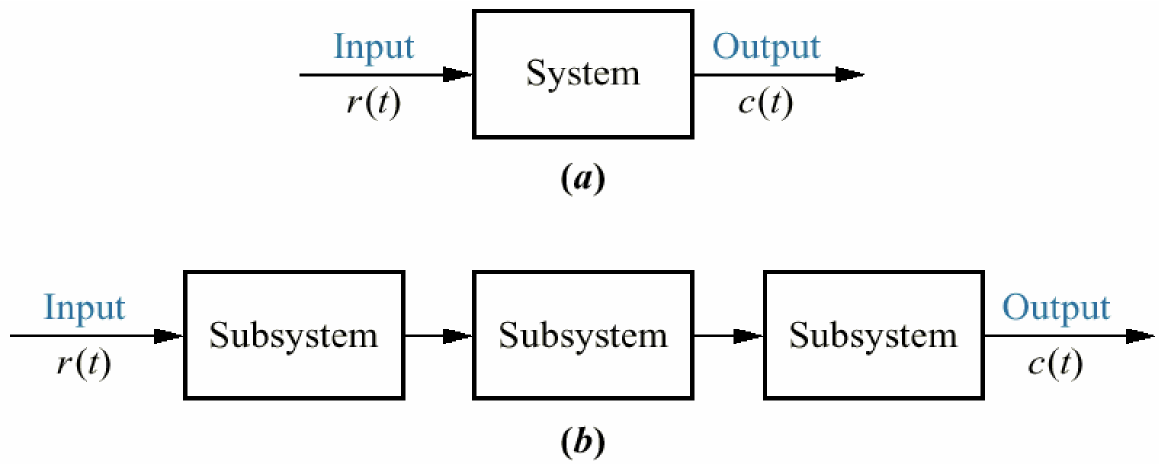
Figure 2.1

a. Block diagram representation of a system;

b. block diagram representation of an interconnection of subsystems



Note: The input, $r(t)$, stands for *reference input*.
The output, $c(t)$, stands for *controlled variable*.

REVIEW OF THE LAPLACE TRANSFORM

Table 2.1
Laplace transform table

Item no.	*f(t)*	*F(s)*
1.	$\delta(t)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5.	$e^{-at}u(t)$	$\frac{1}{s+a}$
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
7.	$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

Table 2.2
Laplace transform theorems

Item no.	Theorem	Name
1.	$\mathscr{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$	Definition
2.	$\mathscr{L}[kf(t)] = kF(s)$	Linearity theorem
3.	$\mathscr{L}[f_1(t)+f_2(t)] = F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathscr{L}[e^{-at}f(t)] = F(s+a)$	Frequency shift theorem
5.	$\mathscr{L}[f(t-T)] = e^{-sT}F(s)$	Time shift theorem
6.	$\mathscr{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathscr{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$	Differentiation theorem
8.	$\mathscr{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0-) - \dot{f}(0-)$	Differentiation theorem
9.	$\mathscr{L}\left[\frac{d^nf}{dt^n}\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k}f^{k-1}(0-)$	Differentiation theorem
10.	$\mathscr{L}\left[\int_{0-}^{t} f(\tau)\,d\tau\right] = \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty) = \lim_{s\to 0} sF(s)$	Final value theorem[1]
12.	$f(0+) = \lim_{s\to\infty} sF(s)$	Initial value theorem[2]

[1] For this theorem to yield correct finite results, all roots of the denominator of $F(s)$ must have negative real parts and no more than one can be at the origin.
[2] For this theorem to be valid, $f(t)$ must be continuous or have a step discontinuity at $t = 0$ (i.e., no impulses or their derivatives at $t = 0$).

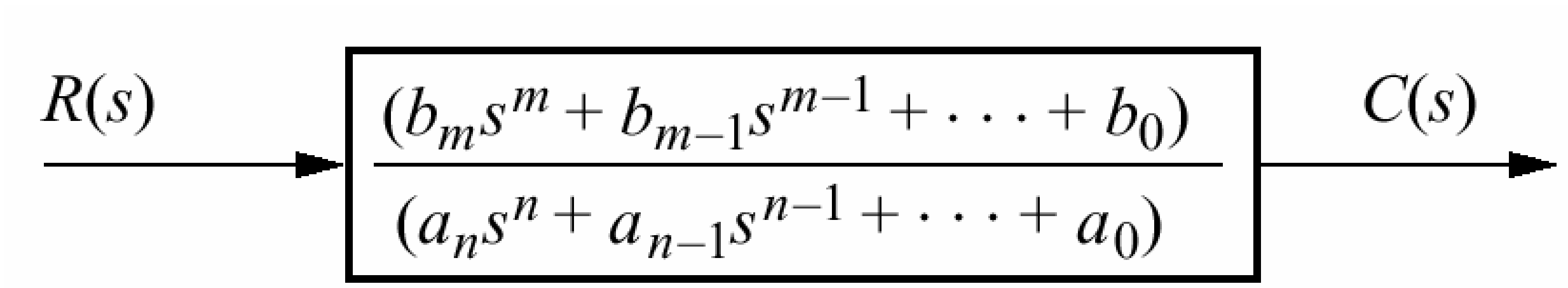


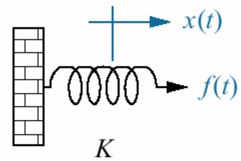
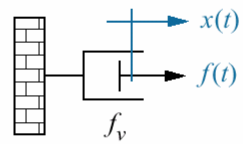
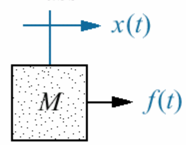
Figure 2.2

Block diagram of a transfer function

MECHANICAL SYSTEMS TRANSFER FUNTIONS

Table 2.4

Force-velocity, force-displacement, and impedance translational relationships
for springs, viscous dampers, and mass

Component	Force-velocity	Force-displacement	Impedance $Z_M(s) = F(s)/X(s)$
Spring 	$f(t) = K\int_0^t v(\tau)\,d\tau$	$f(t) = Kx(t)$	K
Viscous damper 	$f(t) = f_v v(t)$	$f(t) = f_v \dfrac{dx(t)}{dt}$	$f_v s$
Mass 	$f(t) = M\dfrac{dv(t)}{dt}$	$f(t) = M\dfrac{d^2x(t)}{dt^2}$	Ms^2

Note: The following set of symbols and units is used throughout this book: $f(t)$ = N (newtons), $x(t)$ = m (meters), $v(t)$ = m/s (meters/second), K = N/m (newtons/meter), f_v = N-s/m (newton-seconds/meter), M = kg (kilograms = newton-seconds^2/meter).

Example: *Find the transfer function X(s) / F(s) for the system given below*

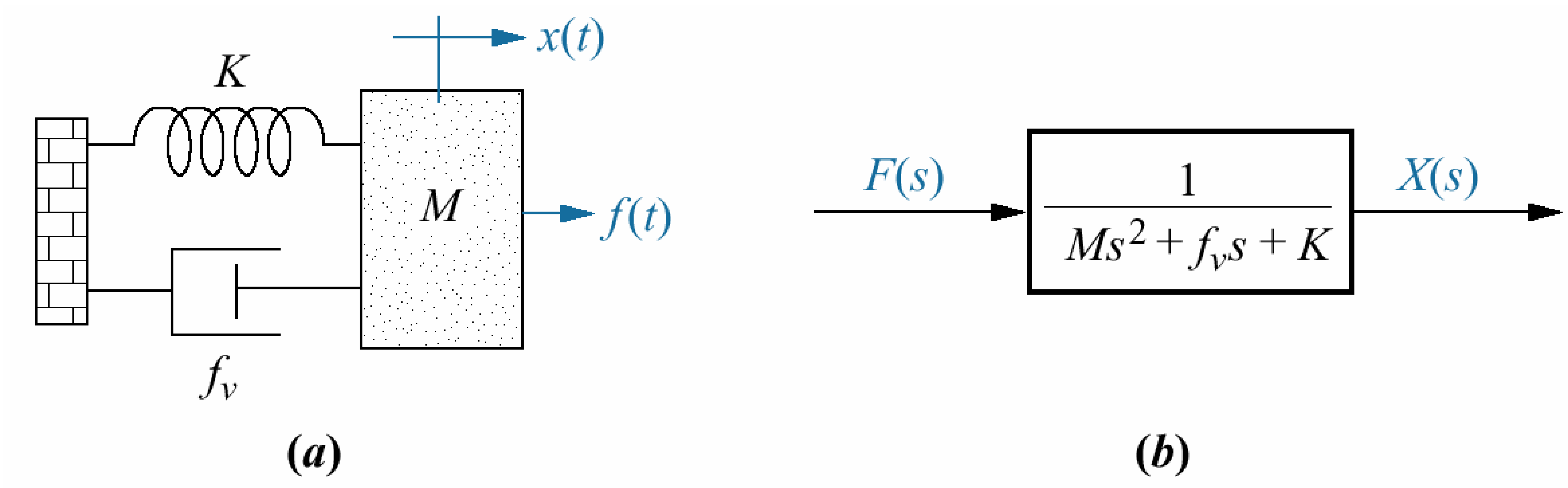
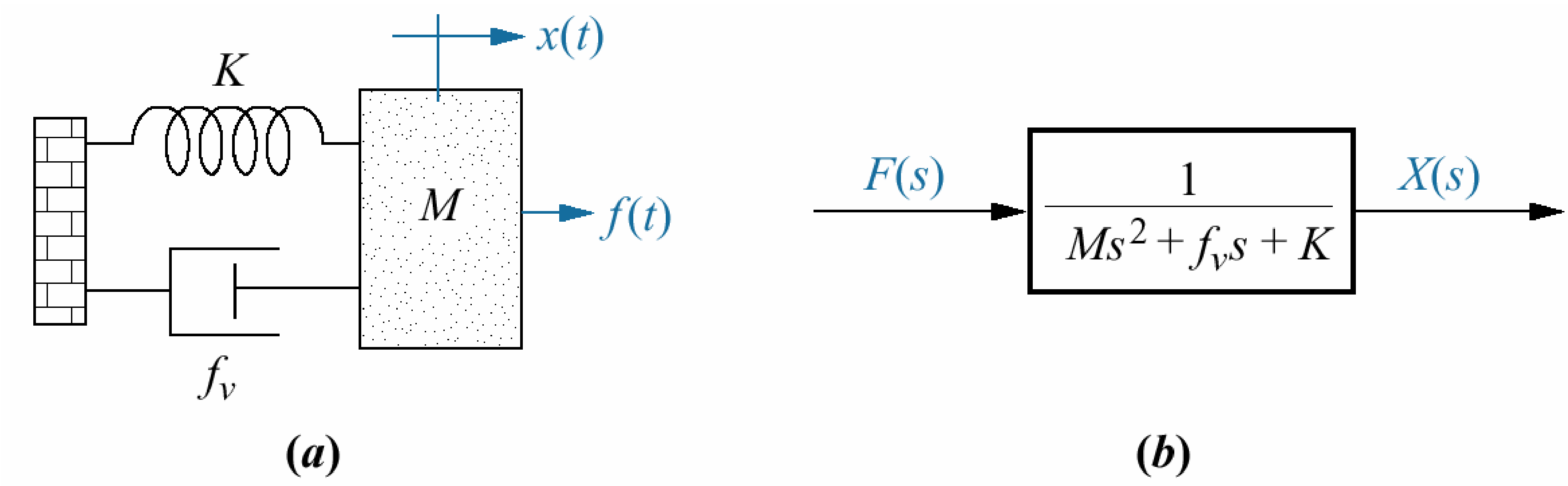


Figure 2.15
a. Mass, spring, and damper system;
b. block diagram



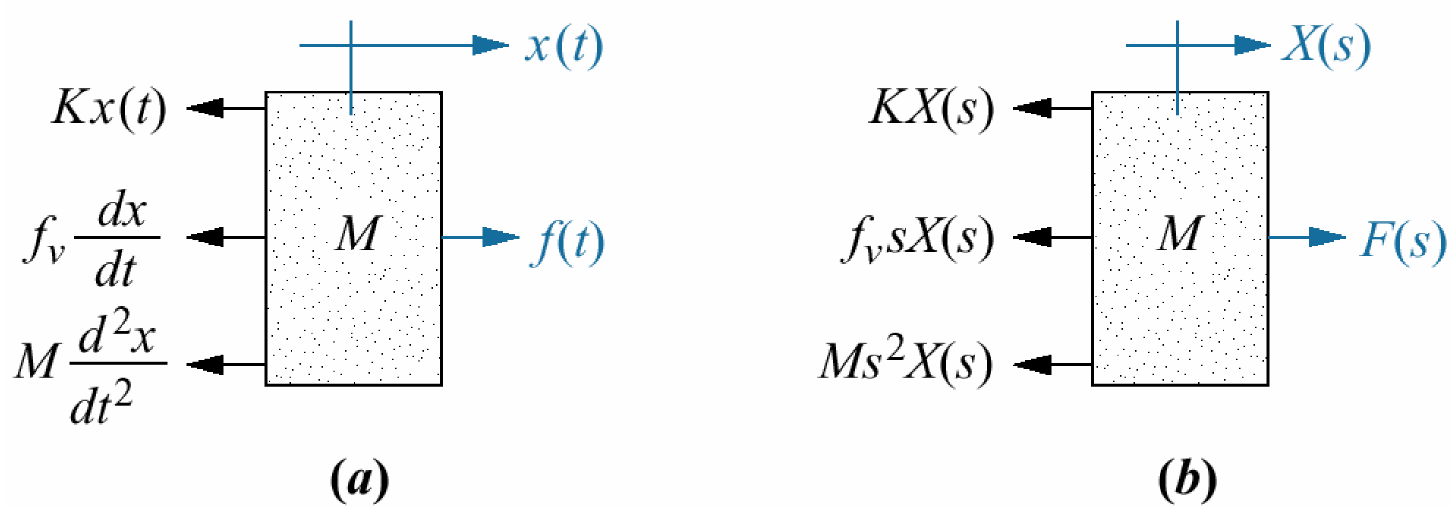
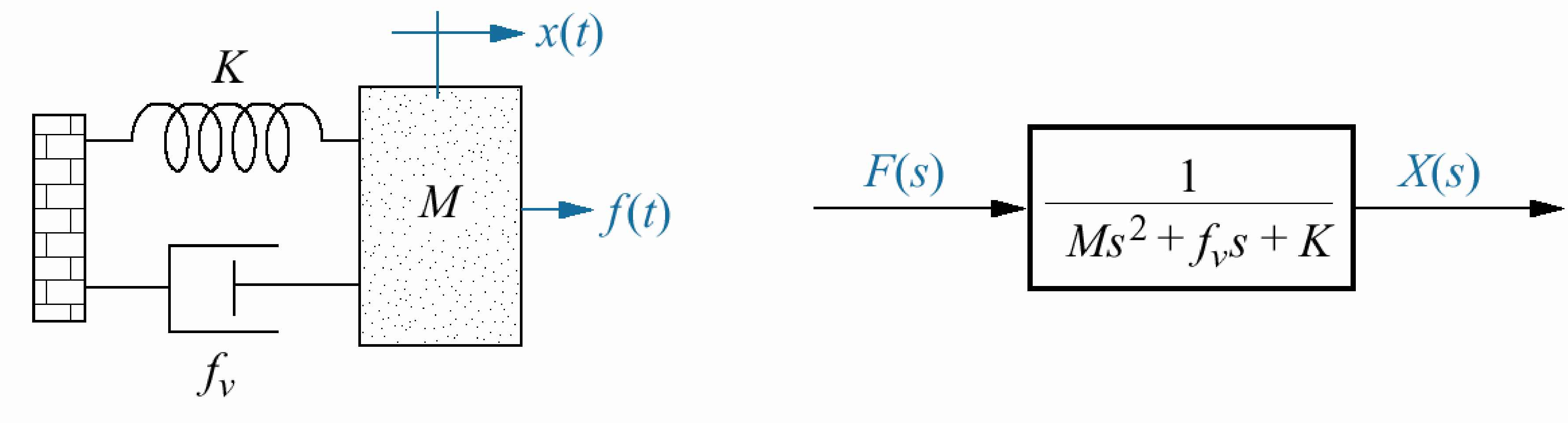


Figure 2.16
a. Free-body diagram of mass, spring, and damper system;
b. transformed free-body diagram



(a) **(b)**

We now write the differential equation of motion using Newton's law

$$M\frac{d^2x(t)}{dt^2} + f_v\frac{dx(t)}{dt} + Kx(t) = f(t)$$

Taking the Laplace transform, assuming zero initial conditions

$$Ms^2X(s) + f_vsX(s) + KX(s) = F(s)$$

$$(Ms^2 + f_vs + K)X(s) = F(s)$$

Solving for transfer function yields

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + f_v + K}$$

[Sum of impedance]$X(s)$ = [Sum of applied forces]

Note that the number of equation of motion required is equal to number of *linearly independent* motion. Linear independence implies that a point of motion in a system can still move if all other points of motion are held still. Another name of the number of linearly independent motion is the number of *degrees of freedom*.

<u>Example</u> : Find the transfer function $X_2(s)/F(s)$

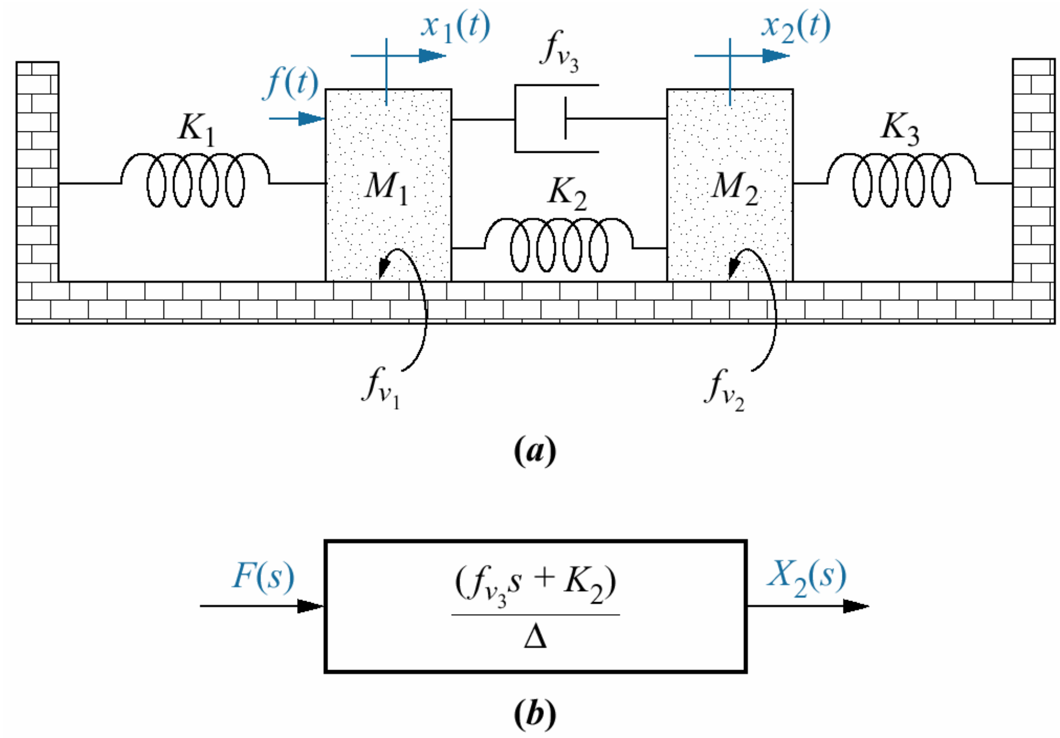
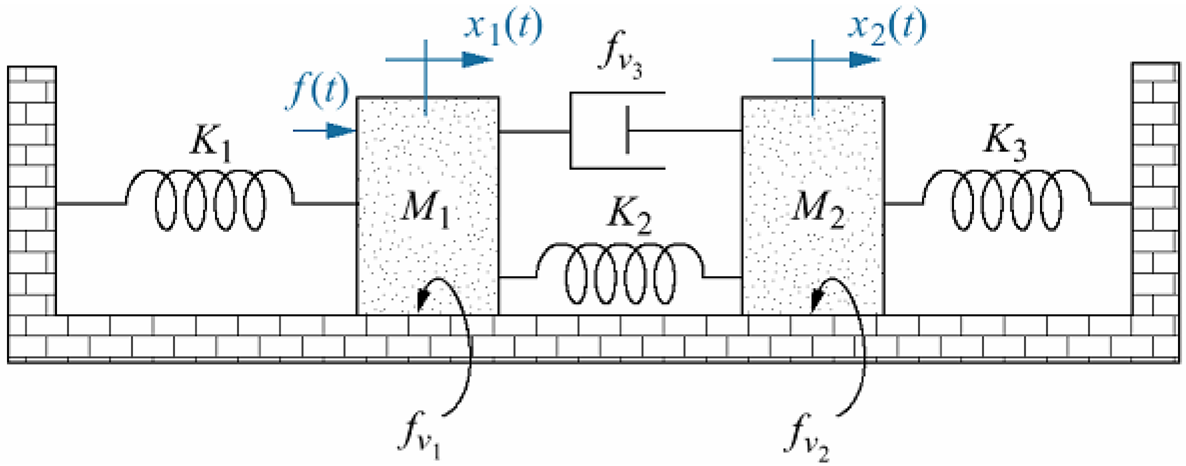
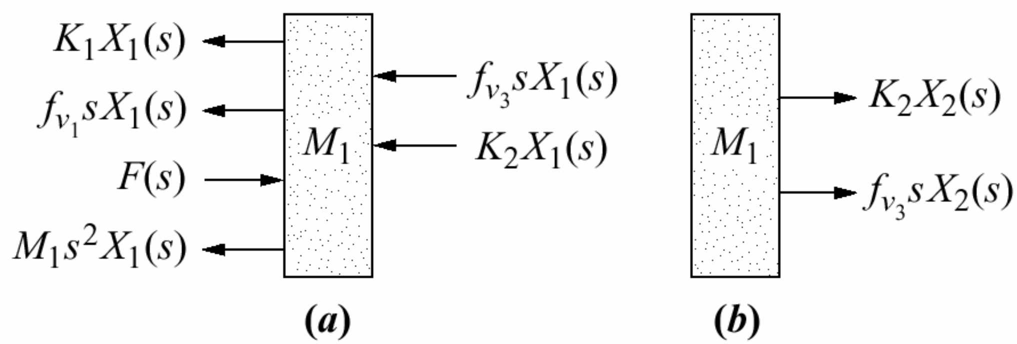


