ATHENS

PROGRAMME

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Single Particle Motion

Nonuniform and constant electromagnetic field

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In this notebook we analyse the movement of individual particles under nonuniform electromagnetic fields. We deduce the expressions for the grad-B drift, curvature drift and analise the effect of a gradient along the B field direction. We show that the magnetic moment is invariant and we analyse the application of these effects on magnetic mirrors. Finally we study a non uniform E field.

1 Introduction

In this notebook we will need two of the Maxwell equations in vacuum (in SI units):

$$\nabla \times \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{j} \qquad \text{(Ampere's law)}$$

$$\nabla \cdot \vec{B} = 0 \qquad \text{(Gauss's law for magnetism)}$$

and, off course, we the force equation,

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \qquad \text{(Lorentz force)}$$

it is also convenient to write the divergence of a vector in cylindrical coordinates:

$$\nabla \cdot \vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{\partial A_\theta}{r\partial \theta} + \frac{\partial A_z}{\partial z}$$

We will use also the Gauss theorem,

$$\int_V \nabla \cdot \vec{A}\, dV = \oint_S \vec{A} \cdot \vec{n}\, dS,$$

and also the concept of magnetic moment, μ, which, for a closed loop of area A and current I, has the value $\mu = IA$.

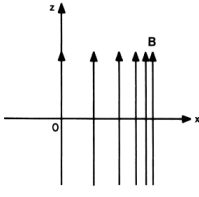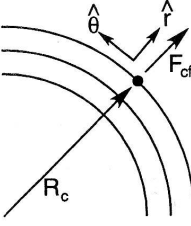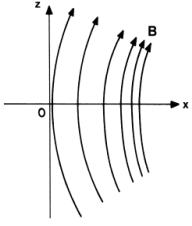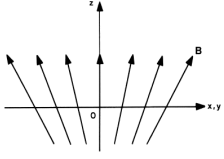The subject of this notebook is covered in the bibliography in the following chapters:

- Chen[1]: chapter Two, section 2.3
- Nicholson[2]: chapter 2, section 2.3, 2.4 and 2.6
- Bittencourt[3]: chapter 3
- Goldston[4]: chapter 3

The examples are prepared with the help of two scientific software packages, *Scipy*[5] and *IPython*[6].

2 Type of nonuniformities

We start by studying four types of space changes in $\vec{B}$:

Types of nonuniform magnetic field

Grad B	Curvature	Grad B and curvature	Divergence B
1) $\partial B_z/\partial x$, $\partial B_z/\partial y$	2) $\partial B_x/\partial z$, $\partial B_y/\partial z$	1)+2)	3) $\partial B_i/\partial i$, $i = x, y, z$

3 $\nabla\vec{B} \perp \vec{B}$: Grad B drift

Important: *Assumption* :$r_L \ll B/|\nabla B|$ and

$$\vec{B} = \vec{B}_0 + (\vec{r} \cdot \nabla)\vec{B}_0 + \cdots$$

To simplify, we consider that **B** varies only with y:

$$\vec{B} = B_{gc,0}\vec{u}_z + (y - y_{gc})\frac{dB}{dy}\vec{u}_z$$

Note 1: The components of $\vec{B}$ along x or y are still 0!

Note 2: according to Maxwell equations, as $\nabla \times \vec{B} \neq \vec{0}$, to have this field we need distributed volume currents...

From the Lorentz equation (with $\vec{E} = 0$), on average on the gyroperiods, $\overline{F_x} = 0$ as, in the y direction, the particles take as much time going back and forward. For the y component, we have $F_y = -qv_x B_z(y)$. using the results of the previous notebook for v_x and $y - y_0$:

$$F_y = -qv_\perp \cos(\omega_c t)\left[B_{gc,0} \pm r_L \cos(\omega_c t)\frac{\partial B_z}{\partial y}\right]$$

Averaging in a gyroperiod, the first term becomes zero and we obtain

$$\overline{F_y} = \mp\frac{1}{2}qv_\perp r_L\frac{\partial B_z}{\partial y}$$

(*Note*: $\overline{\cos^2(\omega_c t)} = 1/2$).

