ECON381 Fall 2024
Homework Assignment 4

1. Explain what a minimum spanning tree (MST) is and then create an MST for the above graph. You can use Kruskal's algorithm or any similar algorithm. Show all your steps. This may take some space.
2. We need to know if the MST you created is unique or not. Do some research on the conditions for uniqueness of a generated MST. Is the above graph suitable for guaranteed uniqueness?
3. Calculate the shortest paths from node A to all other nodes using Dijkstra's algorithm. Show all your steps. This may take some space.
4. Explain what a critical edge in a graph is. Then try to find critical edges in this graph. Show detailed steps for a single edge removal. If you can not find any critical edge, explain why.
5. Explain what an articulation point in a graph is. Then try to find articulation points in this graph. Show detailed steps for a single vertex removal. If you can not find any articulation point, explain why.
6. Suppose you want to go from A to E, the path you are given is A-B-C-E (based on Dijkstra's algorithm). You are at B and learn that C is now unavailable. Given that you know there are no critical edges or articulation points beforehand, can you be sure that there is now another path towards E without any calculations?
7. Do some research on the concept of graph robustness, and explain it using critical edges and articulation points.

Definition of MST:

An MST is a subset of edges of a connected, weighted, and undirected graph that:

1. Spans all vertices (connects them all).
2. Minimizes the sum of the edge weights.
3. Contains no cycles.

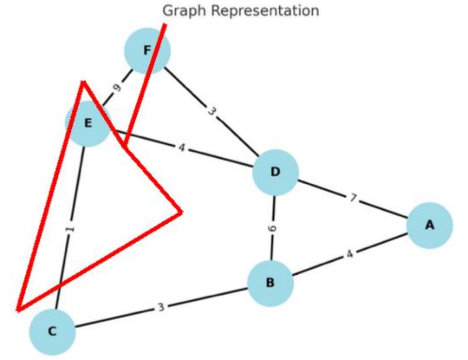

Figure 1: Reference Graph

Steps to Create MST Using Kruskal's Algorithm:

1. Sort edges in ascending order of weights.
    - Sorted edges: (E, C) - 1, (B, C) - 3, (D, F) - 3, (D, B) - 4, (E, D) - 4, (A, B) - 4, (A, D) - 7, (E, F) - 9.

2. Add edges one by one to the MST, ensuring no cycles.
    - Add (E, C): 1 → No cycle → Include.
    - Add (B, C): 3 → No cycle → Include.
    - Add (D, F): 3 → No cycle → Include.
    - Add (D, B): 4 → No cycle → Include.
    - Add (E, D): 4 → No cycle → Include.
3. Stop when we have V−1=5V-1 = 5V−1=5 edges (for 6 vertices).

MST Edges: (E, C), (B, C), (D, F), (D, B), (E, D).
Total Weight: 1 + 3 + 3 + 4 + 4 = 15.

2. Is the MST Unique?

Conditions for MST Uniqueness:

An MST is unique if **all edge weights in the graph are distinct**.

- If two or more edges have the same weight, there can be multiple MSTs, depending on which edge is chosen during the algorithm.

Application to the Graph:

- The graph has duplicate weights (e.g., (D, B) and (A, B) both have weight 4).
- Hence, the MST is **not guaranteed to be unique**. Different valid MSTs could exist depending on edge selection order.

3. Shortest Paths from Node A Using Dijkstra's Algorithm

Steps:

1. Initialize distances from AAA: dist(A)=0\text{dist}(A) = 0dist(A)=0, all others ∞\infty∞.
2. Visit the unvisited node with the smallest distance, update neighbors.
3. Repeat until all nodes are visited.

Execution:

- Start at AAA:
  Distances: A=0, B=4, others=∞.
- Visit BBB:
  Update CCC: dist(C)=4+3=7\text{dist}(C) = 4+3 = 7dist(C)=4+3=7.
  Update DDD: dist(D)=4+6=10\text{dist}(D) = 4+6 = 10dist(D)=4+6=10.
- Visit CCC:
  Update EEE: dist(E)=7+1=8\text{dist}(E) = 7+1 = 8dist(E)=7+1=8.
- Visit EEE:
  DDD: Already smaller distance.
- Visit DDD:
  Update FFF: dist(F)=10+3=13\text{dist}(F) = 10+3 = 13dist(F)=10+3=13.
- Visit FFF: Done.

Final Shortest Distances:
A=0,B=4,C=7,D=10,E=8,F=13A=0, B=4, C=7, D=10, E=8, F=13A=0,B=4,C=7,D=10,E=8,F=13.

4. What is a Critical Edge? Find Critical Edges.

Definition:

A critical edge is one whose removal increases the number of connected components in the graph, i.e., it disconnects parts of the graph.

Steps:

- Remove each edge one by one.
- Check if the graph remains connected.

Analysis for One Edge Removal (e.g., (E, C)):

- Removing (E, C) does not disconnect the graph as there is an alternative path through C−B−D−EC-B-D-EC−B−D−E.
- **Result:** No edge in this graph is critical because multiple paths exist between all vertices.

5. What is an Articulation Point? Find Articulation Points.

Definition:

An articulation point is a vertex whose removal increases the number of connected components in the graph.

Steps:

- Remove each vertex and its edges.
- Check if the graph remains connected.

Analysis for One Vertex Removal (e.g., DDD):

- Removing DDD leaves the graph connected via other paths.
- **Result:** No articulation point exists as the graph is well-connected.

6. Path from AAA to EEE with Blocked CCC.

Question:

If CCC is blocked, is there another path to EEE?

Answer:

Since there are no critical edges or articulation points, the graph is robust. Blocking CCC does not disconnect AAA and EEE.
Alternate path: A−D−EA-D-EA−D−E.

7. Graph Robustness and Its Relation to Critical Edges and Articulation Points

Definition of Robustness:

Graph robustness measures its resilience to node or edge failures. A robust graph has:

- No critical edges or articulation points.
- Multiple independent paths between vertices.

Application:

The given graph is robust:

1. No critical edges exist.
2. No articulation points exist.
3. Multiple paths exist between all vertices.

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