Figure 2.17
a. Two-degrees-of-freedom translational mechanical system[8];
b. block diagram





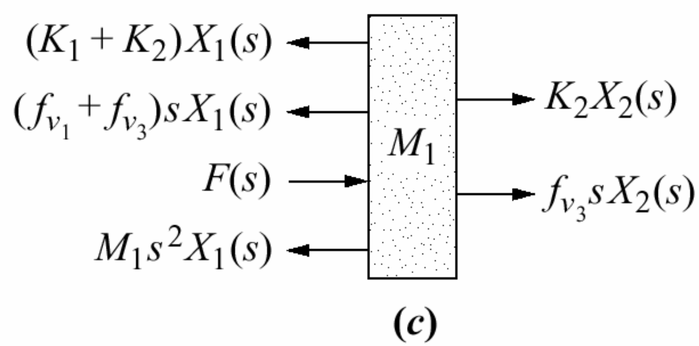
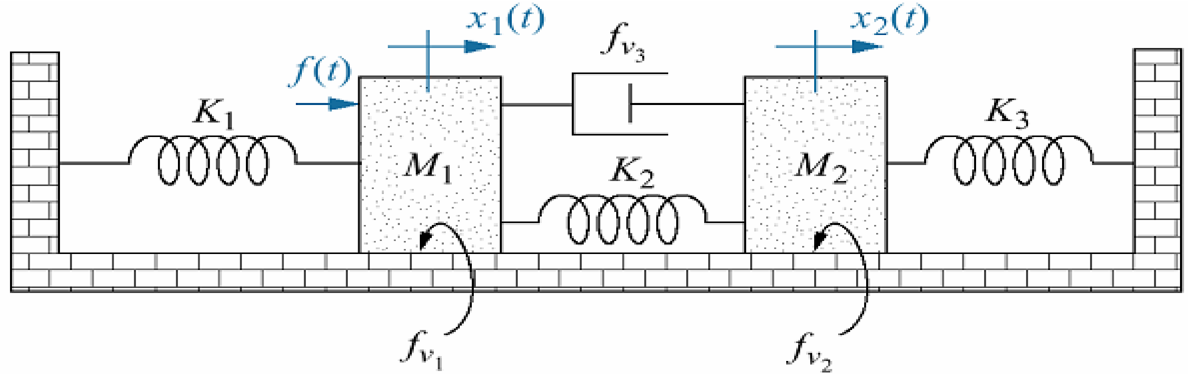


Figure 2.18

a. Forces on M_1 due only to motion of M_1

b. forces on M_1 due only to motion of M_2

c. all forces on M_1



$$[M_1 s^2 + (f_{v1}+f_{v3})s + (K_1+K_2)]\,\mathbf{X_1(s)} - (f_{v3}s+K_2)\mathbf{X_2(s)} = F(s)$$

$$-(f_{v3}s + K_2)\mathbf{X_1(s)} + [M_2 s^2 + (f_{v2}+f_{v3})s + (K_2 + K_3)]\mathbf{X_2(s)} = 0$$

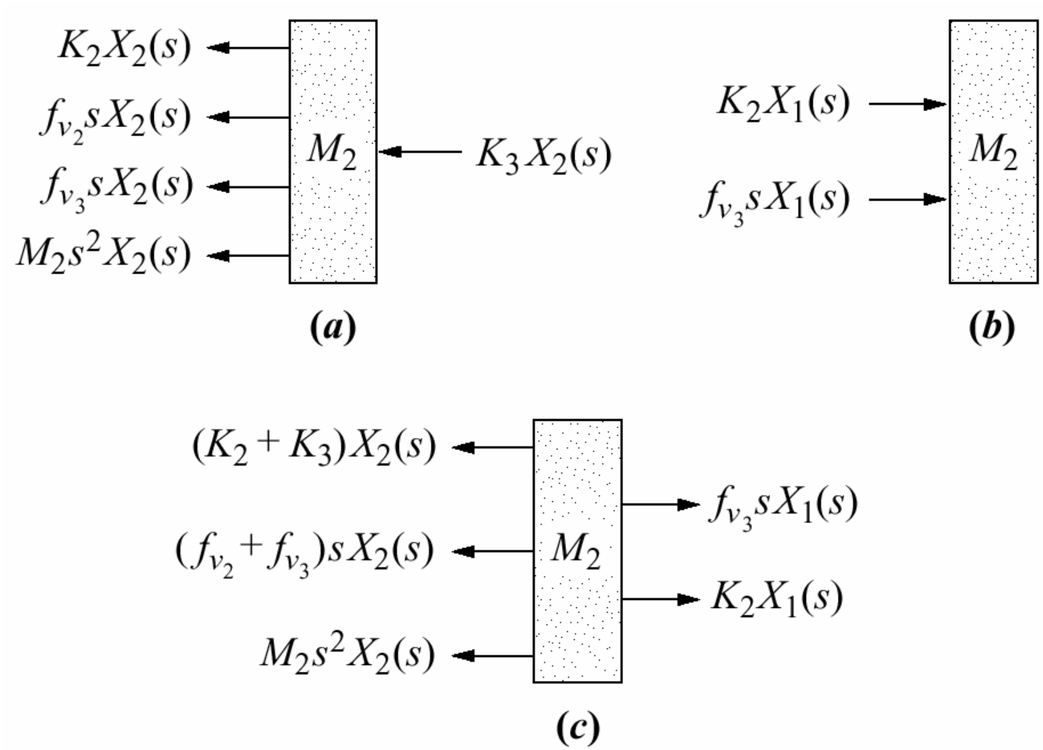
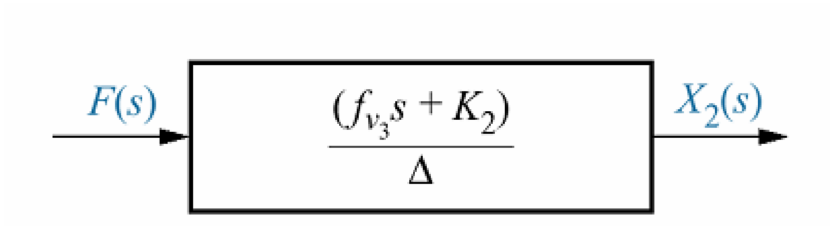


Figure 2.19
a. Forces on M_2 due only to motion of M_2;
b. forces on M_2 due only to motion of M_1;
c. all forces on M_2

The transfer function $X_2(s)/F(s)$ is



$$\Delta = \begin{vmatrix} [M_1 s^2 + (f_{v1}+f_{v3})s + (K_1+K_2)] & -(f_{v3}s+K_2) \\ -(f_{v3}s+K_2) & [M_2 s^2 + (f_{v2}+f_{v3})s + (K_2+K_3)] \end{vmatrix}$$

Note that

$$\begin{pmatrix} \textit{Sum of} \\ \textit{impedance} \\ \textit{connected} \\ \textit{to the} \\ \textit{motion at} \\ X_1 \end{pmatrix} X_1(s) - \begin{pmatrix} \textit{Sum of} \\ \textit{impedance} \\ \textit{between} \\ X_1 \textit{ and } X_2 \end{pmatrix} X_2(s) = \begin{pmatrix} \textit{Sum of applied} \\ \textit{forces at} \\ X_1 \end{pmatrix}$$

$$\begin{pmatrix} \textit{Sum of} \\ \textit{impedance} \\ \textit{connected} \\ \textit{to the} \\ \textit{motion at} \\ X_2 \end{pmatrix} X_2(s) - \begin{pmatrix} \textit{Sum of} \\ \textit{impedance} \\ \textit{between} \\ X_1 \textit{ and } X_2 \end{pmatrix} X_1(s) = \begin{pmatrix} \textit{Sum of applied} \\ \textit{forces at} \\ X_2 \end{pmatrix}$$

Equations of Motion by Inspection

Problem : Write, but not solve, the equation of motion for the mechanical system given below.

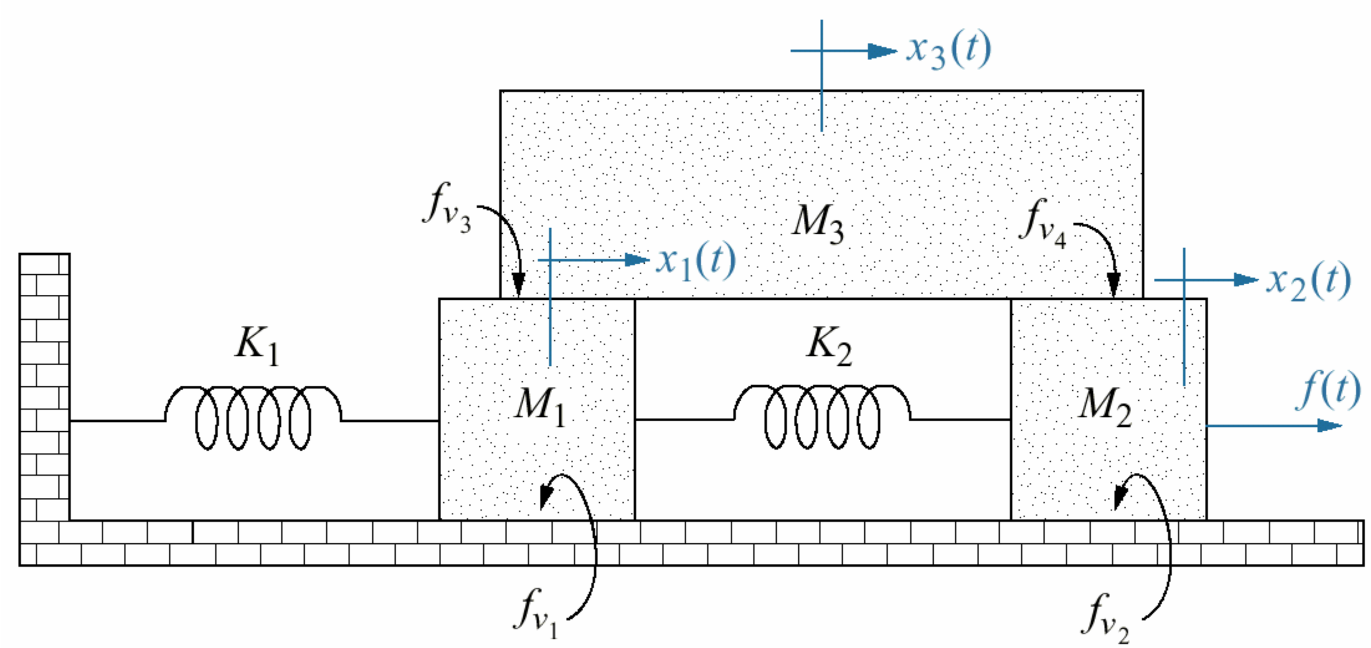


Figure 2.20
Three-degrees-of-freedom translational mechanical system

Using same logic, for M_1

$$\begin{pmatrix} \text{Sum of} \\ \text{impedance} \\ \text{connected} \\ \text{to the} \\ \text{motion at} \\ X_1 \end{pmatrix} X_1(s) - \begin{pmatrix} \text{Sum of} \\ \text{impedance} \\ \text{between} \\ X_1 \text{ and } X_2 \end{pmatrix} X_2(s) - \begin{pmatrix} \text{Sum of} \\ \text{impedance} \\ \text{between} \\ X_1 \text{ and } X_3 \end{pmatrix} X_3(s) = \begin{pmatrix} \text{Sum of} \\ \text{applied} \\ \text{forces at} \\ X_1 \end{pmatrix}$$

and for M_2

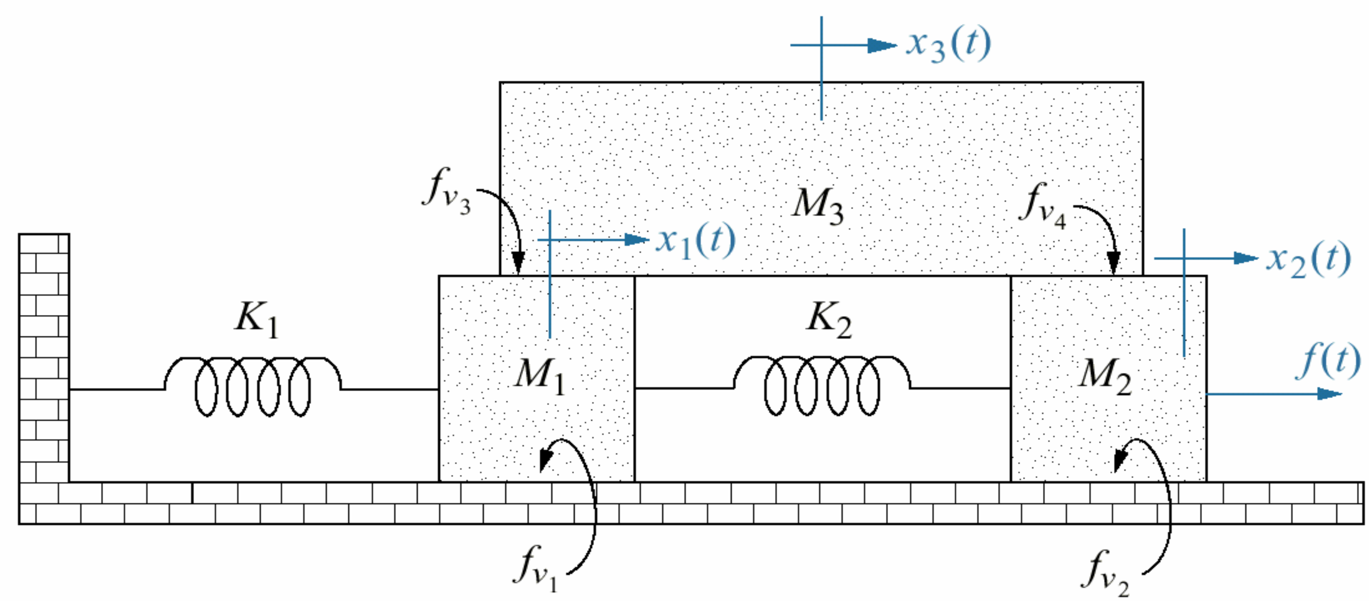
$$-\begin{pmatrix} \text{Sum of} \\ \text{impedance} \\ \text{between} \\ X_1 \text{ and } X_2 \end{pmatrix} X_1(s) + \begin{pmatrix} \text{Sum of} \\ \text{impedance} \\ \text{connected to} \\ \text{the motion at} \\ X_2 \end{pmatrix} X_2(s) - \begin{pmatrix} \text{Sum of} \\ \text{impedance} \\ \text{between} \\ X_2 \text{ and } X_3 \end{pmatrix} X_3(s) = \begin{pmatrix} \text{Sum of} \\ \text{applied} \\ \text{forces at} \\ X_2 \end{pmatrix}$$

Similarly, for M_3

$$-\begin{pmatrix} \text{Sum of} \\ \text{impedance} \\ \text{between} \\ X_1 \text{ and } X_3 \end{pmatrix} X_1(s) - \begin{pmatrix} \text{Sum of} \\ \text{impedance} \\ \text{between} \\ X_2 \text{ and } X_3 \end{pmatrix} X_2(s) + \begin{pmatrix} \text{Sum of} \\ \text{impedance} \\ \text{connected to} \\ \text{the motion at} \\ X_3 \end{pmatrix} X_3(s) = \begin{pmatrix} \text{Sum of} \\ \text{applied} \\ \text{forces at} \\ X_3 \end{pmatrix}$$

Anymore we can write the equations for M_1, M_2 and M_3

Note that M_1 has two springs, two viscous damper and mass associated with its motion. There is one spring between M_1 and M_2 and one viscous damper between M_1 and M_3.



Equation for M_1

$$[M_1 s^2 + (f_{v1} + f_{v3})s + (K_1 + K_2)]X_1(s) - K_2 X_2(s) - f_{v3} s X_3(s) = 0$$

for M_2

$$-K_2 X_1(s) + [M_2 s^2 + (f_{v2} + f_{v4})s + K_2]X_2(s) - f_{v4} s X_3(s) = F(s)$$

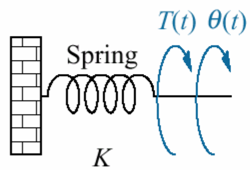
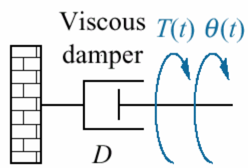
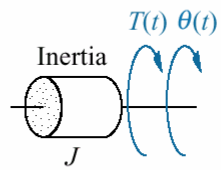
and for M_3

$$-f_{v3} s X_1(s) - f_{v4} s X_2(s) + [M_3 s^2 + (f_{v3} + f_{v4})s]X_3(s) = 0$$

These equations are the equations of motion. We can solve them for any displacement $X_1(s)$, $X_2(s)$ or $X_3(s)$, or transfer function.

ROTATIONAL MECHANICAL SYSTEM TRANSFER FUNTIONS

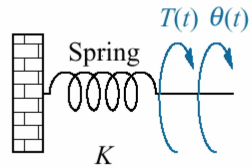
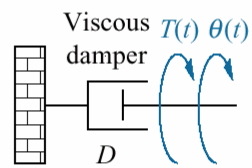
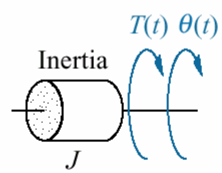
Table 2.5
Torque-angular velocity, torque-angular displacement, and impedance
rotational relationships for springs, viscous dampers, and inertia

Component	Torque-angular velocity	Torque-angular displacement	Impedance $Z_M(s) = T(s)/\theta(s)$
	$T(t) = K\int_0^t \omega(\tau)\,d\tau$	$T(t) = K\theta(t)$	K
	$T(t) = D\omega(t)$	$T(t) = D\dfrac{d\theta(t)}{dt}$	Ds
	$T(t) = J\dfrac{d\omega(t)}{dt}$	$T(t) = J\dfrac{d^2\theta(t)}{dt^2}$	Js^2

Note: The following set of symbols and units is used throughout this book: $T(t)$ = N-m (newton-meters), $\theta(t)$ = rad (radians), $\omega(t)$ = rad/s (radians/second), K = N-m/rad (newton-meters/radian), D = N-m-s/rad (newton-meters-seconds/radian), J = kg-m^2 (kilogram-meters^2 = newton-meters-seconds^2/radian).

