Exercise 4 in Introduction to Computer Graphics 2023

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Reminder: This exercise is submitted individually.

Question 1

Write the matrices representing the following transformations. For each transformation write its specific type (linear/rigid/similarity/affine/projective)

a. Scale uniformly by 15, then rotate by 20° counterclockwise around the x-axis.
b. Translate by the vector (2,1,3), rotate by 60° counterclockwise around the line $l(t) = (1,-2,1) + t\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$ and scale by a factor of 2 along the x-axis
c. Scale by 3 in the direction of $(-4,1,0)$ then shear by a factor of 0.5 in the Z direction (meaning after the shearing $x' = x + 0.5z$, $y' = y + 0.5z$)
d. Reflect around the YZ plain and project (cabinet projection) on the XY plane with an angle of 30°

Solution

a.

$$T_1 = \begin{bmatrix} 15 & 0 & 0 & 0 \\ 0 & 15 & 0 & 0 \\ 0 & 0 & 15 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(60°) & -\sin(60°) & 0 \\ 0 & -\sin(60°) & \cos(60°) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformation is a similarity transformation (linear is also accepted) and equals:

$$T_2 \cdot T_1$$

b.

$$T_1 = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2 = \begin{bmatrix} \cos(45°) & 0 & \sin(45°) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(45°) & 0 & \cos(45°) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2^{-1} = \begin{bmatrix} \cos(-45°) & 0 & \sin(-45°) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(-45°) & 0 & \cos(-45°) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_3 = \begin{bmatrix} \cos(60°) & -\sin(60°) & 0 & 0 \\ \sin(60°) & \cos(60°) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_4 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformation is affine and equals:

$$T_4 \cdot T_2^{-1} T_3 T_2 T_1$$

c.

First, we need to align $(-4,1,0)$ with one of the axes.

$$v_1 = \frac{(-4,1,0)}{\|(-4,1,0)\|} = \frac{1}{\sqrt{17}}(-4,1,0)$$

$$v_2 = \frac{\frac{1}{\sqrt{17}}(-4,1,0) \times (1,0,0)}{\left\|\frac{1}{\sqrt{17}}(-4,1,0) \times (1,0,0)\right\|} = (0,0,-1)$$

$$v_3 = \frac{v_1 \times v_2}{\|v_1 \times v_2\|} = \frac{\frac{1}{\sqrt{17}}(-4,1,0) \times (0,0,-1)}{\left\|\frac{1}{\sqrt{17}}(-4,1,0 \times (0,0,-1)\right\|} = \frac{1}{\sqrt{17}}(-1,-4,0)$$

We will align the vector with the x axis using the matrix:

$$T_1 = \begin{bmatrix} -\frac{4}{\sqrt{17}} & \frac{1}{\sqrt{17}} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -\frac{1}{\sqrt{17}} & -\frac{4}{\sqrt{17}} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_1^{-1} = \begin{bmatrix} -\frac{4}{\sqrt{17}} & 0 & -\frac{1}{\sqrt{17}} & 0 \\ \frac{1}{\sqrt{17}} & 0 & -\frac{4}{\sqrt{17}} & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Scaling the x-axis:

$$T_2 = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Shearing in the Z direction:

$$T_3 = \begin{bmatrix} 1 & 0 & 0.5 & 0 \\ 0 & 1 & 0.5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformation is linear and equals:

$$T_3 \cdot T_1^{-1} \cdot T_2 \cdot T_1$$

d.

$$T_1 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2 = \begin{bmatrix} 1 & 0 & 0.5\cos(45°) & 0 \\ 0 & 1 & 0.5\sin(45°) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformation is linear and equals:

$$T_2 \cdot T_1$$

Question 2

In this question we will prove that shear transformations in two dimensions with the operation of composition are a group.

a. Closure: Let A, B be two shearing transformations in direction $v = (x, y)$, prove that $A \circ B$ is also a shearing in direction v
b. Associativity: Let A, B, C be three shearing transformations in direction $v = (x, y)$, prove that $(A \circ B) \circ C = A \circ (B \circ C)$
c. Identity element: Prove that there exists a shearing transformation in direction $v = (x, y)$, S_{id} s.t. for all shearing transformation in direction v, A, it holds that $A \circ S_{id} = S_{id} \circ A = A$
d. Inverse element: Prove that for every shearing transformation in direction $v = (x, y)$, A, there exists A^{-1} which is also a shearing in the same direction s.t. $A \cdot A^{-1} = A^{-1} \cdot A = S_{id}$. Given A, what is A^{-1}?
e. Are shearing transformations in the same direction commutative? That is, Let A, B be two shearing transformations in direction $v = (x, y)$, does it hold that $A \circ B = B \circ A$

