* transforms a function of the form $a(s)/b(s)$, where both $a(s)$ and $b(s)$ are polynomials in s, into the sum of other fractions such that the denominator of each new fraction is either a first-degree or a quadratic polynomial raised to some power. The method requires only that (1) the degree of $a(s)$ be less than the degree of $b(s)$ (if this is not the case, first perform long division, and consider the remainder term) and (2) $b(s)$ be factored into the product of distinct linear and quadratic polynomials raised to various powers.

The method is carried out as follows. To each factor of $b(s)$ of the form $(s-a)^m$, assign a sum of m fractions, of the form

$$\frac{A_1}{s-a}+\frac{A_2}{(s-a)^2}+\cdots+\frac{A_m}{(s-a)^m}$$

To each factor of $b(s)$ of the form $(s^2+bs+c)^p$, assign a sum of p fractions, of the form

$$\frac{B_1s+C_1}{s^2+bs+c}+\frac{B_2s+C_2}{(s^2+bs+c)^2}+\cdots+\frac{B_ps+C_p}{(s^2+bs+c)^p}$$

Here A_i, B_j, and C_k $(i=1, 2,\ldots, m; j, k=1, 2,\ldots, p)$ are constants which still must be determined.

Set the original fraction $a(s)/b(s)$ equal to the sum of the new fractions just constructed. Clear the resulting equation of fractions and then equate coefficients of like powers of s, thereby obtaining a set of simultaneous linear equations in the unknown constants A_i, B_j, and C_k. Finally, solve these equations for A_i, B_j, and C_k. (See Problems 22.11 through 22.14.)

MANIPULATING NUMERATORS

A factor $s-a$ in the numerators may be written in terms of the factor $s-b$, where both a and b are constants, through the identity $s-a=(s-b)+(b-a)$. The multiplicative constant a in the numerator may be written explicitly in terms of the multiplicative constant b through the identity

$$a=\frac{a}{b}(b)$$

Both identities generate recognizable inverse Laplace transforms when they are combined with:

Property 22.1. (Linearity). If the inverse Laplace transforms of two functions $F(s)$ and $G(s)$ exist, then for any constants c_1 and c_2,

$$\mathscr{L}^{-1}\{c_1F(s)+c_2G(s)\}=c_1\mathscr{L}^{-1}\{F(s)\}+c_2\mathscr{L}^{-1}\{G(s)\}$$

(See Problems 22.4 through 22.7.)

Solved Problems

22.1. Find $\mathscr{L}^{-1}\left\{\frac{1}{s}\right\}$.

Here $F(s)=1/s$. From either Problem 21.4 or entry 1 of Appendix A, we have $\mathscr{L}\{1\}=1/s$. Therefore, $\mathscr{L}^{-1}\{1/s\}=1$.

22.2. Find $\mathscr{L}^{-1}\left\{\frac{1}{s-8}\right\}$.

From either Problem 21.6 or entry 7 of Appendix A with $a=8$, we have

$$\mathscr{L}\{e^{8x}\}=\frac{1}{s-8}$$

Therefore,
$$\mathscr{L}^{-1}\left\{\frac{1}{s-8}\right\}=e^{8x}$$

22.3. Find $\mathcal{L}^{-1}\left\{\frac{s}{s^2+6}\right\}$.

From entry 9 of Appendix A with $a=\sqrt{6}$, we have

$$\mathcal{L}\{\cos\sqrt{6}x\} = \frac{s}{s^2+(\sqrt{6})^2} = \frac{s}{s^2+6}$$

Therefore,

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+6}\right\} = \cos\sqrt{6}x$$

22.4. Find $\mathcal{L}^{-1}\left\{\frac{5s}{(s^2+1)^2}\right\}$.

The given function is similar in form to entry 12 of Appendix A. The denominators become identical if we take $a=1$. Manipulating the numerator of the given function and using Property 22.1, we obtain

$$\mathcal{L}^{-1}\left\{\frac{5s}{(s^2+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{\frac{5}{2}(2s)}{(s^2+1)^2}\right\} = \frac{5}{2}\mathcal{L}^{-1}\left\{\frac{2s}{(s^2+1)^2}\right\} = \frac{5}{2}x\sin x$$

22.5. Find $\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}}\right\}$.

The given function is similar in form to entry 5 of Appendix A. Their denominators are identical; manipulating the numerator of the given function and using Property 22.1, we obtain

$$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{\sqrt{\pi}}\frac{\sqrt{\pi}}{\sqrt{s}}\right\} = \frac{1}{\sqrt{\pi}}\mathcal{L}^{-1}\left\{\frac{\sqrt{\pi}}{\sqrt{s}}\right\} = \frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{x}}$$