This force is responsible for a drift of the guiding center. Using the result from the previous notebook for the drift from a general force, we have

$$\vec{v}_{grad} = \mp \frac{1}{2} \frac{v_\perp r_L}{B_{gc}} \frac{\partial B}{\partial y} \vec{u}_x$$

Finally, generalizing for a gradient $\nabla B \perp \vec{B}$, we reach at the result

$$\vec{v}_{\nabla B} = \pm \frac{1}{2} v_\perp r_L \frac{\vec{B} \times \nabla B}{B^2}$$

3.1 Practice:

Let's see how important is the assumption $r_L \ll B/|\nabla B|$. We can compute the trajectories for an arbitrary value of $\partial B_z/\partial y$ and see what happens... Besides, we don't need to be limit to positive values and can include also a $\partial B_z/\partial x$ gradient.

We start by importing some libraries and define some common values as we have done in the previous notebook. Besides, the only thing different from the case analysed on the first notebook is the magnetic field. We can use the same integration routine and similar plotting functions that we have moved to an external file, *trajectories*.

```
In [1]: %matplotlib inline
        import numpy as np
        from IPython.html.widgets import interact

        from trajectories import *
```

Now we need to include the space dependency of the magnetic field to compute the acceleration in each point. And we make the code interactive to test different values of gradient.

```
In [2]: def gradB(Q, t, qbym, E0, B0, keywords):
            """Equations of movement for a grad-B magnetic field.

            Positional arguments:
            Q -- 6-dimension array with (x,y,z) values of position and velocity on t-
            t -- next time (not used here but passed by odeint)
            qbym -- q/m
            E0, B0 -- arrays with electric and magnetic field values
            Keyword arguments:
            dBdx, dBdy -- The gradient values in x and y, respectively.

            Return value:
            Array with dr/dt and dv/dt values."""

            gradx, grady = "grad" in keywords.keys() and keywords["grad"] or [0, 0]
            x, y = Q[:2]
            B = B0*np.array([1,1,1+x*gradx+y*grady])

            v = Q[3:]                              # Velocity
            dvdt = qbym*np.cross(v,B)              # Acceleration
```
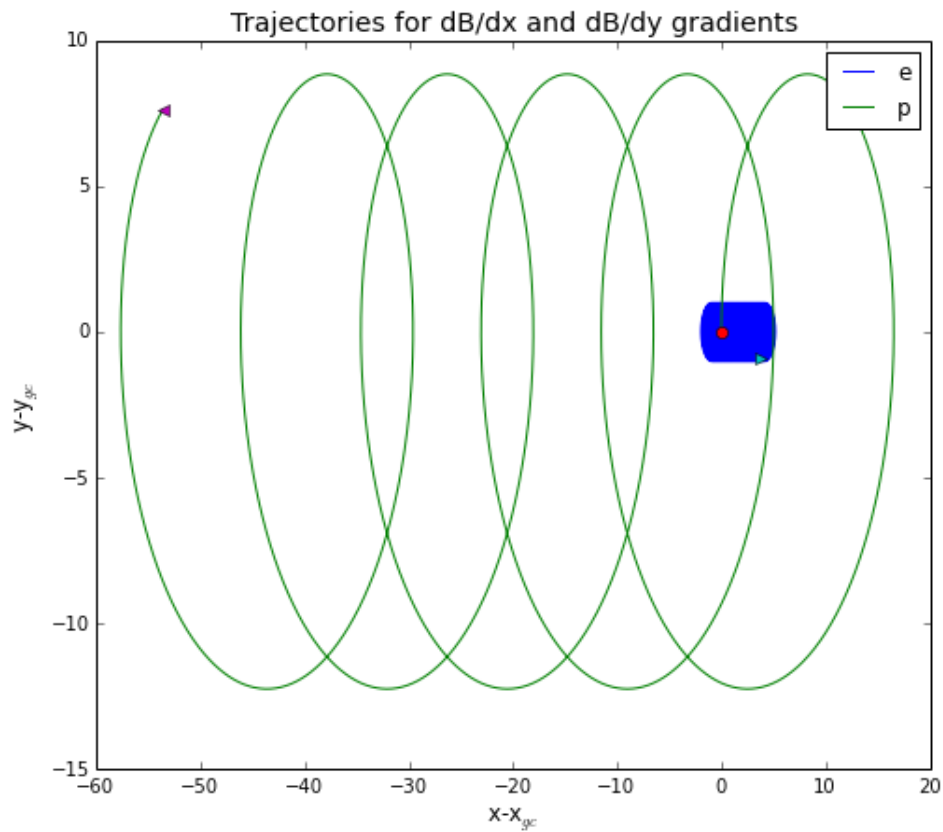
```
        return np.concatenate((v,dvdt))

    def gradBdrift(gradx=0.0, grady=0.0):
        """Movement under a grad B perpendicular to B"""
        re, rp = computeTrajectories(gradB, grad=[gradx/10,grady/10]) # NOTE the /
        plotGradB(re,rp)

    dummy = interact(gradBdrift, gradx=(-2.5,2.5), grady=(-2.5,2.5))
```


