1062CH08

INTRODUCTION TO TRIGONOMETRY 8

There is perhaps nothing which so occupies the middle position of mathematics as trigonometry.

– J.F. Herbart (1890)

8.1 Introduction

You have already studied about triangles, and in particular, right triangles, in your earlier classes. Let us take some examples from our surroundings where right triangles can be imagined to be formed. For instance :

1. Suppose the students of a school are visiting Qutub Minar. Now, if a student is looking at the top of the Minar, a right triangle can be imagined to be made, as shown in Fig 8.1. Can the student find out the height of the Minar, without actually measuring it?

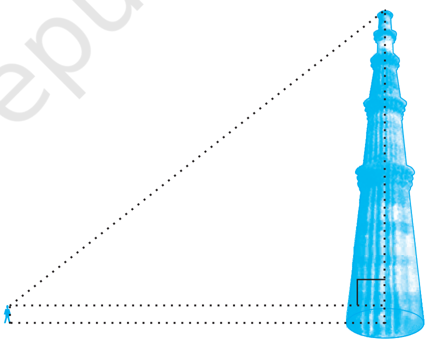


Fig. 8.1

2. Suppose a girl is sitting on the balcony of her house located on the bank of a river. She is looking down at a flower pot placed on a stair of a temple situated nearby on the other bank of the river. A right triangle is imagined to be made in this situation as shown in Fig.8.2. If you know the height at which the person is sitting, can you find the width of the river?

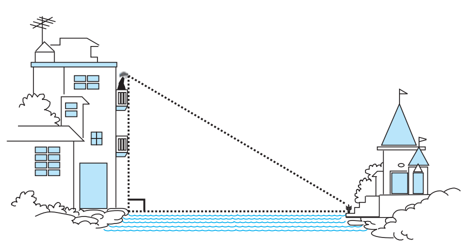


Fig. 8.2

3. Suppose a hot air balloon is flying in the air. A girl happens to spot the balloon in the sky and runs to her mother to tell her about it. Her mother rushes out of the house to look at the balloon.Now when the girl had spotted the balloon intially it was at point A. When both the mother and daughter came out to see it, it had already travelled to another point B. Can you find the altitude of B from the ground?

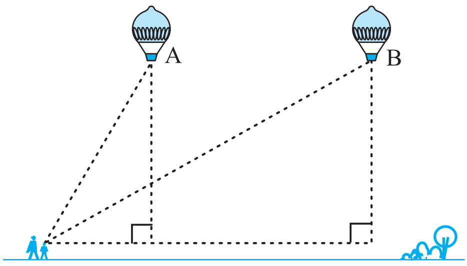


Fig. 8.3

In all the situations given above, the distances or heights can be found by using some mathematical techniques, which come under a branch of mathematics called 'trigonometry'. The word 'trigonometry' is derived from the Greek words 'tri' (meaning three), 'gon' (meaning sides) and 'metron' (meaning measure). In fact, **trigonometry** is the study of relationships between the sides and angles of a triangle. The earliest known work on trigonometry was recorded in Egypt and Babylon. Early astronomers used it to find out the distances of the stars and planets from the Earth. Even today, most of the technologically advanced methods used in Engineering and Physical Sciences are based on trigonometrical concepts.

In this chapter, we will study some ratios of the sides of a right triangle with respect to its acute angles, called **trigonometric ratios of the angle.** We will restrict our discussion to acute angles only. However, these ratios can be extended to other angles also. We will also define the trigonometric ratios for angles of measure 0° and 90°. We will calculate trigonometric ratios for some specific angles and establish some identities involving these ratios, called **trigonometric identities.**

8.2 Trigonometric Ratios

In Section 8.1, you have seen some right triangles imagined to be formed in different situations.

Let us take a right triangle ABC as shown in Fig. 8.4.

Here, ∠ CAB (or, in brief, angle A) is an acute angle. Note the position of the side BC with respect to angle A. It faces ∠ A. We call it the *side opposite* to angle A. AC is the *hypotenuse* of the right triangle and the side AB is a part of ∠ A. So, we call it the *side adjacent* to angle A.

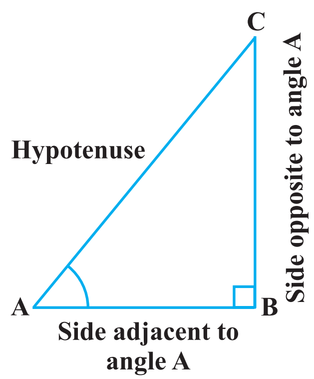


Fig. 8.4

Note that the position of sides change when you consider angle C in place of A (see Fig. 8.5).

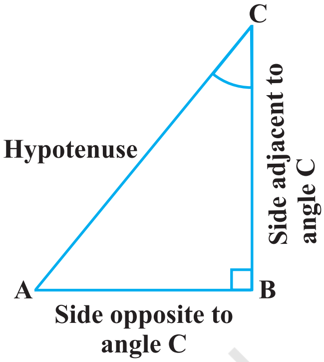


Fig. 8.5

You have studied the concept of 'ratio' in your earlier classes. We now define certain ratios involving the sides of a right triangle, and call them trigonometric ratios.

The trigonometric ratios of the angle A in right triangle ABC (see Fig. 8.4) are defined as follows :

$$\text{sine of } \angle A = \frac{\text{side opposite to angle A}}{\text{hypotenuse}} = \frac{BC}{AC}$$

$$\text{cosine of } \angle A = \frac{\text{side adjacent to angle A}}{\text{hypotenuse}} = \frac{AB}{AC}$$

$$\text{tangent of } \angle A = \frac{\text{side opposite to angle A}}{\text{side adjacent to angle A}} = \frac{BC}{AB}$$

$$\text{cosecant of } \angle A = \frac{1}{\text{sine of } \angle A} = \frac{\text{hypotenuse}}{\text{side opposite to angle A}} = \frac{AC}{BC}$$

$$\text{secant of } \angle A = \frac{1}{\text{cosine of } \angle A} = \frac{\text{hypotenuse}}{\text{side adjacent to angle A}} = \frac{AC}{AB}$$

$$\text{cotangent of } \angle A = \frac{1}{\text{tangent of } \angle A} = \frac{\text{side adjacent to angle A}}{\text{side opposite to angle A}} = \frac{AB}{BC}$$

The ratios defined above are abbreviated as sin A, cos A, tan A, cosec A, sec A and cot A respectively. Note that the ratios **cosec A, sec A and cot A** are respectively, the reciprocals of the ratios sin A, cos A and tan A.

Also, observe that $\tan A = \dfrac{BC}{AB} = \dfrac{\frac{BC}{AC}}{\frac{AB}{AC}} = \dfrac{\sin A}{\cos A}$ and $\cot A = \dfrac{\cos A}{\sin A}$.

So, the **trigonometric ratios** of an acute angle in a right triangle express the relationship between the angle and the length of its sides.

Why don't you try to define the trigonometric ratios for angle C in the right triangle? (See Fig. 8.5)

The first use of the idea of **'sine'** in the way we use it today was in the work *Aryabhatiyam* by Aryabhata, in A.D. 500. Aryabhata used the word ***ardha-jya*** for the half-chord, which was shortened to ***jya*** or ***jiva*** in due course. When the *Aryabhatiyam* was translated into Arabic, the word ***jiva*** was retained as it is. The word ***jiva*** was translated into ***sinus,*** which means curve, when the Arabic version was translated into Latin. Soon the word ***sinus***, also used as ***sine***, became common in mathematical texts throughout Europe. An English Professor of astronomy Edmund Gunter (1581–1626), first used the abbreviated notation **'*sin*'.**

Aryabhata
C.E. 476 – 550

The origin of the terms **'cosine'** and **'tangent'** was much later. The cosine function arose from the need to compute the sine of the complementary angle. Aryabhatta called it **kotijya**. The name ***cosinus*** originated with Edmund Gunter. In 1674, the English Mathematician Sir Jonas Moore first used the abbreviated notation **'*cos*'**.

Remark : Note that the symbol sin A is used as an abbreviation for 'the sine of the angle A'. sin A is *not* the product of 'sin' and A. 'sin' separated from A has no meaning. Similarly, cos A is *not* the product of 'cos' and A. Similar interpretations follow for other trigonometric ratios also.

Now, if we take a point P on the hypotenuse AC or a point Q on AC extended, of the right triangle ABC and draw PM perpendicular to AB and QN perpendicular to AB extended (see Fig. 8.6), how will the trigonometric ratios of $\angle$ A in Δ PAM differ from those of $\angle$ A in Δ CAB or from those of $\angle$ A in Δ QAN?

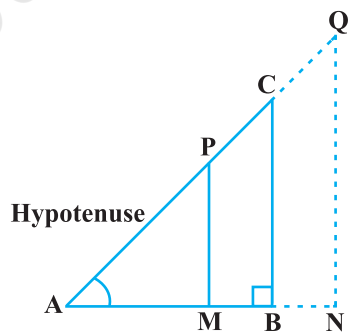


Fig. 8.6

To answer this, first look at these triangles. Is Δ PAM similar to Δ CAB? From Chapter 6, recall the AA similarity criterion. Using the criterion, you will see that the triangles PAM and CAB are similar. Therefore, by the property of similar triangles, the corresponding sides of the triangles are proportional.

So, we have
$$\frac{AM}{AB} = \frac{AP}{AC} = \frac{MP}{BC}.$$

From this, we find $\frac{\text{MP}}{\text{AP}} = \frac{\text{BC}}{\text{AC}} = \sin A$.

Similarly, $\frac{\text{AM}}{\text{AP}} = \frac{\text{AB}}{\text{AC}} = \cos A$, $\frac{\text{MP}}{\text{AM}} = \frac{\text{BC}}{\text{AB}} = \tan A$ and so on.

This shows that the trigonometric ratios of angle A in Δ PAM not differ from those of angle A in Δ CAB.

In the same way, you should check that the value of sin A (and also of other trigonometric ratios) remains the same in Δ QAN also.

From our observations, it is now clear that **the values of the trigonometric ratios of an angle do not vary with the lengths of the sides of the triangle, if the angle remains the same.**

Note : For the sake of convenience, we may write $\sin^2 A$, $\cos^2 A$, etc., in place of $(\sin A)^2$, $(\cos A)^2$, etc., respectively. But $\text{cosec}\, A = (\sin A)^{-1} \neq \sin^{-1} A$ (it is called sine inverse A). $\sin^{-1} A$ has a different meaning, which will be discussed in higher classes. Similar conventions hold for the other trigonometric ratios as well. Sometimes, the Greek letter θ (theta) is also used to denote an angle.

We have defined six trigonometric ratios of an acute angle. If we know any one of the ratios, can we obtain the other ratios? Let us see.

If in a right triangle ABC, $\sin A = \frac{1}{3}$, then this means that $\frac{\text{BC}}{\text{AC}} = \frac{1}{3}$, i.e., the lengths of the sides BC and AC of the triangle ABC are in the ratio 1 : 3 (see Fig. 8.7). So if BC is equal to k, then AC will be $3k$, where k is any positive number. To determine other trigonometric ratios for the angle A, we need to find the length of the third side AB. Do you remember the Pythagoras theorem? Let us use it to determine the required length AB.

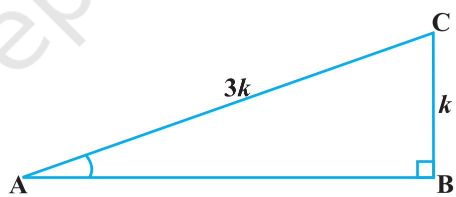


Fig. 8.7

$$\text{AB}^2 = \text{AC}^2 - \text{BC}^2 = (3k)^2 - (k)^2 = 8k^2 = (2\sqrt{2}\,k)^2$$

Therefore, $\text{AB} = \pm 2\sqrt{2}\,k$

So, we get $\text{AB} = 2\sqrt{2}\,k$ (Why is AB not $-2\sqrt{2}\,k$?)

Now, $$\cos A = \frac{\text{AB}}{\text{AC}} = \frac{2\sqrt{2}\,k}{3k} = \frac{2\sqrt{2}}{3}$$

Similarly, you can obtain the other trigonometric ratios of the angle A.

Remark : Since the hypotenuse is the longest side in a right triangle, the value of sin A or cos A is always less than 1 (or, in particular, equal to 1).

Let us consider some examples.

Example 1 : Given tan A = $\frac{4}{3}$, find the other trigonometric ratios of the angle A.

Solution : Let us first draw a right Δ ABC (see Fig 8.8).

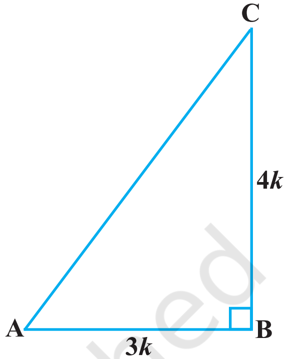


Fig. 8.8

Now, we know that $\tan A = \frac{BC}{AB} = \frac{4}{3}$.

Therefore, if BC = 4k, then AB = 3k, where k is a positive number.

Now, by using the Pythagoras Theorem, we have

$$AC^2 = AB^2 + BC^2 = (4k)^2 + (3k)^2 = 25k^2$$

So, $$AC = 5k$$

Now, we can write all the trigonometric ratios using their definitions.

$$\sin A = \frac{BC}{AC} = \frac{4k}{5k} = \frac{4}{5}$$

$$\cos A = \frac{AB}{AC} = \frac{3k}{5k} = \frac{3}{5}$$

Therefore, $\cot A = \frac{1}{\tan A} = \frac{3}{4}$, $\text{cosec } A = \frac{1}{\sin A} = \frac{5}{4}$ and $\sec A = \frac{1}{\cos A} = \frac{5}{3}$.

Example 2 : If ∠ B and ∠ Q are acute angles such that sin B = sin Q, then prove that ∠ B = ∠ Q.

Solution : Let us consider two right triangles ABC and PQR where sin B = sin Q (see Fig. 8.9).

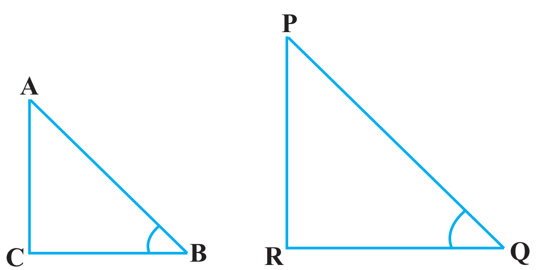


Fig. 8.9

We have $$\sin B = \frac{AC}{AB}$$

and $$\sin Q = \frac{PR}{PQ}$$

Then $$\frac{AC}{AB} = \frac{PR}{PQ}$$

Therefore, $$\frac{AC}{PR} = \frac{AB}{PQ} = k, \text{ say} \qquad (1)$$

Now, using Pythagoras theorem,

$$BC = \sqrt{AB^2 - AC^2}$$

and $$QR = \sqrt{PQ^2 - PR^2}$$

So, $$\frac{BC}{QR} = \frac{\sqrt{AB^2 - AC^2}}{\sqrt{PQ^2 - PR^2}} = \frac{\sqrt{k^2PQ^2 - k^2PR^2}}{\sqrt{PQ^2 - PR^2}} = \frac{k\sqrt{PQ^2 - PR^2}}{\sqrt{PQ^2 - PR^2}} = k \qquad (2)$$

From (1) and (2), we have

$$\frac{AC}{PR} = \frac{AB}{PQ} = \frac{BC}{QR}$$

Then, by using Theorem 6.4, Δ ACB ~ Δ PRQ and therefore, $\angle$ B = $\angle$ Q.

Example 3 : Consider Δ ACB, right-angled at C, in which AB = 29 units, BC = 21 units and $\angle$ ABC = θ (see Fig. 8.10). Determine the values of

(i) $\cos^2 \theta + \sin^2 \theta$,

(ii) $\cos^2 \theta - \sin^2 \theta$.

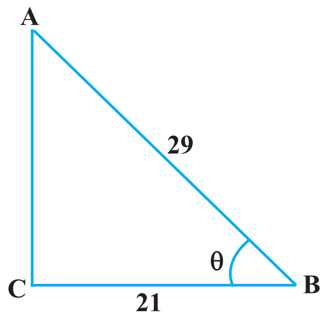


Fig. 8.10

Solution : In Δ ACB, we have

$$AC = \sqrt{AB^2 - BC^2} = \sqrt{(29)^2 - (21)^2}$$

$$= \sqrt{(29-21)(29+21)} = \sqrt{(8)(50)} = \sqrt{400} = 20 \text{ units}$$

So, $$\sin\theta = \frac{AC}{AB} = \frac{20}{29}, \cos\theta = \frac{BC}{AB} = \frac{21}{29}.$$

Now, (i) $$\cos^2\theta + \sin^2\theta = \left(\frac{20}{29}\right)^2 + \left(\frac{21}{29}\right)^2 = \frac{20^2 + 21^2}{29^2} = \frac{400+441}{841} = 1,$$

and (ii) $$\cos^2\theta - \sin^2\theta = \left(\frac{21}{29}\right)^2 - \left(\frac{20}{29}\right)^2 = \frac{(21+20)(21-20)}{29^2} = \frac{41}{841}.$$

Example 4 : In a right triangle ABC, right-angled at B, if tan A = 1, then verify that

2 sin A cos A = 1.

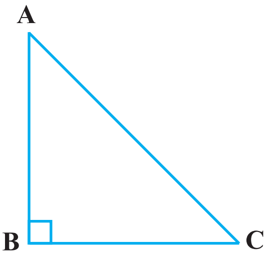


Fig. 8.11

Solution : In Δ ABC, $\tan A = \frac{BC}{AB} = 1$ (see Fig 8.11)

i.e., $BC = AB$

Let AB = BC = k, where k is a positive number.

Now, $AC = \sqrt{AB^2 + BC^2}$

$= \sqrt{(k)^2 + (k)^2} = k\sqrt{2}$

Therefore, $\sin A = \frac{BC}{AC} = \frac{1}{\sqrt{2}}$ and $\cos A = \frac{AB}{AC} = \frac{1}{\sqrt{2}}$

So, $2 \sin A \cos A = 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = 1$, which is the required value.

Example 5 : In Δ OPQ, right-angled at P, OP = 7 cm and OQ – PQ = 1 cm (see Fig. 8.12). Determine the values of sin Q and cos Q.

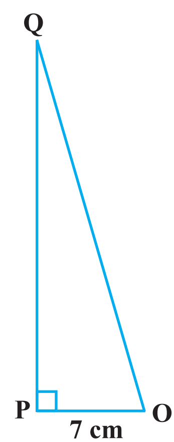


Fig. 8.12

Solution : In Δ OPQ, we have

$OQ^2 = OP^2 + PQ^2$

i.e., $(1 + PQ)^2 = OP^2 + PQ^2$ (Why?)

i.e., $1 + PQ^2 + 2PQ = OP^2 + PQ^2$

i.e., $1 + 2PQ = 7^2$ (Why?)

i.e., $PQ = 24$ cm and $OQ = 1 + PQ = 25$ cm

So, $\sin Q = \frac{7}{25}$ and $\cos Q = \frac{24}{25}$.

EXERCISE 8.1

1. In Δ ABC, right-angled at B, AB = 24 cm, BC = 7 cm. Determine :

(i) sin A, cos A

(ii) sin C, cos C

2. In Fig. 8.13, find tan P – cot R.

3. If $\sin A = \frac{3}{4}$, calculate cos A and tan A.

4. Given 15 cot A = 8, find sin A and sec A.

5. Given $\sec \theta = \frac{13}{12}$, calculate all other trigonometric ratios.

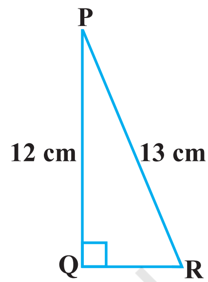


Fig. 8.13

6. If $\angle$ A and $\angle$ B are acute angles such that cos A = cos B, then show that $\angle$ A = $\angle$ B.

7. If $\cot \theta = \frac{7}{8}$, evaluate : (i) $\frac{(1 + \sin \theta)(1 - \sin \theta)}{(1 + \cos \theta)(1 - \cos \theta)}$, (ii) $\cot^2 \theta$

8. If 3 cot A = 4, check whether $\frac{1 - \tan^2 A}{1 + \tan^2 A} = \cos^2 A - \sin^2 A$ or not.

9. In triangle ABC, right-angled at B, if $\tan A = \frac{1}{\sqrt{3}}$, find the value of:

(i) sin A cos C + cos A sin C

(ii) cos A cos C – sin A sin C

10. In Δ PQR, right-angled at Q, PR + QR = 25 cm and PQ = 5 cm. Determine the values of sin P, cos P and tan P.

11. State whether the following are true or false. Justify your answer.

(i) The value of tan A is always less than 1.

(ii) $\sec A = \frac{12}{5}$ for some value of angle A.

(iii) cos A is the abbreviation used for the cosecant of angle A.

(iv) cot A is the product of cot and A.

(v) $\sin \theta = \frac{4}{3}$ for some angle θ.

8.3 Trigonometric Ratios of Some Specific Angles

From geometry, you are already familiar with the construction of angles of 30°, 45°, 60° and 90°. In this section, we will find the values of the trigonometric ratios for these angles and, of course, for 0°.

Trigonometric Ratios of 45°

In Δ ABC, right-angled at B, if one angle is 45°, then the other angle is also 45°, i.e., ∠ A = ∠ C = 45° (see Fig. 8.14).

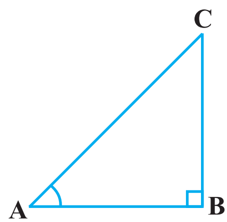


Fig. 8.14

So, BC = AB (Why?)

Now, Suppose BC = AB = a.

Then by Pythagoras Theorem, $AC^2 = AB^2 + BC^2 = a^2 + a^2 = 2a^2$,

and, therefore, $AC = a\sqrt{2}$.

Using the definitions of the trigonometric ratios, we have :

$$\sin 45° = \frac{\text{side opposite to angle } 45°}{\text{hypotenuse}} = \frac{BC}{AC} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\cos 45° = \frac{\text{side adjacent to angle } 45°}{\text{hypotenuse}} = \frac{AB}{AC} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\tan 45° = \frac{\text{side opposite to angle } 45°}{\text{side adjacent to angle } 45°} = \frac{BC}{AB} = \frac{a}{a} = 1$$

Also, $\operatorname{cosec} 45° = \frac{1}{\sin 45°} = \sqrt{2}$, $\sec 45° = \frac{1}{\cos 45°} = \sqrt{2}$, $\cot 45° = \frac{1}{\tan 45°} = 1$.

Trigonometric Ratios of 30° and 60°

Let us now calculate the trigonometric ratios of 30° and 60°. Consider an equilateral triangle ABC. Since each angle in an equilateral triangle is 60°, therefore, ∠ A = ∠ B = ∠ C = 60°.

Draw the perpendicular AD from A to the side BC (see Fig. 8.15).

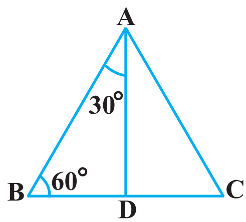


Fig. 8.15

Now Δ ABD ≅ Δ ACD (Why?)

Therefore, BD = DC

and ∠ BAD = ∠ CAD (CPCT)

Now observe that:

Δ ABD is a right triangle, right-angled at D with ∠ BAD = 30° and ∠ ABD = 60° (see Fig. 8.15).

As you know, for finding the trigonometric ratios, we need to know the lengths of the sides of the triangle. So, let us suppose that AB = $2a$.

Then, $$\text{BD} = \frac{1}{2}\text{BC} = a$$

and $$\text{AD}^2 = \text{AB}^2 - \text{BD}^2 = (2a)^2 - (a)^2 = 3a^2,$$

Therefore, $$\text{AD} = a\sqrt{3}$$

Now, we have :

$$\sin 30° = \frac{\text{BD}}{\text{AB}} = \frac{a}{2a} = \frac{1}{2}, \cos 30° = \frac{\text{AD}}{\text{AB}} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}$$

$$\tan 30° = \frac{\text{BD}}{\text{AD}} = \frac{a}{a\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

Also, $$\text{cosec } 30° = \frac{1}{\sin 30°} = 2, \quad \sec 30° = \frac{1}{\cos 30°} = \frac{2}{\sqrt{3}}$$

$$\cot 30° = \frac{1}{\tan 30°} = \sqrt{3}.$$

Similarly,

$$\sin 60° = \frac{\text{AD}}{\text{AB}} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}, \cos 60° = \frac{1}{2}, \tan 60° = \sqrt{3},$$

$$\text{cosec } 60° = \frac{2}{\sqrt{3}}, \sec 60° = 2 \text{ and } \cot 60° = \frac{1}{\sqrt{3}}.$$

Trigonometric Ratios of 0° and 90°

Let us see what happens to the trigonometric ratios of angle A, if it is made smaller and smaller in the right triangle ABC (see Fig. 8.16), till it becomes zero. As $\angle$ A gets smaller and smaller, the length of the side BC decreases. The point C gets closer to point B, and finally when $\angle$ A becomes very close to 0°, AC becomes almost the same as AB (see Fig. 8.17).

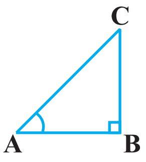


Fig. 8.16

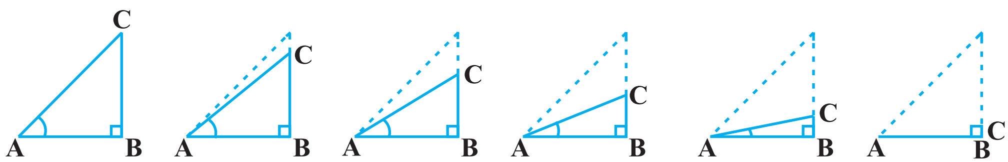


Fig. 8.17

When $\angle$ A is very close to 0°, BC gets very close to 0 and so the value of $\sin A = \frac{BC}{AC}$ is very close to 0. Also, when $\angle$ A is very close to 0°, AC is nearly the same as AB and so the value of $\cos A = \frac{AB}{AC}$ is very close to 1.

This helps us to see how we can define the values of sin A and cos A when A = 0°. We define : **sin 0° = 0 and cos 0° = 1.**

Using these, we have :

$$\tan 0° = \frac{\sin 0°}{\cos 0°} = 0,\ \cot 0° = \frac{1}{\tan 0°},$$ which is not defined. (Why?)

$$\sec 0° = \frac{1}{\cos 0°} = 1 \text{ and } \operatorname{cosec} 0° = \frac{1}{\sin 0°},$$ which is again not defined.(Why?)

Now, let us see what happens to the trigonometric ratios of $\angle$ A, when it is made larger and larger in Δ ABC till it becomes 90°. As $\angle$ A gets larger and larger, $\angle$ C gets smaller and smaller. Therefore, as in the case above, the length of the side AB goes on decreasing. The point A gets closer to point B. Finally when $\angle$ A is very close to 90°, $\angle$ C becomes very close to 0° and the side AC almost coincides with side BC (see Fig. 8.18).

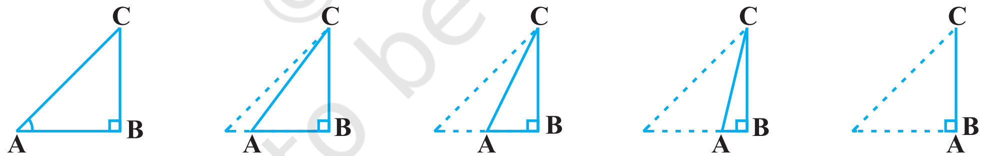


Fig. 8.18

When $\angle$ C is very close to 0°, $\angle$ A is very close to 90°, side AC is nearly the same as side BC, and so sin A is very close to 1. Also when $\angle$ A is very close to 90°, $\angle$ C is very close to 0°, and the side AB is nearly zero, so cos A is very close to 0.

So, we define : **sin 90° = 1 and cos 90° = 0.**

Now, why don't you find the other trigonometric ratios of 90°?

We shall now give the values of all the trigonometric ratios of 0°, 30°, 45°, 60° and 90° in Table 8.1, for ready reference.

Table 8.1

$\angle$ A	0°	30°	45°	60°	90°
sin A	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos A	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan A	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Not defined
cosec A	Not defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
sec A	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	Not defined
cot A	Not defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

Remark : From the table above you can observe that as $\angle$ A increases from 0° to 90°, sin A increases from 0 to 1 and cos A decreases from 1 to 0.

Let us illustrate the use of the values in the table above through some examples.

Example 6 : In Δ ABC, right-angled at B, AB = 5 cm and $\angle$ ACB = 30° (see Fig. 8.19). Determine the lengths of the sides BC and AC.

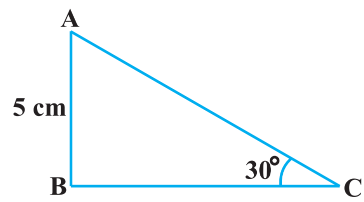


Fig. 8.19

Solution : To find the length of the side BC, we will choose the trigonometric ratio involving BC and the given side AB. Since BC is the side adjacent to angle C and AB is the side opposite to angle C, therefore

$$\frac{AB}{BC} = \tan C$$

i.e.,
$$\frac{5}{BC} = \tan 30° = \frac{1}{\sqrt{3}}$$

which gives
$$BC = 5\sqrt{3} \text{ cm}$$

To find the length of the side AC, we consider

$$\sin 30° = \frac{AB}{AC} \qquad \text{(Why?)}$$

i.e., $$\frac{1}{2} = \frac{5}{AC}$$

i.e., $$AC = 10 \text{ cm}$$

Note that alternatively we could have used Pythagoras theorem to determine the third side in the example above,

i.e., $$AC = \sqrt{AB^2 + BC^2} = \sqrt{5^2 + (5\sqrt{3})^2} \text{ cm} = 10 \text{cm}.$$

Example 7 : In Δ PQR, right-angled at Q (see Fig. 8.20), PQ = 3 cm and PR = 6 cm. Determine ∠ QPR and ∠ PRQ.

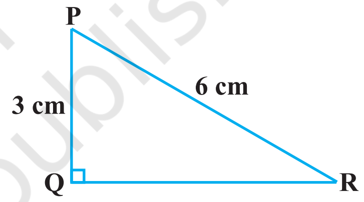


Fig. 8.20

Solution : Given PQ = 3 cm and PR = 6 cm.

Therefore, $$\frac{PQ}{PR} = \sin R$$

or $$\sin R = \frac{3}{6} = \frac{1}{2}$$

So, $$\angle PRQ = 30°$$

and therefore, $$\angle QPR = 60°. \quad \text{(Why?)}$$

You may note that if one of the sides and any other part (either an acute angle or any side) of a right triangle is known, the remaining sides and angles of the triangle can be determined.

Example 8 : If $\sin (A - B) = \frac{1}{2}$, $\cos (A + B) = \frac{1}{2}$, $0° < A + B \le 90°$, $A > B$, find A and B.

Solution : Since, $\sin (A - B) = \frac{1}{2}$, therefore, $A - B = 30°$ (Why?) (1)

Also, since $\cos (A + B) = \frac{1}{2}$, therefore, $A + B = 60°$ (Why?) (2)

Solving (1) and (2), we get : $A = 45°$ and $B = 15°$.

EXERCISE 8.2

1. Evaluate the following :

(i) $\sin 60° \cos 30° + \sin 30° \cos 60°$ (ii) $2 \tan^2 45° + \cos^2 30° - \sin^2 60°$

(iii) $\dfrac{\cos 45°}{\sec 30° + \operatorname{cosec} 30°}$ (iv) $\dfrac{\sin 30° + \tan 45° - \operatorname{cosec} 60°}{\sec 30° + \cos 60° + \cot 45°}$

(v) $\dfrac{5\cos^2 60° + 4\sec^2 30° - \tan^2 45°}{\sin^2 30° + \cos^2 30°}$

2. Choose the correct option and justify your choice :

(i) $\dfrac{2\tan 30°}{1+\tan^2 30°} =$

(A) $\sin 60°$ (B) $\cos 60°$ (C) $\tan 60°$ (D) $\sin 30°$

(ii) $\dfrac{1-\tan^2 45°}{1+\tan^2 45°} =$

(A) $\tan 90°$ (B) 1 (C) $\sin 45°$ (D) 0

(iii) $\sin 2A = 2 \sin A$ is true when $A =$

(A) 0° (B) 30° (C) 45° (D) 60°

(iv) $\dfrac{2\tan 30°}{1-\tan^2 30°} =$

(A) $\cos 60°$ (B) $\sin 60°$ (C) $\tan 60°$ (D) $\sin 30°$

3. If $\tan(A + B) = \sqrt{3}$ and $\tan(A - B) = \dfrac{1}{\sqrt{3}}$; $0° < A + B \le 90°$; $A > B$, find A and B.

4. State whether the following are true or false. Justify your answer.

(i) $\sin(A + B) = \sin A + \sin B$.

(ii) The value of $\sin\theta$ increases as θ increases.

(iii) The value of $\cos\theta$ increases as θ increases.

(iv) $\sin\theta = \cos\theta$ for all values of θ.

(v) $\cot A$ is not defined for $A = 0°$.

8.4 Trigonometric Ratios of Complementary Angles

Recall that two angles are said to be complementary if their sum equals 90°. In Δ ABC, right-angled at B, do you see any pair of complementary angles? (See Fig. 8.21)

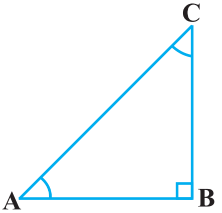


Fig. 8.21

Since $\angle A + \angle C = 90°$, they form such a pair. We have:

$$\left.\begin{array}{lll} \sin A = \dfrac{BC}{AC} & \cos A = \dfrac{AB}{AC} & \tan A = \dfrac{BC}{AB} \\[2ex] \operatorname{cosec} A = \dfrac{AC}{BC} & \sec A = \dfrac{AC}{AB} & \cot A = \dfrac{AB}{BC} \end{array}\right\} \quad (1)$$

Now let us write the trigonometric ratios for $\angle C = 90° - \angle A$.

For convenience, we shall write $90° - A$ instead of $90° - \angle A$.

What would be the side opposite and the side adjacent to the angle $90° - A$?

You will find that AB is the side opposite and BC is the side adjacent to the angle $90° - A$. Therefore,

$$\left.\begin{array}{lll} \sin(90° - A) = \dfrac{AB}{AC}, & \cos(90° - A) = \dfrac{BC}{AC}, & \tan(90° - A) = \dfrac{AB}{BC} \\[2ex] \operatorname{cosec}(90° - A) = \dfrac{AC}{AB}, & \sec(90° - A) = \dfrac{AC}{BC}, & \cot(90° - A) = \dfrac{BC}{AB} \end{array}\right\} \quad (2)$$

Now, compare the ratios in (1) and (2). Observe that :

$$\sin(90° - A) = \frac{AB}{AC} = \cos A \text{ and } \cos(90° - A) = \frac{BC}{AC} = \sin A$$

Also, $$\tan(90° - A) = \frac{AB}{BC} = \cot A, \quad \cot(90° - A) = \frac{BC}{AB} = \tan A$$

$$\sec(90° - A) = \frac{AC}{BC} = \operatorname{cosec} A, \quad \operatorname{cosec}(90° - A) = \frac{AC}{AB} = \sec A$$

So, $\mathbf{\sin(90° - A) = \cos A,}$ $\mathbf{\cos(90° - A) = \sin A,}$

$\mathbf{\tan(90° - A) = \cot A,}$ $\mathbf{\cot(90° - A) = \tan A,}$

$\mathbf{\sec(90° - A) = \operatorname{cosec} A,}$ $\mathbf{\operatorname{cosec}(90° - A) = \sec A,}$

for all values of angle A lying between 0° and 90°. Check whether this holds for $A = 0°$ or $A = 90°$.

Note : $\tan 0° = 0 = \cot 90°$, $\sec 0° = 1 = \operatorname{cosec} 90°$ and $\sec 90°$, $\operatorname{cosec} 0°$, $\tan 90°$ and $\cot 0°$ are not defined.

Now, let us consider some examples.

Example 9 : Evaluate $\dfrac{\tan 65^\circ}{\cot 25^\circ}$.

Solution : We know :

$$\cot A = \tan (90^\circ - A)$$

So,

$$\cot 25^\circ = \tan (90^\circ - 25^\circ) = \tan 65^\circ$$

i.e.,

$$\frac{\tan 65^\circ}{\cot 25^\circ} = \frac{\tan 65^\circ}{\tan 65^\circ} = 1$$

Example 10 : If sin 3A = cos (A – 26°), where 3A is an acute angle, find the value of A.

Solution : We are given that $\sin 3A = \cos (A - 26^\circ)$. (1)

Since $\sin 3A = \cos (90^\circ - 3A)$, we can write (1) as

$$\cos (90^\circ - 3A) = \cos (A - 26^\circ)$$

Since $90^\circ - 3A$ and $A - 26^\circ$ are both acute angles, therefore,

$$90^\circ - 3A = A - 26^\circ$$

which gives

$$A = 29^\circ$$

Example 11 : Express cot 85° + cos 75° in terms of trigonometric ratios of angles between 0° and 45°.

Solution :

$$\cot 85^\circ + \cos 75^\circ = \cot (90^\circ - 5^\circ) + \cos (90^\circ - 15^\circ)$$
$$= \tan 5^\circ + \sin 15^\circ$$

EXERCISE 8.3

1. Evaluate :

(i) $\dfrac{\sin 18^\circ}{\cos 72^\circ}$ (ii) $\dfrac{\tan 26^\circ}{\cot 64^\circ}$ (iii) $\cos 48^\circ - \sin 42^\circ$ (iv) $\operatorname{cosec} 31^\circ - \sec 59^\circ$

2. Show that :

(i) $\tan 48^\circ \tan 23^\circ \tan 42^\circ \tan 67^\circ = 1$

(ii) $\cos 38^\circ \cos 52^\circ - \sin 38^\circ \sin 52^\circ = 0$

3. If $\tan 2A = \cot (A - 18^\circ)$, where 2A is an acute angle, find the value of A.

4. If $\tan A = \cot B$, prove that $A + B = 90^\circ$.

5. If $\sec 4A = \operatorname{cosec} (A - 20^\circ)$, where 4A is an acute angle, find the value of A.

6. If A, B and C are interior angles of a triangle ABC, then show that

$$\sin\left(\frac{B+C}{2}\right) = \cos\frac{A}{2}.$$

7. Express sin 67° + cos 75° in terms of trigonometric ratios of angles between 0° and 45°.

8.5 Trigonometric Identities

You may recall that an equation is called an identity when it is true for all values of the variables involved. Similarly, an equation involving trigonometric ratios of an angle is called a **trigonometric identity,** if it is true for all values of the angle(s) involved.

In this section, we will prove one trigonometric identity, and use it further to prove other useful trigonometric identities.

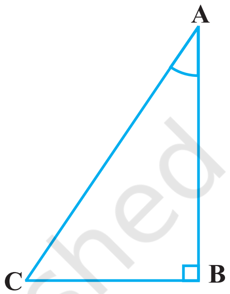


Fig. 8.22

In Δ ABC, right-angled at B (see Fig. 8.22), we have:

$$AB^2 + BC^2 = AC^2 \qquad (1)$$

Dividing each term of (1) by AC^2, we get

$$\frac{AB^2}{AC^2} + \frac{BC^2}{AC^2} = \frac{AC^2}{AC^2}$$

i.e., $$\left(\frac{AB}{AC}\right)^2 + \left(\frac{BC}{AC}\right)^2 = \left(\frac{AC}{AC}\right)^2$$

i.e., $$(\cos A)^2 + (\sin A)^2 = 1$$

i.e., $$\mathbf{\cos^2 A + \sin^2 A = 1} \qquad (2)$$

This is true for all A such that 0° ≤ A ≤ 90°. So, this is a trigonometric identity.

Let us now divide (1) by AB^2. We get

$$\frac{AB^2}{AB^2} + \frac{BC^2}{AB^2} = \frac{AC^2}{AB^2}$$

or, $$\left(\frac{AB}{AB}\right)^2 + \left(\frac{BC}{AB}\right)^2 = \left(\frac{AC}{AB}\right)^2$$

i.e., $$\mathbf{1 + \tan^2 A = \sec^2 A} \qquad (3)$$

Is this equation true for A = 0°? Yes, it is. What about A = 90°? Well, tan A and sec A are not defined for A = 90°. So, (3) is true for all A such that 0° ≤ A < 90°.

Let us see what we get on dividing (1) by BC^2. We get

$$\frac{AB^2}{BC^2}+\frac{BC^2}{BC^2} = \frac{AC^2}{BC^2}$$

i.e.,
$$\left(\frac{AB}{BC}\right)^2+\left(\frac{BC}{BC}\right)^2 = \left(\frac{AC}{BC}\right)^2$$

i.e.,
$$\mathbf{\cot^2 A + 1 = \operatorname{cosec}^2 A} \qquad (4)$$

Note that cosec A and cot A are not defined for A = 0°. Therefore (4) is true for all A such that 0° < A ≤ 90°.

Using these identities, we can express each trigonometric ratio in terms of other trigonometric ratios, i.e., if any one of the ratios is known, we can also determine the values of other trigonometric ratios.

Let us see how we can do this using these identities. Suppose we know that $\tan A = \frac{1}{\sqrt{3}}\cdot$ Then, $\cot A = \sqrt{3}$.

Since, $\sec^2 A = 1 + \tan^2 A = 1+\frac{1}{3}=\frac{4}{3}$, $\sec A = \frac{2}{\sqrt{3}}$, and $\cos A = \frac{\sqrt{3}}{2}\cdot$

Again, $\sin A = \sqrt{1-\cos^2 A} = \sqrt{1-\frac{3}{4}}=\frac{1}{2}$. Therefore, cosec A = 2.

Example 12 : Express the ratios cos A, tan A and sec A in terms of sin A.

Solution : Since $\cos^2 A + \sin^2 A = 1$, therefore,

$$\cos^2 A = 1 - \sin^2 A, \text{ i.e., } \cos A = \pm\sqrt{1-\sin^2 A}$$

This gives
$$\cos A = \sqrt{1-\sin^2 A} \qquad \text{(Why?)}$$

Hence,
$$\tan A = \frac{\sin A}{\cos A} = \frac{\sin A}{\sqrt{1-\sin^2 A}} \text{ and } \sec A = \frac{1}{\cos A} = \frac{1}{\sqrt{1-\sin^2 A}}$$

Example 13 : Prove that sec A (1 – sin A)(sec A + tan A) = 1.

Solution :

$$\text{LHS} = \sec A\,(1 - \sin A)(\sec A + \tan A) = \left(\frac{1}{\cos A}\right)(1 - \sin A)\left(\frac{1}{\cos A} + \frac{\sin A}{\cos A}\right)$$

$$= \frac{(1 - \sin A)(1 + \sin A)}{\cos^2 A} = \frac{1 - \sin^2 A}{\cos^2 A}$$

$$= \frac{\cos^2 A}{\cos^2 A} = 1 = \text{RHS}$$

Example 14 : Prove that $\dfrac{\cot A - \cos A}{\cot A + \cos A} = \dfrac{\operatorname{cosec} A - 1}{\operatorname{cosec} A + 1}$

Solution : $\text{LHS} = \dfrac{\cot A - \cos A}{\cot A + \cos A} = \dfrac{\dfrac{\cos A}{\sin A} - \cos A}{\dfrac{\cos A}{\sin A} + \cos A}$

$$= \frac{\cos A\left(\dfrac{1}{\sin A} - 1\right)}{\cos A\left(\dfrac{1}{\sin A} + 1\right)} = \frac{\left(\dfrac{1}{\sin A} - 1\right)}{\left(\dfrac{1}{\sin A} + 1\right)} = \frac{\operatorname{cosec} A - 1}{\operatorname{cosec} A + 1} = \text{RHS}$$

Example 15 : Prove that $\dfrac{\sin\theta - \cos\theta + 1}{\sin\theta + \cos\theta - 1} = \dfrac{1}{\sec\theta - \tan\theta}$, using the identity $\sec^2\theta = 1 + \tan^2\theta$.

Solution : Since we will apply the identity involving sec θ and tan θ, let us first convert the LHS (of the identity we need to prove) in terms of sec θ and tan θ by dividing numerator and denominator by cos θ.

$$\text{LHS} = \frac{\sin\theta - \cos\theta + 1}{\sin\theta + \cos\theta - 1} = \frac{\tan\theta - 1 + \sec\theta}{\tan\theta + 1 - \sec\theta}$$

$$= \frac{(\tan\theta + \sec\theta) - 1}{(\tan\theta - \sec\theta) + 1} = \frac{\{(\tan\theta + \sec\theta) - 1\}\,(\tan\theta - \sec\theta)}{\{(\tan\theta - \sec\theta) + 1\}\,(\tan\theta - \sec\theta)}$$

$$= \frac{(\tan^2\theta - \sec^2\theta) - (\tan\theta - \sec\theta)}{\{\tan\theta - \sec\theta + 1\}\,(\tan\theta - \sec\theta)}$$

$$= \frac{-1 - \tan\theta + \sec\theta}{(\tan\theta - \sec\theta + 1)\,(\tan\theta - \sec\theta)}$$

$$= \frac{-1}{\tan\theta - \sec\theta} = \frac{1}{\sec\theta - \tan\theta},$$

which is the RHS of the identity, we are required to prove.

EXERCISE 8.4

1. Express the trigonometric ratios sin A, sec A and tan A in terms of cot A.
2. Write all the other trigonometric ratios of $\angle$A in terms of sec A.
3. Evaluate :

   (i) $\dfrac{\sin^2 63° + \sin^2 27°}{\cos^2 17° + \cos^2 73°}$

   (ii) $\sin 25° \cos 65° + \cos 25° \sin 65°$
4. Choose the correct option. Justify your choice.

   (i) $9\sec^2 A - 9\tan^2 A =$

   (A) 1 (B) 9 (C) 8 (D) 0

   (ii) $(1 + \tan\theta + \sec\theta)(1 + \cot\theta - \text{cosec}\,\theta) =$

   (A) 0 (B) 1 (C) 2 (D) –1

   (iii) $(\sec A + \tan A)(1 - \sin A) =$

   (A) sec A (B) sin A (C) cosec A (D) cos A

   (iv) $\dfrac{1 + \tan^2 A}{1 + \cot^2 A} =$

   (A) $\sec^2 A$ (B) –1 (C) $\cot^2 A$ (D) $\tan^2 A$
5. Prove the following identities, where the angles involved are acute angles for which the expressions are defined.

   (i) $(\text{cosec}\,\theta - \cot\theta)^2 = \dfrac{1 - \cos\theta}{1 + \cos\theta}$ (ii) $\dfrac{\cos A}{1 + \sin A} + \dfrac{1 + \sin A}{\cos A} = 2\sec A$

(iii) $\dfrac{\tan\theta}{1-\cot\theta}+\dfrac{\cot\theta}{1-\tan\theta}=1+\sec\theta\,\operatorname{cosec}\theta$

[**Hint :** Write the expression in terms of sin θ and cos θ]

(iv) $\dfrac{1+\sec A}{\sec A}=\dfrac{\sin^2 A}{1-\cos A}$ [**Hint :** Simplify LHS and RHS separately]

(v) $\dfrac{\cos A-\sin A+1}{\cos A+\sin A-1}=\operatorname{cosec} A+\cot A$, using the identity $\operatorname{cosec}^2 A = 1 + \cot^2 A$.

