4.2. Solve $y' = y^2x^3$.

We first rewrite this equation in the differential form (see Chapter 3) $x^3\,dx - (1/y^2)\,dy = 0$. Then $A(x) = x^3$ and $B(y) = -1/y^2$. Substituting these values into Eq. (*4.2*), we have

$$\int x^3\,dx + \int (-1/y^2)\,dy = c$$

or, by performing the indicated integrations, $x^4/4 + 1/y = c$. Solving explicitly for y, we obtain the solution as

$$y = \frac{-4}{x^4 + k}, \quad k = -4c$$

4.3. Solve $\dfrac{dy}{dx} = \dfrac{x^2+2}{y}$

This equation may be rewritten in the differential form

$$(x^2 + 2)\,dx - y\,dy = 0$$

which is separable with $A(x) = x^2 + 2$ and $B(y) = -y$. Its solution is

$$\int (x^2 + 2)\,dx - \int y\,dy = c$$

or

$$\frac{1}{3}x^3 + 2x - \frac{1}{2}y^2 = c$$

Solving for y, we obtain the solution in implicit form as

$$y^2 = \frac{2}{3}x^3 + 4x + k$$

with $k = -2c$. Solving for y implicitly, we obtain the two solutions

$$y = \sqrt{\frac{2}{3}x^3 + 4x + k} \quad \text{and} \quad y = -\sqrt{\frac{2}{3}x^3 + 4x + k}$$

4.4. Solve $y' = 5y$.

First rewrite this equation in the differential form $5\,dx - (1/y)\,dy = 0$, which is separable. Its solution is

$$\int 5\,dx + \int (-1/y)\,dy = c$$

or, by evaluating, $5x - \ln|y| = c$.

To solve for y explicitly, we first rewrite the solution as $\ln|y| = 5x - c$ and then take the exponential of both sides. Thus, $e^{\ln|y|} = e^{5x-c}$. Noting that $e^{\ln|y|} = |y|$, we obtain $|y| = e^{5x}e^{-c}$, or $y = \pm\, e^{-c}e^{5x}$. The solution is given explicitly by $y = ke^{5x}$, $k = \pm\, e^{-c}$.

Note that the presence of the term $(-1/y)$ in the differential form of the differential equation requires the restriction $y \neq 0$ in our derivation of the solution. This restriction is equivalent to the restriction $k \neq 0$, since $y = ke^{5x}$. However, by inspection, $y \equiv 0$ is a solution of the differential equation as originally given. Thus, $y = ke^{5x}$ is the solution for all k.

The differential equation as originally given is also linear. See Problem 6.9 for an alternate method of solution.

4.5. Solve $y' = \dfrac{x+1}{y^4+1}$.

This equation, in differential form, is $(x + 1)\,dx + (-y^4 - 1)\,dy = 0$, which is separable. Its solution is

$$\int (x+1)\,dx + \int (-y^4 - 1)\,dy = c$$

or, by evaluating,

$$\frac{x^2}{2} + x - \frac{y^5}{5} - y = c$$

Since it is impossible algebraically to solve this equation explicitly for y, the solution must be left in its present implicit form.

4.6. Solve $dy = 2t(y^2 + 9)\,dt$.

This equation may be rewritten as

$$\frac{dy}{y^2 + 9} - 2t\,dt = 0$$

which is separable in variables y and t. Its solution is

$$\int \frac{dy}{y^2 + 9} - \int 2t\,dt = c$$

or, upon evaluating the given integrals,

$$\frac{1}{3}\arctan\left(\frac{y}{3}\right) - t^2 = c$$

Solving for y, we obtain

$$\arctan\left(\frac{y}{3}\right) = 3(t^2 + c)$$

$$\frac{y}{3} = \tan\,(3t^2 + 3c)$$

or $$y = 3\tan\,(3t^2 + \mathrm{k})$$

with $k = 3c$.

4.7. Solve $\dfrac{dx}{dt} = x^2 - 2x + 2$.

This equation may be rewritten in differential form

$$\frac{dx}{x^3 - 2x + 2} - dt = 0$$

which is separable in the variables x and t. Its solution is

$$\int \frac{dx}{x^3 - 2x + 2} - \int dt = c$$

Evaluating the first integral by first completing the square, we obtain

$$\int \frac{dx}{(x-1)^2 + 1} - \int dt = c$$

or $$\arctan\,(x - 1) - t = c$$

Solving for x as a function of t, we obtain

$$\arctan\,(x - 1) = t + c$$

$$x - 1 = \tan\,(t + c)$$

or $$x = 1 + \tan\,(t + c)$$

4.8. Solve $e^x\,dx - y\,dy = 0$; $y(0) = 1$.

The solution to the differential equation is given by Eq. (*4.2*) as

$$\int e^x\,dx + \int (-y)\,dy = c$$

or, by evaluating, as $y^2 = 2e^x + k$, $k = -2c$. Applying the initial condition, we obtain $(1)^2 = 2e^0 + k$, $1 = 2 + k$, or $k = -1$. Thus, the solution to the initial-value problem is

$$y^2 = 2e^x - 1 \quad \text{or} \quad y = \sqrt{2e^x - 1}$$

[Note that we cannot choose the negative square root, since then $y(0) = -1$, which violates the initial condition.]

To ensure that y remains real, we must restrict x so that $2e^x - 1 \geq 0$. To guarantee that y' exists [note that $y'(x) = dy/dx = e^x/y$], we must restrict x so that $2e^x - 1 \neq 0$. Together these conditions imply that $2e^x - 1 > 0$, or $x > \ln\frac{1}{2}$.

4.9. Use Eq. (*4.4*) to solve Problem 4.8.

For this problem, $x_0 = 0$, $y_0 = 1$, $A(x) = e^x$, and $B(y) = -y$. Substituting these values into Eq. (*4.4*), we obtain

$$\int_0^x e^x\,dx + \int_1^y (-y)\,dy = 0$$

Evaluating these integrals, we have

$$e^x\Big|_0^x + \left(\frac{-y^2}{2}\right)\Bigg|_1^y = 0 \quad \text{or} \quad e^x - e^0 + \left(\frac{-y^2}{2}\right) - \left(-\frac{1}{2}\right) = 0$$

Thus, $y^2 = 2e^x - 1$, and, as in Problem 4.8, $y = \sqrt{2e^x - 1}$, $\quad x > \ln\frac{1}{2}$.

4.10. Solve $x\cos x\,dx + (1 - 6y^5)\,dy = 0$; $y(\pi) = 0$.

Here, $x_0 = \pi$, $y_0 = 0$, $A(x) = x\cos x$, and $B(y) = 1 - 6y^5$. Substituting these values into Eq. (*4.4*), we obtain

$$\int_\pi^x x\cos x\,dx + \int_0^y (1 - 6y^5)\,dy = 0$$

Evaluating these integrals (the first one by integration by parts), we find

$$x\sin x\Big|_\pi^x + \cos x\Big|_\pi^x + (y - y^6)\Big|_0^y = 0$$

or

$$x\sin x + \cos x + 1 = y^6 - y$$

Since we cannot solve this last equation for y explicitly, we must be content with the solution in its present implicit form.

4.11. Solve $y' = \dfrac{y + x}{x}$.

This differential equation is not separable, but it is homogeneous as shown in Problem 3.9(*a*). Substituting Eqs. (*4.6*) and (*4.7*) into the equation, we obtain

$$v + x\frac{dv}{dx} = \frac{xv + x}{x}$$

which can be algebraically simplified to

$$x\frac{dv}{dx} = 1 \quad \text{or} \quad \frac{1}{x}dx - dv = 0$$

This last equation is separable; its solution is

$$\int \frac{1}{x}\,dx - \int dv = c$$

which, when evaluated, yields $v = \ln|x| - c$, or

$$v = \ln|kx| \tag{1}$$

where we have set $c = -\ln|k|$ and have noted that $\ln|x| + \ln|k| = \ln|kx|$. Finally, substituting $v = y/x$ back into (*1*), we obtain the solution to the given differential equation as $y = x\ln|kx|$.