22.6. Find $\mathcal{L}^{-1}\left\{\frac{s+1}{s^2-9}\right\}$.

The denominator of this function is identical to the denominator of entries 10 and 11 of Appendix A with $a=3$. Using Property 22.1 followed by a simple algebraic manipulation, we obtain

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2-9}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2-9}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2-9}\right\} = \cosh 3x + \mathcal{L}^{-1}\left\{\frac{1}{3}\left(\frac{3}{s^2-(3)^2}\right)\right\}$$

$$= \cosh 3x + \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2-(3)^2}\right\} = \cosh 3x + \frac{1}{3}\sinh 3x$$

22.7. Find $\mathcal{L}^{-1}\left\{\frac{s}{(s-2)^2+9}\right\}$.

The denominator of this function is identical to the denominators of entries 15 and 16 of Appendix A with $a=3$ and $b=2$. Both the given function and entry 16 have the *variable s* in their numerators, so they are the most closely matched. Manipulating the numerator of the given function and using Property 22.1, we obtain

$$\mathcal{L}^{-1}\left\{\frac{s}{(s-2)^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{(s-2)+2}{(s-2)^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{2}{(s-2)^2+9}\right\}$$

$$= e^{2x}\cos 3x + \mathcal{L}^{-1}\left\{\frac{2}{(s-2)^2+9}\right\} = e^{2x}\cos 3x + \mathcal{L}^{-1}\left\{\frac{2}{3}\left(\frac{3}{(s-2)^2+9}\right)\right\}$$

$$= e^{2x}\cos 3x + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{3}{(s-2)^2+9}\right\} = e^{2x}\cos 3x + \frac{2}{3}e^{2x}\sin 3x$$

22.8. Find $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2-2s+9}\right\}$.

No function of this form appears in Appendix A. But, by completing the square, we obtain

$$s^2-2s+9=(s^2-2s+1)+(9-1)=(s-1)^2+(\sqrt{8})^2$$

Hence,
$$\frac{1}{s^2-2s+9}=\frac{1}{(s-1)^2+(\sqrt{8})^2}=\left(\frac{1}{\sqrt{8}}\right)\frac{\sqrt{8}}{(s-1)^2+(\sqrt{8})^2}$$

Then, using Property 22.1 and entry 15 of Appendix A with $a=\sqrt{8}$ and $b=1$, we find that

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2-2s+9}\right\}=\frac{1}{\sqrt{8}}\mathscr{L}^{-1}\left\{\frac{\sqrt{8}}{(s-1)^2+(\sqrt{8})^2}\right\}=\frac{1}{\sqrt{8}}e^x\sin\sqrt{8}x$$

22.9. Find $\mathscr{L}^{-1}\left\{\dfrac{s+4}{s^2+4s+8}\right\}$.

No function of this form appears in Appendix A. Completing the square in the denominator, we have

$$s^2+4s+8=(s^2+4s+4)+(8-4)=(s+2)^2+(2)^2$$

Hence,
$$\frac{s+4}{s^2+4s+8}=\frac{s+4}{(s+2)^2+(2)^2}$$

This expression also is not found in Appendix A. However, if we rewrite the numerator as $s+4=(s+2)+2$ and then decompose the fraction, we have

$$\frac{s+4}{s^2+4s+8}=\frac{s+2}{(s+2)^2+(2)^2}+\frac{2}{(s+2)^2+(2)^2}$$

Then, from entries 15 and 16 of Appendix A,

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{s+4}{s^2+4s+8}\right\}&=\mathscr{L}^{-1}\left\{\frac{s+2}{(s+2)^2+(2)^2}\right\}+\mathscr{L}^{-1}\left\{\frac{2}{(s+2)^2+(2)^2}\right\}\\&=e^{-2x}\cos 2x+e^{-2x}\sin 2x\end{aligned}$$

22.10. Find $\mathscr{L}^{-1}\left\{\dfrac{s+2}{s^2-3s+4}\right\}$.

No function of this form appears in Appendix A. Completing the square in the denominator, we obtain

$$s^2-3s+4=\left(s^2-3s+\frac{9}{4}\right)+\left(4-\frac{9}{4}\right)=\left(s-\frac{3}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2$$

so that
$$\frac{s+2}{s^2-3s+4}=\frac{s+2}{\left(s-\frac{3}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}$$