(vi) $\sqrt{\dfrac{1+\sin A}{1-\sin A}}=\sec A+\tan A$ (vii) $\dfrac{\sin\theta-2\sin^3\theta}{2\cos^3\theta-\cos\theta}=\tan\theta$

(viii) $(\sin A+\operatorname{cosec} A)^2+(\cos A+\sec A)^2=7+\tan^2 A+\cot^2 A$

(ix) $(\operatorname{cosec} A-\sin A)(\sec A-\cos A)=\dfrac{1}{\tan A+\cot A}$

[**Hint :** Simplify LHS and RHS separately]

(x) $\left(\dfrac{1+\tan^2 A}{1+\cot^2 A}\right)=\left(\dfrac{1-\tan A}{1-\cot A}\right)^2=\tan^2 A$

8.6 Summary

In this chapter, you have studied the following points :

1. In a right triangle ABC, right-angled at B,

$\sin A=\dfrac{\text{side opposite to angle A}}{\text{hypotenuse}}$, $\cos A=\dfrac{\text{side adjacent to angle A}}{\text{hypotenuse}}$

$\tan A=\dfrac{\text{side opposite to angle A}}{\text{side adjacent to angle A}}$.

2. $\operatorname{cosec} A=\dfrac{1}{\sin A}$; $\sec A=\dfrac{1}{\cos A}$; $\tan A=\dfrac{1}{\cot A}$, $\tan A=\dfrac{\sin A}{\cos A}$.

3. If one of the trigonometric ratios of an acute angle is known, the remaining trigonometric ratios of the angle can be easily determined.

4. The values of trigonometric ratios for angles 0°, 30°, 45°, 60° and 90°.

5. The value of sin A or cos A never exceeds 1, whereas the value of sec A or cosec A is always greater than or equal to 1.

6. $\sin(90°-A)=\cos A$, $\cos(90°-A)=\sin A$;

$\tan(90°-A)=\cot A$, $\cot(90°-A)=\tan A$;

$\sec(90°-A)=\operatorname{cosec} A$, $\operatorname{cosec}(90°-A)=\sec A$.

7. $\sin^2 A+\cos^2 A=1$,

$\sec^2 A-\tan^2 A=1$ for $0°\le A<90°$,

$\operatorname{cosec}^2 A=1+\cot^2 A$ for $0°<A\le 90°$.

3

Rules for Finding Derivatives

It is tedious to compute a limit every time we need to know the derivative of a function. Fortunately, we can develop a small collection of examples and rules that allow us to compute the derivative of almost any function we are likely to encounter. Many functions involve quantities raised to a constant power, such as polynomials and more complicated combinations like $y = (\sin x)^4$. So we start by examining powers of a single variable; this gives us a building block for more complicated examples.

3.1 The Power Rule

We start with the derivative of a power function, $f(x) = x^n$. Here n is a number of any kind: integer, rational, positive, negative, even irrational, as in x^π. We have already computed some simple examples, so the formula should not be a complete surprise:

$$\frac{d}{dx}x^n = nx^{n-1}.$$

It is not easy to show this is true for any n. We will do some of the easier cases now, and discuss the rest later.

The easiest, and most common, is the case that n is a positive integer. To compute the derivative we need to compute the following limit:

$$\frac{d}{dx}x^n = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}.$$

For a specific, fairly small value of n, we could do this by straightforward algebra.

EXAMPLE 3.1.1 Find the derivative of $f(x) = x^3$.

$$\begin{aligned}
\frac{d}{dx}x^3 &= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^3 - x^3}{\Delta x}. \\
&= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 - x^3}{\Delta x}. \\
&= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x\Delta x^2 + \Delta x^3}{\Delta x}. \\
&= \lim_{\Delta x \to 0} 3x^2 + 3x\Delta x + \Delta x^2 = 3x^2.
\end{aligned}$$

□

The general case is really not much harder as long as we don't try to do too much. The key is understanding what happens when $(x + \Delta x)^n$ is multiplied out:

$$(x+\Delta x)^n = x^n + nx^{n-1}\Delta x + a_2 x^{n-2}\Delta x^2 + \cdots + +a_{n-1}x\Delta x^{n-1} + \Delta x^n.$$

We know that multiplying out will give a large number of terms all of the form $x^i \Delta x^j$, and in fact that $i + j = n$ in every term. One way to see this is to understand that one method for multiplying out $(x + \Delta x)^n$ is the following: In every $(x + \Delta x)$ factor, pick either the x or the Δx, then multiply the n choices together; do this in all possible ways. For example, for $(x + \Delta x)^3$, there are eight possible ways to do this:

$$\begin{aligned}
(x+\Delta x)(x+\Delta x)(x+\Delta x) &= xxx + xx\Delta x + x\Delta x x + x\Delta x\Delta x \\
&\quad + \Delta x x x + \Delta x x \Delta x + \Delta x\Delta x x + \Delta x\Delta x\Delta x \\
&= x^3 + x^2\Delta x + x^2\Delta x + x\Delta x^2 \\
&\quad + x^2\Delta x + x\Delta x^2 + x\Delta x^2 + \Delta x^3 \\
&= x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3
\end{aligned}$$

No matter what n is, there are n ways to pick Δx in one factor and x in the remaining $n-1$ factors; this means one term is $nx^{n-1}\Delta x$. The other coefficients are somewhat harder to understand, but we don't really need them, so in the formula above they have simply been called a_2, a_3, and so on. We know that every one of these terms contains Δx to at least the power 2. Now let's look at the limit:

$$\begin{aligned}
\frac{d}{dx}x^n &= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^n + nx^{n-1}\Delta x + a_2 x^{n-2}\Delta x^2 + \cdots + a_{n-1}x\Delta x^{n-1} + \Delta x^n - x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{nx^{n-1}\Delta x + a_2 x^{n-2}\Delta x^2 + \cdots + a_{n-1}x\Delta x^{n-1} + \Delta x^n}{\Delta x} \\
&= \lim_{\Delta x \to 0} nx^{n-1} + a_2 x^{n-2}\Delta x + \cdots + a_{n-1}x\Delta x^{n-2} + \Delta x^{n-1} = nx^{n-1}.
\end{aligned}$$

Now without much trouble we can verify the formula for negative integers. First let's look at an example:

EXAMPLE 3.1.2 Find the derivative of $y = x^{-3}$. Using the formula, $y' = -3x^{-3-1} = -3x^{-4}$. □

Here is the general computation. Suppose n is a negative integer; the algebra is easier to follow if we use $n = -m$ in the computation, where m is a positive integer.

$$\begin{aligned}
\frac{d}{dx}x^n = \frac{d}{dx}x^{-m} &= \lim_{\Delta x\to 0} \frac{(x+\Delta x)^{-m} - x^{-m}}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{\frac{1}{(x+\Delta x)^m} - \frac{1}{x^m}}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{x^m - (x+\Delta x)^m}{(x+\Delta x)^m x^m \Delta x} \\
&= \lim_{\Delta x\to 0} \frac{x^m - (x^m + mx^{m-1}\Delta x + a_2 x^{m-2}\Delta x^2 + \cdots + a_{m-1}x\Delta x^{m-1} + \Delta x^m)}{(x+\Delta x)^m x^m \Delta x} \\
&= \lim_{\Delta x\to 0} \frac{-mx^{m-1} - a_2 x^{m-2}\Delta x - \cdots - a_{m-1}x\Delta x^{m-2} - \Delta x^{m-1})}{(x+\Delta x)^m x^m} \\
&= \frac{-mx^{m-1}}{x^m x^m} = \frac{-mx^{m-1}}{x^{2m}} = -mx^{m-1-2m} = nx^{-m-1} = nx^{n-1}.
\end{aligned}$$

We will later see why the other cases of the power rule work, but from now on we will use the power rule whenever n is any real number. Let's note here a simple case in which the power rule applies, or almost applies, but is not really needed. Suppose that $f(x) = 1$; remember that this "1" is a function, not "merely" a number, and that $f(x) = 1$ has a graph that is a horizontal line, with slope zero everywhere. So we know that $f'(x) = 0$. We might also write $f(x) = x^0$, though there is some question about just what this means at $x = 0$. If we apply the power rule, we get $f'(x) = 0x^{-1} = 0/x = 0$, again noting that there is a problem at $x = 0$. So the power rule "works" in this case, but it's really best to just remember that the derivative of any constant function is zero.

Exercises 3.1.

Find the derivatives of the given functions.

1. x^{100} ⇒

2. x^{-100} ⇒

3. $\dfrac{1}{x^5}$ ⇒

4. x^{π} ⇒

5. $x^{3/4}$ ⇒

6. $x^{-9/7}$ ⇒

3.2 Linearity of the Derivative

An operation is linear if it behaves "nicely" with respect to multiplication by a constant and addition. The name comes from the equation of a line through the origin, $f(x) = mx$, and the following two properties of this equation. First, $f(cx) = m(cx) = c(mx) = cf(x)$, so the constant c can be "moved outside" or "moved through" the function f. Second, $f(x + y) = m(x + y) = mx + my = f(x) + f(y)$, so the addition symbol likewise can be moved through the function.

The corresponding properties for the derivative are:

$$(cf(x))' = \frac{d}{dx}cf(x) = c\frac{d}{dx}f(x) = cf'(x),$$

and

$$(f(x) + g(x))' = \frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x) = f'(x) + g'(x).$$

It is easy to see, or at least to believe, that these are true by thinking of the distance/speed interpretation of derivatives. If one object is at position $f(t)$ at time t, we know its speed is given by $f'(t)$. Suppose another object is at position $5f(t)$ at time t, namely, that it is always 5 times as far along the route as the first object. Then it "must" be going 5 times as fast at all times.

The second rule is somewhat more complicated, but here is one way to picture it. Suppose a flatbed railroad car is at position $f(t)$ at time t, so the car is traveling at a speed of $f'(t)$ (to be specific, let's say that $f(t)$ gives the position on the track of the rear end of the car). Suppose that an ant is crawling from the back of the car to the front so that its position *on the car* is $g(t)$ and its speed *relative to the car* is $g'(t)$. Then in reality, at time t, the ant is at position $f(t) + g(t)$ along the track, and its speed is "obviously" $f'(t) + g'(t)$.

We don't want to rely on some more-or-less obvious physical interpretation to determine what is true mathematically, so let's see how to verify these rules by computation.

We'll do one and leave the other for the exercises.

$$\begin{aligned}
\frac{d}{dx}(f(x)+g(x)) &= \lim_{\Delta x\to 0} \frac{f(x+\Delta x)+g(x+\Delta x)-(f(x)+g(x))}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{f(x+\Delta x)+g(x+\Delta x)-f(x)-g(x)}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)+g(x+\Delta x)-g(x)}{\Delta x} \\
&= \lim_{\Delta x\to 0} \left(\frac{f(x+\Delta x)-f(x)}{\Delta x}+\frac{g(x+\Delta x)-g(x)}{\Delta x}\right) \\
&= \lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}+\lim_{\Delta x\to 0} \frac{g(x+\Delta x)-g(x)}{\Delta x} \\
&= f'(x)+g'(x)
\end{aligned}$$

This is sometimes called the **sum rule** for derivatives.

EXAMPLE 3.2.1 Find the derivative of $f(x)=x^5+5x^2$. We have to invoke linearity twice here:

$$f'(x)=\frac{d}{dx}(x^5+5x^2)=\frac{d}{dx}x^5+\frac{d}{dx}(5x^2)=5x^4+5\frac{d}{dx}(x^2)=5x^4+5\cdot 2x^1=5x^4+10x.$$

□

Because it is so easy with a little practice, we can usually combine all uses of linearity into a single step. The following example shows an acceptably detailed computation.

EXAMPLE 3.2.2 Find the derivative of $f(x)=3/x^4-2x^2+6x-7$.

$$f'(x)=\frac{d}{dx}\left(\frac{3}{x^4}-2x^2+6x-7\right)=\frac{d}{dx}(3x^{-4}-2x^2+6x-7)=-12x^{-5}-4x+6.$$

□

Exercises 3.2.

Find the derivatives of the functions in 1–6.

1. $5x^3+12x^2-15$ ⇒

2. $-4x^5+3x^2-5/x^2$ ⇒

3. $5(-3x^2+5x+1)$ ⇒

4. $f(x)+g(x)$, where $f(x)=x^2-3x+2$ and $g(x)=2x^3-5x$ ⇒

5. $(x+1)(x^2+2x-3)$ ⇒

6. $\sqrt{625-x^2}+3x^3+12$ (See section 2.1.) ⇒

7. Find an equation for the tangent line to $f(x)=x^3/4-1/x$ at $x=-2$. ⇒

8. Find an equation for the tangent line to $f(x) = 3x^2 - \pi^3$ at $x = 4$. ⇒

9. Suppose the position of an object at time t is given by $f(t) = -49t^2/10 + 5t + 10$. Find a function giving the speed of the object at time t. The acceleration of an object is the rate at which its speed is changing, which means it is given by the derivative of the speed function. Find the acceleration of the object at time t. ⇒

10. Let $f(x) = x^3$ and $c = 3$. Sketch the graphs of f, cf, f', and $(cf)'$ on the same diagram.

11. The general polynomial P of degree n in the variable x has the form $P(x) = \sum_{k=0}^{n} a_k x^k = a_0 + a_1 x + \ldots + a_n x^n$. What is the derivative (with respect to x) of P? ⇒

12. Find a cubic polynomial whose graph has horizontal tangents at $(-2, 5)$ and $(2, 3)$. ⇒

13. Prove that $\frac{d}{dx}(cf(x)) = cf'(x)$ using the definition of the derivative.

14. Suppose that f and g are differentiable at x. Show that $f - g$ is differentiable at x using the two linearity properties from this section.

3.3 The Product Rule

Consider the product of two simple functions, say $f(x) = (x^2 + 1)(x^3 - 3x)$. An obvious guess for the derivative of f is the product of the derivatives of the constituent functions: $(2x)(3x^2 - 3) = 6x^3 - 6x$. Is this correct? We can easily check, by rewriting f and doing the calculation in a way that is known to work. First, $f(x) = x^5 - 3x^3 + x^3 - 3x = x^5 - 2x^3 - 3x$, and then $f'(x) = 5x^4 - 6x^2 - 3$. Not even close! What went "wrong"? Well, nothing really, except the guess was wrong.

So the derivative of $f(x)g(x)$ is NOT as simple as $f'(x)g'(x)$. Surely there is some rule for such a situation? There is, and it is instructive to "discover" it by trying to do the general calculation even without knowing the answer in advance.

$$\begin{aligned}
\frac{d}{dx}(f(x)g(x)) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x+\Delta x)g(x) + f(x+\Delta x)g(x) - f(x)g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x+\Delta x)g(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x) - f(x)g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} f(x+\Delta x)\frac{g(x+\Delta x) - g(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} g(x) \\
&= f(x)g'(x) + f'(x)g(x)
\end{aligned}$$

A couple of items here need discussion. First, we used a standard trick, "add and subtract the same thing", to transform what we had into a more useful form. After some rewriting, we realize that we have two limits that produce $f'(x)$ and $g'(x)$. Of course, $f'(x)$ and

$g'(x)$ must actually exist for this to make sense. We also replaced $\lim_{\Delta x \to 0} f(x + \Delta x)$ with $f(x)$—why is this justified?

What we really need to know here is that $\lim_{\Delta x \to 0} f(x + \Delta x) = f(x)$, or in the language of section 2.5, that f is continuous at x. We already know that $f'(x)$ exists (or the whole approach, writing the derivative of fg in terms of f' and g', doesn't make sense). This turns out to imply that f is continuous as well. Here's why:

$$\begin{aligned}\lim_{\Delta x \to 0} f(x + \Delta x) &= \lim_{\Delta x \to 0} (f(x + \Delta x) - f(x) + f(x)) \\ &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \Delta x + \lim_{\Delta x \to 0} f(x) \\ &= f'(x) \cdot 0 + f(x) = f(x)\end{aligned}$$

To summarize: the product rule says that

$$\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + f'(x)g(x).$$

Returning to the example we started with, let $f(x) = (x^2 + 1)(x^3 - 3x)$. Then $f'(x) = (x^2 + 1)(3x^2 - 3) + (2x)(x^3 - 3x) = 3x^4 - 3x^2 + 3x^2 - 3 + 2x^4 - 6x^2 = 5x^4 - 6x^2 - 3$, as before. In this case it is probably simpler to multiply $f(x)$ out first, then compute the derivative; here's an example for which we really need the product rule.

EXAMPLE 3.3.1 Compute the derivative of $f(x) = x^2\sqrt{625 - x^2}$. We have already computed $\frac{d}{dx}\sqrt{625 - x^2} = \frac{-x}{\sqrt{625 - x^2}}$. Now

$$f'(x) = x^2 \frac{-x}{\sqrt{625 - x^2}} + 2x\sqrt{625 - x^2} = \frac{-x^3 + 2x(625 - x^2)}{\sqrt{625 - x^2}} = \frac{-3x^3 + 1250x}{\sqrt{625 - x^2}}.$$

□

Exercises 3.3.

In 1–4, find the derivatives of the functions using the product rule.

1. $x^3(x^3 - 5x + 10) \Rightarrow$
2. $(x^2 + 5x - 3)(x^5 - 6x^3 + 3x^2 - 7x + 1) \Rightarrow$
3. $\sqrt{x}\sqrt{625 - x^2} \Rightarrow$
4. $\dfrac{\sqrt{625 - x^2}}{x^{20}} \Rightarrow$
5. Use the product rule to compute the derivative of $f(x) = (2x - 3)^2$. Sketch the function. Find an equation of the tangent line to the curve at $x = 2$. Sketch the tangent line at $x = 2$. $\Rightarrow$

6. Suppose that f, g, and h are differentiable functions. Show that $(fgh)'(x) = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$.

7. State and prove a rule to compute $(fghi)'(x)$, similar to the rule in the previous problem.

Product notation. Suppose $f_1, f_2, \ldots f_n$ are functions. The product of all these functions can be written

$$\prod_{k=1}^{n} f_k.$$

This is similar to the use of $\sum$ to denote a sum. For example,

$$\prod_{k=1}^{5} f_k = f_1 f_2 f_3 f_4 f_5$$

and

$$\prod_{k=1}^{n} k = 1 \cdot 2 \cdot \ldots \cdot n = n!.$$

We sometimes use somewhat more complicated conditions; for example

$$\prod_{k=1, k\neq j}^{n} f_k$$

denotes the product of f_1 through f_n except for f_j. For example,

$$\prod_{k=1, k\neq 4}^{5} x^k = x \cdot x^2 \cdot x^3 \cdot x^5 = x^{11}.$$

8. The **generalized product rule** says that if $f_1, f_2, \ldots, f_n$ are differentiable functions at x then

$$\frac{d}{dx}\prod_{k=1}^{n} f_k(x) = \sum_{j=1}^{n}\left(f'_j(x) \prod_{k=1, k\neq j}^{n} f_k(x)\right).$$

Verify that this is the same as your answer to the previous problem when $n = 4$, and write out what this says when $n = 5$.

3.4 The Quotient Rule

What is the derivative of $(x^2 + 1)/(x^3 - 3x)$? More generally, we'd like to have a formula to compute the derivative of $f(x)/g(x)$ if we already know $f'(x)$ and $g'(x)$. Instead of attacking this problem head-on, let's notice that we've already done part of the problem: $f(x)/g(x) = f(x) \cdot (1/g(x))$, that is, this is "really" a product, and we can compute the derivative if we know $f'(x)$ and $(1/g(x))'$. So really the only new bit of information we need is $(1/g(x))'$ in terms of $g'(x)$. As with the product rule, let's set this up and see how far we can get:

$$\begin{aligned}
\frac{d}{dx}\frac{1}{g(x)} &= \lim_{\Delta x \to 0} \frac{\frac{1}{g(x+\Delta x)} - \frac{1}{g(x)}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{g(x)-g(x+\Delta x)}{g(x+\Delta x)g(x)}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{g(x)-g(x+\Delta x)}{g(x+\Delta x)g(x)\Delta x} \\
&= \lim_{\Delta x \to 0} -\frac{g(x+\Delta x)-g(x)}{\Delta x}\frac{1}{g(x+\Delta x)g(x)} \\
&= -\frac{g'(x)}{g(x)^2}
\end{aligned}$$

Now we can put this together with the product rule:

$$\frac{d}{dx}\frac{f(x)}{g(x)} = f(x)\frac{-g'(x)}{g(x)^2} + f'(x)\frac{1}{g(x)} = \frac{-f(x)g'(x)+f'(x)g(x)}{g(x)^2} = \frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}.$$

EXAMPLE 3.4.1 Compute the derivative of $(x^2+1)/(x^3-3x)$.

$$\frac{d}{dx}\frac{x^2+1}{x^3-3x} = \frac{2x(x^3-3x)-(x^2+1)(3x^2-3)}{(x^3-3x)^2} = \frac{-x^4-6x^2+3}{(x^3-3x)^2}.$$

□

It is often possible to calculate derivatives in more than one way, as we have already seen. Since every quotient can be written as a product, it is always possible to use the product rule to compute the derivative, though it is not always simpler.

EXAMPLE 3.4.2 Find the derivative of $\sqrt{625-x^2}/\sqrt{x}$ in two ways: using the quotient rule, and using the product rule.

Quotient rule:

$$\frac{d}{dx}\frac{\sqrt{625-x^2}}{\sqrt{x}} = \frac{\sqrt{x}(-x/\sqrt{625-x^2}) - \sqrt{625-x^2}\cdot 1/(2\sqrt{x})}{x}.$$

Note that we have used $\sqrt{x} = x^{1/2}$ to compute the derivative of $\sqrt{x}$ by the power rule.

Product rule:

$$\frac{d}{dx}\sqrt{625-x^2}x^{-1/2} = \sqrt{625-x^2}\frac{-1}{2}x^{-3/2} + \frac{-x}{\sqrt{625-x^2}}x^{-1/2}.$$

With a bit of algebra, both of these simplify to

$$-\frac{x^2+625}{2\sqrt{625-x^2}x^{3/2}}.$$

□

Occasionally you will need to compute the derivative of a quotient with a constant numerator, like $10/x^2$. Of course you can use the quotient rule, but it is usually not the easiest method. If we do use it here, we get

$$\frac{d}{dx}\frac{10}{x^2} = \frac{x^2 \cdot 0 - 10 \cdot 2x}{x^4} = \frac{-20}{x^3},$$

since the derivative of 10 is 0. But it is simpler to do this:

$$\frac{d}{dx}\frac{10}{x^2} = \frac{d}{dx}10x^{-2} = -20x^{-3}.$$

Admittedly, x^2 is a particularly simple denominator, but we will see that a similar calculation is usually possible. Another approach is to remember that

$$\frac{d}{dx}\frac{1}{g(x)} = \frac{-g'(x)}{g(x)^2},$$

but this requires extra memorization. Using this formula,

$$\frac{d}{dx}\frac{10}{x^2} = 10\frac{-2x}{x^4}.$$

Note that we first use linearity of the derivative to pull the 10 out in front.

Exercises 3.4.

Find the derivatives of the functions in 1–4 using the quotient rule.

1. $\dfrac{x^3}{x^3 - 5x + 10}$ ⇒

2. $\dfrac{x^2 + 5x - 3}{x^5 - 6x^3 + 3x^2 - 7x + 1}$ ⇒

3. $\dfrac{\sqrt{x}}{\sqrt{625 - x^2}}$ ⇒

4. $\dfrac{\sqrt{625 - x^2}}{x^{20}}$ ⇒

5. Find an equation for the tangent line to $f(x) = (x^2 - 4)/(5 - x)$ at $x = 3$. ⇒

6. Find an equation for the tangent line to $f(x) = (x - 2)/(x^3 + 4x - 1)$ at $x = 1$. ⇒

7. Let P be a polynomial of degree n and let Q be a polynomial of degree m (with Q not the zero polynomial). Using sigma notation we can write

$$P = \sum_{k=0}^{n} a_k x^k, \qquad Q = \sum_{k=0}^{m} b_k x^k.$$

Use sigma notation to write the derivative of the **rational function** P/Q.

8. The curve $y = 1/(1 + x^2)$ is an example of a class of curves each of which is called a **witch of Agnesi**. Sketch the curve and find the tangent line to the curve at $x = 5$. (The word *witch* here is a mistranslation of the original Italian, as described at

`http://mathworld.wolfram.com/WitchofAgnesi.html`

and

`http://witchofagnesi.org/`

⇒

9. If $f'(4) = 5$, $g'(4) = 12$, $(fg)(4) = f(4)g(4) = 2$, and $g(4) = 6$, compute $f(4)$ and $\frac{d}{dx}\frac{f}{g}$ at 4. $\Rightarrow$

3.5 The Chain Rule

So far we have seen how to compute the derivative of a function built up from other functions by addition, subtraction, multiplication and division. There is another very important way that we combine simple functions to make more complicated functions: function composition, as discussed in section 2.3. For example, consider $\sqrt{625 - x^2}$. This function has many simpler components, like 625 and x^2, and then there is that square root symbol, so the square root function $\sqrt{x} = x^{1/2}$ is involved. The obvious question is: can we compute the derivative using the derivatives of the constituents $625 - x^2$ and $\sqrt{x}$? We can indeed. In general, if $f(x)$ and $g(x)$ are functions, we can compute the derivatives of $f(g(x))$ and $g(f(x))$ in terms of $f'(x)$ and $g'(x)$.

EXAMPLE 3.5.1 Form the two possible compositions of $f(x) = \sqrt{x}$ and $g(x) = 625 - x^2$ and compute the derivatives. First, $f(g(x)) = \sqrt{625 - x^2}$, and the derivative is $-x/\sqrt{625 - x^2}$ as we have seen. Second, $g(f(x)) = 625 - (\sqrt{x})^2 = 625 - x$ with derivative -1. Of course, these calculations do not use anything new, and in particular the derivative of $f(g(x))$ was somewhat tedious to compute from the definition. □

Suppose we want the derivative of $f(g(x))$. Again, let's set up the derivative and play some algebraic tricks:

$$\begin{aligned}\frac{d}{dx}f(g(x)) &= \lim_{\Delta x\to 0}\frac{f(g(x+\Delta x)) - f(g(x))}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{f(g(x+\Delta x)) - f(g(x))}{g(x+\Delta x) - g(x)}\frac{g(x+\Delta x) - g(x)}{\Delta x}\end{aligned}$$

Now we see immediately that the second fraction turns into $g'(x)$ when we take the limit. The first fraction is more complicated, but it too looks something like a derivative. The denominator, $g(x + \Delta x) - g(x)$, is a change in the value of g, so let's abbreviate it as $\Delta g = g(x + \Delta x) - g(x)$, which also means $g(x + \Delta x) = g(x) + \Delta g$. This gives us

$$\lim_{\Delta x\to 0}\frac{f(g(x)+\Delta g) - f(g(x))}{\Delta g}.$$

As Δx goes to 0, it is also true that Δg goes to 0, because $g(x + \Delta x)$ goes to $g(x)$. So we can rewrite this limit as

$$\lim_{\Delta g\to 0}\frac{f(g(x)+\Delta g) - f(g(x))}{\Delta g}.$$

Now this looks exactly like a derivative, namely $f'(g(x))$, that is, the function $f'(x)$ with x replaced by $g(x)$. If this all withstands scrutiny, we then get

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x).$$

Unfortunately, there is a small flaw in the argument. Recall that what we mean by $\lim_{\Delta x\to 0}$ involves what happens when Δx is close to 0 *but not equal to 0.* The qualification is very important, since we must be able to divide by Δx. But when Δx is close to 0 but not equal to 0, $\Delta g = g(x+\Delta x)) - g(x)$ is close to 0 *and possibly equal to 0.* This means it doesn't really make sense to divide by Δg. Fortunately, it is possible to recast the argument to avoid this difficulty, but it is a bit tricky; we will not include the details, which can be found in many calculus books. Note that many functions g do have the property that $g(x+\Delta x) - g(x) \neq 0$ when Δx is small, and for these functions the argument above is fine.

The chain rule has a particularly simple expression if we use the Leibniz notation for the derivative. The quantity $f'(g(x))$ is the derivative of f with x replaced by g; this can be written df/dg. As usual, $g'(x) = dg/dx$. Then the chain rule becomes

$$\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx}.$$

This looks like trivial arithmetic, but it is not: dg/dx is not a fraction, that is, not literal division, but a single symbol that means $g'(x)$. Nevertheless, it turns out that what looks like trivial arithmetic, and is therefore easy to remember, is really true.

It will take a bit of practice to make the use of the chain rule come naturally—it is more complicated than the earlier differentiation rules we have seen.

EXAMPLE 3.5.2 Compute the derivative of $\sqrt{625-x^2}$. We already know that the answer is $-x/\sqrt{625-x^2}$, computed directly from the limit. In the context of the chain rule, we have $f(x) = \sqrt{x}$, $g(x) = 625 - x^2$. We know that $f'(x) = (1/2)x^{-1/2}$, so $f'(g(x)) = (1/2)(625-x^2)^{-1/2}$. Note that this is a two step computation: first compute $f'(x)$, then replace x by $g(x)$. Since $g'(x) = -2x$ we have

$$f'(g(x))g'(x) = \frac{1}{2\sqrt{625-x^2}}(-2x) = \frac{-x}{\sqrt{625-x^2}}.$$

□

EXAMPLE 3.5.3 Compute the derivative of $1/\sqrt{625-x^2}$. This is a quotient with a constant numerator, so we could use the quotient rule, but it is simpler to use the chain rule. The function is $(625-x^2)^{-1/2}$, the composition of $f(x)=x^{-1/2}$ and $g(x)=625-x^2$. We compute $f'(x)=(-1/2)x^{-3/2}$ using the power rule, and then

$$f'(g(x))g'(x)=\frac{-1}{2(625-x^2)^{3/2}}(-2x)=\frac{x}{(625-x^2)^{3/2}}.$$

□

In practice, of course, you will need to use more than one of the rules we have developed to compute the derivative of a complicated function.

EXAMPLE 3.5.4 Compute the derivative of

$$f(x)=\frac{x^2-1}{x\sqrt{x^2+1}}.$$

The "last" operation here is division, so to get started we need to use the quotient rule first. This gives

$$\begin{aligned} f'(x) &= \frac{(x^2-1)'x\sqrt{x^2+1}-(x^2-1)(x\sqrt{x^2+1})'}{x^2(x^2+1)} \\ &= \frac{2x^2\sqrt{x^2+1}-(x^2-1)(x\sqrt{x^2+1})'}{x^2(x^2+1)}. \end{aligned}$$

Now we need to compute the derivative of $x\sqrt{x^2+1}$. This is a product, so we use the product rule:

$$\frac{d}{dx}x\sqrt{x^2+1}=x\frac{d}{dx}\sqrt{x^2+1}+\sqrt{x^2+1}.$$

Finally, we use the chain rule:

$$\frac{d}{dx}\sqrt{x^2+1}=\frac{d}{dx}(x^2+1)^{1/2}=\frac{1}{2}(x^2+1)^{-1/2}(2x)=\frac{x}{\sqrt{x^2+1}}.$$

And putting it all together:

$$\begin{aligned} f'(x) &= \frac{2x^2\sqrt{x^2+1}-(x^2-1)(x\sqrt{x^2+1})'}{x^2(x^2+1)} \\ &= \frac{2x^2\sqrt{x^2+1}-(x^2-1)\left(x\frac{x}{\sqrt{x^2+1}}+\sqrt{x^2+1}\right)}{x^2(x^2+1)}. \end{aligned}$$

This can be simplified of course, but we have done all the calculus, so that only algebra is left. □

EXAMPLE 3.5.5 Compute the derivative of $\sqrt{1+\sqrt{1+\sqrt{x}}}$. Here we have a more complicated chain of compositions, so we use the chain rule twice. At the outermost "layer" we have the function $g(x)=1+\sqrt{1+\sqrt{x}}$ plugged into $f(x)=\sqrt{x}$, so applying the chain rule once gives

$$\frac{d}{dx}\sqrt{1+\sqrt{1+\sqrt{x}}}=\frac{1}{2}\left(1+\sqrt{1+\sqrt{x}}\right)^{-1/2}\frac{d}{dx}\left(1+\sqrt{1+\sqrt{x}}\right).$$

Now we need the derivative of $\sqrt{1+\sqrt{x}}$. Using the chain rule again:

$$\frac{d}{dx}\sqrt{1+\sqrt{x}}=\frac{1}{2}\left(1+\sqrt{x}\right)^{-1/2}\frac{1}{2}x^{-1/2}.$$

So the original derivative is

$$\begin{aligned}\frac{d}{dx}\sqrt{1+\sqrt{1+\sqrt{x}}}&=\frac{1}{2}\left(1+\sqrt{1+\sqrt{x}}\right)^{-1/2}\frac{1}{2}\left(1+\sqrt{x}\right)^{-1/2}\frac{1}{2}x^{-1/2}.\\&=\frac{1}{8\sqrt{x}\sqrt{1+\sqrt{x}}\sqrt{1+\sqrt{1+\sqrt{x}}}}\end{aligned}$$

□

Using the chain rule, the power rule, and the product rule, it is possible to avoid using the quotient rule entirely.

EXAMPLE 3.5.6 Compute the derivative of $f(x)=\dfrac{x^3}{x^2+1}$. Write $f(x)=x^3(x^2+1)^{-1}$, then

$$\begin{aligned}f'(x)&=x^3\frac{d}{dx}(x^2+1)^{-1}+3x^2(x^2+1)^{-1}\\&=x^3(-1)(x^2+1)^{-2}(2x)+3x^2(x^2+1)^{-1}\\&=-2x^4(x^2+1)^{-2}+3x^2(x^2+1)^{-1}\\&=\frac{-2x^4}{(x^2+1)^2}+\frac{3x^2}{x^2+1}\\&=\frac{-2x^4}{(x^2+1)^2}+\frac{3x^2(x^2+1)}{(x^2+1)^2}\\&=\frac{-2x^4+3x^4+3x^2}{(x^2+1)^2}=\frac{x^4+3x^2}{(x^2+1)^2}\end{aligned}$$

Note that we already had the derivative on the second line; all the rest is simplification. It is easier to get to this answer by using the quotient rule, so there's a trade off: more work for fewer memorized formulas. □

Exercises 3.5.

Find the derivatives of the functions. For extra practice, and to check your answers, do some of these in more than one way if possible.

1. $x^4 - 3x^3 + (1/2)x^2 + 7x - \pi \Rightarrow$

2. $x^3 - 2x^2 + 4\sqrt{x} \Rightarrow$

3. $(x^2+1)^3 \Rightarrow$

4. $x\sqrt{169 - x^2} \Rightarrow$

5. $(x^2 - 4x + 5)\sqrt{25 - x^2} \Rightarrow$

6. $\sqrt{r^2 - x^2}$, r is a constant $\Rightarrow$

7. $\sqrt{1 + x^4} \Rightarrow$

8. $\dfrac{1}{\sqrt{5 - \sqrt{x}}}$. $\Rightarrow$

9. $(1+3x)^2 \Rightarrow$

10. $\dfrac{x^2 + x + 1}{1 - x} \Rightarrow$

11. $\dfrac{\sqrt{25 - x^2}}{x} \Rightarrow$

12. $\sqrt{\dfrac{169}{x} - x} \Rightarrow$

13. $\sqrt{x^3 - x^2 - (1/x)} \Rightarrow$

14. $100/(100 - x^2)^{3/2} \Rightarrow$

15. $\sqrt[3]{x + x^3} \Rightarrow$

16. $\sqrt{(x^2+1)^2 + \sqrt{1 + (x^2+1)^2}} \Rightarrow$

17. $(x+8)^5 \Rightarrow$

18. $(4-x)^3 \Rightarrow$

19. $(x^2+5)^3 \Rightarrow$

20. $(6 - 2x^2)^3 \Rightarrow$

21. $(1 - 4x^3)^{-2} \Rightarrow$

22. $5(x + 1 - 1/x) \Rightarrow$

23. $4(2x^2 - x + 3)^{-2} \Rightarrow$

24. $\dfrac{1}{1 + 1/x} \Rightarrow$

25. $\dfrac{-3}{4x^2 - 2x + 1} \Rightarrow$

26. $(x^2+1)(5-2x)/2 \Rightarrow$

27. $(3x^2+1)(2x-4)^3 \Rightarrow$

28. $\dfrac{x+1}{x-1} \Rightarrow$

29. $\dfrac{x^2-1}{x^2+1} \Rightarrow$

30. $\dfrac{(x-1)(x-2)}{x-3} \Rightarrow$

31. $\dfrac{2x^{-1} - x^{-2}}{3x^{-1} - 4x^{-2}} \Rightarrow$

32. $3(x^2+1)(2x^2-1)(2x+3) \Rightarrow$

33. $\dfrac{1}{(2x+1)(x-3)} \Rightarrow$

34. $((2x+1)^{-1} + 3)^{-1} \Rightarrow$

35. $(2x+1)^3(x^2+1)^2 \Rightarrow$

36. Find an equation for the tangent line to $f(x) = (x-2)^{1/3}/(x^3 + 4x - 1)^2$ at $x = 1$. $\Rightarrow$

37. Find an equation for the tangent line to $y = 9x^{-2}$ at $(3, 1)$. $\Rightarrow$

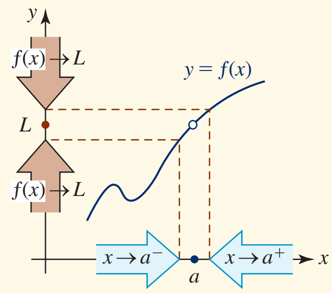
38. Find an equation for the tangent line to $(x^2 - 4x + 5)\sqrt{25 - x^2}$ at $(3, 8)$. $\Rightarrow$

39. Find an equation for the tangent line to $\dfrac{(x^2 + x + 1)}{(1 - x)}$ at $(2, -7)$. $\Rightarrow$

40. Find an equation for the tangent line to $\sqrt{(x^2+1)^2 + \sqrt{1 + (x^2+1)^2}}$ at $(1, \sqrt{4 + \sqrt{5}})$. $\Rightarrow$

Chapter 2

Limit of a Function



In This Chapter Many topics are included in a typical course in calculus. But the three most fundamental topics in this study are the concepts of *limit*, *derivative*, and *integral*. Each of these concepts deals with functions, which is why we began this text by first reviewing some important facts about functions and their graphs.

Historically, two problems are used to introduce the basic tenets of calculus. These are the *tangent line problem* and the *area problem*. We will see in this and the subsequent chapters that the solutions to both problems involve the limit concept.

2.1 Limits—An Informal Approach

Introduction The two broad areas of calculus known as *differential* and *integral calculus* are built on the foundation concept of a *limit*. In this section our approach to this important concept will be intuitive, concentrating on understanding *what* a limit is using numerical and graphical examples. In the next section, our approach will be analytical, that is, we will use algebraic methods to *compute* the value of a limit of a function.

Limit of a Function–Informal Approach Consider the function

$$f(x) = \frac{16 - x^2}{4 + x} \tag{1}$$

whose domain is the set of all real numbers except -4. Although f cannot be evaluated *at* -4 because substituting -4 for x results in the undefined quantity $0/0$, $f(x)$ can be calculated at any number x that is very *close* to -4. The two tables

x	-4.1	-4.01	-4.001
$f(x)$	8.1	8.01	8.001

x	-3.9	-3.99	-3.999
$f(x)$	7.9	7.99	7.999

(2)

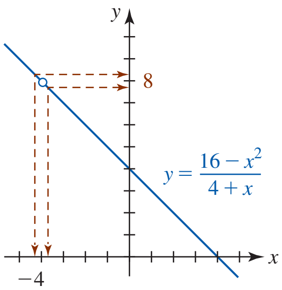


FIGURE 2.1.1 When x is near -4, $f(x)$ is near 8

show that as x approaches -4 from either the left or right, the function values $f(x)$ appear to be approaching 8, in other words, when x is near -4, $f(x)$ is near 8. To interpret the numerical information in (1) graphically, observe that for every number $x \neq -4$, the function f can be simplified by cancellation:

$$f(x) = \frac{16 - x^2}{4 + x} = \frac{(4 + x)(4 - x)}{4 + x} = 4 - x.$$

As seen in FIGURE 2.1.1, the graph of f is essentially the graph of $y = 4 - x$ with the exception that the graph of f has a *hole* at the point that corresponds to $x = -4$. For x sufficiently close to -4, represented by the two arrowheads on the x-axis, the two arrowheads on the y-axis, representing function values $f(x)$, simultaneously get closer and closer to the number 8. Indeed, in view of the numerical results in (2), the arrowheads can be made as *close as we like* to the number 8. We say 8 is the **limit** of $f(x)$ as x approaches -4.

Informal Definition Suppose L denotes a finite number. The notion of $f(x)$ approaching L as x approaches a number a can be defined informally in the following manner.

- *If $f(x)$ can be made arbitrarily close to the number L by taking x sufficiently close to but different from the number a, from both the left and right sides of a, then the **limit** of $f(x)$ as x approaches a is L.*

Notation The discussion of the limit concept is facilitated by using a special notation. If we let the arrow symbol $\rightarrow$ represent the word *approach*, then the symbolism

$x \rightarrow a^-$ *indicates that x approaches a number a from the **left**,*

that is, through numbers that are less than a, and

$x \rightarrow a^+$ *signifies that x approaches a from the **right**,*

that is, through numbers that are greater than a. Finally, the notation

$x \rightarrow a$ *signifies that x approaches a from **both sides**,*

in other words, from the left and the right sides of a on a number line. In the left-hand table in (2) we are letting $x \rightarrow -4^-$ (for example, -4.001 is to the left of -4 on the number line), whereas in the right-hand table $x \rightarrow -4^+$.

One-Sided Limits In general, if a function $f(x)$ can be made arbitrarily close to a number L_1 by taking x sufficiently close to, but not equal to, a number a from the *left*, then we write

$$f(x) \rightarrow L_1 \text{ as } x \rightarrow a^- \qquad \text{or} \qquad \lim_{x \rightarrow a^-} f(x) = L_1. \tag{3}$$

The number L_1 is said to be the **left-hand limit of $f(x)$ as x approaches a**. Similarly, if $f(x)$ can be made arbitrarily close to a number L_2 by taking x sufficiently close to, but not equal to, a number a from the *right*, then L_2 is the **right-hand limit of $f(x)$ as x approaches a** and we write

$$f(x) \to L_2 \text{ as } x \to a^+ \qquad \text{or} \qquad \lim_{x \to a^+} f(x) = L_2. \tag{4}$$

The quantities in (3) and (4) are also referred to as **one-sided limits**.

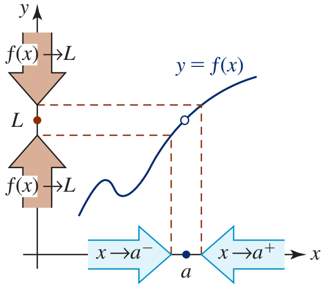


FIGURE 2.1.2 $f(x) \to L$ as $x \to a$ if and only if $f(x) \to L$ as $x \to a^-$ *and* $f(x) \to L$ as $x \to a^+$

Two-Sided Limits If both the left-hand limit $\lim_{x \to a^-} f(x)$ and the right-hand limit $\lim_{x \to a^+} f(x)$ exist and have a common value L,

$$\lim_{x \to a^-} f(x) = L \qquad \text{and} \qquad \lim_{x \to a^+} f(x) = L,$$

then we say that L is the **limit of $f(x)$ as x approaches a** and write

$$\lim_{x \to a} f(x) = L. \tag{5}$$

A limit such as (5) is said to be a **two-sided limit**. See FIGURE 2.1.2. Since the numerical tables in (2) suggest that

$$f(x) \to 8 \text{ as } x \to -4^- \qquad \textit{and} \qquad f(x) \to 8 \text{ as } x \to -4^+, \tag{6}$$

we can replace the two symbolic statements in (6) by the statement

$$f(x) \to 8 \text{ as } x \to -4 \qquad \text{or equivalently} \qquad \lim_{x \to -4} \frac{16 - x^2}{4 + x} = 8. \tag{7}$$

Existence and Nonexistence Of course a limit (one-sided or two-sided) does not have to exist. But it is important that you keep firmly in mind:

- *The existence of a limit of a function f as x approaches a (from one side or from both sides), does not depend on whether f is defined at a but only on whether f is defined for x near the number a.*

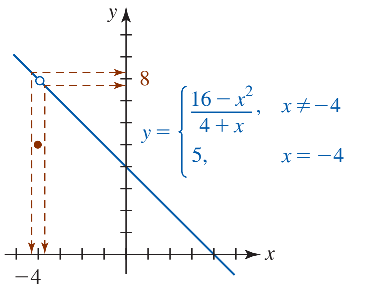


FIGURE 2.1.3 Whether f is defined at a or is not defined at a has no bearing on the existence of the limit of $f(x)$ as $x \to a$

For example, if the function in (1) is modified in the following manner

$$f(x) = \begin{cases} \dfrac{16 - x^2}{4 + x}, & x \neq -4 \\ 5, & x = -4, \end{cases}$$

then $f(-4)$ is defined and $f(-4) = 5$, but still $\lim_{x \to -4} \dfrac{16 - x^2}{4 + x} = 8$. See FIGURE 2.1.3. In general, the two-sided limit $\lim_{x \to a} f(x)$ **does not exist**

- if either of the one-sided limits $\lim_{x \to a^-} f(x)$ or $\lim_{x \to a^+} f(x)$ fails to exist, or
- if $\lim_{x \to a^-} f(x) = L_1$ and $\lim_{x \to a^+} f(x) = L_2$, but $L_1 \neq L_2$.

EXAMPLE 1 A Limit That Exists

The graph of the function $f(x) = -x^2 + 2x + 2$ is shown in FIGURE 2.1.4. As seen from the graph and the accompanying tables, it seems plausible that

$$\lim_{x \to 4^-} f(x) = -6 \qquad \text{and} \qquad \lim_{x \to 4^+} f(x) = -6$$

and consequently $\lim_{x \to 4} f(x) = -6$.

$x \to 4^-$	3.9	3.99	3.999
$f(x)$	-5.41000	-5.94010	-5.99400

$x \to 4^+$	4.1	4.01	4.001
$f(x)$	-6.61000	-6.06010	-6.00600

■

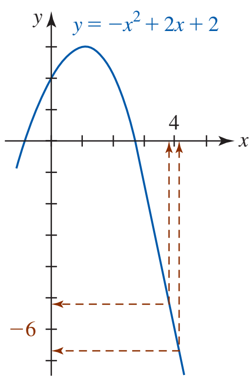


FIGURE 2.1.4 Graph of function in Example 1

Note that in Example 1 the given function is certainly defined at 4, but at no time did we substitute $x = 4$ into the function to find the value of $\lim_{x \to 4} f(x)$.