Rotational mechanical systems are handled the same way as translational mechanical systems, except that *torque* replaces *force* and angular displacement replaces translational displacement. Also notice that the term associated with the *mass* is replaced by *inertia*. The values of K, D and J are called *spring constant, coefficient of viscous friction* and *moment of inertia*, respectively.

Writing the equations of motion for rotational systems is similar to writing them for translational system; the only difference is that the free body diagram consist of *torques* rather than *forces*.

Component	Torque-angular velocity	Torque-angular displacement	Impedance $Z_M(s) = T(s)/\theta(s)$
	$T(t) = K\int_0^t \omega(\tau)\,d\tau$	$T(t) = K\theta(t)$	K
	$T(t) = D\omega(t)$	$T(t) = D\frac{d\theta(t)}{dt}$	Ds
	$T(t) = J\frac{d\omega(t)}{dt}$	$T(t) = J\frac{d^2\theta(t)}{dt^2}$	Js^2

Note: The following set of symbols and units is used throughout this book: $T(t)$ = N-m (newton-meters), $\theta(t)$ = rad (radians), $\omega(t)$ = rad/s (radians/second), K = N-m/rad (newton-meters/radian), D = N-m-s/rad (newton-meters-seconds/radian), J = kg-m^2 (kilogram-meters^2 = newton-meters-seconds^2/radian).

Example : Find the transfer function $\theta_2(s)$ / $T(s)$ for the rotational system given below.

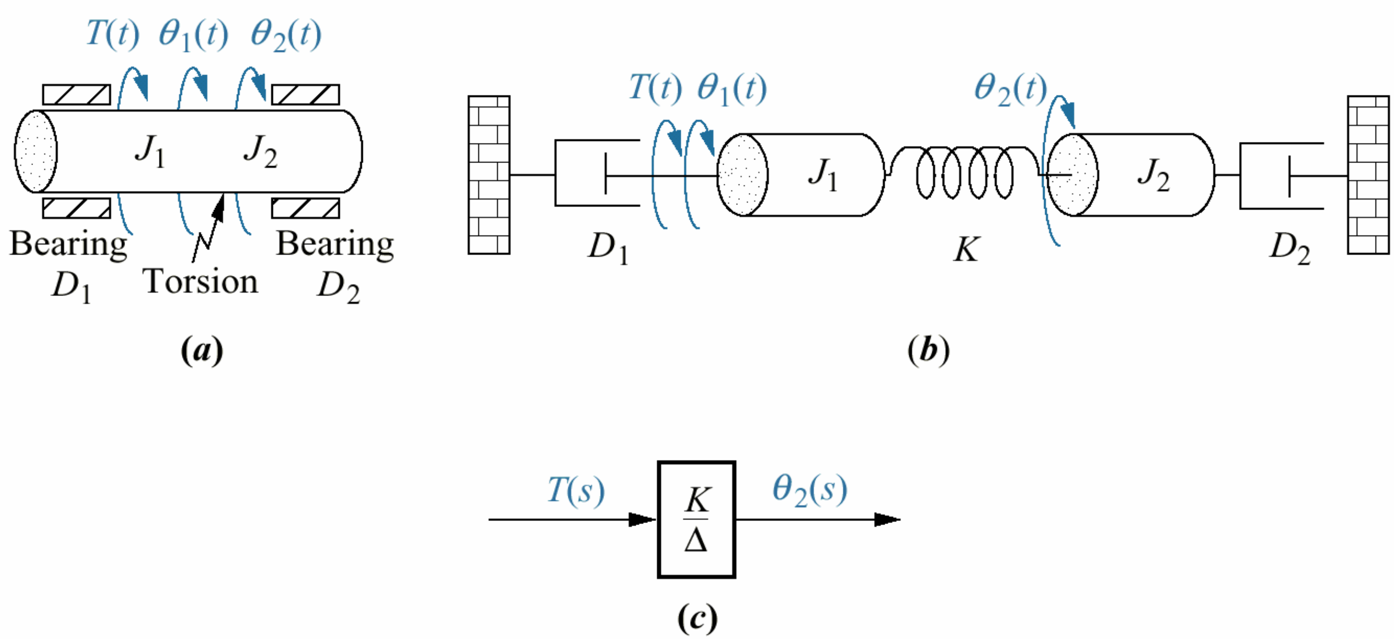


Figure 2.22
a. Physical system;
b. schematic;
c. block diagram

First, obtain the schematic from the physical system.

Draw free-body diagram for J_1 and J_2 using superposition.

Let's start with J_1 :

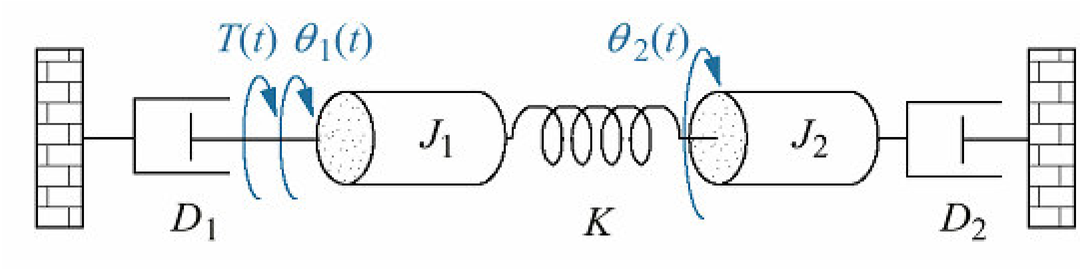
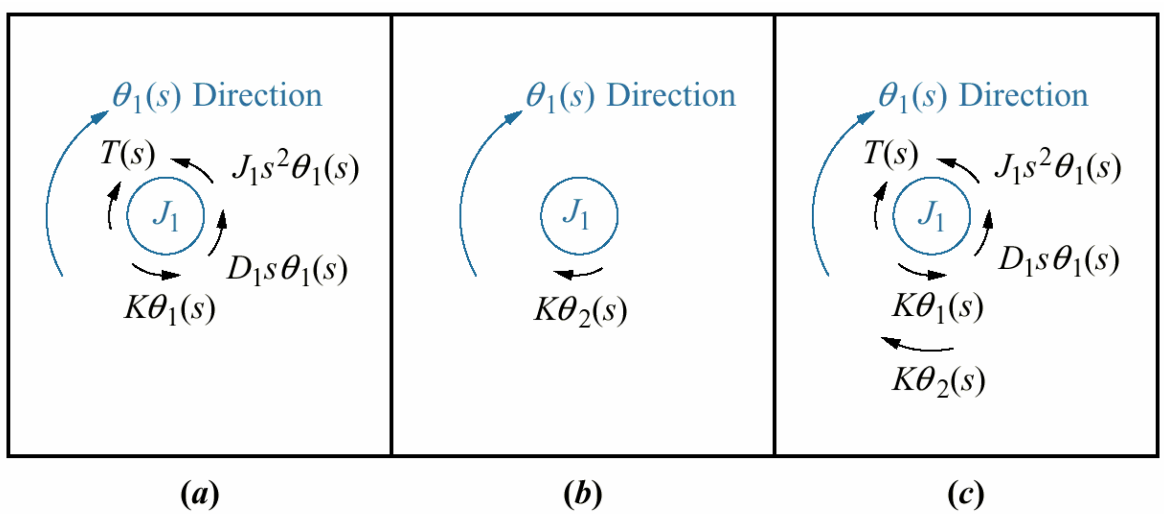


Figure 2.23
a. Torques on J_1 due only to the motion of J_1
b. torques on J_1 due only to the motion of J_2
c. final free-body diagram for J_1



For J_2 :

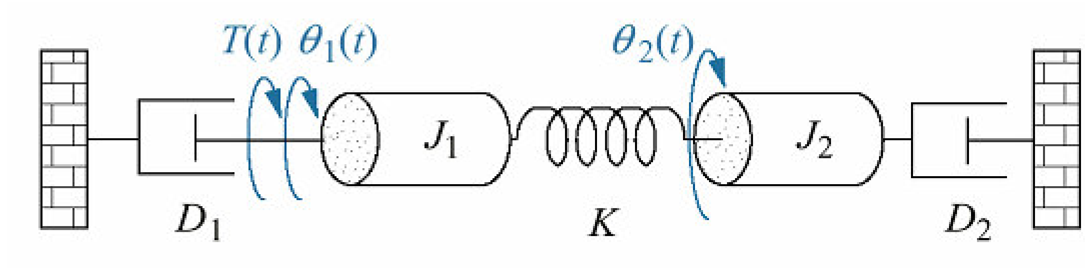
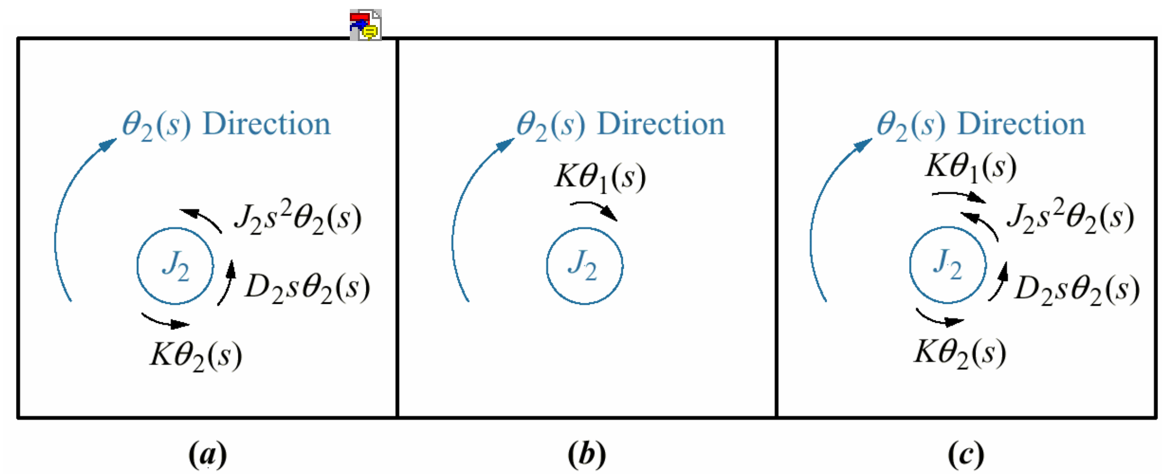
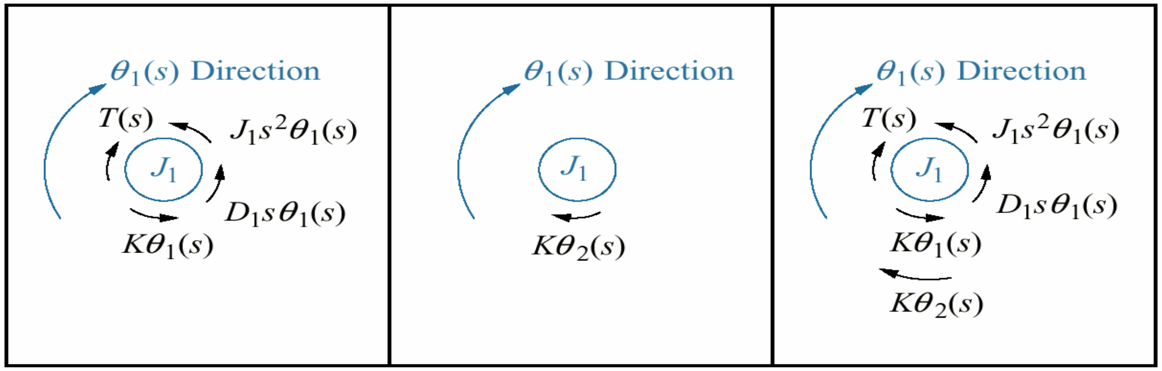


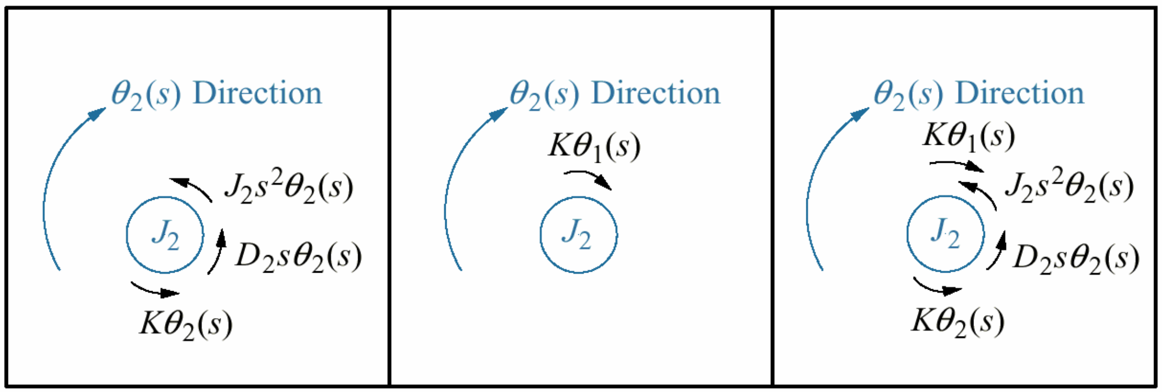
Figure 2.24
a. Torques on J_2 due only to the motion of J_2;
b. torques on J_2 due only to the motion of J_1
c. final free-body diagram for J_2



Now, write the equations of motion by summing the torques on J_1 and J_2



(a) (b) (c)



(a) (b) (c)

$$(J_1 s^2 + D_1 s + K)\theta_1(s) - K\theta_2(s) = T(s)$$

$$-K\theta_1(s) + (J_2 s^2 + D_2 s + K)\theta_2(s) = 0$$

$$(J_1 s^2 + D_1 s + K)\theta_1(s) - K\theta_2(s) = T(s)$$

$$-K\theta_1(s) + (J_2 s^2 + D_2 s + K)\theta_2(s) = 0$$

The required transfer function is found to be

$$\frac{\theta_2(s)}{T(s)} = \frac{K}{\Delta}$$

$$\Delta = \begin{vmatrix} (J_1 s^2 + D_1 s + K) & -K \\ -K & (J_2 s^2 + D_2 s + K) \end{vmatrix}$$

These equations have that now well-known form

$$\begin{pmatrix} \textit{Sum of} \\ \textit{impedance} \\ \textit{connected} \\ \textit{to the} \\ \textit{motion at} \\ \theta_1 \end{pmatrix} \theta_1(s) - \begin{pmatrix} \textit{Sum of} \\ \textit{impedance} \\ \textit{between} \\ \theta_1 \textit{ and } \theta_2 \end{pmatrix} \theta_2(s) = \begin{pmatrix} \textit{Sum of applied} \\ \textit{torques at} \\ \theta_1 \end{pmatrix}$$

$$\begin{pmatrix} \textit{Sum of} \\ \textit{impedance} \\ \textit{connected} \\ \textit{to the} \\ \textit{motion at} \\ \theta_2 \end{pmatrix} \theta_2(s) - \begin{pmatrix} \textit{Sum of} \\ \textit{impedance} \\ \textit{between} \\ \theta_1 \textit{ and } \theta_2 \end{pmatrix} \theta_1(s) = \begin{pmatrix} \textit{Sum of applied} \\ \textit{torques at} \\ \theta_2 \end{pmatrix}$$

Equations of Motion by Inspection

Problem : Write, but not solve, the equation of motion for the mechanical system given below.

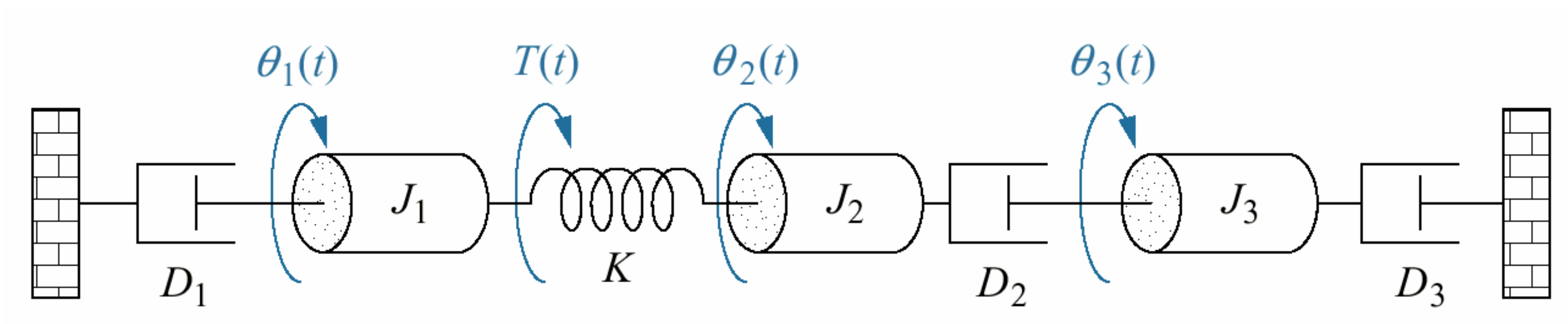


Figure 2.25

Three-degrees-of-freedom rotational system

$$\begin{pmatrix}\text{Sum of}\\ \text{impedance}\\ \text{connected}\\ \text{to the}\\ \text{motion at}\\ \theta_1\end{pmatrix}\theta_1(s) - \begin{pmatrix}\text{Sum of}\\ \text{impedance}\\ \text{between}\\ \theta_1 \text{ and } \theta_2\end{pmatrix}\theta_2(s) - \begin{pmatrix}\text{Sum of}\\ \text{impedance}\\ \text{between}\\ \theta_1 \text{ and } \theta_3\end{pmatrix}\theta_3(s) = \begin{pmatrix}\text{Sum of}\\ \text{applied}\\ \text{torques at}\\ \theta_1\end{pmatrix}$$

$$-\begin{pmatrix}\text{Sum of}\\ \text{impedance}\\ \text{between}\\ \theta_1 \text{ and } \theta_2\end{pmatrix}\theta_1(s) + \begin{pmatrix}\text{Sum of}\\ \text{impedance}\\ \text{connected to}\\ \text{the motion at}\\ \theta_2\end{pmatrix}\theta_2(s) - \begin{pmatrix}\text{Sum of}\\ \text{impedance}\\ \text{between}\\ \theta_2 \text{ and } \theta_3\end{pmatrix}\theta_3(s) = \begin{pmatrix}\text{Sum of}\\ \text{applied}\\ \text{torques at}\\ \theta_2\end{pmatrix}$$

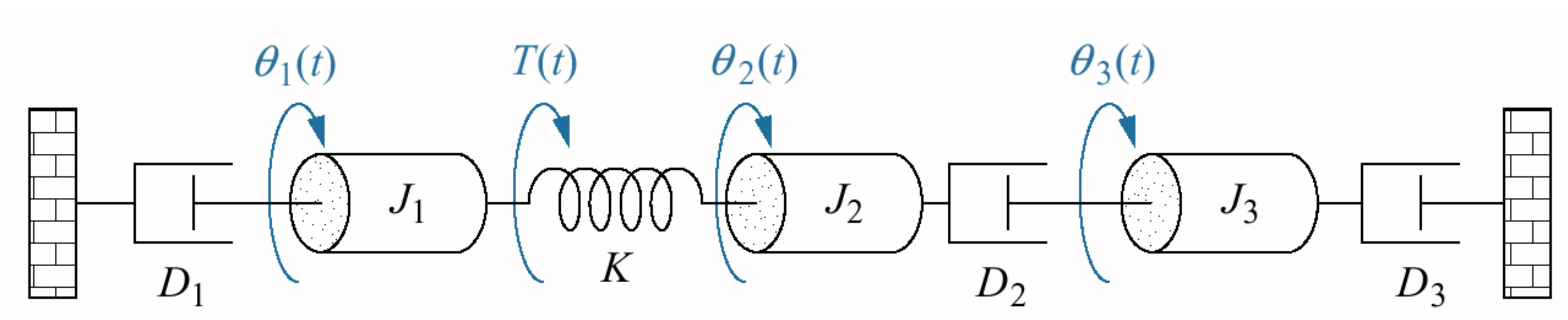
$$-\begin{pmatrix}\text{Sum of}\\ \text{impedance}\\ \text{between}\\ \theta_1 \text{ and } \theta_3\end{pmatrix}\theta_1(s) - \begin{pmatrix}\text{Sum of}\\ \text{impedance}\\ \text{between}\\ \theta_2 \text{ and } \theta_3\end{pmatrix}\theta_2(s) + \begin{pmatrix}\text{Sum of}\\ \text{impedance}\\ \text{connected to}\\ \text{the motion at}\\ \theta_3\end{pmatrix}\theta_3(s) = \begin{pmatrix}\text{Sum of}\\ \text{applied}\\ \text{toques at}\\ \theta_3\end{pmatrix}$$

Hence,

$$(J_1 s^2 + D_1 s + K)\theta_1(s) - K\theta_2(s) = T(s)$$

$$-K\theta_1(s) + (J_2 s^2 + D_2 s + K)\theta_2(s) - D_2 s\theta_3(s) = 0$$

$$-D_2 s\theta_2(s) + (J_3 s^2 D_3 s + D_2 s)\theta_3(s) = 0$$



Transfer Functions for Systems with Gears

In industrial applications, generally gears associate to a motor which drives the load. Gears are used to obtain more speed and less torque or less speed and more torque. The interaction between two gears is depicted in the figure below.