4.12. Solve $y' = \dfrac{2y^4 + x^4}{xy^3}$.

This differential equation is not separable. Instead it has the form $y' = f(x, y)$, with

$$f(x, y) = \frac{2y^4 + x^4}{xy^3}$$

where
$$f(tx, ty) = \frac{2(ty)^4 + (tx)^4}{(tx)(ty)^3} = \frac{t^4(2y^4 + x^4)}{t^4(xy^3)} = \frac{2y^4 + x^4}{xy^3} = f(x, y)$$

so it is homogeneous. Substituting Eqs. (*4.6*) and (*4.7*) into the differential equation as originally given, we obtain

$$v + x\frac{dv}{dx} = \frac{2(xv)^4 + x^4}{x(xv)^3}$$

which can be algebraically simplified to

$$x\frac{dv}{dx} = \frac{v^4 + 1}{v^3} \quad \text{or} \quad \frac{1}{x}dx - \frac{v^3}{v^4 + 1}dv = 0$$

This last equation is separable; its solution is

$$\int \frac{1}{x}\,dx - \int \frac{v^3}{v^4 + 1}\,dv = c$$

Integrating, we obtain in $\ln|x| - \frac{1}{4}\ln(v^4 + 1) = c$, or

$$v^4 + 1 = (kx)^4 \tag{1}$$

where we have set $c = -\ln|k|$ and then used the identities

$$\ln|x| + \ln|k| = \ln|kx| \quad \text{and} \quad 4\ln|kx| = \ln(kx)^4$$

Finally, substituting $v = y/x$ back into (*1*), we obtain

$$y^4 = c_1x^8 - x^4 \quad (c_1 = k^4) \tag{2}$$

4.13. Solve the differential equation of Problem 4.12 by using Eqs. (*4.9*) and (*4.10*).

We first rewrite the differential equation as

$$\frac{dx}{dy} = \frac{xy^3}{2y^4 + x^4}$$

Then substituting (*4.9*) and (*4.10*) into this new differential equation, we obtain

$$u + y\frac{du}{dy} = \frac{(yu)y^3}{2y^4 + (yu)^4}$$

which can be algebraically simplified to

$$y\frac{du}{dy} = -\frac{u+u^5}{2+u^4}$$

or

$$\frac{1}{y}dy + \frac{2+u^4}{u+u^5}du = 0 \tag{1}$$

Equation (*1*) is separable; its solution is

$$\int\frac{1}{y}dy + \int\frac{2+u^4}{u+u^5}du = c$$

The first integral is in ln |y|. To evaluate the second integral, we use partial fractions on the integrand to obtain

$$\frac{2+u^4}{u+u^5} = \frac{2+u^4}{u(1+u^4)} = \frac{2}{u} - \frac{u^3}{1+u^4}$$

Therefore,

$$\int\frac{2+u^4}{u+u^5}\,du = \int\frac{2}{u}\,du - \int\frac{u^3}{1+u^4}\,du = 2\ln|u| - \frac{1}{4}\ln(1+u^4)$$

The solution to (*1*) is in $\ln|y| + 2\ln|u| - \frac{1}{4}\ln(1+u^4) = c$, which can be rewritten as

$$ky^4u^8 = 1 + u^4 \tag{2}$$

where $c = -\frac{1}{4}\ln|k|$. Substituting $u = x/y$ back into (*2*), we once again have (*2*) of Problem 4.12.

4.14. Solve $y' = \dfrac{2xy}{x^2 - y^2}$.

This differential equation is not separable. Instead it has the form $y' = f(x, y)$, with

$$f(x,y) = \frac{2xy}{x^2 - y^2}$$

where

$$f(tx, ty) = \frac{2(tx)(ty)}{(tx)^2 - (ty)^2} = \frac{t^2(2xy)}{t^2(x^2 - y^2)} = \frac{2xy}{x^2 - y^2} = f(x, y)$$

so it is homogenous. Substituting Eqs. (*4.6*) and (*4.7*) into the differential equation as originally given, we obtain

$$v + x\frac{dv}{dx} = \frac{2x(xv)}{x^2 - (xv)^2}$$

which can be algebraically simplified to

$$x\frac{dv}{dx} = -\frac{v(v^2+1)}{v^2-1}$$

or

$$\frac{1}{x}dx + \frac{v^2-1}{v(v^2+1)}dv = 0 \tag{1}$$

Using partial fractions, we can expand (*1*) to

$$\frac{1}{x}\,dx + \left(-\frac{1}{v} + \frac{2v}{v^2+1}\right)dv = 0 \tag{2}$$

The solution to this separable equation is found by integrating both sides of (*2*). Doing so, we obtain $\ln|x| - \ln|v| + \ln(v^2+1) = c$, which can be simplified to

$$x(v^2+1) = kv \quad (c = \ln|k|) \tag{3}$$

Substituting $v = y/x$ into (*3*), we find the solution of the given differential equation is $x^2 + y^2 = ky$.

4.15. Solve $y' = \dfrac{x^2 + y^2}{xy}$.

This differential equation is homogeneous. Substituting Eqs. (*4.6*) and (*4.7*) into it, we obtain

$$v + x\frac{dv}{dx} = \frac{x^2 + (xv)^2}{x(xv)}$$

which can be algebraically simplified to

$$x\frac{dv}{dx} = \frac{1}{v} \quad \text{or} \quad \frac{1}{x}dx - v\,dv = 0$$

The solution to this separable equation is $\ln|x| - v^2/2 = c$, or equivalently

$$v^2 = \ln x^2 + k \qquad (k = -2c) \tag{1}$$

Substituting $v = y/x$ into (*1*), we find that the solution to the given differential equation is

$$y^2 = x^2 \ln x^2 + kx^2$$

4.16. Solve $y' = \dfrac{x^2 + y^2}{xy}$; $y(1) = -2$.

The solution to the differential equation is given in Problem 3.15 as $y^2 = x^2 \ln x^2 + kx^2$. Applying the initial condition, we obtain $(-2)^2 = (1)^2 \ln (1)^2 + k(1)^2$, or $k = 4$. (Recall that $\ln 1 = 0$.) Thus, the solution to the initial-value problem is

$$y^2 = x^2 \ln x^2 + 4x^2 \quad \text{or} \quad y = -\sqrt{x^2 \ln x^2 + 4x^2}$$

The negative square root is taken, to be consistent with the initial condition.

4.17. Solve $y' = \dfrac{2xye^{(x/y)^2}}{y^2 + y^2e^{(x/y)^2} + 2x^2e^{(x/y)^2}}$.

This differential equation is not separable, but it is homogeneous. Noting the (x/y)-term in the exponential, we try the substitution $u = x/y$, which is an equivalent form of (*4.9*). Rewriting the differential equation as

$$\frac{dx}{dy} = \frac{y^2 + y^2e^{(x/y)^2} + 2x^2e^{(x/y)^2}}{2xye^{(x/y)^2}}$$

we have upon using substitutions (*4.9*) and (*4.10*) and simplifying,

$$y\frac{du}{dy} = \frac{1 + e^{u^2}}{2ue^{u^2}} \quad \text{or} \quad \frac{1}{y}dy - \frac{2ue^{u^2}}{1 + e^{u^2}}du = 0$$

This equation is separable; its solution is

$$\ln|y| - \ln(1 + e^{u^2}) = c$$

which can be rewritten as

$$y = k(1 + e^{u^2}) \quad (c = \ln|k|) \tag{1}$$

Substituting $u = x/y$ into (*1*), we obtain the solution of the given differential equation as

$$y = k[1 + e^{(x/y)^2}]$$

4.18. Prove that every solution of Eq. (*4.2*) satisfies Eq. (*4.1*).

Rewrite (*4.1*) as $A(x) + B(y)y' = 0$. If $y(x)$ is a solution, it must satisfy this equation identically in x; hence,

$$A(x) + B[y(x)]y'(x) = 0$$

Integrating both sides of this last equation with respect to x, we obtain

$$\int A(x)\,dx + \int B[y(x)]\,y'(x)\,dx = c$$

In the second integral, make the change of variables $y = y(x)$, hence $dy = y'(x)\,dx$. The result of this substitution is (*4.2*).