EXAMPLE 2 A Limit That Exists

The graph of the piecewise-defined function

$$f(x) = \begin{cases} x^2, & x < 2 \\ -x + 6, & x > 2 \end{cases}$$

is given in FIGURE 2.1.5. Notice that $f(2)$ is not defined, but that is of no consequence when considering $\lim_{x \to 2} f(x)$. From the graph and the accompanying tables,

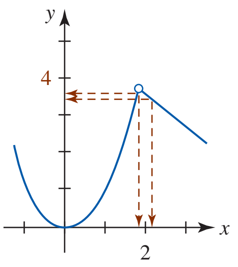


FIGURE 2.1.5 Graph of function in Example 2

$x \to 2^-$	1.9	1.99	1.999
$f(x)$	3.61000	3.96010	3.99600

$x \to 2^+$	2.1	2.01	2.001
$f(x)$	3.90000	3.99000	3.99900

we see that when we make x close to 2, we can make $f(x)$ arbitrarily close to 4, and so

$$\lim_{x \to 2^-} f(x) = 4 \quad \text{and} \quad \lim_{x \to 2^+} f(x) = 4.$$

That is, $\lim_{x \to 2} f(x) = 4$. ■

EXAMPLE 3 A Limit That Does Not Exist

The graph of the piecewise-defined function

$$f(x) = \begin{cases} x + 2, & x \le 5 \\ -x + 10, & x > 5 \end{cases}$$

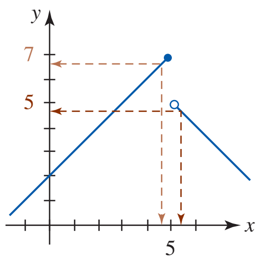


FIGURE 2.1.6 Graph of function in Example 3

is given in FIGURE 2.1.6. From the graph and the accompanying tables, it appears that as x approaches 5 through numbers less than 5 that $\lim_{x \to 5^-} f(x) = 7$. Then as x approaches 5 through numbers greater than 5 it appears that $\lim_{x \to 5^+} f(x) = 5$. But since

$$\lim_{x \to 5^-} f(x) \neq \lim_{x \to 5^+} f(x),$$

we conclude that $\lim_{x \to 5} f(x)$ does not exist.

$x \to 5^-$	4.9	4.99	4.999
$f(x)$	6.90000	6.99000	6.99900

$x \to 5^+$	5.1	5.01	5.001
$f(x)$	4.90000	4.99000	4.99900

■

EXAMPLE 4 A Limit That Does Not Exist

The greatest integer function was discussed in Section 1.1.

Recall, the **greatest integer function** or **floor function** $f(x) = \lfloor x \rfloor$ is defined to be the greatest integer that is less than or equal to x. The domain of f is the set of real numbers $(-\infty, \infty)$. From the graph in FIGURE 2.1.7 we see that $f(n)$ is defined for every integer n; nonetheless, for each integer n, $\lim_{x \to n} f(x)$ does not exist. For example, as x approaches, say, the number 3, the two one-sided limits exist but have different values:

$$\lim_{x \to 3^-} f(x) = 2 \quad \text{whereas} \quad \lim_{x \to 3^+} f(x) = 3. \tag{8}$$

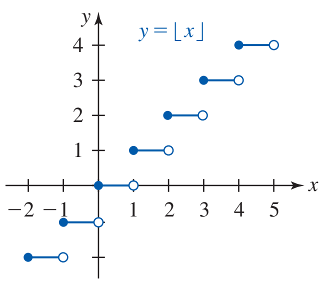


FIGURE 2.1.7 Graph of function in Example 4

In general, for an integer n,

$$\lim_{x \to n^-} f(x) = n - 1 \quad \text{whereas} \quad \lim_{x \to n^+} f(x) = n.$$ ■

EXAMPLE 5 A Right-Hand Limit

From FIGURE 2.1.8 it should be clear that $f(x) = \sqrt{x} \to 0$ as $x \to 0^+$, that is

$$\lim_{x \to 0^+} \sqrt{x} = 0.$$

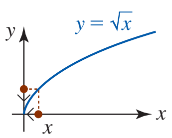


FIGURE 2.1.8 Graph of function in Example 5

It would be incorrect to write $\lim_{x \to 0} \sqrt{x} = 0$ since this notation carries with it the connotation that the limits from the left and from the right exist and are equal to 0. In this case $\lim_{x \to 0^-} \sqrt{x}$ does not exist since $f(x) = \sqrt{x}$ is not defined for $x < 0$. ■

If $x = a$ is a vertical asymptote for the graph of $y = f(x)$, then $\lim_{x\to a} f(x)$ will always fail to exist because the function values $f(x)$ must become unbounded from at least one side of the line $x = a$.

EXAMPLE 6 A Limit That Does Not Exist

A vertical asymptote always corresponds to an infinite break in the graph of a function f. In FIGURE 2.1.9 we see that the y-axis or $x = 0$ is a vertical asymptote for the graph of $f(x) = 1/x$. The tables

$x \to 0^-$	-0.1	-0.01	-0.001
$f(x)$	-10	-100	-1000

$x \to 0^+$	0.1	0.01	0.001
$f(x)$	10	100	1000

clearly show that the function values $f(x)$ become unbounded in absolute value as we get close to 0. In other words, $f(x)$ is not approaching a real number as $x \to 0^-$ nor as $x \to 0^+$. Therefore, neither the left-hand nor the right-hand limit exists as x approaches 0. Thus we conclude that $\lim_{x\to 0} f(x)$ does not exist. ■

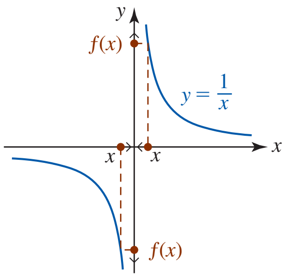


FIGURE 2.1.9 Graph of function in Example 6

EXAMPLE 7 An Important Trigonometric Limit

To do the calculus of the trigonometric functions $\sin x$, $\cos x$, $\tan x$, and so on, it is important to realize that the variable x is either a real number or an angle measured in radians. With that in mind, consider the numerical values of $f(x) = (\sin x)/x$ as $x \to 0^+$ given in the table that follows.

$x \to 0^+$	0.1	0.01	0.001	0.0001
$f(x)$	0.99833416	0.99998333	0.99999983	0.99999999

It is easy to see that the same results given in the table hold as $x \to 0^-$. Because $\sin x$ is an odd function, for $x > 0$ and $-x < 0$ we have $\sin(-x) = -\sin x$ and as a consequence

$$f(-x) = \frac{\sin(-x)}{-x} = \frac{\sin x}{x} = f(x).$$

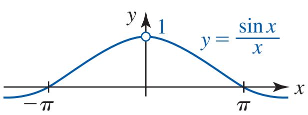


FIGURE 2.1.10 Graph of function in Example 7

As can be seen in FIGURE 2.1.10, f is an even function. The table of numerical values as well as the graph of f strongly suggest the following result:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1. \tag{9}$$ ■

The limit in (9) is a very important result and will be used in Section 3.4. Another trigonometric limit that you are asked to verify as an exercise is given by

$$\lim_{x\to 0} \frac{1 - \cos x}{x} = 0. \tag{10}$$

See Problem 43 in Exercises 2.1. Because of their importance, both (9) and (10) will be proven in Section 2.4.

An Indeterminate Form A limit of a quotient $f(x)/g(x)$, where both the numerator and the denominator approach 0 as $x \to a$, is said to have the **indeterminate form 0/0**. The limit (7) in our initial discussion has this indeterminate form. Many important limits, such as (9) and (10), and the limit

$$\lim_{h\to 0} \frac{f(x + h) - f(x)}{h},$$

which forms the backbone of differential calculus, also have the indeterminate form 0/0.

EXAMPLE 8 An Indeterminate Form

The limit $\lim_{x \to 0} |x|/x$ has the indeterminate form $0/0$, but unlike (7), (9), and (10) this limit fails to exist. To see why, let us examine the graph of the function $f(x) = |x|/x$. For $x \neq 0$, $|x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$ and so we recognize f as the piecewise-defined function

$$f(x) = \frac{|x|}{x} = \begin{cases} 1, & x > 0 \\ -1, & x < 0. \end{cases} \tag{11}$$

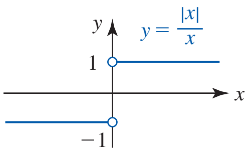


FIGURE 2.1.11 Graph of function in Example 8

From (11) and the graph of f in FIGURE 2.1.11 it should be apparent that both the left-hand and right-hand limits of f exist and

$$\lim_{x \to 0^-} \frac{|x|}{x} = -1 \qquad \text{and} \qquad \lim_{x \to 0^+} \frac{|x|}{x} = 1.$$

Because these one-sided limits are different, we conclude that $\lim_{x \to 0} |x|/x$ does not exist. ■

$\lim_{x \to a}$ NOTES FROM THE CLASSROOM

While graphs and tables of function values may be convincing for determining whether a limit does or does not exist, you are certainly aware that all calculators and computers work only with approximations and that graphs can be drawn inaccurately. A blind use of a calculator can also lead to a false conclusion. For example, $\lim_{x \to 0} \sin(\pi/x)$ is known not to exist, but from the table of values

$x \to 0$	± 0.1	± 0.01	± 0.001
$f(x)$	0	0	0

one would naturally conclude that $\lim_{x \to 0} \sin(\pi/x) = 0$. On the other hand, the limit

$$\lim_{x \to 0} \frac{\sqrt{x^2 + 4} - 2}{x^2} \tag{12}$$

can be shown to exist and equals $\frac{1}{4}$. See Example 11 in Section 2.2. One calculator gives

$x \to 0$	± 0.00001	± 0.000001	± 0.0000001
$f(x)$	0.200000	0.000000	0.000000

The problem in calculating (12) for x very close to 0 is that $\sqrt{x^2 + 4}$ is correspondingly very close to 2. When subtracting two numbers of nearly equal values on a calculator a loss of significant digits may occur due to round-off error.

Exercises 2.1

Answers to selected odd-numbered problems begin on page ANS-000.

≡ Fundamentals

In Problems 1–14, sketch the graph of the function to find the given limit, or state that it does not exist.

1. $\lim_{x \to 2} (3x + 2)$

2. $\lim_{x \to 2} (x^2 - 1)$

3. $\lim_{x \to 0} \left(1 + \frac{1}{x}\right)$

4. $\lim_{x \to 5} \sqrt{x - 1}$

5. $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$

6. $\lim_{x \to 0} \frac{x^2 - 3x}{x}$

7. $\lim_{x \to 3} \frac{|x - 3|}{x - 3}$

8. $\lim_{x \to 0} \frac{|x| - x}{x}$

9. $\lim_{x \to 0} \frac{x^3}{x}$

10. $\lim_{x \to 1} \frac{x^4 - 1}{x^2 - 1}$

11. $\lim_{x \to 0} f(x)$ where $f(x) = \begin{cases} x + 3, & x < 0 \\ -x + 3, & x \geq 0 \end{cases}$

12. $\lim_{x \to 2} f(x)$ where $f(x) = \begin{cases} x, & x < 2 \\ x + 1, & x \geq 2 \end{cases}$

13. $\lim_{x \to 2} f(x)$ where $f(x) = \begin{cases} x^2 - 2x, & x < 2 \\ 1, & x = 2 \\ x^2 - 6x + 8, & x > 2 \end{cases}$

14. $\lim_{x \to 0} f(x)$ where $f(x) = \begin{cases} x^2, & x < 0 \\ 2, & x = 0 \\ \sqrt{x} - 1, & x > 0 \end{cases}$

In Problems 15–18, use the given graph to find the value of each quantity, or state that it does not exist.

(a) $f(1)$ **(b)** $\lim_{x \to 1^+} f(x)$ **(c)** $\lim_{x \to 1^-} f(x)$ **(d)** $\lim_{x \to 1} f(x)$

15.

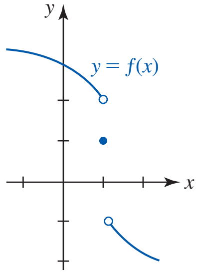


FIGURE 2.1.12 Graph for Problem 15

16.

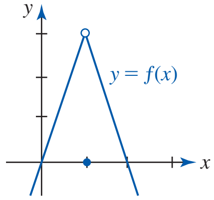


FIGURE 2.1.13 Graph for Problem 16

17.

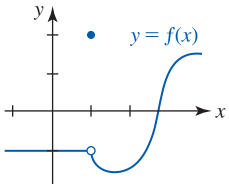


FIGURE 2.1.14 Graph for Problem 17

18.

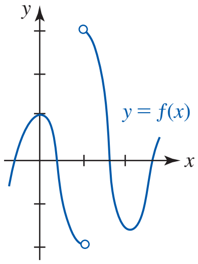


FIGURE 2.1.15 Graph for Problem 18

In Problems 19–28, each limit has the value 0, but some of the notation is incorrect. If the notation is incorrect, give the correct statement.

19. $\lim_{x \to 0} \sqrt[3]{x} = 0$ **20.** $\lim_{x \to 0} \sqrt[4]{x} = 0$

21. $\lim_{x \to 1} \sqrt{1 - x} = 0$ **22.** $\lim_{x \to -2^+} \sqrt{x + 2} = 0$

23. $\lim_{x \to 0^-} \lfloor x \rfloor = 0$ **24.** $\lim_{x \to \frac{1}{2}} \lfloor x \rfloor = 0$

25. $\lim_{x \to \pi} \sin x = 0$ **26.** $\lim_{x \to 1} \cos^{-1} x = 0$

27. $\lim_{x \to 3^+} \sqrt{9 - x^2} = 0$ **28.** $\lim_{x \to 1} \ln x = 0$

In Problems 29 and 30, use the given graph to find each limit, or state that it does not exist.

29. **(a)** $\lim_{x \to -4^+} f(x)$ **(b)** $\lim_{x \to -2} f(x)$
(c) $\lim_{x \to 0} f(x)$ **(d)** $\lim_{x \to 1} f(x)$
(e) $\lim_{x \to 3} f(x)$ **(f)** $\lim_{x \to 4^-} f(x)$

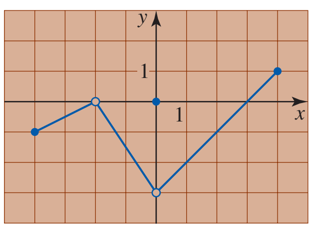


FIGURE 2.1.16 Graph for Problem 29

30. **(a)** $\lim_{x \to -5} f(x)$ **(b)** $\lim_{x \to -3^-} f(x)$
(c) $\lim_{x \to -3^+} f(x)$ **(d)** $\lim_{x \to -3} f(x)$
(e) $\lim_{x \to 0} f(x)$ **(f)** $\lim_{x \to 1} f(x)$

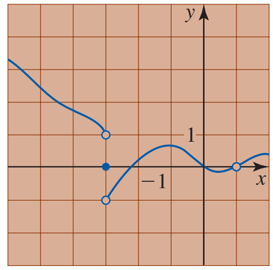


FIGURE 2.1.17 Graph for Problem 30

In Problems 31–34, sketch a graph of a function f with the given properties.

31. $f(-1) = 3, f(0) = -1, f(1) = 0, \lim_{x \to 0} f(x)$ does not exist

32. $f(-2) = 3, \lim_{x \to 0^-} f(x) = 2, \lim_{x \to 0^+} f(x) = -1, f(1) = -2$

33. $f(0) = 1, \lim_{x \to 1^-} f(x) = 3, \lim_{x \to 1^+} f(x) = 3, f(1)$ is undefined, $f(3) = 0$

34. $f(-2) = 2, f(x) = 1, -1 \leq x \leq 1, \lim_{x \to -1} f(x) = 1, \lim_{x \to 1} f(x)$ does not exist, $f(2) = 3$

Calculator/CAS Problems

In Problems 35–40, use a calculator or CAS to obtain the graph of the given function f on the interval $[-0.5, 0.5]$. Use the graph to conjecture the value of $\lim_{x \to 0} f(x)$, or state that the limit does not exist.

35. $f(x) = \cos\dfrac{1}{x}$ **36.** $f(x) = x\cos\dfrac{1}{x}$

37. $f(x) = \dfrac{2 - \sqrt{4 + x}}{x}$

38. $f(x) = \dfrac{9}{x}\left[\sqrt{9 - x} - \sqrt{9 + x}\right]$

39. $f(x) = \dfrac{e^{-2x} - 1}{x}$ **40.** $f(x) = \dfrac{\ln|x|}{x}$

In Problems 41–50, proceed as in Examples 3, 6, and 7 and use a calculator to construct tables of function values. Conjecture the value of each limit, or state that it does not exist.

41. $\lim_{x \to 1} \dfrac{6\sqrt{x} - 6\sqrt{2x - 1}}{x - 1}$ **42.** $\lim_{x \to 1} \dfrac{\ln x}{x - 1}$

43. $\lim_{x \to 0} \dfrac{1 - \cos x}{x}$ **44.** $\lim_{x \to 0} \dfrac{1 - \cos x}{x^2}$

45. $\lim_{x \to 0} \dfrac{x}{\sin 3x}$ **46.** $\lim_{x \to 0} \dfrac{\tan x}{x}$

47. $\lim_{x \to 4} \dfrac{\sqrt{x} - 2}{x - 4}$ **48.** $\lim_{x \to 3}\left[\dfrac{6}{x^2 - 9} - \dfrac{6\sqrt{x - 2}}{x^2 - 9}\right]$

49. $\lim_{x \to 1} \dfrac{x^4 + x - 2}{x - 1}$ **50.** $\lim_{x \to -2} \dfrac{x^3 + 8}{x + 2}$

2.2 Limit Theorems

Introduction The intention of the informal discussion in Section 2.1 was to give you an intuitive grasp of when a limit does or does not exist. However, it is neither desirable nor practical, in every instance, to reach a conclusion about the existence of a limit based on a graph or on a table of numerical values. We must be able to evaluate a limit, or discern its non-existence, in a somewhat mechanical fashion. The theorems that we shall consider in this section establish such a means. The proofs of some of these results are given in the *Appendix*.

The first theorem gives two basic results that will be used throughout the discussion of this section.

Theorem 2.2.1 Two Fundamental Limits

(*i*) $\lim_{x \to a} c = c$, where c is a constant

(*ii*) $\lim_{x \to a} x = a$

Although both parts of Theorem 2.2.1 require a formal proof, Theorem 2.2.1(*ii*) is almost tautological when stated in words:

- *The limit of x as x is approaching a is a.*

See the *Appendix* for a proof of Theorem 2.2.1(*i*).

EXAMPLE 1 Using Theorem 2.2.1

(a) From Theorem 2.2.1(*i*),

$$\lim_{x \to 2} 10 = 10 \qquad \text{and} \qquad \lim_{x \to 6} \pi = \pi.$$

(b) From Theorem 2.1.1(*ii*),

$$\lim_{x \to 2} x = 2 \qquad \text{and} \qquad \lim_{x \to 0} x = 0.$$

■

The limit of a constant multiple of a function f is the constant times the limit of f as x approaches a number a.

Theorem 2.2.2 Limit of a Constant Multiple

If c is a constant, then

$$\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x).$$

We can now start using theorems in conjunction with each other.

EXAMPLE 2 Using Theorems 2.2.1 and 2.2.2

From Theorems 2.2.1 (*ii*) and 2.2.2,

(a) $\lim_{x \to 8} 5x = 5 \lim_{x \to 8} x = 5 \cdot 8 = 40$

(b) $\lim_{x \to -2} \left(-\frac{3}{2}x\right) = -\frac{3}{2} \lim_{x \to -2} x = \left(-\frac{3}{2}\right) \cdot (-2) = 3.$ ■

The next theorem is particularly important because it gives us a way of computing limits in an algebraic manner.

Theorem 2.2.3 Limit of a Sum, Product, and Quotient

Suppose a is a real number and $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. If $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$, then

(*i*) $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L_1 \pm L_2$,

(*ii*) $\lim_{x\to a}[f(x)g(x)] = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right) = L_1L_2$, and

(*iii*) $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{L_1}{L_2}$, $L_2 \neq 0$.

Theorem 2.2.3 can be stated in words:

- *If both limits exist, then*
  - (*i*) *the limit of a sum is the sum of the limits,*
  - (*ii*) *the limit of a product is the product of the limits, and*
  - (*iii*) *the limit of a quotient is the quotient of the limits provided the limit of the denominator is not zero.*

Note: If all limits exist, then Theorem 2.2.3 is also applicable to one-sided limits, that is, the symbolism $x \to a$ in Theorem 2.2.3 can be replaced by either $x \to a^-$ or $x \to a^+$. Moreover, Theorem 2.2.3 extends to differences, sums, products, and quotients that involve more than two functions. See the *Appendix* for a proof of Theorem 2.2.3.

EXAMPLE 3 Using Theorem 2.2.3

Evaluate $\lim_{x\to 5}(10x + 7)$.

Solution From Theorems 2.2.1 and 2.2.2, we know that $\lim_{x\to 5} 7$ and $\lim_{x\to 5} 10x$ exist. Hence, from Theorem 2.2.3(*i*),

$$\begin{aligned}\lim_{x\to 5}(10x + 7) &= \lim_{x\to 5} 10x + \lim_{x\to 5} 7\\ &= 10\lim_{x\to 5} x + \lim_{x\to 5} 7\\ &= 10 \cdot 5 + 7 = 57.\end{aligned}$$

■

Limit of a Power Theorem 2.2.3(*ii*) can be used to calculate the limit of a positive integer power of a function. For example, if $\lim_{x\to a} f(x) = L$, then from Theorem 2.2.3(*ii*) with $g(x) = f(x)$,

$$\lim_{x\to a}[f(x)]^2 = \lim_{x\to a}[f(x)\cdot f(x)] = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} f(x)\right) = L^2.$$

By the same reasoning we can apply Theorem 2.2.3(*ii*) to the general case where $f(x)$ is a factor n times. This result is stated as the next theorem.

Theorem 2.2.4 Limit of a Power

Let $\lim_{x\to a} f(x) = L$ and n be a positive integer. Then

$$\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n = L^n.$$

For the special case $f(x) = x$, the result given in Theorem 2.2.4 yields

$$\lim_{x\to a} x^n = a^n. \tag{1}$$

EXAMPLE 4 Using (1) and Theorem 2.2.3

Evaluate

(a) $\lim_{x\to 10} x^3$ **(b)** $\lim_{x\to 4} \frac{5}{x^2}$.

Solution

(a) From (1),

$$\lim_{x\to 10} x^3 = 10^3 = 1000.$$

(b) From Theorem 2.2.1 and (1) we know that $\lim_{x\to 4} 5 = 5$ and $\lim_{x\to 4} x^2 = 16 \neq 0$. Therefore by Theorem 2.2.3(*iii*),

$$\lim_{x\to 4} \frac{5}{x^2} = \frac{\lim_{x\to 4} 5}{\lim_{x\to 4} x^2} = \frac{5}{4^2} = \frac{5}{16}. \qquad \blacksquare$$

EXAMPLE 5 Using Theorem 2.2.3

Evaluate $\lim_{x\to 3}(x^2 - 5x + 6)$.

Solution In view of Theorem 2.2.1, Theorem 2.2.2, and (1) all limits exist. Therefore by Theorem 2.2.3(*i*),

$$\lim_{x\to 3}(x^2 - 5x + 6) = \lim_{x\to 3} x^2 - \lim_{x\to 3} 5x + \lim_{x\to 3} 6 = 3^2 - 5\cdot 3 + 6 = 0. \qquad \blacksquare$$

EXAMPLE 6 Using Theorems 2.2.3 and 2.2.4

Evaluate $\lim_{x\to 1}(3x - 1)^{10}$.

Solution First, we see from Theorem 2.2.3(*i*) that

$$\lim_{x\to 1}(3x - 1) = \lim_{x\to 1} 3x - \lim_{x\to 1} 1 = 2.$$

It then follows from Theorem 2.2.4 that

$$\lim_{x\to 1}(3x - 1)^{10} = \left[\lim_{x\to 1}(3x - 1)\right]^{10} = 2^{10} = 1024. \qquad \blacksquare$$

Limit of a Polynomial Function Some limits can be evaluated by *direct substitution*. We can use (1) and Theorem 2.2.3(*i*) to compute the limit of a general polynomial function. If

$$f(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$$

is a polynomial function, then

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a}\left(c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0\right)\\ &= \lim_{x\to a} c_n x^n + \lim_{x\to a} c_{n-1}x^{n-1} + \cdots + \lim_{x\to a} c_1 x + \lim_{x\to a} c_0\\ &= c_n a^n + c_{n-1}a^{n-1} + \cdots + c_1 a + c_0. \leftarrow \text{f is defined at $x = a$ and this limit is $f(a)$}\end{aligned}$$

In other words, to evaluate a limit of a polynomial function f as x approaches a real number a, *we need only evaluate the function at* $x = a$:

$$\lim_{x\to a} f(x) = f(a). \tag{2}$$

A reexamination of Example 5 shows that $\lim_{x\to 3} f(x)$, where $f(x) = x^2 - 5x + 6$, is given by $f(3) = 0$.

Because a rational function f is a quotient of two polynomials $p(x)$ and $q(x)$, it follows from (2) and Theorem 2.2.3(*iii*) that a limit of a rational function $f(x) = p(x)/q(x)$ can also be found by evaluating f at $x = a$:

$$\lim_{x\to a} f(x) = \lim_{x\to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}. \tag{3}$$

Of course we must add to (3) the all-important requirement that the limit of the denominator is not 0, that is, $q(a) \neq 0$.

EXAMPLE 7 Using (2) and (3)

Evaluate $\lim_{x\to-1} \dfrac{3x-4}{8x^2+2x-2}$.

Solution $f(x) = \dfrac{3x-4}{8x^2+2x-2}$ is a rational function and so if we identify the polynomials $p(x) = 3x - 4$ and $q(x) = 8x^2 + 2x - 2$, then from (2),

$$\lim_{x\to-1} p(x) = p(-1) = -7 \qquad \text{and} \qquad \lim_{x\to-1} q(x) = q(-1) = 4.$$

Since $q(-1) \neq 0$ it follows from (3) that

$$\lim_{x\to-1} \frac{3x-4}{8x^2+2x-2} = \frac{p(-1)}{q(-1)} = \frac{-7}{4} = -\frac{7}{4}.$$

You should not get the impression that we can *always* find a limit of a function by substituting the number *a directly into the function*.

EXAMPLE 8 Using Theorem 2.2.3

Evaluate $\lim_{x\to1} \dfrac{x-1}{x^2+x-2}$.

Solution The function in this limit is rational, but if we substitute $x = 1$ into the function we see that this limit has the indeterminate form $0/0$. However, by simplifying *first*, we can then apply Theorem 2.2.3(*iii*):

$$\begin{aligned} \lim_{x\to1} \frac{x-1}{x^2+x-2} &= \lim_{x\to1} \frac{x-1}{(x-1)(x+2)} \quad \leftarrow \text{cancellation is valid provided that } x \neq 1 \\ &= \lim_{x\to1} \frac{1}{x+2} \\ &= \frac{\lim_{x\to1} 1}{\lim_{x\to1}(x+2)} = \frac{1}{3}. \end{aligned}$$

If a limit of a rational function has the indeterminate form $0/0$ as $x \to a$, then by the Factor Theorem of algebra $x - a$ must be a factor of both the numerator and the denominator. Factor those quantities and cancel the factor $x - a$.

Sometimes you can tell at a glance when *a limit does not exist*.

Theorem 2.2.5 A Limit That Does Not Exist

Let $\lim_{x\to a} f(x) = L_1 \neq 0$ and $\lim_{x\to a} g(x) = 0$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

does not exist.

PROOF We will give an indirect proof of this result based on Theorem 2.2.3. Suppose $\lim_{x\to a} f(x) = L_1 \neq 0$ and $\lim_{x\to a} g(x) = 0$ and suppose further that $\lim_{x\to a}(f(x)/g(x))$ exists and equals L_2. Then

$$\begin{aligned} L_1 = \lim_{x\to a} f(x) &= \lim_{x\to a}\left(g(x)\cdot\frac{f(x)}{g(x)}\right), \qquad g(x) \neq 0, \\ &= \left(\lim_{x\to a} g(x)\right)\left(\lim_{x\to a}\frac{f(x)}{g(x)}\right) = 0 \cdot L_2 = 0. \end{aligned}$$

By contradicting the assumption that $L_1 \neq 0$, we have proved the theorem.

EXAMPLE 9 Using Theorems 2.2.3 and 2.2.5

Evaluate

(a) $\lim_{x\to 5}\dfrac{x}{x-5}$ **(b)** $\lim_{x\to 5}\dfrac{x^2-10x-25}{x^2-4x-5}$ **(c)** $\lim_{x\to 5}\dfrac{x-5}{x^2-10x+25}$.

Solution Each function in the three parts of the example is rational.

(a) Since the limit of the numerator x is 5, but the limit of the denominator $x-5$ is 0, we conclude from Theorem 2.2.5 that the limit does not exist.

(b) Substituting $x=5$ makes both the numerator and denominator 0, and so the limit has the indeterminate form $0/0$. By the Factor Theorem of algebra, $x-5$ is a factor of both the numerator and denominator. Hence,

$$\begin{aligned}\lim_{x\to 5}\frac{x^2-10x-25}{x^2-4x-5} &= \lim_{x\to 5}\frac{(x-5)^2}{(x-5)(x+1)} && \leftarrow \text{cancel the factor } x-5\\ &= \lim_{x\to 5}\frac{x-5}{x+1}\\ &= \frac{0}{6}=0. && \leftarrow \text{limit exists}\end{aligned}$$

(c) Again, the limit has the indeterminate form $0/0$. After factoring the denominator and canceling the factors we see from the algebra

$$\begin{aligned}\lim_{x\to 5}\frac{x-5}{x^2-10x+25} &= \lim_{x\to 5}\frac{x-5}{(x-5)^2}\\ &= \lim_{x\to 5}\frac{1}{x-5}\end{aligned}$$

that the limit does not exist since the limit of the numerator in the last expression is now 1 but the limit of the denominator is 0. ■

Limit of a Root The limit of the nth root of a function is the nth root of the limit whenever the limit exists and has a real nth root. The next theorem summarizes this fact.

Theorem 2.2.6 Limit of a Root

Let $\lim_{x\to a} f(x) = L$ and n be a positive integer. Then

$$\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L},$$

provided that $L \ge 0$ when n is even.

An immediate special case of Theorem 2.2.6 is

$$\lim_{x\to a}\sqrt[n]{x} = \sqrt[n]{a}, \tag{4}$$

provided $a \ge 0$ when n is even. For example, $\lim_{x\to 9}\sqrt{x} = \left[\lim_{x\to 9} x\right]^{1/2} = 9^{1/2} = 3$.

EXAMPLE 10 Using (4) and Theorem 2.2.3

Evaluate $\lim_{x\to -8}\dfrac{x-\sqrt[3]{x}}{2x+10}$.

Solution Since $\lim_{x\to -8}(2x+10) = -6 \ne 0$, we see from Theorem 2.2.3(*iii*) and (4) that

$$\lim_{x\to -8}\frac{x-\sqrt[3]{x}}{2x+10} = \frac{\lim_{x\to -8} x - \left[\lim_{x\to -8} x\right]^{1/3}}{\lim_{x\to -8}(2x+10)} = \frac{-8-(-8)^{1/3}}{-6} = \frac{-6}{-6} = 1.$$ ■

When a limit of an algebraic function involving radicals has the indeterminate form $0/0$, rationalization of the numerator or the denominator may be something to try.

EXAMPLE 11 Rationalization of a Numerator

Evaluate $\lim_{x\to 0}\dfrac{\sqrt{x^2+4}-2}{x^2}$.

We have seen this limit in (12) in *Notes from the Classroom* at the end of Section 2.1.

Solution Because $\lim_{x\to 0}\sqrt{x^2+4} = \sqrt{\lim_{x\to 0}(x^2+4)} = 2$ we see by inspection that the given limit has the indeterminate form $0/0$. However, by rationalization of the numerator we obtain

$$\begin{aligned}
\lim_{x\to 0}\frac{\sqrt{x^2+4}-2}{x^2} &= \lim_{x\to 0}\frac{\sqrt{x^2+4}-2}{x^2}\cdot\frac{\sqrt{x^2+4}+2}{\sqrt{x^2+4}+2}\\
&= \lim_{x\to 0}\frac{(x^2+4)-4}{x^2\left(\sqrt{x^2+4}+2\right)}\\
&= \lim_{x\to 0}\frac{x^2}{x^2\left(\sqrt{x^2+4}+2\right)} && \leftarrow \text{cancel } x\text{'s}\\
&= \lim_{x\to 0}\frac{1}{\sqrt{x^2+4}+2}. && \leftarrow \text{this limit is no longer } 0/0
\end{aligned}$$

We are now in a position to use Theorems 2.2.3 and 2.2.6:

$$\begin{aligned}
\lim_{x\to 0}\frac{\sqrt{x^2+4}-2}{x^2} &= \lim_{x\to 0}\frac{1}{\sqrt{x^2+4}+2}\\
&= \frac{\lim_{x\to 0}1}{\sqrt{\lim_{x\to 0}(x^2+4)}+\lim_{x\to 0}2}\\
&= \frac{1}{2+2} = \frac{1}{4}.
\end{aligned}$$

∎

In case anyone is wondering whether there can be more than one limit of a function $f(x)$ as $x\to a$, we state the last theorem for the record.

Theorem 2.2.7 Existence Implies Uniqueness

If $\lim_{x\to a} f(x)$ exists, then it is unique.

$\lim_{x\to a}$ NOTES FROM THE CLASSROOM

In mathematics it is just as important to be aware of what a definition or a theorem does *not* say as what it says.

(*i*) Property (*i*) of Theorem 2.2.3 does not say that the limit of a sum is *always* the sum of the limits. For example, $\lim_{x\to 0}(1/x)$ does not exist, so

$$\lim_{x\to 0}\left[\frac{1}{x}-\frac{1}{x}\right] \neq \lim_{x\to 0}\frac{1}{x} - \lim_{x\to 0}\frac{1}{x}.$$

Nevertheless, since $1/x - 1/x = 0$ for $x \neq 0$, the limit of the difference exists

$$\lim_{x\to 0}\left[\frac{1}{x}-\frac{1}{x}\right] = \lim_{x\to 0} 0 = 0.$$

(*ii*) Similarly, the limit of a product could exist and yet not be equal to the product of the limits. For example, $x/x = 1$, for $x \neq 0$, and so

$$\lim_{x\to 0}\left(x\cdot\frac{1}{x}\right) = \lim_{x\to 0} 1 = 1$$

but

$$\lim_{x\to 0}\left(x\cdot\frac{1}{x}\right) \neq \left(\lim_{x\to 0} x\right)\left(\lim_{x\to 0}\frac{1}{x}\right)$$

because $\lim_{x\to 0}(1/x)$ does not exist.

(*iii*) Theorem 2.2.5 does not say that the limit of a quotient fails to exist whenever the limit of the denominator is zero. Example 8 provides a counterexample to that interpretation. However, Theorem 2.2.5 states that a limit of a quotient does not exist whenever the limit of the denominator is zero *and* the limit of the numerator is not zero.

Exercises 2.2

Answers to selected odd-numbered problems begin on page ANS-000.

Fundamentals

In Problems 1–52, find the given limit, or state that it does not exist.

1. $\lim_{x\to -4} 15$
2. $\lim_{x\to 0} \cos \pi$
3. $\lim_{x\to 3} (-4)x$
4. $\lim_{x\to 2} (3x - 9)$
5. $\lim_{x\to -2} x^2$
6. $\lim_{x\to 5} (-x^3)$
7. $\lim_{x\to -1} (x^3 - 4x + 1)$
8. $\lim_{x\to 6} (-5x^2 + 6x + 8)$
9. $\lim_{x\to 2} \dfrac{2x + 4}{x - 7}$
10. $\lim_{x\to 0} \dfrac{x + 5}{3x}$
11. $\lim_{t\to 1} (3t - 1)(5t^2 + 2)$
12. $\lim_{t\to -2} (t + 4)^2$
13. $\lim_{s\to 7} \dfrac{s^2 - 21}{s + 2}$
14. $\lim_{x\to 6} \dfrac{x^2 - 6x}{x^2 - 7x + 6}$
15. $\lim_{x\to -1} (x + x^2 + x^3)^{135}$
16. $\lim_{x\to 2} \dfrac{(3x - 4)^{40}}{(x^2 - 2)^{36}}$
17. $\lim_{x\to 6} \sqrt{2x - 5}$
18. $\lim_{x\to 8} (1 + \sqrt[3]{x})$
19. $\lim_{t\to 1} \dfrac{\sqrt{t}}{t^2 + t - 2}$
20. $\lim_{x\to 2} x^2\sqrt{x^2 + 5x + 2}$
21. $\lim_{y\to -5} \dfrac{y^2 - 25}{y + 5}$
22. $\lim_{u\to 8} \dfrac{u^2 - 5u - 24}{u - 8}$
23. $\lim_{x\to 1} \dfrac{x^3 - 1}{x - 1}$
24. $\lim_{t\to -1} \dfrac{t^3 + 1}{t^2 - 1}$
25. $\lim_{x\to 10} \dfrac{(x - 2)(x + 5)}{(x - 8)}$
26. $\lim_{x\to -3} \dfrac{2x + 6}{4x^2 - 36}$
27. $\lim_{x\to 2} \dfrac{x^3 + 3x^2 - 10x}{x - 2}$
28. $\lim_{x\to 1.5} \dfrac{2x^2 + 3x - 9}{x - 1.5}$
29. $\lim_{t\to 1} \dfrac{t^3 - 2t + 1}{t^3 + t^2 - 2}$
30. $\lim_{x\to 0} x^3(x^4 + 2x^3)^{-1}$
31. $\lim_{x\to 0^+} \dfrac{(x + 2)(x^5 - 1)^3}{(\sqrt{x} + 4)^2}$
32. $\lim_{x\to -2} x\sqrt{x + 4}\,\sqrt[3]{x - 6}$
33. $\lim_{x\to 0} \left[\dfrac{x^2 + 3x - 1}{x} + \dfrac{1}{x}\right]$
34. $\lim_{x\to 2} \left[\dfrac{1}{x - 2} - \dfrac{6}{x^2 + 2x - 8}\right]$
35. $\lim_{x\to 3^+} \dfrac{(x + 3)^2}{\sqrt{x - 3}}$
36. $\lim_{x\to 3} (x - 4)^{99}(x^2 - 7)^{10}$
37. $\lim_{x\to 10} \sqrt{\dfrac{10x}{2x + 5}}$
38. $\lim_{r\to 1} \dfrac{\sqrt{(r^2 + 3r - 2)^3}}{\sqrt[3]{(5r - 3)^2}}$
39. $\lim_{h\to 4} \sqrt{\dfrac{h}{h + 5}}\left(\dfrac{h^2 - 16}{h - 4}\right)^2$
40. $\lim_{t\to 2} (t + 2)^{3/2}(2t + 4)^{1/3}$
41. $\lim_{x\to 0^-} \sqrt[5]{\dfrac{x^3 - 64x}{x^2 + 2x}}$
42. $\lim_{x\to 1^+} \left(8x + \dfrac{2}{x}\right)^5$
43. $\lim_{t\to 1} (at^2 - bt)^2$
44. $\lim_{x\to -1} \sqrt{u^2x^2 + 2xu + 1}$
45. $\lim_{h\to 0} \dfrac{(8 + h)^2 - 64}{h}$
46. $\lim_{h\to 0} \dfrac{1}{h}[(1 + h)^3 - 1]$
47. $\lim_{h\to 0} \dfrac{1}{h}\left(\dfrac{1}{x + h} - \dfrac{1}{x}\right)$
48. $\lim_{h\to 0} \dfrac{\sqrt{x + h} - \sqrt{x}}{h}\ (x > 0)$
49. $\lim_{t\to 1} \dfrac{\sqrt{t} - 1}{t - 1}$
50. $\lim_{u\to 5} \dfrac{\sqrt{u + 4} - 3}{u - 5}$
51. $\lim_{v\to 0} \dfrac{\sqrt{25 + v} - 5}{\sqrt{1 + v} - 1}$
52. $\lim_{x\to 1} \dfrac{4 - \sqrt{x + 15}}{x^2 - 1}$

In Problems 53–60, assume that $\lim_{x\to a} f(x) = 4$ and $\lim_{x\to a} g(x) = 2$. Find the given limit, or state that it does not exist.

53. $\lim_{x\to a} [5f(x) + 6g(x)]$
54. $\lim_{x\to a} [f(x)]^3$
55. $\lim_{x\to a} \dfrac{1}{g(x)}$
56. $\lim_{x\to a} \sqrt{\dfrac{f(x)}{g(x)}}$
57. $\lim_{x\to a} \dfrac{f(x)}{f(x) - 2g(x)}$
58. $\lim_{x\to a} \dfrac{[f(x)]^2 - 4[g(x)]^2}{f(x) - 2g(x)}$
59. $\lim_{x\to a} xf(x)g(x)$
60. $\lim_{x\to a} \dfrac{6x + 3}{xf(x) + g(x)},\ a \neq -\frac{1}{2}$

Think About It

In Problems 61 and 62, use the first result to find the limits in parts (a)–(c). Justify each step in your work citing the appropriate property of limits.

61. $\lim_{x\to 1} \dfrac{x^{100} - 1}{x - 1} = 100$

   (a) $\lim_{x\to 1} \dfrac{x^{100} - 1}{x^2 - 1}$ (b) $\lim_{x\to 1} \dfrac{x^{50} - 1}{x - 1}$ (c) $\lim_{x\to 1} \dfrac{(x^{100} - 1)^2}{(x - 1)^2}$

62. $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$

   (a) $\lim_{x\to 0} \dfrac{2x}{\sin x}$ (b) $\lim_{x\to 0} \dfrac{1 - \cos^2 x}{x^2}$ (c) $\lim_{x\to 0} \dfrac{8x^2 - \sin x}{x}$

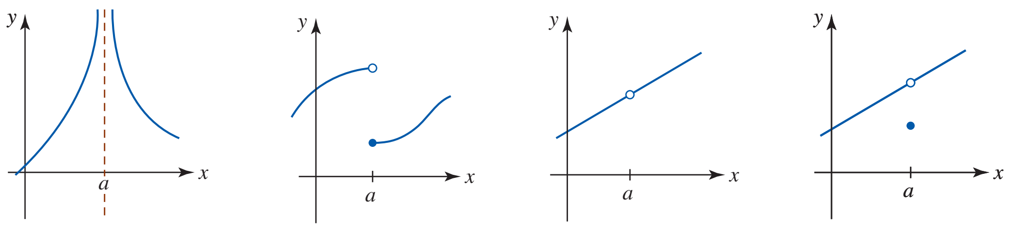
63. Using $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$, show that $\lim_{x\to 0} \sin x = 0$.

64. If $\lim_{x\to 2} \dfrac{2f(x) - 5}{x + 3} = 4$, find $\lim_{x\to 2} f(x)$.

2.3 Continuity

Introduction In the discussion in Section 1.1 on graphing functions, we used the phrase "connect the points with a smooth curve." This phrase invokes the image of a graph that is a nice *continuous* curve—in other words, a curve with no breaks, gaps, or holes in it. Indeed, a continuous function is often described as one whose graph can be drawn without lifting pencil from paper.

In Section 2.2 we saw that the function value $f(a)$ played no part in determining the existence of $\lim_{x\to a} f(x)$. But we did see in Section 2.2 that limits as $x \to a$ of polynomial functions and certain rational functions could be found by simply evaluating the function *at* $x = a$. The reason we can do that in some instances is the fact that the function is *continuous* at a number a. In this section we will see that both the value $f(a)$ and the limit of f as x approaches a number a play major roles in defining the notion of continuity. Before giving the definition, we illustrate in FIGURE 2.3.1 some intuitive examples of graphs of functions that are *not* continuous at a.



(a) $\lim_{x\to a} f(x)$ does not exist and $f(a)$ is not defined

(b) $\lim_{x\to a} f(x)$ does not exist but $f(a)$ is defined

(c) $\lim_{x\to a} f(x)$ exists but $f(a)$ is not defined

(d) $\lim_{x\to a} f(x)$ exists, $f(a)$ is defined, but $\lim_{x\to a} f(x) \neq f(a)$

FIGURE 2.3.1 Four examples of f *not* continuous at a

Continuity at a Number Figure 2.3.1 suggests the following threefold condition of continuity of a function f at a number a.

Definition 2.3.1 Continuity at a

A function f is said to be **continuous** at a number a if

(i) $f(a)$ is defined, (ii) $\lim_{x\to a} f(x)$ exists, and (iii) $\lim_{x\to a} f(x) = f(a)$.

If any one of the three conditions in Definition 2.3.1 fails, then f is said to be **discontinuous** at the number a.

EXAMPLE 1 Three Functions

Determine whether each of the functions is continuous at 1.

(a) $f(x) = \dfrac{x^3 - 1}{x - 1}$ **(b)** $g(x) = \begin{cases} \dfrac{x^3 - 1}{x - 1}, & x \neq 1 \\ 2, & x = 1 \end{cases}$ **(c)** $h(x) = \begin{cases} \dfrac{x^3 - 1}{x - 1}, & x \neq 1 \\ 3, & x = 1 \end{cases}$.

Solution

(a) f is discontinuous at 1 since substituting $x = 1$ into the function results in $0/0$. We say that $f(1)$ is not defined and so the first condition of continuity in Definition 2.3.1 is violated.

(b) Because g is defined at 1, that is, $g(1) = 2$, we next determine whether $\lim_{x\to 1} g(x)$ exists. From

$$\lim_{x\to 1} \frac{x^3 - 1}{x - 1} = \lim_{x\to 1} \frac{(x - 1)(x^2 + x + 1)}{x - 1} = \lim_{x\to 1}(x^2 + x + 1) = 3 \tag{1}$$

Recall from algebra that $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

we conclude $\lim_{x\to 1} g(x)$ exists and equals 3. Since this value is not the same as $g(1) = 2$, the second condition of Definition 2.3.1 is violated. The function g is discontinuous at 1.

(c) First, $h(1)$ is defined, in this case, $h(1) = 3$. Second, $\lim_{x\to 1} h(x) = 3$ from (1) of part (b). Third, we have $\lim_{x\to 1} h(x) = h(1) = 3$. Thus *all* three conditions in Definition 2.3.1 are satisfied and so the function h is continuous at 1.

The graphs of the three functions are compared in FIGURE 2.3.2.