Trajectories for dB/dx and dB/dy gradients

- Electrons and positive ions drift in oposite directions ⇒ **net current**!

4 Curvature drift

Assumption : B field lines locally curved (but constant) with radius of curvature $\vec{R}_c$.

Achieved also with volume currents.

Centrifugal force in the radial direction:

$$\vec{F}_{cf} = \frac{mv_\parallel^2}{R_c}\vec{u}_r = mv_\parallel^2 \frac{\vec{R}_c}{R_c^2}$$

and using the equation above for a generic force:

$$\vec{v}_{curv} = \frac{mv_\parallel^2}{qB^2}\frac{\vec{R}_c \times \vec{B}}{R_c^2}$$

In 'vacuum fields' (without volume currents) B must fall in the perpendicular direction as $|B| \propto 1/R_c$ and $\nabla|B|/B = -\vec{R}_c/R_c^2$. Thus,

$$\vec{v}_{curv} = \pm\frac{v_\parallel^2}{\omega_c}\frac{\vec{B}\times\nabla B}{B^2}$$

5 Total drift

- The curvature and grad drift add;
- In opposite directions for charges of opposite signs;
- Proportional to the particle energy.

$$\vec{v}_d = \left(\frac{1}{2}mv_\perp^2 + mv_\parallel^2\right)\frac{1}{qB^2}\frac{\vec{R}_c\times\vec{B}}{R_c^2}$$

- For a Maxwellian isotropic distribution both terms give the same contribution.

6 Gradient along B

Let us analyse the case where the magnetic field increases along his direction, i.e. for $\vec{B} = B_z\vec{u}_z$, we have $\partial B_z/\partial z > 0$.

As we must have $\nabla\cdot\vec{B} = 0$, (assuming, to simplify, that $B_\theta = 0$) then $\frac{1}{r}\frac{\partial}{\partial r}(rB_r) < 0$.

Assuming again that $r_L \ll B/|\nabla B|$, we can compute an *average* value for $\overline{B_r}$. Taking a small cylindrical volume centered along a magnetic field line, integrating and using the Gauss theorem, we can write $\pi(\delta r)^2\delta l(dB/dz) + 2\pi\delta r\delta l\overline{B_r} = 0$ and

$$\overline{B_r} = -\frac{\delta r}{2}\frac{dB}{dz}.$$

Now, if δr is the Larmor radius, it is this field that intervenes in the Lorentz force. Taking the time average over the gyro-period, we obtain

$$\overline{F_\parallel} = -\frac{|q|v_\perp^2}{2\omega_c}\frac{dB}{dz} = -\frac{W_\perp}{B}\frac{dB}{dz},$$

with $W_\perp = \frac{v_\perp^2}{2m}$.

- We have a force in the direction opposite to the field gradient for **both** positive and negative charges.

6.1 Magnetic moment

It is time to introduce the magnetic moment, μ, of the girating particle. The magnetic moment is defined as a vector relating the torque, τ, on the object from an externally applied magnetic field to the field vector itself:

$$\vec{\tau} = \vec{\mu}\times\vec{B},$$

with intensity $\mu = IA$ where I is the current loop covering area A.