4.19. Prove that every solution of system (*4.3*) is a solution of (*4.4*).

Following the same reasoning as in Problem 4.18, except now integrating from $x = x_0$ to $x = x$, we obtain

$$\int_{x_0}^{x} A(x)\,dx + \int_{x_0}^{x} B[y(x)]y'(x)\,dx = 0$$

The substitution $y = y(x)$ again gives the desired result. Note that as x varies from x_0 to x, y will vary from $y(x_0) = y_0$ to $y(x) = y$.

4.20. Prove that if $y' = f(x, y)$ is homogeneous, then the differential equation can be rewritten as $y' = g(y/x)$, where $g(y/x)$ depends only on the quotient y/x.

We have that $f(x, y) = f(tx, ty)$. Since this equation is valid for all t, it must be true, in particular, for $t = 1/x$. Thus, $f(x, y) = f(1, y/x)$. If we now define $g(y/x) = f(1, y/x)$, we then have $y' = f(x, y) = f(1, y/x) = g(y/x)$ as required.

Note that this form suggests the substitution $v = y/x$ which is equivalent to (*4.6*). If, in the above, we had set $t = 1/y$, then $f(x, y) = f(x/y, 1) = h(x/y)$, which suggests the alternate substitution (*4.9*).

4.21. A function $g(x, y)$ is *homogeneous of degree n* if $g(tx, ty) = t^n g(x, y)$ for all t. Determine whether the following functions are homogeneous, and, if so, find their degree:

(*a*) $xy + y^2$, (*b*) $x + y \sin (y/x)^2$, (*c*) $x^3 + xy^2 e^{x/y}$, and (*d*) $x + xy$.

(*a*) $(tx)(ty) + (ty)^2 = t^2(xy + y^2)$; homogeneous of degree two.

(*b*) $tx + ty\sin\left(\dfrac{ty}{tx}\right)^2 = t\left[x + y\sin\left(\dfrac{y}{x}\right)^2\right]$; homogeneous of degree one.

(*c*) $(tx)^3 + (tx)(ty)^2 e^{tx/ty} = t^3(x^3 + xy^2 e^{x/y})$; homogeneous of degree three.

(*d*) $tx + (tx)(ty) = tx + t^2 xy$; not homogeneous.

4.22. An alternate definition of a homogeneous differential equation is as follows: A differential equation $M(x, y)\,dx + N(x, y)\,dy = 0$ is *homogenous* if both $M(x, y)$ and $N(x, y)$ are homogeneous of the same degree (see Problem 4.21). Show that this definition implies the definition given in Chapter 3.

If $M(x, y)$ and $N(x, y)$ are homogeneous of degree n, then

$$f(tx, ty) = \frac{M(tx, ty)}{-N(tx, ty)} = \frac{t^n M(x, y)}{-t^n N(x, y)} = \frac{M(x, y)}{-N(x, y)} = f(x, y)$$

Supplementary Problems

In Problems 4.23 through 4.45, solve the given differential equations or initial-value problems.

4.23. $x\,dx + y\,dy = 0$

4.24. $x\,dx - y^3\,dy = 0$

4.25. $dx + \dfrac{1}{y^4}\,dy = 0$

4.26. $(t + 1)\,dt - \dfrac{1}{y^2}\,dy = 0$

4.27. $\frac{1}{x}dx - \frac{1}{y}dy = 0$

4.28. $\frac{1}{x}dx + dy = 0$

4.29. $x\,dx + \frac{1}{y}dy = 0$

4.30. $(t^2+1)\,dt + (y^2+y)\,dy = 0$

4.31. $\frac{4}{t}dt - \frac{y-3}{y}dy = 0$

4.32. $dx - \frac{1}{1+y^2}dy = 0$

4.33. $dx - \frac{1}{y^2-6y+13}dy = 0$

4.34. $y' = \frac{y}{x^2}$

4.35. $y' = \frac{xe^x}{2y}$

4.36. $\frac{dy}{dx} = \frac{x+1}{y}$

4.37. $\frac{dy}{dx} = y^2$

4.38. $\frac{dx}{dt} = x^2t^2$

4.39. $\frac{dx}{dt} = \frac{x}{t}$

4.40. $\frac{dy}{dt} = 3 + 5y$

4.41. $\sin x\,dx + y\,dy = 0; \quad y(0) = -2$

4.42. $(x^2+1)\,dx + \frac{1}{y}dy = 0; \quad y(-1) = 1$

4.43. $xe^{x^2}dx + (y^5-1)\,dy = 0; \quad y(0) = 0$

4.44. $y' = \frac{x^2y - y}{y+1}; \quad y(3) = -1$

4.45. $\frac{dx}{dt} = 8 - 3x; \quad x(0) = 4$

In Problems 4.46 through 4.54, determine whether the given differential equations are homogenous and, if so, solve them.

4.46. $y' = \frac{y-x}{x}$

4.47. $y' = \frac{2y+x}{x}$

4.48. $y' = \frac{x^2+2y^2}{xy}$

4.49. $y' = \frac{2x+y^2}{xy}$

4.50. $y' = \frac{x^2+y^2}{2xy}$

4.51. $y' = \frac{2xy}{y^2-x^2}$

4.52. $y' = \frac{y}{x+\sqrt{xy}}$

4.53. $y' = \frac{y^2}{xy+(xy^2)^{1/3}}$

4.54. $y' = \frac{x^4+3x^2y^2+y^4}{x^3y}$

If $M = yf(xy)$ and $N = xg(xy)$, then

$$I(x, y) = \frac{1}{xM - yN} \tag{5.10}$$

In general, integrating factors are difficult to uncover. If a differential equation does not have one of the forms given above, then a search for an integrating factor likely will not be successful, and other methods of solution are recommended.

Solved Problems

5.1. Determine whether the differential equation $2xy\,dx + (1 + x^2)dy = 0$ is exact.

This equation has the form of Eq. (*5.1*) with $M(x, y) = 2xy$ and $N(x, y) = 1 + x^2$. Since $\partial M/\partial y = \partial N/\partial x = 2x$, the differential equation is exact.

5.2. Solve the differential equation given in Problem 5.1.

This equation was shown to be exact. We now determine a function $g(x, y)$ that satisfies Eqs. (*5.4*) and (*5.5*). Substituting $M(x, y) = 2xy$ into (*5.4*), we obtain $\partial g/\partial x = 2xy$. Integrating both sides of this equation with respect to x, we find

$$\int \frac{\partial g}{\partial x}dx = \int 2xy\,dx$$

or

$$g(x, y) = x^2y + h(y) \tag{1}$$

Note that when integrating with respect to x, the constant (*with respect to* x) of integration can depend on y.

