A Private Approximation of the 2nd-Moment Matrix of Any Subsamplable Input

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Introduction

Estimating the second moment matrix is central to statistics and machine learning. We focus on achieving differential privacy guarantees even for worst-case inputs under a new assumption: **subsamplability**.

Subsamplability Assumption

A dataset X is (m, α, β)-subsamplable if a random subsample of $m' \geq m$ points yields a spectral approximation of the second moment:

$\Pr[(1-\alpha)\Sigma \preceq \hat{\Sigma} \preceq (1+\alpha)\Sigma] \geq 1-\beta.$

This property enables robust DP estimation with only minimal assumptions.

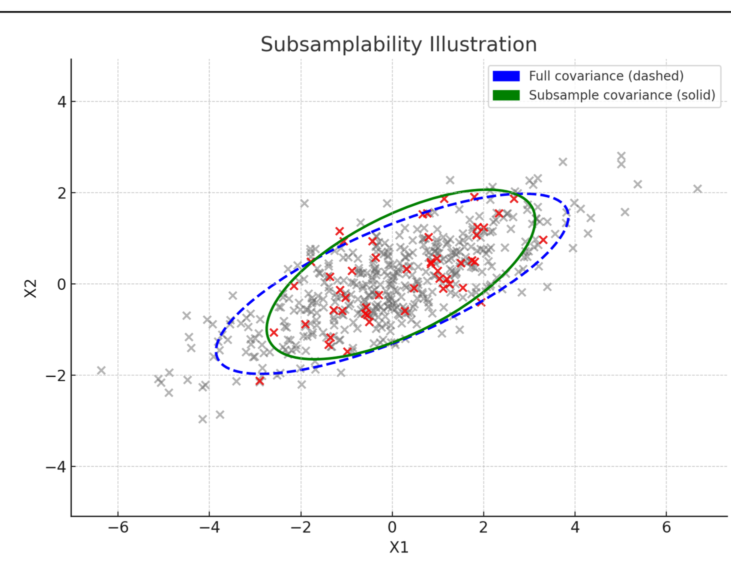


Figure 1. Subsamplability Illustration

Our Contributions

We Present a new algorithm that achieve strong privacy-utility trade-offs even for worst-case inputs under subsamplability assumptions on the data.

Building upon *subsamplability*, we give a recursive algorithmic framework similar to Kamath et al. [3] that abides zero-Concentrated Differential Privacy (zCDP) while preserving w.h.p the accuracy of the second moment estimation upto an arbitrary factor of $(1 \pm \gamma)$.

We then show how to apply our algorithm to approximate the second moment matrix of a distribution $\mathcal{D}$, even when a noticeable *fraction* of the input are outliers.

Guarantees

Let $P_{\text{tail}} = \left\{x \in X : \ \exists u \in \mathbb{R}^d : \ \langle x, u\rangle^2 > m(1+\alpha) \cdot \frac{1}{n}\Sigma_{x\in X}\langle x, u\rangle^2\right\}$.

If the input is (m, α, β)-subsamplable with $\beta = O(\alpha/\log(R))$ and $\alpha \leq 1/2$, then w.h.p.:

$$(1-\gamma)\Sigma_{\text{eff}} \preceq \tilde{\Sigma} \preceq (1+\gamma)\Sigma$$

where $\Sigma_{\text{eff}} = \frac{1}{n}\Sigma_{x\in X\setminus P_{\text{tail}}} xx^T$.