An input gear with radius r_1 and N_1 teeth is rotated through angle $\theta_1(t)$ due to a torque, $T_1(t)$. An output gear with radius r_2 and N_2 teeth responds by rotating through angle $\theta_2(t)$ and delivering a torque, $T_2(t)$.

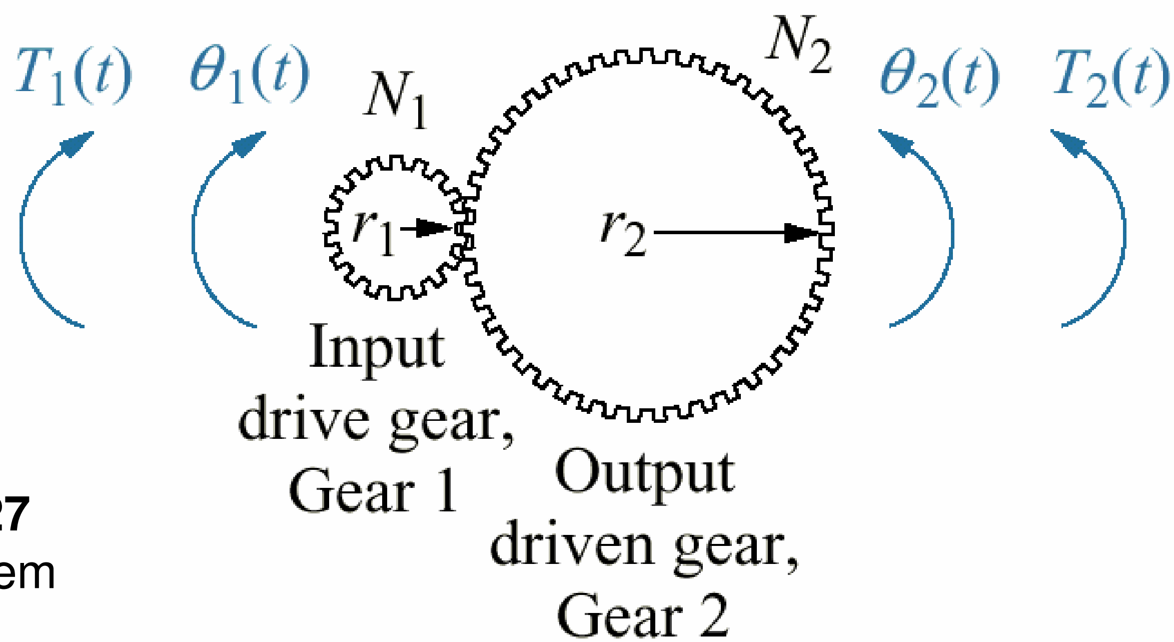
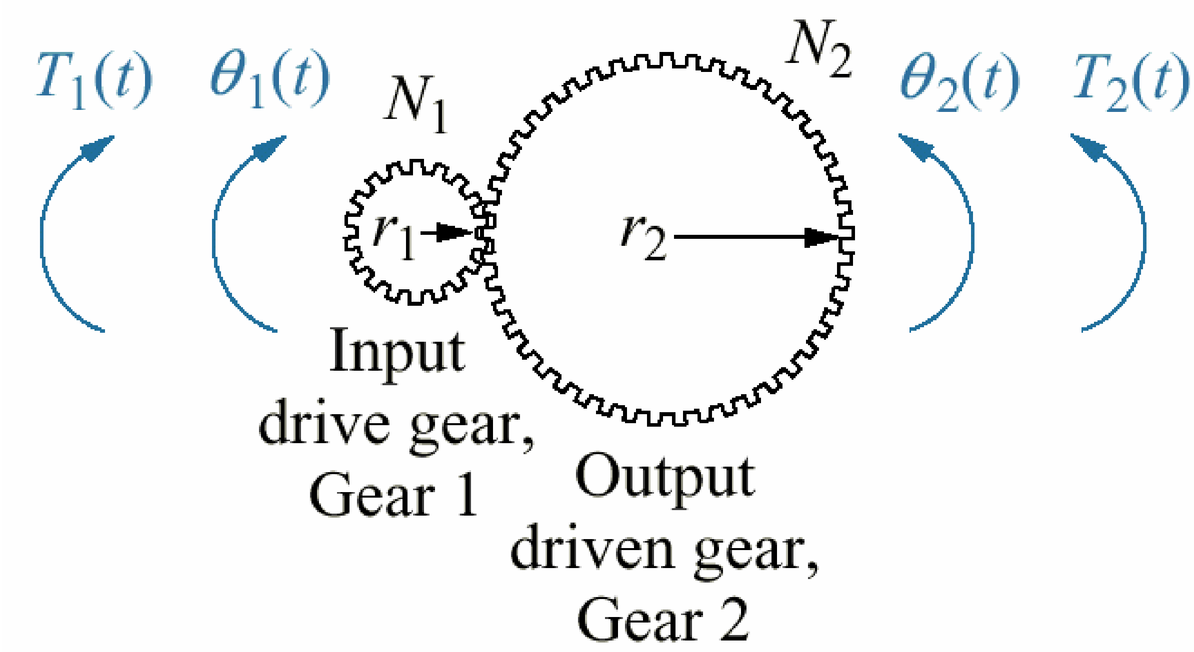


Figure 2.27
A gear system

Let us find the relationship between the rotation of Gear 1, $\theta_1(t)$ and Gear 2, $\theta_2(t)$.



As the gears turn, the distance traveled along each each gear's circumference is the same. Thus,

$$r_1\theta_1 = r_2\theta_2 \qquad \text{or}$$

$$\frac{\theta_2}{\theta_1} = \frac{r_1}{r_2} = \frac{N_1}{N_2}$$

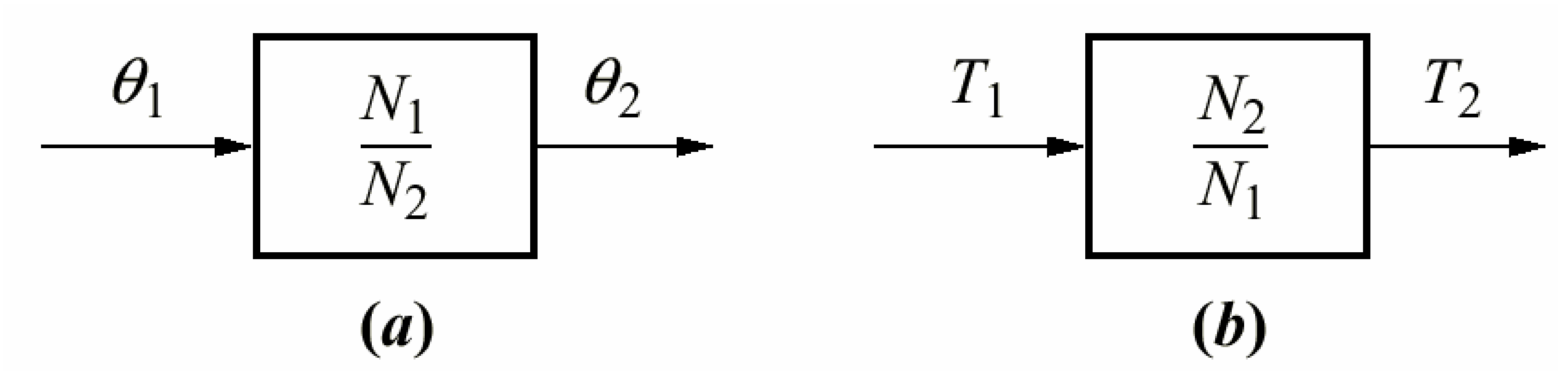
What is the relationship between the input torque, T_1 and the delivered torque, T_2 ?

If we assume the gears do not absorb or store energy into Gear 1 equals the energy out of Gear 2.

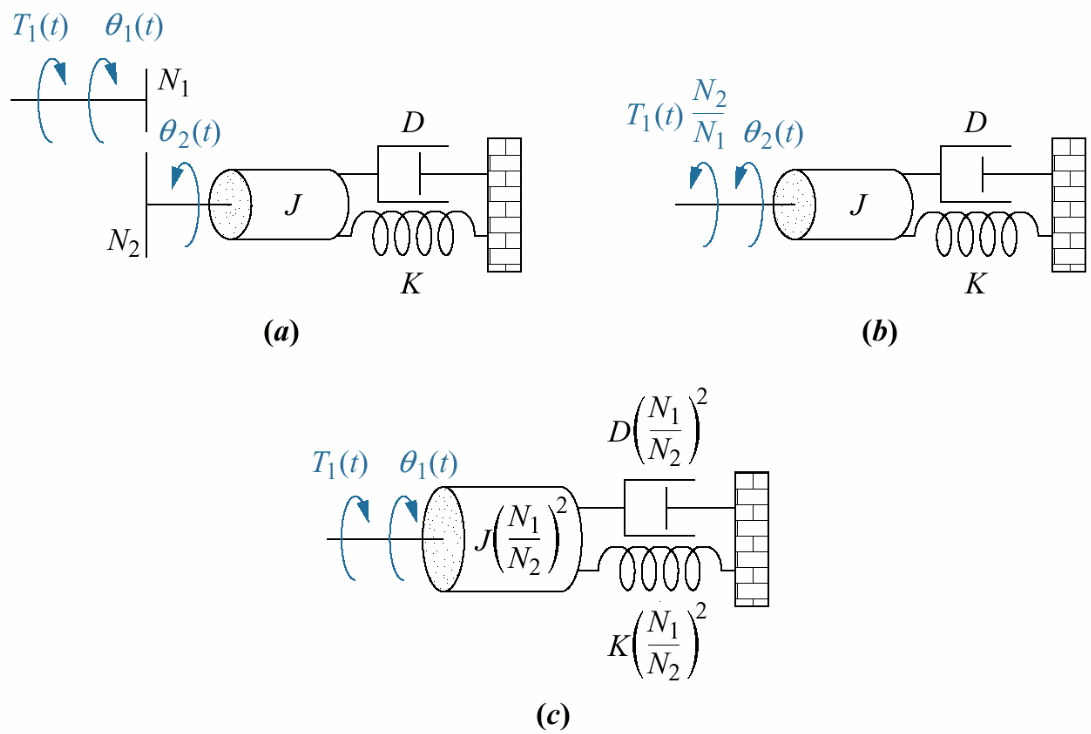
$$T_1\theta_1 = T_2\theta_2 \qquad \text{and we get}$$

$$\frac{T_2}{T_1} = \frac{\theta_1}{\theta_2} = \frac{N_2}{N_1}$$

All results are summarized in the figure below



Let us see what happens to mechanical impedance that are driven by gears. The figure above shows gears driving a rotational inertia, spring and viscous damper. For clarity, the gears are shown by an end-on view. We want to represent Figure 2.29(a) as an equivalent system at θ_1 without the gears. In other words, can the mechanical impedances be reflected from the output to the input, thereby eliminating the gears?



We know T_1 can be reflected to the output by multiplying by N_2/N_1. the result is shown in Figure 2.29(b). We write the equation of motion as

$$(Js^2 + Ds + K)\theta_2(s) = T_1(s)\frac{N_2}{N_1}$$

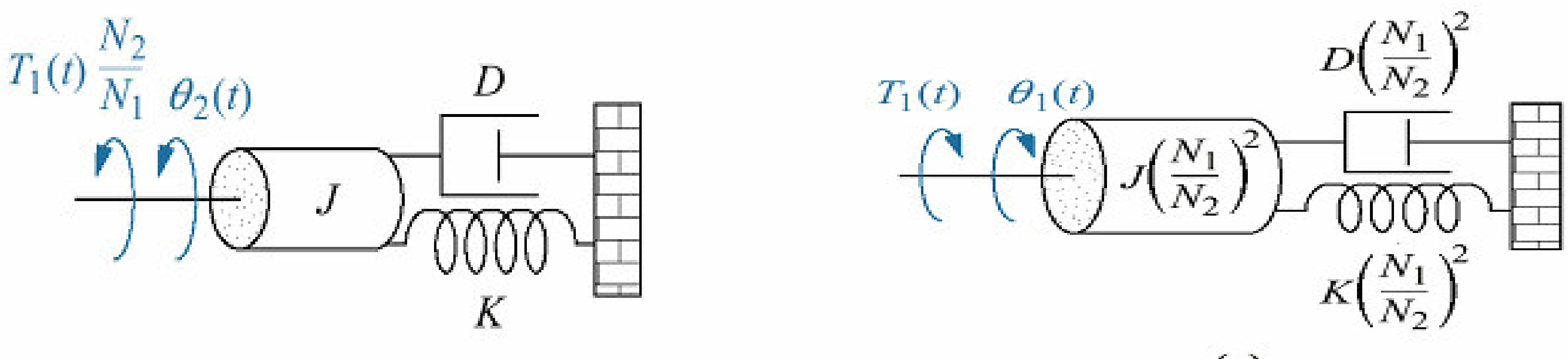
Now convert $\theta_2(s)$ into an equivalent $\theta_1(s)$,

$$(Js^2 + Ds + K)\frac{N_1}{N_2}\theta_1(s) = T_1(s)\frac{N_2}{N_1}$$

After simplification,

$$\left[J\left(\frac{N_2}{N_1}\right)^2 s^2 + D\left(\frac{N_1}{N_2}\right)^2 s + K\left(\frac{N_1}{N_2}\right)^2\right]\theta_1(s) = T_1(s)$$

we get the equivalent system shown in Figure 2.29(c)



(b) (c)

Generalizing the results, we can make the following statement : Rotational mechanical impedances can be reflected through gear trains by multiplying the mechanical impedances by the ratio

$$\left(\frac{\text{Number of teeth of gear on } \textbf{\textit{destination}} \text{ shaft}}{\text{Number of teeth of gear on } \textbf{\textit{source}} \text{ shaft}} \right)^2$$

The next example demonstrates the application of the concept of a rotational mechanical system with gears.

<u>Example</u> : Find the transfer function $\theta_2(s)$ / $T_1(s)$ for the system of Figure 2.30(a)

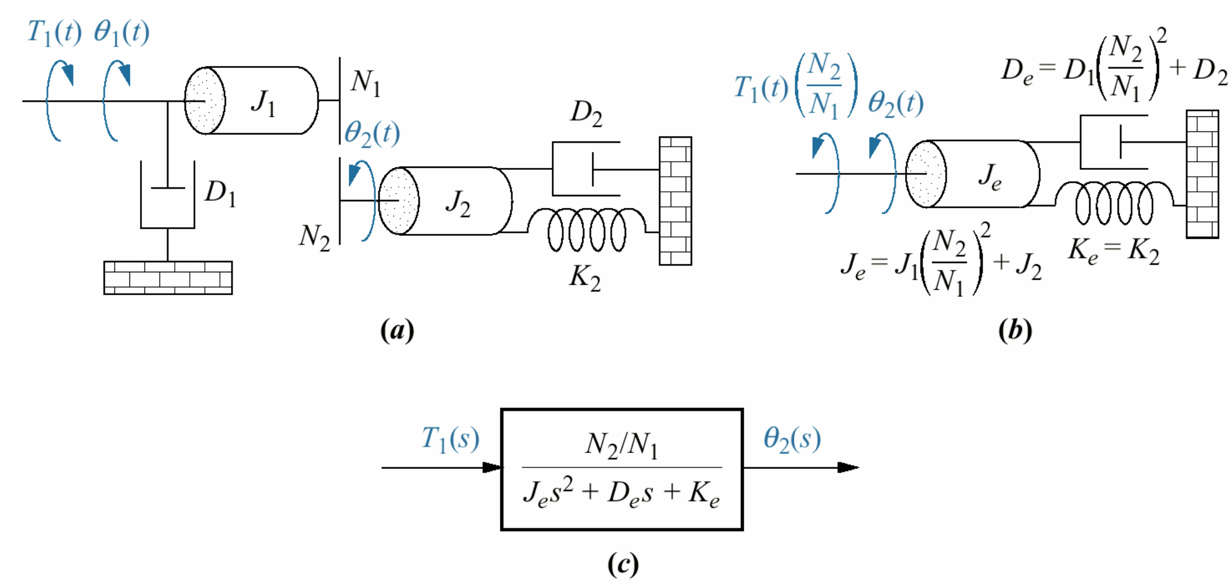


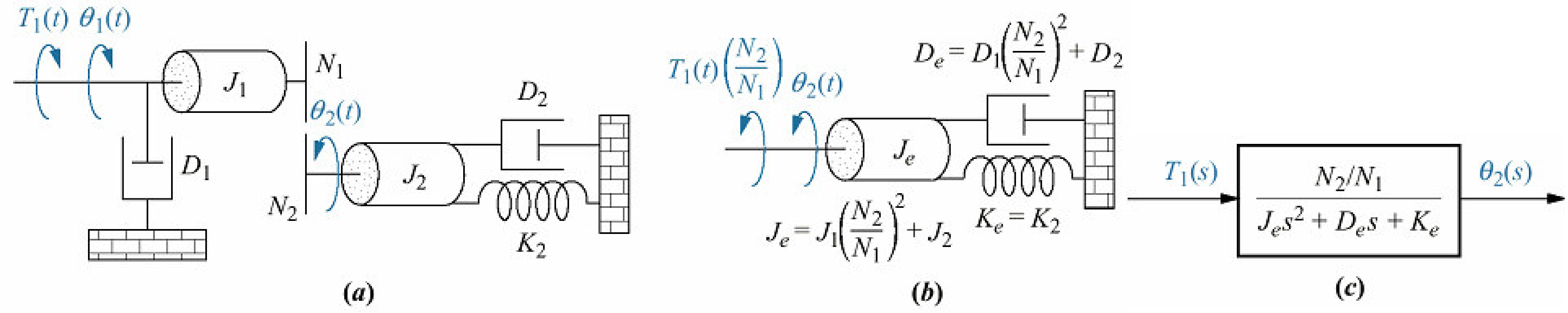
Figure 2.30

a. Rotational mechanical system with gears;

b. system after reflection of torques and impedances to the output shaft;

c. block diagram

Let us first reflect the impedances (J_1 and D_1) and the torque (T_1) on the input shaft to the output as shown in Figure 2.30(b), where the impedances are reflected by $(N_2/N_1)^2$ and the torque is reflected by (N_2/N_1).