We now determine $h(y)$. Differentiating (*1*) with respect to y, we obtain $\partial g/\partial y = x^2 + h'(y)$. Substituting this equation along with $N(x, y) = 1 + x^2$ into (*5.5*), we have

$$x^2 + h'(y) = 1 + x^2 \qquad \text{or} \qquad h'(y) = 1$$

Integrating this last equation with respect to y, we obtain $h(y) = y + c_1$ (c_1 = constant). Substituting this expression into (*1*) yields

$$g(x, y) = x^2y + y + c_1$$

The solution to the differential equation, which is given implicitly by (*5.6*) as $g(x, y) = c$, is

$$x^2y + y = c_2 \qquad (c_2 = c - c_1)$$

Solving for y explicitly, we obtain the solution as $y = c_2/(x^2 + 1)$.

5.3. Determine whether the differential equation $y\,dx - x\,dy = 0$ is exact.

This equation has the form of Eq. (*5.1*) with $M(x, y) = y$ and $N(x, y) = -x$. Here

$$\frac{\partial M}{\partial y} = 1 \quad \text{and} \quad \frac{\partial N}{\partial x} = -1$$

which are not equal, so the differential equation as given is not exact.

5.4. Determine whether the differential equation

$$(x + \sin y)\, dx + (x \cos y - 2y)\, dy = 0$$

is exact.

Here $M(x, y) = x + \sin y$ and $N(x, y) = x \cos y - 2y$. Thus, $\partial M/\partial y = \partial N/\partial x = \cos y$, and the differential equation is exact.

5.5. Solve the differential equation given in Problem 5.4.

This equation was shown to be exact. We now seek a function $g(x, y)$ that satisfies (*5.4*) and (*5.5*). Substituting $M(x, y)$ into (*5.4*), we obtain $\partial g/\partial x = x + \sin y$. Integrating both sides of this equation with respect to x, we find

$$\int \frac{\partial g}{\partial x}\, dx = \int (x + \sin y)\, dx$$

or

$$g(x, y) = \frac{1}{2}x^2 + x \sin y + h(y) \tag{1}$$

To find $h(y)$, we differentiate (*1*) with respect to y, yielding $\partial g/\partial y = x \cos y + h'(y)$, and then substitute this result along with $N(x, y) = x \cos y - 2y$ into (*5.5*). Thus we find

$$x \cos y + h'(y) = x \cos y - 2y \qquad \text{or} \qquad h'(y) = -2y$$

from which it follows that $h(y) = -y^2 + c_1$. Substituting this $h(y)$ into (*1*), we obtain

$$g(x, y) = \frac{1}{2}x^2 + x \sin y - y^2 + c_1$$

The solution of the differential equation is given implicitly by (*5.6*) as

$$\frac{1}{2}x^2 + x \sin y - y^2 = c_2 \quad (c_2 = c - c_1)$$

5.6. Solve $y' = \dfrac{2 + ye^{xy}}{2y - xe^{xy}}$.

Rewriting this equation in differential form, we obtain

$$(2 + ye^{xy})\, dx + (xe^{xy} - 2y)\, dy = 0$$

Here, $M(x, y) = 2 + ye^{xy}$ and $N(x, y) = xe^{xy} - 2y$ and, since $\partial M/\partial y = \partial N/\partial x = e^{xy} + xye^{xy}$, the differential equation is exact. Substituting $M(x, y)$ into (*5.4*), we find $\partial g/\partial x = 2 + ye^{xy}$; then integrating with respect to x, we obtain

$$\int \frac{\partial g}{\partial x}\, dx = \int [2 + ye^{xy}]\, dx$$

or

$$g(x, y) = 2x + e^{xy} + h(y) \tag{1}$$

To find $h(y)$, first differentiate (*1*) with respect to y, obtaining $\partial g/\partial y = xe^{xy} + h'(y)$; then substitute this result along with $N(x, y)$ into (*5.5*) to obtain

$$xe^{xy} + h'(y) = xe^{xy} - 2y \quad \text{or} \quad h'(y) = -2y$$

It follows that $h(y) = -y^2 + c_1$. Substituting this $h(y)$ into (*1*), we obtain

$$g(x, y) = 2x + e^{xy} - y^2 + c_1$$

The solution to the differential equation is given implicitly by (*5.6*) as

$$2x + e^{xy} - y^2 = c_2 \quad (c_2 = c - c_1)$$

5.7. **Determine whether the differential equation $y^2\,dt + (2yt + 1)\,dy = 0$ is exact.**

This is an equation for the unknown function $y(t)$. In terms of the variables t and y, we have $M(t, y) = y^2$, $N(t, y) = 2yt + 1$, and

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(y^2) = 2y = \frac{\partial}{\partial t}(2yt + 1) = \frac{\partial N}{\partial t}$$

so the differential equation is exact.

5.8. **Solve the differential equation given in Problem 5.7.**

This equation was shown to be exact, so the solution procedure given by Eqs. (*5.4*) through (*5.6*), with t replacing x, is applicable. Here

$$\frac{\partial g}{\partial t} = y^2$$

Integrating both sides with respect to t, we have

$$\int \frac{\partial g}{\partial t}\,dt = \int y^2\,dt$$

or
$$g(y, t) = y^2 t + h(y) \tag{1}$$

Differentiating (*1*) with respect to y, we obtain

$$\frac{\partial g}{\partial y} = 2yt + \frac{dh}{dy}$$

Hence,
$$2yt + \frac{dh}{dy} = 2yt + 1$$

where the right side of this last equation is the coefficient of dy in the original differential equation. It follows that

$$\frac{dh}{dy} = 1$$

$h(y) = y + c_1$, and (*1*) becomes $g(t, y) = y^2 t + y + c_1$. The solution to the differential equation is given implicitly by (*5.6*) as

$$y^2 t + y = c_2 \qquad (c_2 = c - c_1) \tag{2}$$

We can solve for y explicitly with the quadratic formula, whence

$$y = \frac{-1 \pm \sqrt{1 + 4c_2 t}}{2t}$$

5.9. **Determine whether the differential equation**

$$(2x^2 t - 2x^3)\,dt + (4x^3 - 6x^2 t + 2xt^2)\,dx = 0$$

is exact.

This is an equation for the unknown function $x(t)$. In terms of the variables t and x, we find

$$\frac{\partial}{\partial x}(2x^2 t - 2x^3) = 4xt - 6x^2 = \frac{\partial}{\partial t}(4x^3 - 6x^2 t + 2xt^2)$$

so the differential equation is exact.

5.10. Solve the differential equation given in Problem 5.9.

This equation was shown to be exact, so the solution procedure given by Eqs. (*5.4*) through (*5.6*), with t and x replacing x and y, respectively, is applicable. We seek a function $g(t, x)$ having the property that dg is the right side of the given differential equation. Here

$$\frac{\partial g}{\partial t} = 2x^2t - 2x^3$$

Integrating both sides with respect to t, we have

$$\int \frac{\partial g}{\partial t}\,dt = \int (2x^2t - 2x^3)\,dt$$

or

$$g(x, t) = x^2t - 2x^3t + h(x) \tag{1}$$

Differentiating (*1*) with respect to x, we obtain

$$\frac{\partial g}{\partial x} = 2xt^2 - 6x^2t + \frac{dh}{dx}$$

Hence,

$$2xt^2 - 6x^2t + \frac{dh}{dx} = 4x^3 - 6x^2t + 2xt^2$$

where the right side of this last equation is the coefficient of dx in the original differential equation. It follows that

$$\frac{dh}{dx} = 4x^3$$

Now $h(x) = x^4 + c_1$, and (*1*) becomes

$$g(t, x) = x^2t^2 - 2x^3t + x^4 + c_1 = (x^2 - xt)^2 + c_1$$

The solution to the differential equation is given implicitly by (*5.6*) as

$$(x^2 - xt)^2 = c_2 \qquad (c_2 = c - c_1)$$

or, by taking the square roots of both sides of this last equation, as

$$x^2 - xt = c_3 \quad c_3 = \pm\sqrt{c_2} \tag{2}$$

We can solve for x explicitly with the quadratic formula, whence

$$x = \frac{t \pm \sqrt{t^2 + 4c_3}}{2}$$

5.11. Solve $y' = \dfrac{-2xy}{1 + x^2}$; $y(2) = -5$.

The differential equation has the differential form given in Problem 5.1. Its solution is given in (2) of Problem 5.2 as $x^2y + y = c_2$. Using the initial condition, $y = -5$ when $x = 2$, we obtain $(2)^2(-5) + (-5) = c_2$, or $c_2 = -25$. The solution to the initial-value problem is therefore $x^2y + y = -25$ or $y = -25/(x^2 + 1)$.