Solution

a. In order to shear in direction $v = (x, y)$ we first must align it with one of the axes, let us say the x-axis. To do that we need a rotation R around the origin. As we know rotations to be invertible, there exists R^{-1}. Thus, we can represent A as $R^{-1}A'R$ where $A' = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and a is the factor by which A shears. Similarly, $B = R^{-1}B'R$ where $B' = \begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and b is the factor by which B shears. Their composition

$$A \circ B = R^{-1}A'RR^{-1}B'R = R^{-1}A'B'R = R^{-1}\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}R$$
$$= R^{-1}\begin{bmatrix} 1 & 0 & 0 \\ a+b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}R$$

This is clearly a shearing transformation with factor $a+b$.

b. The composition of transformation is equivalent to the multiplication of the matrix representing them. As matrix multiplication is associative, so is this composition and it holds that that $(A \circ B) \circ C = A \circ (B \circ C)$

c. The identity matrix I represents a shearing transformation by a factor of 0. By the definition of matrix multiplication $I = S_{id}$ as defined in the question.

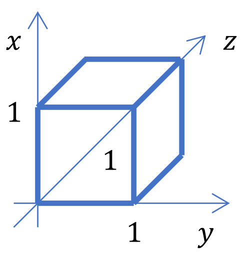
d. To shear in direction $v = (x, y)$ we first must align it with one of the axes, let us say the x-axis. To do that we need a rotation R around the origin. As we know rotations to be invertible, there exists R^{-1}. Thus, we can represent A as $R^{-1}A'R$ where $A' = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and a is the factor by which A shears. Define $A^{-1} = R^{-1}\begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}R$. As proven above $A \circ A^{-1} = R^{-1}\begin{bmatrix} 1 & 0 & 0 \\ a-a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}R = R^{-1}IR = R^{-1}R = I = S_{id}$.

e. To shear in direction $v = (x, y)$ we first must align it with one of the axes, let us say the x-axis. To do that we need a rotation R around the origin. As we know rotations to be invertible, there exists R^{-1}. Thus, we can represent A as $R^{-1}A'R$ where $A' = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and a is the factor by which A shears. Similarly, $B = R^{-1}B'R$ where $B' = \begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and b is the factor by which B shears. As proven above $A \circ B = R^{-1}\begin{bmatrix} 1 & 0 & 0 \\ a+b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}R =$

$$R^{-1}\begin{bmatrix} 1 & 0 & 0 \\ b+a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}R = R^{-1}\begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}R =$$
$$R^{-1}\begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}RR^{-1}\begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}R = R^{-1}B'RR^{-1}A'R = B \circ A$$

Thus, they are commutative.