We can easily compute his value for the gyrating particle: The particle covers an area $A = \pi r_L^2 = \pi v_\perp^2/\omega_c^2$ and represents a current $I = |q|\omega_c/(2\pi)$. I.e.

$$\boxed{\mu = \frac{|q|v_\perp^2}{2\omega_c} = \frac{mv_\perp^2}{2B} = \frac{W_\perp}{B}}.$$

Now, to see the importance of the magnetic moment, let's look at the conservation laws for this case:

- Angular momentum: As the force $\overline{F_\parallel}$ is constant, the angular momemtum, $\vec{L} = m\vec{r}_L \times \vec{v}$ is constant. Note that $\vec{L} = mr_L v_\perp \vec{u}_\parallel$;
- Kinetic energy: In the presence of a static magnetic field, the total kinetic energy must be conserved.

Both laws imply that the μ is a constant of motion as we will show:

- Conservation of the angular momentum: $\vec{L} = mr_L v_\perp \vec{u}_\parallel \equiv (2m/|q|)\vec{\mu}$ as $d\vec{L}/dt = 0 \Rightarrow \vec{\mu}$ is constant;
- Conservation of the kinetic energy: We start by using the above result we write

$$m\frac{dv_\parallel}{dt} = -\mu\frac{dB}{ds},$$

where we have parameterized the distance along the field line, s. Multiplying both sides by $ds/dt = v_\parallel$, we obtain

$$mv_\parallel\frac{dv_\parallel}{dt} = -\mu\frac{dB}{dt}.$$

For the total kinetic energy, we write

$$\frac{d}{dt}\left(\frac{mv_\parallel^2}{2} + \frac{mv_\perp^2}{2}\right) = \frac{d}{dt}\left(\frac{mv_\parallel^2}{2} + \mu B\right) = 0$$

Using the above result we have

$$-\mu\frac{dB}{dt} + \frac{d}{dt}(\mu B) = 0 \Rightarrow \frac{d\mu}{dt} = 0.$$

- μ is an invariant!
- $v_\perp$ has to increase when the particle moves to higher B regions;
- $v_\parallel$ decreases as $v_\perp$ increases.

6.2 Magnetic mirrors

This effect is used to confine plasmas. Let's supose that we have the a magnetic field with the following configuration of field lines: The parallel velocity of a particle with energy W and magnetic moment μ has to obey the equation

$$\frac{mv_\parallel^2}{2} = W - \mu B(z)$$

and will have a reflection point at $z_{max} : B(z_{max}) = W/\mu$.

Are all particles trapped?

Let as consider a particle moving around a field line with and minimum value, B_{min}, in the midplane, m, and B_{max} at the mirror throat. The limiting conditions to trap particles are (using the result above):

$$W_\perp|_m = \mu B_{min} = W B_{min}/B_{max} \tag{1}$$
$$W_\parallel|_m = W(1 - B_{min}/B_{max}) \tag{2}$$

Particles with higher value of $W_\parallel|_m/W$ can escape the trap! This condition defines a 'loss cone'.

- Exercise: Show that the equation for the 'loss cone' is

$$\frac{|v_{\parallel}|}{|v_{\perp}|} = \left(\frac{B_{max}}{B_{min}} - 1\right)^{1/2}$$

and this defines a critical angle.

6.3 Practice:

Note: The following code is still under development!

In [10]:
```python
#%matplotlib qt4
%matplotlib inline

def divB(Q, t, qbym, E0, B0, keywords):
    """Equations of movement for a dBz/dz gradient"""
    global eMu, pMu, zMax

    gamma = "gm" in keywords.keys() and keywords["gm"] or 0
    z = Q[2]; v = Q[3:]                          # Position and velocity
    vperp = np.sqrt(v[0]**2+v[1]**2)             # Perpendicular|B velocity
    if np.abs(z) <= zMax:
        Bz = B0[2]*(1+gamma*z**2) #*np.abs(qbym)
        omega_c = qbym*Bz                        # Cyclotron frequency (we keep
        rL = vperp/np.abs(omega_c)               # Larmor radius
        Br = -rL*B0[2]*gamma*np.abs(z)
        theta = omega_c*t
        B = np.array([Br*np.cos(theta),-Br*np.sin(theta),Bz])
        dvdt = qbym*np.cross(v,B)                # Acceleration
        if np.sign(qbym) == -1:
            #dvdt[2] = mue/qbym*2*B0[2]*gamma*z
            dvdt[2] = - eMu*2*B0[2]*gamma*z
        else:
            #dvdt[2] = - mup/qbym*2*B0[2]*gamma*z
            dvdt[2] = - pMu*2*B0[2]*gamma*z
    else:
        dvdt = np.zeros(3)
    return np.concatenate((v,dvdt))

def mirrorB(Bratio=1.5, escape=0.99, gamma=1e-4):
    """Movement with a grad B parallel to B"""
    global eMu, pMu, zMax
    # Initial values
    if escape == 0:
        vperp0 = 1; vpar0 = 0
    else:
        vratio = np.sqrt(Bratio-1)*escape
        vpar0 = 1; vperp0 = vpar0/vratio

    # Initial values:
    rLe0 = vperp0/(q/me*B0[2])          # e-Larmor radius
```