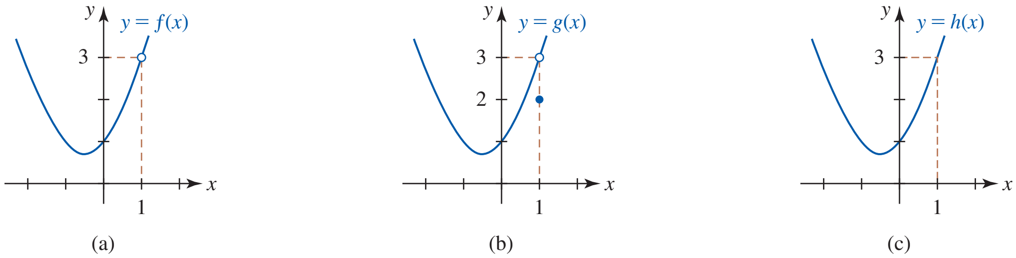


FIGURE 2.3.2 Graphs of functions in Example 1 ■

EXAMPLE 2 Piecewise-Defined Function

Determine whether the piecewise-defined function is continuous at 2.

$$f(x) = \begin{cases} x^2, & x < 2 \\ 5, & x = 2 \\ -x + 6, & x > 2. \end{cases}$$

Solution First, observe that $f(2)$ is defined and equals 5. Next, we see from

$$\left.\begin{aligned} \lim_{x\to 2^-} f(x) &= \lim_{x\to 2^-} x^2 = 4 \\ \lim_{x\to 2^+} f(x) &= \lim_{x\to 2^+} (-x + 6) = 4 \end{aligned}\right\} \text{implies } \lim_{x\to 2} f(x) = 4$$

that the limit of f as $x \to 2$ exists. Finally, because $\lim_{x\to 2} f(x) \neq f(2) = 5$, it follows from (*iii*) of Definition 2.3.1 that f is discontinuous at 2. The graph of f is shown in FIGURE 2.3.3. ■

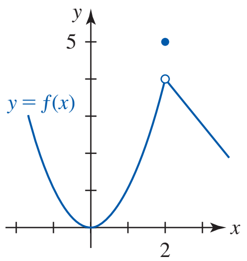


FIGURE 2.3.3 Graph of function in Example 2

Continuity on an Interval We will now extend the notion of continuity at a number a to **continuity on an interval**.

Definition 2.3.2 Continuity on an Interval

A function f is continuous

(*i*) on an **open interval** (a, b) if it is continuous at every number in the interval; and
(*ii*) on a **closed interval** $[a, b]$ if it is continuous on (a, b) and, in addition,

$$\lim_{x\to a^+} f(x) = f(a) \qquad \text{and} \qquad \lim_{x\to b^-} f(x) = f(b).$$

If the right-hand limit condition $\lim_{x\to a^+} f(x) = f(a)$ given in (*ii*) of Definition 2.3.1 is satisfied, we say that ***f* is continuous from the right at *a***; if $\lim_{x\to b^-} f(x) = f(b)$, then ***f* is continuous from the left at *b***.

Extensions of these concepts to intervals such as $[a, b)$, $(a, b]$, (a, ∞), $(-\infty, b)$, $(-\infty, \infty)$, $[a, \infty)$, and $(-\infty, b]$ are made in the expected manner. For example, f is continuous on $[1, 5)$ if it is continuous on the open interval $(1, 5)$ and continuous from the right at 1.

EXAMPLE 3 Continuity on an Interval

(a) As we see from FIGURE 2.3.4(a), $f(x) = 1/\sqrt{1 - x^2}$ is continuous on the open interval $(-1, 1)$ but is not continuous on the closed interval $[-1, 1]$, since neither $f(-1)$ nor $f(1)$ is defined.

(b) $f(x) = \sqrt{1 - x^2}$ is continuous on $[-1, 1]$. Observe from Figure 2.3.4(b) that

$$\lim_{x \to -1^+} f(x) = f(-1) = 0 \text{ and } \lim_{x \to 1^-} f(x) = f(1) = 0.$$

(c) $f(x) = \sqrt{x - 1}$ is continuous on the unbounded interval $[1, \infty)$, because

$$\lim_{x \to a} f(x) = \sqrt{\lim_{x \to a}(x - 1)} = \sqrt{a - 1} = f(a),$$

for any real number a satisfying $a > 1$, and f is continuous from the right at 1 since

$$\lim_{x \to 1^+} \sqrt{x - 1} = f(1) = 0.$$

See Figure 2.3.4(c). ■

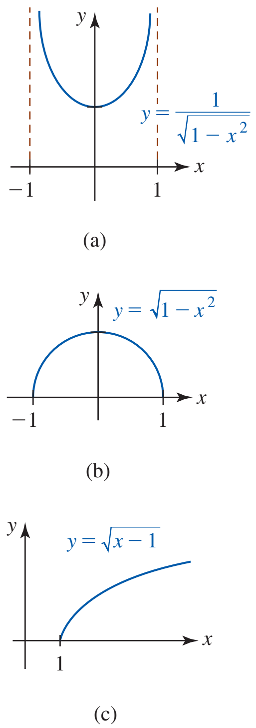


FIGURE 2.3.4 Graphs of functions in Example 3

A review of the graphs in Figures 1.4.1 and 1.4.2 shows that $y = \sin x$ and $y = \cos x$ are continuous on $(-\infty, \infty)$. Figures 1.4.3 and 1.4.5 show that $y = \tan x$ and $y = \sec x$ are discontinuous at $x = (2n + 1)\pi/2$, $n = 0, \pm 1, \pm 2, \ldots$, whereas Figures 1.4.4 and 1.4.6 show that $y = \cot x$ and $y = \csc x$ are discontinuous at $x = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$. The inverse trigonometric functions $y = \sin^{-1} x$ and $y = \cos^{-1} x$ are continuous on the closed interval $[-1, 1]$. See Figures 1.5.9 and 1.5.12. The natural exponential function $y = e^x$ is continuous on $(-\infty, \infty)$, whereas the natural logarithmic function $y = \ln x$ is continuous on $(0, \infty)$. See Figures 1.6.5 and 1.6.6.

Continuity of a Sum, Product, and Quotient When two functions f and g are continuous at a number a, then the combinations of functions formed by addition, multiplication, and division are also continuous at a. In the case of division f/g we must, of course, require that $g(a) \neq 0$.

Theorem 2.3.1 Continuity of a Sum, Product, and Quotient

If the functions f and g are continuous at a number a, then the sum $f + g$, the product fg, and the quotient f/g $(g(a) \neq 0)$ are continuous at $x = a$.

PROOF OF CONTINUITY OF THE PRODUCT *fg* As a consequence of the assumption that the functions f and g are continuous at a number a, we can say that both functions are defined at $x = a$, the limits of both functions as x approaches a exist, and

$$\lim_{x \to a} f(x) = f(a) \qquad \text{and} \qquad \lim_{x \to a} g(x) = g(a).$$

Because the limits exist, we know that the limit of a product is the product of the limits:

$$\lim_{x \to a}(f(x)g(x)) = \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right) = f(a)g(a).$$

The proofs of the remaining parts of Theorem 2.3.1 are obtained in a similar manner. ■

Since Definition 2.3.1 implies that $f(x) = x$ is continuous at any real number x, we see from successive applications of Theorem 2.3.1 that the functions $x, x^2, x^3, \ldots, x^n$ are also continuous for every x in the interval $(-\infty, \infty)$. Because a polynomial function is just a sum of powers of x, another application of Theorem 2.3.1 shows:

- *A polynomial function f is continuous on* $(-\infty, \infty)$.

Functions, such as polynomials and the sine and cosine, that are continuous for *all* real numbers, that is, on the interval $(-\infty, \infty)$, are said to be **continuous everywhere**. A function that is continuous everywhere is also just said to be **continuous**. Now, if $p(x)$ and $q(x)$ are polynomial functions, it also follows directly from Theorem 2.3.1 that:

- *A rational function $f(x) = p(x)/q(x)$ is continuous except at numbers at which the denominator $q(x)$ is zero.*

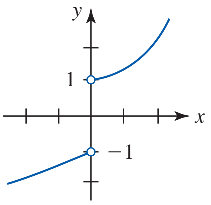
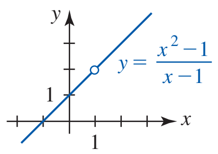
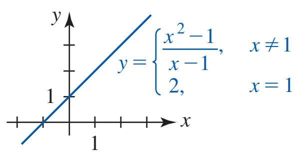


FIGURE 2.3.5 Jump discontinuity at $x = 0$



(a) Not continuous at 1



(b) Continuous at 1

FIGURE 2.3.6 Removable discontinuity at $x = 1$

Terminology

A discontinuity of a function f is often given a special name.

- If $x = a$ is a vertical asymptote for the graph of $y = f(x)$, then f is said to have an **infinite discontinuity** at a.

Figure 2.3.1(a) illustrates a function with an infinite discontinuity at a.

- If $\lim_{x\to a^-} f(x) = L_1$ and $\lim_{x\to a^+} f(x) = L_2$ and $L_1 \neq L_2$, then f is said to have a **finite discontinuity** or a **jump discontinuity** at a.

The function $y = f(x)$ given in FIGURE 2.3.5 has a jump discontinuity at 0, since $\lim_{x\to 0^-} f(x) = -1$ and $\lim_{x\to 0^+} f(x) = 1$. The greatest integer function $f(x) = \lfloor x \rfloor$ has a jump discontinuity at every integer value of x.

- If $\lim_{x\to a} f(x)$ exists but either f is not defined at $x = a$ or $f(a) \neq \lim_{x\to a} f(x)$, then f is said to have a **removable discontinuity** at a.

For example, the function $f(x) = (x^2 - 1)/(x - 1)$ is not defined at $x = 1$ but $\lim_{x\to 1} f(x) = 2$. By *defining* $f(1) = 2$, the new function

$$f(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ 2, & x = 1 \end{cases}$$

is continuous everywhere. See FIGURE 2.3.6.

Continuity of f^{-1}

The plausibility of the next theorem follows from the fact that the graph of an inverse function f^{-1} is a reflection of the graph of f in the line $y = x$.

Theorem 2.3.2 Continuity of an Inverse Function

If f is a continuous one-to-one function on an interval $[a, b]$, then f^{-1} is continuous on either $[f(a), f(b)]$ or $[f(b), f(a)]$.

The sine function, $f(x) = \sin x$, is continuous on $[-\pi/2, \pi/2]$ and, as noted previously, the inverse of f, $y = \sin^{-1} x$, is continuous on the closed interval $[f(-\pi/2), f(\pi/2)] = [-1, 1]$.

Limit of a Composite Function

The next theorem tells us that if a function f is continuous, then the limit of the function is the function of the limit. The proof of Theorem 2.3.3 is given in the *Appendix*.

Theorem 2.3.3 Limit of a Composite Function

If $\lim_{x\to a} g(x) = L$ and f is continuous at L, then

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(L).$$

Theorem 2.3.3 is useful in proving other theorems. If the function g is continuous at a and f is continuous at $g(a)$, then we see that

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(g(a)).$$

We have just proved that the composite of two continuous functions is continuous.

Theorem 2.3.4 Continuity of a Composite Function

If g is continuous at a number a and f is continuous at $g(a)$, then the composite function $(f \circ g)(x) = f(g(x))$ is continuous at a.

EXAMPLE 4 Continuity of a Composite Function

$f(x) = \sqrt{x}$ is continuous on the interval $[0, \infty)$ and $g(x) = x^2 + 2$ is continuous on $(-\infty, \infty)$. But, since $g(x) \geq 0$ for all x, the composite function

$$(f \circ g)(x) = f(g(x)) = \sqrt{x^2 + 2}$$

is continuous everywhere. ■

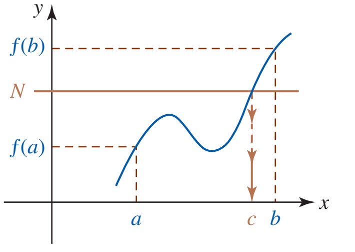


FIGURE 2.3.7 A continuous function f takes on all values between $f(a)$ and $f(b)$

If a function f is continuous on a closed interval $[a, b]$, then, as illustrated in FIGURE 2.3.7, f takes on all values between $f(a)$ and $f(b)$. Put another way, a continuous function f does not "skip" any values.

Theorem 2.3.5 Intermediate Value Theorem

If f denotes a function continuous on a closed interval $[a, b]$ for which $f(a) \neq f(b)$, and if N is any number between $f(a)$ and $f(b)$, then there exists at least one number c between a and b such that $f(c) = N$.

EXAMPLE 5 Consequence of Continuity

The polynomial function $f(x) = x^2 - x - 5$ is continuous on the interval $[-1, 4]$ and $f(-1) = -3, f(4) = 7$. For any number N for which $-3 \leq N \leq 7$, Theorem 2.3.5 guarantees that there is a solution to the equation $f(c) = N$, that is, $c^2 - c - 5 = N$ in $[-1, 4]$. Specifically, if we choose $N = 1$, then $c^2 - c - 5 = 1$ is equivalent to

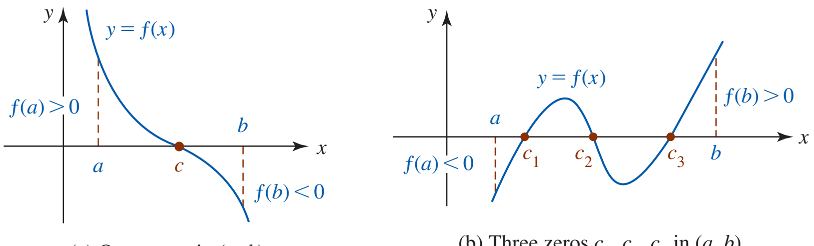
$$c^2 - c - 6 = 0 \quad \text{or} \quad (c - 3)(c + 2) = 0.$$

Although the latter equation has two solutions, only the value $c = 3$ is between -1 and 4. ■

The foregoing example suggests a corollary to the Intermediate Value Theorem.

- *If f satisfies the hypotheses of Theorem 2.3.5 and $f(a)$ and $f(b)$ have opposite algebraic signs, then there exists a number x between a and b for which $f(x) = 0$.*

This fact is often used in locating real zeros of a continuous function f. If the function values $f(a)$ and $f(b)$ have opposite signs, then by identifying $N = 0$, we can say that there is at least one number c in (a, b) for which $f(c) = 0$. In other words, if either $f(a) > 0, f(b) < 0$ or $f(a) < 0, f(b) > 0$, then $f(x)$ has at least one zero c in the interval (a, b). The plausibility of this conclusion is illustrated in FIGURE 2.3.8.



(a) One zero c in (a, b)

(b) Three zeros c_1, c_2, c_3 in (a, b)

FIGURE 2.3.8 Locating zeros of functions using the Intermediate Value Theorem

Bisection Method As a direct consequence of the Intermediate Value Theorem, we can devise a means of approximating the zeros of a continuous function to any degree of accuracy. Suppose $y = f(x)$ is continuous on the closed interval $[a, b]$ such that $f(a)$ and $f(b)$ have opposite algebraic signs. Then, as we have just seen, f has a zero in $[a, b]$. Suppose we bisect the interval $[a, b]$ by finding its midpoint $m_1 = (a + b)/2$. If $f(m_1) = 0$, then m_1 is a zero of f and we proceed no further, but if $f(m_1) \neq 0$, then we can say that:

- If $f(a)$ and $f(m_1)$ have opposite algebraic signs, then f has a zero c in $[a, m_1]$.
- If $f(m_1)$ and $f(b)$ have opposite algebraic signs, then f has a zero c in $[m_1, b]$.

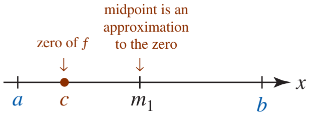


FIGURE 2.3.9 The number m_1 is an approximation to the number c

That is, if $f(m_1) \neq 0$, then f has a zero in an interval that is one-half the length of the original interval. See FIGURE 2.3.9. We now repeat the process by bisecting this new interval by finding its midpoint m_2. If m_2 is a zero of f, we stop, but if $f(m_2) \neq 0$, we have located a zero in an interval that is one-fourth the length of $[a, b]$. We continue this process of locating a zero of f in shorter and shorter intervals indefinitely. This method of approximating a zero of a continuous function by a sequence of midpoints is called the **bisection method**. Reinspection of Figure 2.3.9 shows that the error in an approximation to a zero in an interval is less than one-half the length of the interval.

EXAMPLE 6 Zeros of a Polynomial Function

(a) Show that the polynomial function $f(x) = x^6 - 3x - 1$ has a real zero in $[-1, 0]$ and in $[1, 2]$.
(b) Approximate the zero in $[1, 2]$ to two decimal places.

Solution

(a) Observe that $f(-1) = 3 > 0$ and $f(0) = -1 < 0$. This change in sign indicates that the graph of f must cross the x-axis at least once in the interval $[-1, 0]$. In other words, there is at least one zero of f in $[-1, 0]$.

Similarly, $f(1) = -3 < 0$ and $f(2) = 57 > 0$ implies that there is at least one zero of f in the interval $[1, 2]$.

(b) A first approximation to the zero in $[1, 2]$ is the midpoint of the interval:

$$m_1 = \frac{1+2}{2} = \frac{3}{2} = 1.5, \qquad \text{error} < \frac{1}{2}(2 - 1) = 0.5.$$

Now since $f(m_1) = f\left(\frac{3}{2}\right) > 0$ and $f(1) < 0$, we know that the zero lies in the interval $\left[1, \frac{3}{2}\right]$.

The second approximation is the midpoint of $\left[1, \frac{3}{2}\right]$:

$$m_2 = \frac{1+\frac{3}{2}}{2} = \frac{5}{4} = 1.25, \qquad \text{error} < \frac{1}{2}\left(\frac{3}{2} - 1\right) = 0.25.$$

Since $f(m_2) = f\left(\frac{5}{4}\right) < 0$, the zero lies in the interval $\left[\frac{5}{4}, \frac{3}{2}\right]$.

The third approximation is the midpoint of $\left[\frac{5}{4}, \frac{3}{2}\right]$:

$$m_3 = \frac{\frac{5}{4}+\frac{3}{2}}{2} = \frac{11}{8} = 1.375, \qquad \text{error} < \frac{1}{2}\left(\frac{3}{2} - \frac{5}{4}\right) = 0.125.$$

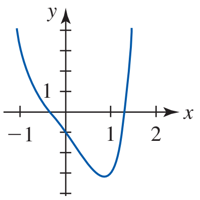


FIGURE 2.3.10 Graph of function in Example 6

If we wish the approximation to be accurate to *three* decimal places, we continue until the error becomes less

After eight calculations, we find that $m_8 = 1.300781$ with error less than 0.005. Hence, 1.30 is an approximation to the zero of f in $[1, 2]$ that is accurate to two decimal places. The graph of f is given in FIGURE 2.3.10. ■

Exercises 2.3 Answers to selected odd-numbered problems begin on page ANS-000.

Fundamentals

In Problems 1–12, determine the numbers, if any, at which the given function f is discontinuous.

1. $f(x) = x^3 - 4x^2 + 7$

2. $f(x) = \dfrac{x}{x^2 + 4}$

3. $f(x) = (x^2 - 9x + 18)^{-1}$

4. $f(x) = \dfrac{x^2 - 1}{x^4 - 1}$

5. $f(x) = \dfrac{x - 1}{\sin 2x}$

6. $f(x) = \dfrac{\tan x}{x + 3}$

7. $f(x) = \begin{cases} x, & x < 0 \\ x^2, & 0 \le x < 2 \\ x, & x > 2 \end{cases}$ **8.** $f(x) = \begin{cases} \dfrac{|x|}{x}, & x \ne 0 \\ 1, & x = 0 \end{cases}$

9. $f(x) = \begin{cases} \dfrac{x^2 - 25}{x - 5}, & x \ne 5 \\ 10, & x = 5 \end{cases}$

10. $f(x) = \begin{cases} \dfrac{x - 1}{\sqrt{x} - 1}, & x \ne 1 \\ \dfrac{1}{2}, & x = 1 \end{cases}$

11. $f(x) = \dfrac{1}{2 + \ln x}$ **12.** $f(x) = \dfrac{2}{e^x - e^{-x}}$

In Problems 13–24, determine whether the given function f is continuous on the indicated intervals.

13. $f(x) = x^2 + 1$
 (a) $[-1, 4]$ **(b)** $[5, \infty)$

14. $f(x) = \dfrac{1}{x}$
 (a) $(-\infty, \infty)$ **(b)** $(0, \infty)$

15. $f(x) = \dfrac{1}{\sqrt{x}}$
 (a) $(0, 4]$ **(b)** $[1, 9]$

16. $f(x) = \sqrt{x^2 - 9}$
 (a) $[-3, 3]$ **(b)** $[3, \infty)$

17. $f(x) = \tan x$
 (a) $[0, \pi]$ **(b)** $[-\pi/2, \pi/2]$

18. $f(x) = \csc x$
 (a) $(0, \pi)$ **(b)** $(2\pi, 3\pi)$

19. $f(x) = \dfrac{x}{x^3 + 8}$
 (a) $[-4, -3]$ **(b)** $(-\infty, \infty)$

20. $f(x) = \dfrac{1}{|x| - 4}$
 (a) $(-\infty, -1]$ **(b)** $[1, 6]$

21. $f(x) = \dfrac{x}{2 + \sec x}$
 (a) $(-\infty, \infty)$ **(b)** $[\pi/2, 3\pi/2]$

22. $f(x) = \sin\dfrac{1}{x}$
 (a) $[1/\pi, \infty)$ **(b)** $[-2/\pi, 2/\pi]$

23.

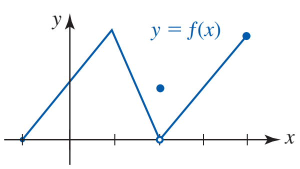


FIGURE 2.3.11 Graph for Problem 23

 (a) $[-1, 3]$ **(b)** $(2, 4]$

24.

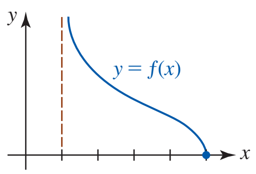


FIGURE 2.3.12 Graph for Problem 24

 (a) $[2, 4]$ **(b)** $[1, 5]$

In Problems 25–28, find values of m and n so that the given function f is continuous.

25. $f(x) = \begin{cases} mx, & x < 4 \\ x^2, & x \ge 4 \end{cases}$

26. $f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \ne 2 \\ m, & x = 2 \end{cases}$

27. $f(x) = \begin{cases} mx, & x < 3 \\ n, & x = 3 \\ -2x + 9, & x > 3 \end{cases}$

28. $f(x) = \begin{cases} mx - n, & x < 1 \\ 5, & x = 1 \\ 2mx + n, & x > 1 \end{cases}$

In Problems 29 and 30, $\lfloor x \rfloor$ denotes the greatest integer not exceeding x. Sketch a graph to determine the points at which the given function is discontinuous.

29. $f(x) = \lfloor 2x - 1 \rfloor$ **30.** $f(x) = \lfloor x \rfloor - x$

In Problems 31 and 32, determine whether the given function has a removable discontinuity at the given number a. If the discontinuity is removable, define a new function that is continuous at a.

31. $f(x) = \dfrac{x - 9}{\sqrt{x} - 3}, \quad a = 9$ **32.** $f(x) = \dfrac{x^4 - 1}{x^2 - 1}, \quad a = 1$

In Problems 33–42, use Theorem 2.3.3 to find the given limit.

33. $\lim_{x \to \pi/6} \sin(2x + \pi/3)$ **34.** $\lim_{x \to \pi^2} \cos\sqrt{x}$

35. $\lim_{x \to \pi/2} \sin(\cos x)$ **36.** $\lim_{x \to \pi/2} (1 + \cos(\cos x))$

37. $\lim_{t \to \pi} \cos\left(\dfrac{t^2 - \pi^2}{t - \pi}\right)$ **38.** $\lim_{t \to 0} \tan\left(\dfrac{\pi t}{t^2 + 3t}\right)$

39. $\lim_{t \to \pi} \sqrt{t - \pi + \cos^2 t}$ **40.** $\lim_{t \to 1} (4t + \sin 2\pi t)^3$

41. $\lim_{x \to -3} \sin^{-1}\left(\dfrac{x + 3}{x^2 + 4x + 3}\right)$ **42.** $\lim_{x \to \pi} e^{\cos 3x}$

In Problems 43 and 44, determine the interval(s) where $f \circ g$ is continuous.

43. $f(x) = \dfrac{1}{\sqrt{x - 1}}, \quad g(x) = x + 4$

44. $f(x) = \dfrac{5x}{x - 1}, \quad g(x) = (x - 2)^2$

In Problems 45–48, verify the Intermediate Value Theorem for f on the given interval. Find a number c in the interval for the indicated value of N.

45. $f(x) = x^2 - 2x$, $[1, 5]$; $N = 8$

46. $f(x) = x^2 + x + 1$, $[-2, 3]$; $N = 6$

47. $f(x) = x^3 - 2x + 1$, $[-2, 2]$; $N = 1$

48. $f(x) = \dfrac{10}{x^2 + 1}$, $[0, 1]$; $N = 8$

49. Given that $f(x) = x^5 + 2x - 7$, show that there is a number c such that $f(c) = 50$.

50. Given that f and g are continuous on $[a, b]$ such that $f(a) > g(a)$ and $f(b) < g(b)$, show that there is a number c in (a, b) such that $f(c) = g(c)$. [*Hint*: Consider the function $f - g$.]

In Problems 51–54, show that the given equation has a solution in the indicated interval.

51. $2x^7 = 1 - x$, $(0, 1)$

52. $\dfrac{x^2 + 1}{x + 3} + \dfrac{x^4 + 1}{x - 4} = 0$, $(-3, 4)$

53. $e^{-x} = \ln x$, $(1, 2)$

54. $\dfrac{\sin x}{x} = \dfrac{1}{2}$, $(\pi/2, \pi)$

Calculator/CAS Problems

In Problems 55 and 56, use a calculator or CAS to obtain the graph of the given function. Use the bisection method to approximate, to an accuracy of two decimal places, the real zeros of f that you discover from the graph.

55. $f(x) = 3x^5 - 5x^3 - 1$ **56.** $f(x) = x^5 + x - 1$

57. Use the bisection method to approximate the value of c in Problem 49 to an accuracy of two decimal places.

58. Use the bisection method to approximate the solution in Problem 51 to an accuracy of two decimal places.

59. Use the bisection method to approximate the solution in Problem 52 to an accuracy of two decimal places.

60. Suppose a closed right-circular cylinder has a given volume V and surface area S (lateral side, top, and bottom).

(a) Show that the radius r of the cylinder must satisfy the equation $2\pi r^3 - Sr + 2V = 0$.

(b) Suppose $V = 3000$ ft^3 and $S = 1800$ ft^2. Use a calculator or CAS to obtain the graph of

$$f(r) = 2\pi r^3 - 1800r + 6000.$$

(c) Use the graph in part (b) and the bisection method to find the dimensions of the cylinder corresponding to the volume and surface area given in part (b). Use an accuracy of two decimal places.

Think About It

61. Given that f and g are continuous at a number a, prove that $f + g$ is continuous at a.

62. Given that f and g are continuous at a number a and $g(a) \neq 0$, prove that f/g is continuous at a.

63. Let $f(x) = \lfloor x \rfloor$ be the greatest integer function and $g(x) = \cos x$. Determine the points at which $f \circ g$ is discontinuous.

64. Consider the functions

$$f(x) = |x| \quad \text{and} \quad g(x) = \begin{cases} x + 1, & x < 0 \\ x - 1, & x \geq 0. \end{cases}$$

Sketch the graphs of $f \circ g$ and $g \circ f$. Determine whether $f \circ g$ and $g \circ f$ are continuous at 0.

65. A Mathematical Classic The **Dirichlet function**

$$f(x) = \begin{cases} 1, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}$$

is named after the German mathematician **Johann Peter Gustav Lejeune Dirichlet** (1805–1859). Dirichlet is responsible for the definition of a function as we know it today.

(a) Show that f is discontinuous at every real number a. In other words, f is a *nowhere continuous function*.

(b) What does the graph of f look like?

(c) If r is a positive rational number, show that f is r-periodic, that is, $f(x + r) = f(x)$.

2.4 Trigonometric Limits

Introduction In this section we examine limits that involve trigonometric functions. As the examples in this section will illustrate, computation of trigonometric limits entails both algebraic manipulations and knowledge of some basic trigonometric identities. We begin with some simple limit results that are consequences of continuity.

Using Continuity We saw in the preceding section that the sine and cosine functions are everywhere continuous. It follows from Definition 2.3.1 that for any real number a,

$$\lim_{x \to a} \sin x = \sin a, \tag{1}$$

$$\lim_{x \to a} \cos x = \cos a. \tag{2}$$

Similarly, for a number a in the domain of the given trigonometric function

$$\lim_{x\to a} \tan x = \tan a, \qquad \lim_{x\to a} \cot x = \cot a, \tag{3}$$

$$\lim_{x\to a} \sec x = \sec a, \qquad \lim_{x\to a} \csc x = \csc a. \tag{4}$$

EXAMPLE 1 Using (1) and (2)

From (1) and (2) we have

$$\lim_{x\to 0} \sin x = \sin 0 = 0 \quad \text{and} \quad \lim_{x\to 0} \cos x = \cos 0 = 1. \tag{5}$$ ■

We will draw on the results in (5) in the following discussion on computing other trigonometric limits. But first, we consider a theorem that is particularly useful when working with trigonometric limits.

Squeeze Theorem The next theorem has many names: **Squeeze Theorem**, **Pinching Theorem**, **Sandwiching Theorem**, **Squeeze Play Theorem**, and **Flyswatter Theorem** are just a few of them. As shown in FIGURE 2.4.1, if the graph of $f(x)$ is "squeezed" between the graphs of two other functions $g(x)$ and $h(x)$ for all x close to a, and if the functions g and h have a common limit L as $x \to a$, it stands to reason that f also approaches L as $x \to a$. The proof of Theorem 2.4.1 is given in the *Appendix*.

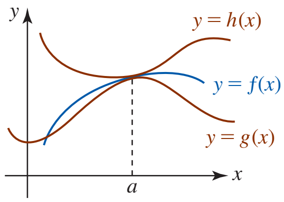


FIGURE 2.4.1 Graph of f squeezed between the graphs g and h

Theorem 2.4.1 Squeeze Theorem

Suppose f, g, and h are functions for which $g(x) \le f(x) \le h(x)$ for all x in an open interval that contains a number a, except possibly at a itself. If

$$\lim_{x\to a} g(x) = L \quad \text{and} \quad \lim_{x\to a} h(x) = L,$$

then $\lim_{x\to a} f(x) = L$.

A colleague from Russia said this result was called the **Two Soldiers Theorem** when he was in school. Think about it.

Before applying Theorem 2.4.1, let us consider a trigonometric limit that does not exist.

EXAMPLE 2 A Limit That Does Not Exist

The limit $\lim_{x\to 0} \sin(1/x)$ does not exist. The function $f(x) = \sin(1/x)$ is odd but is not periodic. The graph f oscillates between -1 and 1 as $x \to 0$:

$$\sin\frac{1}{x} = \pm 1 \quad \text{for} \quad \frac{1}{x} = \frac{\pi}{2} + n\pi, \quad n = 0, \pm 1, \pm 2, \ldots.$$

For example, $\sin(1/x) = 1$ for $n = 500$ or $x \approx 0.00064$, and $\sin(1/x) = -1$ for $n = 501$ or $x \approx 0.00063$. This means that near the origin the graph of f becomes so compressed that it appears to be one continuous smear of color. See FIGURE 2.4.2. ■

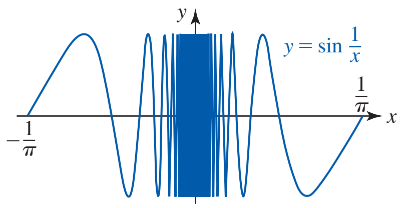


FIGURE 2.4.2 Graph of function in Example 2

EXAMPLE 3 Using the Squeeze Theorem

Find the limit $\lim_{x\to 0} x^2 \sin\frac{1}{x}$.

Solution First observe that

$$\lim_{x\to 0} x^2 \sin\frac{1}{x} \ne \left(\lim_{x\to 0} x^2\right)\left(\lim_{x\to 0} \sin\frac{1}{x}\right)$$

because we have just seen in Example 2 that $\lim_{x\to 0} \sin(1/x)$ does not exist. But for $x \ne 0$ we have $-1 \le \sin(1/x) \le 1$. Therefore,

$$-x^2 \le x^2 \sin\frac{1}{x} \le x^2.$$

Now if we make the identifications $g(x) = -x^2$ and $h(x) = x^2$, it follows from (1) of Section 2.2 that $\lim_{x\to 0} g(x) = 0$ and $\lim_{x\to 0} h(x) = 0$. Hence, from the Squeeze Theorem we conclude that

$$\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0.$$

In FIGURE 2.4.3 note the small scale on the x- and y-axes.

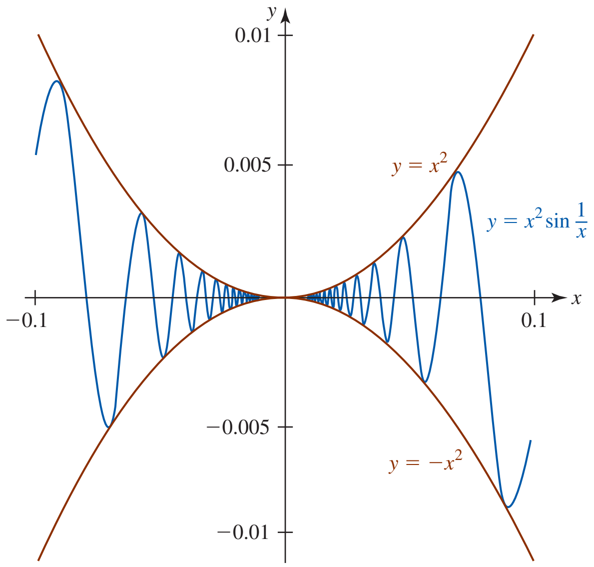


FIGURE 2.4.3 Graph of function in Example 3 ■

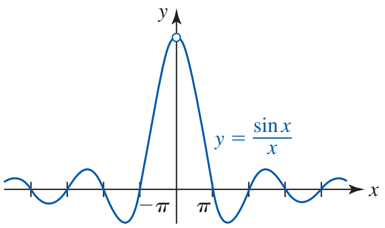


FIGURE 2.4.4 Graph of $f(x) = (\sin x)/x$

An Important Trigonometric Limit Although the function $f(x) = (\sin x)/x$ is not defined at $x = 0$, the numerical table in Example 7 of Section 2.1 and the graph in FIGURE 2.4.4 suggests that $\lim_{x\to 0}(\sin x)/x$ exists. We are now able to prove this conjecture using the Squeeze Theorem.

Consider a circle centered at the origin O with radius 1. As shown in FIGURE 2.4.5(a), let the shaded region OPR be a sector of the circle with central angle t such that $0 < t < \pi/2$. We see from parts (b), (c), and (d) of Figure 2.4.5 that

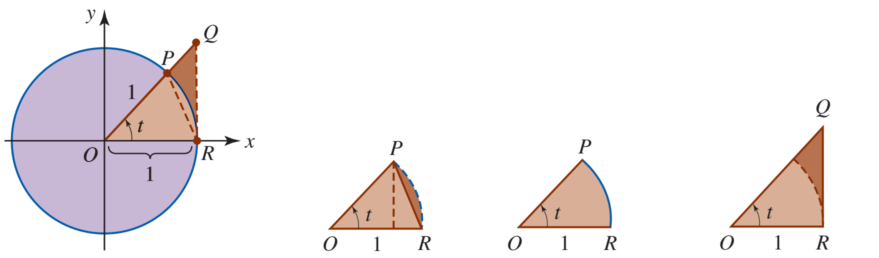
$$\text{area of } \triangle OPR \leq \text{area of sector } OPR \leq \text{area of } \triangle OQR. \tag{6}$$

From Figure 2.4.5(b) the height of $\triangle OPR$ is $\overline{OP}\sin t = 1 \cdot \sin t = \sin t$, and so

$$\text{area of } \triangle OPR = \frac{1}{2}\overline{OR}\cdot(\text{height}) = \frac{1}{2}\cdot 1 \cdot \sin t = \frac{1}{2}\sin t. \tag{7}$$

From Figure 2.4.5(d), $\overline{QR}/\overline{OR} = \tan t$ or $\overline{QR} = \tan t$, so that

$$\text{area of } \triangle OQR = \frac{1}{2}\overline{OR}\cdot\overline{QR} = \frac{1}{2}\cdot 1 \cdot \tan t = \frac{1}{2}\tan t. \tag{8}$$



(a) Unit circle (b) Triangle OPR (c) Sector OPR (d) Right triangle OQR

FIGURE 2.4.5 Unit circle along with two triangles and a circular sector

Finally, the area of a sector of a circle is $\frac{1}{2}r^2\theta$, where r is its radius and θ is the central angle measured in radians. Thus,

$$\text{area of sector } OPR = \frac{1}{2}\cdot 1^2 \cdot t = \frac{1}{2}t. \tag{9}$$

Using (7), (8), and (9) in the inequality (6) gives

$$\frac{1}{2}\sin t < \frac{1}{2}t < \frac{1}{2}\tan t \qquad \text{or} \qquad 1 < \frac{t}{\sin t} < \frac{1}{\cos t}.$$

From the properties of inequalities, the last inequality can be written

$$\cos t < \frac{\sin t}{t} < 1.$$

We now let $t \to 0^+$ in the last result. Since $(\sin t)/t$ is "squeezed" between 1 and $\cos t$ (which we know from (5) is approaching 1), it follows from Theorem 2.4.1 that $(\sin t)/t \to 1$. While we have assumed $0 < t < \pi/2$, the same result holds for $t \to 0^-$ when $-\pi/2 < t < 0$. Using the symbol x in place of t, we summarize the result:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1. \tag{10}$$

As the following examples illustrate, the results in (1), (2), (3), and (10) are used often to compute other limits. Note that the limit (10) is the indeterminate form $0/0$.

EXAMPLE 4 Using (10)

Find the limit $\lim_{x\to 0}\dfrac{10x - 3\sin x}{x}$.

Solution We rewrite the fractional expression as two fractions with the same denominator x:

$$\begin{aligned}\lim_{x\to 0}\frac{10x - 3\sin x}{x} &= \lim_{x\to 0}\left[\frac{10x}{x} - \frac{3\sin x}{x}\right]\\ &= \lim_{x\to 0}\frac{10x}{x} - 3\lim_{x\to 0}\frac{\sin x}{x} \quad \leftarrow \text{since both limits exist, also cancel the } x \text{ in the first expression}\\ &= \lim_{x\to 0}10 - 3\lim_{x\to 0}\frac{\sin x}{x} \quad \leftarrow \text{now use (10)}\\ &= 10 - 3\cdot 1\\ &= 7.\end{aligned}$$

■

EXAMPLE 5 Using the Double-Angle Formula

Find the limit $\lim_{x\to 0}\dfrac{\sin 2x}{x}$.

Solution To evaluate the given limit we make use of the double-angle formula $\sin 2x = 2\sin x\cos x$ of Section 1.4, and the fact the limits exist:

$$\begin{aligned}\lim_{x\to 0}\frac{\sin 2x}{x} &= \lim_{x\to 0}\frac{2\cos x\sin x}{x}\\ &= 2\lim_{x\to 0}\left(\cos x\cdot\frac{\sin x}{x}\right)\\ &= 2\left(\lim_{x\to 0}\cos x\right)\left(\lim_{x\to 0}\frac{\sin x}{x}\right).\end{aligned}$$

From (5) and (10) we know that $\cos x \to 1$ and $(\sin x)/x \to 1$ as $x \to 0$, and so the preceding line becomes

$$\lim_{x\to 0}\frac{\sin 2x}{x} = 2\cdot 1\cdot 1 = 2.$$

■

EXAMPLE 6 Using (5) and (10)

Find the limit $\lim_{x\to 0} \dfrac{\tan x}{x}$.

Solution Using $\tan x = (\sin x)/\cos x$ and the fact that the limits exist we can write

$$\begin{aligned}\lim_{x\to 0} \frac{\tan x}{x} &= \lim_{x\to 0} \frac{(\sin x)/\cos x}{x}\\ &= \lim_{x\to 0} \frac{1}{\cos x}\cdot\frac{\sin x}{x}\\ &= \left(\lim_{x\to 0}\frac{1}{\cos x}\right)\left(\lim_{x\to 0}\frac{\sin x}{x}\right)\\ &= \frac{1}{1}\cdot 1 = 1. \quad \leftarrow \text{from (5) and (10)}\end{aligned}$$

Using a Substitution We are often interested in limits similar to that considered in Example 5. But if we wish to find, say, $\lim_{x\to 0} \dfrac{\sin 5x}{x}$ the procedure employed in Example 5 breaks down at a practical level since we do not have a readily available trigonometric identity for $\sin 5x$. There is an alternative procedure that allows us to quickly find $\lim_{x\to 0}\dfrac{\sin kx}{x}$, where $k \neq 0$ is any real constant, by simply changing the variable by means of a **substitution**. If we let $t = kx$, then $x = t/k$. Notice that as $x \to 0$ then necessarily $t \to 0$. Thus we can write

$$\lim_{x\to 0}\frac{\sin kx}{x} = \lim_{t\to 0}\frac{\sin t}{t/k} = \lim_{t\to 0}\left(\frac{\sin t}{1}\cdot\frac{k}{t}\right) = k\overset{\text{this limit is 1 from (10)}}{\overset{\downarrow}{\lim_{t\to 0}\frac{\sin t}{t}}} = k.$$

Thus we have proved the general result

$$\lim_{x\to 0}\frac{\sin kx}{x} = k. \tag{11}$$

From (11), with $k = 2$, we get the same result $\lim_{x\to 0}\dfrac{\sin 2x}{x} = 2$ obtained in Example 5.

EXAMPLE 7 Using a Substitution

Find the limit $\lim_{x\to 1}\dfrac{\sin(x-1)}{x^2+2x-3}$.

Solution Before beginning observe that the limit has the indeterminate form $0/0$ as $x \to 1$. By factoring $x^2 + 2x - 3 = (x+3)(x-1)$ the given limit can be expressed as a limit of a product:

$$\lim_{x\to 1}\frac{\sin(x-1)}{x^2+2x-3} = \lim_{x\to 1}\frac{\sin(x-1)}{(x+3)(x-1)} = \lim_{x\to 1}\left[\frac{1}{x+3}\cdot\frac{\sin(x-1)}{x-1}\right]. \tag{12}$$

Now if we let $t = x - 1$, we see that $x \to 1$ implies $t \to 0$. Therefore,

$$\lim_{x\to 1}\frac{\sin(x-1)}{x-1} = \lim_{t\to 0}\frac{\sin t}{t} = 1. \quad \leftarrow \text{from (10)}$$

Returning to (12), we can write

$$\begin{aligned}\lim_{x\to 1}\frac{\sin(x-1)}{x^2+2x-3} &= \lim_{x\to 1}\left[\frac{1}{x+3}\cdot\frac{\sin(x-1)}{x-1}\right]\\ &= \left(\lim_{x\to 1}\frac{1}{x+3}\right)\left(\lim_{x\to 1}\frac{\sin(x-1)}{x-1}\right)\\ &= \left(\lim_{x\to 1}\frac{1}{x+3}\right)\left(\lim_{t\to 0}\frac{\sin t}{t}\right)\end{aligned}$$

since both limits exist. Thus,

$$\lim_{x\to1}\frac{\sin(x-1)}{x^2+2x-3}=\left(\lim_{x\to1}\frac{1}{x+3}\right)\left(\lim_{t\to0}\frac{\sin t}{t}\right)=\frac{1}{4}\cdot1=\frac{1}{4}. \qquad \blacksquare$$

EXAMPLE 8 Using a Pythagorean Identity

Find the limit $\lim_{x\to0}\frac{1-\cos x}{x}$.

Solution To compute this limit we start with a bit of algebraic cleverness by multiplying the numerator and denominator by the conjugate factor of the numerator. Next we use the fundamental Pythagorean identity $\sin^2 x+\cos^2 x=1$ in the form $1-\cos^2 x=\sin^2 x$:

$$\begin{aligned}\lim_{x\to0}\frac{1-\cos x}{x}&=\lim_{x\to0}\frac{1-\cos x}{x}\cdot\frac{1+\cos x}{1+\cos x}\\&=\lim_{x\to0}\frac{1-\cos^2 x}{x(1+\cos x)}\\&=\lim_{x\to0}\frac{\sin^2 x}{x(1+\cos x)}.\end{aligned}$$

For the next step we resort back to algebra to rewrite the fractional expression as a product, then use the results in (5):

$$\begin{aligned}\lim_{x\to0}\frac{1-\cos x}{x}&=\lim_{x\to0}\frac{\sin^2 x}{x(1+\cos x)}\\&=\lim_{x\to0}\left(\frac{\sin x}{x}\cdot\frac{\sin x}{1+\cos x}\right)\\&=\left(\lim_{x\to0}\frac{\sin x}{x}\right)\cdot\left(\lim_{x\to0}\frac{\sin x}{1+\cos x}\right).\end{aligned}$$

Because $\lim_{x\to0}(\sin x)/(1+\cos x)=0/2=0$ we have

$$\lim_{x\to0}\frac{1-\cos x}{x}=0. \qquad (13) \quad \blacksquare$$

Since the limit in (13) is equal to 0, we can write

$$\lim_{x\to0}\frac{1-\cos x}{x}=\lim_{x\to0}\frac{-(\cos x-1)}{x}=(-1)\lim_{x\to0}\frac{\cos x-1}{x}=0.$$

Dividing by -1 then gives another important trigonometric limit:

$$\lim_{x\to0}\frac{\cos x-1}{x}=0. \qquad (14)$$

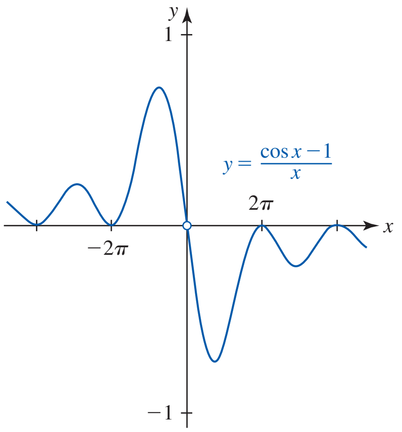


FIGURE 2.4.6 Graph of $f(x)=(\cos x-1)/x$

FIGURE 2.4.6 shows the graph of $f(x)=(\cos x-1)/x$. We will use the results in (10) and (14) in Exercises 2.7 and again in Section 3.4.

Exercises 2.4

Answers to selected odd-numbered problems begin on page ANS-000.

≡ Fundamentals

In Problems 1–36, find the given limit, or state that it does not exist.

