Question 1

Answer all the following questions. Each question stands by itself and is unrelated to the others.

a) Let l_1 be a line that passes through points (1,2,4) and (2,2,6) and l_2 be a line that passes through points (1,3,7) and (5,2,8). Find if these lines intersect, and if so find the intersection point.

b) Given the sphere equation:

$$S: (x-2)^2 + (y+1)^2 + (z-3)^2 - 61 = 0$$

b1) Describe what is the shape of all points on the surface of the sphere whose z value is -3?

b2) How many points in the shape defined in b1 have a y value 2?

b3) Choose one point from b2 (if any) and find an implicit representation of the plane that passes through the point and is tangent to the sphere

c) Let $l = (3,2,4) + t(2,2,5)$ be a line. Find the projection of $p = (1,-2,1)$ on the line.

d) Find if the lines $l_1 = (3,-1,-1) + t(0,1,2)$ and $l_2 = (1,1,1) + s(4,-1,2)$ are on the same plane, and if so, find the implicit representation of that plane.

e) Find the parametric representation of a plane that passes through the point (0,1,5) and is parallel to the xy plane.

f) A line lies on the plane $x + 2y - 8z + 1 = 0$. Its direction is $(0, a, 1)$. For some value $a \in \mathbb{R}$.
   a. Is it possible to determine the parametric representation of the line? Find it or prove it's not possible.
   b. If we also know that the line passes through the point $(3,-10,-2)$. Can you find the parametric representation of the line now? Find it or prove it's not possible.

Solution

a)

$$l_1 = (1,2,4) + t(1,0,2)$$
$$l_2 = (1,3,7) + s(4,-1,1)$$

We need to find where $l_1 = l_2$

$$(1,2,4) + t(1,0,2) = (1,3,7) + s(4,-1,1)$$

Thus, we need to solve the following system of equations

$$1 + t = 1 + 4s$$
$$2 = 3 - s$$
$$4 + 2t = 7 + s$$

From the second equation we get that $s = 1$

From the first equation we get: $1 + t = 1 + 4 \rightarrow t = 4$

Checking the third equation we get: $4 + 2 \cdot 4 = 12 \neq 7 + 1 = 8$

Thus, there is no intersection between the lines.

b)

b1) All points that lie on the sphere must fulfil the sphere equation, and given that the z value of the points in our shape is -3, all these points must satisfy the equation:

$$(x-2)^2 + (y+1)^2 + (-3-3)^2 - 61 = 0$$

Or

$$(x-2)^2 + (y+1)^2 - 25 = 0$$

Which is a circle equation, meaning these points create a circle.

b2) Given that the y coordinate must be 2 we insert to the equation and get:

$$(x-2)^2+(2+1)^2-25=(x-2)^2-16=0$$

This is a quadratic equation with two solutions: $x-2=4$ and $x-2=-4$
Meaning there are only two points that have the y value of 2 and they are:
$(6,2,-3)$ and $(-2,2,-3)$

b3) We find the plane that passes through the point $q=(6,2,-3)$ and is tangent to the sphere. The vector connecting the point q and the center of the sphere $p_0=(2,-1,3)$ must be perpendicular to this plane, hence it is normal (not with length 1) to the plane.
This vector is

$$(6,2,-3)-(2,-1,3)=(4,3,-6)$$

So now we can write the (un-normalized) plane equation as:

$$4x+3y-6z+d=0$$

But we know that the point $q=(6,2,-3)$ lies on the plane so we can insert it to the equation and get d:

$$4\cdot 6+3\cdot 2-6\cdot(-3)+d=0\rightarrow d=-48$$

So the plane is

$$4x+3y-6z-48=0$$

c)