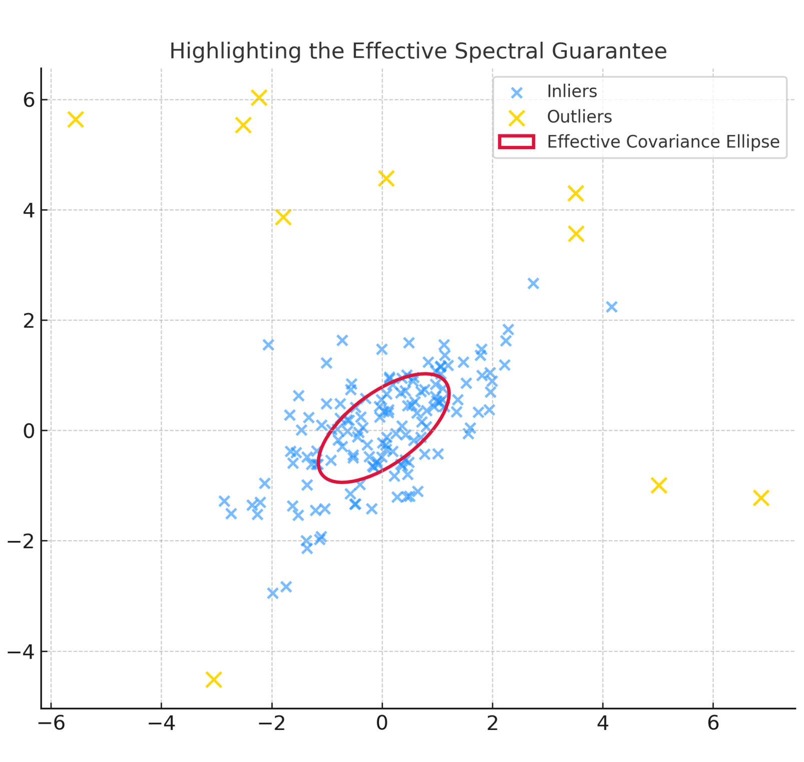
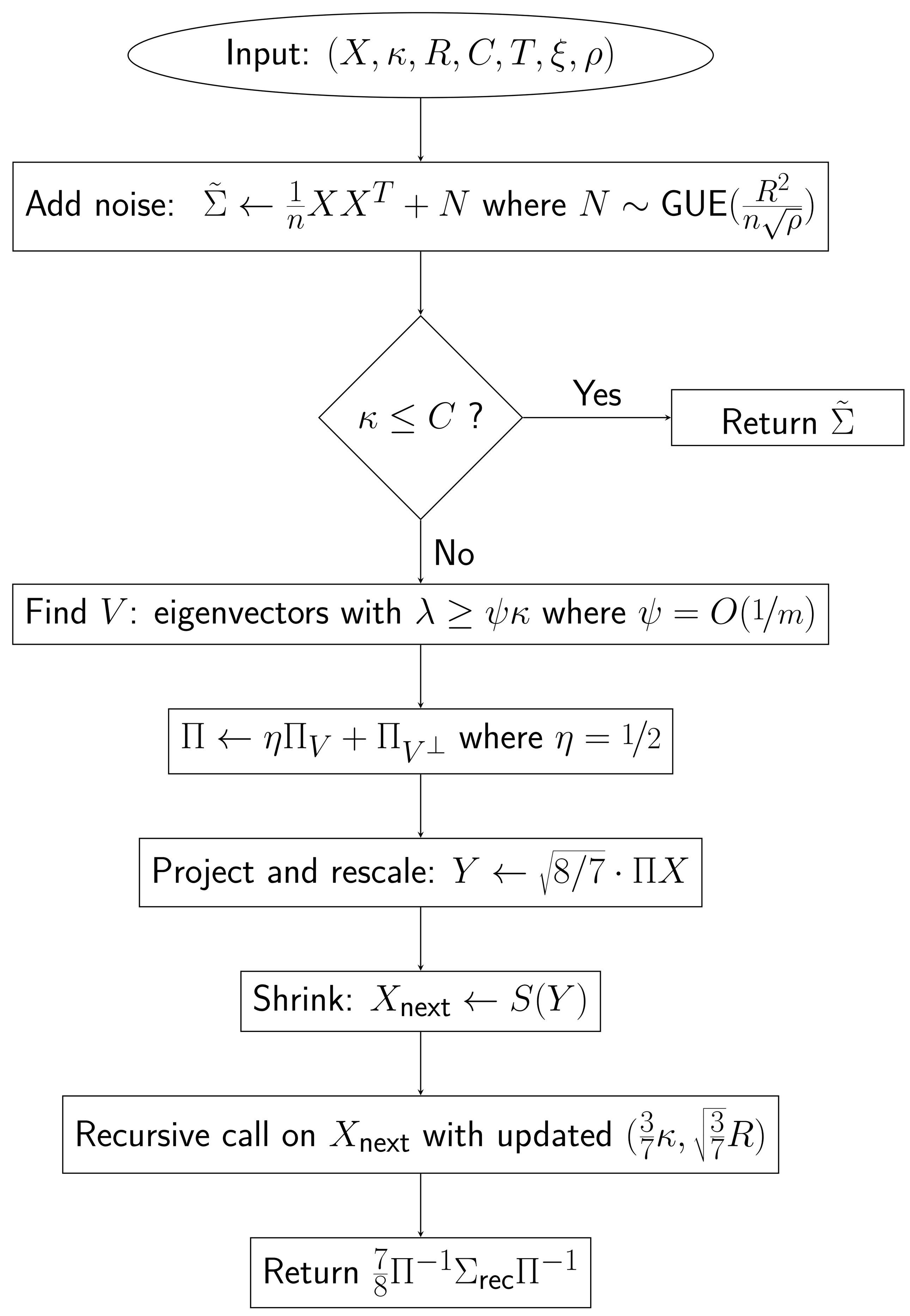