1. $\lim_{t\to0}\frac{\sin 3t}{2t}$
2. $\lim_{t\to0}\frac{\sin(-4t)}{t}$
3. $\lim_{x\to0}\frac{\sin x}{4+\cos x}$
4. $\lim_{x\to0}\frac{1+\sin x}{1+\cos x}$
5. $\lim_{x\to0}\frac{\cos 2x}{\cos 3x}$
6. $\lim_{x\to0}\frac{\tan x}{3x}$
7. $\lim_{t\to0}\frac{1}{t\sec t\csc 4t}$
8. $\lim_{t\to0}5t\cot 2t$
9. $\lim_{t\to0}\frac{2\sin^2 t}{t\cos^2 t}$
10. $\lim_{t\to0}\frac{\sin^2(t/2)}{\sin t}$
11. $\lim_{t\to0}\frac{\sin^2 6t}{t^2}$
12. $\lim_{t\to0}\frac{t^3}{\sin^2 3t}$
13. $\lim_{x\to1}\frac{\sin(x-1)}{2x-2}$
14. $\lim_{x\to2\pi}\frac{x-2\pi}{\sin x}$

15. $\lim_{x\to 0} \dfrac{\cos x}{x}$

16. $\lim_{\theta\to\pi/2} \dfrac{1+\sin\theta}{\cos\theta}$

17. $\lim_{x\to 0} \dfrac{\cos(3x-\pi/2)}{x}$

18. $\lim_{x\to -2} \dfrac{\sin(5x+10)}{4x+8}$

19. $\lim_{t\to 0} \dfrac{\sin 3t}{\sin 7t}$

20. $\lim_{t\to 0} \sin 2t \csc 3t$

21. $\lim_{t\to 0^+} \dfrac{\sin t}{\sqrt{t}}$

22. $\lim_{t\to 0^+} \dfrac{1-\cos\sqrt{t}}{\sqrt{t}}$

23. $\lim_{t\to 0} \dfrac{t^2-5t\sin t}{t^2}$

24. $\lim_{t\to 0} \dfrac{\cos 4t}{\cos 8t}$

25. $\lim_{x\to 0^+} \dfrac{(x+2\sqrt{\sin x})^2}{x}$

26. $\lim_{x\to 0} \dfrac{(1-\cos x)^2}{x}$

27. $\lim_{x\to 0} \dfrac{\cos x - 1}{\cos^2 x - 1}$

28. $\lim_{x\to 0} \dfrac{\sin x + \tan x}{x}$

29. $\lim_{x\to 0} \dfrac{\sin 5x^2}{x^2}$

30. $\lim_{t\to 0} \dfrac{t^2}{1-\cos t}$

31. $\lim_{x\to 2} \dfrac{\sin(x-2)}{x^2+2x-8}$

32. $\lim_{x\to 3} \dfrac{x^2-9}{\sin(x-3)}$

33. $\lim_{x\to 0} \dfrac{2\sin 4x + 1 - \cos x}{x}$

34. $\lim_{x\to 0} \dfrac{4x^2 - 2\sin x}{x}$

35. $\lim_{x\to\pi/4} \dfrac{1-\tan x}{\cos x - \sin x}$

36. $\lim_{x\to\pi/4} \dfrac{\cos 2x}{\cos x - \sin x}$

37. Suppose $f(x) = \sin x$. Use (10) and (14) of this section along with (17) of Section 1.4 to find the limit:

$$\lim_{h\to 0} \frac{f\left(\frac{\pi}{4}+h\right) - f\left(\frac{\pi}{4}\right)}{h}.$$

38. Suppose $f(x) = \cos x$. Use (10) and (14) of this section along with (18) of Section 1.4 to find the limit:

$$\lim_{h\to 0} \frac{f\left(\frac{\pi}{6}+h\right) - f\left(\frac{\pi}{6}\right)}{h}.$$

In Problems 39 and 40, use the Squeeze Theorem to establish the given limit.

39. $\lim_{x\to 0} x \sin\dfrac{1}{x} = 0$

40. $\lim_{x\to 0} x^2 \cos\dfrac{\pi}{x} = 0$

41. Use the properties of limits given in Theorem 2.2.3 to show that

(a) $\lim_{x\to 0} x^3 \sin\dfrac{1}{x} = 0$ **(b)** $\lim_{x\to 0} x^2 \sin^2\dfrac{1}{x} = 0.$

42. If $|f(x)| \le B$ for all x in an interval containing 0, show that $\lim_{x\to 0} x^2 f(x) = 0$.

In Problems 43 and 44, use the Squeeze Theorem to evaluate the given limit.

43. $\lim_{x\to 2} f(x)$ where $2x - 1 \le f(x) \le x^2 - 2x + 3$, $x \ne 2$

44. $\lim_{x\to 0} f(x)$ where $|f(x) - 1| \le x^2$, $x \ne 0$

☰ Think About It

In Problems 45–48, use an appropriate substitution to find the given limit.

45. $\lim_{x\to\pi/4} \dfrac{\sin x - \cos x}{x - \pi/4}$

46. $\lim_{x\to\pi} \dfrac{x-\pi}{\tan 2x}$

47. $\lim_{x\to 1} \dfrac{\sin(\pi/x)}{x-1}$

48. $\lim_{x\to 2} \dfrac{\cos(\pi/x)}{x-2}$

49. Discuss: Is the function

$$f(x) = \begin{cases} \dfrac{\sin x}{x}, & x \ne 0 \\ 1, & x = 0 \end{cases}$$

continuous at 0?

50. The existence of $\lim_{x\to 0} \dfrac{\sin x}{x}$ does not imply the existence of $\lim_{x\to 0} \dfrac{\sin|x|}{x}$. Explain why the second limit fails to exist.

2.5 Limits That Involve Infinity

Introduction In Sections 1.2 and 1.3 we considered some functions whose graphs possessed asymptotes. We will see in this section that vertical and horizontal asymptotes of a graph are defined in terms of limits involving the concept of *infinity*. Recall, the **infinity symbols**, $-\infty$ ("minus infinity") and ∞ ("infinity"), are notational devices used to indicate, in turn, that a quantity becomes unbounded in the negative direction (in the Cartesian plane this means to the left for x and downward for y) and in the positive direction (to the right for x and upward for y).

Some texts use the symbol $+\infty$ and the words *plus infinity* instead of ∞ and *infinity*.

Although the terminology and notation used when working with $\pm\infty$ is standard, it is nevertheless a bit unfortunate and can be confusing. So let us make it clear at the outset that we are going to consider two kinds of limits. First, we are going to examine

- *infinite limits*.

The words *infinite limit* always refer to *a limit that does not exist* because the function f exhibits unbounded behavior: $f(x) \to -\infty$ or $f(x) \to \infty$. Next, we will consider

- *limits at infinity*.

The words *at infinity* mean that we are trying to determine whether a function f possesses a limit when the variable x is allowed to become unbounded: $x \to -\infty$ or $x \to \infty$. Such limits may or may not exist.

Throughout the discussion, bear in mind that $-\infty$ and ∞ do not represent real numbers and should *never* be manipulated arithmetically like a number.

Infinite Limits The limit of a function f will fail to exist as x approaches a number a whenever the function values increase or decrease without bound. The fact that the function values $f(x)$ increase without bound as x approaches a is denoted symbolically by

$$f(x) \to \infty \text{ as } x \to a \qquad \text{or} \qquad \lim_{x \to a} f(x) = \infty. \tag{1}$$

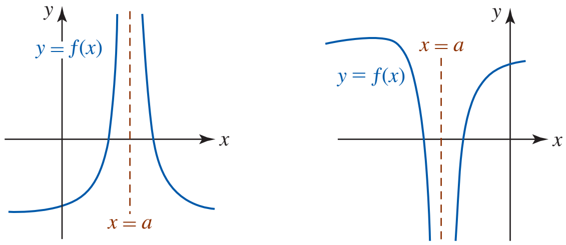
If the function values decrease without bound as x approaches a, we write

$$f(x) \to -\infty \text{ as } x \to a \qquad \text{or} \qquad \lim_{x \to a} f(x) = -\infty. \tag{2}$$

Recall, the use of the symbol $x \to a$ signifies that f exhibits the same behavior—in this instance, unbounded behavior—from both sides of the number a on the x-axis. For example, the notation in (1) indicates that

$$f(x) \to \infty \text{ as } x \to a^- \qquad \textit{and} \qquad f(x) \to \infty \text{ as } x \to a^+.$$

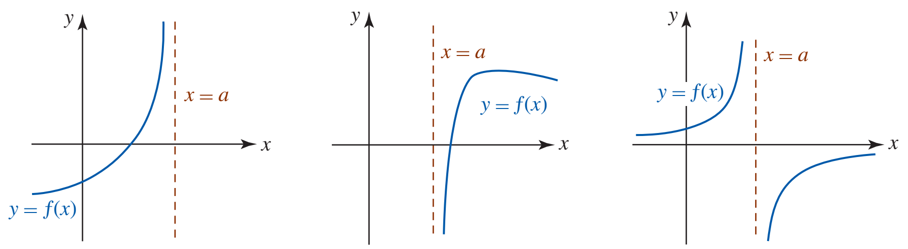
See FIGURE 2.5.1.



(a) $\lim_{x \to a} f(x) = \infty$ (b) $\lim_{x \to a} f(x) = -\infty$

FIGURE 2.5.1 Two types of infinite limits

Similarly, FIGURE 2.5.2 shows the unbounded behavior of a function f as x approaches a from one side. Note in Figure 2.5.2(c), we cannot describe the behavior of f near a using just one limit symbol.



(a) $\lim_{x \to a^-} f(x) = \infty$ (b) $\lim_{x \to a^+} f(x) = -\infty$ (c) $\lim_{x \to a^-} f(x) = \infty$ and $\lim_{x \to a^+} f(x) = -\infty$

FIGURE 2.5.2 Three more types of infinite limits

In general, any limit of the six types

$$\begin{aligned} &\lim_{x \to a^-} f(x) = -\infty, \qquad &&\lim_{x \to a^-} f(x) = \infty, \\ &\lim_{x \to a^+} f(x) = -\infty, &&\lim_{x \to a^+} f(x) = \infty, \\ &\lim_{x \to a} f(x) = -\infty, &&\lim_{x \to a} f(x) = \infty, \end{aligned} \tag{3}$$

is called an **infinite limit**. Again, in each case of (3) we are simply describing in a symbolic manner the behavior of a function f near the number a. *None of the limits in* (3) *exist*.

In Section 1.3 we reviewed how to identify a vertical asymptote for the graph of a rational function $f(x) = p(x)/q(x)$. We are now in a position to define a vertical asymptote of any function in terms of the limit concept.

Definition 2.5.1 Vertical Asymptote

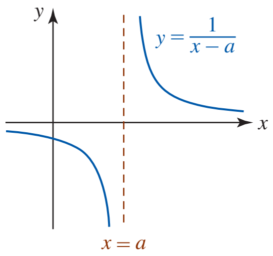
A line $x = a$ is said to be a **vertical asymptote** for the graph of a function f if at least one of the six statements in (3) is true.

See Figure 1.2.1.

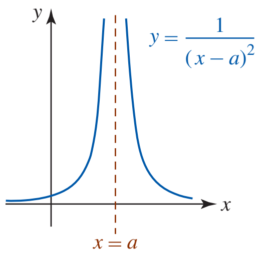
In the review of functions in Chapter 1 we saw that the graphs of rational functions often possess asymptotes. We saw that the graphs of the rational functions $y = 1/x$ and $y = 1/x^2$ were similar to the graphs in Figure 2.5.2(c) and Figure 2.5.1(a), respectively. The y-axis, that is, $x = 0$, is a vertical asymptote for each of these functions. The graphs of

$$y = \frac{1}{x - a} \quad \text{and} \quad y = \frac{1}{(x - a)^2} \tag{4}$$

are obtained by shifting the graphs of $y = 1/x$ and $y = 1/x^2$ horizontally $|a|$ units. As seen in FIGURE 2.5.3, $x = a$ is a vertical asymptote for the rational functions in (4). We have



(a)



(b)

FIGURE 2.5.3 Graphs of functions in (4)

$$\lim_{x \to a^-} \frac{1}{x - a} = -\infty \quad \text{and} \quad \lim_{x \to a^+} \frac{1}{x - a} = \infty \tag{5}$$

and

$$\lim_{x \to a} \frac{1}{(x - a)^2} = \infty. \tag{6}$$

The infinite limits in (5) and (6) are just special cases of the following general result:

$$\lim_{x \to a^-} \frac{1}{(x - a)^n} = -\infty \quad \text{and} \quad \lim_{x \to a^+} \frac{1}{(x - a)^n} = \infty, \tag{7}$$

for n an odd positive integer, and

$$\lim_{x \to a} \frac{1}{(x - a)^n} = \infty, \tag{8}$$

for n an even positive integer. As a consequence of (7) and (8), the graph of a rational function $y = 1/(x - a)^n$ either resembles the graph in Figure 2.5.3(a) for n odd or that in Figure 2.5.3(b) for n even.

For a general rational function $f(x) = p(x)/q(x)$, where p and q have no common factors, it should be clear from this discussion that when q contains a factor $(x - a)^n$, n a positive integer, then the shape of the graph near the vertical line $x = a$ must be either one of those shown in Figure 2.5.3 or its reflection in the x-axis.

EXAMPLE 1 Vertical Asymptotes of a Rational Function

Inspection of the rational function

$$f(x) = \frac{x + 2}{x^2(x + 4)}$$

shows that $x = -4$ and $x = 0$ are vertical asymptotes for the graph of f. Since the denominator contains the factors $(x - (-4))^1$ and $(x - 0)^2$ we expect the graph of f near the line $x = -4$ to resemble Figure 2.5.3(a) or its reflection in the x-axis, and the graph near $x = 0$ to resemble Figure 2.5.3(b) or its reflection in the x-axis.

For x close to 0, from either side of 0, it is easily seen that $f(x) > 0$. But, for x close to -4, say $x = -4.1$ and $x = -3.9$, we have $f(x) > 0$ and $f(x) < 0$, respectively. Using the additional information that there is only a single x-intercept $(-2, 0)$, we obtain the graph of f in FIGURE 2.5.4. ■

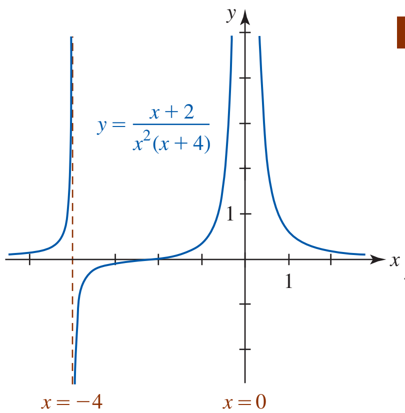


FIGURE 2.5.4 Graph of function in Example 1

EXAMPLE 2 One-Sided Limit

In Figure 1.6.6 we saw that the y-axis, or the line $x = 0$, is a vertical asymptote for the natural logarithmic function $f(x) = \ln x$ since

$$\lim_{x \to 0^+} \ln x = -\infty.$$

The graph of the logarithmic function $y = \ln(x + 3)$ is the graph of $f(x) = \ln x$ shifted 3 units to the left. Thus $x = -3$ is a vertical asymptote for the graph of $y = \ln(x + 3)$ since $\lim_{x \to -3^+} \ln(x + 3) = -\infty$. ■

EXAMPLE 3 One-Sided Limit

Graph the function $f(x) = \dfrac{x}{\sqrt{x + 2}}$.

Solution Inspection of f reveals that its domain is the interval $(-2, \infty)$ and the y-intercept is $(0, 0)$. From the accompanying table we conclude that f decreases

$x \to -2^+$	-1.9	-1.99	-1.999	-1.9999
$f(x)$	-6.01	-19.90	-63.21	-199.90

without bound as x approaches -2 from the right:

$$\lim_{x \to -2^+} f(x) = -\infty.$$

Hence, the line $x = -2$ is a vertical asymptote. The graph of f is given in FIGURE 2.5.5. ■

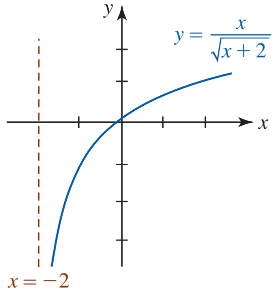


FIGURE 2.5.5 Graph of function in Example 3

Limits at Infinity If a function f approaches a constant value L as the independent variable x increases without bound $(x \to \infty)$ or as x decreases $(x \to -\infty)$ without bound, then we write

$$\lim_{x \to -\infty} f(x) = L \quad \text{or} \quad \lim_{x \to \infty} f(x) = L \tag{9}$$

and say that f possesses a **limit at infinity**. Here are all the possibilities for limits at infinity $\lim_{x \to -\infty} f(x)$ and $\lim_{x \to \infty} f(x)$:

- One limit exists but the other does not,
- Both $\lim_{x \to -\infty} f(x)$ and $\lim_{x \to \infty} f(x)$ exist and equal the same number,
- Both $\lim_{x \to -\infty} f(x)$ and $\lim_{x \to \infty} f(x)$ exist but are different numbers,
- Neither $\lim_{x \to -\infty} f(x)$ nor $\lim_{x \to \infty} f(x)$ exists.

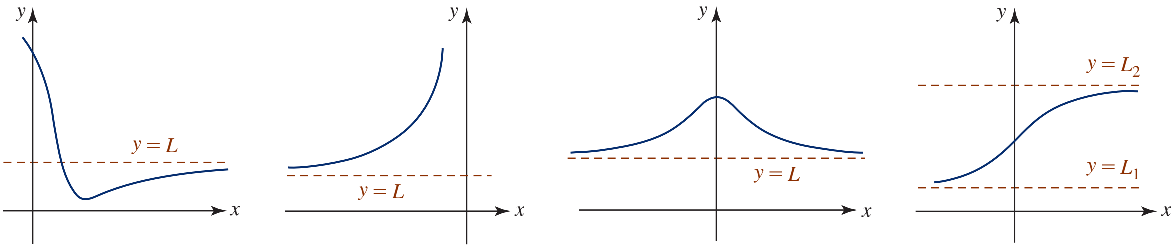
If at least one of the limits exists, say, $\lim_{x \to \infty} f(x) = L$, then the graph of f can be made arbitrarily close to the line $y = L$ as x increases in the positive direction.

Definition 2.5.2 Horizontal Asymptote

A line $y = L$ is said to be a **horizontal asymptote** for the graph of a function f if at least one of the two statements in (9) is true.

In FIGURE 2.5.6 we have illustrated some typical horizontal asymptotes. We note, in conjunction with Figure 2.5.6(d) that, in general, the graph of a function can have at most *two* horizontal asymptotes but the graph of a *rational function* $f(x) = p(x)/q(x)$ can have at most *one*. If the graph of a rational function f possesses a horizontal asymptote $y = L$, then its end behavior is as shown in Figure 2.5.6(c), that is:

$$f(x) \to L \text{ as } x \to -\infty \qquad \textit{and} \qquad f(x) \to L \text{ as } x \to \infty.$$



(a) $f(x) \to L$ as $x \to \infty$ (b) $f(x) \to L$ as $x \to -\infty$ (c) $f(x) \to L$ as $x \to -\infty$, $f(x) \to L$ as $x \to \infty$ (d) $f(x) \to L_1$ as $x \to -\infty$, $f(x) \to L_2$ as $x \to \infty$

FIGURE 2.5.6 $y = L$ is a horizontal asymptote in (a), (b), and (c); $y = L_1$ and $y = L_2$ are horizontal asymptotes in (d)

For example, if x becomes unbounded in either the positive or negative direction, the functions in (4) decrease to 0 and we write

$$\lim_{x\to-\infty}\frac{1}{x-a}=0,\ \lim_{x\to\infty}\frac{1}{x-a}=0 \quad\text{and}\quad \lim_{x\to-\infty}\frac{1}{(x-a)^2}=0,\ \lim_{x\to\infty}\frac{1}{(x-a)^2}=0. \tag{10}$$

In general, if r is a positive rational number and if $(x-a)^r$ is defined, then

These results are also true when $x-a$ is replaced by $a-x$, provided $(a-x)^r$ is defined.

$$\lim_{x\to-\infty}\frac{1}{(x-a)^r}=0 \quad\text{and}\quad \lim_{x\to\infty}\frac{1}{(x-a)^r}=0. \tag{11}$$

EXAMPLE 4 Horizontal and Vertical Asymptotes

The domain of the function $f(x)=\dfrac{4}{\sqrt{2-x}}$ is the interval $(-\infty, 2)$. In view of (11) we can write

$$\lim_{x\to-\infty}\frac{4}{\sqrt{2-x}}=0.$$

Note that we cannot consider the limit of f as $x\to\infty$ because the function is not defined for $x\geq 2$. Nevertheless $y=0$ is a horizontal asymptote. Now from infinite limit

$$\lim_{x\to2^-}\frac{4}{\sqrt{2-x}}=\infty$$

we conclude that $x=2$ is a vertical asymptote for the graph of f. See FIGURE 2.5.7. ■

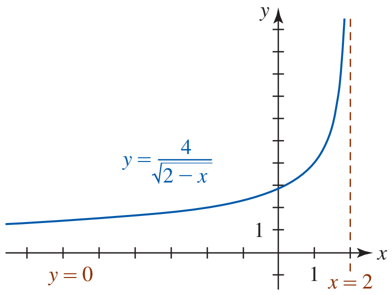


FIGURE 2.5.7 Graph of function in Example 4

In general, if $F(x)=f(x)/g(x)$, then the following table summarizes the limit results for the forms $\lim\limits_{x\to a}F(x)$, $\lim\limits_{x\to\infty}F(x)$, and $\lim\limits_{x\to-\infty}F(x)$. The symbol L denotes a real number.

limit form: $x\to a,\ \infty,\ -\infty$	$\dfrac{L}{\pm\infty}$	$\dfrac{\pm\infty}{L},\ L\neq 0$	$\dfrac{L}{0},\ L\neq 0$
limit is:	0	infinite	infinite

(12)

Limits of the form $\lim\limits_{x\to\infty}F(x)=\pm\infty$ or $\lim\limits_{x\to-\infty}F(x)=\pm\infty$ are said to be **infinite limits at infinity**. Furthermore, the limit properties given in Theorem 2.2.3 hold by replacing the symbol a by ∞ or $-\infty$ provided the limits exist. For example,

$$\lim_{x\to\infty}f(x)g(x)=\left(\lim_{x\to\infty}f(x)\right)\left(\lim_{x\to\infty}g(x)\right) \quad\text{and}\quad \lim_{x\to\infty}\frac{f(x)}{g(x)}=\frac{\lim\limits_{x\to\infty}f(x)}{\lim\limits_{x\to\infty}g(x)}, \tag{13}$$

whenever $\lim\limits_{x\to\infty}f(x)$ and $\lim\limits_{x\to\infty}g(x)$ exist. In the case of the limit of a quotient we must also have $\lim\limits_{x\to\infty}g(x)\neq 0$.

End Behavior In Section 1.3 we saw that how a function f behaves when $|x|$ is very large is its **end behavior**. As already discussed, if $\lim\limits_{x\to\infty}f(x)=L$, then the graph of f can be made arbitrarily close to the line $y=L$ for large positive values of x. The graph of a polynomial function,

$$f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_2x^2+a_1x+a_0,$$

resembles the graph of $y=a_nx^n$ for $|x|$ very large. In other words, for

$$f(x)=a_nx^n\ \boxed{+\,a_{n-1}x^{n-1}+\cdots+a_1x+a_0} \tag{14}$$

the terms enclosed in the blue rectangle in (14) are irrelevant when we look at a graph of a polynomial globally—that is, for $|x|$ large. Thus we have

$$\lim_{x\to\pm\infty}a_nx^n=\lim_{x\to\pm\infty}\left(a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0\right), \tag{15}$$

where (15) is either ∞ or $-\infty$ depending on a_n and n. In other words, the limit in (15) is an example of an infinite limit at infinity.

EXAMPLE 5 Limit at Infinity

Evaluate $\lim\limits_{x\to\infty} \dfrac{-6x^4 + x^2 + 1}{2x^4 - x}$.

Solution We cannot apply the limit quotient law in (13) to the given function, since $\lim\limits_{x\to\infty}(-6x^4 + x^2 + 1) = -\infty$ and $\lim\limits_{x\to\infty}(2x^4 - x) = \infty$. However, by dividing the numerator and the denominator by x^4, we can write

$$\lim_{x\to\infty} \frac{-6x^4 + x^2 + 1}{2x^4 - x} = \lim_{x\to\infty} \frac{-6 + \left(\dfrac{1}{x^2}\right) + \left(\dfrac{1}{x^4}\right)}{2 - \left(\dfrac{1}{x^3}\right)}$$

$$= \frac{\lim\limits_{x\to\infty}\left[-6 + \left(\dfrac{1}{x^2}\right) + \left(\dfrac{1}{x^4}\right)\right]}{\lim\limits_{x\to\infty}\left[2 - \left(\dfrac{1}{x^3}\right)\right]} \leftarrow$$

Limit of the numerator and denominator both exist and the limit of the denominator is not zero

$$= \frac{-6 + 0 + 0}{2 - 0} = -3.$$

This means the line $y = -3$ is a horizontal asymptote for the graph of the function.

Alternative Solution In view of (14), we can discard all powers of x other than the highest:

discard terms in the blue boxes

$$\lim_{x\to\infty} \frac{-6x^4 \boxed{+ x^2 + 1}}{2x^4 \boxed{- x}} = \lim_{x\to\infty} \frac{-6x^4}{2x^4} = \lim_{x\to\infty} \frac{-6}{2} = -3.$$ ■

EXAMPLE 6 Infinite Limit at Infinity

Evaluate $\lim\limits_{x\to\infty} \dfrac{1 - x^3}{3x + 2}$.

Solution By (14),

$$\lim_{x\to\infty} \frac{1 - x^3}{3x + 2} = \lim_{x\to\infty} \frac{-x^3}{3x} = -\frac{1}{3}\lim_{x\to\infty} x^2 = -\infty.$$

In other words, the limit does not exist. ■

EXAMPLE 7 Graph of a Rational Function

Graph the function $f(x) = \dfrac{x^2}{1 - x^2}$.

Solution Inspection of the function f reveals that its graph is symmetric with respect to the y-axis, the y-intercept is $(0, 0)$, and the vertical asymptotes are $x = -1$ and $x = 1$. Now, from the limit

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2}{1 - x^2} = \lim_{x\to\infty} \frac{x^2}{-x^2} = -\lim_{x\to\infty} 1 = -1$$

we conclude that the line $y = -1$ is a horizontal asymptote. The graph of f is given in FIGURE 2.5.8. ■

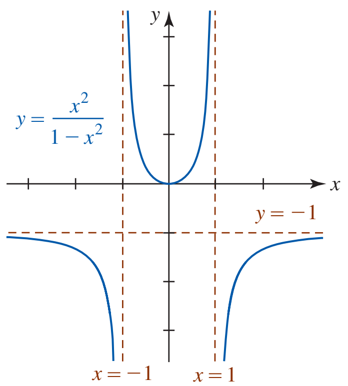


FIGURE 2.5.8 Graph of function in Example 7

Another limit law that holds true for limits at infinity is that the limit of an nth root of a function is the nth root of the limit, whenever the limit exists and the nth root is defined. In symbols, if $\lim\limits_{x\to\infty} g(x) = L$, then

$$\lim_{x\to\infty} \sqrt[n]{g(x)} = \sqrt[n]{\lim_{x\to\infty} g(x)} = \sqrt[n]{L}, \tag{16}$$

provided $L \geq 0$ when n is even. The result also holds for $x \to -\infty$.

EXAMPLE 8 Limit of a Square Root

Evaluate $\lim_{x\to\infty}\sqrt{\dfrac{2x^3-5x^2+4x-6}{6x^3+2x}}$.

Solution Because the limit of the rational function inside the radical exists and is positive, we can write

$$\lim_{x\to\infty}\sqrt{\frac{2x^3-5x^2+4x-6}{6x^3+2x}}=\sqrt{\lim_{x\to\infty}\frac{2x^3-5x^2+4x-6}{6x^3+2x}}=\sqrt{\lim_{x\to\infty}\frac{2x^3}{6x^3}}=\sqrt{\frac{1}{3}}=\frac{1}{\sqrt{3}}.\quad\blacksquare$$

EXAMPLE 9 Graph with Two Horizontal Asymptotes

Determine whether the graph of $f(x)=\dfrac{5x}{\sqrt{x^2+4}}$ has any horizontal asymptotes.

Solution Since the function is not rational, we must investigate the limit of f as $x\to\infty$ and as $x\to-\infty$. First, recall from algebra that $\sqrt{x^2}$ is nonnegative, or more to the point,

$$\sqrt{x^2}=|x|=\begin{cases}x, & x\geq 0\\ -x, & x<0.\end{cases}$$

We then rewrite f as

$$f(x)=\frac{\dfrac{5x}{\sqrt{x^2}}}{\dfrac{\sqrt{x^2+4}}{\sqrt{x^2}}}=\frac{\dfrac{5x}{|x|}}{\dfrac{\sqrt{x^2+4}}{\sqrt{x^2}}}=\frac{\dfrac{5x}{|x|}}{\sqrt{1+\dfrac{4}{x^2}}}.$$

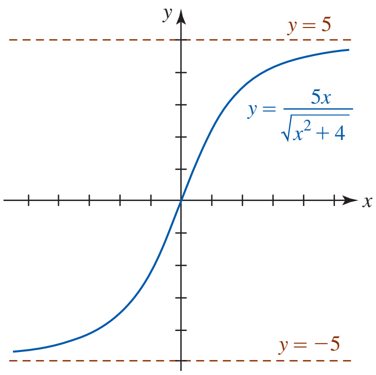


FIGURE 2.5.9 Graph of function in Example 9

The limits of f as $x\to\infty$ and as $x\to-\infty$ are, respectively,

$$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{\dfrac{5x}{|x|}}{\sqrt{1+\dfrac{4}{x^2}}}=\lim_{x\to\infty}\frac{\dfrac{5x}{x}}{\sqrt{1+\dfrac{4}{x^2}}}=\frac{\lim_{x\to\infty}5}{\sqrt{\lim_{x\to\infty}\left(1+\dfrac{4}{x^2}\right)}}=\frac{5}{1}=5,$$

and $$\lim_{x\to-\infty}f(x)=\lim_{x\to-\infty}\frac{\dfrac{5x}{|x|}}{\sqrt{1+\dfrac{4}{x^2}}}=\lim_{x\to-\infty}\frac{\dfrac{5x}{-x}}{\sqrt{1+\dfrac{4}{x^2}}}=\frac{\lim_{x\to-\infty}(-5)}{\sqrt{\lim_{x\to-\infty}\left(1+\dfrac{4}{x^2}\right)}}=\frac{-5}{1}=-5.$$

Thus the graph of f has two horizontal asymptotes $y=5$ and $y=-5$. The graph of f, which is similar to Figure 2.5.6(d), is given in FIGURE 2.5.9. ■

In the next example we see that the form of the given limit is $\infty-\infty$, but the limit exists and is *not* 0.

EXAMPLE 10 Using Rationalization

Evaluate $\lim_{x\to\infty}\left(x^2-\sqrt{x^4+7x^2+1}\right)$.

Solution Because $f(x)=x^2-\sqrt{x^4+7x^2+1}$ is an even function (verify that $f(-x)=f(x)$) with domain $(-\infty,\infty)$, if $\lim_{x\to\infty}f(x)$ exists it must be the same as $\lim_{x\to-\infty}f(x)$. We first rationalize the numerator:

$$\begin{aligned}\lim_{x\to\infty}\left(x^2-\sqrt{x^4+7x^2+1}\right)&=\lim_{x\to\infty}\frac{\left(x^2-\sqrt{x^4+7x^2+1}\right)}{1}\cdot\left(\frac{x^2+\sqrt{x^4+7x^2+1}}{x^2+\sqrt{x^4+7x^2+1}}\right)\\&=\lim_{x\to\infty}\frac{x^4-(x^4+7x^2+1)}{x^2+\sqrt{x^4+7x^2+1}}\\&=\lim_{x\to\infty}\frac{-7x^2-1}{x^2+\sqrt{x^4+7x^2+1}}.\end{aligned}$$

Next, we divide the numerator and denominator by $\sqrt{x^4} = x^2$:

$$\lim_{x\to\infty} \frac{-7x^2 - 1}{x^2 + \sqrt{x^4 + 7x^2 + 1}} = \lim_{x\to\infty} \frac{\dfrac{-7x^2}{\sqrt{x^4}} - \dfrac{1}{\sqrt{x^4}}}{\dfrac{x^2 + \sqrt{x^4 + 7x^2 + 1}}{\sqrt{x^4}}}$$

$$= \lim_{x\to\infty} \frac{-7 - \dfrac{1}{x^2}}{1 + \sqrt{1 + \dfrac{7}{x^2} + \dfrac{1}{x^4}}}$$

$$= \frac{\lim\limits_{x\to\infty}\left(-7 - \dfrac{1}{x^2}\right)}{\lim\limits_{x\to\infty} 1 + \sqrt{\lim\limits_{x\to\infty}\left(1 + \dfrac{7}{x^2} + \dfrac{1}{x^4}\right)}}$$

$$= \frac{-7}{1+1} = -\frac{7}{2}.$$

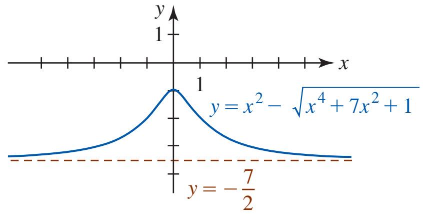


FIGURE 2.5.10 Graph of function in Example 10

With the help of a CAS, the graph of the function f is given in FIGURE 2.5.10. The line $y = -\frac{7}{2}$ is a horizontal asymptote. Note the symmetry of the graph with respect to the y-axis. ■

When working with functions containing the natural exponential function, the following four limits merit special attention:

$$\lim_{x\to\infty} e^x = \infty, \quad \lim_{x\to-\infty} e^x = 0, \quad \lim_{x\to\infty} e^{-x} = 0, \quad \lim_{x\to-\infty} e^{-x} = \infty. \tag{17}$$

As discussed in Section 1.6 and verified by the second and third limit in (17), $y = 0$ is a horizontal asymptote for the graphs of $y = e^x$ and $y = e^{-x}$. See FIGURE 2.5.11.

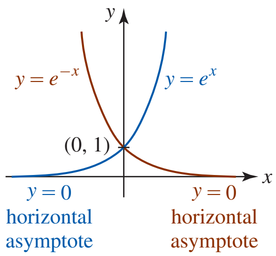


FIGURE 2.5.11 Graphs of exponential functions

EXAMPLE 11 Graph with Two Horizontal Asymptotes

Determine whether the graph of $f(x) = \dfrac{6}{1 + e^{-x}}$ has any horizontal asymptotes.

Solution Because f is not a rational function, we must examine $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$. First, in view of the third result given in (17) we can write

$$\lim_{x\to\infty} \frac{6}{1 + e^{-x}} = \frac{\lim\limits_{x\to\infty} 6}{\lim\limits_{x\to\infty}(1 + e^{-x})} = \frac{6}{1+0} = 6.$$

Thus $y = 6$ is a horizontal asymptote. Now, because $\lim\limits_{x\to-\infty} e^{-x} = \infty$ it follows from the table in (12) that

$$\lim_{x\to-\infty} \frac{6}{1 + e^{-x}} = 0.$$

Therefore $y = 0$ is a horizontal asymptote. The graph of f is given in FIGURE 2.5.12. ■

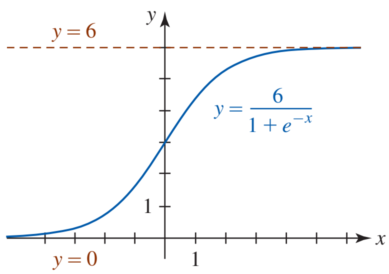


FIGURE 2.5.12 Graph of function in Example 11

Composite Functions Theorem 2.3.3, the limit of a composite function, holds when a is replaced by $-\infty$ or ∞ and the limit exists. For example, if $\lim\limits_{x\to\infty} g(x) = L$ and f is continuous at L, then

$$\lim_{x\to\infty} f(g(x)) = f\left(\lim_{x\to\infty} g(x)\right) = f(L). \tag{18}$$

The limit result in (16) is just a special case of (18) when $f(x) = \sqrt[n]{x}$. The result in (18) also holds for $x \to -\infty$. Our last example illustrates (18) involving a limit at ∞.

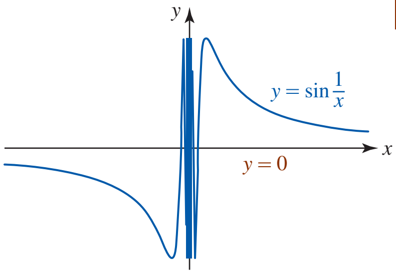


FIGURE 2.5.13 Graph of function in Example 12

EXAMPLE 12 A Trigonometric Function Revisited

In Example 2 of Section 2.4 we saw that $\lim_{x\to 0} \sin(1/x)$ does not exist. However, the limit at infinity, $\lim_{x\to\infty} \sin(1/x)$, exists. By (18) we can write

$$\lim_{x\to\infty} \sin\frac{1}{x} = \sin\left(\lim_{x\to\infty}\frac{1}{x}\right) = \sin 0 = 0.$$

As we see in FIGURE 2.5.13, $y = 0$ is a horizontal asymptote for the graph of $f(x) = \sin(1/x)$. You should compare this graph with that given in Figure 2.4.2. ■

Exercises 2.5

Answers to selected odd-numbered problems begin on page ANS-000.

Fundamentals

In Problems 1–24, express the given limit as a number, as $-\infty$, or as ∞.

1. $\lim_{x\to 5^-} \dfrac{1}{x-5}$
2. $\lim_{x\to 6} \dfrac{4}{(x-6)^2}$
3. $\lim_{x\to -4^+} \dfrac{2}{(x+4)^3}$
4. $\lim_{x\to 2^-} \dfrac{10}{x^2-4}$
5. $\lim_{x\to 1} \dfrac{1}{(x-1)^4}$
6. $\lim_{x\to 0^+} \dfrac{-1}{\sqrt{x}}$
7. $\lim_{x\to 0^+} \dfrac{2+\sin x}{x}$
8. $\lim_{x\to \pi^+} \csc x$
9. $\lim_{x\to\infty} \dfrac{x^2-3x}{4x^2+5}$
10. $\lim_{x\to\infty} \dfrac{x^2}{1+x^{-2}}$
11. $\lim_{x\to\infty} \left(5 - \dfrac{2}{x^4}\right)$
12. $\lim_{x\to -\infty} \left(\dfrac{6}{\sqrt[3]{x}} + \dfrac{1}{\sqrt[5]{x}}\right)$
13. $\lim_{x\to\infty} \dfrac{8-\sqrt{x}}{1+4\sqrt{x}}$
14. $\lim_{x\to -\infty} \dfrac{1+7\sqrt[3]{x}}{2\sqrt[3]{x}}$
15. $\lim_{x\to\infty} \left(\dfrac{3x}{x+2} - \dfrac{x-1}{2x+6}\right)$
16. $\lim_{x\to\infty} \left(\dfrac{x}{3x+1}\right)\left(\dfrac{4x^2+1}{2x^2+x}\right)^3$
17. $\lim_{x\to\infty} \sqrt{\dfrac{3x+2}{6x-8}}$
18. $\lim_{x\to -\infty} \sqrt[3]{\dfrac{2x-1}{7-16x}}$
19. $\lim_{x\to\infty} \left(x - \sqrt{x^2+1}\right)$
20. $\lim_{x\to\infty} \left(\sqrt{x^2+5x} - x\right)$
21. $\lim_{x\to\infty} \cos\left(\dfrac{5}{x}\right)$
22. $\lim_{x\to -\infty} \sin\left(\dfrac{\pi x}{3-6x}\right)$
23. $\lim_{x\to -\infty} \sin^{-1}\left(\dfrac{x}{\sqrt{4x^2+1}}\right)$
24. $\lim_{x\to\infty} \ln\left(\dfrac{x}{x+8}\right)$

In Problems 25–32, find $\lim_{x\to -\infty} f(x)$ and $\lim_{x\to\infty} f(x)$ for the given function f.

25. $f(x) = \dfrac{4x+1}{\sqrt{x^2+1}}$
26. $f(x) = \dfrac{\sqrt{9x^2+6}}{5x-1}$
27. $f(x) = \dfrac{2x+1}{\sqrt{3x^2+1}}$
28. $f(x) = \dfrac{-5x^2+6x+3}{\sqrt{x^4+x^2+1}}$
29. $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$
30. $f(x) = 1 + \dfrac{2e^{-x}}{e^x + e^{-x}}$
31. $f(x) = \dfrac{|x-5|}{x-5}$
32. $f(x) = \dfrac{|4x| + |x-1|}{x}$

In Problems 33–42, find all vertical and horizontal asymptotes for the graph of the given function. Sketch the graph.

33. $f(x) = \dfrac{1}{x^2+1}$
34. $f(x) = \dfrac{x}{x^2+1}$
35. $f(x) = \dfrac{x^2}{x+1}$
36. $f(x) = \dfrac{x^2-x}{x^2-1}$
37. $f(x) = \dfrac{1}{x^2(x-2)}$
38. $f(x) = \dfrac{4x^2}{x^2+4}$
39. $f(x) = \sqrt{\dfrac{x}{x-1}}$
40. $f(x) = \dfrac{1-\sqrt{x}}{\sqrt{x}}$
41. $f(x) = \dfrac{x-2}{\sqrt{x^2+1}}$
42. $f(x) = \dfrac{x+3}{\sqrt{x^2-1}}$

In Problems 43–46, use the given graph to find:

(a) $\lim_{x\to 2^-} f(x)$ (b) $\lim_{x\to 2^+} f(x)$
(c) $\lim_{x\to -\infty} f(x)$ (d) $\lim_{x\to\infty} f(x)$

43.

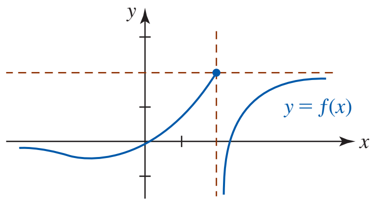


FIGURE 2.5.14 Graph for Problem 43

44.

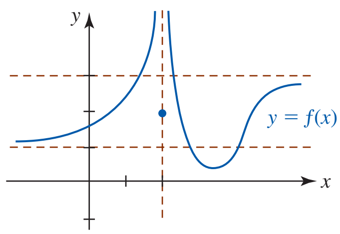


FIGURE 2.5.15 Graph for Problem 44

45.

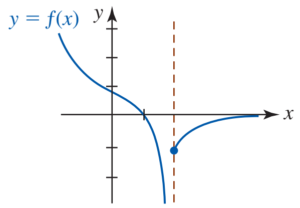


FIGURE 2.5.16 Graph for Problem 45

46.

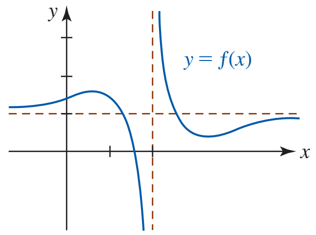


FIGURE 2.5.17 Graph for Problem 46

In Problems 47–50, sketch a graph of a function f that satisfies the given conditions.

47. $\lim_{x\to1^+} f(x) = -\infty$, $\lim_{x\to1^-} f(x) = -\infty$, $f(2) = 0$, $\lim_{x\to\infty} f(x) = 0$

48. $f(0) = 1$, $\lim_{x\to-\infty} f(x) = 3$, $\lim_{x\to\infty} f(x) = -2$

49. $\lim_{x\to2} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = \infty$, $\lim_{x\to\infty} f(x) = 1$

50. $\lim_{x\to1^-} f(x) = 2$, $\lim_{x\to1^+} f(x) = -\infty$, $f\left(\frac{3}{2}\right) = 0$, $f(3) = 0$,

$\lim_{x\to-\infty} f(x) = 0$, $\lim_{x\to\infty} f(x) = 0$

51. Use an appropriate substitution to evaluate

$$\lim_{x\to\infty} x \sin\frac{3}{x}.$$

52. According to Einstein's theory of relativity, the mass m of a body moving with velocity v is $m = m_0/\sqrt{1 - v^2/c^2}$, where m_0 is the initial mass and c is the speed of light. What happens to m as $v \to c^-$?

☰ Calculator/CAS Problems

In Problems 53 and 54, use a calculator or CAS to investigate the given limit. Conjecture its value.

53. $\lim_{x\to\infty} x^2 \sin\dfrac{2}{x^2}$ **54.** $\lim_{x\to\infty} \left(\cos\dfrac{1}{x}\right)^x$

55. Use a calculator or CAS to obtain the graph of $f(x) = (1 + x)^{1/x}$. Use the graph to conjecture the values of $f(x)$ as **(a)** $x \to -1^+$, **(b)** $x \to 0$, and **(c)** $x \to \infty$.

56. **(a)** A regular n-gon is an n-sided polygon inscribed in a circle; the polygon is formed by n equally spaced points on the circle. Suppose the polygon shown in FIGURE 2.5.18 represents a regular n-gon inscribed in a circle of radius r. Use trigonometry to show that the area $A(n)$ of the n-gon is given by

$$A(n) = \frac{n}{2} r^2 \sin\left(\frac{2\pi}{n}\right).$$

(b) It stands to reason that the area $A(n)$ approaches the area of the circle as the number of sides of the n-gon increases. Use a calculator to compute $A(100)$ and $A(1000)$.

(c) Let $x = 2\pi/n$ in $A(n)$ and note that as $n \to \infty$ then $x \to 0$. Use (10) of Section 2.4 to show that $\lim_{n\to\infty} A(n) = \pi r^2$.

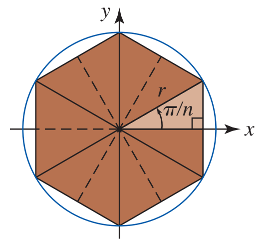


FIGURE 2.5.18 Inscribed n-gon for Problem 56

☰ Think About It

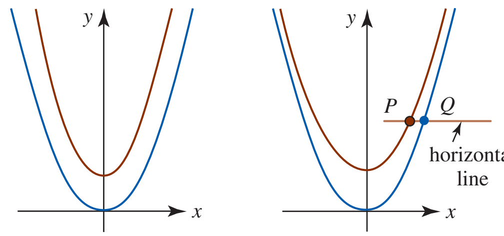
57. **(a)** Suppose $f(x) = x^2/(x + 1)$ and $g(x) = x - 1$. Show that

$$\lim_{x\to\pm\infty} [f(x) - g(x)] = 0.$$

(b) What does the result in part (a) indicate about the graphs of f and g where $|x|$ is large?

(c) If possible, give a name to the function g.

58. Very often students and even instructors will sketch vertically shifted graphs incorrectly. For example, the graphs of $y = x^2$ and $y = x^2 + 1$ are incorrectly drawn in FIGURE 2.5.19(a) but are correctly drawn in Figure 2.5.19(b). Demonstrate that Figure 2.5.19(b) is correct by showing that the horizontal distance between the two points P and Q shown in the figure approaches 0 as $x \to \infty$.