Let $u=(2,2,5)$. $u'=\frac{u}{\|u\|}=\frac{1}{\sqrt{33}}(2,2,5)$. Denote $q=(3,2,4)$. The projection is thus $q+\big((p-q)u'\big)u'=(3,2,4)+\left(\big((1,-2,1)-(3,2,4)\big)\cdot\frac{1}{\sqrt{33}}(2,2,5)\right)\frac{1}{\sqrt{33}}(2,2,5)=(3,2,4)+\left((-2,-4,-3)\cdot\frac{1}{\sqrt{33}}(2,2,5)\right)\frac{1}{\sqrt{33}}(2,2,5)=(3,2,4)-\frac{27}{33}(2,2,5)=\left(\frac{15}{11},\frac{4}{11},-\frac{1}{11}\right)$

d)

$$3=1+4s$$
$$-1+t=1-s$$
$$-1+2t=1+2s$$

From the first equation $s=\frac{1}{2}$, and so by the second equation

$$-1+t=1-\frac{1}{2}\rightarrow t=\frac{3}{2}$$

Check: $-1+2\cdot\frac{3}{2}=2=1+1=1+2\cdot\frac{1}{2}$
Thus, the two lines intersect and are on a plane.

$$u=(0,1,2)$$
$$v=(4,-1,2)$$
$$u'=\frac{u}{\|u\|}=\frac{1}{\sqrt{5}}(0,1,2)$$
$$v'=\frac{v}{\|v\|}=\frac{1}{\sqrt{21}}(4,-1,2)$$
$$u'\times v'=\begin{bmatrix}u_2v_3-u_3v_2\\u_3v_1-u_1v_3\\u_1v_2-u_2v_1\end{bmatrix}=\frac{1}{\sqrt{105}}\begin{bmatrix}2+2\\8-0\\(0-4)\end{bmatrix}=\frac{1}{\sqrt{105}}\begin{bmatrix}4\\8\\-4\end{bmatrix}$$

Thus the plane is $\frac{4}{\sqrt{105}}x + \frac{8}{\sqrt{105}}y - \frac{4}{\sqrt{105}}z + d = 0$.

Point (1,1,1) lies on one of the lines so also on the plane so it must fulfil the equation:

$$\frac{4}{\sqrt{105}} + \frac{8}{\sqrt{105}} - \frac{4}{\sqrt{105}} + d = 0 \to d = -\frac{8}{\sqrt{105}}$$

Thus, the parametric representation of the plane:

$$\frac{4}{\sqrt{105}} + \frac{8}{\sqrt{105}} - \frac{4}{\sqrt{105}} - \frac{8}{\sqrt{105}} = 0$$

e) Since the plane is parallel to the xy axis we know that $(0,0,1)$ is perpendicular to it. Thus, the implicit representation of the plane is $z + d = 0$.

$$5 + d = 0 \to d = -5$$

We can thus say that the plane is $z - 5 = 0$.

We will find two lines on the plane to get its parametric representation:$(0,1,5) + t(1,2,0)$ and $(0,1,5) + s(4,7,0)$. Both of these line lie on the plane as for each point on them it holds that $z - 5 = 0$.

The parametric representation of the plane is:

$$(0,1,5) + t(1,2,0) + s(4,7,0)$$

(Note: any two lines where the direction vector has 0 in their z-coordinate would work)

f)

a. Since we do not know a point the line passes through, we can stretch the line from any point. As the direction is $(0, a, 1)$ and it's on the plane it must hold that $2y - 8z = -x - 1$. There is no advancement in the x direction, thus there is a line for each value of x. There are infinite such lines, and so we cannot determine the line equation. We can find a as the line must be perpendicular to the normal vector to the plane. Thus:

$$(0, a, 1)(1,2,-8) = 0 \to 2a - 8 = 0 \to a = 4$$

b. When we have a point that lies on the line we can find a in a different way:
We will define the line $(3, -10, -2) + t(0, a, 1)$. From the plane equation we know that

$$3 + 2(-10 + ta) - 8(-2 + t) + 1 = 0$$
$$2ta = 8t$$

So, for $t \neq 0$ we get $a = 4$.
Anyway the line is $(3, -10, -2) + t(0,4,1)$.