The equation of motion can now be written as

$$(J_e s^2 + D_e s + K_e)\theta_2(s) = T_1(s)\frac{N_2}{N_1}$$ where

$$J_e = J_1\left(\frac{N_2}{N_1}\right)^2 + J_2 \qquad D_e = D_1\left(\frac{N_2}{N_1}\right)^2 + D_2 \qquad \text{and} \qquad K_e = K_2$$

Solving for **$\theta_2(s) / T_1(s)$**, the transfer function is found to be

$$G(s) = \frac{\theta_2(s)}{T_1(s)} = \frac{N_2 / N_1}{J_e s^2 + D_e s + K_e}$$

as shown in Figure 2.30(c)

In order to eliminate gears with the large radii, a ***gear train*** is used to implement large gear ratios by cascading smaller gear ratios. A schematic diagram of a ***gear train*** is shown in Figure 2.31.

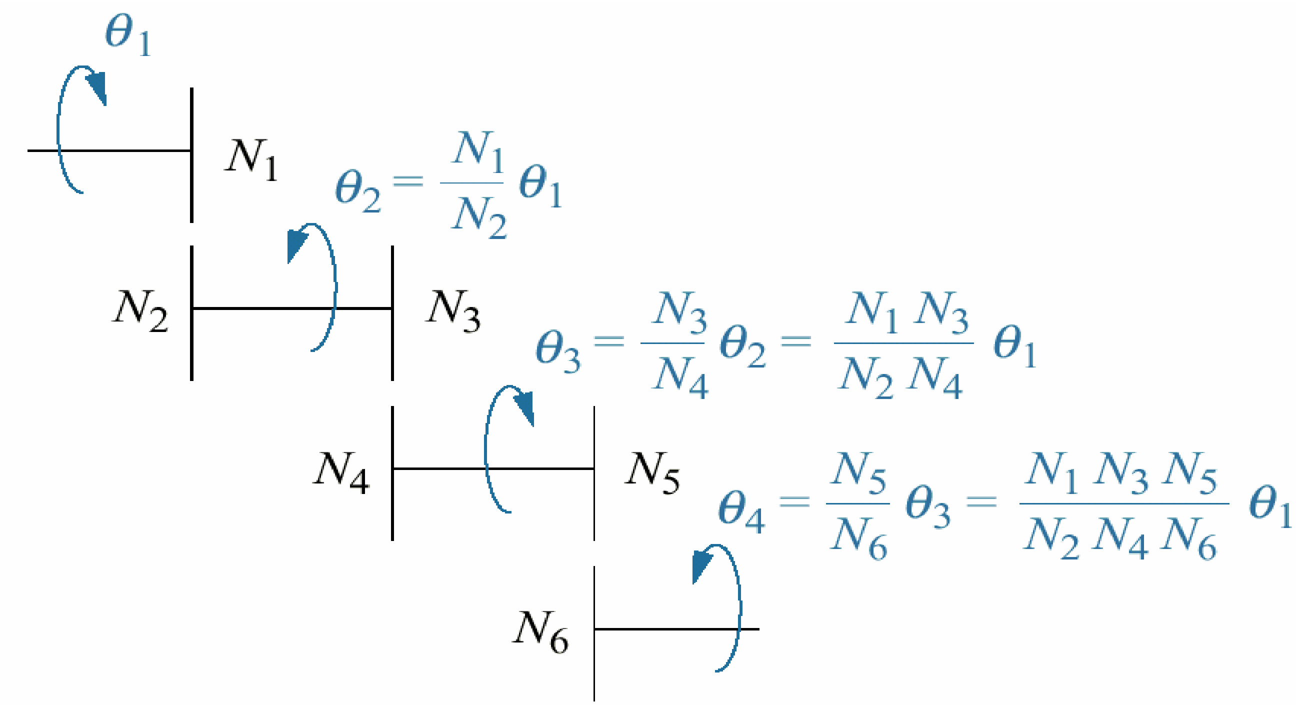


Figure 2.31
Gear train

Next to each rotation, the angular displacement relative to θ_1 has been calculated as

$$\theta_4 = \frac{N_1 N_3 N_5}{N_2 N_4 N_6}\theta_1$$

For gear trains, we conclude that the equivalent gear ratio is the product of individual gear ratio. We now apply this result to solve for the transfer function of a system that does not have lossless gears.

<u>Example</u> : Find the transfer function $\theta_1(s)$ / $T_1(s)$ for the system of Figure 2.32(a)

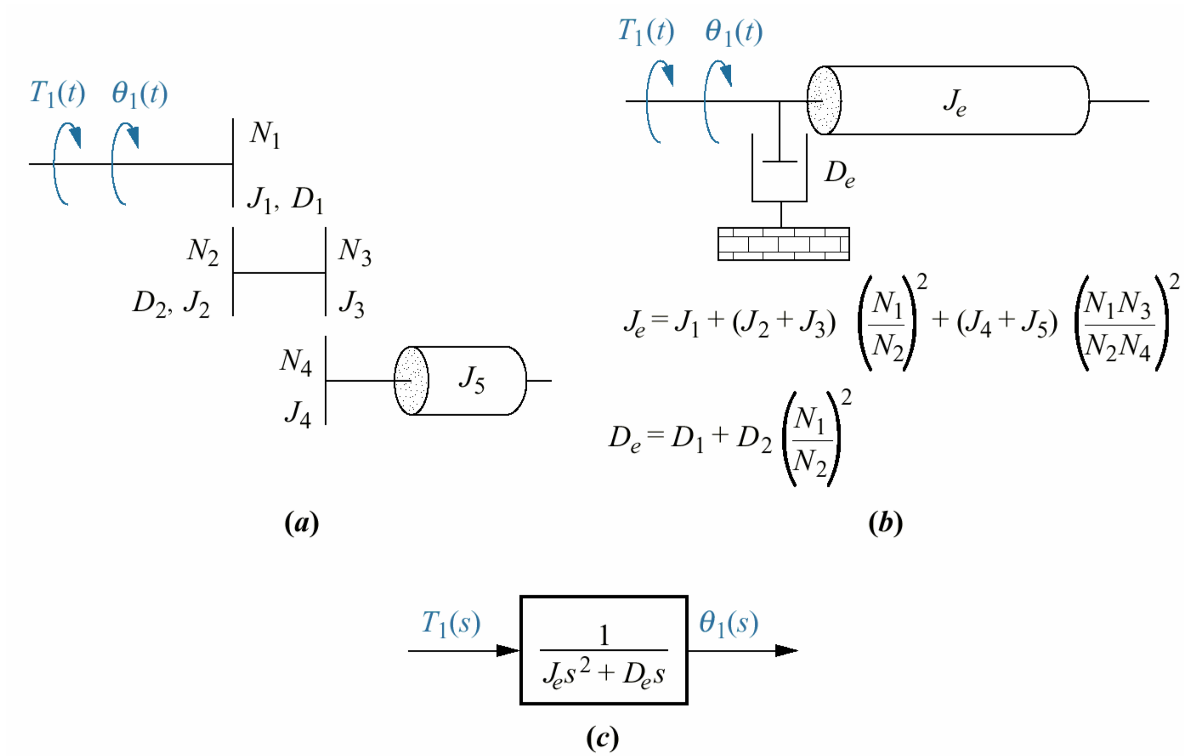
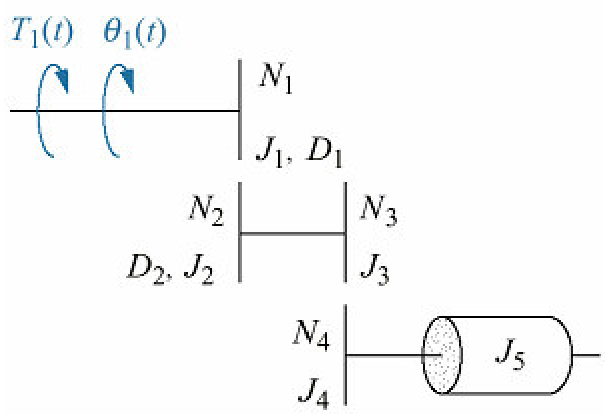
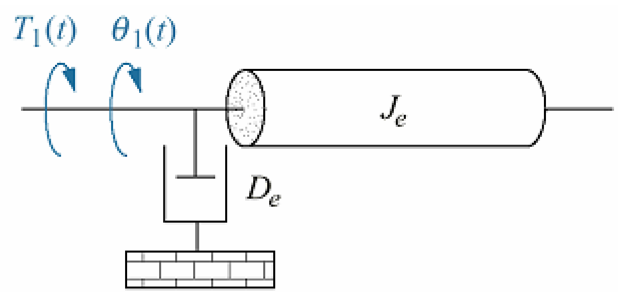


Figure 2.32
a. System using a gear train;
b. equivalent system at the input;
c. block diagram

This system, which uses a gear train, does not have lossless gears. All of the gears have inertia and for some shafts there is viscous friction. To solve the problem, we want to reflect all of the impedances to the input shaft, θ_1. The gear ratio is not same for all impedances. For example, D_2 is reflected only through one gear ratio as $\mathbf{D_2(N_1/N_2)^2}$, whereas J_4 plus J_5 is reflected through two gear ratio as $\mathbf{(J_4+J_5)[(N_3/N_4)(N_1/N_2)]^2}$. The results of reflected all impedances to θ_1 is shown in Figure 2.32(b)





The equation of motion is

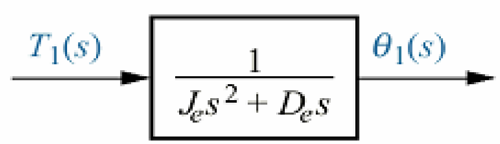
$$(J_e s^2 + D_e s)\theta_1(s) = T_1(s)$$

$$J_e = J_1 + (J_2 + J_3)\left(\frac{N_1}{N_2}\right)^2 + (J_4 + J_5)\left(\frac{N_1 N_3}{N_2 N_4}\right)^2$$

$$D_e = D_1 + D_2\left(\frac{N_1}{N_2}\right)^2$$

The transfer function is $G(s) = \dfrac{\theta_1(s)}{T_1(s)} = \dfrac{1}{J_e s^2 + D_e s}$

as shown in Figure 2.32(c)



(c)

ELECTROMECHANICAL SYSTEM TRANSFER FUNCTIONS

We talked about mechanical system by now. You have already known the electrical systems. Now, we move to systems that are hybrids of electrical and mechanical variables, the *electromechanical systems*. An application of electromechanical systems is robot controls. A robot have both electrical and mechanical parameters. A robot arm as an example of control system that uses electromechanical components is shown in Figure below.

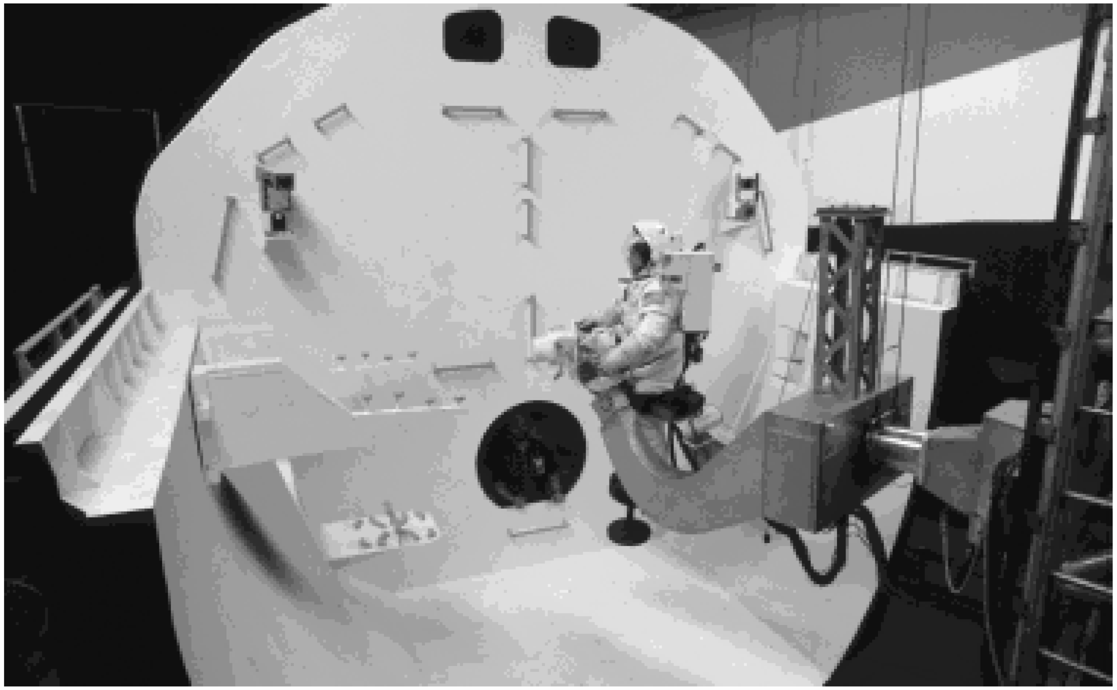
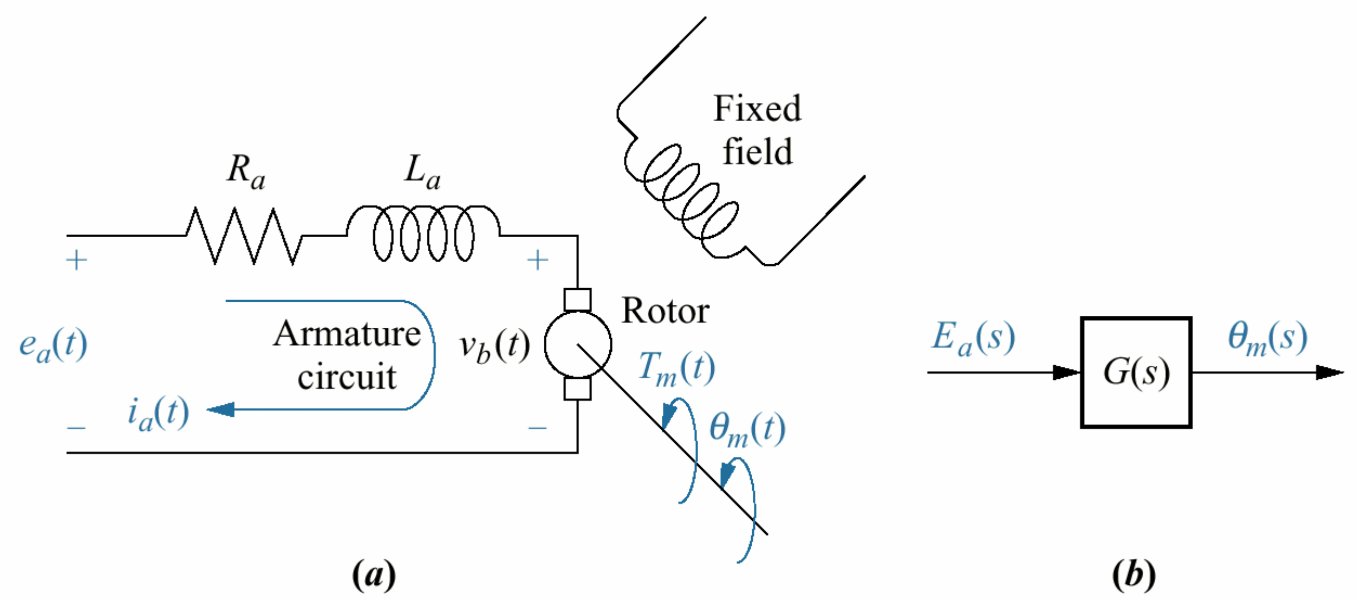


Figure 2.34
NASA flight simulator robot arm with electromechanical control system components

A motor is an electromechanical component that yields a displacement output for a voltage input, that is, a mechanical output generated by an electrical input. We will derive the transfer function for one particular kind of electromechanical system, the armature-controlled dc servomotor. Schematic of motor is shown in Figure below.



In this figure, a magnetic field developed by stationary permanent magnet or a stationary electromagnet called the *fixed field*. A rotating circuit called the *armature*, through which current $i_a(t)$ flows, passes through this magnetic field at right angles and feels a force, $F=Bli_a(t)$, where B is the magnetic field strength and l is the length of the conductor. The resulting torque turns the *rotor*, the rotating member of the motor.

Modeling of the Permanent Magnet Dc Motor

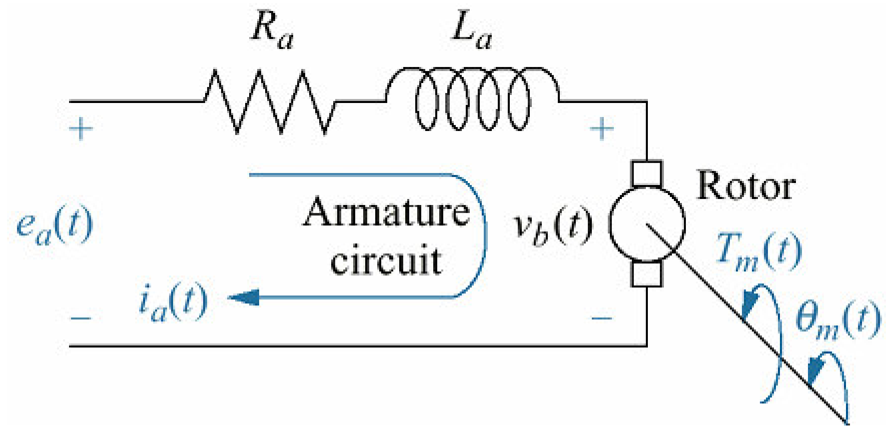
$$V_b(t) = K_b \frac{d\theta_m(t)}{dt}$$

We call $V_b(t)$ the *back electromotive force* (back emf); K_b is a constant of proportionality called the back emf constant ; and $d\theta_m(t)/dt=\omega_m(t)$ is the angular velocity of the motor. Taking the Laplace transform, we get

$$V_b(s) = K_b s \theta_m(s)$$

The relationship between the armature current $i_a(t)$, the applied armature voltage $e_a(t)$ and the back emf $V_b(t)$ is found by writing a loop equation around the Laplace transformed armature circuit

$$R_a I_a(s) + L_a s I_a(s) + V_b(s) = E_a(s)$$



The torque developed by the motor is proportional to the armature current; thus,

$$T_m(s) = K_t I_a(s)$$

where T_m is the torque developed by the motor, and K_t is a constant of proportionality, called the motor torque constant, which depends on the motor and magnetic field characteristics.