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```
            rLp0 = rLe0*Mp/me                   # p-Larmor radius
            re0 = np.array([rLe0,0,0])          # e-Position
            rp0 = np.array([-rLp0,0,0])         # p-Position
            v0 = np.array([0,vperp0,vpar0])     # Velocity
            eWperp0 = me*vperp0**2/2            # Perpendicular kinetic energy
            eW = eWperp0 + me/2*vpar0**2        # e-Kinetic energy
            pW = eW*Mp/me                       # p-Kinetic energy

            eMu = eWperp0/B0[2]; pMu = eMu*Mp/me   # Magnetic moment
            BMax = Bratio*B0[2]                     # Max. magnetic field
            Bconf = eW/eMu                       # Mag. field for v|| = 0
            zMax = np.sqrt((Bratio-1)/gamma)

            re, rp = computeTrajectories(divB, ri=[re0,rp0], vi=v0, gm=gamma)

            plotMirror(re0,rp0,re,rp,zMax)

            print('Magnetic moment  Kinetic energy  B_Max    B_conf  \
        zM_conf\n   eMu = {:.3f}        We = {:.3f}       {:.3f}     {:.3f} \
        {:.3f}'.format(eMu,eW,BMax,Bconf,zMax))
            print('  pMu = {:.3f}       Wp = {:.3f}'.format(pMu,pW))