<u>Question 2</u>

a. Let l be a line, p_1 and p_2 be two points and let p_1' and p_2' be their projection on l, respectively. Prove <u>in two different ways</u> that $\|p_1' - p_2'\|_2 \leq \|p_1 - p_2\|_2$
Hint: algebraic and geometric.

b. Let p_1 and p_2 be two points and let p_1' and p_2' be their projection on a 2D plane π. Prove or disprove: $\|p_1' - p_2'\| \leq \|p_1 - p_2\|$. You may assume that π is given by the unit normal n and a point q.

<u>Solution</u>

a.

First way:

Denote:

$$l: q + vt$$

Where q is a point on the line and v is a unit vector.

From the definition of projection:

$$p_1' = q + \big((p_1 - q) \cdot v\big)v$$
$$p_2' = q + \big((p_2 - q) \cdot v\big)v$$

$$\begin{aligned}
\|p_1' - p_2'\| &= \left\|q + \big((p_1 - q) \cdot v\big)v - \big(q + \big((p_2 - q) \cdot v\big)v\big)\right\|_2 \\
&= \left\|\big((p_1 - q) \cdot v\big)v - \big((p_2 - q) \cdot v\big)v\right\|_2 \\
&= \left\|\Big(\big((p_1 - q) \cdot v\big) - \big((p_2 - q) \cdot v\big)\Big)v\right\|_2 \\
&= \left|\big((p_1 - q) \cdot v\big) - \big((p_2 - q) \cdot v\big)\right| \cdot \|v\|_2 \\
&= \left|\big((p_1 - q) \cdot v\big) - \big((p_2 - q) \cdot v\big)\right| \cdot \|v\|_2 \\
&= \left|\big((p_1 - q) - (p_2 - q)\big) \cdot v\right| \cdot \|v\|_2 = |(p_1 - p_2) \cdot v| \cdot \|v\|_2 \\
&= \big|\|p_1 - p_2\|_2 \cdot \|v\|_2 cos\theta\big| \underset{|cos\theta| \le 1}{\le} \|p_1 - p_2\|_2
\end{aligned}$$

Second way:

Let l be a line, p_1 and p_2 be two points and let p_1' and p_2' be their projection on l, respectively.

Note that the lines $p_1' - p_1$ and $p_2' - p_2$ are parallel, and that l is perpendicular to both. By definition of the distance between parallel lines, it is the length of the segment between them of a line that is perpendicular to both, and it is shorter than the distance between any two points on the lines. Hence, $\|p_1' - p_2'\|_2$ is the distance and it holds that $\|p_1' - p_2'\|_2 \le \|p_1 - p_2\|$.

b.

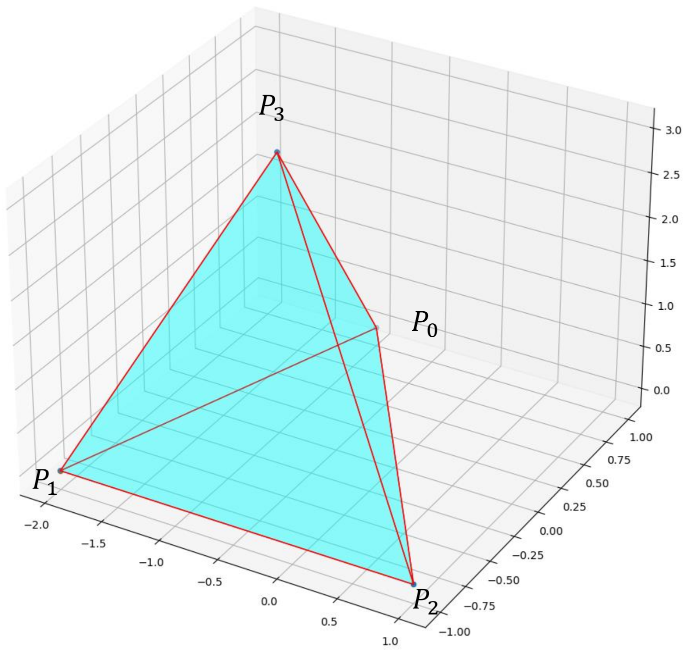
We notice that the projection of the line that connects p_1 and p_2 is the line that connects p_1' and p_2'. From there it holds by what we proved in (a). We can also repeat the second way from here.