To find the transfer function of the motor, we use the equation

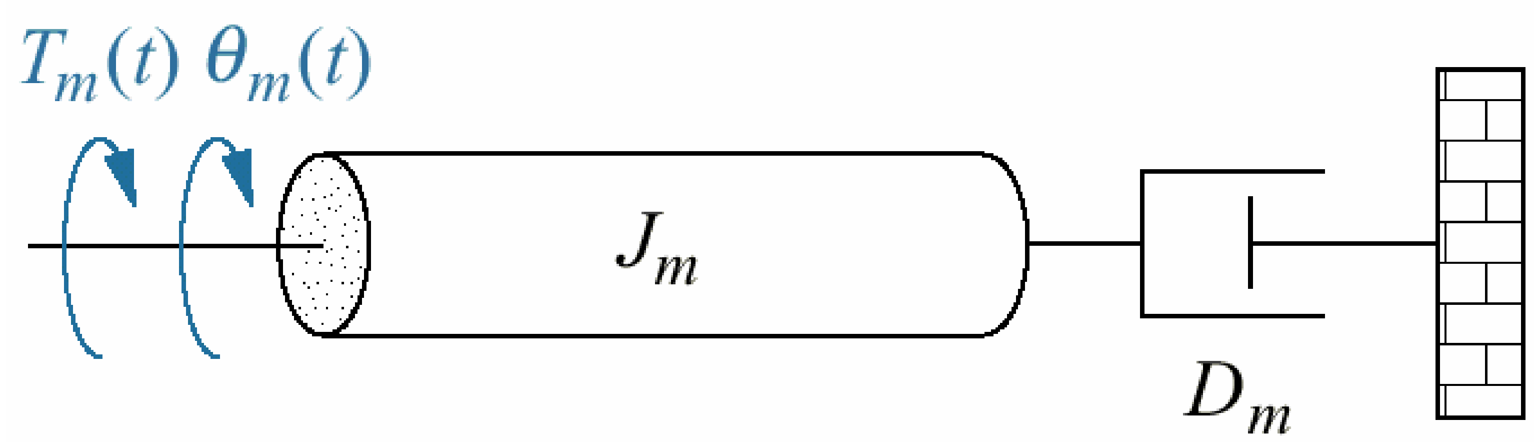
$$R_a I_a(s) + L_a s I_a(s) + V_b(s) = E_a(s)$$

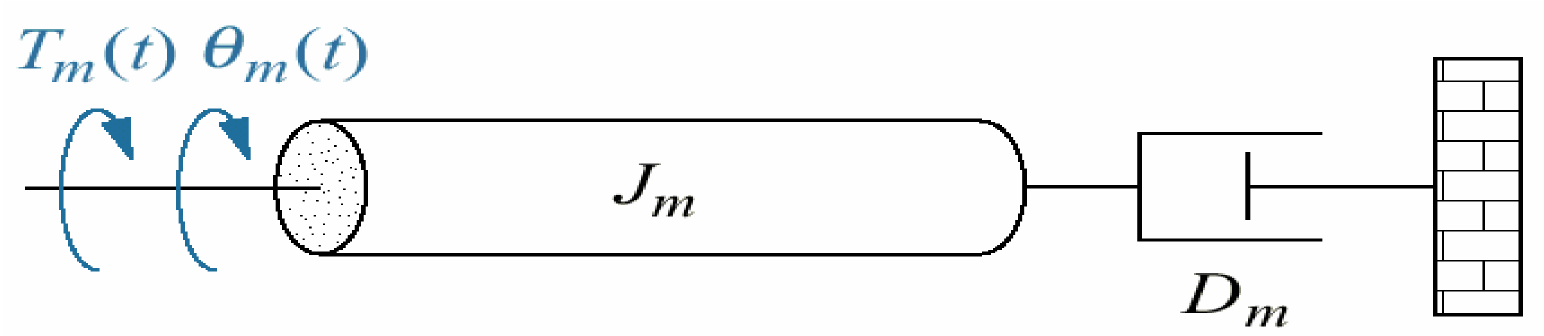
Rearranging this equation yields

$$\frac{(R_a + L_a s) T_m(s)}{K_t} + K_b s \theta_m(s) = E_a(s)$$

Now we must find $T_m(s)$ in terms of $\theta_m(s)$ if we are to seperate the input and output variables and obtain the transfer function $\theta_m(s) / E_a(s)$.

Following figure shows typical eqivalent mechanical loading on a motor. J_m is the equivalent inertia at the armature and includes both the armature inertia and, as we will see later, the load inertia reflected to the armature. D_m is the equivalent viscous damping at the armature and includes armature viscous damping and, as we will see later, the load viscous damping reflected to the armature.





$$\frac{(R_a + L_a s)T_m(s)}{K_t} + K_b s\theta_m(s) = E_a(s)$$

$$T_m(s) = (J_m s^2 + D_m s)\theta_m(s)$$

Substituting the second Eq. into the first one yields

$$\frac{(R_a + L_a s)(J_m s^2 + D_m s)\theta_m(s)}{K_t} + K_b s\theta_m(s) = E_a(s)$$

If we assume that the armature inductance, L_a, is small compared to the armature resistance, R_a, which is usual for a dc motor, the last equation becomes

$$\left[\frac{R_a}{K_t}(J_m s + D_m) + K_b\right] s\theta_m(s) = E_a(s)$$

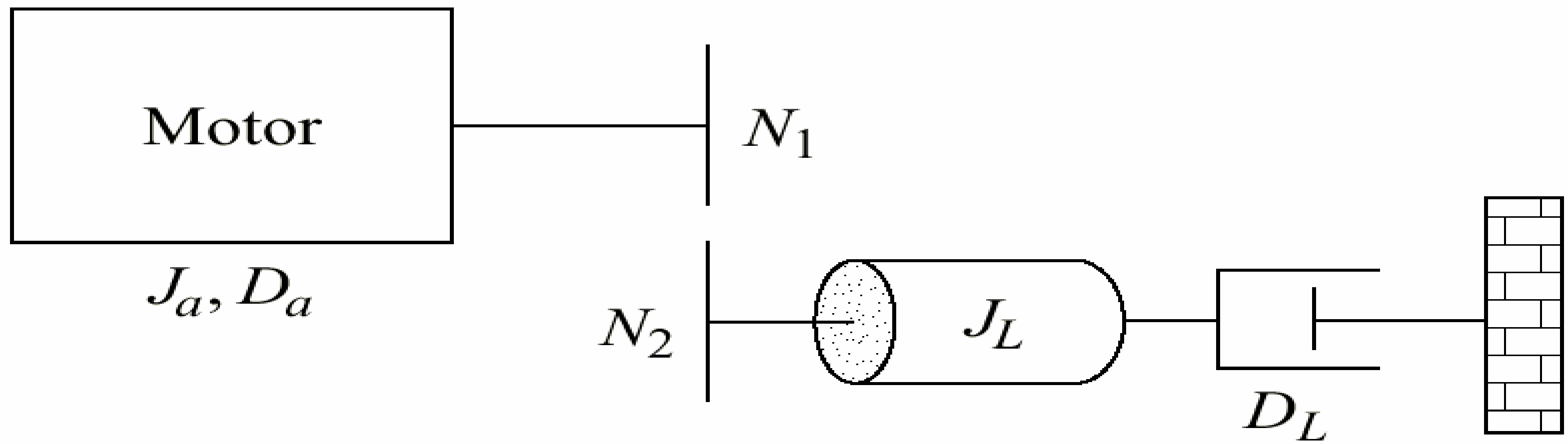
After simplification, the desired transfer function, $\theta_m(s) / E_a(s)$, is found to be

$$\frac{\theta_m(s)}{E_a(s)} = \frac{K_t/(R_a J_m)}{s\left[s + \frac{1}{J_m}\left(D_m + \frac{K_t K_b}{R_a}\right)\right]}$$

This form is relatively simple;

$$\frac{\theta_m(s)}{E_a(s)} = \frac{K}{s(s+\alpha)}$$

Let us first discuss the mechanical constants, J_m and D_m. Consider the Figure 2.37, which shows a motor which inertia J_a and damping D_a at the armature driving a load consisting of inertia J_L and damping D_L.



Assuming that all inertia and damping values shown are known. J_L and D_L can be reflected back to the armature as some iquivalent inertia and damping to be added to J_a and D_a respectively. Thus the iquivalent inertia J_m and equivalent damping D_m at the armature are

$$J_m = J_a + J_L \left(\frac{N_1}{N_2} \right)^2 \qquad D_m = D_a + D_L \left(\frac{N_1}{N_2} \right)^2$$

Now that how we evaluated the mechanical constant, J_m and D_m, what about the electrical constant in the transfer function? We will show that this costants can be obtained through a dynamometer test of the motor, where a dynamometer measures the torque and speed of a motor under condition of a constant applied voltage.

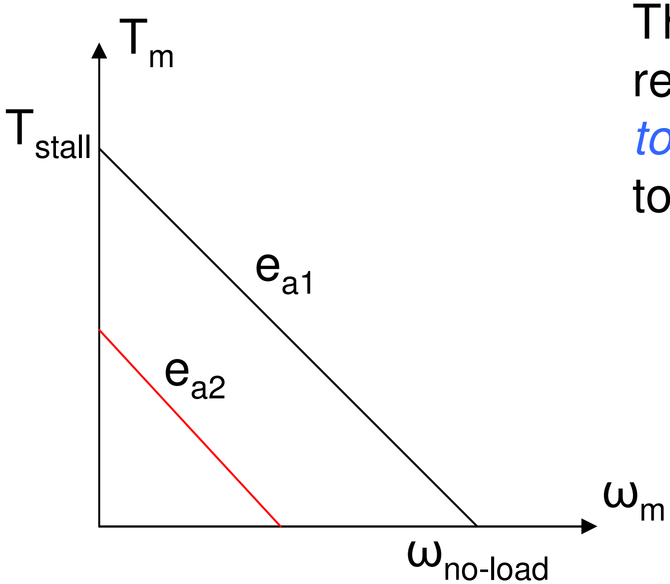
Let us first develop the relationship that dictate the use of a dynamometer. Taking L_a=0,

$$\frac{R_a}{K_t}T_m(s)+K_b s\theta_m(s)=E_a(s)\xrightarrow{\text{inverse Laplace tr.}}\frac{R_a}{K_t}T_m(t)+K_b\omega_m(t)=e_a(t)$$

If dc voltage e_a is applied, the motor will turn at a constant angular velocity ω_m with a constant torque T_m. Hence, dropping the functional relationship based on the time, the following relationship exists when the motor is operating at steady-state with a dc voltage input:

$$\frac{R_a}{K_t}T_m+K_b\omega_m=e_a \quad , \quad \text{Solving for } T_m \text{ yields;} \qquad T_m=-\frac{K_bK_t}{R_a}\omega_m+\frac{K_t}{R_a}e_a$$

From this equation, we get the torque-speed curve, T_m versus ω_m which is shown below.



The torque axis intercept occurs when the angular velocity reaches the zero. That value of torque is called the *stall torque, T_{stall}*. The angular velocity occuring when the torque is zero is called the *no-load speed, $\omega_{no,load}$*. Thus,

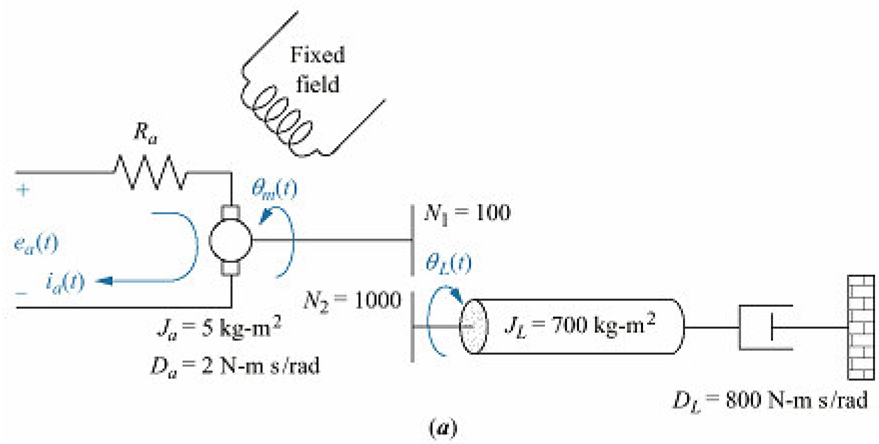
$$T_{stall}=\frac{K_t}{R_a}e_a \qquad \text{and} \qquad \omega_{no-load}=\frac{e_a}{K_b}$$

The electrical constants of the motor can now be found from the equations of

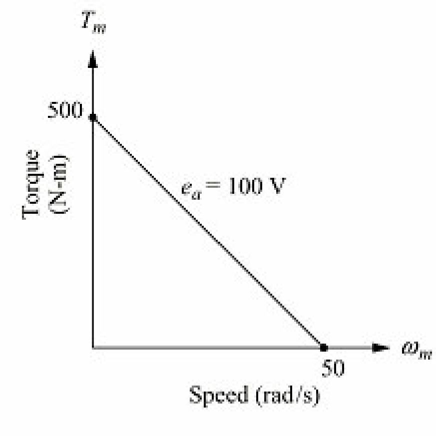
$$\frac{K_t}{R_a}=\frac{T_{stall}}{e_a} \qquad \text{and} \qquad K_b=\frac{e_a}{\omega_{no-load}}$$

The electrical constants, K_t/R_a and K_b, can be found from a dynamaeter test of the motor, which would yield T_{stall} and $\omega_{no\text{-}load}$ for a given e_a .

Example : Given the system and torque-speed curve of the following figure, find the transfer function $\theta_L(s)\ /\ E_a(s)$.



(a)



(b)

Solution : Begin by finding the mechanical constant J_m and D_m. The total inertia and the total damping at the armature of the motor is

$$J_m = J_a + J_L\left(\frac{N_1}{N_2}\right)^2 = 5 + 700\left(\frac{1}{10}\right)^2 = 12 \qquad D_m = D_a + D_L\left(\frac{N_1}{N_2}\right)^2 = 2 + 800\left(\frac{1}{10}\right)^2 = 10$$

Now we will find the electrical constants $K_t\ /\ R_a$ and K_b using torque-speed curve. T_{stall}=500 Nm , $\omega_{no\text{-}load}$=50 rad/sn , e_a=100 V . Hence the electrical constants are

$$\frac{K_t}{R_a} = \frac{T_{stall}}{e_a} = \frac{500}{100} = 5 \qquad K_b = \frac{e_a}{\omega_{no-load}} = \frac{100}{50} = 2$$

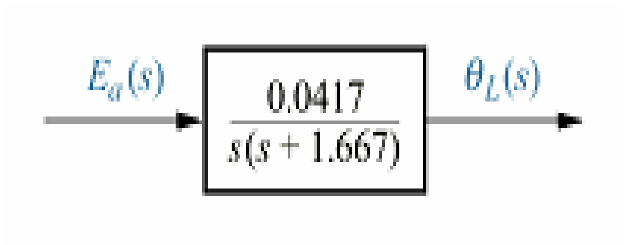
We know the transfer function formulation for $\theta_m(s) / E_a(s)$

$$\frac{\theta_m(s)}{E_a(s)} = \frac{K_t/(R_a J_m)}{s\left[s + \frac{1}{J_m}\left(D_m + \frac{K_t K_b}{R_a}\right)\right]} \qquad \frac{\theta_m(s)}{E_a(s)} = \frac{5/12}{s\left[s + \frac{1}{12}(10+5\times 2)\right]} = \frac{0.417}{s(s+1.667)}$$

In order to find $\theta_L(s) / E_a(s)$, we use the gear ratio, $N_1 / N_2 = 1 / 10$, and find

$$\frac{\theta_L(s)}{E_a(s)} = \frac{0.0417}{s(s+1.667)}$$

as shown in figure below.



ELECTRIC CIRCUIT ANALOGS

In this section, we show the commonality of systems from the various disciplines by demonstrating that the mechanical system with which we worked can be represented by equivalent electric circuits.

An electric circuit that is analogous to a system from the another discipline is called an electric circuit *analog*.

Series Analog : Consider the translational mechanical system shown in Figure(a) whose the equation of motion **$(Ms^2 + f_v + K)\,X(s) = F(s)$**. Kirchhof's mesh equation for the simple RLC network shown in Figure(b) is

$$\left(Ls + R + \frac{1}{Cs}\right) I(s) = E(s)$$

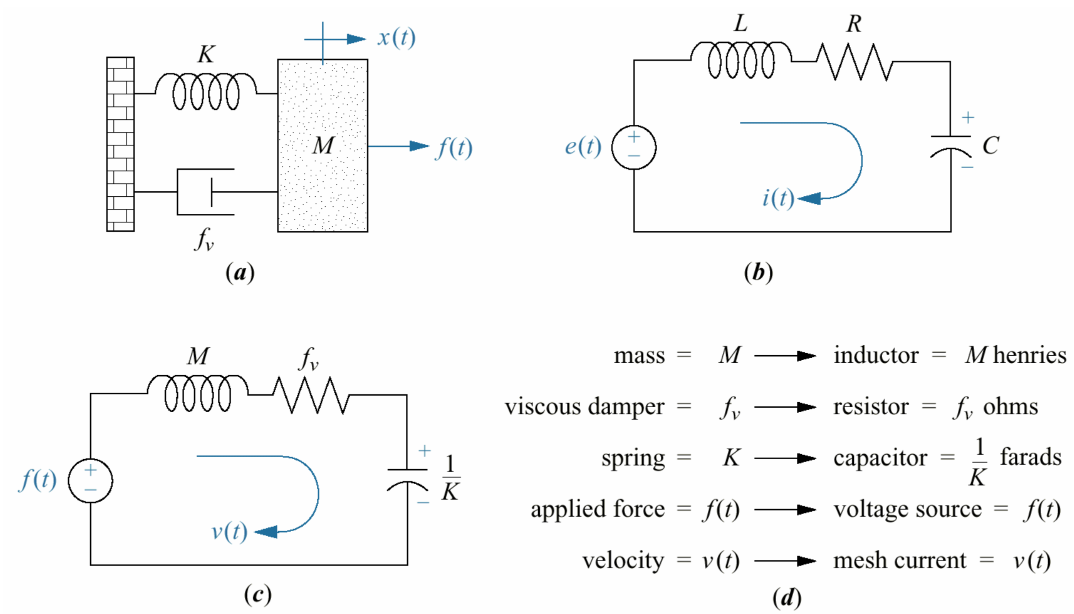


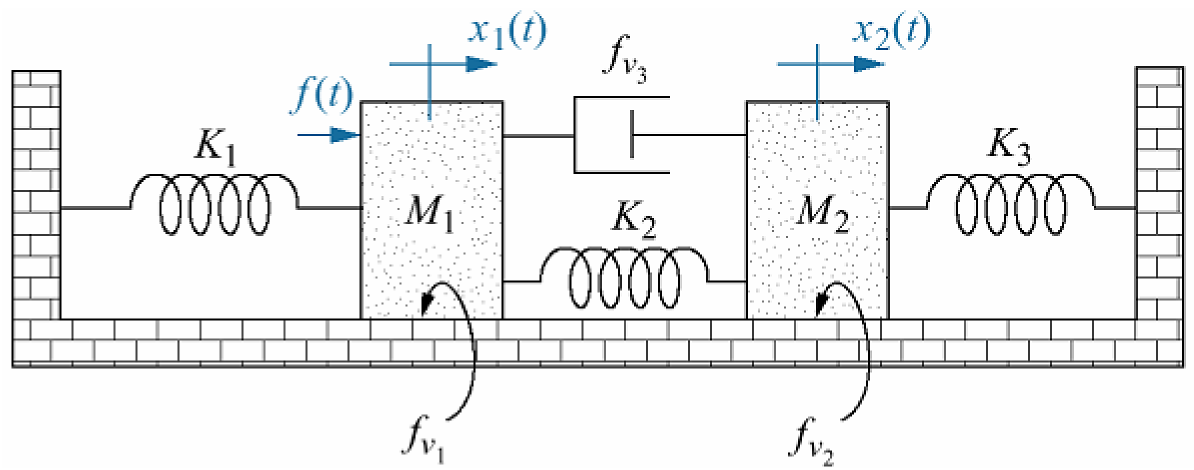
Figure 2.41 Development of series analog: **a.** mechanical system; **b.** desired electrical representation; **c.** series analog; **d.** parameters for series analog

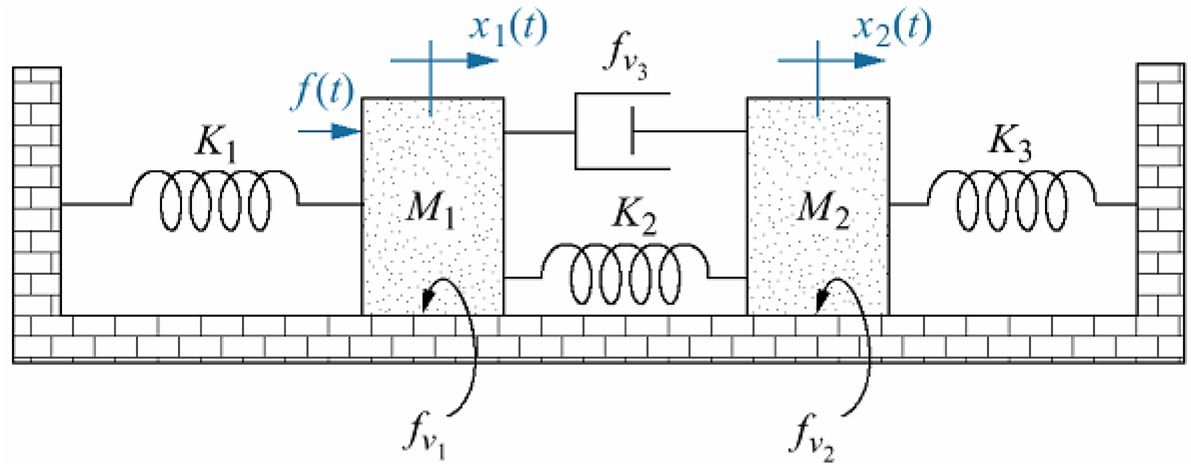
As we previously pointed out, these two mechanical and electrical equations is not directly analogous because displacement and current are not analogous. We can create a direct analogy by operating on the mechanical equation to convert displacement to velocity by dividing and multiplying the left-hand side by s, yielding

$$\frac{Ms^2 + f_v s + K}{s} sX(s) = \left(Ms + f_v + \frac{K}{s} \right) V(s) = F(s)$$

When we have more than one degree of freedom, the impedance associated with the motion appear as the serial electrical elements in a mesh, but the impedances between adjacent motions are drawn as a series electrical impedances between the corresponding meshes. We demonstrate with an example.