        mirrorB()
```
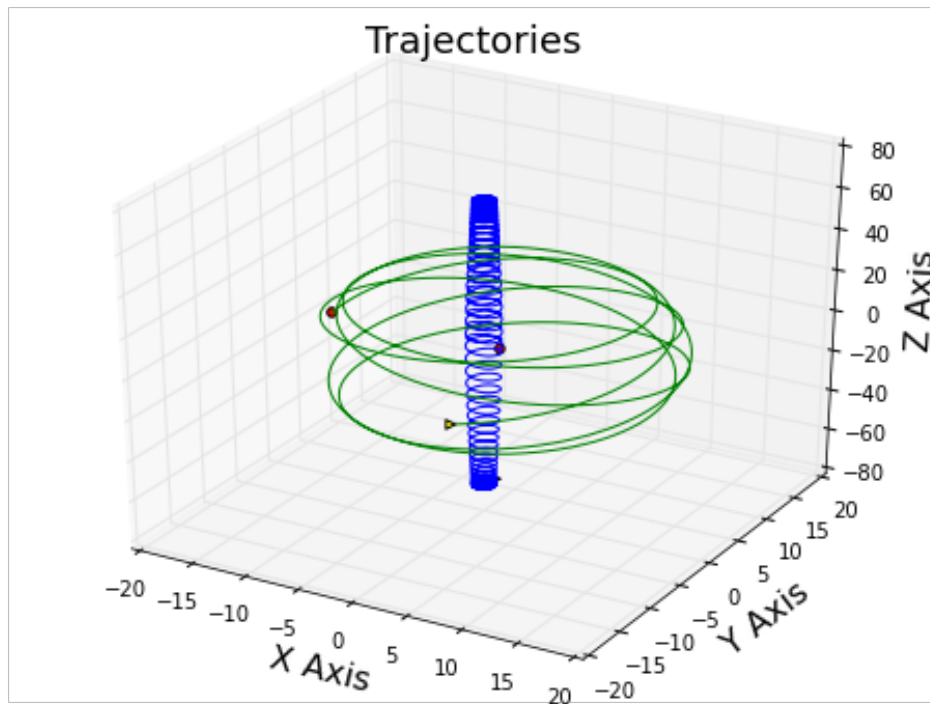
```
Magnetic moment  Kinetic energy  B_Max    B_conf  zM_conf
   eMu = 1.020        We = 1.520       1.500     1.490  70.711
   pMu = 10.203        Wp = 15.203
```

6.4 Applications

6.4.1 In plasma machines

6.4.2 In Nature

1 - Cyclotronic motion; 2 - Mirror effect; 3 - Curvature drift.
   New effect discovered in September 2012! - the Van Allen belts:

7 Nonuniform $\vec{E}$ field

To finish this notebook we still have to look at the case of a nonuniform electric field. We consider again a B field along zz. To simplify, let us assume a sinusoidal variation in the x direction:
$$\vec{E} = E_0 \cos(kx)\vec{u}_x,$$
and we take k such that the wavelength $\lambda = 2\pi/k$ is large comparing with r_L.
   From the Lorentz force equation we have

$$\dot{v}_x = \frac{q}{m}\left(E_x(x) + v_y B\right) \tag{3}$$

$$\dot{v}_y = -\frac{qB}{m}v_x \tag{4}$$

$$\tag{5}$$

and, we focus our attention in the equation for $\ddot{v}_y$:

$$\ddot{v}_y = -\omega_c^2 v_y - \omega_c^2 \frac{E_x(x)}{B}.$$

To solve this equation we need to know the field at x but this depends on the orbit equation that we are trying to obtain... That's when our approximation is needed: We use the *undisturbed orbit*, $x = x_0 + r_L \sin\omega_c t$.

   As before, we expect to have a drift, v_E, superimposed to a gyration movement. As we have done before, to find v_E we average on the gyroperiods to obtain $\bar{\bar{v}}_{x,y} = 0$. As this implies $\overline{v}_x = 0$ (*Question*: why?) we only have a drift along y.

   To reach to a value for v_E we need to average the $\overline{\cos(x_0 + r_L \sin\omega_c t)}$ term. After expanding the cosine, we need again our assumption, $kr_L \ll 1$ to use a Taylor expansion. After some mathematics we obtain

$$\overline{v}_y = -\frac{E_x(x_0)}{B}\left(1 - \frac{1}{4}k^2 r_L^2\right),$$

where we have put $E_0 \cos(kx_0) = E_x(x_0)$.
   Or, in vector form,

$$\vec{v}_E = \frac{\vec{E}\times\vec{B}}{B^2}\left(1 - \frac{1}{4}k^2 r_L^2\right).$$

This result introduces a correction from the inhomogeneity on our previous result for the $\vec{E}\times\vec{B}$ drift.

   For an arbitrary variation of $\vec{E}$, this term has the value

$$\vec{v}_E = \left(1 + \frac{1}{4}r_L^2 \nabla^2\right)\frac{\vec{E}\times\vec{B}}{B^2}.$$

Electrons and ions have different Larmor radius $\Rightarrow$ charge separation $\Rightarrow$ possibility of plasma instabilities (*drift instability*).

References

[1] Francis Chen. *Single-particle Motions*, chapter 2, pages 19–52. Plenum, 1974.

[2] Dwight R. Nicholson. *Single Particle Motion*, chapter 2, pages 17–36. Wiley series in plasma physics. John Wiley & Sons, 1983.

[3] J. A. Bittencourt. *Charged particle motion in constant and uniform electromagnetic fields*, chapter 2, 3, pages 33–89. Springer, 2004.

[4] R. L. Goldston and P. H. Rutherford. *Single-particle Motion*, pages 21–68. IOP Publishing, Bristol and Philadelphia, 1995.

[5] Eric Jones, Travis Oliphant, Pearu Peterson, et al. SciPy: Open source scientific tools for Python, 2001–. [Online].

[6] Fernando Pérez and Brian E. Granger. IPython: a system for interactive scientific computing. *Computing in Science and Engineering*, 9(3):21–29, May 2007.