Question 3

A tetrahedral is given by the points:

$$P_0 = (-1,1,0), P_1 = (-2,-1,0), P_2 = (1,-1,0), P_3 = (-1,0,3)$$

This is s sketch of the shape (this is only an illustration):



a. Find the outwards facing unit normal vector of each face of the tetrahedral.
b. Find the implicit representation of the planes the contain each of the tetrahedral faces.
c. For each of the following points find if it is inside or outside the tetrahedral
   i. $(-1,0,1)$
   ii. $(-3,1,1)$
   iii. (0, 0.5, 2)

Solution

a.

$$n_1 = (P_2 - P_0) \times (P_1 - P_0) = (2,-2,0) \times (-1,-2,0) = \begin{bmatrix} 0-0 \\ 0-0 \\ -4-2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -6 \end{bmatrix}$$

$$\widehat{n_1} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$n_2 = (P_1 - P_0) \times (P_3 - P_0) = (-1,-2,0) \times (0,-1,3) = \begin{bmatrix} -6-0 \\ 0+3 \\ 1-0 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \\ 1 \end{bmatrix}$$

$$\hat{n}_2 = \frac{1}{\sqrt{46}} \begin{bmatrix} -6 \\ 3 \\ 1 \end{bmatrix}$$

$$n_3 = (P_3 - P_0) \times (P_2 - P_0) = (0,-1,3) \times (2,-2,0) = \begin{bmatrix} 0+6 \\ 6-0 \\ 0+2 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 2 \end{bmatrix}$$

$$\hat{n}_3 = \frac{1}{\sqrt{19}} \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$$

$$n_4 = (P_2 - P_1) \times (P_3 - P_1) = (3,0,0) \times (1,1,3) = \begin{bmatrix} 0-0 \\ 0-9 \\ 3-0 \end{bmatrix} = \begin{bmatrix} 0 \\ -9 \\ 3 \end{bmatrix}$$

$$\hat{n}_4 = \frac{1}{\sqrt{10}} \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$$

b.

For π_1

$$0x + 0y - 1z + d = 0$$
$$-1 \cdot 0 + d = 0$$
$$\pi_1 : -z = 0$$

For π_2

$$\frac{-6x + 3y + z + d}{\sqrt{46}} = 0$$
$$-6 \cdot -1 + 3 \cdot 1 + 0 + d = 0$$
$$d = -9$$
$$\pi_2 : \frac{-6x + 3y + z - 9}{\sqrt{46}} = 0$$

For π_3

$$\frac{3x + 3y + z + d}{\sqrt{19}} = 0$$
$$3 \cdot -1 + 3 \cdot 1 + 0 + d = 0$$
$$d = 0$$
$$\pi_3 : \frac{3x + 3y + z}{\sqrt{19}} = 0$$

For π_4

$$\frac{0x - 3y + z + d}{\sqrt{10}} = 0$$
$$0 \cdot 1 - 3 \cdot -1 + 0 + d = 0$$
$$d = -3$$
$$\pi_4 : \frac{-3y + z - 3}{\sqrt{10}} = 0$$

c.

i.

Distance from π_1: $-1 < 0$

Distance from π_2: $-6 \cdot -1 + 3 \cdot 0 + 1 - 9 = -2 < 0$

Distance from π_3: $3 \cdot -1 + 3 \cdot 0 + 1 = -2 < 0$

Distance from π_4: $-3 \cdot 0 + 1 - 4 = -3 < 0$

Since the distance is negative from all the planes containing the sides of the polygon, the point is inside the polygon.

ii.

Distance from π_2: $-6 \cdot -3 + 3 \cdot 1 + 1 - 9 = 13 > 0$

The distance is larger than 0, thus it is outside the tetrahedral.

iii.

Distance from π_3: $3 \cdot 0 + 3 \cdot 0.5 + 2 = 3.5 > 0$

The distance is larger than 0, thus it is outside the tetrahedral.