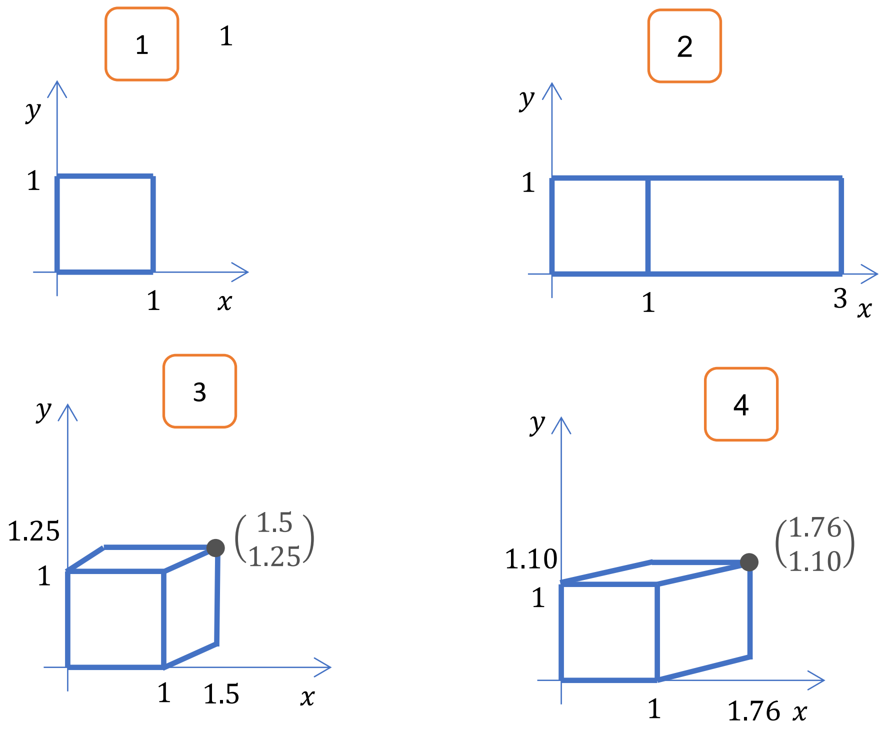
Question 3



Given a cube with vertices at:

$(0,0,0), (0,0,1), (1,0,0), (0,1,0), (1,1,0), (1,0,1), (0,1,1), (1,1,1)$

1. For each of the following diagrams, find the parallel projection matrix that will project the cube upon the given shape, in the xy plane (in which $z = 0$).



2. Answer, for each of the parallel projections you found in (1) : We use this parallel projection to project two points : (3,4,3), (3,4,6). What is the distance between the projected points? Answer <u>without</u> calculating the projected points.

<u>Solution</u>

1.

For 1:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For 2:

Note that $(1,1,1)$ was projected to $(3,1,0)$thus:

$$\begin{bmatrix} 1 & 0 & a\cos(\phi) & 0 \\ 0 & 1 & a\sin(\phi) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

So:

$$1 + \frac{\cos(\phi)}{\tan(\alpha)} = 3$$

$$1 + \frac{\sin(\phi)}{\tan(\alpha)} = 1$$

$$\tan(\alpha) = \frac{\cos(\phi)}{2}$$

$$\frac{\sin(\phi)}{\tan(\alpha)} = 0$$

Thus:

$$\sin(\phi) = 0$$

And $\phi = 0, \tan(\alpha) = \frac{\cos(0)}{2} = \frac{1}{2}$

$$\alpha = 26.57$$

The projective matrix:

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For 3:

Note that $(1,1,1)$ was projected to $(1.5, 1.25,0)$ thus:

Thus:

$$\begin{bmatrix} 1 & 0 & a\cos(\phi) & 0 \\ 0 & 1 & a\sin(\phi) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 1.25 \\ 0 \\ 1 \end{bmatrix}$$

$$1 + a\cos(\phi) = 1.5$$

$$a = \frac{1}{2\cos(\phi)}$$

$$1 + a\sin(\phi) = 1.25$$

$$1 + \frac{\sin(\phi)}{2\cos(\phi)} = 1.25$$

$$\frac{1}{2}\tan(\phi) = 0.25$$

$$\phi = 26.56$$

$$a = \frac{1}{\tan(\alpha)} = \frac{1}{2\cos(\phi)} = 0.56$$

$$\begin{bmatrix} 1 & 0 & 0.56\cos(26.56) & 0 \\ 0 & 1 & 0.56\ \sin(26.56) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0.5 & 0 \\ 0 & 1 & 0.25 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For 4:

Note that $(1,1,1)$ was projected to $(1.76,1.1,0)$ thus:

$$\begin{bmatrix} 1 & 0 & a\cos(\phi) & 0 \\ 0 & 1 & a\sin(\phi) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.76 \\ 1.1 \\ 0 \\ 1 \end{bmatrix}$$

$$1 + a\cos(\phi) = 1.76$$

$$a = \frac{0.76}{\cos(\phi)}$$

$$1 + a\sin(\phi) = 1.1$$

$$1 + \frac{0.76}{\cos(\phi)}\sin(\phi) = 1.1$$

$$0.76\tan(\phi) = 0.1$$

$$\phi = 7.49$$

$$a = \frac{0.76}{\cos(7.49)} = 0.77$$

$$\begin{bmatrix} 1 & 0 & 0.77\cos(7.49) & 0 \\ 0 & 1 & 0.77\sin(7.49) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0.76 & 0 \\ 0 & 1 & 0.1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.

The original distance between the points is 3.

For 1:

The two points project to the same point, and thus the distance between them is 0.