We now rewrite the numerator as

$$s+2=s-\frac{3}{2}+\frac{7}{2}=\left(s-\frac{3}{2}\right)+\sqrt{7}\left(\frac{\sqrt{7}}{2}\right)$$

so that
$$\frac{s+2}{s^2-3s+4}=\frac{s-\frac{3}{2}}{\left(s-\frac{3}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}+\sqrt{7}\frac{\frac{\sqrt{7}}{2}}{\left(s-\frac{3}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}$$

Then,

$$\mathscr{L}^{-1}\left\{\frac{s+2}{s^2-3s+4}\right\}=\mathscr{L}^{-1}\left\{\frac{s-\frac{3}{2}}{\left(s-\frac{3}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}\right\}+\sqrt{7}\mathscr{L}^{-1}\left\{\frac{\frac{\sqrt{7}}{2}}{\left(s-\frac{3}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}\right\}$$
$$=e^{(3/2)x}\cos\frac{\sqrt{7}}{2}x+\sqrt{7}e^{(3/2)x}\sin\frac{\sqrt{7}}{2}x$$

22.11. Use partial function to decompose $\dfrac{1}{(s+1)(s^2+1)}$.

To the linear factor $s+1$, we associate the fraction $A/(s+1)$; whereas to the quadratic factor s^2+1, we associate the fraction $(Bs+C)/(s^2+1)$. We then set

$$\frac{1}{(s+1)(s^2+1)}\equiv\frac{A}{s+1}+\frac{Bs+C}{s^2+1} \tag{1}$$

Clearing fractions, we obtain

$$1\equiv A(s^2+1)+(Bs+C)\,(s+1) \tag{2}$$

or
$$s^2(0)+s(0)+1\equiv s^2(A+B)+s(B+C)+(A+C)$$

Equating coefficients of like powers of s, we conclude that $A+B=0$, $B+C=0$, and $A+C=1$. The solution of this set of equations is $A=\frac{1}{2}$, $B=-\frac{1}{2}$, and $C=\frac{1}{2}$. Substituting these values into (*1*), we obtain the partial-fractions decomposition

$$\frac{1}{(s+1)(s^2+1)}\equiv\frac{\frac{1}{2}}{s+1}+\frac{-\frac{1}{2}s+\frac{1}{2}}{s^2+1}$$

The following is an alternative procedure for finding the constants A, B, and C in (*1*). Since (*2*) must hold for all s, it must in particular hold $s=-1$. Substituting this value into (*2*), we immediately find $A=\frac{1}{2}$. Equation (*2*) must also hold for $s=0$. Substituting this value along with $A=\frac{1}{2}$ into (*2*), we obtain $C=\frac{1}{2}$. Finally, substituting any other value of s into (*2*), we find that $B=-\frac{1}{2}$.

22.12. Use partial fractions to decompose $\dfrac{1}{(s^2+1)(s^2+4s+8)}$.

To the quadratic factors s^2+1 and s^2+4s+8, we associate the fractions $(As+B)/(s^2+1)$ and $(Cs+D)/(s^2+4s+8)$. We set

$$\frac{1}{(s^2+1)(s^2+4s+8)}\equiv\frac{As+B}{s^2+1}+\frac{Cs+D}{s^2+4s+8} \tag{1}$$

and clear fractions to obtain

$$1\equiv(As+B)(s^2+4s+8)+(Cs+D)(s^2+1)$$

or
$$s^3(0)+s^2(0)+s(0)+1\equiv s^3(A+C)+s^2(4A+B+D)+s(8A+4B+C)+(8B+D)$$

Equating coefficients of like powers of s, we obtain $A+C=0$, $4A+B+D=0$, $8A+4B+C=0$, and $8B+D=1$. The solution of this set of equation is

$$A=-\frac{4}{65}\quad B=\frac{7}{65}\quad C=\frac{4}{65}\quad D=\frac{9}{65}$$

Therefore,
$$\frac{1}{(s^2+1)(s^2+4s+8)} \equiv \frac{-\frac{4}{65}s+\frac{7}{65}}{s^2+1}+\frac{\frac{4}{65}s+\frac{9}{65}}{s^2+4s+8}$$