5.12. Solve $\dot{y} = \dfrac{-y^2}{2yt + 1}$; $y(1) = -2$.

This differential equation in standard form has the differential form of Problem 5.7. Its solution is given in (2) of Problem 5.8 as $y^2t + y = c_2$. Using the initial condition $y = -2$ when $t = 1$, we obtain $(-2)^2(1) + (-2) = c_2$, or $c_2 = 2$.

The solution to the initial-value problem is $y^2t + y = 2$, in implicit form. Solving for y directly, using the quadratic formula, we have

$$y = \frac{-1 - \sqrt{1+8t}}{2t}$$

where the negative sign in front of the radical was chosen to be consistent with the given initial condition.

5.13. Solve $\dot{x} = \dfrac{2x^2(x-t)}{4x^3 - 6x^2t + 2xt^2}$; $x(2) = 3$.

This differential equation in standard form has the differential form of Problem 5.9. Its solution is given in (*2*) of Problem 5.10 as $x^2 - xt = c_3$. Using the initial condition $x = 3$ when $t = 2$, we obtain $(3)^2 - 3(2) = c_3$, or $c_3 = 3$. The solution to the initial-value problem is $x^2 + xt = 3$, in implicit form. Solving for x directly, using the quadratic formula, we have

$$x = \frac{1}{2}(t + \sqrt{t^2 + 12})$$

where the positive sign in front of the radical was chosen to be consistent with the given initial condition.

5.14. Determine whether $-1/x^2$ is an integrating factor for the differential equation $y\,dx - x\,dy = 0$.

It was shown in Problem 5.3 that the differential equation is not exact. Multiplying it by $-1/x^2$, we obtain

$$\frac{-1}{x^2}(y\,dx - x\,dy) = 0 \quad \text{or} \quad \frac{-y}{x^2}dx + \frac{1}{x}dy = 0 \tag{1}$$

Equation (*1*) has the form of Eq. (*5.1*) with $M(x, y) = -y/x^2$ and $N(x, y) = 1/x$. Now

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{-y}{x^2}\right) = \frac{-1}{x^2} = \frac{\partial}{\partial x}\left(\frac{1}{x}\right) = \frac{\partial N}{\partial x}$$

so (*1*) is exact, which implies that $-1/x^2$ is an integrating factor for the original differential equation.

5.15. Solve $y\,dx - x\,dy = 0$.

Using the results of Problem 5.14, we can rewrite the given differential equation as

$$\frac{x\,dy - y\,dx}{x^2} = 0$$

which is exact. Equation (*1*) can be solved using the steps described in Eqs. (*5.4*) through (*5.6*).

Alternatively, we note from Table 5-1 that (*1*) can be rewritten as $d(y/x) = 0$. Hence, by direct integration, we have $y/x = c$, or $y = cx$, as the solution.

5.16. Determine whether $-1/(xy)$ is also an integrating factor for the differential equation defined in Problem 5.14.

Multiplying the differential equation $y\,dx - x\,dy = 0$ by $-1/(xy)$, we obtain

$$\frac{-1}{xy}(y\,dx - x\,dy) = 0 \quad \text{or} \quad -\frac{1}{x}dx + \frac{1}{y}dy = 0 \tag{1}$$

Equation (*1*) has the form of Eq. (*5.1*) with $M(x, y) = -1/x$ and $N(x, y) = 1/y$. Now

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{1}{x}\right) = 0 = \frac{\partial}{\partial x}\left(\frac{1}{y}\right) = \frac{\partial N}{\partial x}$$

so (*1*) is exact, which implies that $-1/xy$ is also an integrating factor for the original differential equation.

5.17. Solve Problem 5.15 using the integrating factor given in Problem 5.16.

Using the results of Problem 5.16, we can rewrite the given differential equation as

$$\frac{x\,dy - y\,dx}{xy} = 0 \tag{1}$$

which is exact. Equation (*1*) can be solved using the steps described in Eqs. (*5.4*) through (*5.6*).

Alternatively, we note from Table 5-1 that (*1*) can be rewritten as $d[\ln(y/x)] = 0$. Then, by direct integration, $\ln(y/x) = c_1$. Taking the exponential of both sides, we find $y/x = e^{c_1}$, or finally,

$$y = cx \quad (c = e^{c_1})$$

5.18. Solve $(y^2 - y)\,dx + x\,dy = 0$.

This differential equation is not exact, and no integrating factor is immediately apparent. Note, however, that if terms are strategically regrouped, the differential equation can be rewritten as

$$-(y\,dx - x\,dy) + y^2\,dx = 0 \tag{1}$$

The group of terms in parentheses has many integrating factors (see Table 5-1). Trying each integrating factor separately, we find that the only one that makes the entire equation exact is $I(x, y) = 1/y^2$. Using this integrating factor, we can rewrite (*1*) as

$$-\frac{y\,dx - x\,dy}{y^2} + 1\,dx = 0 \tag{2}$$

Since (2) is exact, it can be solved using the steps described in Eqs. (*5.4*) through (*5.6*).

Alternatively, we note from Table 5-1 that (2) can be rewritten as $-d(x/y) + 1dx = 0$, or as $d(x/y) = 1dx$. Integrating, we obtain the solution

$$\frac{x}{y} = x + c \quad \text{or} \quad y = \frac{x}{x + c}$$

5.19. Solve $(y - xy^2)\,dx + (x + x^2y^2)\,dy = 0$.

This differential equation is not exact, and no integrating factor is immediately apparent. Note, however, that the differential equation can be rewritten as

$$(y\,dx + x\,dy) + (-xy^2\,dx + x^2y^2\,dy) = 0 \tag{1}$$

The first group of terms has many integrating factors (see Table 5-1). One of these factors, namely $I(x, y) = 1/(xy)^2$, is an integrating factor for the entire equation. Multiplying (*1*) by $1/(xy)^2$, we find

$$\frac{y\,dx + x\,dy}{(xy)^2} + \frac{-xy^2\,dx + x^2y^2\,dy}{(xy)^2} = 0$$

or equivalently,

$$\frac{y\,dx + x\,dy}{(xy)^2} = \frac{1}{x}dx - 1\,dy \tag{2}$$

Since (2) is exact, it can be solved using the steps described in Eqs. (*5.4*) through (*5.6*).

Alternatively, we note from Table 5-1

$$\frac{y\,dx + x\,dy}{(xy)^2} = d\left(\frac{-1}{xy}\right)$$

so that (2) can be rewritten as

$$d\left(\frac{-1}{xy}\right) = \frac{1}{x}dx - 1\,dy$$

Integrating both sides of this last equation, we find

$$\frac{-1}{xy} = \ln|x| - y + c$$

which is the solution in implicit form.