(a) Incorrect (b) Correct

FIGURE 2.5.19 Graphs for Problem 58

2.6 Limits—A Formal Approach

Introduction In the discussion that follows we will consider an alternative approach to the notion of a limit that is based on analytical concepts rather than on intuitive concepts. A **proof** of the existence of a limit can never be based on one's ability to sketch graphs or on tables of numerical values. Although a good intuitive understanding of $\lim_{x\to a} f(x)$ is sufficient for proceeding with the study of the calculus in this text, an intuitive understanding is admittedly too vague to be of any use in proving theorems. To give a rigorous demonstration of the existence of a limit, or to prove the important theorems of Section 2.2, we must start with a precise definition of a limit.

Limit of a Function Let us try to prove that $\lim_{x\to 2}(2x+6) = 10$ by elaborating on the following idea: "If $f(x) = 2x + 6$ can be made arbitrarily close to 10 by taking x sufficiently close to 2, from either side but different from 2, then $\lim_{x\to 2} f(x) = 10$." We need to make the concepts of *arbitrarily close* and *sufficiently close* precise. In order to set a standard of arbitrary closeness, let us demand that the distance between the numbers $f(x)$ and 10 be less than 0.1; that is,

$$|f(x) - 10| < 0.1 \qquad \text{or} \qquad 9.9 < f(x) < 10.1. \tag{1}$$

Then, how close must x be to 2 to accomplish (1)? To find out, we can use ordinary algebra to rewrite the inequality

$$9.9 < 2x + 6 < 10.1$$

as $1.95 < x < 2.05$. Adding -2 across this simultaneous inequality then gives

$$-0.05 < x - 2 < 0.05.$$

Using absolute values and remembering that $x \neq 2$, we can write the last inequality as $0 < |x - 2| < 0.05$. Thus, for an "arbitrary closeness to 10" of 0.1, "sufficiently close to 2" means within 0.05. In other words, if x is a number different from 2 such that its distance from 2 satisfies $|x - 2| < 0.05$, then the distance of $f(x)$ from 10 is guaranteed to satisfy $|f(x) - 10| < 0.1$. Expressed in yet another way, when x is a number different from 2 but in the open interval $(1.95, 2.05)$ on the x-axis, then $f(x)$ is in the interval $(9.9, 10.1)$ on the y-axis.

Using the same example, let us try to generalize. Suppose ε (the Greek letter *epsilon*) denotes an arbitrary *positive number* that is our measure of arbitrary closeness to the number 10. If we demand that

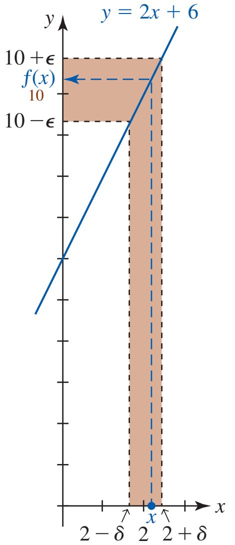


FIGURE 2.6.1 $f(x)$ is in $(10 - \varepsilon, 10 + \varepsilon)$ whenever x is in $(2 - \delta, 2 + \delta)$, $x \neq 2$

$$|f(x) - 10| < \varepsilon \qquad \text{or} \qquad 10 - \varepsilon < f(x) < 10 + \varepsilon, \tag{2}$$

then from $10 - \varepsilon < 2x + 6 < 10 + \varepsilon$ and algebra, we find

$$2 - \frac{\varepsilon}{2} < x < 2 + \frac{\varepsilon}{2} \qquad \text{or} \qquad -\frac{\varepsilon}{2} < x - 2 < \frac{\varepsilon}{2}. \tag{3}$$

Again using absolute values and remembering that $x \neq 2$, we can write the last inequality in (3) as

$$0 < |x - 2| < \frac{\varepsilon}{2}. \tag{4}$$

If we denote $\varepsilon/2$ by the new symbol δ (the Greek letter *delta*), (2) and (4) can be written as

$$|f(x) - 10| < \varepsilon \qquad \text{whenever} \qquad 0 < |x - 2| < \delta.$$

Thus, for a new value for ε, say $\varepsilon = 0.001$, $\delta = \varepsilon/2 = 0.0005$ tells us the corresponding closeness to 2. For any number x different from 2 in $(1.9995, 2.0005)$,* we can be sure $f(x)$ is in $(9.999, 10.001)$. See FIGURE 2.6.1.

A Definition The foregoing discussion leads us to the so-called $\varepsilon - \delta$ **definition of a limit**.

Definition 2.6.1 Definition of a Limit

Suppose a function f is defined everywhere on an open interval, except possibly at a number a in the interval. Then

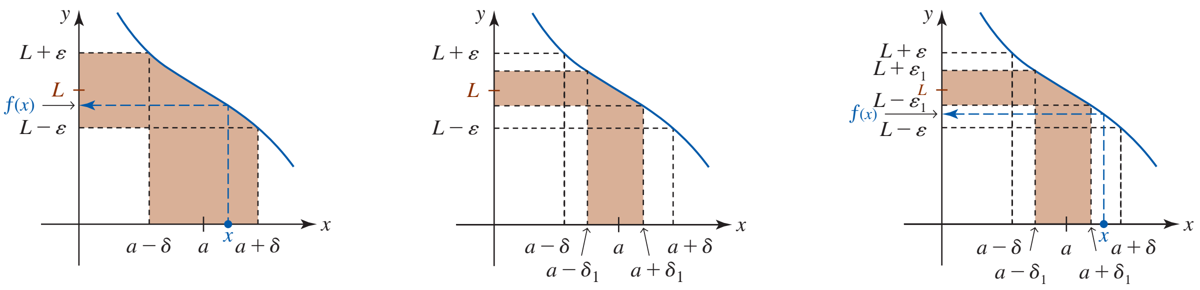
$$\lim_{x\to a} f(x) = L$$

means that for every $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \qquad \text{whenever} \qquad 0 < |x - a| < \delta.$$

*For this reason, we use $0 < |x - 2| < \delta$ rather than $|x - 2| < \delta$. Keep in mind when considering $\lim_{x\to 2} f(x)$, we do not care about f *at* 2.

Let $\lim_{x \to a} f(x) = L$ and suppose $\delta > 0$ is the number that "works" in the sense of Definition 2.6.1 for a given $\varepsilon > 0$. As shown in FIGURE 2.6.2(a), every x in $(a - \delta, a + \delta)$, with the possible exception of a itself, will then have an image $f(x)$ in $(L - \varepsilon, L + \varepsilon)$. Furthermore, as in Figure 2.6.2(b), a choice $\delta_1 < \delta$ for the same ε also "works" in that every x not equal to a in $(a - \delta_1, a + \delta_1)$ gives $f(x)$ in $(L - \varepsilon, L + \varepsilon)$. However, Figure 2.6.2(c) shows that choosing a smaller ε_1, $0 < \varepsilon_1 < \varepsilon$, will demand finding a new value of δ. Observe in Figure 2.6.2(c) that x is in $(a - \delta, a + \delta)$ but not in $(a - \delta_1, a + \delta_1)$, and so $f(x)$ is not necessarily in $(L - \varepsilon_1, L + \varepsilon_1)$.



(a) A δ that works for a given ε

(b) A smaller δ_1 will also work for the same ε

(c) A smaller ε_1 will require a $\delta_1 < \delta$. For x in $(a - \delta, a + \delta)$, $f(x)$ is not necessarily in $(L - \varepsilon_1, L + \varepsilon_1)$

FIGURE 2.6.2 $f(x)$ is in $(L - \varepsilon, L + \varepsilon)$ whenever x is in $(a - \delta, a + \delta)$, $x \neq a$

EXAMPLE 1 Using Definition 2.6.1

Prove that $\lim_{x \to a} (5x + 2) = 17$.

Solution For any arbitrary $\varepsilon > 0$, regardless how small, we wish to find a δ so that

$$|(5x + 2) - 17| < \varepsilon \qquad \text{whenever} \qquad 0 < |x - 3| < \delta.$$

To do this consider

$$|(5x + 2) - 17| = |5x - 15| = 5|x - 3|.$$

Thus, to make $|(5x + 2) - 17| = 5|x - 3| < \varepsilon$, we need only make $0 < |x - 3| < \varepsilon/5$; that is, choose $\delta = \varepsilon/5$.

Verification If $0 < |x - 3| < \varepsilon/5$, then $5|x - 3| < \varepsilon$ implies

$$|5x - 15| < \varepsilon \qquad \text{or} \qquad |(5x + 2) - 17| < \varepsilon \qquad \text{or} \qquad |f(x) - 17| < \varepsilon. \qquad \blacksquare$$

EXAMPLE 2 Using Definition 2.6.1

Prove that $\lim_{x \to -4} \dfrac{16 - x^2}{4 + x} = 8$.

We examined this limit in (1) and (2) of Section 2.1.

Solution For $x \neq -4$,

$$\left| \frac{16 - x^2}{4 + x} - 8 \right| = |4 - x - 8| = |-x - 4| = |x + 4| = |x - (-4)|$$

Thus,

$$\left| \frac{16 - x^2}{4 + x} - 8 \right| = |x - (-4)| < \varepsilon$$

whenever we have $0 < |x - (-4)| < \varepsilon$; that is, choose $\delta = \varepsilon$. $\blacksquare$

EXAMPLE 3 A Limit That Does Not Exist

Consider the function

$$f(x) = \begin{cases} 0, & x \leq 1 \\ 2, & x > 1. \end{cases}$$

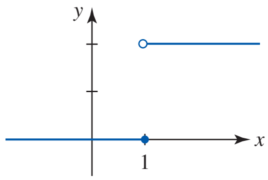


FIGURE 2.6.3 Limit of f does not exist as x approaches 1 in Example 3

We recognize in FIGURE 2.6.3 that f has a jump discontinuity at 1 and so $\lim_{x\to 1} f(x)$ does not exist. However, to *prove* this last fact, we shall proceed indirectly. Assume that the limit exists, namely, $\lim_{x\to 1} f(x) = L$. Then from Definition 2.6.1 we know that for the choice $\varepsilon = \frac{1}{2}$ there must exist a $\delta > 0$ so that

$$|f(x) - L| < \frac{1}{2} \qquad \text{whenever} \qquad 0 < |x - 1| < \delta.$$

Now to the right of 1, let us choose $x = 1 + \delta/2$. Since

$$0 < \left|1 + \frac{\delta}{2} - 1\right| = \left|\frac{\delta}{2}\right| < \delta$$

we must have

$$\left|f\left(1 + \frac{\delta}{2}\right) - L\right| = |2 - L| < \frac{1}{2}. \tag{5}$$

To the left of 1, choose $x = 1 - \delta/2$. But

$$0 < \left|1 - \frac{\delta}{2} - 1\right| = \left|-\frac{\delta}{2}\right| < \delta$$

implies

$$\left|f\left(1 - \frac{\delta}{2}\right) - L\right| = |0 - L| = |L| < \frac{1}{2}. \tag{6}$$

Solving the absolute-value inequalities (5) and (6) gives, respectively,

$$\frac{3}{2} < L < \frac{5}{2} \qquad \text{and} \qquad -\frac{1}{2} < L < \frac{1}{2}.$$

Since no number L can satisfy both of these inequalities, we conclude that $\lim_{x\to 1} f(x)$ does not exist. ■

In the next example we consider the limit of a quadratic function. We shall see that finding the δ in this case requires a bit more ingenuity than in Examples 1 and 2.

EXAMPLE 4 Using Definition 2.6.1

We examined this limit in Example 1 of Section 2.1.

Prove that $\lim_{x\to 4}(-x^2 + 2x + 2) = -6$.

Solution For an arbitrary $\varepsilon > 0$ we must find a $\delta > 0$ so that

$$|-x^2 + 2x + 2 - (-6)| < \varepsilon \qquad \text{whenever} \qquad 0 < |x - 4| < \delta.$$

Now,

$$\begin{aligned} |-x^2 + 2x + 2 - (-6)| &= |(-1)(x^2 - 2x - 8)| \\ &= |(x + 2)(x - 4)| \\ &= |x + 2||x - 4|. \end{aligned} \tag{7}$$

In other words, we want to make $|x + 2||x - 4| < \varepsilon$. But since we have agreed to examine values of x *near* 4, let us consider only those values for which $|x - 4| < 1$. This last inequality gives $3 < x < 5$ or equivalently $5 < x + 2 < 7$. Consequently we can write $|x + 2| < 7$. Hence from (7),

$$0 < |x - 4| < 1 \qquad \text{implies} \qquad |-x^2 + 2x + 2 - (-6)| < 7|x - 4|.$$

If we now choose δ to be the minimum of the two numbers, 1 and $\varepsilon/7$, written $\delta = \min\{1, \varepsilon/7\}$ we have

$$0 < |x - 4| < \delta \qquad \text{implies} \qquad |-x^2 + 2x + 2 - (-6)| < 7|x - 4| < 7 \cdot \frac{\varepsilon}{7} = \varepsilon. \quad ■$$

The reasoning in Example 4 is subtle. Consequently it is worth a few minutes of your time to reread the discussion immediately following Definition 2.6.1, reexamine

Figure 2.3.2(b), and then think again about why $\delta = \min\{1, \varepsilon/7\}$ is the δ that "works" in the example. Remember, you can pick the ε arbitrarily; think about δ for, say, $\varepsilon = 8$, $\varepsilon = 6$, and $\varepsilon = 0.01$.

One-Sided Limits We state next the definitions of the **one-sided limits**, $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$.

Definition 2.6.2 Left-Hand Limit

Suppose a function f is defined on an open interval (c, a). Then

$$\lim_{x \to a^-} f(x) = L$$

means for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \qquad \text{whenever} \qquad a - \delta < x < a.$$

Definition 2.6.3 Right-Hand Limit

Suppose a function f is defined on an open interval (a, c). Then

$$\lim_{x \to a^+} f(x) = L$$

means for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \qquad \text{whenever} \qquad a < x < a + \delta.$$

EXAMPLE 5 Using Definition 2.6.3

Prove that $\lim_{x \to 0^+} \sqrt{x} = 0$.

Solution First, we can write

$$|\sqrt{x} - 0| = |\sqrt{x}| = \sqrt{x}.$$

Then, $|\sqrt{x} - 0| < \varepsilon$ whenever $0 < x < 0 + \varepsilon^2$. In other words, we choose $\delta = \varepsilon^2$.

Verification If $0 < x < \varepsilon^2$, then $0 < \sqrt{x} < \varepsilon$ implies

$$|\sqrt{x}| < \varepsilon \qquad \text{or} \qquad |\sqrt{x} - 0| < \varepsilon. \qquad \blacksquare$$

Limits Involving Infinity The two concepts of **infinite limits**

$$f(x) \to \infty \text{ (or } -\infty) \quad \text{as} \quad x \to a$$

and a **limit at infinity**

$$f(x) \to L \quad \text{as} \quad x \to \infty \text{ (or } -\infty)$$

are formalized in the next two definitions.

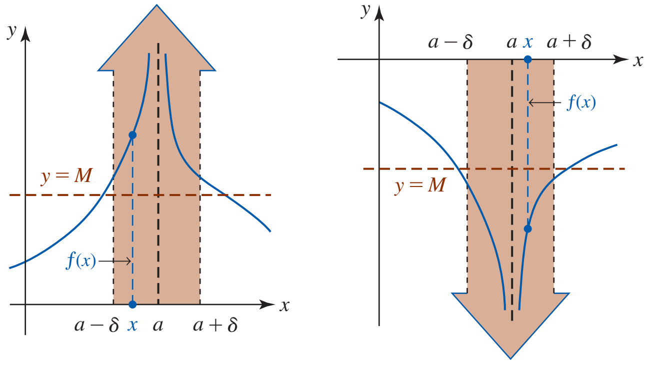
Recall, an infinite limit is a limit that does not exist as $x \to a$.

Definition 2.6.4 Infinite Limits

(*i*) $\lim_{x \to a} f(x) = \infty$ means for each $M > 0$, there exists a $\delta > 0$ such that $f(x) > M$ whenever $0 < |x - a| < \delta$.

(*ii*) $\lim_{x \to a} f(x) = -\infty$ means for each $M < 0$, there exists a $\delta > 0$ such that $f(x) < M$ whenever $0 < |x - a| < \delta$.

Parts (i) and (ii) of Definition 2.6.4 are illustrated in FIGURE 2.6.4(a) and Figure 2.6.4(b), respectively. Recall, if $f(x) \to \infty$ (or $-\infty$) as $x \to a$, then $x = a$ is a vertical asymptote for the graph of f. In the case when $f(x) \to \infty$ as $x \to a$, then $f(x)$ can be made larger than any arbitrary positive number (that is, $f(x) > M$) by taking x sufficiently close to a (that is, $0 < |x - a| < \delta$).



(a) For a given M, whenever $a - \delta < x < a + \delta$, $x \neq a$, then $f(x) > M$

(b) For a given M, whenever $a - \delta < x < a + \delta$, $x \neq a$, then $f(x) < M$

FIGURE 2.6.4 Infinite limits as $x \to a$

The four one-sided infinite limits

$$f(x) \to \infty \text{ as } x \to a^-, \qquad f(x) \to -\infty \text{ as } x \to a^-$$
$$f(x) \to \infty \text{ as } x \to a^+, \qquad f(x) \to -\infty \text{ as } x \to a^+$$

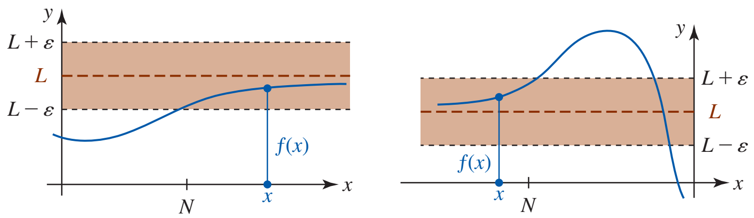
are defined in a manner analogous to that given in Definitions 2.6.2 and 2.6.3.

Definition 2.6.5 Limits at Infinity

(i) $\lim_{x \to \infty} f(x) = L$ if for each $\varepsilon > 0$, there exists an $N > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x > N$.

(ii) $\lim_{x \to -\infty} f(x) = L$ if for each $\varepsilon > 0$, there exists an $N < 0$ such that $|f(x) - L| < \varepsilon$ whenever $x < N$.

Parts (i) and (ii) of Definition 2.6.5 are illustrated in FIGURE 2.6.5(a) and Figure 2.6.5(b), respectively. Recall, if $f(x) \to L$ as $x \to \infty$ (or $-\infty$), then $y = L$ is a horizontal asymptote for the graph of f. In the case when $f(x) \to L$ as $x \to \infty$, then the graph of f can be made arbitrarily close to the line $y = L$ (that is, $|f(x) - L| < \varepsilon$) by taking x sufficiently far out on the positive x-axis (that is, $x > N$).



(a) For a given ε, $x > N$ implies $L - \varepsilon < f(x) < L + \varepsilon$

(b) For a given ε, $x < N$ implies $L - \varepsilon < f(x) < L + \varepsilon$

FIGURE 2.6.5 Limits at infinity

EXAMPLE 6 Using Definition 2.6.5(i)

Prove that $\lim_{x\to\infty} \dfrac{3x}{x+1} = 3$.

Solution By Definition 2.6.5(i), for any $\varepsilon > 0$, we must find a number $N > 0$ such that

$$\left|\frac{3x}{x+1} - 3\right| < \varepsilon \qquad \text{whenever} \qquad x > N.$$

Now, by considering $x > 0$, we have

$$\left|\frac{3x}{x+1} - 3\right| = \left|\frac{-3}{x+1}\right| = \frac{3}{x+1} < \frac{3}{x} < \varepsilon$$

whenever $x > 3/\varepsilon$. Hence, choose $N = 3/\varepsilon$. For example, if $\varepsilon = 0.01$, then $N = 3/(0.01) = 300$ will guarantee that $|f(x) - 3| < 0.01$ whenever $x > 300$. ■

Postscript—A Bit of History After this section you may agree with English philosopher, priest, historian, and scientist William Whewell (1794–1866), who wrote in 1858 that "A limit is a peculiar . . . conception." For many years after the invention of calculus in the seventeenth century, mathematicians argued and debated the nature of a limit. There was an awareness that intuition, graphs, and numerical examples of ratios of vanishing quantities provide at best a shaky foundation for such a fundamental concept. As you will see beginning in the next chapter, the limit concept plays a central role in calculus. The study of calculus went through several periods of increased mathematical rigor beginning with the French mathematician Augustin-Louis Cauchy and continuing later with the German mathematician Karl Wilhelm Weierstrass.


Cauchy

Augustin-Louis Cauchy (1789–1857) was born during an era of upheaval in French history. Cauchy was destined to initiate a revolution of his own in mathematics. For many contributions, but especially for his efforts in clarifying mathematical obscurities, his incessant demand for satisfactory definitions and rigorous proofs of theorems, Cauchy is often called "the father of modern analysis." A prolific writer whose output has been surpassed by only a few, Cauchy produced nearly 800 papers in astronomy, physics, and mathematics. But the same mind that was always open and inquiring in science and mathematics was also narrow and unquestioning in many other areas. Outspoken and arrogant, Cauchy's passionate stands on political and religious issues often alienated him from his colleagues.


Weierstrass

Karl Wilhelm Weierstrass (1815–1897) One of the foremost mathematical analysts of the nineteenth century never earned an academic degree! After majoring in law at the University of Bonn, but concentrating in fencing and beer drinking for four years, Weierstrass "graduated" to real life with no degree. In need of a job, Weierstrass passed a state examination and received a teaching certificate in 1841. During a period of 15 years as a secondary school teacher, his dormant mathematical genius blossomed. Although the quantity of his research publications was modest, especially when compared with that of Cauchy, the quality of these works so impressed the German mathematical community that he was awarded a doctorate, *honoris causa*, from the University of Königsberg and eventually was appointed a professor at the University of Berlin. While there, Weierstrass achieved worldwide recognition both as a mathematician and as a teacher of mathematics. One of his students was Sonja Kowalewski, the greatest female mathematician of the nineteenth century. It was Karl Wilhelm Weierstrass who was responsible for putting the concept of a limit on a firm foundation with the ε-δ definition.

Exercises 2.6

Answers to selected odd-numbered problems begin on page ANS-000.

Fundamentals

In Problems 1–24, use Definitions 2.6.1, 2.6.2, or 2.6.3 to prove the given limit result.

1. $\lim_{x\to 5} 10 = 10$

2. $\lim_{x\to -2} \pi = \pi$

3. $\lim_{x\to 3} x = 3$

4. $\lim_{x\to 4} 2x = 8$

5. $\lim_{x\to -1} (x+6) = 5$

6. $\lim_{x\to 0} (x-4) = -4$

7. $\lim_{x\to 0} (3x+7) = 7$

8. $\lim_{x\to 1} (9-6x) = 3$

9. $\lim_{x\to 2} \dfrac{2x-3}{4} = \dfrac{1}{4}$

10. $\lim_{x\to 1/2} 8(2x+5) = 48$

11. $\lim_{x\to -5} \dfrac{x^2-25}{x+5} = -10$

12. $\lim_{x\to 3} \dfrac{x^2-7x+12}{2x-6} = -\dfrac{1}{2}$

13. $\lim_{x\to 0} \dfrac{8x^5+12x^4}{x^4} = 12$

14. $\lim_{x\to 1} \dfrac{2x^3+5x^2-2x-5}{x^2-1} = 7$

15. $\lim_{x\to 0} x^2 = 0$

16. $\lim_{x\to 0} 8x^3 = 0$

17. $\lim_{x\to 0^+} \sqrt{5x} = 0$

18. $\lim_{x\to (1/2)^+} \sqrt{2x-1} = 0$

19. $\lim_{x\to 0^-} f(x) = -1, \quad f(x) = \begin{cases} 2x-1, & x<0 \\ 2x+1, & x>0 \end{cases}$

20. $\lim_{x\to 1^+} f(x) = 3, \quad f(x) = \begin{cases} 0, & x\le 1 \\ 3, & x>1 \end{cases}$

21. $\lim_{x\to 3} x^2 = 9$

22. $\lim_{x\to 2} (2x^2+4) = 12$

23. $\lim_{x\to 1} (x^2-2x+4) = 3$

24. $\lim_{x\to 5} (x^2+2x) = 35$

25. For $a > 0$, use the identity.

$$|\sqrt{x}-\sqrt{a}| = |\sqrt{x}-\sqrt{a}|\cdot\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}} = \frac{|x-a|}{\sqrt{x}+\sqrt{a}}$$

and the fact that $\sqrt{x}\ge 0$ to prove that $\lim_{x\to a}\sqrt{x} = \sqrt{a}$.

26. Prove that $\lim_{x\to 2}(1/x) = \frac{1}{2}$. [*Hint*: Consider only those numbers x for which $1 < x < 3$.]

In Problems 27–30, prove that $\lim_{x\to a} f(x)$ does not exist.

27. $f(x) = \begin{cases} 2, & x<1 \\ 0, & x\ge 1 \end{cases}; \quad a = 1$

28. $f(x) = \begin{cases} 1, & x\le 3 \\ -1, & x>3 \end{cases}; \quad a = 3$

29. $f(x) = \begin{cases} x, & x\le 0 \\ 2-x, & x>0 \end{cases}; \quad a = 0$

30. $f(x) = \dfrac{1}{x}; \quad a = 0$

In Problems 31–34, use Definition 2.6.5 to prove the given limit result.

31. $\lim_{x\to\infty} \dfrac{5x-1}{2x+1} = \dfrac{5}{2}$

32. $\lim_{x\to\infty} \dfrac{2x}{3x+8} = \dfrac{2}{3}$

33. $\lim_{x\to -\infty} \dfrac{10x}{x-3} = 10$

34. $\lim_{x\to -\infty} \dfrac{x^2}{x^2+3} = 1$

Think About It

35. Prove that $\lim_{x\to 0} f(x) = 0$, where $f(x) = \begin{cases} x, & x \text{ rational} \\ 0, & x \text{ irrational}. \end{cases}$

2.7 The Tangent Line Problem

Introduction In a calculus course you will study many different things, but as mentioned in the introduction to Section 2.1, the subject "calculus" is roughly divided into two broad but related areas known as **differential calculus** and **integral calculus**. The discussion of each of these topics invariably begins with a motivating problem involving the graph of a function. Differential calculus is motivated by the problem

- *Find a tangent line to the graph of a function f,*

whereas integral calculus is motivated by the problem

- *Find the area under the graph of a function f.*

The first problem will be addressed in this section; the second problem will be discussed in Section 5.3.

Tangent Line to a Graph The word *tangent* stems from the Latin verb *tangere*, meaning "to touch." You might remember from the study of plane geometry that a tangent to a circle is a line L that intersects, or touches, the circle in exactly one point P. See FIGURE 2.7.1. It is not quite as easy to define a tangent line to the graph of a function f. The idea of *touching* carries over to the notion of a tangent line to the graph of a function, but the idea of *intersecting the graph in one point* does not carry over.

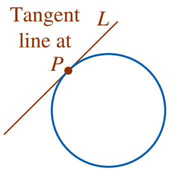


FIGURE 2.7.1 Tangent line L touches a circle at point P

Suppose $y = f(x)$ is a continuous function. If, as shown in FIGURE 2.7.2, f possesses a line L tangent to its graph at a point P, then what is an equation of this line? To answer this question, we need the coordinates of P and the slope $m_{\tan}$ of L. The coordinates of P pose no difficulty, since a point on the graph of a function f is obtained by specifying a value of x in the domain of f. The coordinates of the point of tangency at $x = a$ are then $(a, f(a))$. Therefore, the problem of finding a tangent line comes down to the problem of finding the slope $m_{\tan}$ of the line. As a means of *approximating* $m_{\tan}$, we can readily find the slopes $m_{\sec}$ of *secant lines* (from the Latin verb *secare*, meaning "to cut") that pass through the point P and any other point Q on the graph. See FIGURE 2.7.3.

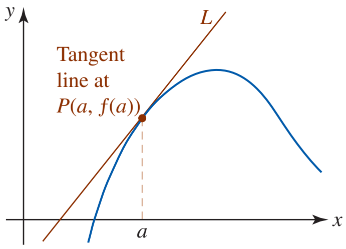


FIGURE 2.7.2 Tangent line L to a graph at point P

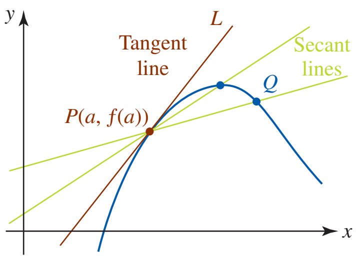


FIGURE 2.7.3 Slopes of secant lines approximate the slope $m_{\tan}$ of L

Slope of Secant Lines If P has coordinates $(a, f(a))$ and if Q has coordinates $(a + h, f(a + h))$, then as shown in FIGURE 2.7.4, the slope of the secant line through P and Q is

$$m_{\sec} = \frac{\text{change in } y}{\text{change in } x} = \frac{f(a + h) - f(a)}{(a + h) - a}$$

or
$$m_{\sec} = \frac{f(a + h) - f(a)}{h}. \tag{1}$$

The expression on the right-hand side of the equality in (1) is called a **difference quotient**. When we let h take on values that are closer and closer to zero, that is, as $h \to 0$, then the points $Q(a + h, f(a + h))$ move along the curve closer and closer to the point $P(a, f(a))$. Intuitively, we expect the secant lines to approach the tangent line L, and that $m_{\sec} \to m_{\tan}$ as $h \to 0$. That is,

$$m_{\tan} = \lim_{h \to 0} m_{\sec}$$

provided this limit exists. We summarize this conclusion in an equivalent form of the limit using the difference quotient (1).

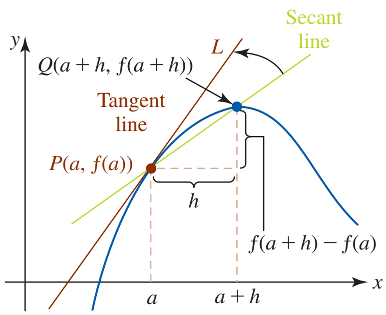


FIGURE 2.7.4 Secant lines swing into the tangent line L as $h \to 0$

Definition 2.7.1 Tangent Line with Slope

Let $y = f(x)$ be continuous at the number a. If the limit

$$m_{\tan} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \tag{2}$$

exists, then the **tangent line** to the graph of f at $(a, f(a))$ is that line passing through the point $(a, f(a))$ with slope $m_{\tan}$.

Just like many of the problems discussed earlier in this chapter, observe that the limit in (2) has the indeterminate form $0/0$ as $h \to 0$.

If the limit in (2) exists, the number $m_{\tan}$ is also called the **slope of the curve** $y = f(x)$ at $(a, f(a))$.

The computation of (2) is essentially a *four-step process*; three of these steps involve only precalculus mathematics: algebra and trigonometry. If the first three steps are done accurately, the fourth step, or the calculus step, *may* be the easiest part of the problem.

Guidelines for Computing (2)

(*i*) Evaluate $f(a)$ and $f(a + h)$.
(*ii*) Evaluate the difference $f(a + h) - f(a)$. Simplify.
(*iii*) Simplify the difference quotient

$$\frac{f(a + h) - f(a)}{h}.$$

(*iv*) Compute the limit of the difference quotient

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

The computation of the difference $f(a+h) - f(a)$ in step (*ii*) is in most instances the most important step. It is imperative that you simplify this step as much as possible. Here is a tip: In *many* problems involving the computation of (2) you will be able to factor h from the difference $f(a+h) - f(a)$.

Note ▶

EXAMPLE 1 The Four-Step Process

Find the slope of the tangent line to the graph of $y = x^2 + 2$ at $x = 1$.

Solution We use the four-step procedure outlined above with the number 1 playing the part of the symbol a.

(*i*) The initial step is the computation of $f(1)$ and $f(1+h)$. We have $f(1) = 1^2 + 2 = 3$, and

$$\begin{aligned} f(1+h) &= (1+h)^2 + 2 \\ &= (1 + 2h + h^2) + 2 \\ &= 3 + 2h + h^2. \end{aligned}$$

(*ii*) Next, from the result in the preceding step the difference is:

$$\begin{aligned} f(1+h) - f(1) &= 3 + 2h + h^2 - 3 \\ &= 2h + h^2 \\ &= h(2+h). \leftarrow \text{notice the factor of } h \end{aligned}$$

(*iii*) The computation of the difference quotient $\dfrac{f(1+h) - f(1)}{h}$ is now straightforward. Again, we use the results from the preceding step:

$$\frac{f(1+h) - f(1)}{h} = \frac{h(2+h)}{h} = 2 + h. \leftarrow \text{cancel the } h\text{'s}$$

(*iv*) The last step is now easy. The limit in (2) is seen to be

$$m_{\tan} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \overset{\text{from the preceding step}}{=} \lim_{h \to 0} (2 + h) = 2.$$

The slope of the tangent line to the graph of $y = x^2 + 2$ at $(1, 3)$ is 2. ■

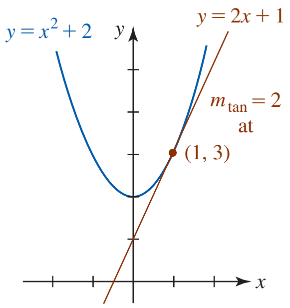


FIGURE 2.7.5 Tangent line in Example 2

EXAMPLE 2 Equation of Tangent Line

Find an equation of the tangent line whose slope was found in Example 1.

Solution We know the point of tangency $(1, 3)$ and the slope $m_{\tan} = 2$, and so from the point–slope equation of a line we find

$$y - 3 = 2(x - 1) \quad \text{or} \quad y = 2x + 1.$$

Observe that the last equation is consistent with the x- and y-intercepts of the red line in FIGURE 2.7.5. ■

EXAMPLE 3 Equation of Tangent Line

Find an equation of the tangent line to the graph of $f(x) = 2/x$ at $x = 2$.

Solution We start by using (2) to find $m_{\tan}$ with a identified as 2. In the second of the four steps, we will have to combine two symbolic fractions by means of a common denominator.

(*i*) We have $f(2) = 2/2 = 1$ and $f(2+h) = 2/(2+h)$.

(*ii*)
$$\begin{aligned} f(2+h) - f(2) &= \frac{2}{2+h} - 1 \\ &= \frac{2}{2+h} - \frac{1}{1} \cdot \frac{2+h}{2+h} \leftarrow \text{a common denominator is } 2+h \\ &= \frac{2 - 2 - h}{2+h} \\ &= \frac{-h}{2+h}. \leftarrow \text{here is the factor of } h \end{aligned}$$

(*iii*) The last result is to be divided by h or more precisely $\frac{h}{1}$. We invert and multiply by $\frac{1}{h}$:

$$\frac{f(2+h)-f(2)}{h} = \frac{\dfrac{-h}{2+h}}{\dfrac{h}{1}} = \frac{-h}{2+h}\cdot\frac{1}{h} = \frac{-1}{2+h}. \quad \leftarrow \text{cancel the } h\text{'s}$$

(*iv*) From (2) $m_{\tan}$ is

$$m_{\tan} = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\frac{-1}{2+h} = -\frac{1}{2}.$$

From $f(2) = 1$ the point of tangency is (2, 1) and the slope of the tangent line at (2, 1) is $m_{\tan} = -\frac{1}{2}$. From the point–slope equation of a line, the tangent line is

$$y - 1 = \frac{1}{2}(x-2) \qquad \text{or} \qquad y = -\frac{1}{2}x + 2.$$

The graphs of $y = 2/x$ and the tangent line at (2, 1) are shown in FIGURE 2.7.6. ■

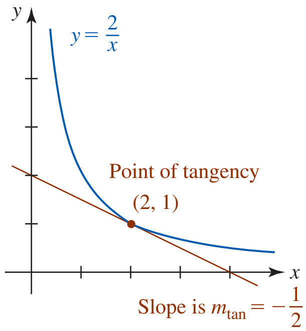


FIGURE 2.7.6 Tangent line in Example 3

EXAMPLE 4 Slope of Tangent Line

Find the slope of the tangent line to the graph of $f(x) = \sqrt{x-1}$ at $x = 5$.

Solution Replacing a by 5 in (2), we have:

(*i*) $f(5) = \sqrt{5-1} = \sqrt{4} = 2$, and

$$f(5+h) = \sqrt{5+h-1} = \sqrt{4+h}.$$

(*ii*) The difference is

$$f(5+h) - f(5) = \sqrt{4+h} - 2.$$

Because we expect to find a factor of h in this difference, we proceed to rationalize the numerator:

$$\begin{aligned} f(5+h) - f(5) &= \frac{\sqrt{4+h}-2}{1}\cdot\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2} \\ &= \frac{(4+h)-4}{\sqrt{4+h}+2} \\ &= \frac{h}{\sqrt{4+h}+2}. \quad \leftarrow \text{here is the factor of } h \end{aligned}$$

(*iii*) The difference quotient $\dfrac{f(5+h)-f(5)}{h}$ is then:

$$\begin{aligned} \frac{f(5+h)-f(5)}{h} &= \frac{\dfrac{h}{\sqrt{4+h}+2}}{h} \\ &= \frac{h}{h(\sqrt{4+h}+2)} \\ &= \frac{1}{\sqrt{4+h}+2}. \end{aligned}$$

(*iv*) The limit in (2) is

$$m_{\tan} = \lim_{h\to 0}\frac{f(5+h)-f(5)}{h} = \lim_{h\to 0}\frac{1}{\sqrt{4+h}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{4}.$$

The slope of the tangent line to the graph of $f(x) = \sqrt{x-1}$ at (5, 2) is $\frac{1}{4}$. ■

The result obtained in the next example should come as no surprise.

EXAMPLE 5 Tangent Line to a Line

For any linear function $y = mx + b$, the tangent line to its graph coincides with the line itself. Not unexpectedly then, the slope of the tangent line for any number $x = a$ is

$$m_{\tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{m(a+h) + b - (ma+b)}{h} = \lim_{h \to 0} \frac{mh}{h} = \lim_{h \to 0} m = m. \quad ■$$

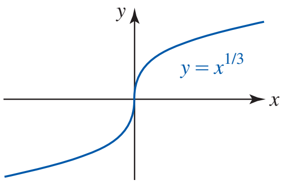


FIGURE 2.7.7 Vertical tangent in Example 6

Vertical Tangents The limit in (2) can fail to exist for a function f at $x = a$ and yet there may be a tangent at the point $(a, f(a))$. The tangent line to a graph can be **vertical**, in which case its slope is undefined. We will consider the concept of vertical tangents in more detail in Section 3.1.

EXAMPLE 6 Vertical Tangent Line

Although we will not pursue the details at this time, it can be shown that the graph of $f(x) = x^{1/3}$ possesses a vertical tangent line at the origin. In FIGURE 2.7.7 we see that the y-axis, that is, the line $x = 0$, is tangent to the graph at the point $(0, 0)$. ■

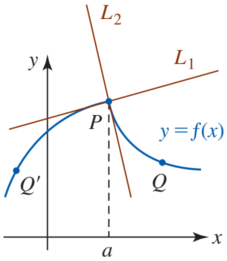


FIGURE 2.7.8 Tangent fails to exist at $(a, f(a))$

A Tangent May Not Exist The graph of a function f that is continuous at a number a does not have to possess a tangent line at the point $(a, f(a))$. A tangent line will not exist whenever the graph of f has a sharp corner at $(a, f(a))$. FIGURE 2.7.8 indicates what can go wrong when the graph of a function f has a "corner." In this case f is continuous at a, but the secant lines through P and Q approach L_2 as $Q \to P$, and the secant lines through P and Q' approach a different line L_1 as $Q' \to P$. In other words, the limit in (2) fails to exist because the one-sided limits of the difference quotient (as $h \to 0^+$ and as $h \to 0^-$) are different.

EXAMPLE 7 Graph with a Corner

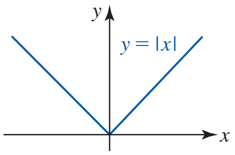


FIGURE 2.7.9 Function in Example 7

Show that the graph of $f(x) = |x|$ does not have a tangent at $(0, 0)$.

Solution The graph of the absolute-value function in FIGURE 2.7.9 has a corner at the origin. To prove that the graph of f does not possess a tangent line at the origin we must examine

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|0+h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h}.$$

From the definition of absolute value

$$|h| = \begin{cases} h, & h > 0 \\ -h, & h < 0 \end{cases}$$

we see that

$$\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1 \qquad \text{whereas} \qquad \lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1.$$

Since the right-hand and left-hand limits are not equal we conclude that the limit (2) does not exist. Even though the function $f(x) = |x|$ is continuous at $x = 0$, the graph of f possesses no tangent at $(0, 0)$. ■

Average Rate of Change In different contexts the difference quotient in (1) and (2), or slope of the secant line, is written in terms of alternative symbols. The symbol h in (1) and (2) is often written as Δx and the difference $f(a + \Delta x) - f(a)$ is denoted by Δy, that is, the difference quotient is

$$\frac{\text{change in } y}{\text{change in } x} = \frac{f(a+\Delta x) - f(a)}{(a+\Delta x) - a} = \frac{f(a+\Delta x) - f(a)}{\Delta x} = \frac{\Delta y}{\Delta x}. \tag{3}$$

Moreover, if $x_1 = a + \Delta x$, $x_0 = a$, then $\Delta x = x_1 - x_0$ and (3) is the same as

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{\Delta y}{\Delta x}. \tag{4}$$

The slope $\Delta y/\Delta x$ of the secant line through the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ is called the **average rate of change of the function** f over the interval $[x_0, x_1]$. The limit $\lim_{\Delta x \to 0} \Delta y/\Delta x$ is then called the **instantaneous rate of change of the function** with respect to x at x_0.

Almost everyone has an intuitive notion of speed as a rate at which a distance is covered in a certain length of time. When, say, a bus travels 60 mi in 1 h, the *average speed* of the bus must have been 60 mi/h. Of course, it is difficult to maintain the rate of 60 mi/h for the entire trip because the bus slows down for towns and speeds up when it passes cars. In other words, the speed changes with time. If a bus company's schedule demands that the bus travel the 60 mi from one town to another in 1 h, the driver knows instinctively that he or she must compensate for speeds less than 60 mi/h by traveling at speeds greater than this at other points in the journey. Knowing that the average velocity is 60 mi/h does not, however, answer the question: What is the velocity of the bus at a particular instant?

Average Velocity In general, the **average velocity** or **average speed** of a moving object is defined by

$$v_{\text{ave}} = \frac{\text{change of distance}}{\text{change in time}}. \tag{5}$$

Consider a runner who finishes a 10-km race in an elapsed time of 1 h 15 min (1.25 h). The runner's average velocity or average speed for the race was

$$v_{\text{ave}} = \frac{10 - 0}{1.25 - 0} = 8 \text{ km/h}.$$

But suppose we now wish to determine the runner's *exact* velocity v at the instant the runner is one-half hour into the race. If the distance run in the time interval from 0 h to 0.5 h is measured to be 5 km, then

$$v_{\text{ave}} = \frac{5}{0.5} = 10 \text{ km/h}.$$

Again, this number is not a measure, or necessarily even a good indicator, of the instantaneous rate v at which the runner is moving 0.5 h into the race. If we determine that at 0.6 h the runner is 5.7 km from the starting line, then the average velocity from 0 h to 0.6 h is $v_{\text{ave}} = 5.7/0.6 = 9.5$ km/h. However, during the time interval from 0.5 h to 0.6 h,

$$v_{\text{ave}} = \frac{5.7 - 5}{0.6 - 0.5} = 7 \text{ km/h}.$$

The latter number is a more realistic measure of the rate v. See FIGURE 2.7.10. By "shrinking" the time interval between 0.5 h and the time that corresponds to a measured position close to 5 km, we expect to obtain even better approximations to the runner's velocity at time 0.5 h.

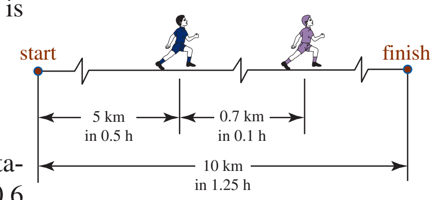


FIGURE 2.7.10 Runner in a 10-km race

Rectilinear Motion To generalize the preceding discussion, let us suppose an object, or particle, at point P moves along either a vertical or horizontal coordinate line as shown in FIGURE 2.7.11. Furthermore, let the particle move in such a manner that its position, or coordinate, on the line is given by a function $s = s(t)$, where t represents time. The values of s are directed distances measured from O in units such as centimeters, meters, feet, or miles. When P is either to the right of or above O, we take $s > 0$, whereas $s < 0$ when P is either to the left of or below O. Motion in a straight line is called **rectilinear motion**.

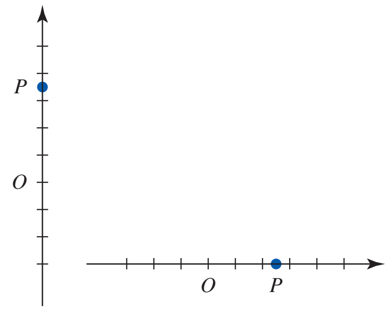


FIGURE 2.7.11 Coordinate lines

If an object, such as a toy car moving on a horizontal coordinate line, is at point P at time t_0 and at point P' at time t_1, then the coordinates of the points, shown in FIGURE 2.7.12, are $s(t_0)$ and $s(t_1)$. By (4) the **average velocity** of the object in the time interval $[t_0, t_1]$ is

$$v_{\text{ave}} = \frac{\text{change in position}}{\text{change in time}} = \frac{s(t_1) - s(t_0)}{t_1 - t_0}. \tag{6}$$

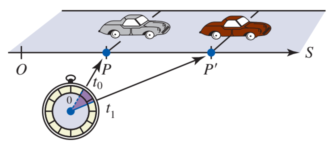


FIGURE 2.7.12 Position of toy car on a coordinate line at two times

EXAMPLE 8 Average Velocity

The height s above ground of a ball dropped from the top of the St. Louis Gateway Arch is given by $s(t) = -16t^2 + 630$, where s is measured in feet and t in seconds. See FIGURE 2.7.13. Find the average velocity of the falling ball between the time the ball is released and the time it hits the ground.

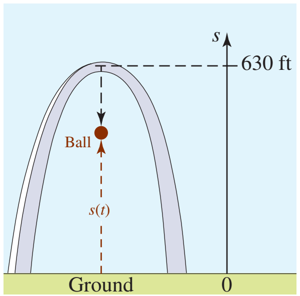


FIGURE 2.7.13 Falling ball in Example 8

Solution The time at which the ball is released is determined from the equation $s(t) = 630$ or $-16t^2 + 630 = 630$. This gives $t = 0$ s. When the ball hits the ground then $s(t) = 0$ or

$-16t^2 + 630 = 0$. The last equation gives $t = \sqrt{315/8} \approx 6.27$ s. Thus from (6) the average velocity in the time interval $\left[0, \sqrt{315/8}\right]$ is

$$v_{\text{ave}} = \frac{s\left(\sqrt{351/8}\right) - s(0)}{\sqrt{351/8} - 0} = \frac{0 - 630}{\sqrt{351/8} - 0} \approx -100.40 \text{ ft/s.} \qquad \blacksquare$$

If we let $t_1 = t_0 + \Delta t$, or $\Delta t = t_1 - t_0$, and $\Delta s = s(t_0 + \Delta t) - s(t_0)$, then (6) is equivalent to

$$v_{\text{ave}} = \frac{\Delta s}{\Delta t}. \tag{7}$$

This suggests that the limit of (7) as $\Delta t \to 0$ gives the **instantaneous rate of change** of $s(t)$ at $t = t_0$ or the **instantaneous velocity**.