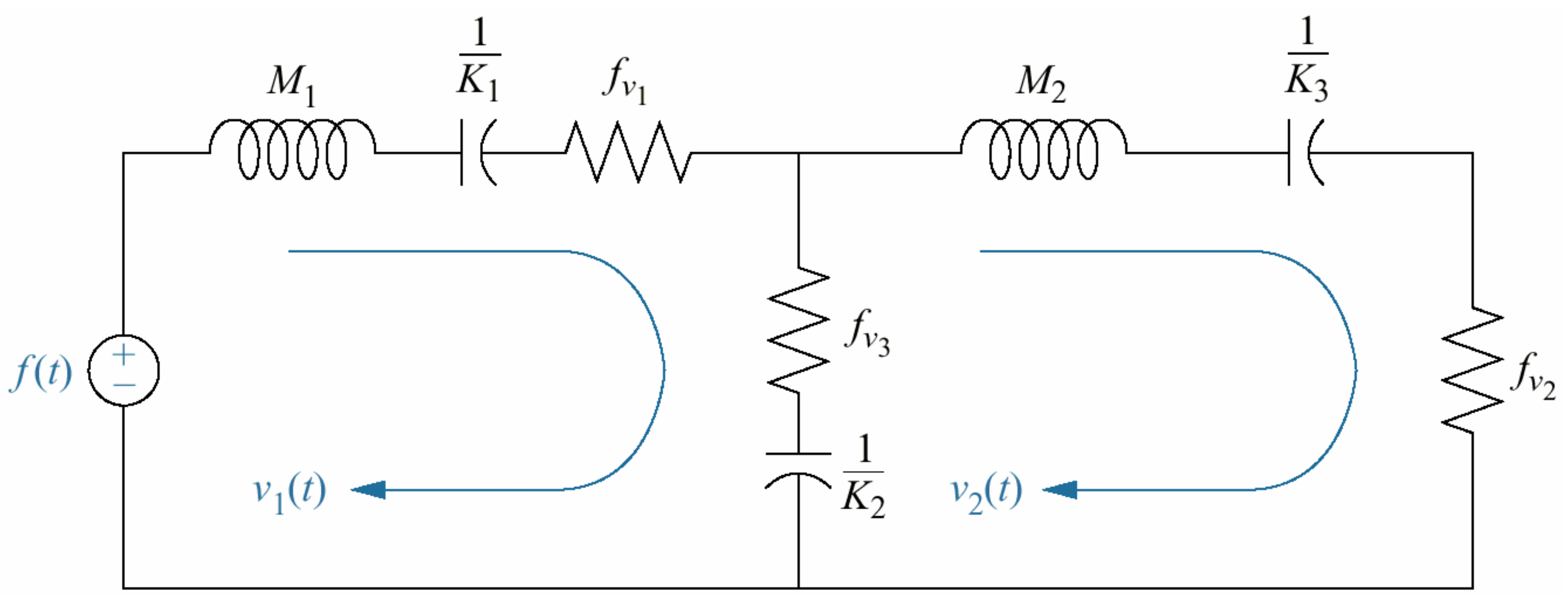
Example : Draw a series analog for the mechanical system of figure below.





$$\left(M_1 s + (f_{v1} + f_{v3}) + \frac{(K_1 + K_2)}{s}\right)V_1(s) - \left(f_{v3} + \frac{K_2}{s}\right)V_2(s) = F(s)$$

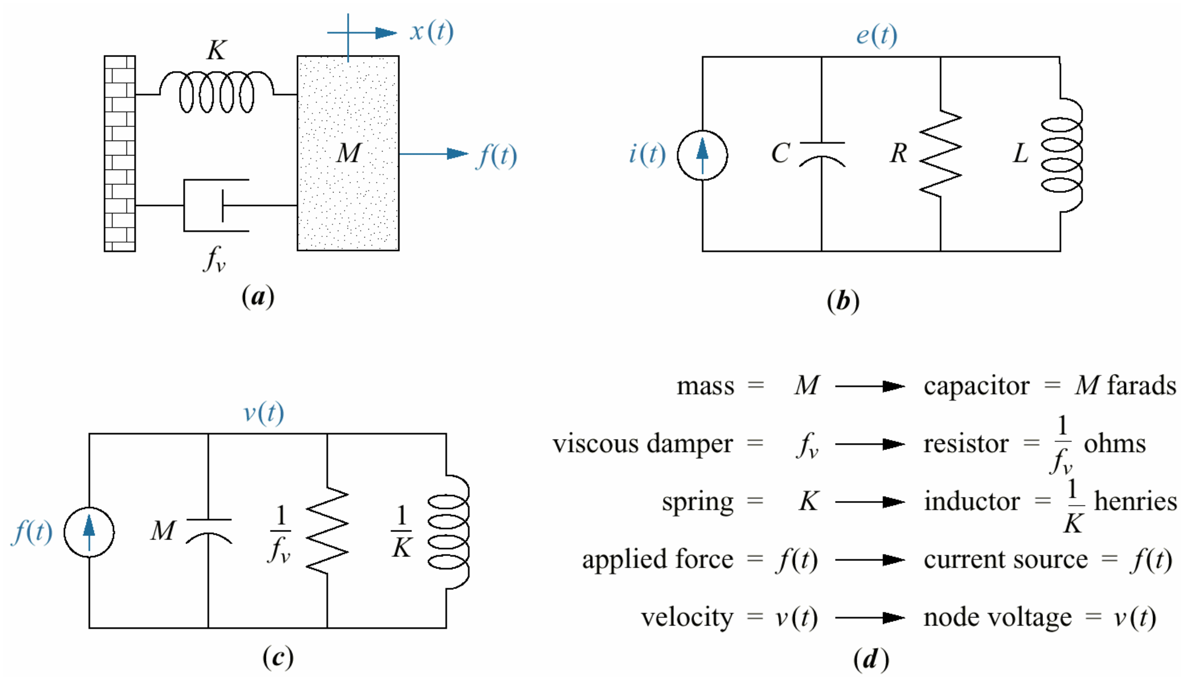
$$-\left(f_{v3} + \frac{K_2}{s}\right)V_1(s) + \left[M_2 s + (f_{v2} + f_{v3}) + \frac{(K_2 + K_3)}{s}\right]V_2(s) = 0$$



Paralel Analog : A system can be converted an equivalent paralel analog. Consider the translational mechanical system in the Figure (a), whose the equation of motion is given by

$$\frac{Ms^2+f_v s+K}{s}sX(s)=\left(Ms+f_v+\frac{K}{s}\right)V(s)=F(s)$$

Figure 2.43 Development of parallel analog: **a.** mechanical system; **b.** desired electrical representation; **c.** parallel analog; **d.** parameters for parallel analog

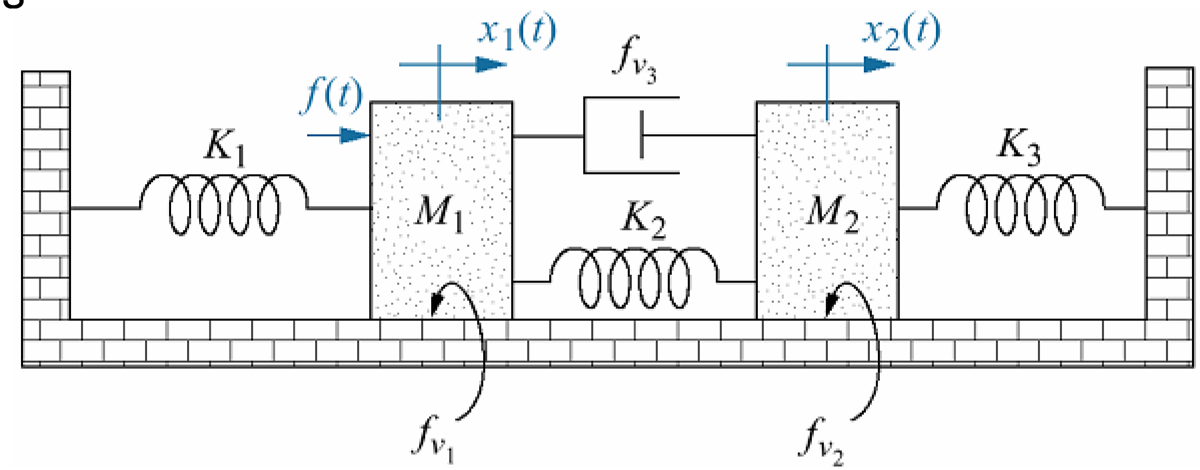


Kirchhoff's nodal equation for the simple paralel RLC network shown in Figure(b) is

$$\left(Cs+\frac{1}{R}+\frac{1}{Ls}\right)E(s)=I(s)$$

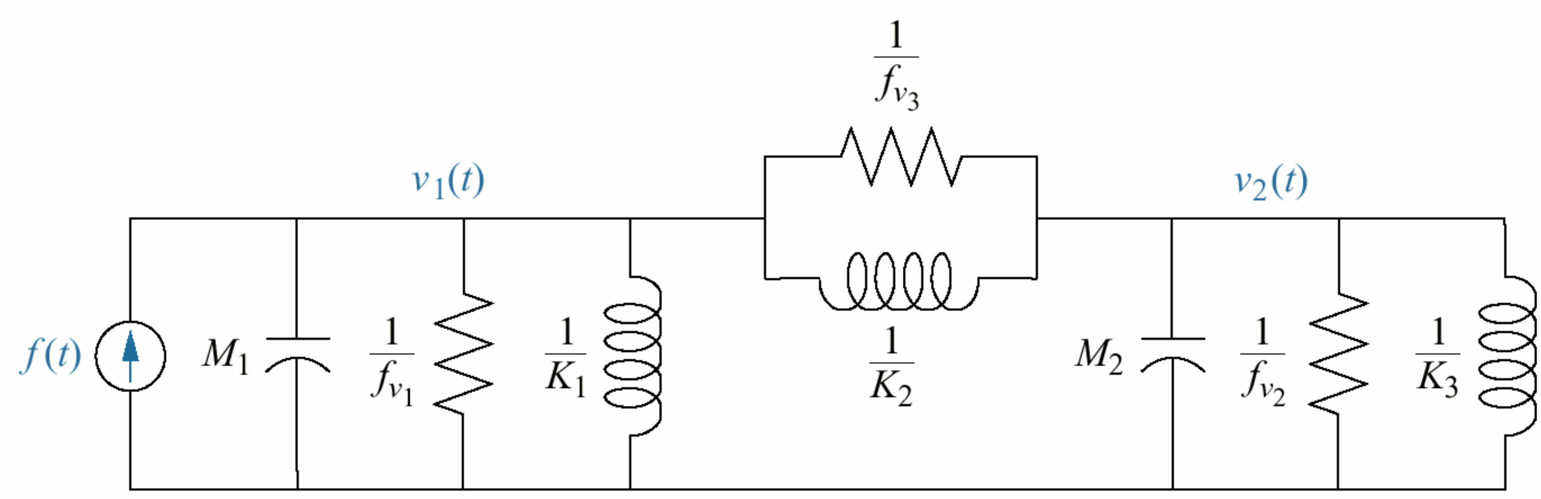
Clearly, Figure(c) is equivalent to Figure(a) in the sense of analogy.

Example : Draw a parallel analog for the same mechanical system with previous example which is



$$\left(M_1 s + (f_{v1} + f_{v3}) + \frac{(K_1 + K_2)}{s}\right)V_1(s) - \left(f_{v3} + \frac{K_2}{s}\right)V_2(s) = F(s)$$

$$-\left(f_{v3} + \frac{K_2}{s}\right)V_1(s) + \left[M_2 s + (f_{v2} + f_{v3}) + \frac{(K_2 + K_3)}{s}\right]V_2(s) = 0$$



NONLINEARITIES

In this section, we formally define the terms *linear* and *nonlinear* and how to distinguish between the two. We will show how to approximate a nonlinear system as a linear system.

A linear system possesses two properties : *Superposition* and *Homogeneity*. The property of *superposition* means that the output response of a system to the sum of inputs is the sum of responses to the individual inputs. Thus, if a input of $r_1(t)$ yield an output of $c_1(t)$ and an input of $r_2(t)$ yields an output of $c_2(t)$, then an input of $r_1(t)+r_2(t)$ yields an output of $c_1(t)+c_2(t)$. The property of *homogeneity* describes the response of the system to multiplication of the system by a scalar. Specifially, in a linear system, the property of homogeneity is demanstrated if for an input of $r_1(t)$ that yields an output of $c_1(t)$, an input $Ar_1(t)$ yields an output of $Ac_1(t)$; that is, multiplication of an input by a scalar yields a response that is multiplied by the same scalar. We can visualize the linearity as shown in Figure 2.45.

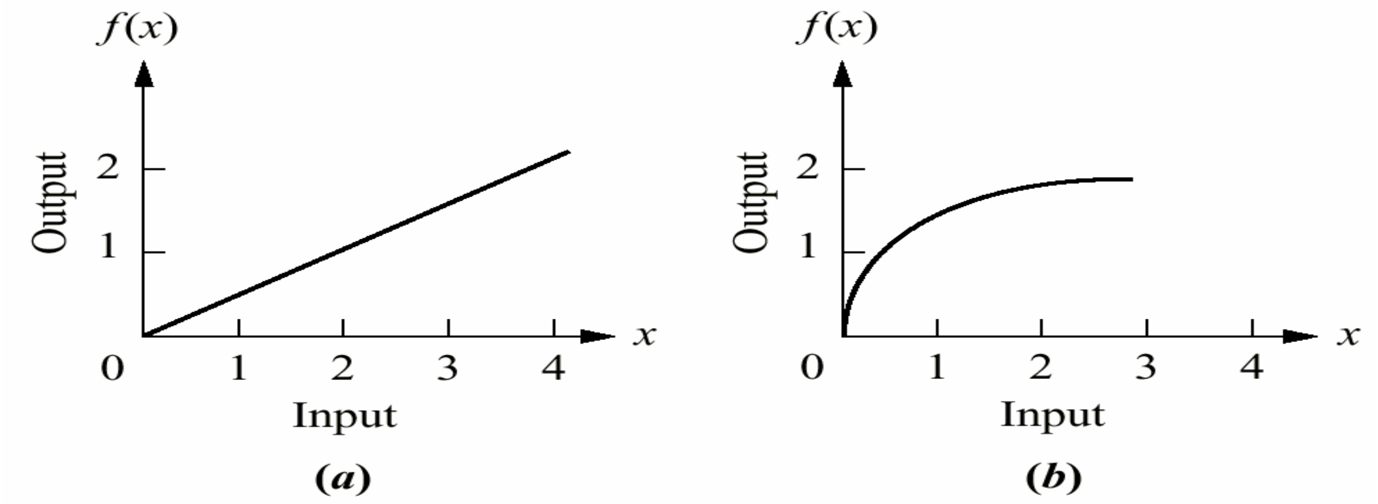
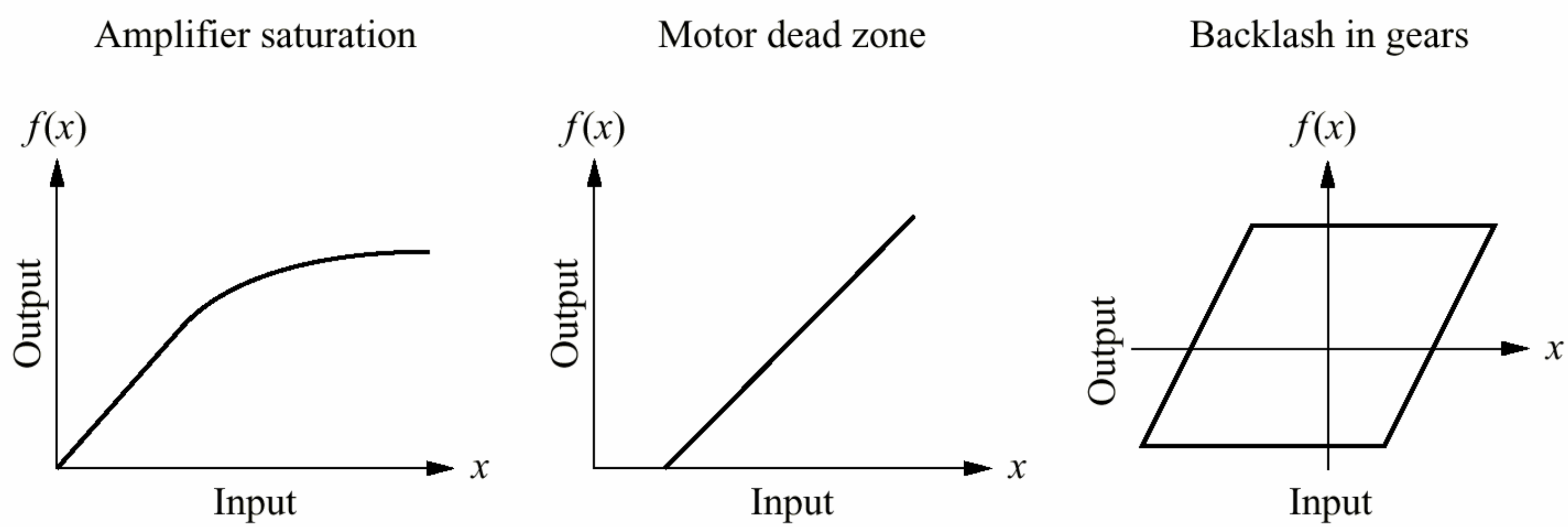


Figure 2.45
a. Linear system;
b. Nonlinear system

Figure 2.46 shows some example of physical nonlinearities.

Figure 2.46
Some physical nonlinearities



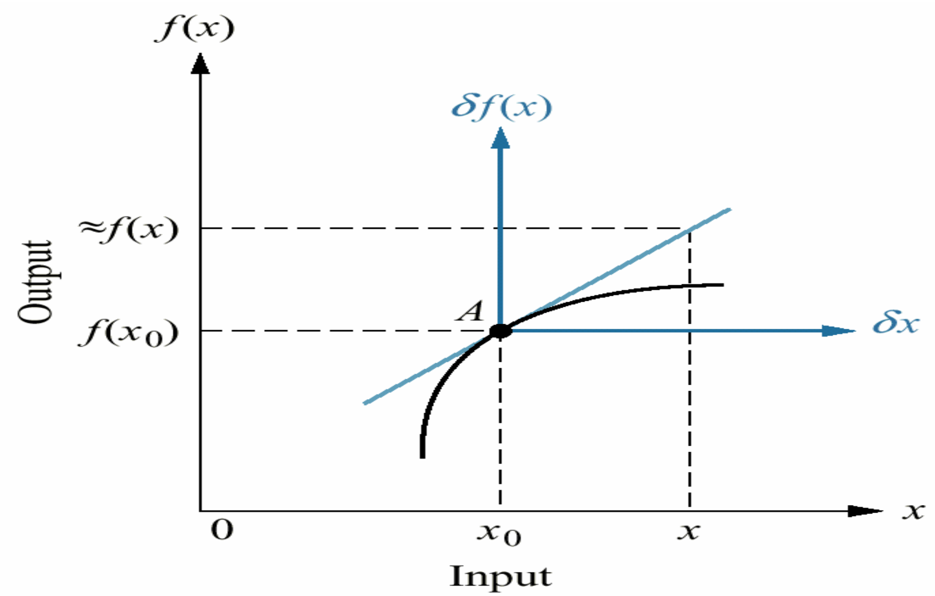
A designer can often make a linear approximation to a nonlinear system. Linear approximation simplfy the analysis and design of a system and are used as long as the results yield a good approximation to reality. For example, a linear relationship can be established as a point on the nonlinear curve if the range of input values about that point is small and the origin is translated to that point. Electronic amplifiers are an example of physical devices that perform linear amplification with small excursion about a point.