5.20. Solve $y' = \dfrac{3yx^2}{x^3 + 2y^4}$.

Rewriting this equation in differential form, we have

$$(3yx^2)\,dx + (-x^3 - 2y^4)\,dy = 0$$

which is not exact. Furthermore, no integrating factor is immediately apparent. We can, however, rearrange this equation as

$$x^2(3y\,dx - x\,dy) - 2y^4\,dy = 0 \tag{1}$$

The group in parentheses is of the form $ay\,dx + bx\,dy$, where $a = 3$ and $b = -1$, which has an integrating factor x^2y^{-2}. Since the expression in parentheses is already multiplied by x^2, we try an integrating factor of the form $I(x, y) = y^{-2}$. Multiplying (*1*) by y^{-2}, we have

$$x^2y^{-2}(3y\,dx - x\,dy) - 2y^2\,dy = 0$$

which can be simplified (see Table 5-1) to

$$d(x^3y^{-1}) = 2y^2\,dy \tag{2}$$

Integrating both sides of (*2*), we obtain

$$x^3y^{-1} = \frac{2}{3}y^3 + c$$

as the solution in implicit form.

5.21. Convert $y' = 2xy - x$ into an exact differential equation.

Rewriting this equation in differential form, we have

$$(-2xy + x)dx + dy = 0 \tag{1}$$

Here $M(x, y) = -2xy + x$ and $N(x, y) = 1$. Since

$$\frac{\partial M}{\partial y} = -2x \quad \text{and} \quad \frac{\partial N}{\partial x} = 0$$

are not equal, (*1*) is not exact. But

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{(-2x) - (0)}{1} = -2x$$

is a function of x alone. Using Eq. (*5.8*), we have $I(x, y) = e^{\int -2x\,dx} = e^{-x^2}$ as an integrating factor. Multiplying (*1*) by e^{-x^2}, we obtain

$$(-2xye^{-x^2} + xe^{-x^2})\,dx + e^{-x^2}\,dy = 0 \tag{2}$$

which is exact.

5.22. Convert $y^2\,dx + xy\,dy = 0$ into an exact differential equation.

Here $M(x, y) = y^2$ and $N(x, y) = xy$. Since

$$\frac{\partial M}{\partial y} = 2y \quad \text{and} \quad \frac{\partial N}{\partial x} = y$$

are not equal, (*1*) is not exact. But

$$\frac{1}{M}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{2y-y}{y^2}=\frac{1}{y}$$

is a function of y alone. Using Eq. (*5.9*), we have as an integrating factor $I(x, y) = e^{-\int(1/y)dy} = e^{-\ln y} = 1/y$. Multiplying the given differential equation by $I(x, y) = 1/y$, we obtain the exact equation $y\,dx + x\,dy = 0$.

5.23. **Convert** $y' = \dfrac{xy^2 - y}{x}$ **into an exact differential equation.**

Rewriting this equation in differential form, we have

$$y(1 - xy)\,dx + x\,dy = 0 \tag{1}$$

Here $M(x, y) = y(1 - xy)$ and $N(x, y) = x$. Since

$$\frac{\partial M}{\partial y}=1-2xy \quad \text{and} \quad \frac{\partial N}{\partial x}=1$$

are not equal, (*1*) is not exact. Equation (*5.10*), however, is applicable and provides the integrating factor

$$I(x, y)=\frac{1}{x[y(1-xy)]-yx}=\frac{-1}{(xy)^2}$$

Multiplying (*1*) by $I(x, y)$, we obtain

$$\frac{xy-1}{x^2y}dx-\frac{1}{xy^2}dy=0$$

which is exact.

Supplementary Problems

In Problems 5.24 through 5.40, test whether the differential equations are exact and solve those that are.

5.24. $(y + 2xy^3)\,dx + (1 + 3x^2y^2 + x)\,dy = 0$

5.25. $(xy + 1)\,dx + (xy - 1)\,dy = 0$

5.26. $e^{x^3}(3x^2y - x^2)\,dx + e^{x^3}\,dy = 0$

5.27. $3x^2y^2\,dx + (2x^3y + 4y^3)\,dy = 0$

5.28. $y\,dx + x\,dy = 0$

5.29. $(x - y)\,dx + (x + y)\,dy = 0$

5.30. $(y \sin x + xy \cos x)\,dx + (x \sin x + 1)\,dy = 0$

5.31. $-\dfrac{y^2}{t^2}dt+\dfrac{2y}{t}dy=0$

5.32. $-\dfrac{2y}{t^3}dt+\dfrac{1}{t^2}dy=0$

5.33. $y^2\,dt + t^2\,dy = 0$

5.34. $(4t^3y^3 - 2ty)\,dt + (3t^4y^2 - t^2)\,dy = 0$

5.35. $\dfrac{ty-1}{t^2y}dt-\dfrac{1}{ty^2}dy=0$

5.36. $(t^2 - x)\,dt - t\,dx = 0$

5.37. $(t^2 + x^2)\,dt + (2tx - x)\,dx = 0$

5.38. $2xe^{2t}\,dt + (1 + e^{2t})\,dx = 0$

5.39. $\sin t \cos x\,dt - \sin x \cos t\,dx = 0$

5.40. $(\cos x + x \cos t)\,dt + (\sin t - t \sin x)\,dx = 0$

In Problems 5.41 through 5.55, find an appropriate integrating factor for each differential equation and solve.

5.41. $(y+1)\,dx - x\,dy = 0$

5.42. $y\,dx + (1-x)\,dy = 0$

5.43. $(x^2+y+y^2)\,dx - x\,dy = 0$

5.44. $(y+x^3y^3)\,dx + x\,dy = 0$

5.45. $(y+x^4y^2)\,dx + x\,dy = 0$

5.46. $(3x^2y - x^2)\,dx + dy = 0$

5.47. $dx - 2xy\,dy = 0$

5.48. $2xy\,dx + y^2\,dy = 0$

5.49. $y\,dx + 3x\,dy = 0$

5.50. $\left(2xy^2 + \dfrac{x}{y^2}\right)dx + 4x^2y\,dy = 0$

5.51. $xy^2\,dx + (x^2y^2 + x^2y)\,dy = 0$

5.52. $xy^2\,dx + x^2y\,dy = 0$

5.53. $(y+x^3+xy^2)\,dx - x\,dy = 0$

5.54. $(x^3y^2 - y)\,dx + (x^2y^4 - x)\,dy = 0$

5.55. $3x^2y^2\,dx + (2x^3y + x^3y^4)\,dy = 0$

In Problems 5.56 through 5.65, solve the initial-value problems.