Definition 2.7.2 Instantaneous Velocity

Let $s = s(t)$ be a function that gives the position of an object moving in a straight line. Then the **instantaneous velocity** at time $t = t_0$ is

$$v(t_0) = \lim_{\Delta t \to 0} \frac{s(t_0 + \Delta t) - s(t_0)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}, \tag{8}$$

whenever the limit exists.

Note: Except for notation and interpretation, there is no mathematical difference between (2) and (8). Also, the word *instantaneous* is often dropped, and so one often speaks of the *rate of change* of a function or the *velocity* of a moving particle.

EXAMPLE 9 Example 8 Revisited

Find the instantaneous velocity of the falling ball in Example 8 at $t = 3$ s.

Solution We use the same four-step procedure as in the earlier examples with $s = s(t)$ given in Example 8.

(*i*) $s(3) = -16(9) + 630 = 486$. For any $\Delta t \neq 0$,

$$s(3 + \Delta t) = -16(3 + \Delta t)^2 + 630 = -16(\Delta t)^2 - 96\Delta t + 486.$$

(*ii*) $$s(3 + \Delta t) - s(3) = [-16(\Delta t)^2 - 96\Delta t + 486] - 486 = -16(\Delta t)^2 - 96\Delta t = \Delta t(-16\Delta t - 96)$$

(*iii*) $$\frac{\Delta s}{\Delta t} = \frac{\Delta t(-16\Delta t - 96)}{\Delta t} = -16\Delta t - 96$$

(*iv*) From (8),

$$v(3) = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0}(-16\Delta t - 96) = -96 \text{ ft/s.} \tag{9}$$ $\blacksquare$

In Example 9, the number $s(3) = 486$ ft is the height of the ball above ground at 3 s. The minus sign in (9) is significant because the ball is moving opposite to the positive or upward direction.

Exercises 2.7

Answers to selected odd-numbered problems begin on page ANS-000.

Fundamentals

In Problems 1–6, sketch the graph of the function and the tangent line at the given point. Find the slope of the secant line through the points that correspond to the indicated values of x.

1. $f(x) = -x^2 + 9$, $(2, 5)$; $x = 2$, $x = 2.5$

2. $f(x) = x^2 + 4x$, $(0, 0)$; $x = -\frac{1}{4}$, $x = 0$

3. $f(x) = x^3$, $(-2, -8)$; $x = -2$, $x = -1$

4. $f(x) = 1/x$, $(1, 1)$; $x = 0.9$, $x = 1$

5. $f(x) = \sin x$, $(\pi/2, 1)$; $x = \pi/2$, $x = 2\pi/3$

6. $f(x) = \cos x$, $\left(-\pi/3, \frac{1}{2}\right)$; $x = -\pi/2$, $x = -\pi/3$

In Problems 7–8, use (2) to find the slope of the tangent line to the graph of the function at the given value of x. Find an equation of the tangent line at the corresponding point.

7. $f(x) = x^2 - 6, x = 3$

8. $f(x) = -3x^2 + 10, x = -1$

9. $f(x) = x^2 - 3x, x = 1$

10. $f(x) = -x^2 + 5x - 3, x = -2$

11. $f(x) = -2x^3 + x, x = 2$

12. $f(x) = 8x^3 - 4, x = \frac{1}{2}$

13. $f(x) = \frac{1}{2x}, x = -1$

14. $f(x) = \frac{4}{x - 1}, x = 2$

15. $f(x) = \frac{1}{(x - 1)^2}, x = 0$

16. $f(x) = 4 - \frac{8}{x}, x = -1$

17. $f(x) = \sqrt{x}, x = 4$

18. $f(x) = \frac{1}{\sqrt{x}}, x = 1$

In Problems 19 and 20, use (2) to find the slope of the tangent line to the graph of the function at the given value of x. Find an equation of the tangent line at the corresponding point. Before starting, review the limits in (10) and (14) of Section 2.4 and the sum formulas (17) and (18) in Section 1.4.

19. $f(x) = \sin x, x = \pi/6$

20. $f(x) = \cos x, x = \pi/4$

In Problems 21 and 22, determine whether the line that passes through the red point is tangent to the graph of $f(x) = x^2$ at the blue point.

21.

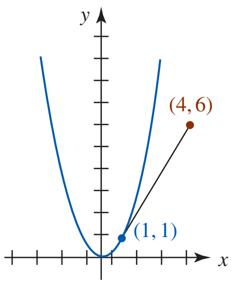


FIGURE 2.7.14 Graph for Problem 21

22.

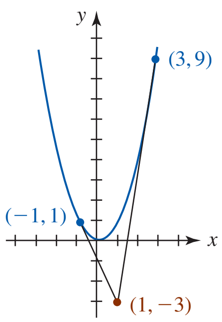


FIGURE 2.7.15 Graph for Problem 22

23. In FIGURE 2.7.16, the red line is tangent to the graph of $y = f(x)$ at the indicated point. Find an equation of the tangent line. What is the y-intercept of the tangent line?

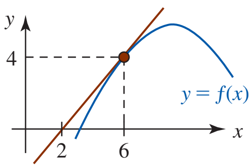


FIGURE 2.7.16 Graph for Problem 23

24. In FIGURE 2.7.17, the red line is tangent to the graph of $y = f(x)$ at the indicated point. Find $f(-5)$.

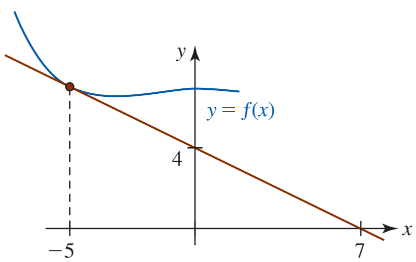


FIGURE 2.7.17 Graph for Problem 24

In Problems 25–28, use (2) to find a formula for $m_{\tan}$ at a general point $(x, f(x))$ on the graph of f. Use the formula for $m_{\tan}$ to determine the points where the tangent line to the graph is horizontal.

25. $f(x) = -x^2 + 6x + 1$

26. $f(x) = 2x^2 + 24x - 22$

27. $f(x) = x^3 - 3x$

28. $f(x) = -x^3 + x^2$

Applications

29. A car travels the 290 mi between Los Angeles and Las Vegas in 5 h. What is its average velocity?

30. Two marks on a straight highway are $\frac{1}{2}$ mi apart. A highway patrol plane observes that a car traverses the distance between the marks in 40 s. Assuming a speed limit of 60 mi/h, will the car be stopped for speeding?

31. A jet airplane averages 920 km/h to fly the 3500 km between Hawaii and San Francisco. How many hours does the flight take?

32. A marathon race is run over a straight 26-mi course. The race begins at noon. At 1:30 P.M. a contestant passes the 10-mi mark and at 3:10 P.M. the contestant passes the 20-mi mark. What is the contestant's average running speed between 1:30 P.M. and 3:10 P.M.?

In Problems 33 and 34, the position of a particle moving on a horizontal coordinate line is given by the function. Use (8) to find the instantaneous velocity of the particle at the indicated time.

33. $s(t) = -4t^2 + 10t + 6, t = 3$

34. $s(t) = t^2 + \frac{1}{5t + 1}, t = 0$

35. The height above ground of a ball dropped from an initial altitude of 122.5 m is given by $s(t) = -4.9t^2 + 122.5$, where s is measured in meters and t in seconds.

(a) What is the instantaneous velocity at $t = \frac{1}{2}$?
(b) At what time does the ball hit the ground?
(c) What is the impact velocity?

36. Ignoring air resistance, if an object is dropped from an initial height h, then its height above ground at time $t > 0$ is given by $s(t) = -\frac{1}{2}gt^2 + h$, where g is the acceleration of gravity.

(a) At what time does the object hit the ground?
(b) If $h = 100$ ft, compare the impact times for Earth ($g = 32$ ft/s^2), for Mars ($g = 12$ ft/s^2), and for the Moon ($g = 5.5$ ft/s^2).
(c) Use (8) to find a formula for the instantaneous velocity v at a general time t.
(d) Using the times found in part (b) and the formula found in part (c), find the corresponding impact velocities for Earth, Mars, and the Moon.

37. The height of a projectile shot from ground level is given by $s = -16t^2 + 256t$, where s is measured in feet and t in seconds.

(a) Determine the height of the projectile at $t = 2$, $t = 6$, $t = 9$, and $t = 10$.
(b) What is the average velocity of the projectile between $t = 2$ and $t = 5$?
(c) Show that the average velocity between $t = 7$ and $t = 9$ is zero. Interpret physically.
(d) At what time does the projectile hit the ground?

(e) Use (8) to find a formula for instantaneous velocity v at a general time t.

(f) Using the time found in part (d) and the formula found in part (e), find the corresponding impact velocity.

(g) What is the maximum height the projectile attains?

38. Suppose the graph shown in FIGURE 2.7.18 is that of position function $s = s(t)$ of a particle moving in a straight line, where s is measured in meters and t in seconds.

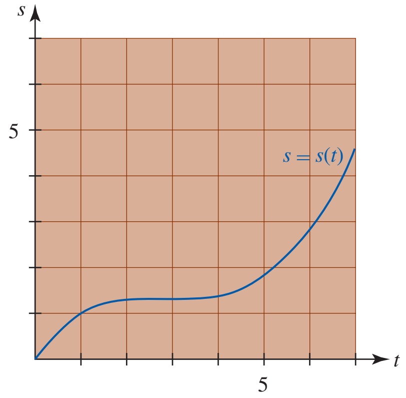


FIGURE 2.7.18 Graph for Problem 38

(a) Estimate the position of the particle at $t = 4$ and at $t = 6$.

(b) Estimate the average velocity of the particle between $t = 4$ and $t = 6$.

(c) Estimate the initial velocity of the particle—that is, its velocity at $t = 0$.

(d) Estimate a time at which the velocity of the particle is zero.

(e) Determine an interval on which the velocity of the particle is decreasing.

(f) Determine an interval on which the velocity of the particle is increasing.

☰ Think About It

39. Let $y = f(x)$ be an even function whose graph possesses a tangent line with slope m at $(a, f(a))$. Show that the slope of the tangent line at $(-a, f(a))$ is $-m$. [*Hint*: Explain why $f(-a + h) = f(a - h)$.]

40. Let $y = f(x)$ be an odd function whose graph possesses a tangent line with slope m at $(a, f(a))$. Show that the slope of the tangent line at $(-a, -f(a))$ is m.

41. Proceed as in Example 7 and show that there is no tangent line to graph of $f(x) = x^2 + |x|$ at $(0, 0)$.

Chapter 2 in Review

Answers to selected odd-numbered problems begin on page ANS-000.

A. True/False

In Problems 1–22, indicate whether the given statement is true or false.

1. $\lim_{x\to 2} \dfrac{x^3 - 8}{x - 2} = 12$ _____

2. $\lim_{x\to 5} \sqrt{x - 5} = 0$ _____

3. $\lim_{x\to 0} \dfrac{|x|}{x} = 1$ _____

4. $\lim_{x\to\infty} e^{2x - x^2} = \infty$ _____

5. $\lim_{x\to 0^+} \tan^{-1}\left(\dfrac{1}{x}\right)$ does not exist. _____

6. $\lim_{z\to 1} \dfrac{z^3 + 8z - 2}{z^2 + 9z - 10}$ does not exist. _____

7. If $\lim_{x\to a} f(x) = 3$ and $\lim_{x\to a} g(x) = 0$, then $\lim_{x\to a} f(x)/g(x)$ does not exist. _____

8. If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} g(x)$ does not exist, then $\lim_{x\to a} f(x)g(x)$ does not exist. _____

9. If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a} f(x)/g(x) = 1$. _____

10. If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a} [f(x) - g(x)] = 0$. _____

11. If f is a polynomial function, then $\lim_{x\to\infty} f(x) = \infty$. _____

12. Every polynomial function is continuous on $(-\infty, \infty)$. _____

13. For $f(x) = x^5 + 3x - 1$ there exists a number c in $[-1, 1]$ such that $f(c) = 0$. _____

14. If f and g are continuous at the number 2, then f/g is continuous at 2. _____

15. The greatest integer function $f(x) = \lfloor x \rfloor$ is not continuous on the interval $[0, 1]$. _____

16. If $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist, then $\lim_{x\to a} f(x)$ exists. _____

17. If a function f is discontinuous at the number 3, then $f(3)$ is not defined. _____

18. If a function f is continuous at the number a, then $\lim_{x\to a} (x - a)f(x) = 0$. _____

19. If f is continuous and $f(a)f(b) < 0$, there is a root of $f(x) = 0$ in the interval $[a, b]$. _____

20. The function $f(x) = \begin{cases} \dfrac{x^2 - 6x + 5}{x - 5}, & x \neq 5 \\ 4, & x = 5 \end{cases}$ is discontinuous at 5. _____

21. The function $f(x) = \dfrac{\sqrt{x}}{x + 1}$ has a vertical asymptote at $x = -1$. _____

22. If $y = x - 2$ is a tangent line to the graph of a function $y = f(x)$ at $(3, f(3))$, then $f(3) = 1$. _____

B. Fill in the Blanks

In Problems 1–22, fill in the blanks.

1. $\lim_{x \to 2}(3x^2 - 4x) =$ _____

2. $\lim_{x \to 3}(5x^2)^0 =$ _____

3. $\lim_{t \to \infty}\dfrac{2t - 1}{3 - 10t} =$ _____

4. $\lim_{x \to -\infty}\dfrac{\sqrt{x^2 + 1}}{2x + 1} =$ _____

5. $\lim_{t \to 1}\dfrac{1 - \cos^2(t - 1)}{t - 1} =$ _____

6. $\lim_{x \to 0}\dfrac{\sin 3x}{5x} =$ _____

7. $\lim_{x \to 0^+} e^{1/x} =$ _____

8. $\lim_{x \to 0^-} e^{1/x} =$ _____

9. $\lim_{x \to \infty} e^{1/x} =$ _____

10. $\lim_{x \to -\infty}\dfrac{1 + 2e^x}{4 + e^x} =$ _____

11. $\lim_{x \to \_\_}\dfrac{1}{x - 3} = -\infty$

12. $\lim_{x \to \_\_}(5x + 2) = 22$

13. $\lim_{x \to \_\_} x^3 = -\infty$

14. $\lim_{x \to \_\_}\dfrac{1}{\sqrt{x}} = \infty$

15. If $f(x) = 2(x - 4)/|x - 4|$, $x \neq 4$, and $f(4) = 9$, then $\lim_{x \to 4^-} f(x) =$ _____.

16. Suppose $x^2 - x^4/3 \le f(x) \le x^2$ for all x. Then $\lim_{x \to 0} f(x)/x^2 =$ _____.

17. If f is continuous at a number a and $\lim_{x \to a} f(x) = 10$, then $f(a) =$ _____.

18. If f is continuous at $x = 5$, $f(5) = 2$, and $\lim_{x \to 5} g(x) = 10$, then $\lim_{x \to 5}[g(x) - f(x)] =$ _____.

19. $f(x) = \begin{cases} \dfrac{2x - 1}{4x^2 - 1}, & x \neq \frac{1}{2} \\ 0.5, & x = \frac{1}{2} \end{cases}$ is _________ (continuous/discontinuous) at the number $\frac{1}{2}$.

20. The equation $e^{-x^2} = x^2 - 1$ has precisely _____ roots in the interval $(-\infty, \infty)$.

21. The function $f(x) = \dfrac{10}{x} + \dfrac{x^2 - 4}{x - 2}$ has a removable discontinuity at $x = 2$. To remove the discontinuity, $f(2)$ should be defined to be _____.

22. If $\lim_{x \to -5} g(x) = -9$ and $f(x) = x^2$, then $\lim_{x \to -5} f(g(x)) =$ _____.

C. Exercises

In Problems 1–4, sketch a graph of a function f that satisfies the given conditions.

1. $f(0) = 1, f(4) = 0, f(6) = 0, \lim_{x \to 3^-} f(x) = 2, \lim_{x \to 3^+} f(x) = \infty, \lim_{x \to -\infty} f(x) = 0, \lim_{x \to \infty} f(x) = 2$

2. $\lim_{x \to -\infty} f(x) = 0, f(0) = 1, \lim_{x \to 4^-} f(x) = \infty, \lim_{x \to 4^+} f(x) = \infty, f(5) = 0, \lim_{x \to \infty} f(x) = -1$

3. $\lim_{x \to -\infty} f(x) = 2, f(-1) = 3, f(0) = 0, f(-x) = -f(x)$

4. $\lim_{x \to \infty} f(x) = 0, f(0) = -3, f(1) = 0, f(-x) = f(x)$

In Problems 5–10, state which of the conditions (a)–(j) are applicable to the graph of $y = f(x)$.

(a) $f(a)$ is not defined	**(b)** $f(a) = L$	**(c)** f is continuous at $x = a$	**(d)** f is continuous on $[0, a]$	**(e)** $\lim_{x \to a^+} f(x) = L$
(f) $\lim_{x \to a} f(x) = L$	**(g)** $\lim_{x \to a} \lvert f(x) \rvert = \infty$	**(h)** $\lim_{x \to \infty} f(x) = L$	**(i)** $\lim_{x \to \infty} f(x) = -\infty$	**(j)** $\lim_{x \to \infty} f(x) = 0$

5.

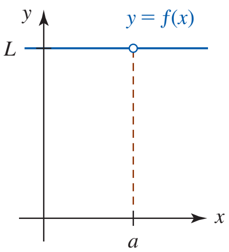


FIGURE 2.R.1 Graph for Problem 5

6.

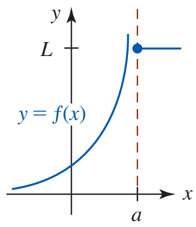


FIGURE 2.R.2 Graph for Problem 6

7.

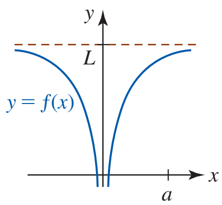


FIGURE 2.R.3 Graph for Problem 7

8.

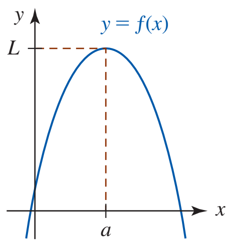


FIGURE 2.R.4 Graph for Problem 8

9.

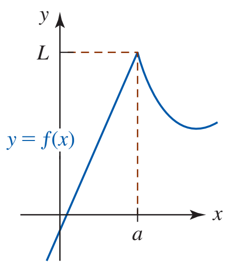


FIGURE 2.R.5 Graph for Problem 9

10.

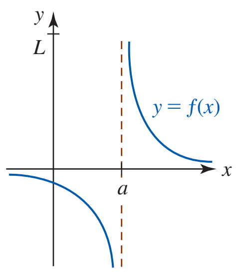


FIGURE 2.R.6 Graph for Problem 10

In Problems 11 and 12, sketch the graph of the given function. Determine the numbers, if any, at which f is discontinuous.

11. $f(x) = |x| + x$

12. $f(x) = \begin{cases} x + 1, & x < 2 \\ 3, & 2 < x < 4 \\ -x + 7, & x > 4 \end{cases}$

In Problems 13–16, determine intervals on which the given function is continuous.

13. $f(x) = \dfrac{x + 6}{x^3 - x}$

14. $f(x) = \dfrac{\sqrt{4 - x^2}}{x^2 - 4x + 3}$

15. $f(x) = \dfrac{x}{\sqrt{x^2 - 5}}$

16. $f(x) = \dfrac{\csc x}{\sqrt{x}}$

17. Find a number k so that

$$f(x) = \begin{cases} kx + 1, & x \le 3 \\ 2 - kx, & x > 3 \end{cases}$$

is continuous at the number 3.

18. Find numbers a and b so that

$$f(x) = \begin{cases} x + 4, & x \le 1 \\ ax + b, & 1 < x \le 3 \\ 3x - 8, & x > 3 \end{cases}$$

is continuous everywhere.

In Problems 19–22, find the slope of the tangent line to the graph of the function at the given value of x. Find an equation of the tangent line at the corresponding point.

19. $f(x) = -3x^2 + 16x + 12, \quad x = 2$

20. $f(x) = x^3 - x^2, \quad x = -1$

21. $f(x) = \dfrac{-1}{2x^2}, \quad x = \dfrac{1}{2}$

22. $f(x) = x + 4\sqrt{x}, \quad x = 4$

23. Find an equation of the line that is perpendicular to the tangent line at the point (1, 2) on the graph of $f(x) = -4x^2 + 6x$.

24. Suppose $f(x) = 2x + 5$ and $\varepsilon = 0.01$. Find a $\delta > 0$ that will guarantee that $|f(x) - 7| < \varepsilon$ when $0 < |x - 1| < \delta$. What limit has been proved by finding δ?

Chapter 4: Vectors, Matrices, and Linear Algebra

Scott Owen & Greg Corrado

Linear Algebra is strikingly similar to the algebra you learned in high school, except that in the place of ordinary single numbers, it deals with vectors. Many of the same algebraic operations you're used to performing on ordinary numbers (a.k.a. scalars), such as addition, subtraction and multiplication, can be generalized to be performed on vectors. We'll better start by defining what we mean by *scalars* and *vectors*.

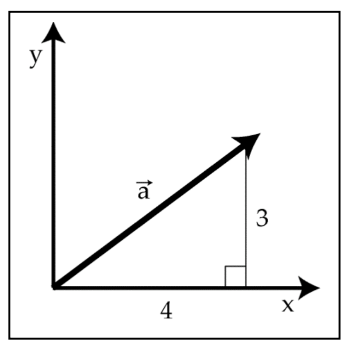
Definition: A scalar is a number. Examples of scalars are temperature, distance, speed, or mass – all quantities that have a magnitude but no "direction", other than perhaps positive or negative.

Okay, so scalars are what you're used to. In fact we could go so far as to describe the algebra you learned in grade school as *scalar* algebra, or the calculus many of us learned in high school as *scalar* calculus, be cause both dealt almost exclusively with scalars. This is to be contrasted with *vector* calculus or *vector* algebra, that most of us either only got in college if at all. So what *is* a vector?

Definition: A vector is *a list of numbers*. There are (at least) two ways to interpret what this list of numbers mean: One way to think of the vector as being *a point in a space*. Then this list of numbers is a way of identifying that point in space, where each number represents the vector's component that dimension. Another way to think of a vector is *a magnitude and a direction*, e.g. a quantity like velocity ("the fighter jet's velocity is 250 mph north-by-northwest"). In this way of think of it, a vector is a directed arrow pointing from the origin to the end point given by the list of numbers.

In this class we'll denote vectors by including a small arrow overtop of the symbol like so: $\vec{a}$. Another common convention you might encounter in other texts and papers is to denote vectors by use of a boldface font ($\boldsymbol{a}$). An example of a vector is $\vec{a} = [4,3]$. Graphically, you can think of this vector as an arrow in the *x-y* plane, pointing from the origin to the point at *x*=3, *y*=4 (see illustration.)

In this example, the list of numbers was only two elements long, but in principle it could be any length. The dimensionality of a vector is the length of the list. So, our example $\vec{a}$ is 2-dimensional because it is a list of two numbers. Not surprisingly all 2-dimentional vectors live in a plane. A 3-dimensional vector would be a list of three numbers, and they live in a 3-D volume. A 27-dimensional vector would be a list of twenty-seven numbers, and would live in a space only Ilana's dad could visualize.



Magnitudes and direction

The “magnitude” of a vector is the distance from the endpoint of the vector to the origin – in a word, it’s length. Suppose we want to calculate the magnitude of the vector $\vec{a} = [4,3]$. This vector extends 4 units along the x-axis, and 3 units along the y-axis. To calculate the magnitude $\|\vec{a}\|$ of the vector we can use the Pythagorean theorem ($x^2 + y^2 = z^2$).

$$\|\vec{a}\| = \sqrt{x^2 + y^2} = \sqrt{4^2 + 3^2} = 5$$

The magnitude of a vector is a scalar value – a number representing the length of the vector independent of the direction. There are a lot of examples were the magnitudes of vectors are important to us: velocities are vectors, speeds are their magnitudes; displacements are vectors, distances are their magnitudes.

To complement the magnitude, which represents the length independent of direction, one might wish for a way of representing the direction of a vector independent of its length. For this purpose, we use “unit vectors,” which are quite simply vectors with a magnitude of 1. A unit vector is denoted by a small “carrot” or “hat” above the symbol. For example, $\hat{a}$ represents the unit vector associated with the vector $\vec{a}$. To calculate the unit vector associated with a particular vector, we take the original vector and divide it by its magnitude. In mathematical terms, this process is written as:

$$\hat{a} = \frac{\vec{a}}{||\vec{a}||}$$

Definition: A unit vector is a vector of magnitude 1. Unit vectors can be used to express the direction of a vector independent of its magnitude.

Returning to the previous example of $\vec{a} = [4,3]$, recall $||\vec{a}|| = 5$. When dividing a vector ($\vec{a}$) by a scalar ($||\vec{a}||$), we divide each component of the vector individually by the scalar. In the same way, when multiplying a vector by a scalar we will proceed component by component. Note that this will be very different when multiplying a vector by another vector, as discussed below. But for now, in the case of dividing a vector by a scalar we arrive at:

$$\hat{a} = \frac{[4,3]}{5}$$

$$\hat{a} = \left[\frac{4}{5}, \frac{3}{5}\right]$$

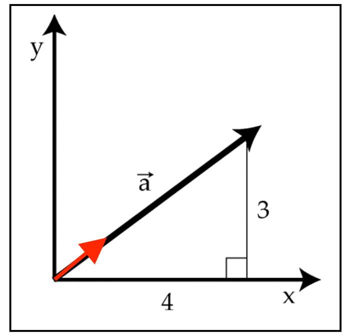
As shown in red in the figure, by dividing each component of the vector by the same number, we leave the direction of the vector unchanged, while we change the magnitude. If we have done this correctly, then the magnitude of the unit vector must be equal to 1 (otherwise it would not be a unit vector). We can verify this by calculating the magnitude of the unit vector $||\hat{a}||$.

$$\|\hat{a}\|^2 = \left(\frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2$$

$$\|\hat{a}\|^2 = \left(\frac{16}{25}\right) + \left(\frac{9}{25}\right)$$

$$\|\hat{a}\|^2 = \left(\frac{25}{25}\right) = 1$$

$$\|\hat{a}\| = 1$$



So we have demonstrated how to create a unit vector $\hat{a}$ that has a magnitude of 1 but a direction identical to the vector $\vec{a}$. Taking together the magnitude $\|\vec{a}\|$ and the unit vector $\hat{a}$ we have all of the information contained in the vector $\vec{a}$, but neatly separated into its magnitude and direction components. We can use these two components to re-create the vector $\vec{a}$ by multiplying the vector $\hat{a}$ by the scalar $\|\vec{a}\|$ like so:

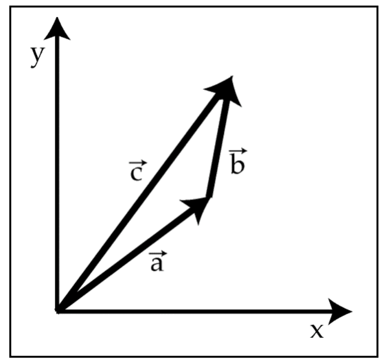
$$\vec{a} = \hat{a} * \|\vec{a}\|$$

Vector addition and subtraction

Vectors can be added and subtracted. Graphically, we can think of adding two vectors together as placing two line segments end-to-end, maintaining distance and direction. An example of this is shown in the illustration, showing the addition of two vectors $\vec{a}$ and $\vec{b}$ to create a third vector $\vec{c}$.

$$\vec{a} + \vec{b} = \vec{c}$$

Numerically, we add vectors component-by-component. That is to say, we add the x components together, and then separately we add the y components together. For example, if $\vec{a} = [4,3]$ and $\vec{b} = [1,2]$, then:



$$\vec{c} = \vec{a} + \vec{b}$$

$$\vec{c} = [4,3] + [1,2]$$

$$\vec{c} = [4 + 1, 3 + 2]$$

$$\vec{c} = [5,5]$$

Similarly, in vector subtraction:

$$\vec{c} = \vec{a} - \vec{b}$$

$$\vec{c} = [4,3] - [1,2]$$

$$\vec{c} = [3,1]$$

Vector addition has a very simple interpretation in the case of things like displacement. If in the morning a ship sailed 4 miles east and 3 miles north, and then in the afternoon it sailed a further 1 mile east and 2 miles north, what was the total displacement for the whole day? 5 miles east and 5 miles north – vector addition at work.

Linear independence

If two vectors point in different directions, even if they are not very different directions, then the two vectors are said to be *linearly independent*. If vectors $\vec{a}$ and $\vec{b}$ point in the same direction, then you can multiply vector $\vec{a}$ by a constant, scalar value and get vector $\vec{b}$, and vice versa to get from $\vec{b}$ to $\vec{a}$. If the two vectors point in different directions, then this is not possible to make one out of the other because multiplying a vector by a scalar will never change the direction of the vector, it will only change the magnitude. This concept generalizes to families of more than two vectors. Three vectors are said to be linearly independent if there is no way to construct one vector by combining scaled versions of the other two. The same definition applies to families of four or more vectors by applying the same rules.

The vectors in the previous figure provide a graphical example of linear independence. Vectors $\vec{a}$ and $\vec{c}$ point in slightly different directions. There is no way to change the length of vector $\vec{a}$ and generate vector $\vec{c}$, nor vice-versa to get from $\vec{c}$ to $\vec{a}$. If, on the other hand, we consider the family of vectors that contains $\vec{a}$, $\vec{b}$ and $\vec{c}$, it is now possible, as shown, to add vectors $\vec{a}$ and $\vec{b}$ to generate vector $\vec{c}$. So the family of vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is *not* linearly independent, but is instead said to be linearly dependent. Incidentally, you could change the length of any or all of these three vectors and they would still be linearly dependent.

Definition: A family of vectors is linearly independent if no one of the vectors can be created by any linear combination of the other vectors in the family. For example, $\vec{c}$ is linearly independent of $\vec{a}$ and $\vec{b}$ if and only if it is *impossible* to find scalar values of α and β such that $\vec{c} = \alpha\vec{a} + \beta\vec{b}$

Vector multiplication: dot products

Next we move into the world of vector multiplication. There are two principal ways of multiplying vectors, called *dot products* (a.k.a. *scalar products*) and *cross products*. The dot product:

$$d = \vec{a} \cdot \vec{b}$$

generates a scalar value from the product of two vectors and will be discussed in greater detail below. Do not confuse the dot product with the cross product:

$$\vec{c} = \vec{a} \times \vec{b}$$

which is an entirely different beast. The cross product generates a vector from the product of two vectors. Cross products so up in physics sometimes, such as when describing the interaction between electrical and magnetic fields (ask your local fMRI expert), but we'll set those aside for now and just focus on dot products in this course. The dot product is calculated by multiplying the x components, then separately multiplying the y components (and so on for z, etc… for products in more than 2 dimensions) and then adding these products together. To do an example using the vectors above:

$$\vec{a} \cdot \vec{b} = [4,3] \cdot [1,2]$$

$$\vec{a} \cdot \vec{b} = (4 * 1) + (3 \ * 2)$$

$$\vec{a} \cdot \vec{b} = 11$$

Another way of calculating the dot product of two vectors is to use a geometric means. The dot product can be expressed geometrically as:

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos\theta$$

where θ represents the angle between the two vectors. Believe it or not, calculation of the dot product by either procedure will yield exactly the same result. Recall, again from high school geometry, that $\cos 0° = 1$, and that $\cos 90° = 0$. If the angle between $\vec{a}$ and $\vec{b}$ is nearly $0°$ (i.e. if the vectors point in nearly the same direction), then the dot product of the two vectors will be nearly $\|\vec{a}\| \|\vec{b}\|$.

Definition: A dot product (or scalar product) is the numerical product of the lengths of two vectors, multiplied by the cosine of the angle between them.

Orthogonality

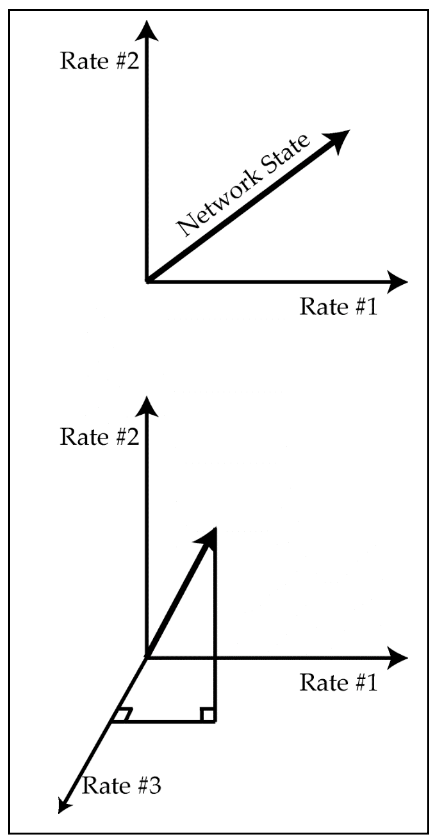
As the angle between the two vectors opens up to approach $90°$, the dot product of the two vectors will approach 0, regardless of the vector magnitudes $\|\vec{a}\|$ and $\|\vec{b}\|$. In the special case that the angle between the two vectors is exactly $90°$, the dot product of the two vectors will be 0 regardless of the magnitude of the vectors. In this case, the two vectors are said to be *orthogonal*.

Definition: Two vectors are orthogonal to one another if the dot product of those two vectors is equal to zero.

Orthogonality is an important and general concept, and is a more mathematically precise way of saying “perpendicular.” In two- or three-dimensional space, orthogonality is identical to perpendicularity and the two ideas can be thought of interchangeably. Whereas perpendicularity is restricted to spatial representations of things, orthogonality is a more general term. In the context of neural networks, neuroscientists will often talk in terms of two patterns of neuronal firing being orthogonal to one another. Orthogonality can also apply to functions as well as to things like vectors and firing rates. As we will discuss later in this class in the context of *fourier transforms*, the sin and cos functions can be said to be orthogonal functions. In any of these contexts, orthogonality will always mean something akin to “totally independent” and is specifically refering to two things having a dot product of zero.

Vector spaces

All vectors live within a *vector space*. A vector space is exactly what it sounds like – the space in which vectors live. When talking about spatial vectors, for instance the direction and speed with which a person is walking through a room, the vector space is intuitively spatial since all available directions of motion can be plotted directly onto a spatial map of the room.



A less spatially intuitive example of a vector space might be all available states of a neural network. Imagine a very simple network, consisting of only five neurons which we will call n_1, n_2, n_3, n_4, and n_5. At each point in time, each neuron might not fire any action potentials at all, in which case we write $n_i = 0$, where i denotes the neuron number. Alternatively, the neuron might be firing action potentials at a rate of up to 100 Hz, in which case we write that $n_i = x$, where $0 \leq x \leq 100$. The state of this network at any moment in time can be depicted by a vector that describes the firing rates of all five neurons:

$$s = [n_1, n_2, n_3, n_4, n_5]$$

The set of all possible firing rates for all five neurons represents a vector space that is every bit as real as the vector space represented by a person walking through a room. The vector space represented by the neurons, however, is a 5-dimensional vector space. The math is identical to the two dimensional situation, but in this case we must trust the math because our graphical intuition fails us.

If we call state 1 the state in which neuron #1 is firing at a rate of 1 Hz and all others are silent, we can write this as:

$$s_1 = [1, 0, 0, 0, 0]$$

We may further define states 2, 3, 4, and 5 as follows:

$$s_2 = [0, 1, 0, 0, 0]$$

$$s_3 = [0, 0, 1, 0, 0]$$

$$s_4 = [0, 0, 0, 1, 0]$$

$$s_5 = [0, 0, 0, 0, 1]$$

By taking combinations of these five *basis vectors*, and multiplying them by scalar constants, we can describe any state of the network in the entire vector space. For example, to generate the network state [0, 3, 0, 9, 0] we could write:

$$(3 * s_2) + (9 * s_4) = [0, 3, 0, 9, 0]$$

If any one of the basis vectors is removed from the set, however, there will be some states of the network we will be unable to describe. For example, no combination of the vectors s_1, s_2, s_3, and s_4 can describe the network state [1, 0, 5, 3, 2] without also making use of s_5. Every vector space has a set of basis vectors. The definition of a set of basis vectors is twofold: (1) linear combinations (meaning addition, subtraction and multiplication by scalars) of the basis vectors can describe any vector in the vector space, and (2) every one of the basis vectors must be required in order to be able to describe all of the vectors in the vector space. It is also worth noting that the vectors s_1, s_2, s_3, s_4, and s_5 are all orthogonal to one another. You can test this for yourself by calculating the dot product of any two of these five basis vectors and verifying that it is zero. Basis vectors are not always orthogonal to one another, but they must always be linearly independent. The vector space that is defined by the set of all vectors you can possibly generate with different combinations of the basis vectors is called the *span* of the basis vectors.

Definition: A basis set is a linearly independent set of vectors that, when used in linear combination, can represent every vector in a given vector space.

It's worth taking a moment to consider what vector space is actually defined by these basis vectors. It's not all of 5-dimensional space, because we don't allow negative firing rates. So it's sort of the positive "quadrant" of 5-dimensional space. But we also said that we don't allow firing rates over 100Hz, so it's really only the part of that quadrant that fits between 0 and 100. This is a 5-dimensional hypercube. Doesn't that just sound so cool.

Matrices

A matrix, like a vector, is also a collection of numbers. The difference is that a matrix is a table of numbers rather than a list. Many of the same rules we just outlined for vectors above apply equally well to matrices. (In fact, you can think of vectors as matrices that happen to only have one column or one row.) First, let's consider matrix addition and subtraction. This part is uncomplicated. You can add and subtract matrices the same way you add vectors – element by element:

$$A + B = C$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{pmatrix}$$

Matrix multiplication gets a bit more complicated, since multiple elements in the first matrix interact with multiple elements in the second to generate each element in the product matrix. This means that matrix multiplication can be a tedious task to carry out by hand, and can be time consuming on a computer for very large matrices. But as we shall see, this also means that, depending on the matrices we are multiplying, we will be able to perform a vast array of computations. Shown here is a simple example of multiplying a pair of 2×2 matrices, but this same procedure can generalize to matrices of arbitrarily large size, as will be discussed below:

$$AB = D$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

If you look at what's going on here, you may notice that what's going on it that we're taking dot products between rows of the ***A*** matrix with columns of the ***B*** matrix. for example to find the entry in our new matrix that's in the first-row-second-column position, we take the first row of ***A*** and compute it's dot product with the second column of ***B***. Matrix multiplication and dot products are intimately linked.

When multiplying two matrices together, the two matrices do not need to be the same size. For example, the vector $\vec{a} = (4,3)$ from earlier in this chapter represents a very simple matrix. If we multiply this matrix by another simple matrix the result is:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

Linear Transformations

Linear transformations are functions (functions that take vectors as inputs), but their a special subset of functions. Linear transformations are the set of all functions that can be written as a matrix multiplication:

$$\vec{y} = A\vec{x}$$

This sees like a pretty restrictive class of function, and indeed it is. But as it turns out, these simple functions can accomplish some surprisingly interesting things.

Consider the matrix we used for matrix multiplication in our last example.

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

When multiplied the vector [4,3] by this matrix, we got the vector [-3,4]. If you draw those to vectors you'll notice that the two vectors have the same length, but the second vector is rotated 90º counter clockwise from the first vector. Coincidence? Pick another vector, say [100, 0]; that vector gets transformed to the vector [0, 100], rotating 90º from the 3 o'clock position to the 12 o'clock position. As it turns out this is true for any vector $\vec{x}$ you pick: $\vec{y}$ will always be rotated 90º counter clockwise.

As it turns out our matrix $\boldsymbol{A}$ is a very special matrix. It's a *rotation matrix* – it never changes the length of a vector, but changes every vector's direction by a certain fixed amount – in this case 90º. This is just one special case of a rotation matrix. We can generate a rotation matrix that will allow us to rotate a vector by any angle we want. The general rotation matrix to rotate a vector by an angle θ looks like this:

$$R_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

First you plug in the desired value for θ to calculate the different elements of the matrix, then you multiply the matrix by the vector and you have a rotated vector.

This may sound like an awful lot to go through when a protractor would do just as well. The reason this is worth all the trouble is that it doesn't stop at rotations, it doesn't even stop at two dimensions. Above we defined a rotation matrix $\boldsymbol{R}$ for two dimensions. We can just as well define a rotation matrix in three dimensions, it would just be a 3×3 matrix rather than the 2×2 one above. In the case above, we think of the rotation matrix as "operating on" the vector to perform a rotation. Following this terminology, we call the rotation matrix the *operator*. We can make all sorts of other linear operators (matrixes) to do all sorts of other transformations. A few other useful linear transformations include identity $(\boldsymbol{I})$, Stretch and squash $(\boldsymbol{S})$, Skew $(\boldsymbol{W})$ and Flip $(\boldsymbol{F})$.

You may notice that since a linear operator is anything that can be expressed as a matrix, that means that you can stick any series of linear operators together to make a new function, and that function will also be a linear operator. For example if we want to rotate a vector and then stretch it we could, premultiply those to matrices together, and the resulting matrix operator will still be linear and will still follow all the same rules. So if $\boldsymbol{S} \cdot \boldsymbol{R} = \boldsymbol{X}$, then we can apply $\boldsymbol{X}$ as its own operator. Extending this, any set of linear transformations, no mater how long can be multiplied together and condensed into a single operator. For instance, if we wanted to stretch-skew-rotate-skew-stretch-flip-rotate ($\boldsymbol{S} \cdot \boldsymbol{W} \cdot \boldsymbol{R} \cdot \boldsymbol{W} \cdot \boldsymbol{S} \cdot \boldsymbol{F} \cdot \boldsymbol{R}$), we can define all of that into a single new operator $\boldsymbol{Y}$.

CHAPTER 2

POLYNOMIALS

2.1 Introduction

You have studied algebraic expressions, their addition, subtraction, multiplication and division in earlier classes. You also have studied how to factorise some algebraic expressions. You may recall the algebraic identities :

$$(x + y)^2 = x^2 + 2xy + y^2$$

$$(x - y)^2 = x^2 - 2xy + y^2$$

and

$$x^2 - y^2 = (x + y)\,(x - y)$$

and their use in factorisation. In this chapter, we shall start our study with a particular type of algebraic expression, called *polynomial*, and the terminology related to it. We shall also study the *Remainder Theorem* and *Factor Theorem* and their use in the factorisation of polynomials. In addition to the above, we shall study some more algebraic identities and their use in factorisation and in evaluating some given expressions.

2.2 Polynomials in One Variable

Let us begin by recalling that a variable is denoted by a symbol that can take any real value. We use the letters x, y, z, etc. to denote variables. Notice that $2x, 3x, -x, -\frac{1}{2}x$ are algebraic expressions. All these expressions are of the form (a constant) $\times x$. Now suppose we want to write an expression which is (a constant) $\times$ (a variable) and we do not know what the constant is. In such cases, we write the constant as a, b, c, etc. So the expression will be ax, say.

However, there is a difference between a letter denoting a constant and a letter denoting a variable. The values of the constants remain the same throughout a particular situation, that is, the values of the constants do not change in a given problem, but the value of a variable can keep changing.

Now, consider a square of side 3 units (see Fig. 2.1). What is its perimeter? You know that the perimeter of a square is the sum of the lengths of its four sides. Here, each side is 3 units. So, its perimeter is 4×3, i.e., 12 units. What will be the perimeter if each side of the square is 10 units? The perimeter is 4×10, i.e., 40 units. In case the length of each side is x units (see Fig. 2.2), the perimeter is given by $4x$ units. So, as the length of the side varies, the perimeter varies.

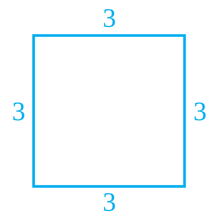


Fig. 2.1

Can you find the area of the square PQRS? It is $x \times x = x^2$ square units. x^2 is an algebraic expression. You are also familiar with other algebraic expressions like $2x$, $x^2 + 2x$, $x^3 - x^2 + 4x + 7$. Note that, all the algebraic expressions we have considered so far have only whole numbers as the exponents of the variable. Expressions of this form are called *polynomials in one variable*. In the examples above, the variable is x. For instance, $x^3 - x^2 + 4x + 7$ is a polynomial in x. Similarly, $3y^2 + 5y$ is a polynomial in the variable y and $t^2 + 4$ is a polynomial in the variable t.

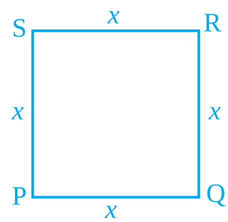


Fig. 2.2

In the polynomial $x^2 + 2x$, the expressions x^2 and $2x$ are called the **terms** of the polynomial. Similarly, the polynomial $3y^2 + 5y + 7$ has three terms, namely, $3y^2$, $5y$ and 7. Can you write the terms of the polynomial $-x^3 + 4x^2 + 7x - 2$? This polynomial has 4 terms, namely, $-x^3$, $4x^2$, $7x$ and -2.

Each term of a polynomial has a **coefficient**. So, in $-x^3 + 4x^2 + 7x - 2$, the coefficient of x^3 is -1, the coefficient of x^2 is 4, the coefficient of x is 7 and -2 is the coefficient of x^0 (Remember, $x^0 = 1$). Do you know the coefficient of x in $x^2 - x + 7$? It is -1.

2 is also a polynomial. In fact, 2, –5, 7, etc. are examples of *constant polynomials*. The constant polynomial 0 is called the **zero polynomial**. This plays a very important role in the collection of all polynomials, as you will see in the higher classes.

Now, consider algebraic expressions such as $x + \dfrac{1}{x}$, $\sqrt{x} + 3$ and $\sqrt[3]{y} + y^2$. Do you know that you can write $x + \dfrac{1}{x} = x + x^{-1}$? Here, the exponent of the second term, i.e., x^{-1} is -1, which is not a whole number. So, this algebraic expression is not a polynomial.

Again, $\sqrt{x} + 3$ can be written as $x^{\frac{1}{2}} + 3$. Here the exponent of x is $\dfrac{1}{2}$, which is not a whole number. So, is $\sqrt{x} + 3$ a polynomial? No, it is not. What about $\sqrt[3]{y} + y^2$? It is also not a polynomial (Why?).

If the variable in a polynomial is x, we may denote the polynomial by $p(x)$, or $q(x)$, or $r(x)$, etc. So, for example, we may write :

$$p(x) = 2x^2 + 5x - 3$$
$$q(x) = x^3 - 1$$
$$r(y) = y^3 + y + 1$$
$$s(u) = 2 - u - u^2 + 6u^5$$

A polynomial can have any (finite) number of terms. For instance, $x^{150} + x^{149} + ... + x^2 + x + 1$ is a polynomial with 151 terms.