22.13. Use partial fractions to decompose $\dfrac{s+3}{(s-2)(s+1)}$.

To the linear factors $s-2$ and $s+1$, we associate respectively the fractions $A/(s-2)$ and $B/(s+1)$. We set

$$\frac{s+3}{(s-2)(s+1)} \equiv \frac{A}{s-2}+\frac{B}{s+1}$$

and, upon clearing fractions, obtain

$$s+3 \equiv A(s+1)+B(s-2) \tag{1}$$

To find A and B, we use the alternative procedure suggested in Problem 22.11. Substituting $s=-1$ and then $s=2$ into (*1*), we immediately obtain $A=5/3$ and $B=-2/3$. Thus,

$$\frac{s+3}{(s-2)(s+1)} \equiv \frac{5/3}{s-2}-\frac{2/3}{s+1}$$

22.14. Use partial fractions to decompose $\dfrac{8}{s^3(s^2-s-2)}$.

Note that s^2-s-2 factors into $(s-2)(s+1)$. To the factor $s^3=(s-0)^3$, which is a linear polynomial raised to the third power, we associate the sum $A_1/s+A_2/s^2+A_3/s^3$. To the linear factors $(s-2)$ and $(s+1)$, we associate the fractions $B/(s-2)$ and $C/(s+1)$. Then

$$\frac{8}{s^3(s^2-s-2)} \equiv \frac{A_1}{s}+\frac{A_2}{s^2}+\frac{A_3}{s^3}+\frac{B}{s-2}+\frac{C}{s+1}$$

or, clearing fractions,

$$8 \equiv A_1s^2(s-2)(s+1)+A_2s(s-2)(s+1)+A_3(s-2)(s+1)+Bs^3(s+1)+Cs^3(s-2)$$

Letting $s=-1$, 2, and 0, consecutively, we obtain, respectively, $C=8/3$, $B=1/3$, and $A_3=-4$. Then choosing $s=1$ and $s=-2$, and simplifying, we obtain the equations $A_1+A_2=-1$ and $2A_1-A_2=-8$, which have the solutions $A_1=-3$ and $A_2=2$. Note that any other two values for s (not -1, 2, or 0) will also do; the resulting equations may be different, but the solution will be identical. Finally,

$$\frac{2}{s^3(s^2-s-2)} \equiv -\frac{3}{s}+\frac{2}{s^2}-\frac{4}{s^3}+\frac{1/3}{s-2}+\frac{8/3}{s+1}$$

22.15. Find $\mathscr{L}^{-1}\left\{\dfrac{s+3}{(s-2)(s+1)}\right\}$.

No function of this form appears in Appendix A. Using the results of Problem 22.13 and Property 22.1, we obtain

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{s+3}{(s-2)(s+1)}\right\} &= \frac{5}{3}\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\}-\frac{2}{3}\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\}\\ &= \frac{5}{3}e^{2x}-\frac{2}{3}e^{-x}\end{aligned}$$

22.16. Find $\mathscr{L}^{-1}\left\{\dfrac{8}{s^3(s^2-s-2)}\right\}$.

No function of this form appears in Appendix A. Using the results of Problem 22.14 and Property 22.1, we obtain

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{8}{s^3(s^2-s-2)}\right\} &= -3\mathscr{L}^{-1}\left\{\frac{1}{s}\right\}+2\mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\}\\ &\quad -2\mathscr{L}^{-1}\left\{\frac{2}{s^3}\right\}+\frac{1}{3}\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\}+\frac{8}{3}\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\}\\ &= -3+2x-2x^2+\frac{1}{3}e^{2x}+\frac{8}{3}e^{-x}\end{aligned}$$

22.17. Find $\mathscr{L}^{-1}\left\{\dfrac{1}{(s+1)(s^2+1)}\right\}$.

Using the result of Problem 22.11, and noting that

$$\frac{-\frac{1}{2}s+\frac{1}{2}}{s^2+1}=-\frac{1}{2}\left(\frac{s}{s^2+1}\right)+\frac{1}{2}\left(\frac{1}{s^2+1}\right)$$

we find that

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{1}{(s+1)(s^2+1)}\right\} &= \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\}-\frac{1}{2}\mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\right\}+\frac{1}{2}\mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\right\}\\ &= \frac{1}{2}e^{-x}-\frac{1}{2}\cos x+\frac{1}{2}\sin x\end{aligned}$$

22.18. Find $\mathscr{L}^{-1}\left\{\dfrac{1}{(s^2+1)(s^2+4s+8)}\right\}$.

From Problem 22.12, we have

$$\mathscr{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4s+8)}\right\}=\mathscr{L}^{-1}\left\{\frac{-\frac{4}{65}s+\frac{7}{65}}{s^2+1}\right\}+\mathscr{L}^{-1}\left\{\frac{\frac{4}{65}s+\frac{9}{65}}{s^2+4s+8}\right\}$$