5.56. Problem 5.10 with $x(0) = 2$

5.57. Problem 5.10 with $x(2) = 0$

5.58. Problem 5.10 with $x(1) = -5$

5.59. Problem 5.24 with $y(1) = -5$

5.60. Problem 5.26 with $y(0) = -1$

5.61. Problem 5.31 with $y(0) = -2$

5.62. Problem 5.31 with $y(2) = -2$

5.63. Problem 5.32 with $y(2) = -2$

5.64. Problem 5.36 with $x(1) = 5$

5.65. Problem 5.38 with $x(1) = -2$

Solved Problems

6.1. Find an integrating factor for $y' - 3y = 6$.

The differential equation has the form of Eq. (*6.1*), with $p(x) = -3$ and $q(x) = 6$, and is linear. Here

$$\int p(x)\,dx = \int -3\,dx = -3x$$

so (*6.2*) becomes

$$I(x) = e^{\int p(x)\,dx} = e^{-3x} \tag{1}$$

6.2. Solve the differential equation in the previous problem.

Multiplying the differential equation by the integrating factor defined by (*1*) of Problem 6.1, we obtain

$$e^{-3x}y' - 3e^{-3x}y = 6e^{-3x} \quad \text{or} \quad \frac{d}{dx}(ye^{-3x}) = 6e^{-3x}$$

Integrating both sides of this last equation with respect to x, we have

$$\int \frac{d}{dx}(ye^{-3x})\,dx = \int 6e^{-3x}dx$$
$$ye^{-3x} = -2e^{-3x} + c$$
$$y = ce^{3x} - 2$$

6.3. Find an integrating factor for $y' - 2xy = x$.

The differential equation has the form of Eq. (*6.1*), with $p(x) = -2x$ and $q(x) = x$, and is linear. Here

$$\int p(x)\,dx = \int(-2x)\,dx = -x^2$$

so (*6.2*) becomes

$$I(x) = e^{\int p(x)dx} = e^{-x^2} \tag{1}$$

6.4. Solve the differential equation in the previous problem.

Multiplying the differential equation by the integrating factor defined by (*1*) of Problem 6.3, we obtain

$$e^{-x^2}y' - 2xe^{-x^2}y = xe^{-x^2} \quad \text{or} \quad \frac{d}{dx}[ye^{-x^2}] = xe^{-x^2}$$

Integrating both sides of this last equation with respect to x, we find that

$$\int \frac{d}{dx}(ye^{-x^2})\,dx = \int xe^{-x^2}\,dx$$
$$ye^{-x^2} = -\frac{1}{2}e^{-x^2} + c$$
$$y = ce^{x^2} - \frac{1}{2}$$

6.5. Find an integrating factor for $y' + (4/x)y = x^4$.

The differential equation has the form of Eq. (*6.1*), with $p(x) = 4/x$ and $q(x) = x^4$, and is linear. Here

$$\int p(x)\,dx = \int \frac{4}{x}\,dx = 4\ln|x| = \ln x^4$$

so (*6.2*) becomes

$$I(x) = e^{\int p(x)\,dx} = e^{\ln x^4} = x^4 \tag{1}$$

6.6. Solve the differential equation in the previous problem.

Multiplying the differential equation by the integrating factor defined by (*1*) of Problem 6.5, we obtain

$$x^4 y' + 4x^3 y = x^8 \quad \text{or} \quad \frac{d}{dx}(yx^4) = x^8$$

Integrating both sides of this last equation with respect to x, we obtain

$$yx^4 = \frac{1}{9}x^9 + c \quad \text{or} \quad y = \frac{c}{x^4} + \frac{1}{9}x^5$$

6.7. Solve $y' + y = \sin x$.

Here $p(x) = 1$; hence $I(x) = e^{\int 1\,dx} = e^x$. Multiplying the differential equation by $I(x)$, we obtain

$$e^x y' + e^x y = e^x \sin x \quad \text{or} \quad \frac{d}{dx}(ye^x) = e^x \sin x$$

Integrating both sides of the last equation with respect to x (to integrate the right side, we use integration by parts twice), we find

$$ye^x = \frac{1}{2}e^x(\sin x - \cos x) + c \quad \text{or} \quad y = ce^{-x} + \frac{1}{2}\sin x - \frac{1}{2}\cos x$$

6.8. Solve the initial-value problem $y' + y = \sin x$; $y(\pi) = 1$.

From Problem 6.7, the solution to the differential equation is

$$y = ce^{-x} + \frac{1}{2}\sin x - \frac{1}{2}\cos x$$

Applying the initial condition directly, we obtain

$$1 = y(\pi) = ce^{-\pi} + \frac{1}{2} \quad \text{or} \quad c = \frac{1}{2}e^{\pi}$$

Thus

$$y = \frac{1}{2}e^{\pi}e^{-x} + \frac{1}{2}\sin x - \frac{1}{2}\cos x = \frac{1}{2}(e^{\pi - x} + \sin x - \cos x)$$

6.9. Solve $y' - 5y = 0$.

Here $p(x) = -5$ and $I(x) = e^{\int(-5)\,dx} = e^{-5x}$. Multiplying the differential equation by $I(x)$, we obtain

$$e^{-5x}y' - 5e^{-5x}y = 0 \quad \text{or} \quad \frac{d}{dx}(ye^{-5x}) = 0$$

Integrating, we obtain $ye^{-5x} = c$ or $y = ce^{5x}$.

Note that the differential equation is also separable. (See Problem 4.4.)

6.10. Solve $\dfrac{dz}{dx} - xz = -x$.

This is a linear differential equation for the unknown function $z(x)$. It has the form of Eq. (*6.1*) with y replaced by z and $p(x) = q(x) = -x$. The integrating factor is

$$I(x) = e^{\int(-x)dx} = e^{-x^2/2}$$

Multiplying the differential equation by $I(x)$, we obtain

$$e^{-x^2/2}\frac{dz}{dx} - xe^{-x^2/2}z = -xe^{-x^2/2}$$

or
$$\frac{d}{dx}(ze^{-x^2/2}) = -xe^{-x^2/2}$$

Upon integrating both sides of this last equation, we have

$$ze^{-x^2/2} = e^{-x^2/2} + c$$

whereupon
$$z(x) = ce^{x^2/2} + 1$$

6.11. Solve the initial-value problem $z' - xz = -x$; $z(0) = -4$.

The solution to this differential equation is given in Problem 6.10 as

$$z(x) = 1 + ce^{x^2/2}$$

Applying the initial condition directly, we have

$$-4 = z(0) = 1 + ce^0 = 1 + c$$

or $c = -5$. Thus,

$$z(x) = 1 - 5e^{x^2/2}$$

6.12. Solve $z' - \frac{2}{x}z = \frac{2}{3}x^4$.

This is a linear differential equation for the unknown function $z(x)$. It has the form of Eq. (*6.1*) with y replaced by z. The integrating factor is

$$I(x) = e^{\int(-2/x)dx} = e^{-2\ln|x|} = e^{\ln x^{-2}} = x^{-2}$$

Multiplying the differential equation by $I(x)$, we obtain

$$x^{-2}z' - 2x^{-3}z = \frac{2}{3}x^2$$

or
$$\frac{d}{dx}(x^{-2}z) = \frac{2}{3}x^2$$

Upon integrating both sides of this last equation, we have

$$x^{-2}z = \frac{2}{9}x^3 + c$$

whereupon
$$z(x) = cx^2 + \frac{2}{9}x^5$$

6.13. Solve $\frac{dQ}{dt} + \frac{2}{10+2t}Q = 4$.

This is a linear differential equation for the unknown function $Q(t)$. It has the form of Eq. (*6.1*) with y replaced by Q, x replaced by t, $p(t) = 2/(10 + 2t)$, and $q(t) = 4$. The integrating factor is

$$I(t) = e^{\int[2/(10+2t)]\,dt} = e^{\ln[10+2t]} = 10 + 2t \quad (t > -5)$$

Multiplying the differential equation by $I(t)$, we obtain

$$(10+2t)\frac{dQ}{dt}+2Q=40+8t$$

or
$$\frac{d}{dt}[(10+2t)Q]=40+8t$$

Upon integrating both sides of this last equation, we have

$$(10+2t)Q=40t+4t^2+c$$

whereupon
$$Q(t)=\frac{40t+4t^2+c}{10+2t}\quad(t>-5)$$

6.14. Solve the initial-value problem $\dfrac{dQ}{dt}+\dfrac{2}{10+2t}Q=4;\ Q(2)=100.$

The solution to this differential equation is given in Problem 6.13 as

$$Q(t)=\frac{40t+4t^2+c}{10+2t}\quad(t>-5)$$

Applying the initial condition directly, we have

$$100=Q(2)=\frac{40(2)+4(4)+c}{10+2(2)}$$

or $c=1304$. Thus,

$$Q(t)=\frac{4t^2+40t+1304}{2t+10}\quad(t>-5)$$

6.15. Solve $\dfrac{dT}{dt}+kT=100k$, where k denotes a constant.

This is a linear differential equation for the unknown function $T(t)$. It has the form of Eq. *(6.1)* with y replaced by T, x replaced by t, $p(t)=k$, and $q(t)=100k$. The integrating factor is