LINEARIZATION

In this section, we show how to obtain linear approximations to nonlinear systems in order to obtain transfer function.

The first step is to recognize the nonlinear component and write the nonlinear differential equation, we linearize it for small-signal inputs about the steady-state solution when the small-signal input is equal to zero. This steadystate solution is called *equilibrium* and is selcted as the second step in the linearization process. Next we linearize the nonlinear differential equation, and then we take the Laplace transform of the linearized differential equaiton, assuming all zero initial condition. Finally we seperate input and output variables and form the transfer function. Let us first see how to linearize a function; later, we will apply the method to the linearization of differential equation.

If we assume a nonlinear system operating at point **A**, $[x_0, f(x_0)]$ in Figure 2.47.



Small changes in the input can be related to changes in tje output about the point by way of the slope of the curve at a point **A**. Thus, if the slope of the curve at point **A** is m_a, then small excursion of the input about point **A**, δx, yields small changes in the output, $\delta f(x)$, related by the slope at point **A**.

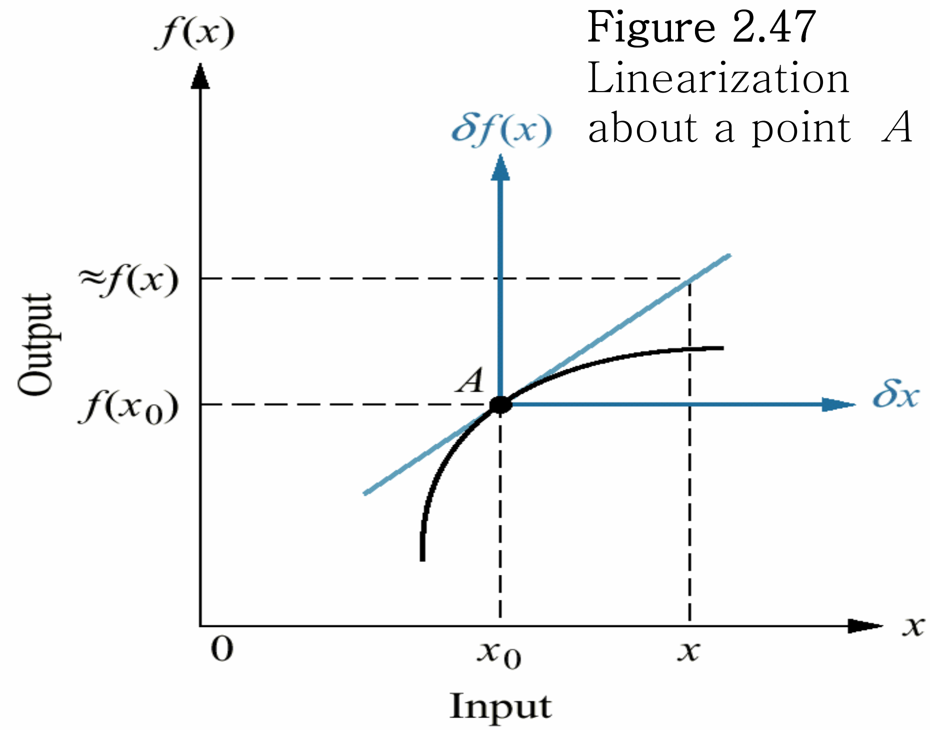


Figure 2.47
Linearization
about a point A

Thus,

$$[f(x) - f(x_o)] \approx m_a (x - x_0)$$

from which

$$\delta f(x) \approx m_a \delta x$$

and

$$f(x) \approx f(x_o)] + m_a (x - x_0) \approx f(x_0) + m_a \delta x$$

This relationship is shown graphically in Figure 2.47, where a new set of axes, δx and δf(x), is created at the point **A**, and f(x) is approximately equal to $f(x_0)$, the ordinate of the new origin, plus small excursion, $m_a\delta x$, away from point **A**. Let us look an example.

Example : Linearize f(x) = 5cosx about x=π/2

Solution : We first find that the derivative of f(x) is df/dx = (-5sinx). At x = π/2, the derivative is -5. Also $f(x_0)=f(\pi/2)=5\cos(\pi/2)=0$. Thus, from the equation

$$f(x) \approx f(x_o)] + m_a(x - x_0) \approx f(x_0) + m_a \delta x$$

The system can be represented as f(x)= -5δx for small excursions of x about x= π/2. The process is shown graphically in the figure, where the cosine curve does not indeed look like a straight line of slope -5 near π/2.

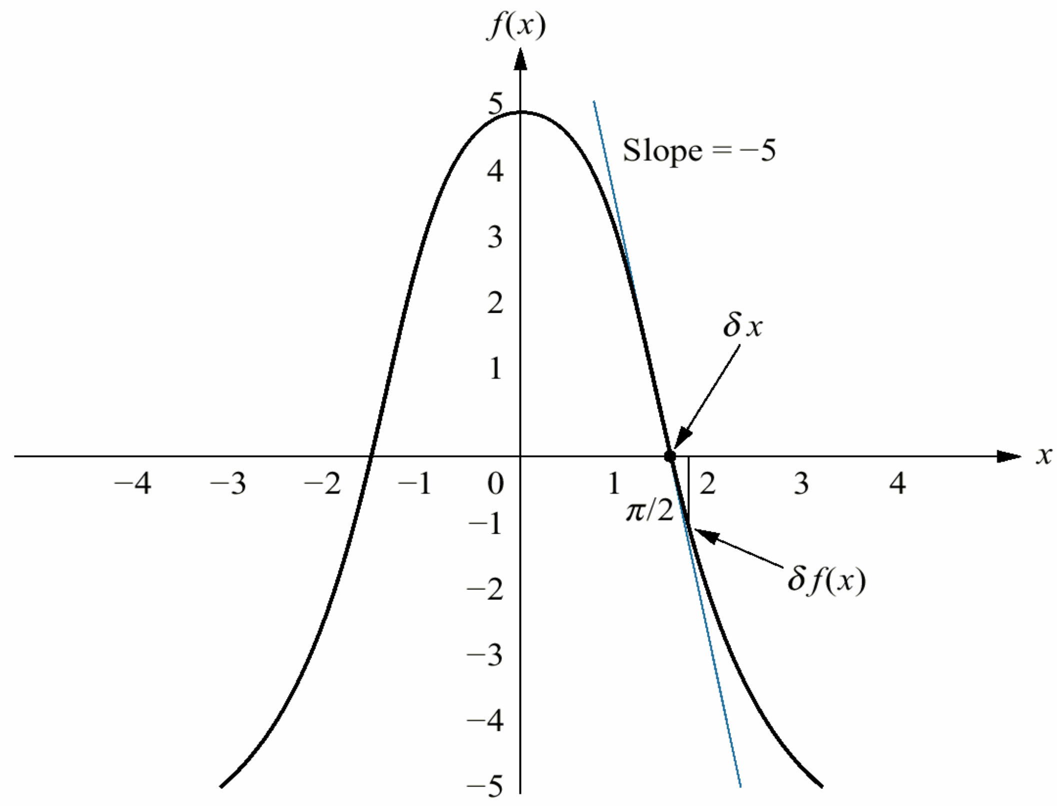


Figure 2.48
Linearization
of 5 cos *x* about
$x = \pi/2$

Taylor Series Expansion : Another approach to linearization is Taylor series. The previous discussion can be formalized using the Taylor series expansion, which expresses the value of a function in terms of value of this function at a particular point. The Taylor series of f(x) is

$$f(x) = f(x_o) + \left(\frac{df}{dx}\right)_{x=x_o}(x - x_o) + \frac{1}{2!}\left(\frac{d^2 f}{dx^2}\right)_{x=x_o}(x - x_o)^2 + \cdots + \frac{1}{k!}\left(\frac{d^k f}{dx^k}\right)_{x=x_o}(x - x_o)^k + \cdots$$

For small excursion of x from x_0, we can neglect high order terms. The resulting approximation yields a straight-line relationship between the change in f(x) and the excursion away from x_0. Neglecting the high order terms in the equation above, we get

$$f(x) - f(x_0) = \left.\frac{df}{dx}\right|_{x=x_0}(x - x_0) \qquad \text{or} \qquad \delta f(x) \approx m_a \delta x$$

Which is a linear relationship between δf(x) and δx for small excursion away from x_0. The following examples demonstrates linearization.The first example demonstrates linearization of a differential equation and the second example applies linearization to finding a transfer function.

Example : Linearize the following differential equation for small excursion about x=π/4

$$\frac{d^2x}{dt^2}+2\frac{dx}{dt}+\cos x=0$$

Solution : The presence of the term cosx makes this equation nonlinear. Since we want to linearize the equation about x=π/ 4, we let x=δ+ π/ 4, where δx is the small excursion about π/ 4, and substitue x into given equation,

$$\frac{d^2\left(\delta x+\frac{\pi}{4}\right)}{dt^2}+2\frac{d\left(\delta x+\frac{\pi}{4}\right)}{dt}+\cos\left(\delta x+\frac{\pi}{4}\right)=0$$

But $\frac{d^2\left(\delta x+\frac{\pi}{4}\right)}{dt^2}=\frac{d^2\delta x}{dt^2}$ and $\frac{d\left(\delta x+\frac{\pi}{4}\right)}{dt}=\frac{d\,\delta x}{dt}$

Finally, the term cos(δx+(π/ 4)) can be linearized with the truncated Taylor series. Substituting f(x)=cos(δx+(π/ 4)) , $f(x_0)$=f(π/ 4)=cos (π/ 4), and $(x-x_0)$= δx into the equation

$$f(x)-f(x_0)=\left.\frac{df}{dx}\right|_{x=x_0}(x-x_0)$$

yields

$$\cos\left(\delta x+\frac{\pi}{4}\right)-\cos\left(\frac{\pi}{4}\right)=\left.\frac{d\cos x}{dx}\right|_{x=\frac{\pi}{4}}\delta x=-\sin\left(\frac{\pi}{4}\right)\delta x$$

Solving this equation for cos(δx+(π/ 4)) yields

$$\cos\left(\delta x+\frac{\pi}{4}\right)=\cos\left(\frac{\pi}{4}\right)-\sin\left(\frac{\pi}{4}\right)\delta x=\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\delta x$$

Note that we have obtained by now three equations which are

$$\frac{d^2\left(\delta x+\frac{\pi}{4}\right)}{dt^2}=\frac{d^2\delta x}{dt^2} \qquad \frac{d\left(\delta x+\frac{\pi}{4}\right)}{dt}=\frac{d\,\delta x}{dt} \qquad \cos\left(\delta x+\frac{\pi}{4}\right)=\cos\left(\frac{\pi}{4}\right)-\sin\left(\frac{\pi}{4}\right)\delta x=\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\delta x$$

Substituting these three equations into the first equation we wrote which is

$$\frac{d^2\left(\delta x+\frac{\pi}{4}\right)}{dt^2}+2\frac{d\left(\delta x+\frac{\pi}{4}\right)}{dt}+\cos\left(\delta x+\frac{\pi}{4}\right)=0$$

yields the following **linearized differential equation**

$$\boxed{\frac{d^2\delta x}{dt^2}+2\frac{d\delta x}{dt}-\frac{\sqrt{2}}{2}\delta x=-\frac{\sqrt{2}}{2}}$$

Example : **Nonlinear Electrical Network**

Find the transfer function $V_L(s)/V(s)$ for electrical network shown in the figure which contains a nonlinear resistor whose voltage-current relationship is defined by

$$i_r = 0.1e^{0.1V_r}$$

Where i_r and V_r are the resistor current and voltage, respectively. Also v(t) in the figure is small-signal source.

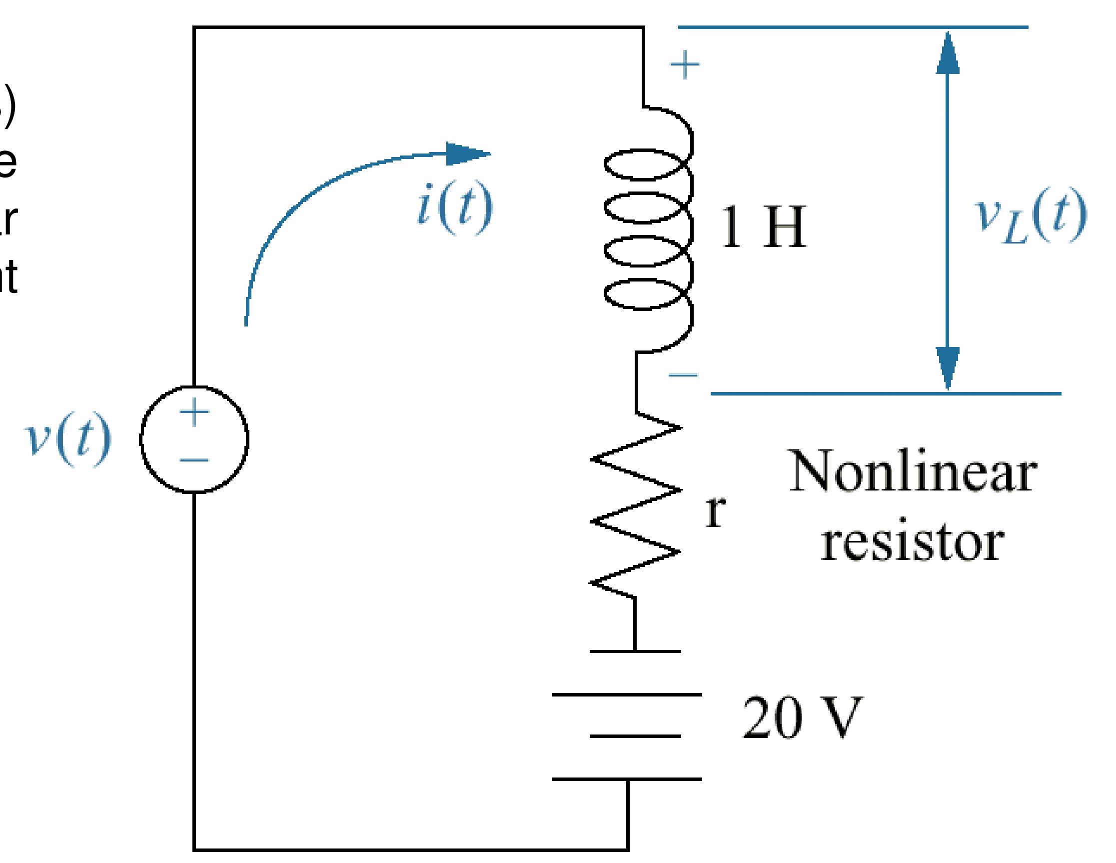
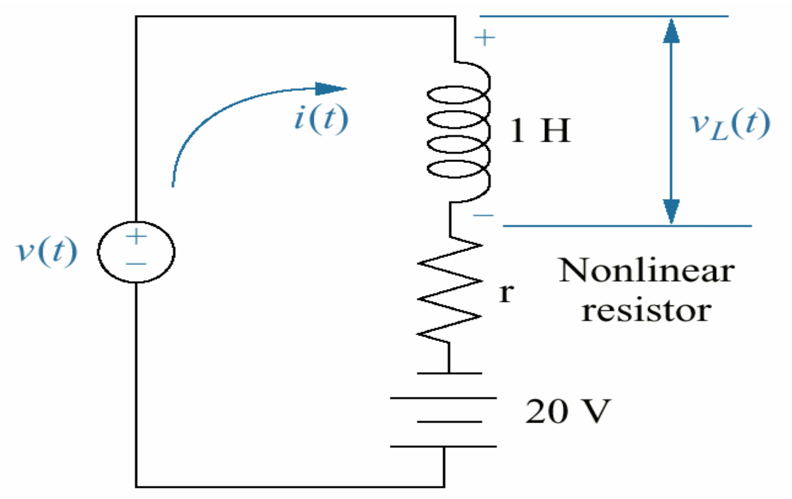


Figure 2.49
Nonlinear
electrical
network



We will use the Kirchhoff's voltage law to sum the voltages in the loop to obtain the nonlinear differential equation, but first we must solve for the voltage across the nonlinear resistor. Taking the natural log of resistor's current-voltage relationship, we get

$$V_r = 10 \ln \frac{1}{2} i_r$$

Applying the Kirchhoff's voltage law around the loop, where $i_r = i$, yields

$$L\frac{di}{dt} + 10\ln\frac{1}{2}i - 20 = v(t) \qquad \textbf{(*)}$$

Next, let us evaluate the equilibrium solution. First, set the small-signal source, v(t), equal to zero. Now evaluate the steady-state current. With v(t)=0, the circuit consist of 20 V battery in series with the inductor will be zero, since v_L=L(di/dt) and (di/dt) is zero in the steady,state, given aconstant battrey source. Hence the resistor voltage v_r is 20 V. Using the characteristics of the resistor,

$$i_r = 2e^{0.1V_r}$$

we find that i_r=i=14.78 amps. This current, i_0, is the equilibrium value of the network current. Hence i=i_0+δi. Substituting this current into **(*)** equation yields

$$L\frac{d(i_0+\delta i)}{dt}+10\frac{1}{2}(i_0+\delta i)-20=v(t) \qquad \textbf{(**)}$$

Using the linearizing equation

$$f(x)-f(x_0)=\left.\frac{df}{dx}\right|_{x=x_0}(x-x_0)$$

to linearize $\ln\frac{1}{2}(i_0+\delta i)$, we get

$$\ln\frac{1}{2}(i_0+\delta i)-\ln\frac{1}{2}i_0=\left.\frac{d\left(\ln\frac{1}{2}i\right)}{di}\right|_{i=i_0}\delta i=\left.\frac{1}{i}\right|_{i=i_0}\delta i=\frac{1}{i_0}\delta i \qquad \textbf{or} \qquad \ln\frac{1}{2}(i_0+\delta i)=\ln\frac{i_0}{2}+\frac{1}{i_0}\delta i$$

Substituting this equation into **(**)**, the linearized equation becomes

$$\boxed{L\frac{d\delta i}{dt}+10(\ln\frac{i_0}{2}+\frac{1}{i_0}\delta i)-20=v(t)}$$

Letting L=1 and i_0=14.78, the final linearized differential equation is $\frac{d\,\delta i}{dt} + 0.6778\ i = v(t)$

Taking the Laplace transform with zero initial conditions and solving for δi(s) yields

$$\delta i(s) = \frac{V(s)}{s + 0.677} \qquad \textbf{(***)}$$

But the voltage across the inductor about the equilibrium point is $V_L(t) = L\frac{d}{dt}(i_0 + \delta i) = L\frac{d\delta i}{dt}$

Taking the Laplace transform $\quad V_L(s)=Ls\delta i(s)=s\ \delta i(s)$

Substituting the **(***)** equation into the last equation yields

$$V_L(s) = s\frac{V(s)}{s + 0.677}$$

from which the final transfer function is

$$\frac{V_L(s)}{V(s)} = \frac{s}{s + 0.677}$$

for small excursions about i=14.78 or, equivalently, about v(t)=0