Consider the polynomials $2x$, 2, $5x^3$, $-5x^2$, y and u^4. Do you see that each of these polynomials has only one term? Polynomials having only one term are called *monomials* ('mono' means 'one').

Now observe each of the following polynomials:

$p(x) = x + 1$, $q(x) = x^2 - x$, $r(y) = y^{30} + 1$, $t(u) = u^{43} - u^2$

How many terms are there in each of these? Each of these polynomials has only two terms. Polynomials having only two terms are called *binomials* ('bi' means 'two').

Similarly, polynomials having only three terms are called *trinomials* ('tri' means 'three'). Some examples of trinomials are

$p(x) = x + x^2 + \pi$, $q(x) = \sqrt{2} + x - x^2$,

$r(u) = u + u^2 - 2$, $t(y) = y^4 + y + 5$.

Now, look at the polynomial $p(x) = 3x^7 - 4x^6 + x + 9$. What is the term with the highest power of x ? It is $3x^7$. The exponent of x in this term is 7. Similarly, in the polynomial $q(y) = 5y^6 - 4y^2 - 6$, the term with the highest power of y is $5y^6$ and the exponent of y in this term is 6. We call the highest power of the variable in a polynomial as the *degree of the polynomial*. So, the degree of the polynomial $3x^7 - 4x^6 + x + 9$ is 7 and the degree of the polynomial $5y^6 - 4y^2 - 6$ is 6. **The degree of a non-zero constant polynomial is zero**.

Example 1 : Find the degree of each of the polynomials given below:

(i) $x^5 - x^4 + 3$ (ii) $2 - y^2 - y^3 + 2y^8$ (iii) 2

Solution : (i) The highest power of the variable is 5. So, the degree of the polynomial is 5.

(ii) The highest power of the variable is 8. So, the degree of the polynomial is 8.

(iii) The only term here is 2 which can be written as $2x^0$. So the exponent of x is 0. Therefore, the degree of the polynomial is 0.

Now observe the polynomials $p(x) = 4x + 5$, $q(y) = 2y$, $r(t) = t + \sqrt{2}$ and $s(u) = 3 - u$. Do you see anything common among all of them? The degree of each of these polynomials is one. A polynomial of degree one is called a *linear polynomial*. Some more linear polynomials in one variable are $2x - 1$, $\sqrt{2}\,y + 1$, $2 - u$. Now, try and find a linear polynomial in x with 3 terms? You would not be able to find it because a linear polynomial in x can have at most two terms. So, any linear polynomial in x will be of the form $ax + b$, where a and b are constants and $a \neq 0$ (why?). Similarly, $ay + b$ is a linear polynomial in y.

Now consider the polynomials :

$$2x^2 + 5,\ \ 5x^2 + 3x + \pi,\ \ x^2 \text{ and } x^2 + \frac{2}{5}x$$

Do you agree that they are all of degree two? A polynomial of degree two is called a *quadratic polynomial*. Some examples of a quadratic polynomial are $5 - y^2$, $4y + 5y^2$ and $6 - y - y^2$. Can you write a quadratic polynomial in one variable with four different terms? You will find that a quadratic polynomial in one variable will have at most 3 terms. If you list a few more quadratic polynomials, you will find that any quadratic polynomial in x is of the form $ax^2 + bx + c$, where $a \neq 0$ and a, b, c are constants. Similarly, quadratic polynomial in y will be of the form $ay^2 + by + c$, provided $a \neq 0$ and a, b, c are constants.

We call a polynomial of degree three a *cubic polynomial*. Some examples of a cubic polynomial in x are $4x^3$, $2x^3 + 1$, $5x^3 + x^2$, $6x^3 - x$, $6 - x^3$, $2x^3 + 4x^2 + 6x + 7$. How many terms do you think a cubic polynomial in one variable can have? It can have at most 4 terms. These may be written in the form $ax^3 + bx^2 + cx + d$, where $a \neq 0$ and a, b, c and d are constants.

Now, that you have seen what a polynomial of degree 1, degree 2, or degree 3 looks like, can you write down a polynomial in one variable of degree n for any natural number n? A polynomial in one variable x of degree n is an expression of the form

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

where $a_0, a_1, a_2, \ldots, a_n$ are constants and $a_n \neq 0$.

In particular, if $a_0 = a_1 = a_2 = a_3 = \ldots = a_n = 0$ (all the constants are zero), we get the **zero polynomial**, which is denoted by 0. What is the degree of the zero polynomial? The degree of the zero polynomial is *not defined*.

So far we have dealt with polynomials in one variable only. We can also have polynomials in more than one variable. For example, $x^2 + y^2 + xyz$ (where variables are x, y and z) is a polynomial in three variables. Similarly $p^2 + q^{10} + r$ (where the variables are p, q and r), $u^3 + v^2$ (where the variables are u and v) are polynomials in three and two variables, respectively. You will be studying such polynomials in detail later.

EXERCISE 2.1

1. Which of the following expressions are polynomials in one variable and which are not? State reasons for your answer.

(i) $4x^2 - 3x + 7$ (ii) $y^2 + \sqrt{2}$ (iii) $3\sqrt{t} + t\sqrt{2}$ (iv) $y + \frac{2}{y}$

(v) $x^{10} + y^3 + t^{50}$

2. Write the coefficients of x^2 in each of the following:

(i) $2 + x^2 + x$ (ii) $2 - x^2 + x^3$ (iii) $\frac{\pi}{2}x^2 + x$ (iv) $\sqrt{2}x - 1$

3. Give one example each of a binomial of degree 35, and of a monomial of degree 100.

4. Write the degree of each of the following polynomials:

(i) $5x^3 + 4x^2 + 7x$ (ii) $4 - y^2$

(iii) $5t - \sqrt{7}$ (iv) 3

5. Classify the following as linear, quadratic and cubic polynomials:

(i) $x^2 + x$ (ii) $x - x^3$ (iii) $y + y^2 + 4$ (iv) $1 + x$

(v) $3t$ (vi) r^2 (vii) $7x^3$

2.3 Zeroes of a Polynomial

Consider the polynomial $p(x) = 5x^3 - 2x^2 + 3x - 2$.

If we replace x by 1 everywhere in $p(x)$, we get

$$
\begin{aligned}
p(1) &= 5 \times (1)^3 - 2 \times (1)^2 + 3 \times (1) - 2\\
&= 5 - 2 + 3 - 2\\
&= 4
\end{aligned}
$$

So, we say that the value of $p(x)$ at $x = 1$ is 4.

Similarly, $p(0) = 5(0)^3 - 2(0)^2 + 3(0) - 2$

$= -2$

Can you find $p(-1)$?

Example 2 : Find the value of each of the following polynomials at the indicated value of variables:

(i) $p(x) = 5x^2 - 3x + 7$ at $x = 1$.

(ii) $q(y) = 3y^3 - 4y + \sqrt{11}$ at $y = 2$.

(iii) $p(t) = 4t^4 + 5t^3 - t^2 + 6$ at $t = a$.

Solution : (i) $p(x) = 5x^2 - 3x + 7$

The value of the polynomial $p(x)$ at $x = 1$ is given by

$$p(1) = 5(1)^2 - 3(1) + 7$$
$$= 5 - 3 + 7 = 9$$

(ii) $q(y) = 3y^3 - 4y + \sqrt{11}$

The value of the polynomial $q(y)$ at $y = 2$ is given by

$$q(2) = 3(2)^3 - 4(2) + \sqrt{11} = 24 - 8 + \sqrt{11} = 16 + \sqrt{11}$$

(iii) $p(t) = 4t^4 + 5t^3 - t^2 + 6$

The value of the polynomial $p(t)$ at $t = a$ is given by

$$p(a) = 4a^4 + 5a^3 - a^2 + 6$$

Now, consider the polynomial $p(x) = x - 1$.

What is $p(1)$? Note that : $p(1) = 1 - 1 = 0$.

As $p(1) = 0$, we say that 1 is a *zero* of the polynomial $p(x)$.

Similarly, you can check that 2 is a *zero* of $q(x)$, where $q(x) = x - 2$.

In general, we say that a *zero* of a polynomial $p(x)$ is a number c such that $p(c) = 0$.

You must have observed that the zero of the polynomial $x - 1$ is obtained by equating it to 0, i.e., $x - 1 = 0$, which gives $x = 1$. We say $p(x) = 0$ is a polynomial equation and 1 is the *root of the polynomial* equation $p(x) = 0$. So we say 1 is the zero of the polynomial $x - 1$, or a *root* of the polynomial equation $x - 1 = 0$.

Now, consider the constant polynomial 5. Can you tell what its zero is? It has no zero because replacing x by any number in $5x^0$ still gives us 5. In fact, *a non-zero constant polynomial has no zero*. What about the zeroes of the zero polynomial? By convention, *every real number is a zero of the zero polynomial*.

Example 3 : Check whether –2 and 2 are zeroes of the polynomial $x + 2$.

Solution : Let $p(x) = x + 2$.

Then $p(2) = 2 + 2 = 4$, $p(-2) = -2 + 2 = 0$

Therefore, –2 is a zero of the polynomial $x + 2$, but 2 is not.

Example 4 : Find a zero of the polynomial $p(x) = 2x + 1$.

Solution : Finding a zero of $p(x)$, is the same as solving the equation

$$p(x) = 0$$

Now, $2x + 1 = 0$ gives us $x = -\frac{1}{2}$

So, $-\frac{1}{2}$ is a zero of the polynomial $2x + 1$.

Now, if $p(x) = ax + b$, $a \neq 0$, is a linear polynomial, how can we find a zero of $p(x)$? Example 4 may have given you some idea. Finding a zero of the polynomial $p(x)$, amounts to solving the polynomial equation $p(x) = 0$.

Now, $p(x) = 0$ means $ax + b = 0, \ a \neq 0$

So, $ax = -b$

i.e., $x = -\frac{b}{a}$.

So, $x = -\frac{b}{a}$ is the only zero of $p(x)$, i.e., a *linear polynomial has one and only one zero.*

Now we can say that 1 is *the* zero of $x - 1$, and -2 is *the* zero of $x + 2$.

Example 5 : Verify whether 2 and 0 are zeroes of the polynomial $x^2 - 2x$.

Solution : Let $p(x) = x^2 - 2x$

Then $p(2) = 2^2 - 4 = 4 - 4 = 0$

and $p(0) = 0 - 0 = 0$

Hence, 2 and 0 are both zeroes of the polynomial $x^2 - 2x$.

Let us now list our observations:

(i) A zero of a polynomial need not be 0.

(ii) 0 may be a zero of a polynomial.

(iii) Every linear polynomial has one and only one zero.

(iv) A polynomial can have more than one zero.

EXERCISE 2.2

1. Find the value of the polynomial $5x - 4x^2 + 3$ at

(i) $x = 0$ (ii) $x = -1$ (iii) $x = 2$

2. Find $p(0)$, $p(1)$ and $p(2)$ for each of the following polynomials:

(i) $p(y) = y^2 - y + 1$ (ii) $p(t) = 2 + t + 2t^2 - t^3$

(iii) $p(x) = x^3$ (iv) $p(x) = (x - 1)(x + 1)$

3. Verify whether the following are zeroes of the polynomial, indicated against them.

(i) $p(x) = 3x + 1,\ x = -\frac{1}{3}$ (ii) $p(x) = 5x - \pi,\ x = \frac{4}{5}$

(iii) $p(x) = x^2 - 1,\ x = 1, -1$ (iv) $p(x) = (x + 1)(x - 2),\ x = -1, 2$

(v) $p(x) = x^2,\ x = 0$ (vi) $p(x) = lx + m,\ x = -\frac{m}{l}$

(vii) $p(x) = 3x^2 - 1,\ x = -\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}$ (viii) $p(x) = 2x + 1,\ x = \frac{1}{2}$

4. Find the zero of the polynomial in each of the following cases:

(i) $p(x) = x + 5$ (ii) $p(x) = x - 5$ (iii) $p(x) = 2x + 5$

(iv) $p(x) = 3x - 2$ (v) $p(x) = 3x$ (vi) $p(x) = ax,\ a \neq 0$

(vii) $p(x) = cx + d,\ c \neq 0,\ c, d$ are real numbers.

2.4 Remainder Theorem

Let us consider two numbers 15 and 6. You know that when we divide 15 by 6, we get the quotient 2 and remainder 3. Do you remember how this fact is expressed? We write 15 as

$$15 = (6 \times 2) + 3$$

We observe that the *remainder* 3 is less than the *divisor* 6. Similarly, if we divide 12 by 6, we get

$$12 = (6 \times 2) + 0$$

What is the remainder here? Here the remainder is 0, and we say that 6 is a *factor* of 12 or 12 is a *multiple* of 6.

Now, the question is: can we divide one polynomial by another? To start with, let us try and do this when the divisor is a monomial. So, let us divide the polynomial $2x^3 + x^2 + x$ by the monomial x.

We have $$(2x^3 + x^2 + x) \div x = \frac{2x^3}{x} + \frac{x^2}{x} + \frac{x}{x}$$

$$= 2x^2 + x + 1$$

In fact, you may have noticed that x is common to each term of $2x^3 + x^2 + x$. So we can write $2x^3 + x^2 + x$ as $x(2x^2 + x + 1)$.

We say that x and $2x^2 + x + 1$ are *factors* of $2x^3 + x^2 + x$, and $2x^3 + x^2 + x$ is a *multiple* of x as well as a multiple of $2x^2 + x + 1$.

Consider another pair of polynomials $3x^2 + x + 1$ and x.

Here, $(3x^2 + x + 1) \div x = (3x^2 \div x) + (x \div x) + (1 \div x)$.

We see that we cannot divide 1 by x to get a polynomial term. So in this case we stop here, and note that 1 is the remainder. Therefore, we have

$$3x^2 + x + 1 = \{x \times (3x + 1)\} + 1$$

In this case, $3x + 1$ is the quotient and 1 is the remainder. Do you think that x is a factor of $3x^2 + x + 1$? Since the remainder is not zero, it is not a factor.

Now let us consider an example to see how we can divide a polynomial by any non-zero polynomial.

Example 6 : Divide $p(x)$ by $g(x)$, where $p(x) = x + 3x^2 - 1$ and $g(x) = 1 + x$.

Solution : We carry out the process of division by means of the following steps:

Step 1 : We write the dividend $x + 3x^2 - 1$ and the divisor $1 + x$ in the standard form, i.e., after arranging the terms in the descending order of their degrees. So, the dividend is $3x^2 + x - 1$ and divisor is $x + 1$.

Step 2 : We divide the first term of the dividend by the first term of the divisor, i.e., we divide $3x^2$ by x, and get $3x$. This gives us the first term of the quotient.

$$\frac{3x^2}{x} = 3x = \text{first term of quotient}$$

Step 3 : We multiply the divisor by the first term of the quotient, and subtract this product from the dividend, i.e., we multiply $x + 1$ by $3x$ and subtract the product $3x^2 + 3x$ from the dividend $3x^2 + x - 1$. This gives us the remainder as $-2x - 1$.

$$\begin{array}{r|l} & \;3x \\ \hline x+1 & 3x^2 + x - 1 \\ & 3x^2 + 3x \\ & - \quad\; - \\ \hline & \quad -2x - 1 \end{array}$$

Step 4 : We treat the remainder $-2x - 1$ as the new dividend. The divisor remains the same. We repeat Step 2 to get the next term of the quotient, i.e., we divide the first term $-2x$ of the (new) dividend by the first term x of the divisor and obtain -2. Thus, -2 is the second term in the quotient.

$$\frac{-2x}{x} = -2$$
$= \text{second term of quotient}$

New Quotient $= 3x - 2$

Step 5 : We multiply the divisor by the second term of the quotient and subtract the product from the dividend. That is, we multiply $x + 1$ by -2 and subtract the product $-2x-2$ from the dividend $-2x-1$. This gives us 1 as the remainder.

$$
\begin{array}{l|l}
(x+1)(-2) & -2x-1 \\
= -2x-2 & -2x-2 \\
 & \underline{+ \quad +} \\
 & \quad +1
\end{array}
$$

This process continues till the remainder is 0 or the degree of the new dividend is less than the degree of the divisor. At this stage, this new dividend becomes the remainder and the sum of the quotients gives us the whole quotient.

Step 6 : Thus, the quotient in full is $3x-2$ and the remainder is 1.

Let us look at what we have done in the process above as a whole:

$$
\begin{array}{r|l}
 & \quad 3x - 2 \\ \hline
x+1 & 3x^2 + x - 1 \\
 & 3x^2 + 3x \\
 & \underline{- \qquad -} \\
 & \qquad -2x - 1 \\
 & \qquad -2x - 2 \\
 & \qquad \underline{+ \qquad +} \\
 & \qquad\qquad 1
\end{array}
$$

Notice that $3x^2 + x - 1 = (x+1)(3x-2) + 1$

i.e., **Dividend = (Divisor × Quotient) + Remainder**

In general, if $p(x)$ and $g(x)$ are two polynomials such that degree of $p(x) \geq$ degree of $g(x)$ and $g(x) \neq 0$, then we can find polynomials $q(x)$ and $r(x)$ such that:

$$p(x) = g(x)q(x) + r(x),$$

where $r(x) = 0$ or degree of $r(x) <$ degree of $g(x)$. Here we say that $p(x)$ divided by $g(x)$, gives $q(x)$ as quotient and $r(x)$ as remainder.

In the example above, the divisor was a linear polynomial. In such a situation, let us see if there is any link between the remainder and certain values of the dividend.

In $\quad p(x) = 3x^2 + x - 1$, if we replace x by -1, we have

$$p(-1) = 3(-1)^2 + (-1) - 1 = 1$$

So, the remainder obtained on dividing $p(x) = 3x^2 + x - 1$ by $x + 1$ is the same as the value of the polynomial $p(x)$ at the zero of the polynomial $x + 1$, i.e., -1.

Let us consider some more examples.

Example 7 : Divide the polynomial $3x^4 - 4x^3 - 3x - 1$ by $x - 1$.

Solution : By long division, we have:

$$
\begin{array}{rl}
 & 3x^3 - x^2 - x - 4 \\ \cline{2-2}
x - 1 \,\big) & 3x^4 - 4x^3 - 3x - 1 \\
 & 3x^4 - 3x^3 \\
 & \underline{- \quad\; +} \\
 & \quad -x^3 \qquad - 3x - 1 \\
 & \quad -x^3 + x^2 \\
 & \quad \underline{+ \quad\; -} \\
 & \qquad\quad -x^2 - 3x - 1 \\
 & \qquad\quad -x^2 + x \\
 & \qquad\quad \underline{+ \quad\; -} \\
 & \qquad\qquad\quad -4x - 1 \\
 & \qquad\qquad\quad -4x + 4 \\
 & \qquad\qquad\quad \underline{+ \quad\; -} \\
 & \qquad\qquad\qquad\; -5
\end{array}
$$

Here, the remainder is -5. Now, the zero of $x - 1$ is 1. So, putting $x = 1$ in $p(x)$, we see that

$$
\begin{aligned}
p(1) &= 3(1)^4 - 4(1)^3 - 3(1) - 1 \\
&= 3 - 4 - 3 - 1 \\
&= -5, \text{ which is the remainder.}
\end{aligned}
$$

Example 8 : Find the remainder obtained on dividing $p(x) = x^3 + 1$ by $x + 1$.

Solution : By long division,

$$
\begin{array}{rl}
 & x^2 - x + 1 \\ \cline{2-2}
x + 1 \,\big) & x^3 + 1 \\
 & x^3 + x^2 \\
 & \underline{- \quad\; -} \\
 & \quad -x^2 \qquad + 1 \\
 & \quad -x^2 - x \\
 & \quad \underline{+ \quad\; +} \\
 & \qquad\qquad x + 1 \\
 & \qquad\qquad x + 1 \\
 & \qquad\qquad \underline{- \quad -} \\
 & \qquad\qquad\; 0
\end{array}
$$

So, we find that the remainder is 0.

Here $p(x) = x^3 + 1$, and the root of $x + 1 = 0$ is $x = -1$. We see that

$$\begin{aligned} p(-1) &= (-1)^3 + 1 \\ &= -1 + 1 \\ &= 0, \end{aligned}$$

which is equal to the remainder obtained by actual division.

Is it not a simple way to find the remainder obtained on dividing a polynomial by a *linear polynomial*? We shall now generalise this fact in the form of the following theorem. We shall also show you why the theorem is true, by giving you a proof of the theorem.

Remainder Theorem : *Let $p(x)$ be any polynomial of degree greater than or equal to one and let a be any real number. If $p(x)$ is divided by the linear polynomial $x - a$, then the remainder is $p(a)$.*

Proof : Let $p(x)$ be any polynomial with degree greater than or equal to 1. Suppose that when $p(x)$ is divided by $x - a$, the quotient is $q(x)$ and the remainder is $r(x)$, i.e.,

$$p(x) = (x - a)\, q(x) + r(x)$$

Since the degree of $x - a$ is 1 and the degree of $r(x)$ is less than the degree of $x - a$, the degree of $r(x) = 0$. This means that $r(x)$ is a constant, say r.

So, for every value of x, $r(x) = r$.

Therefore, $$p(x) = (x - a)\, q(x) + r$$

In particular, if $x = a$, this equation gives us

$$\begin{aligned} p(a) &= (a - a)\, q(a) + r \\ &= r, \end{aligned}$$

which proves the theorem.

Let us use this result in another example.

Example 9 : Find the remainder when $x^4 + x^3 - 2x^2 + x + 1$ is divided by $x - 1$.

Solution : Here, $p(x) = x^4 + x^3 - 2x^2 + x + 1$, and the zero of $x - 1$ is 1.

So, $$\begin{aligned} p(1) &= (1)^4 + (1)^3 - 2(1)^2 + 1 + 1 \\ &= 2 \end{aligned}$$

So, by the Remainder Theorem, 2 is the remainder when $x^4 + x^3 - 2x^2 + x + 1$ is divided by $x - 1$.

Example 10 : Check whether the polynomial $q(t) = 4t^3 + 4t^2 - t - 1$ is a multiple of $2t + 1$.

Solution : As you know, $q(t)$ will be a multiple of $2t + 1$ only, if $2t + 1$ divides $q(t)$ leaving remainder zero. Now, taking $2t + 1 = 0$, we have $t = -\frac{1}{2}$.

Also, $$q\left(-\frac{1}{2}\right) = 4\left(-\frac{1}{2}\right)^3 + 4\left(-\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right) - 1 = -\frac{1}{2} + 1 + \frac{1}{2} - 1 = 0$$

So the remainder obtained on dividing $q(t)$ by $2t + 1$ is 0.

So, $2t + 1$ is a factor of the given polynomial $q(t)$, that is $q(t)$ is a multiple of $2t + 1$.

EXERCISE 2.3

1. Find the remainder when $x^3 + 3x^2 + 3x + 1$ is divided by

   (i) $x + 1$ (ii) $x - \frac{1}{2}$ (iii) x (iv) $x + \pi$ (v) $5 + 2x$

2. Find the remainder when $x^3 - ax^2 + 6x - a$ is divided by $x - a$.
3. Check whether $7 + 3x$ is a factor of $3x^3 + 7x$.

2.5 Factorisation of Polynomials

Let us now look at the situation of Example 10 above more closely. It tells us that since the remainder, $q\left(-\frac{1}{2}\right) = 0$, $(2t + 1)$ is a factor of $q(t)$, i.e., $q(t) = (2t + 1)\, g(t)$ for some polynomial $g(t)$. This is a particular case of the following theorem.

Factor Theorem : If $p(x)$ is a polynomial of degree $n \geq 1$ and a is any real number, then (i) $x - a$ is a factor of $p(x)$, if $p(a) = 0$, and (ii) $p(a) = 0$, if $x - a$ is a factor of $p(x)$.

Proof: By the Remainder Theorem, $p(x)=(x - a)\, q(x) + p(a)$.

(i) If $p(a) = 0$, then $p(x) = (x - a)\, q(x)$, which shows that $x - a$ is a factor of $p(x)$.

(ii) Since $x - a$ is a factor of $p(x)$, $p(x) = (x - a)\, g(x)$ for same polynomial $g(x)$. In this case, $p(a) = (a - a)\, g(a) = 0$.

Example 11 : Examine whether $x + 2$ is a factor of $x^3 + 3x^2 + 5x + 6$ and of $2x + 4$.

Solution : The zero of $x + 2$ is -2. Let $p(x) = x^3 + 3x^2 + 5x + 6$ and $s(x) = 2x + 4$

Then, $$p(-2) = (-2)^3 + 3(-2)^2 + 5(-2) + 6$$

$$= -8 + 12 - 10 + 6$$

$$= 0$$

So, by the Factor Theorem, $x + 2$ is a factor of $x^3 + 3x^2 + 5x + 6$.

Again, $s(-2) = 2(-2) + 4 = 0$

So, $x + 2$ is a factor of $2x + 4$. In fact, you can check this without applying the Factor Theorem, since $2x + 4 = 2(x + 2)$.

Example 12 : Find the value of k, if $x - 1$ is a factor of $4x^3 + 3x^2 - 4x + k$.

Solution : As $x - 1$ is a factor of $p(x) = 4x^3 + 3x^2 - 4x + k$, $p(1) = 0$

Now, $p(1) = 4(1)^3 + 3(1)^2 - 4(1) + k$

So, $4 + 3 - 4 + k = 0$

i.e., $k = -3$

We will now use the Factor Theorem to factorise some polynomials of degree 2 and 3. You are already familiar with the factorisation of a quadratic polynomial like $x^2 + lx + m$. You had factorised it by splitting the middle term lx as $ax + bx$ so that $ab = m$. Then $x^2 + lx + m = (x + a)(x + b)$. We shall now try to factorise quadratic polynomials of the type $ax^2 + bx + c$, where $a \neq 0$ and a, b, c are constants.

Factorisation of the polynomial $ax^2 + bx + c$ **by splitting the middle term** is as follows:

Let its factors be $(px + q)$ and $(rx + s)$. Then

$$ax^2 + bx + c = (px + q)(rx + s) = pr\,x^2 + (ps + qr)\,x + qs$$

Comparing the coefficients of x^2, we get $a = pr$.

Similarly, comparing the coefficients of x, we get $b = ps + qr$.

And, on comparing the constant terms, we get $c = qs$.

This shows us that b is the sum of two numbers ps and qr, whose product is $(ps)(qr) = (pr)(qs) = ac$.

Therefore, to factorise $ax^2 + bx + c$, we have to write b as the sum of two numbers whose product is ac. This will be clear from Example 13.

Example 13 : Factorise $6x^2 + 17x + 5$ by splitting the middle term, and by using the Factor Theorem.

Solution 1 : (By splitting method) **:** If we can find two numbers p and q such that $p + q = 17$ and $pq = 6 \times 5 = 30$, then we can get the factors.

So, let us look for the pairs of factors of 30. Some are 1 and 30, 2 and 15, 3 and 10, 5 and 6. Of these pairs, 2 and 15 will give us $p + q = 17$.

$$
\begin{aligned}
\text{So, } 6x^2 + 17x + 5 &= 6x^2 + (2 + 15)x + 5\\
&= 6x^2 + 2x + 15x + 5\\
&= 2x(3x + 1) + 5(3x + 1)\\
&= (3x + 1)(2x + 5)
\end{aligned}
$$

Solution 2 : (Using the Factor Theorem)

$6x^2 + 17x + 5 = 6\left(x^2 + \frac{17}{6}x + \frac{5}{6}\right) = 6\,p(x)$, say. If a and b are the zeroes of $p(x)$, then $6x^2 + 17x + 5 = 6(x - a)(x - b)$. So, $ab = \frac{5}{6}$. Let us look at some possibilities for a and b. They could be $\pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{5}{3}, \pm\frac{5}{2}, \pm 1$. Now, $p\left(\frac{1}{2}\right) = \frac{1}{4} + \frac{17}{6}\left(\frac{1}{2}\right) + \frac{5}{6} \neq 0$. But $p\left(\frac{-1}{3}\right) = 0$. So, $\left(x + \frac{1}{3}\right)$ is a factor of $p(x)$. Similarly, by trial, you can find that $\left(x + \frac{5}{2}\right)$ is a factor of $p(x)$.

$$
\begin{aligned}
\text{Therefore,}\qquad 6x^2 + 17x + 5 &= 6\left(x + \frac{1}{3}\right)\left(x + \frac{5}{2}\right)\\
&= 6\left(\frac{3x + 1}{3}\right)\left(\frac{2x + 5}{2}\right)\\
&= (3x + 1)(2x + 5)
\end{aligned}
$$

For the example above, the use of the splitting method appears more efficient. However, let us consider another example.

Example 14 : Factorise $y^2 - 5y + 6$ by using the Factor Theorem.

Solution : Let $p(y) = y^2 - 5y + 6$. Now, if $p(y) = (y - a)(y - b)$, you know that the constant term will be ab. So, $ab = 6$. So, to look for the factors of $p(y)$, we look at the factors of 6.

The factors of 6 are 1, 2 and 3.

Now, $\quad p(2) = 2^2 - (5 \times 2) + 6 = 0$

So, $y - 2$ is a factor of $p(y)$.

Also, $p(3) = 3^2 - (5 \times 3) + 6 = 0$

So, $y - 3$ is also a factor of $y^2 - 5y + 6$.

Therefore, $y^2 - 5y + 6 = (y - 2)(y - 3)$

Note that $y^2 - 5y + 6$ can also be factorised by splitting the middle term $-5y$.

Now, let us consider factorising cubic polynomials. Here, the splitting method will not be appropriate to start with. We need to find at least one factor first, as you will see in the following example.

Example 15 : Factorise $x^3 - 23x^2 + 142x - 120$.

Solution : Let $p(x) = x^3 - 23x^2 + 142x - 120$

We shall now look for all the factors of -120. Some of these are ±1, ±2, ±3, ±4, ±5, ±6, ±8, ±10, ±12, ±15, ±20, ±24, ±30, ±60.

By trial, we find that $p(1) = 0$. So $x - 1$ is a factor of $p(x)$.

Now we see that $x^3 - 23x^2 + 142x - 120 = x^3 - x^2 - 22x^2 + 22x + 120x - 120$

$$= x^2(x - 1) - 22x(x - 1) + 120(x - 1) \quad \text{(Why?)}$$

$$= (x - 1)(x^2 - 22x + 120) \quad \text{[Taking } (x - 1) \text{ common]}$$

We could have also got this by dividing $p(x)$ by $x - 1$.

Now $x^2 - 22x + 120$ can be factorised either by splitting the middle term or by using the Factor theorem. By splitting the middle term, we have:

$$x^2 - 22x + 120 = x^2 - 12x - 10x + 120$$

$$= x(x - 12) - 10(x - 12)$$

$$= (x - 12)(x - 10)$$

So, $$x^3 - 23x^2 - 142x - 120 = (x - 1)(x - 10)(x - 12)$$

EXERCISE 2.4

1. Determine which of the following polynomials has $(x + 1)$ a factor :

(i) $x^3 + x^2 + x + 1$ (ii) $x^4 + x^3 + x^2 + x + 1$

(iii) $x^4 + 3x^3 + 3x^2 + x + 1$ (iv) $x^3 - x^2 - \left(2 + \sqrt{2}\right)x + \sqrt{2}$

2. Use the Factor Theorem to determine whether $g(x)$ is a factor of $p(x)$ in each of the following cases:

(i) $p(x) = 2x^3 + x^2 - 2x - 1, g(x) = x + 1$

(ii) $p(x) = x^3 + 3x^2 + 3x + 1, g(x) = x + 2$

(iii) $p(x) = x^3 - 4x^2 + x + 6, g(x) = x - 3$

3. Find the value of k, if $x - 1$ is a factor of $p(x)$ in each of the following cases:

(i) $p(x) = x^2 + x + k$ (ii) $p(x) = 2x^2 + kx + \sqrt{2}$

(iii) $p(x) = kx^2 - \sqrt{2}x + 1$ (iv) $p(x) = kx^2 - 3x + k$

4. Factorise :

(i) $12x^2 - 7x + 1$ (ii) $2x^2 + 7x + 3$

(iii) $6x^2 + 5x - 6$ (iv) $3x^2 - x - 4$

5. Factorise :

(i) $x^3 - 2x^2 - x + 2$ (ii) $x^3 - 3x^2 - 9x - 5$

(iii) $x^3 + 13x^2 + 32x + 20$ (iv) $2y^3 + y^2 - 2y - 1$

2.6 Algebraic Identities

From your earlier classes, you may recall that an algebraic identity is an algebraic equation that is true for all values of the variables occurring in it. You have studied the following algebraic identities in earlier classes:

Identity I : $(x + y)^2 = x^2 + 2xy + y^2$

Identity II : $(x - y)^2 = x^2 - 2xy + y^2$

Identity III : $x^2 - y^2 = (x + y)(x - y)$

Identity IV : $(x + a)(x + b) = x^2 + (a + b)x + ab$

You must have also used some of these algebraic identities to factorise the algebraic expressions. You can also see their utility in computations.

Example 16 : Find the following products using appropriate identities:

(i) $(x + 3)(x + 3)$ (ii) $(x - 3)(x + 5)$

Solution : (i) Here we can use Identity I : $(x + y)^2 = x^2 + 2xy + y^2$. Putting $y = 3$ in it, we get

$$(x + 3)(x + 3) = (x + 3)^2 = x^2 + 2(x)(3) + (3)^2$$
$$= x^2 + 6x + 9$$

(ii) Using Identity IV above, i.e., $(x + a)(x + b) = x^2 + (a + b)x + ab$, we have

$$(x - 3)(x + 5) = x^2 + (-3 + 5)x + (-3)(5)$$
$$= x^2 + 2x - 15$$

Example 17 : Evaluate 105 × 106 without multiplying directly.

Solution :

$$\begin{aligned} 105 \times 106 &= (100 + 5) \times (100 + 6) \\ &= (100)^2 + (5 + 6)(100) + (5 \times 6), \text{ using Identity IV} \\ &= 10000 + 1100 + 30 \\ &= 11130 \end{aligned}$$

You have seen some uses of the identities listed above in finding the product of some given expressions. These identities are useful in factorisation of algebraic expressions also, as you can see in the following examples.

Example 18 : Factorise:

(i) $49a^2 + 70ab + 25b^2$ (ii) $\frac{25}{4}x^2 - \frac{y^2}{9}$

Solution : (i) Here you can see that

$$49a^2 = (7a)^2,\ 25b^2 = (5b)^2,\ 70ab = 2(7a)(5b)$$

Comparing the given expression with $x^2 + 2xy + y^2$, we observe that $x = 7a$ and $y = 5b$.

Using Identity I, we get

$$49a^2 + 70ab + 25b^2 = (7a + 5b)^2 = (7a + 5b)(7a + 5b)$$

(ii) We have $\frac{25}{4}x^2 - \frac{y^2}{9} = \left(\frac{5}{2}x\right)^2 - \left(\frac{y}{3}\right)^2$

Now comparing it with Identity III, we get

$$\begin{aligned} \frac{25}{4}x^2 - \frac{y^2}{9} &= \left(\frac{5}{2}x\right)^2 - \left(\frac{y}{3}\right)^2 \\ &= \left(\frac{5}{2}x + \frac{y}{3}\right)\left(\frac{5}{2}x - \frac{y}{3}\right) \end{aligned}$$

So far, all our identities involved products of binomials. Let us now extend the Identity I to a trinomial $x + y + z$. We shall compute $(x + y + z)^2$ by using Identity I.

Let $x + y = t$. Then,

$$\begin{aligned} (x + y + z)^2 &= (t + z)^2 \\ &= t^2 + 2tz + t^2 && \text{(Using Identity I)} \\ &= (x + y)^2 + 2(x + y)z + z^2 && \text{(Substituting the value of } t\text{)} \end{aligned}$$

$$= x^2 + 2xy + y^2 + 2xz + 2yz + z^2 \qquad \text{(Using Identity I)}$$
$$= x^2 + y^2 + z^2 + 2xy + 2yz + 2zx \qquad \text{(Rearranging the terms)}$$

So, we get the following identity:

Identity V : $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$

Remark : We call the right hand side expression **the expanded form** of the left hand side expression. Note that the expansion of $(x + y + z)^2$ consists of three square terms and three product terms.

Example 19 : Write $(3a + 4b + 5c)^2$ in expanded form.

Solution : Comparing the given expression with $(x + y + z)^2$, we find that

$$x = 3a,\ y = 4b \text{ and } z = 5c.$$

Therefore, using Identity V, we have

$$(3a + 4b + 5c)^2 = (3a)^2 + (4b)^2 + (5c)^2 + 2(3a)(4b) + 2(4b)(5c) + 2(5c)(3a)$$
$$= 9a^2 + 16b^2 + 25c^2 + 24ab + 40bc + 30ac$$

Example 20 : Expand $(4a – 2b – 3c)^2$.

Solution : Using Identity V, we have

$$(4a – 2b – 3c)^2 = [4a + (–2b) + (–3c)]^2$$
$$= (4a)^2 + (–2b)^2 + (–3c)^2 + 2(4a)(–2b) + 2(–2b)(–3c) + 2(–3c)(4a)$$
$$= 16a^2 + 4b^2 + 9c^2 – 16ab + 12bc – 24ac$$

Example 21 : Factorise $4x^2 + y^2 + z^2 – 4xy – 2yz + 4xz$.

Solution : We have $4x^2 + y^2 + z^2 – 4xy – 2yz + 4xz = (2x)^2 + (–y)^2 + (z)^2 + 2(2x)(–y)$

$$+ 2(–y)(z) + 2(2x)(z)$$
$$= [2x + (–y) + z]^2 \qquad \text{(Using Identity V)}$$
$$= (2x – y + z)^2 = (2x – y + z)(2x – y + z)$$

So far, we have dealt with identities involving second degree terms. Now let us extend Identity I to compute $(x + y)^3$. We have:

$$(x + y)^3 = (x + y)(x + y)^2$$
$$= (x + y)(x^2 + 2xy + y^2)$$
$$= x(x^2 + 2xy + y^2) + y(x^2 + 2xy + y^2)$$
$$= x^3 + 2x^2y + xy^2 + x^2y + 2xy^2 + y^3$$
$$= x^3 + 3x^2y + 3xy^2 + y^3$$
$$= x^3 + y^3 + 3xy(x + y)$$

So, we get the following identity:

Identity VI : $(x + y)^3 = x^3 + y^3 + 3xy\,(x + y)$

Also, by replacing y by $-y$ in the Identity VI, we get

Identity VII : $(x - y)^3 = x^3 - y^3 - 3xy(x - y)$

$$= x^3 - 3x^2y + 3xy^2 - y^3$$

Example 22 : Write the following cubes in the expanded form:

(i) $(3a + 4b)^3$ (ii) $(5p - 3q)^3$

Solution : (i) Comparing the given expression with $(x + y)^3$, we find that

$$x = 3a \text{ and } y = 4b.$$

So, using Identity VI, we have:

$$\begin{aligned}(3a + 4b)^3 &= (3a)^3 + (4b)^3 + 3(3a)(4b)(3a + 4b)\\ &= 27a^3 + 64b^3 + 108a^2b + 144ab^2\end{aligned}$$

(ii) Comparing the given expression with $(x - y)^3$, we find that

$$x = 5p,\ y = 3q.$$

So, using Identity VII, we have:

$$\begin{aligned}(5p - 3q)^3 &= (5p)^3 - (3q)^3 - 3(5p)(3q)(5p - 3q)\\ &= 125p^3 - 27q^3 - 225p^2q + 135pq^2\end{aligned}$$

Example 23 : Evaluate each of the following using suitable identities:

(i) $(104)^3$ (ii) $(999)^3$

Solution : (i) We have

$$\begin{aligned}(104)^3 &= (100 + 4)^3\\ &= (100)^3 + (4)^3 + 3(100)(4)(100 + 4)\end{aligned}$$

(Using Identity VI)

$$\begin{aligned}&= 1000000 + 64 + 124800\\ &= 1124864\end{aligned}$$

(ii) We have

$$\begin{aligned}(999)^3 &= (1000 - 1)^3\\ &= (1000)^3 - (1)^3 - 3(1000)(1)(1000 - 1)\end{aligned}$$

(Using Identity VII)

$$\begin{aligned}&= 1000000000 - 1 - 2997000\\ &= 997002999\end{aligned}$$

Example 24 : Factorise $8x^3 + 27y^3 + 36x^2y + 54xy^2$

Solution : The given expression can be written as

$$(2x)^3 + (3y)^3 + 3(4x^2)(3y) + 3(2x)(9y^2)$$
$$= (2x)^3 + (3y)^3 + 3(2x)^2(3y) + 3(2x)(3y)^2$$
$$= (2x + 3y)^3 \qquad \text{(Using Identity VI)}$$
$$= (2x + 3y)(2x + 3y)(2x + 3y)$$

Now consider $(x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

On expanding, we get the product as

$$x(x^2 + y^2 + z^2 - xy - yz - zx) + y(x^2 + y^2 + z^2 - xy - yz - zx)$$
$$+ z(x^2 + y^2 + z^2 - xy - yz - zx) = x^3 + xy^2 + xz^2 - x^2y - xyz - zx^2 + x^2y$$
$$+ y^3 + yz^2 - xy^2 - y^2z - xyz + x^2z + y^2z + z^3 - xyz - yz^2 - xz^2$$
$$= x^3 + y^3 + z^3 - 3xyz \qquad \text{(On simplification)}$$

So, we obtain the following identity:

Identity VIII : $\boldsymbol{x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)}$

Example 25 : Factorise : $8x^3 + y^3 + 27z^3 - 18xyz$

Solution : Here, we have

$$8x^3 + y^3 + 27z^3 - 18xyz$$
$$= (2x)^3 + y^3 + (3z)^3 - 3(2x)(y)(3z)$$
$$= (2x + y + 3z)[(2x)^2 + y^2 + (3z)^2 - (2x)(y) - (y)(3z) - (2x)(3z)]$$
$$= (2x + y + 3z)(4x^2 + y^2 + 9z^2 - 2xy - 3yz - 6xz)$$

EXERCISE 2.5

1. Use suitable identities to find the following products:

(i) $(x + 4)(x + 10)$ (ii) $(x + 8)(x - 10)$ (iii) $(3x + 4)(3x - 5)$

(iv) $(y^2 + \frac{3}{2})(y^2 - \frac{3}{2})$ (v) $(3 - 2x)(3 + 2x)$

2. Evaluate the following products without multiplying directly:

(i) 103×107 (ii) 95×96 (iii) 104×96

3. Factorise the following using appropriate identities:

(i) $9x^2 + 6xy + y^2$ (ii) $4y^2 - 4y + 1$ (iii) $x^2 - \frac{y^2}{100}$

4. Expand each of the following, using suitable identities:

(i) $(x + 2y + 4z)^2$ (ii) $(2x - y + z)^2$ (iii) $(-2x + 3y + 2z)^2$

(iv) $(3a - 7b - c)^2$ (v) $(-2x + 5y - 3z)^2$ (vi) $\left[\frac{1}{4}a - \frac{1}{2}b + 1\right]^2$

5. Factorise:

(i) $4x^2 + 9y^2 + 16z^2 + 12xy - 24yz - 16xz$

(ii) $2x^2 + y^2 + 8z^2 - 2\sqrt{2}\,xy + 4\sqrt{2}\,yz - 8xz$

6. Write the following cubes in expanded form:

(i) $(2x + 1)^3$ (ii) $(2a - 3b)^3$ (iii) $\left[\frac{3}{2}x + 1\right]^3$ (iv) $\left[x - \frac{2}{3}y\right]^3$

7. Evaluate the following using suitable identities:

(i) $(99)^3$ (ii) $(102)^3$ (iii) $(998)^3$

8. Factorise each of the following:

(i) $8a^3 + b^3 + 12a^2b + 6ab^2$ (ii) $8a^3 - b^3 - 12a^2b + 6ab^2$

(iii) $27 - 125a^3 - 135a + 225a^2$ (iv) $64a^3 - 27b^3 - 144a^2b + 108ab^2$

(v) $27p^3 - \frac{1}{216} - \frac{9}{2}p^2 + \frac{1}{4}p$

9. Verify : (i) $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$ (ii) $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

10. Factorise each of the following:

(i) $27y^3 + 125z^3$ (ii) $64m^3 - 343n^3$

[**Hint :** See Question 9.]

11. Factorise : $27x^3 + y^3 + z^3 - 9xyz$

12. Verify that $x^3 + y^3 + z^3 - 3xyz = \frac{1}{2}(x + y + z)\left[(x - y)^2 + (y - z)^2 + (z - x)^2\right]$

13. If $x + y + z = 0$, show that $x^3 + y^3 + z^3 = 3xyz$.

14. Without actually calculating the cubes, find the value of each of the following:

(i) $(-12)^3 + (7)^3 + (5)^3$

(ii) $(28)^3 + (-15)^3 + (-13)^3$

15. Give possible expressions for the length and breadth of each of the following rectangles, in which their areas are given:

Area : $25a^2 - 35a + 12$	Area : $35y^2 + 13y - 12$
(i)	(ii)

16. What are the possible expressions for the dimensions of the cuboids whose volumes are given below?

(i)	(ii)
Volume : $3x^2 - 12x$	Volume : $12ky^2 + 8ky - 20k$

2.7 Summary

In this chapter, you have studied the following points:

1. A *polynomial* $p(x)$ in one variable x is an algebraic expression in x of the form

$p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$,

where $a_0, a_1, a_2, \ldots, a_n$ are constants and $a_n \neq 0$.

$a_0, a_1, a_2, \ldots, a_n$ are respectively the *coefficients* of $x^0, x, x^2, \ldots, x^n$, and n is called the *degree of the polynomial*. Each of $a_n x^n$, $a_{n-1} x^{n-1}$, ..., a_0, with $a_n \neq 0$, is called a *term* of the polynomial $p(x)$.

2. A polynomial of one term is called a monomial.

3. A polynomial of two terms is called a binomial.

4. A polynomial of three terms is called a trinomial.

5. A polynomial of degree one is called a linear polynomial.

6. A polynomial of degree two is called a quadratic polynomial.

7. A polynomial of degree three is called a cubic polynomial.

8. A real number 'a' is a *zero* of a polynomial $p(x)$ if $p(a) = 0$. In this case, a is also called a *root* of the equation $p(x) = 0$.

9. Every linear polynomial in one variable has a unique zero, a non-zero constant polynomial has no zero, and every real number is a zero of the zero polynomial.

10. Remainder Theorem : If $p(x)$ is any polynomial of degree greater than or equal to 1 and $p(x)$ is divided by the linear polynomial $x - a$, then the remainder is $p(a)$.

11. Factor Theorem : $x - a$ is a factor of the polynomial $p(x)$, if $p(a) = 0$. Also, if $x - a$ is a factor of $p(x)$, then $p(a) = 0$.

12. $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$

13. $(x + y)^3 = x^3 + y^3 + 3xy(x + y)$

14. $(x - y)^3 = x^3 - y^3 - 3xy(x - y)$

15. $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$