The first term can be evaluated easily if we note that

$$\frac{-\frac{4}{65}s+\frac{7}{65}}{s^2+1}=\left(-\frac{4}{65}\right)\frac{s}{s^2+1}+\left(\frac{7}{65}\right)\frac{1}{s^2+1}$$

To evaluate the second inverse transforms, we must first complete the square in the denominator, $s^2+4s+8=(s+2)^2+(2)^2$, and then note that

$$\frac{\frac{4}{65}s+\frac{9}{65}}{s^2+4s+8}=\frac{4}{65}\left[\frac{s+2}{(s+2)^2+(2)^2}\right]+\frac{1}{130}\left[\frac{2}{(s+2)^2+(2)^2}\right]$$

Therefore,

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{1}{(s^2+1)(s^2+4s+8)}\right\} &= -\frac{4}{65}\mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\right\}+\frac{7}{65}\mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\right\}\\ &\quad +\frac{4}{65}\mathscr{L}^{-1}\left\{\frac{s+2}{(s+2)^2+(2)^2}\right\}+\frac{1}{130}\mathscr{L}^{-1}\left\{\frac{2}{(s+2)^2+(2)^2}\right\}\\ &= -\frac{4}{65}\cos x+\frac{7}{65}\sin x+\frac{4}{65}e^{-2x}\cos 2x+\frac{1}{130}e^{-2x}\sin 2x\end{aligned}$$

22.19. Find $\mathscr{L}^{-1}\left\{\dfrac{1}{s(s^2+4)}\right\}$.

By the method of partial fractions, we obtain

$$\frac{1}{s(s^2+4)} \equiv \frac{1/4}{s} + \frac{(-1/4)s}{s^2+4}$$

Thus,
$$\mathscr{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\} = \frac{1}{4}\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{4}\mathscr{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = \frac{1}{4} - \frac{1}{4}\cos 2x$$

Supplementary Problems

Find the inverse Laplace transforms, as a function of x, of the following functions:

22.20. $\dfrac{1}{s^2}$

22.21. $\dfrac{2}{s^2}$

22.22. $\dfrac{2}{s^3}$

22.23. $\dfrac{1}{s^3}$

22.24. $\dfrac{1}{s^4}$

22.25. $\dfrac{1}{s+2}$

22.26. $\dfrac{-2}{s-2}$

22.27. $\dfrac{12}{3s+9}$

22.28. $\dfrac{1}{2s-3}$

22.29. $\dfrac{1}{(s-2)^3}$

22.30. $\dfrac{12}{(s+5)^4}$

22.31. $\dfrac{3s^2}{(s^2+1)^2}$

22.32. $\dfrac{s^2}{(s^2+3)^2}$

22.33. $\dfrac{1}{s^2+4}$

22.34. $\dfrac{2}{(s-2)^2+9}$

22.35. $\dfrac{s}{(s+1)^2+5}$

22.36. $\dfrac{2s+1}{(s-1)^2+7}$

22.37. $\dfrac{1}{2s^2+1}$

22.38. $\dfrac{1}{s^2-2s+2}$

22.39. $\dfrac{s+3}{s^2+2s+5}$

22.40. $\dfrac{s}{s^2-s+17/4}$

22.41. $\dfrac{s+1}{s^2+3s+5}$

22.42. $\dfrac{2s^2}{(s-1)(s^2+1)}$

22.43. $\dfrac{1}{s^2-1}$

22.44. $\dfrac{2}{(s^2+1)(s-1)^2}$

22.45. $\dfrac{s+2}{s^3}$

22.46. $\dfrac{-s+6}{s^3}$

22.47. $\dfrac{s^3+3s}{s^6}$

22.48. $\dfrac{12+15\sqrt{s}}{s^4}$

22.49. $\dfrac{2s-13}{s(s^2-4s+13)}$

22.50. $\dfrac{2(s-1)}{s^2-s+1}$

22.51. $\dfrac{s}{(s^2+9)^2}$

22.52. $\dfrac{1}{2(s-1)(s^2-s-1)}=\dfrac{1/2}{(s-1)(s^2-s-1)}$

22.53. $\dfrac{s}{2s^2+4s+5/2}=\dfrac{(1/2)s}{s^2+2s+5/4}$