For 2:

As the z coordinate of the two points after the projection is 0, and the difference is only in the x coordinates, where their z coordinate is being added to it. As the z coordinate in the second point is twice that of the first, the distance will be twice the original, 6.

For 3:

α represents the difference between the projection and the orthographic projection, thus we need to multiply the original distance by α and get $0.56 \cdot 3 = 1.68$

For 4:

Same as above, $0.77 \cdot 3 = 2.31$

Question 4

a. We project (perspective) with C.O.P=$(0,0,0)$ and viewing plane at $z = -2$, a scene containing the line $l = (3,2,-8) + t(2,2,0)$. What is l_p, the projected line?
b. A scene contains a billboard of a tree embedded in $z = k$, $k < -5$, whose height is h. We project the scene (perspective) with $C.O.P$=$(0,0,0)$ and a viewing plane at $z = -5$. What is the height of the projected tree?
c. We project (perspective) a scene with C.O.P = (0,0,0) and a viewing plane at $z = -4$. We know that two intersecting lines in our scenes, l_1, l_2, have two different vanishing points, $(2,-1,-4), (1,3,-4)$ accordingly. What is the angle θ between l_1 and l_2?
d. Find the matrix representing the perspective projection where the C.O.P=$(2,-1,4)$ and a projection plane with implicit representation $-x + 2y + 2z = 0$.

Solution

a. The direction remains the same, as the line is parallel to the viewing plane. $x_p = x \cdot \frac{-2}{-8} = \frac{3}{4}$ and $y_p = y \cdot \frac{-2}{-8} = \frac{1}{2}$. Thus, $l_p = \left(\frac{3}{4}, \frac{1}{2}, -2\right) + t(2,2,0)$.
b. Each point on the plane is multiplied by $-\frac{f}{k}$, so the height of the tree is multiplied by $-\frac{f}{k}$ and so the height of the projected tree is $h \cdot -\frac{f}{k}$.
c. Denote $l_1 = p_1 + tv_1$ and $l_2 = p_2 + tv_2$. Their vanishing points are the intersections of the lines tv_1 and tv_2 respectively with the viewing plane. We can the find that $v_1 = \frac{(2,-1,-4)}{||(2,-1,-4)||}$ and $v_2 = \frac{(1,3,-4)}{||(1,3,-4)||}$.

   The angle between the line is thus $\cos\theta = v_1 \cdot v_2 = \ 0.6419$

$$\theta = \cos^{-1}\ 0.6419 = \ 50.06331°$$

d. First, we would find the distance between the C.O.P and the plane.

$$\frac{-2 + 2 \cdot (-1) + 2 \cdot 4}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{4}{\sqrt{9}} > 0$$

   This also shows that the point is in the same direction as the normal.
   We translate the C.O.P to the origin.

$$T_1 = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

   The plane is translated to $-(x + 2) + 2(y - 1) + 2(z + 4) = 0$
   We now align the normal of the plane $v_{towards} = \frac{1}{\sqrt{9}}(-1,2,2)$, with the z axis.

$$v_{up} = \frac{v_{towards} \times (1,0,0)}{\|v_{towards} \times (1,0,0)\|} = \frac{1}{\sqrt{8}}(0,2,-2)$$

$$v_{right} = \frac{v_{towards} \times v_{up}}{\left\|v_{towards} \times v_{up}\right\|} = \frac{1}{\sqrt{18}}(-4,-1,-1)$$

Thus, we can use the following matrix to align the vectors with the axes.

$$T_2 = \begin{bmatrix} v_{right}^x & v_{right}^y & v_{righ}^z & 0 \\ v_{up}^x & v_{up}^y & v_{up}^z & 0 \\ v_{towards}^x & v_{towards}^y & v_{towards}^z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2^{-1} = \begin{bmatrix} v_{right}^x & v_{up}^x & v_{towards}^x & 0 \\ v_{right}^y & v_{up}^y & v_{towards}^y & 0 \\ v_{righ}^z & v_{up}^z & v_{towards}^z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We have already found that $f = \frac{4}{\sqrt{9}}$. Note that we have only applied rigid transformation, thus the distance is maintained. The projection matrix

$$T_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{\sqrt{9}}{4} & 0 \end{bmatrix}$$

The projection in total is therefore:

$$T_1^{-1}T_2^{-1}T_3T_2T_1$$