$$I(t)=e^{\int k\,dt}=e^{kt}$$

Multiplying the differential equation by $I(t)$, we obtain

$$e^{kt}\frac{dT}{dt}+ke^{kt}T=100ke^{kt}$$

or
$$\frac{d}{dt}(Te^{kt})=100ke^{kt}$$

Upon integrating both sides of this last equation, we have

$$Te^{kt}=100e^{kt}+c$$

whereupon
$$T(t)=ce^{-kt}+100$$

6.16 Solve $y'+xy=xy^2$.

This equation is not linear. It is, however, a Bernoulli differential equation having the form of Eq. *(6.4)* with $p(x)=q(x)=x$, and $n=2$. We make the substitution suggested by *(6.5)*, namely, $z=y^{1-2}=y^{-1}$, from which follow

$$y=\frac{1}{z}\quad\text{and}\quad y'=-\frac{z'}{z^2}$$

Substituting these equations into the differential equation, we obtain

$$-\frac{z'}{z^2}+\frac{x}{z}=\frac{x}{z^2} \quad \text{or} \quad z'-xz=-x$$

This last equation is linear. Its solution is found in Problem 6.10 to be $z=ce^{x^2/2}+1$. The solution of the original differential equation is then

$$y=\frac{1}{z}=\frac{1}{ce^{x^2/2}+1}$$

6.17. Solve $y'-\frac{3}{4}y=x^4y^{1/3}$.

This is a Bernoulli differential equation with $p(x)=-3/x$, $q(x)=x^4$, and $n=\frac{1}{3}$. Using Eq. (*6.5*), we make the substitution $z=y^{1-(1/3)}=y^{2/3}$. Thus, $y=z^{3/2}$ and $y'=\frac{3}{2}z^{1/2}z'$. Substituting these values into the differential equation, we obtain

$$\frac{3}{2}z^{1/2}z'-\frac{3}{x}z^{3/2}=x^4z^{1/2} \quad \text{or} \quad z'-\frac{2}{x}z=\frac{2}{3}x^4$$

This last equation is linear. Its solution is found in Problem 6.12 to be $z=cx^2+\frac{2}{9}x^5$. Since $z=y^{2/3}$, the solution of the original problem is given implicitly by $y^{2/3}=cx^2+\frac{2}{9}x^5$, or explicitly by $y=\pm(cx^2+\frac{2}{9}x^5)^{3/2}$.

6.18. Show that the integrating factor found in Problem 6.1 is also an integrating factor as defined in Chapter 5 Eq. (*5.7*).

The differential equation of Problem 6.1 can be rewritten as

$$\frac{dy}{dx}=3y+6$$

which has the differential form

$$dy=(3y+6)\,dx$$

or

$$(3y+6)\,dx+(-1)\,dy=0 \tag{1}$$

Multiplying (*1*) by the integrating factor $I(x)=e^{-3x}$, we obtain

$$(3ye^{-3x}+6e^{-3x})\,dx+(-e^{-3x})\,dy=0 \tag{2}$$

Setting $\quad M(x,y)=3ye^{-3x}+6e^{-3x} \quad$ and $\quad N(x,y)=-e^{-3x}$

we have

$$\frac{\partial M}{\partial y}=3e^{-3x}=\frac{\partial N}{\partial x}$$

from which we conclude that (*2*) is an exact differential equation.

6.19. Find the general form of the solution of Eq. (*6.1*).

Multiplying (*6.1*) by (*6.2*), we have

$$e^{\int p(x)\,dx}y'+e^{\int p(x)\,dx}p(x)y=e^{\int p(x)\,dx}q(x) \tag{1}$$

Since

$$\frac{d}{dx}[e^{\int p(x)dx}]=e^{\int p(x)dx}p(x)$$

it follows from the product rule of differentiation that the left side of (1) equals $\frac{d}{dx}[e^{\int p(x)\,dx}y]$. Thus, (1) can be rewritten as

$$\frac{d}{dx}[e^{\int p(x)\,dx}y] = e^{\int p(x)\,dx}q(x) \tag{2}$$

Integrating both sides of (2) with respect to x, we have

$$\int \frac{d}{dx}[e^{\int p(x)\,dx}y]\,dx = \int e^{\int p(x)\,dx}q(x)\,dx$$

or,

$$e^{\int p(x)\,dx}y + c_1 = \int e^{\int p(x)\,dx}q(x)\,dx \tag{3}$$

Finally, setting $c_1 = -c$ and solving (3) for y, we obtain

$$y = ce^{-\int p(x)dx} + e^{-\int p(x)dx}\int e^{\int p(x)dx}q(x)\,dx \tag{4}$$

Solved Problems

In Problems 6.20 through 6.49, solve the given differential equations.

6.20. $\frac{dy}{dx} + 5y = 0$

6.21. $\frac{dy}{dx} - 5y = 0$

6.22. $\frac{dy}{dx} - 0.01y = 0$

6.23. $\frac{dy}{dx} + 2xy = 0$

6.24. $y' + 3x^2y = 0$

6.25. $y' - x^2y = 0$

6.26. $y' - 3x^4y = 0$

6.27. $y' + \frac{1}{x}y = 0$

6.28. $y' + \frac{2}{x}y = 0$

6.29. $y' - \frac{2}{x}y = 0$

6.30. $y' - \frac{2}{x^2}y = 0$

6.31. $y' - 7y = e^x$

6.32. $y' - 7y = 14x$

6.33. $y' - 7y = \sin 2x$

6.34. $y' + x^2y = x^2$

6.35. $y' - \frac{3}{x^2}y = \frac{1}{x^2}$

6.36. $y' = \cos x$

6.37. $y' + y = y^2$

6.38. $xy' + y = xy^3$

6.39. $y' + xy = 6x\sqrt{y}$

6.40. $y' + y = y^2$

6.41. $y' + y = y^{-2}$

6.42. $y' + y = y^2e^x$

6.43. $\frac{dy}{dt} + 50y = 0$

6.44. $\frac{dz}{dt} - \frac{1}{2t}z = 0$

6.45. $\frac{dN}{dt} = kN$, (k = a constant)

6.46 $\frac{dp}{dt} - \frac{1}{t}p = t^2 + 3t - 2$

6.47. $\frac{dQ}{dt} + \frac{2}{20-t}Q = 4$

6.48. $25\dfrac{dT}{dt} + T = 80e^{-0.04t}$

6.49. $\dfrac{dp}{dz} + \dfrac{2}{z}p = 4$

Solve the following initial-value problems.

6.50. $y' + \dfrac{2}{x}y = x;\ y(1) = 0$

6.51. $y' + 6xy = 0;\ y(\pi) = 5$

6.52. $y' + 2xy = 2x^3;\ y(0) = 1$

6.53. $y' + \dfrac{2}{x}y = -x^9y^5;\ y(-1) = 2$

6.54. $\dfrac{dv}{dt} + 2v = 32;\ v(0) = 0$

6.55. $\dfrac{dq}{dt} + q = 4\cos 2t;\ q(0) = 1$

6.56. $\dfrac{dN}{dt} + \dfrac{1}{t}N = t;\ N(2) = 8$

6.57. $\dfrac{dT}{dt} + 0.069T = 2.07;\ T(0) = -30$