Figure 2. Highlighting the Effective Spectral Guarantee

Comparison to Prior Work

- **Brown et al. [2]:** Require that *every* data point has bounded leverage. Their algorithm fails or degrades under even a small number of high-leverage outliers.
- **Previous Work:** Prior algorithms rely on strong distributional assumptions (e.g., Gaussians, sub-Gaussian tails) or on restrictive bounded-norm constraints.
- **Beyond Subsample and Aggregate:** Our algorithm improves over classical frameworks like Subsample-and-Aggregate, which tolerate far fewer outliers and require smaller contamination rates ($\eta = \tilde{O}(\gamma^2/d)$ vs. ours: $\eta = \tilde{O}(1/d)$).

Main Algorithm



Applications

We study the problem of estimating the second moment matrix of a distribution $\mathcal{D}$, where the input is corrupted by an η-fraction of arbitrary outliers. Consider $\mathcal{D}$ to be a distribution that for any $\alpha, \beta > 0$ is $m(\alpha, \beta)$-subsamplable for $m = O(\frac{d\ln(d/\beta)}{\alpha^2})$. We thus denote the second moment matrix of the input as $\Sigma = (1-\eta)\Sigma_{\mathcal{D}} + \eta\Sigma_{\text{out}}$.

- **Our Guarantee:** Our method succeeds if a small random subsample is a good spectral approximation. It tolerates many high-leverage outliers so long as $\eta = \tilde{O}(1/d)$ and $\Sigma_{\text{out}} \preceq O(1/\eta)\Sigma_{\mathcal{D}}$. Under these conditions, our DP algorithm returns $\tilde{\Sigma} \succeq (1 - O(\gamma))\Sigma_{\mathcal{D}}$ w.p. $\geq 1 - \xi$.
- **Sample Complexity:** baseline: $\tilde{O}(\frac{d^2}{\gamma^3\sqrt{\rho}})$ vs. ours: $\tilde{O}(\frac{d}{\gamma^2} + \frac{d^{3/2}}{\gamma\sqrt{\rho}})$

Want to know more?

https://arxiv.org/abs/2505.14251

References

[1] Hassan Ashtiani and Christopher Liaw. Private and polynomial time algorithms for learning gaussians and beyond. In Po-Ling Loh and Maxim Raginsky, editors, *Proceedings of Thirty Fifth Conference on Learning Theory*, volume 178 of *Proceedings of Machine Learning Research*, pages 1075–1076. PMLR, 02–05 Jul 2022. URL `https://proceedings.mlr.press/v178/ashtiani22a.html`.

[2] Gavin Brown, Samuel Hopkins, and Adam Smith. Fast, sample-efficient, affine-invariant private mean and covariance estimation for subgaussian distributions. In Gergely Neu and Lorenzo Rosasco, editors, *Proceedings of Thirty Sixth Conference on Learning Theory*, volume 195 of *Proceedings of Machine Learning Research*, pages 5578–5579. PMLR, 12–15 Jul 2023. URL `https://proceedings.mlr.press/v195/brown23a.html`.

[3] Gautam Kamath, Jerry Li, Vikrant Singhal, and Jonathan Ullman. Privately learning high-dimensional distributions. In Alina Beygelzimer and Daniel Hsu, editors, *Proceedings of the Thirty-Second Conference on Learning Theory*, volume 99 of *Proceedings of Machine Learning Research*, pages 1853–1902. PMLR, 25–28 Jun 2019. URL `https://proceedings.mlr.press/v99/kamath19a.html`.